

To the Graduate Council:

I am submitting herewith a dissertation written by Sathyanarayanan Rengaswami entitled "The Geometry of Ancient Solutions to Curvature Flows." I have examined the final paper copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Theodora Bourni, Major Professor

Mat Langford, Major Professor

We have read this dissertation
and recommend its acceptance:

Vasileios Maroulas

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Accepted for the Council:

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(Original signatures are on file with official student records.)

The Geometry of Ancient Solutions to Curvature Flows

A Dissertation Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Sathyanarayanan Rengaswami

August 2024

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To my father Rengaswami and mother Chandra for their unwavering belief and support in my explorations; to my brother, cousins, aunts, uncles for their encouragement; to my friends for their support; and to the young ones in the family who will go on to do more impressive things.

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Where there is a will, there is a way...

Abstract

Following the tremendous success of the mean curvature flow, other variants such as the Gauss curvature flow, inverse mean curvature flow have been investigated in great detail, leading to interesting applications to other fields including partial differential equations, convex geometry etc. This calls for an investigation of curvature flow as a general phenomenon. While basic existence and uniqueness results, roundness estimates etc have been obtained, there isn't a substantial body of work that addresses the geometry of solutions of curvature flows and their relation to the choice of speed function used. It is therefore interesting to investigate curvature flows as a general phenomenon from a more abstract and axiomatic point of view.

In this volume, we take such an abstract point of view, and study a certain important class of solutions called *ancient solutions*; in particular, we study translating solutions and the so-called “pancake” solutions. We provide a detailed construction of these solutions, and prove their uniqueness as well. We also provide a description of their geometry. The advantage of such an abstract approach is that we can see the dependence of the solution's geometry on basic properties of the speed function, often just algebraic properties.

Finally, we also take a discrete point of view to curvature, tackling some questions about the curvature of a network (i.e. graph) and its relation to the network topology. In particular, we are interested in community structure. A community in a graph, loosely speaking, is a subset C of nodes with many edges between points in C and few edges from C to the rest of the graph. An edge connecting members of two

different communities is called an intercommunity edge. We examine the Ollivier-Ricci curvature, which is defined on edges of graphs based on optimal transport theory. Using this, we study the relation between the curvature of intercommunity edges and the community structure. In particular, we quantify how the curvature of intercommunity edges is governed by the number of intercommunity edges.

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Chapter 1

Introduction

1.1 Geometric Flows

On an intuitive level, a *geometric flow* is a process by which a manifold changes shape over time. This is accomplished either by evolving the Riemannian metric, or by deforming it in some ambient space. This evolution is governed by a partial differential equation called its *evolution equation*. The fundamental tools of this subject are differential geometry and partial differential equations; there is also a lot of interplay between this discipline and the fields of convex geometry and measure theory.

The fundamental questions in this field are of a geometric nature, and concern the shape, size, and topology of an object and their relationship to its curvature. But since curvature is in some sense a ‘geometric’ second derivative, this field is highly intertwined with second order PDEs.

The goal is to define processes that make the shape “more round” by diffusing curvature away from high-curvature regions into lower curvature regions analogous to heat diffusion. Thus, for general evolution processes (at least the ones we are interested in anyway), the underlying geometric PDE is parabolic, and for objects with certain invariances (e.g. minimal submanifolds, submanifolds of prescribed

curvature, translating solitons, and self-similar shapes like expanders and shrinkers) the PDE reduces to an elliptic one.

Hamilton introduced the Ricci flow to attack the Poincaré conjecture (and more generally, Thurston’s Geometrization conjecture), and eventually this program was completed by Grigori Perelman in 2005 [Tao \(2006\)](#). The underlying idea here is that flowing an object by its curvature tends to diffuse its curvature, making it easier to study. The same philosophy underlies several other remarkable mathematical works, examples of which are proofs of the Riemannian Penrose conjecture (for surfaces) by [Huisken and Ilmanen \(2001\)](#) using the inverse mean curvature flow, the differentiable sphere theorem by [Brendle and Schoen \(2009, 2008\)](#), classification of pinched submanifolds [Andrews and Baker \(2010\)](#), Gromov-Andrews “twisted” 1/4-pinched differentiable sphere theorem [Andrews \(1994b\)](#). The tendency of these flows to make objects “more isoperimetric” has been understood and exploited in the proof of Alexandrov–Fenchel inequalities in [Andrews et al. \(2018\)](#)[Andrews et al. \(2019\)](#). In terms of modeling natural phenomena, the well-studied Gauss curvature flow models the wearing of rocks on the ocean floor [Andrews \(1999\)](#). In real-world applications, geometric flows are used in MRI enhancements, modification of color histograms of images, and image detection [Lasiacka and Morton \(1995\)](#); [Sapiro \(2006\)](#).

1.2 The Formal Definition

An extrinsic geometric flow is a solution \mathbf{X} to the partial differential equation

$$\frac{\partial \mathbf{X}}{\partial t} = -f\nu, \tag{1.1}$$

where \mathbf{X} is a one-parameter family $\mathbf{X} : M^n \times I \rightarrow \mathbb{R}^{n+1}$ of immersed hypersurfaces, ν denotes the (outward) unit normal field and the speed $f = f(\kappa_1, \dots, \kappa_n)$ is a function of the principal curvatures $\kappa_1 \leq \dots \leq \kappa_n$. We denote by $\Sigma_t = \mathbf{X}(M, t)$ the image of M at time t , and sometimes only use the family of images $\{\Sigma_t\}_{t \in I}$ to denote the flow.

While there are no restrictions apriori on the speed function f , it is often chosen to have certain desirable properties, which we summarize below:

Definition 1. A C^1 function $f : \Omega \subset \mathbb{R}^n \rightarrow [0, \infty)$ such that Ω contains $\Gamma^+ := \{x \in \mathbb{R}^n : z_i > 0, i = 1, \dots, n\}$, is an **admissible speed function** if

(i) f is invariant under permutation of its variables z_1, \dots, z_n . (*Symmetry*)

(ii) $\frac{\partial f}{\partial z_i} > 0$ for each $i = 1, \dots, n$. (*Ellipticity*)

(iii) f is α -homogeneous for some $\alpha > 0$ (*Homogeneity*), i.e.

$$f(c\vec{\kappa}) = c^\alpha f(\vec{\kappa}), \quad c > 0$$

This is a very large class of speeds. Examples include the mean curvature $\sum \kappa_i$ and Gauss curvature $\prod \kappa_i$, and for more exotic ones, the commonly studied class of flows by positive powers of one-homogeneous roots of ratios of elementary symmetric polynomials in the principal curvatures [Andrews \(2007\)](#). Note, moreover, that no concavity or regularity beyond C^1 are required.

Let us briefly discuss the purpose of the criteria in Definition 1. The first two conditions are “non-negotiable” in that symmetry is needed to ensure that f may be regarded as a smooth function of the second fundamental form II (which in turn ensures that the composition of f with the principal curvatures/second fundamental form is a smooth function on spacetime), while ellipticity is needed to ensure that (1.1) may be interpreted as a parabolic partial differential equation.

Homogeneity guarantees that (1.1) is invariant under the parabolic rescaling $\Sigma_t \mapsto \lambda \Sigma_{\lambda^{-2\alpha}t}$. Note that, under the milder condition of *asymptotic homogeneity*, meaning that the limit

$$(T\phi)(z) := \lim_{\lambda \rightarrow \infty} \lambda^{-\alpha} \phi(\lambda z)$$

exists, blow-up limits (with respect to the rescaling $\Sigma_t \mapsto \lambda \Sigma_{\lambda^{-2\alpha}t}$) to the flow (1.1) with speed f evolve by (1.1) with f replaced by the homogeneous speed Tf . So our results are relevant also to flows by certain asymptotically homogeneous speeds.

1.3 Special Solutions

The question of classifying all solutions to a geometric flow is intractable, and hence we focus on solutions that exhibit certain special properties. For instance, solutions with certain symmetries such as rotations and reflections are widely considered. Other solutions that evolve by translations (*translators*) and homotheties (*expanders*, *shrinkers*) have also been studied. Another class of interesting solutions is that of *ancient* solutions, i.e. ones that have existed for an infinite amount of time in the past. The understanding of such solutions has played a significant role in the understanding of singularity formations.

1.3.1 Ancient Solutions

Ancient solutions to geometric flows, i.e. solutions that are defined on a time interval of the form $(-\infty, T)$, $-\infty < T \leq \infty$, are fundamental to understanding the global behaviour of these flows, because they arise as limits of rescalings near singularities [Hamilton \(1995a\)](#). While the question of classifying all ancient solutions remains intractable, a great deal is known when we restrict our attention to solutions that satisfy some convexity criteria. For instance, it is known when $f = H$ (the mean curvature flow) that ancient solutions arising in this way in mean-convex mean curvature flows are necessarily convex [Huisken and Sinestrari \(1999a,b\)](#).

In the one-dimensional case of curve shortening flow, there is a complete classification of convex ancient solutions: if they are compact, they are either a family of shrinking circles or *Angenent ovals*, and if they are noncompact, they are either the so-called *grim reapers* or stationary lines. In higher dimensional mean curvature

flow, even though such a finite list has not been given, it is a theorem of Wang (2011) that convex ancient solutions essentially come in two types: they either sweep all of space, or are contained between two parallel stationary hyperplanes.

Thus we make the following definition:

Definition 2. *A solution to a geometric flow $\{\Sigma_t\}_{t \in I}$ is said to be **entire** if the flow “passes through” every point of \mathbb{R}^{n+1} , i.e. $\bigcup_{t \in I} \Sigma_t = \mathbb{R}^{n+1}$*

1.3.2 Translators

A special type of ancient solution is one that evolves under the f -flow by translation with a fixed velocity. More precisely, given a speed f , a translating solution to Equation (1.1), or a f -translator for short, is a solution of the form

$$F(x, t) = F_0(x) + tv.$$

(up to tangential reparametrizations.) Note that by symmetry condition in Definition 1, the f -flow is invariant under the isometries of ambient space and suitable spacetime rescalings. Thus there is no loss of generality in fixing the translation direction to be $v = e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$.

Importantly, translating solutions can be studied by the elliptic PDE points of view, since every time slice Σ_t satisfies the Equation

$$f(\vec{\kappa}) = -\langle \nu, e_{n+1} \rangle. \tag{1.2}$$

Recall that ν is the inward pointing unit normal of Σ_t in \mathbb{R}^{n+1} . In fact, in local coordinates Σ_t can be seen as a graph of a function for which Equation (1.2) correspond to a nonlinear elliptic PDE. Moreover, from the parabolic point of view, f -translators without boundary are examples of noncompact *eternal solutions* of

Equation (1.1), i.e.: solutions that are defined for all $t \in (-\infty, \infty)$, see [Torres-Santaella \(2023\)](#) for details.

Translating solutions arise in the analysis of singularities directly, as blow-up limits [Angenent and Velázquez \(1997\)](#); [Hamilton \(1995b\)](#), and also indirectly, in the sense that convex ancient solutions tend to decompose into configurations of asymptotic translators [Angenent et al. \(2020\)](#); [Bourni et al. \(2022b,c, 2021a, 2020\)](#); [Choi et al. \(2020\)](#). In some cases, it can be shown that translating blow-up limits are necessarily rotationally symmetric [Bourni and Langford \(2017\)](#); [Haslhofer \(2015\)](#).

1.3.3 ‘Pancake’ Solutions

Definition 3. A *pancake* solution is a convex ancient solution that is confined to a region between two parallel hyperplanes

Ancient solutions confined to a slab (the region between two parallel hyperplanes) are natural in view of the slab dichotomy of [Wang \(2011\)](#), which states, for convex ancient mean curvature flows $\{\partial\Omega_t\}_{t \in (-\infty, \omega)}$, that if the region $\cup_{t \in (-\infty, 0)} \Omega_t$ is a strict subset of \mathbb{R}^{n+1} , then it is a slab. When $n = 1$, an ancient solution which sweeps out a slab region in the plane was constructed by Bourni *et al.* in [Bourni et al. \(2022a\)](#) for flows by certain powers of the curvature. On the other hand, while it is known, for a large class of flows, that the shrinking spheres are the only “pinched” ancient solutions [Langford and Lynch \(2020\)](#); [Risa and Sinestrari \(2019\)](#), non-spherical $O(n)$ -invariant ancient solutions (to a somewhat different but nontrivially overlapping class of flows) which sweep out all of space were constructed by Risa and Sinestrari [Risa and Sinestrari \(2022\)](#) (see also [Lu and Zhou \(2021\)](#); [Risa \(2017\)](#)). When a suitable differential Harnack inequality is available (see [Andrews \(1994c\)](#)), these “ancient ovaloid” solutions decompose into a pair of entire “bowl-type” translating solitons joined by a shrinking cylinder. (Chapters 2,4 provide a comprehensive analysis of axially symmetric translating solitons for extrinsic geometric flows. See also [Urbas \(1998a\)](#).)

1.4 Ricci Curvatures of Graphs

Although the study of curvature originates in the study of differentiable manifolds, a great deal of work in the 20th century has shown that an analogous coherent theory exists in the realm of discrete spaces as well. Moreover, following the classical theme of the relation between the curvature of manifolds and their topology (the Gauss-Bonnet theorem, Myers' theorem etc.), there has been a recent interest in the search for such relations in the discrete realm. One such relation is the relation between graph curvature and community structure, resulting in an unsupervised algorithm that can cluster vertices belonging to the same community. This clustering procedure, known as *community detection*, has a lot of practical importance in the sciences, in fields including but not limited to computer science, chemistry, biology, and logistics (Grout and Cunningham (2006); Krishna et al. (1997); Rives and Galitski (2003); Spirin and Mirny (2003); Dunn et al. (2005); Queen et al. (2023); Guimera et al. (2005); O'Kelly (1992)).

One approach to measuring curvature of a graph is Ollivier's definition of Ricci curvature in Ollivier (2009) (which works more generally for Markov chains on metric spaces). Inspired by the success of the Ricci-flow-with-surgery in solving the Poincaré conjecture, a discrete Ricci-flow-with-surgery was proposed in Ni et al. (2019) for identifying (and deleting) intercommunity edges, which reveals the underlying community structure of the graph. We build on a result in this paper, deriving apriori estimates on curvature, and which reveals the relation between curvature and community structure.

1.5 Results

In the case of mean curvature flow, it has been shown in Clutterbuck et al. (2007) that there exists a unique 'bowl' soliton (up to isometries of ambient space and parabolic rescaling) that evolves by translation, it can be written as a graph over all of \mathbb{R}^n

and that it has the asymptotics of a paraboloid. On the other hand, flows by Gauss curvature, harmonic mean curvature etc result in solitons of finite radius that are asymptotic to a cylinder containing them (See Figure 1.1. All figures (including this one) and tables in this dissertation will be in the appendices of their respective chapters.) Under a suitable nondegeneracy condition on our admissible speeds, we show existence, uniqueness and asymptotics.

In addition, we provide criteria for an almost-complete classification of speed functions according to whether the corresponding ‘bowl’-type soliton is entire or contained in a cylinder.

The content of Chapter 3 is the other side of Wang’s dichotomy: the pancake solutions. In fact, Wang himself proved the existence of such pancake objects; the authors in [Bourni et al. \(2021b\)](#) provide a geometric construction of the rotationally symmetric pancake, and a precise description of its asymptotics, culminating in a uniqueness result. Again, under an appropriate nondegeneracy condition on our admissible speed, we prove something similar.

In Chapter 4 we return to the question of translating solitons. The context for this chapter is set in [Martín et al. \(2015\)](#), where the authors prove that the bowl soliton described in [Clutterbuck et al. \(2007\)](#) is unique in its asymptotic class. Thus, by refining the asymptotics we derived in Chapter 2, we prove an analogue of [Martín et al. \(2015\)](#) in the fully nonlinear case.

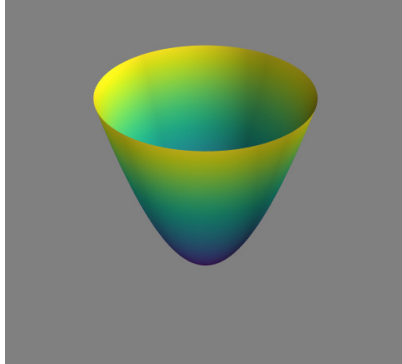
Moreover, we study higher order terms in the asymptotic expansions of several speed functions, and describe how varied the behavior can be when the speed function is nondegenerate.

Altogether, the results in these chapters illustrate extent to which geometric flows are similar to each other and the extent to which they can be different from each other, all based on purely algebraic properties of the speed function f and the precise manner in which it nondegenerates at the boundary of the positive cone.

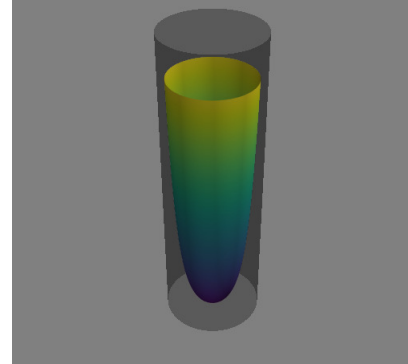
Finally, we delve into the world of graph Ricci curvatures and explore their relation to community structure. More precisely, we derive apriori curvature estimates,

thereby revealing the relation between the number of intercommunity edges and their curvature. In particular, we give precise criteria for when intercommunity edges are guaranteed to have nonpositive curvature. We show that these criteria are sharp from a theoretical point of view. On the other hand, we show empirical evidence that in practice, there is a lot of leniency from the theoretical bounds when intercommunity edges are randomly sampled.

Appendix



(a) The bowl-type soliton of mean curvature flow, low powers of Gauss curvature etc. are entire.



(b) The bowl-type soliton for harmonic MCF, high powers of Gauss curvature etc. are contained in cylinders.

Figure 1.1: Bowl-type solitons come in two varieties

Chapter 2

Translating Solutions

In this chapter, we study a translating solution that is known to arise as a singularity model in curvature flows. It is expressible as a graph and is rotationally symmetric about the e_{n+1} -axis.

Definition 4. A ***bowl-type soliton*** for the flow by f is a complete graph $\{(x, u(|x|))\}$, in \mathbb{R}^{n+1} where $u : [0, R) \rightarrow \mathbb{R}$ and $u'(0) = 0$ such that the 1-parameter family of graphs $y = u(|x|) + t$ is a (graphical) translating solution to the flow.

This is the generalization of the bowl soliton in mean curvature flow studied in [Clutterbuck et al. \(2007\)](#), and is the central object of study in this chapter.

One of the earliest works on them is by [Altschuler and Wu \(1994\)](#), who used elliptic PDE theory to study general translating solutions to the mean curvature flow, and the existence of the bowl soliton was obtained as a corollary. Later, [Clutterbuck et al. \(2007\)](#) approached the axially symmetric case directly using ODE methods, and were able to give precise asymptotics of the solution at infinity. They solved the translator PDE locally near the origin using barriers, and used ODE extension methods to continue the solution. See ([Andrews et al., 2020](#), Theorem 13.38) for a concise exposition of this. There have also been works on flows by other specific speeds, which we summarize here. [Urbas \(1998b\)](#) studied solitons (including bowl-type solitons) to flows by powers of the Gauss curvature, exploiting techniques from

the study of Monge–Ampère-type equations. [Santaella \(2020, 2023\)](#) studied the case $f = Q_k := S_{k+1}/S_k$ of where S_k is the elementary symmetric polynomial of degree k in the principal curvatures. In particular, he constructed a bowl-type soliton for the Q_{n-1} -flow (better known as the *harmonic mean curvature flow*). But a study of the general situation of homogeneous flow speeds had not yet been carried out to our knowledge (even under additional concavity assumptions on the speed) prior to this work. Moreover, in the case of Gauss curvature flows, the analysis is fairly simple due to the separability of the translator ODE. Therefore, not only do we generalize all of the above works, but provide simpler arguments with minimal PDE machinery, and uncover the relation between algebraic properties of f and geometric properties of solutions to the corresponding flow.

Our approach is modeled on [Clutterbuck et al. \(2007\)](#): we construct and analyze $u : I \rightarrow \mathbb{R}$ (where $I = [0, R), 0 < R \leq \infty$) of class C^2 such that the family of surfaces given by the graphs of $y = u(|x|) + t$ ($t \in \mathbb{R}$) are translating solutions to (1.1), under very general hypotheses on f . We do so by deriving and analyzing the second order ODE that u solves. This ODE has a coordinate singularity at the origin, which the translator PDE does not. In [Clutterbuck et al. \(2007\)](#) (or [Andrews et al. \(2020\)](#)), they bypass this singularity by using PDE methods to solve the equation near 0. By contrast, we bypass the singularity using completely elementary means, thereby obviating the need for concavity hypotheses on the speed. Smoothness at the origin can then be inferred from PDE theory, once again without needing concavity.

We now define a crucial degeneracy criterion via the following definition. Let $f : \Gamma \rightarrow [0, \infty)$ be a nonnegative function defined on an open set Γ that contains the positive cone $\Gamma^+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0 \ \forall i = 1, \dots, n\}$ and $\mathbf{e} := (1, \dots, 1) \in \mathbb{R}^{n-1}$.

Definition 5. A speed function f is **degenerate** (respectively, **nondegenerate**) if $f(0, \mathbf{e}) = 0$ (respectively, $f(0, \mathbf{e}) > 0$.)

Note that, although Ω need not contain $(0, \mathbf{e})$, we can define $f(0, \mathbf{e}) := \lim_{s \rightarrow 0} f(s, \mathbf{e})$, since f is increasing in all arguments and nonnegative.

Remark. *The geometric meaning of the above definition is as follows: the cylinder $\mathbf{S}^{n-1} \times \mathbb{R}$ is a stationary solution under flow by a degenerate speed whereas it collapses to its axis under a nondegenerate speed in finite time. Since translators are rigid and eternal, one does not expect translating solitons of nondegenerate speeds to be contained in a cylinder.*

Bowl-type solitons of the nonentire kind (and hence the entire kind) can be equivalently defined in the following way:

Definition 6. *A bowl-type soliton $y = u(|x|)$ is said to be **cylindrical** if its domain is a ball B_R for $R < \infty$, and $\lim_{r \rightarrow R^-} u(r) = \infty$. It is said to be entire if $R = \infty$.*

For the mean curvature flow, it was proved in [Clutterbuck et al. \(2007\)](#) that the bowl soliton is entire, and its asymptotic expansion up to order $|x|^2$ is $u(|x|) = \frac{|x|^2}{2(n-1)} + o(|x|^2)$ as $|x| \rightarrow \infty$. The bowl-type soliton corresponding to $f = (\sum \kappa_i^{-1})^{-1}$ (the harmonic mean curvature) is cylindrical. For flows by powers of Gauss curvature $K^{\alpha/n}$, bowl-type solitons are entire when $\alpha \leq 1/2$ and cylindrical when $\alpha > 1/2$ (see [Figure 1.1](#)). In light of these observations for specific speed functions, we present our results in the following four theorems.

Theorem 2.1. *Let f be an admissible speed function. There exists a unique bowl-type soliton for the corresponding flow. It is the graph of a convex radial function $y = u(|x|)$, where $u \in C^2([0, R))$, $R \in (0, \infty]$. If f is of class $C^{k,\beta}$, $\beta \in (0, 1)$, then $u \in C^{k+2,\beta}([0, R))$. In particular, if f is smooth, so is u .*

If f is non-degenerate, then $R = \infty$ and

$$u(|x|) = C|x|^{\alpha+1} + o(|x|^{\alpha+1}) \text{ as } |x| \rightarrow \infty,$$

where $C := \frac{1}{(\alpha+1)f(0,\mathbf{e})}$, and α is the degree of homogeneity of f .

It is somewhat surprising that the asymptotic expansion in Theorem 2.1 is available under such general hypotheses on the speed. Lower order terms are computed in Chapter 4.

Our next theorem concerns low homogeneities (regardless of whether they are degenerate or not.)

Theorem 2.2. *If f is an admissible speed function with $\alpha \leq 1/2$, then the corresponding bowl-type soliton is entire.*

To understand degenerate speeds of higher homogeneity, we need to understand the constraint equation $f(x, y\mathbf{e}) = 1$ as $y \rightarrow \infty$ (See Figure 2.1 for an illustration.) Note that the monotonicity of f in each slot, together with the implicit function theorem allows us to solve for x as a function of y, z in the equation $f(x, y\mathbf{e}) = z$.

Definition 7. *To each admissible speed f , we associate a function g_f which is defined implicitly by*

$$f(g_f(y, z), y\mathbf{e}) = z.$$

For example, if f is the mean curvature, $g_f = z - (n-1)y$. For the Gauss curvature, it is $zy^{-(n-1)}$ and for the harmonic mean curvature, it is $(z^{-1} - (n-1)y^{-1})^{-1}$.

Remark. *In the remainder of this chapter, we will simply refer to g_f as g .*

Now, define

$$L := \lim_{y \rightarrow \infty} g(y, 1).$$

For example, as demonstrated in Figure 2.1, $L = 0$ in the Gauss curvature case (as well as its powers), whereas in the harmonic mean curvature case, $L = 1$.

Theorem 2.3. *Let f be degenerate admissible speed (i.e. $f(0, \mathbf{e}) = 0$) with $\alpha > 1/2$. If $L > 0$, then the bowl-type soliton corresponding to f is defined on a ball B of finite radius and is asymptotic to the cylinder $\partial B \times \mathbb{R}$.*

Recall that $f(t) = O(g(t))$ as $t \rightarrow \infty$ if there exists $C > 0$ such that, for sufficiently large t , $|f(t)/g(t)| \leq C$.

Theorem 2.4. *Let f be a degenerate admissible speed function with $\alpha > 1/2$. Consider the constraint equation $f(x, y\mathbf{e}) = 1$. Suppose that $L = 0$.*

- (i) *If $x = O(y^{-(2\alpha-1)})$ then the corresponding bowl-type soliton is entire.*
- (ii) *If there exist constants $C > 0, k \in (0, 2\alpha - 1)$ such that $x \geq Cy^{-k}$ for sufficiently large y , then the corresponding bowl-type soliton is defined over a bounded domain.*

This explains the case of the various powers of Gauss curvature. Thus, we have an almost-complete classification of all admissible speeds under consideration (except in case $x = O(y^{-k})$ for all $k < 2\alpha - 1$ but $x \neq O(y^{-(2\alpha-1)})$, which are exceedingly rare. For example $x = y^{-2(\alpha-1)} \log y$, but this does not arise from an admissible speed.)

This chapter is organized as follows:

- Section 1: We derive the ODE satisfied by the rotational translator.
- Section 2: We construct and prove uniqueness of solutions to translator ODE, and show that it is C^1 up to the origin (and smooth elsewhere).
- Section 3: We show that the solution is smooth at the origin using the theory of elliptic PDE.
- Section 4: We provide a proof of Theorem 2.2.
- Section 5: We analyze degenerate speeds and prove the classification theorems 2.3 and 2.4.
- Section 6: We analyze nondegenerate speeds, show that solutions are entire, and prove the asymptotic expansion in 2.1.
- Section 7: We show that the solution we have constructed is convex, thereby completing the proof of Theorem 2.1.

2.1 The rotational translator ODE

For a real-valued C^2 -function u of a single real variable, consider the graph of $y = u(|x|)$, where $x \in \mathbb{R}^n$. The principal curvatures of this hypersurface are κ_1 , the curvature of the profile curve $y = u(r)$, and κ_i ($i = 2, \dots, n$), the rotational curvatures, which are all the same. Let ν be the downward normal. In formulae,

$$\nu = \left(\frac{u'}{\sqrt{1+u'^2}} \frac{x}{|x|}, \frac{-1}{\sqrt{1+u'^2}} \right), \quad \kappa_1 = \frac{u''}{(1+u'^2)^{3/2}}, \quad \kappa_i = \frac{u'}{r\sqrt{1+u'^2}} \quad (2.1)$$

for $i = 2, \dots, n$.

For the flow (1.1), a translator with unit speed in the direction $e_{n+1} := (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ satisfies

$$\langle \nu, e_{n+1} \rangle = -f(\kappa_1, \dots, \kappa_n).$$

Plugging in the principal curvatures from above, we get

$$f \left(\frac{u''}{(1+u'^2)^{3/2}}, \frac{u'}{r\sqrt{1+u'^2}}, \dots, \frac{u'}{r\sqrt{1+u'^2}} \right) = \frac{1}{\sqrt{1+u'^2}}. \quad (2.2)$$

Putting $v := u'$ to reduce the order, using the α -homogeneity to simplify, and setting $\mathbf{e} := \underbrace{(1, \dots, 1)}_{(n-1)\text{-times}}$ we get the first order equation

$$f \left(\frac{v'}{(1+v^2)^{3/2-1/(2\alpha)}}, \frac{v}{r(1+v^2)^{1/2-1/(2\alpha)}} \mathbf{e} \right) = 1.$$

The function v has the geometric significance of being the gradient of the profile curve.

As in Definition 7, we can solve for x using a unique function g , in the sense that $f(g(y, z), y\mathbf{e}) = z$. Similarly, we can solve for y using a function g_1 , i.e. $f(x, g_1(x, z)\mathbf{e}) = z$. Note that by the implicit function theorem, g, g_1 are of class C^1 .

Thus, solving for v' and setting $\beta = \frac{1}{2} - \frac{1}{2\alpha}$, we get

$$v' = (1 + v^2)^{1+\beta} g \left(\frac{v}{r(1 + v^2)^\beta}, 1 \right). \quad (2.3)$$

We shall refer to this equation as the *Translator ODE*. Note that since we seek bowl-type solitons, we have the initial condition $v(0) = u'(0) = 0$. Without loss of generality, we can set $u(0) = 0$ for convenience, so that our translator has its “tip” at the origin. One recovers u from v via the formula

$$u(r) = \int_0^r v(\rho) d\rho.$$

2.2 Existence and uniqueness

Our aim here is to show that there exists a unique solution to the problem

$$\begin{cases} v' = & (1 + v^2)^{1+\beta} g \left(\frac{v}{r(1+v^2)^\beta}, 1 \right), \\ v(0) = & 0 \end{cases} \quad (2.4)$$

on a maximal interval $[0, R)$, $0 < R \leq \infty$, which is of class $C^1([0, R))$. Note that this is non-trivial since the problem is singular at $r = 0$.

We shall obtain a solution to (2.4) as the limit of a sequence of solutions v_n to the approximating problems

$$\begin{aligned} v_n' &= (1 + v_n^2)^{1+\beta} g \left(\frac{v_n}{r(1 + v_n^2)^\beta}, 1 \right), \\ v_n(r_n) &= a_n \end{aligned}$$

with initial values $(r_n, a_n) \rightarrow (0, 0)$.

Our approach is to solve the equation near the origin on some small interval $[0, \delta]$ where δ will be determined later. We first obtain a subsolution and some supersolutions to the ODE which will serve as uniform lower and upper barriers on

$[0, \delta]$. Note that if v satisfies the initial condition and admits a (one-sided) derivative at $r = 0$, then its derivative must satisfy $v'(0) = 1/f(1, \dots, 1)^{1/\alpha}$. We verify this by allowing $r \rightarrow 0$ in (2.4) and observe the following:

$$\begin{aligned}
v'(0) &= g(v'(0), 1) \\
\iff f(v'(0), v'(0)\mathbf{e}) &= 1 \\
\iff v'(0)^\alpha f(1, \dots, 1) &= 1 \\
\iff v'(0) &= 1/f(1, \dots, 1)^{1/\alpha}.
\end{aligned}$$

So we define $\gamma := 1/f(1, \dots, 1)^{1/\alpha}$. Then γ is the unique number that solves the equation $\gamma = g(\gamma, 1)$ due to the above calculation.

Now define a function $w_0(r)$ implicitly by the relation

$$\frac{w_0}{r(1+w_0^2)^\beta} = \gamma. \quad (2.5)$$

Note that w_0 is well-defined by Lemma A.0.7.

Proposition 2.5. *The function w_0 as defined in (2.5) is a subsolution to (2.4).*

Proof. Due to (13),

$$\begin{aligned}
w_0' &= \gamma \frac{(1+w_0^2)^{1+\beta}}{1+(1-2\beta)w_0^2} \\
&= g(\gamma, 1) \frac{(1+w_0^2)^{1+\beta}}{1+(1-2\beta)w_0^2} \\
&\leq (1+w_0^2)^{1+\beta} g(\gamma, 1) \\
&= (1+w_0^2)^{1+\beta} g\left(\frac{w_0}{r(1+w_0^2)^\beta}, 1\right). \quad \square
\end{aligned}$$

Given that the equation $f(x, \gamma\mathbf{e}) = 1$ has a solution $x = \gamma$, the implicit function theorem now guarantees that the equation $f(x, y\mathbf{e}) = 1$ can be solved for x when $y \in [\gamma e^{-\theta}, \gamma e^\theta]$, for some $\theta > 0$. In other words, $[\gamma e^{-\theta}, \gamma e^\theta]$ is in the domain of $g(\cdot, 1)$. Given $\varepsilon \in [0, \theta]$, let $\gamma_\varepsilon = \gamma e^\varepsilon$, and define $w_\varepsilon(r)$ using $\frac{w_\varepsilon}{r(1+w_\varepsilon^2)^\beta} = \gamma_\varepsilon$.

Proposition 2.6. *For each $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that w_ε is a supersolution to (2.4) on $(0, r_\varepsilon)$.*

Proof. Note that

$$w'_\varepsilon = \gamma_\varepsilon \frac{(1 + w_\varepsilon^2)^{1+\beta}}{1 + (1 - 2\beta)w_\varepsilon^2}$$

and

$$(1 + w_\varepsilon^2)^{1+\beta} g\left(\frac{w_\varepsilon}{r(1 + w_\varepsilon^2)^\beta}, 1\right) = (1 + w_\varepsilon^2)^{1+\beta} g(\gamma_\varepsilon, 1).$$

Therefore,

$$\begin{aligned} w'_\varepsilon &> (1 + w_\varepsilon^2)^{1+\beta} g\left(\frac{w_\varepsilon}{r(1 + w_\varepsilon^2)^\beta}, 1\right) \\ \iff \gamma_\varepsilon \frac{(1 + w_\varepsilon^2)^{1+\beta}}{1 + (1 - 2\beta)w_\varepsilon^2} &> (1 + w_\varepsilon^2)^{1+\beta} g(\gamma_\varepsilon, 1) \\ \iff \frac{\gamma e^\varepsilon}{1 + (1 - 2\beta)w_\varepsilon^2} &> g(\gamma e^\varepsilon, 1) \\ \iff \gamma^\alpha e^{\alpha\varepsilon} f\left(\frac{1}{1 + (1 - 2\beta)w_\varepsilon^2}, \mathbf{e}\right) &> 1 \\ \iff f\left(\frac{1}{1 + (1 - 2\beta)w_\varepsilon^2}, \mathbf{e}\right) &> \frac{f(1, \mathbf{e})}{e^{\alpha\varepsilon}}. \end{aligned}$$

The above inequality holds when $r = 0$, hence it holds by continuity on some $[0, r_\varepsilon]$, where r_ε might depend on ε . \square

It is important to note that (by Lemma A.0.8) for any fixed r the function $w_\varepsilon(r)$ is monotone increasing with respect to ε . In particular, $w_\theta \geq w_0$, where θ was defined before the previous proposition.

Now we construct a family of functions that converge to a solution of our ODE on $[0, r_\theta]$. For each $n > 1/r_\theta$, consider the continuous function v_n defined as follows.

- $\frac{v_n}{r(1+v_n^2)^\beta} = \gamma$ on $[0, 1/n]$
- v_n obeys the ODE $v'_n = (1 + v_n^2)^{1+\beta} g\left(\frac{v_n}{r(1+v_n^2)^\beta}, 1\right)$ on $(1/n, r_\theta]$.

The initial data for the ODE is of course $v_n(1/n)$, which is implicitly defined by the first relation. Note that v_n is well-defined on the interval $(1/n, r_\theta]$ due to Theorem

A.0.3, where the compact set K (see Figure 2.2) can be taken to be

$$K = \left\{ (r, w) \in \mathbb{R}^2 : \gamma \leq \frac{w}{r(1+w^2)^\beta} \leq \gamma_\theta, 0 < r \leq r_\theta \right\} \cup \{(0, 0)\}.$$

Now let us define a constant C as follows. First, we define the following constants:

$$\begin{aligned} C_1 &:= \sup \left\{ (1+v^2)^{1+\beta} g \left(\frac{v}{r(1+v^2)^\beta}, 1 \right) : (r, v) \in K - \{(0, 0)\} \right\} \\ C_2 &:= \sup \left\{ 2\beta v(1+v^2)^\beta g \left(\frac{v}{r(1+v^2)^\beta}, 1 \right) : (r, v) \in K - \{(0, 0)\} \right\} \\ C_3 &:= \max\{C_1, C_2\} \\ C_4 &:= e^{C_3 r_\theta} \end{aligned}$$

C_1, C_2 are finite due to the fact that $v, \frac{v}{r(1+v^2)^\beta}$ are bounded in $K - \{(0, 0)\}$. Hence all of the constants above are finite.

Now define $C := \max\{C_i : i = 1, \dots, 4\}$.

Proposition 2.7. *The sequence of functions $\{v_n\}$ converges uniformly on $(0, r_\theta]$.*

Proof. Let $m > n$. By the mean value theorem, we have

$$\begin{aligned} v_m(1/n) &\leq v_m(1/m) + (1/n - 1/m) \sup_{r \in [1/m, 1/n]} v'_m(r) \\ &\leq v_m(1/m) + (1/n)C_1. \end{aligned}$$

Thus,

$$v_m(1/n) - v_n(1/n) \leq v_m(1/m) - v_n(1/n) + C_1/n. \quad (2.6)$$

Now, from the definition of v_n , $v_n(1/n) = w_0(1/n)$ and $v_m(1/m) = w_0(1/m)$, where w is defined implicitly by $\frac{w_0}{r(1+w_0^2)^\beta} = \gamma$. Since w_0 is monotone increasing in r ,

$$v_m(1/m) - v_n(1/n) \leq 0.$$

Thus (2.6) becomes

$$v_m(1/n) - v_n(1/n) \leq C_1/n.$$

Now we estimate $v_m(r) - v_n(r)$ for $r \in (1/n, r_\theta]$. Taking the right hand side of (2.3) as a function of v and applying the mean value theorem, we get

$$\begin{aligned} v'_m(r) - v'_n(r) &= (1 + v_m^2)^{1+\beta} g\left(\frac{v_m}{r(1 + v_m^2)^\beta}, 1\right) - (1 + v_n^2)^{1+\beta} g\left(\frac{v_n}{r(1 + v_n^2)^\beta}, 1\right) \\ &\leq (v_m - v_n) \sup_{v \in [v_n(r), v_m(r)]} \left\{ 2\beta v(1 + v^2)^\beta g\left(\frac{v}{r(1 + v^2)^\beta}, 1\right) \right. \\ &\quad \left. + (1 + v^2) g_y\left(\frac{v}{r(1 + v^2)^\beta}, 1\right) \frac{(1 + (1 - 2\beta)v^2)}{r(1 + v^2)^{\beta+1}} \right\} \\ &\leq (v_m - v_n) \sup_{v \in [v_n(r), v_m(r)]} \left\{ 2\beta v(1 + v^2)^\beta g\left(\frac{v}{r(1 + v^2)^\beta}, 1\right) \right\} \\ &\leq C_2(v_m - v_n). \end{aligned}$$

We used the fact that g is decreasing in the first slot in the penultimate step.

Dividing by the positive quantity $v_m - v_n$ and integrating on $[1/n, r]$,

$$\begin{aligned} v_m(r) - v_n(r) &\leq (v_m(1/n) - v_n(1/n)) e^{C_2(r-1/n)} \\ &\leq C^2/n. \end{aligned}$$

The claim follows by letting $n \rightarrow \infty$. □

Therefore $\{v_n\}_{n=1}^\infty$ converges uniformly to a continuous function on $(0, r_\theta)$. Since the sequence of functions $\{v_n\}$ is uniformly bounded, v'_n has a uniformly continuous dependence on v_n , and hence v'_n converges uniformly as well. Thus we have C^1 -convergence of v_n to some differentiable function v on $(0, r_\theta)$. In particular, v satisfies (2.3).

Proposition 2.8. *The limit function v can be extended continuously to $r = 0$. Moreover, the extended function is continuously differentiable at $r = 0$.*

Proof. Recall that since we have a subsolution w_0 on $(0, r_\theta)$ and supersolutions $w_\varepsilon =$ on $(0, r_\varepsilon)$, we have $w_0 \leq v_n \leq w_\varepsilon$. Combining with Lemma A.0.7

$$\gamma r \leq \frac{v_n}{(1 + v_n^2)^\beta} \leq \gamma e^\varepsilon r \text{ on } (0, r_\varepsilon).$$

Passing to the limit, we have

$$\gamma r \leq \frac{v}{(1 + v^2)^\beta} \leq \gamma e^\varepsilon r \text{ on } (0, r_\varepsilon). \quad (2.7)$$

Letting $r \rightarrow 0$, we see that $v(0) = 0$. Also, dividing the same equation by r , we can evaluate $v'(0) = \lim_{r \rightarrow 0} v/r$.

$$\gamma \leq \frac{v/r}{(1 + v^2)^\beta} \leq \gamma e^\varepsilon \text{ on } (0, r_\varepsilon).$$

Letting $\varepsilon \rightarrow 0$, we get

$$v'(0) = \gamma.$$

Now we show differentiability at $r = 0$. Estimating equation (2.4) from above and below using (2.7),

$$(1 + w^2)g(\gamma e^\varepsilon, 1) \leq v'(r) \leq (1 + w_\varepsilon^2)g(\gamma, 1).$$

Letting $\varepsilon \rightarrow 0$,

$$\lim_{r \rightarrow 0} v'(r) = g(\gamma, 1) = \gamma = v'(0).$$

This completes the proof. □

Regarding the question of uniqueness, we are only interested in solutions v of class C^1 up to the origin, because these correspond to C^2 -solutions u of (2.2).

Proposition 2.9. *The solution v as obtained above is the unique C^1 -solution to the initial value problem (2.4).*

Proof. We first show uniqueness near the origin. The equation $f(x, ye) = 1$ is solvable for x when $y = \gamma$. Thus by the implicit function theorem, there exists $\varepsilon > 0$ such that this equation is solvable for $y \in (\gamma e^{-\varepsilon}, \gamma e^{\varepsilon})$. Suppose that v_1, v_2 are two solutions, with initial conditions $v_1(0) = v_2(0) = 0$. Now, since both solutions are C^1 up to the origin, $v_1'(0) = v_2'(0) = \gamma$, and hence there exists some $\delta > 0$ such that the graphs of v_1, v_2 are contained in the compact set $K' := \{(r, v) : \gamma e^{-\varepsilon} r \leq v \leq \gamma e^{\varepsilon} r, 0 \leq r \leq \delta\}$. The slope field is nonsingular in $K' - \{(0, 0)\}$, and hence integral curves do not intersect. Thus without loss of generality, we may assume $v_1 > v_2$ in $K' - \{0\}$. Define $C' = \sup_{(r,v) \in K' - \{(0,0)\}} \left\{ 2\beta v(1+v^2)^\beta g\left(\frac{v}{r(1+v^2)^\beta}, 1\right) \right\}$. Now let δ', r be real numbers such that $0 < \delta' \leq r \leq \delta$. By an argument similar to Proposition 2.7,

$$|v_2(r) - v_1(r)| \leq C' |v_2(\delta') - v_1(\delta')|$$

Since v_1, v_2 are continuous and agree at $r = 0$, letting $\delta' \rightarrow 0$ shows that the solutions agree on $[0, \delta]$. This gives local uniqueness near the origin. Now by standard ODE theory, the solutions agree for as long as they are both defined. \square

Due to Theorems A.0.2 and A.0.3, we have proved the following.

Lemma 2.9.1. *The initial value problem (2.4) has a unique solution v defined on some maximal interval $[0, R)$, where $0 < R \leq \infty$. If f is of class C^k , then v is of class C^{k+1} everywhere except possibly $r = 0$. Moreover, v and v' are continuous up to $r = 0$, with $v(0) = 0$ and $v'(0) = \gamma$.*

From this, we recover u using the formula $u(r) = \int_0^r v(\rho) d\rho$. Note that u is C^2 at $r = 0$. Thus we have the following existence result for bowl-type solitons.

Theorem 2.10. *Let f be an admissible speed. There exists a unique bowl-type soliton with velocity e_{n+1} corresponding to it. The bowl-type soliton is the graph of a function $u : B_R \rightarrow \mathbb{R}$, where $0 < R \leq \infty$. If f is of class C^k , then u is of class C^{k+2} everywhere except possibly the origin, and at least C^2 at the origin.*

This proves, in particular, the first part of Theorem 2.1.

2.3 Smoothness at the origin

For higher regularity at the origin, we apply the PDE lemma given in Proposition [A.0.6](#). So let us cast our bowl-type soliton (near 0) as the solution to a (fully nonlinear) elliptic PDE.

Recall that, for a graph $M = \text{graph } u$, the component matrix of the Weingarten tensor is given by

$$W = g^{-1} \cdot \text{II},$$

where

$$\text{II} = \frac{D^2u}{\sqrt{1 + |Du|^2}}$$

is the component matrix of the second fundamental form and

$$g^{-1} = \text{I} - \frac{Du \otimes Du}{1 + |Du|^2}$$

is the component matrix of the cometric. Since W is not in general a symmetric matrix, we consider instead the following matrix (see [Urbas \(1991\)](#)):

$$\tilde{W} := P \cdot \text{II} \cdot P,$$

where P , a square root of g^{-1} , is given by

$$P = \text{I} - \frac{Du \otimes Du}{\sqrt{1 + |Du|^2}(1 + \sqrt{1 + |Du|^2})}.$$

Note that \tilde{W} is symmetric and has the same eigenvalues as W . Thus, if M is a translating solution to the flow by speed f , then u is a solution to the equation

$$-\hat{f}(Du, D^2u) = -\frac{1}{\sqrt{1 + |Du|^2}}, \tag{2.8}$$

where $\hat{f} : \mathbb{R}^n \times S_+^{n \times n} \rightarrow \mathbb{R}$ ($S_+^{n \times n}$ are the positive definite symmetric $n \times n$ matrices) is defined by

$$\hat{f}(p, r) := f \left(\left(\mathbb{I} - \frac{p \otimes p}{\sqrt{1 + |p|^2}(1 + \sqrt{1 + |p|^2})} \right) \cdot \frac{r}{\sqrt{1 + |p|^2}} \cdot \left(\mathbb{I} - \frac{p \otimes p}{\sqrt{1 + |p|^2}(1 + \sqrt{1 + |p|^2})} \right) \right),$$

where we treat f as a function of a symmetric matrix Z by evaluating it on the eigenvalues z_1, \dots, z_n of Z .

Observe that \hat{f} is of the same smoothness class as f and

$$\begin{aligned} \left(\frac{\partial \hat{f}}{\partial r_{ij}} \right)_{(Du, D^2u)} &= \left(\frac{\partial f}{\partial r_{pq}} \right)_{(P \cdot \mathbb{I} \cdot P)} P_{pi} P_{qj} \\ \implies \frac{\partial \hat{f}}{\partial r_{ij}} \xi_i \xi_j &= \frac{\partial f}{\partial r_{pq}} P_{pi} P_{qj} \xi_i \xi_j > 0 \end{aligned}$$

for all $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n - \{0\}$, because P is non-degenerate and the eigenvalues of the matrix $\left(\frac{\partial f}{\partial r_{pq}} \right)$, which are equal to f_{z_i} , $i = 1, \dots, n$, are positive by hypothesis. This implies that (2.8) is elliptic.

Finally, since we have proved that the solution u corresponding to our bowl-type soliton is of class C^2 , Proposition A.0.6 yields the following improvement of Theorem 2.10.

Theorem 2.11. *There exists a unique bowl-type soliton for every admissible speed f . The bowl-type soliton is the graph of a function $u : B_R \rightarrow \mathbb{R}$, where $0 < R \leq \infty$. If f is of class $C^{k, \alpha}$, then u is of class $C^{k+2, \alpha}$. In particular, if f is smooth, then so is u .*

This proves the second part of Theorem 2.1.

2.4 Low homogeneities

For any $\alpha > 0$, (2.5) shows that the function v_- implicitly defined by $\frac{v_-}{r(1+v_-^2)^\beta} = \gamma$ is a subsolution to the translator ODE. From §A.3 we infer that the solution v satisfies $\frac{v}{r(1+v^2)^\beta} \geq \gamma > 0$. In addition, $v \geq 1$ for sufficiently large r .

Now if $\alpha \leq 1/2$, we have that for sufficiently large r (such that $v \geq 1$),

$$\begin{aligned} v' &\leq \gamma(1+v^2)^{1+\beta} \\ &\leq \gamma\sqrt{1+v^2} \\ &\leq \sqrt{2}\gamma v. \end{aligned}$$

By comparing with Lemma A.0.5, we see that the corresponding bowl-type soliton is entire. This proves Theorem 2.2.

2.5 Degenerate speeds

We only need to discuss what happens if $\alpha > 1/2$. We formulate the degeneracy condition $f(0, \mathbf{e}) = 0$ as $\lim_{s \rightarrow 0} f(s, \mathbf{e}) = 0$, as f may be undefined when one of its inputs is zero. Note that in the equation $f(x, \mathbf{y}\mathbf{e}) = 1$, x is decreasing with respect to y . If $x \rightarrow 0$ as $y \rightarrow y_0 < \infty$, then $\lim_{y \rightarrow y_0} f(0, \mathbf{y}\mathbf{e}) = 1$ which violates our degeneracy hypothesis that $f(0, \mathbf{e}) = 0$. Thus $L := \lim_{y \rightarrow \infty} x = \lim_{y \rightarrow \infty} g(y, 1) \geq 0$

2.5.1 The $L > 0$ case

If $L > 0$, we have the inequality

$$\begin{aligned} v' &> (1+v^2)^{1+\beta} L \\ &> Lv^{2+2\beta}. \end{aligned}$$

Comparing with Lemma A.0.5 shows that solutions are defined on bounded domains provided $\beta > -1/2$, i.e. $\alpha > 1/2$. This explains what we observed in the harmonic mean curvature case, and proves Theorem 2.3.

2.5.2 The $L = 0$ case

Again we analyse the case of $\alpha > 1/2$. Here it turns out that whether the bowl-type soliton is entire or nonentire depends on the asymptotics of $g(y, 1)$ as $y \rightarrow \infty$.

First, we note that since the function $v_-(r)$ implicitly defined by $\frac{v_-}{r(1+v_-^2)^\beta} = \gamma$ is a subsolution, the solution v satisfies $v \geq v_-$, and hence $\frac{v}{r(1+v^2)^\beta} \geq \gamma > 0$. We claim that $\frac{v}{r(1+v^2)^\beta}$ is in fact unbounded. We prove this by contradiction. Suppose this is not the case. Then there exists $M > 0$ such that $\frac{v}{r(1+v^2)^\beta} \leq M$. This means $v(r)$ exists for all r . Define $g(M, 1) := \varepsilon$. Then, since $L = 0$ and $g(\cdot, 1)$ is strictly decreasing, $\varepsilon > 0$. Since g is monotone decreasing in the first argument, $g\left(\frac{v}{r(1+v^2)^\beta}, 1\right) \geq \varepsilon$. This means

$$v' \geq (1 + v^2)^{1+\beta} \varepsilon$$

but since $\alpha > 1/2$ (i.e. $\beta > -1/2$), this equation blows up at some finite R , leading to a contradiction. This proves our claim. Indeed more is true: given any $N > 0$, there exists $r_1 > 0$ such that $\frac{v}{r(1+v^2)^\beta} \geq N$ for $r \geq r_1$. This is because of the following lemma which holds for both degenerate and nondegenerate speeds f . The “ \sim ” symbol used below, which is an equivalence relation on the asymptotics of two functions, is defined in §A.3.

Lemma 2.11.1. *Suppose $N > 0$ and that $g(N, 1) > 0$. Then the function w_N defined implicitly by $\frac{w_N}{r(1+w_N^2)^\beta} = N$ is a subsolution to the translator ODE (2.3) for sufficiently large r .*

Proof. For large r , by (14) we have $w'_N \sim r^{\alpha-1}$. On the other hand, the asymptotics of the right hand side of the ODE is

$$\begin{aligned} (1 + w_N^2)^{1+\beta} g\left(\frac{w_N}{r(1 + w_N^2)^\beta}, 1\right) &= \varepsilon_N (1 + w_N^2)^{1+\beta} \\ &\sim (r^{2\alpha})^{1+\beta} \\ &= (r^{2\alpha})^{\frac{3}{2} - \frac{1}{2}\alpha} \\ &= r^{3\alpha-1} \end{aligned}$$

where $\varepsilon_N := g(N, 1) > 0$. Thus for sufficiently large r , w_N is a subsolution to (2.3). \square

Now we can consider the asymptotics of $g(y, 1)$ as $y \rightarrow \infty$.

Suppose $g(y, 1) = O(y^{1-2\alpha})$, i.e. there exists a constant $C > 0$ such that $g(y, 1) \leq Cy^{1-2\alpha}$ for sufficiently large y . Then for sufficiently large values of r ,

$$\begin{aligned} v' &= (1 + v^2)^{1+\beta} g\left(\frac{v}{r(1 + v^2)^\beta}, 1\right) \\ &\leq C(1 + v^2)^{1+\beta} \left(\frac{r(1 + v^2)^\beta}{v}\right)^{2\alpha-1} \\ &\leq C'vr^{2\alpha-1}. \end{aligned}$$

Now, by comparing this to Lemma A.0.5 we see that v exists for all $r > 0$. In this case, the bowl-type soliton is entire.

In contrast with the previous case, now suppose that there exist constants $C > 0$ and some $k < 2\alpha - 1$ such that $g(y, 1) > Cy^{-k}$. Define the positive number ε by the

relation $\frac{\varepsilon}{1-2\beta} = 2\alpha - 1 - k$. Then,

$$\begin{aligned} v' &= (1+v^2)^{1+\beta} g\left(\frac{v}{r(1+v^2)^\beta}, 1\right) \\ &\geq C(1+v^2)^{1+\beta} \left(\frac{r(1+v^2)^\beta}{v}\right)^{2\alpha-1-\varepsilon/(1-2\beta)} \\ &\geq C'v^{1+\varepsilon}. \end{aligned}$$

Comparing this to Lemma A.0.5, one sees that the solution v blows up at some finite $r = R$. In this case, the bowl-type soliton exists over the ball B_R . This proves Theorem 2.4.

Remark. *As mentioned in the introduction, this analysis leaves out the case when $x = O(y^{-k})$ for $k < 2\alpha - 1$ but not for $k = 2\alpha - 1$ due to the following indeterminacy. Consider for $p > 0$ the differential equation $v' = v(\log v)^p$ with initial condition $v(r_0) = v_0 > 1$. The right hand side is $O(v^\theta)$ for $\theta > 1$ but not $\theta = 1$. This equation blows up at some finite r if and only if $p > 1$. We believe that a more general integrability condition on $g(\cdot, 1)$ could be used to provide a complete classification, but since typical applications involve algebraic functions of the principal curvatures and transcendental functions are extremely rare, the criteria provided here are more readily applicable.*

2.6 Nondegenerate speeds

Finally, we prove the asymptotic expansion for bowl-type solitons of flows by nondegenerate speeds.

2.6.1 Entireness

As done in previous section, we can extend f to the boundary $\partial\Gamma_+^n$ of the positive cone by taking limits. Suppose $f(0, \mathbf{e}) > 0$ as in Theorem 2.1. Let $\gamma' := f(0, \mathbf{e})$.

Then $g_1(0, 1) = 1/(\gamma')^{1/\alpha}$ is finite as well (Refer to §2.1 for the definition of g_1 .) Let $\gamma_+ := g_1(0, 1)$. Then we have that $g_1(0, 1) = \gamma_+ \iff g(\gamma_+, 1) = 0$. Now, the function $v_+(r)$ defined by $\frac{v_+}{r(1+v_+^2)^\beta} = \gamma_+$ is precisely where the slope field vanishes. Also we see that v_+ is a supersolution to (2.3) because

$$\begin{aligned} v'_+ &> 0 \\ &= (1 + v_+^2)^{1+\beta} g\left(\frac{v_+}{r(1 + v_+^2)^\beta}, 1\right). \end{aligned}$$

We showed in Proposition 2.5 that the function $v_-(r)$ defined by $\frac{v_-}{r(1+v_-^2)^\beta} = \gamma$, where $\gamma := 1/f(1, \dots, 1)^{1/\alpha}$, is a subsolution to (2.3). Since we have a subsolution and a supersolution, both defined on $[0, \infty)$, it follows from Theorem A.0.3 that the solution v is also defined on $[0, \infty)$. Using the formula $u(r) = \int_0^r v(\rho) d\rho$, we see that u is defined on $[0, \infty)$. Therefore we have the following theorem:

Theorem 2.12. *If $f(0, \mathbf{e}) > 0$, then the corresponding bowl-type soliton is entire.*

2.6.2 Asymptotics

Let v be the solution to (2.4). We prove the following:

Proposition 2.13. *Suppose f is a nondegenerate speed. Let v be the solution to (2.4). $\frac{v}{r(1+v^2)^\beta} \rightarrow \gamma_+$ as $r \rightarrow \infty$*

Proof. Let $\varepsilon > 0, r_0 > 0$. We claim that there exists $r_1 > r_0$ such that $\frac{v(r_1)}{r_1(1+v(r_1)^2)^\beta} \geq (1 - \varepsilon)\gamma_+$.

Suppose this was not the case, i.e. for some $\varepsilon > 0$ we have $\frac{v}{r(1+v^2)^\beta} < (1 - \varepsilon)\gamma_+$ for all $r > r_0$. Note that since g is decreasing in the first slot, $g\left(\frac{v}{r(1+v^2)^\beta}, 1\right) > g((1 - \varepsilon)\gamma_+, 1) := \varepsilon'$. Then we have

$$v' > (1 + v^2)^{1+\beta} \varepsilon' > (v^2)^{1+\beta} \varepsilon' = v^{3-1/\alpha} \varepsilon'$$

which implies that

$$v^{1/\alpha-3}v' > \varepsilon'. \quad (2.9)$$

If $\alpha > 1/2$, then $3 - 1/\alpha > 1$ so that v blows up at some finite R . So we focus on the case that $\alpha \in (0, 1/2]$. Let r_0 be any positive number, and define $v_0 := v(r_0)$.

In case $\alpha = 1/2$, the differential equality (2.9) becomes

$$v'/v > \varepsilon' \implies v > v_0 e^{r-r_0}.$$

If $\alpha \in (0, 1/2)$ then the differential inequality becomes

$$\begin{aligned} \frac{(v^{1/\alpha-2})'}{1/\alpha-2} &> \varepsilon' \\ \implies v(r)^{1/\alpha-2} - v_0^{1/\alpha-2} &> (1/\alpha-2)\varepsilon'(r-r_0) \\ \implies v > [C + C'(r-r_0)]^{\frac{1}{1/\alpha-2}}, \end{aligned}$$

where $C = v_0^{1/\alpha-2}$, $C' = (1/\alpha-2)\varepsilon'$.

Note that

$$\frac{1}{1/\alpha-2} > \alpha \iff 1 > \alpha(1/\alpha-2) = 1-2\alpha \iff \alpha > 0$$

provided $\alpha < 1/2$.

We have already shown that $\frac{v}{r(1+v^2)^\beta} = (1-\varepsilon)\gamma_+$ implies v grows like r^α . This is in contradiction to what we have just shown, which is that v grows like $r^{1/\alpha-2}$ when $\alpha \in (0, 1/2)$ and like e^r when $\alpha = 1/2$, both of which are strictly faster than r^α .

Having proved this, now we conclude using Lemma 2.11.1 that for sufficiently large r , $\frac{v}{r(1+v^2)^\beta}$ stays above $(1-\varepsilon)\gamma_+$.

This means that our solution v exceeds w as defined here, and due to upward monotonicity of the expression $\frac{w}{r(1+w^2)^\beta}$ with respect to w , we infer that

$$\frac{v}{r(1+v^2)^\beta} > \frac{w}{r(1+w^2)^\beta} = (1-\varepsilon)\gamma_+$$

The claim follows since ε is arbitrary. □

Therefore, we have that $v = \frac{r^\alpha}{f(0, \mathbf{e})} + o(r^\alpha)$. Integrating, we see that the bowl-type soliton has the following asymptotics as $|x| \rightarrow \infty$:

$$u(|x|) = \frac{|x|^{\alpha+1}}{(\alpha+1)f(0, \mathbf{e})} + o(|x|^{\alpha+1}).$$

This proves the asymptotic expansion that was asserted in Theorem 2.1.

2.7 Convexity

We show here that the bowl-type solitons that we have constructed are convex. Observe that for any admissible speed function, the function $v_-(r)$ defined by $\frac{v_-}{r(1+v_-^2)^\beta} = \gamma$, where $\gamma := 1/f(1, \dots, 1)^{1/\alpha}$ is a subsolution. For degenerate speeds, $\lim_{y \rightarrow \infty} g(y, 1) \geq 0$ and for nondegenerate speeds the function $v_+(r)$ defined by $\frac{v_+}{r(1+v_+^2)^\beta} = \gamma_+$, where $\gamma_+ := g_1(0, 1)$, is a supersolution. Therefore, for our solution v , the expression $g\left(\frac{v}{r(1+v^2)^\beta}, 1\right)$ is positive in both the degenerate and nondegenerate cases for the following reason: v is below this supersolution in the nondegenerate case, and in the degenerate case, $g(\cdot, 1)$ is positive for all positive inputs. Therefore in either case, the solution satisfies $v' = (1+v^2)^{1+\beta} g\left(\frac{v}{r(1+v^2)^\beta}, 1\right) > 0$, which implies that $u'' > 0$ for the profile curve u . Thus, $\kappa_1 = \frac{u''}{(1+u'^2)^{3/2}}$. The remaining curvatures $\kappa_i = \frac{u'}{r\sqrt{1+u'^2}}$ are also positive because our subsolution guarantees that $u' = v \geq v_- > 0$. We conclude that the bowl-type solitons are convex.

Appendix

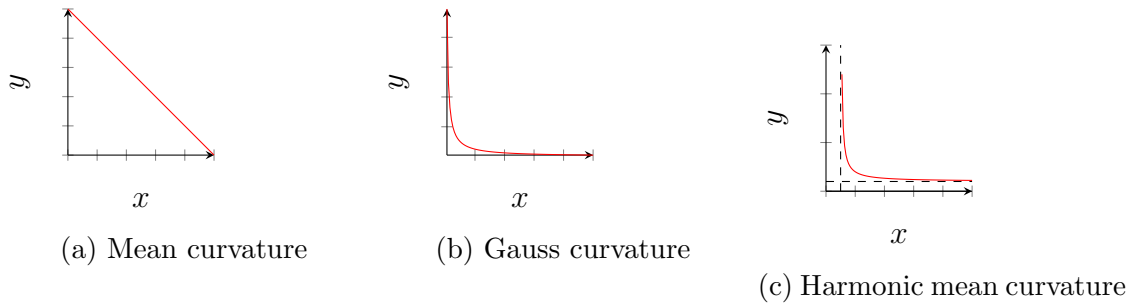


Figure 2.1: Level sets of the equation $f(x, y) = 1$.

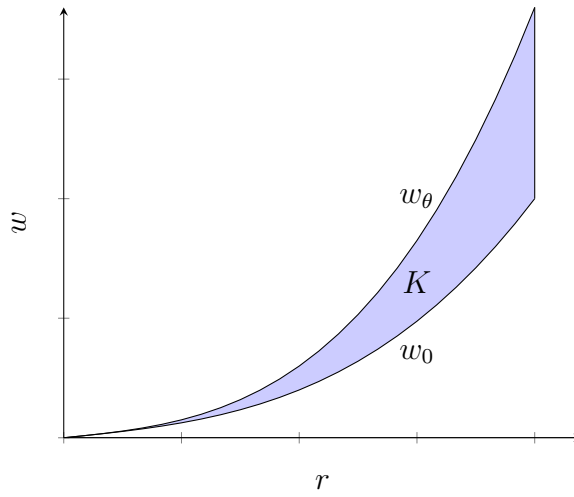


Figure 2.2: The compact set K

Chapter 3

Pancake Solutions

. Our purpose here is to analyze “ancient pancake” solutions (i.e. compact convex ancient solutions which are confined to slab regions) to a large and natural class of extrinsic geometric flows. In particular, we establish the following classification theorem.

Theorem 3.1. *Given any non-degenerate admissible speed function, there exists an $O(1) \times O(n)$ -invariant ancient solution $\{\partial\Omega_t\}_{t \in (-\infty, 0)}$, $\Omega_t \underset{\text{convex}}{\subset} \mathbb{R}^{n+1}$, to the corresponding extrinsic geometric flow exhibiting the following behaviour:*

(a) $\{\lambda\Omega_{\lambda^{-2}t}\}_{t \in (-\infty, 0)} \rightarrow \{B_{\sqrt{-2\phi_1 t}}^{n+1}\}_{t \in (-\infty, 0)}$ locally uniformly in the smooth topology as $\lambda \rightarrow \infty$, where ϕ_1 is the value the speed takes on the unit sphere.

(b) $\cup_{t \in (-\infty, 0)} \Omega_t = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n : |x| < \frac{\pi}{2}\}$, and

(c) $\{\partial\Omega_{t+\tau} - P(w, \tau)\}_{t \in (-\infty, -\tau)} \rightarrow \{\Gamma_t^n\}_{t \in (-\infty, 0)}$ locally uniformly in the smooth topology as $\tau \rightarrow -\infty$ for every $w \in \{0\} \times S^{n-1}$, where $P(w, t)$ is the point on $\partial\Omega_t$ whose outer unit normal is w , and

$$\Gamma_t^n := \{p = (x, \hat{x}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^n : -p \cdot w = \log \sec x + t\}$$

is the Grim hyperplane with velocity w .

Moreover, $\{\partial\Omega_t\}_{t\in(-\infty,0)}$ is unique amongst $O(n)$ -invariant convex ancient solutions to the corresponding flow satisfying property (b).

Remark. For uniqueness, we do not require the solution to be $O(1)$ -symmetric a priori. Indeed, the $O(1)$ -symmetry of $O(n)$ -symmetric slab solutions described as above is automatic (See Theorem 3.5.)

In this chapter, our admissible speeds satisfy, in addition to Definition 1 of Chapter 1, the following conditions as well:

- (a) 1-homogeneous: $\phi(\lambda z_1, \dots, \lambda z_n) = \lambda \phi(z_1, \dots, z_n)$ for every $\lambda > 0$;
- (b) inverse-concave: the function $(r_1, \dots, r_n) \mapsto \phi(r_1^{-1}, \dots, r_n^{-1})^{-1}$ is concave.

We call an admissible speed ϕ *non-degenerate* if it is

- (c) non-degenerate: the function $s \mapsto \phi(1, s, \dots, s)$ is of class $C^2([0, \infty))$ and satisfies $\phi(1, 0, \dots, 0) > 0$.

Remark. Note that in this chapter, we refer to our speed function as ϕ and not f as in the previous chapter. Our definition of nondegeneracy in this chapter is also subtly different from that of the previous chapter.

This class of speeds we consider is large and natural in view of the behaviour described in Theorem 3.1. It includes of course the mean curvature, which is uniformly elliptic, but also many examples whose ellipticity degenerates at the boundary of the positive cone (such as power means in the principal curvatures and linear combinations of these with highly degenerate speeds such as the n -th root of the Gauss curvature).

When the speed is the mean curvature, we recover the main results of [Bourni et al. \(2021b\)](#). Indeed we follow a similar approach here (though there are a number of new difficulties that arise, due to the nonlinearity of the speed function and the degeneration of ellipticity as convexity degenerates): we construct the ancient pancake by taking the limit of a family of “very old” solutions obtained by evolving

rotations of “very old” timeslices of the Angenent oval. A rough asymptotic analysis of arbitrary convex ancient solutions that have $O(n)$ -symmetry (which exploits Andrews’ differential Harnack inequality [Andrews \(1994c\)](#)), shows that such solutions necessarily have the additional $O(1)$ -symmetry as well. With some work, we are then able to establish a fine asymptotic expansion for the radial displacements of these solutions as $t \rightarrow -\infty$, which we are able to exploit to show that they are unique.

Let us briefly discuss the purpose of the hypotheses:

Note that, even for non-degenerate admissible speeds, the derivatives of ϕ with respect to the principal curvatures are allowed to degenerate at $\partial\Gamma^+$, the boundary of the positive cone.

Inverse-concavity is typically invoked to guarantee the applicability of the Krylov–Safanov Harnack inequality; this is not needed here, however, due to the symmetry of the solutions we consider — instead, we require it in order to make use of Andrews’ differential Harnack inequality, which guarantees, e.g., that ancient solutions are asymptotically modelled on translating solitons.

The non-degeneracy condition ensures that ϕ is uniformly elliptic in the principal direction corresponding to the largest principal curvature, even as the other principal curvatures approach zero (while the derivatives of ϕ in the remaining directions are allowed to degenerate). Under the normalization $\phi(1, 0, \dots, 0) = 1$, the non-degeneracy condition guarantees that the cylinder $\{\Gamma_t \times \mathbb{R}^{n-1}\}_{t \in I}$ is a solution to (1.1) whenever $\{\Gamma_t\}_{t \in I}$ is a solution to curve shortening flow. This ensures that the ancient solutions we construct are asymptotically modelled on Grim hyperplanes.

Examples

The class of admissible speeds is very large (see, for example, ([Andrews et al., 2020](#), §18.3)). As mentioned in the introduction, the power means

$$P_r(z_1, \dots, z_n) := \left(\sum_{i=1}^n z_i^r \right)^{\frac{1}{r}}, \quad r > 0$$

give rise to non-degenerate admissible speeds. Further examples are given by convex functions $f \in C^\infty(\Gamma)$ satisfying conditions (a)-(c) such that the domain Γ compactly contains $\Gamma_+ \cap S^n$.

Note also that the class of (non-degenerate) admissible speeds is closed under natural compositions: if $\zeta_1, \dots, \zeta_k \in C^\infty(\Gamma_+^n)$ and $\phi \in C^\infty(\Gamma_+^k)$ are (non-degenerate) admissible speeds, then so is $\phi(\zeta_1, \dots, \zeta_k)$.

3.1 Preliminaries

3.1.1 The Angenent oval

The Angenent oval (a.k.a the paperclip) is a solution to curve shortening flow that lies in the strip $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$. It is defined for $t \in (-\infty, 0)$ and satisfies the implicit equation

$$\cos x = e^t \cosh y. \quad (3.1)$$

In the turning angle parametrization $(x(\theta, t), y(\theta, t))$, where θ is the angle made by the tangent vector and the x -axis when the curve is oriented counterclockwise, the x, y coordinates are given explicitly by

$$x(\theta, t) = \arctan \left(\frac{\sin \theta}{\sqrt{\cos^2 \theta + a^2(t)}} \right)$$

and

$$y(\theta, t) = -t + \log \left(\frac{\sqrt{\cos^2 \theta + a^2(t)} - \cos \theta}{\sqrt{a^2(t) + 1}} \right),$$

where $a^2(t) := \frac{1}{e^{-2t} - 1}$.

We also define the horizontal and vertical displacements $h(t)$ and $\ell(t)$ by

$$h(t) := x\left(\frac{\pi}{2}, t\right)$$

and

$$\ell(t) := y(\pi, t).$$

These displacements satisfy the estimates

$$\begin{aligned} \frac{\pi}{2}(1 - e^{-t}) &\leq h(t) \leq \frac{\pi}{2}, \\ -t &\leq \ell(t) \leq -t + \log 2. \end{aligned} \tag{3.2}$$

3.1.2 $O(n)$ -invariance

Let us give \mathbb{R}^{n+1} the coordinates $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$. By $O(n)$ -invariance of a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$, we will mean that it is invariant under the action of $O(n)$ on the (y, z) hyperplane; this of course means that $\Sigma^n = \{xe_1 + y\eta : xe_1 + ye_2 \in \Sigma^n \cap \mathbb{E}^2, \eta \in S^n \cap \{0\} \times \mathbb{R}^n\}$, where $\mathbb{E}^2 := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^{n+1}$. We will refer to the curve $\Sigma^n \cap \mathbb{E}^2$ as the *profile curve* of Σ^n .

Conversely, given a smooth convex curve $\Gamma \subset \mathbb{R}^2$ that is reflection-symmetric about the x -axis, we can form an $O(n)$ -invariant hypersurface by revolving $\Gamma \subset \mathbb{R}^2 \times \mathbb{R}^{n-1}$ about the x -axis. This hypersurface has only two distinct principal curvatures, κ , the curvature of Γ (multiplicity one), and λ , the “rotational” curvature (multiplicity $n - 1$), which is given by

$$\lambda = \begin{cases} \frac{\cos \theta}{y}, & \theta \neq \frac{\pi}{2}, \frac{3\pi}{2} \\ \kappa, & \theta = \frac{\pi}{2}, \frac{3\pi}{2}. \end{cases}$$

Note that λ is a smooth function on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and satisfies

$$\kappa \lambda_\theta = -\lambda \tan \theta (\kappa - \lambda). \tag{3.3}$$

Notation

Given an $O(n)$ -invariant hypersurface with profile curvature κ and rotational curvatures λ , we shall, by an abuse of notation, write $\phi(\kappa, \lambda)$ to mean $\phi(\kappa, \lambda, \dots, \lambda)$. On an $O(n)$ -invariant flow with profile curvature $\kappa(\theta, t)$ and rotational curvatures

$\lambda(\theta, t)$, we also sometimes use $\phi(\theta, t)$ to mean $\phi(\kappa(\theta, t), \lambda(\theta, t))$; the distinction will be clear from context, however. By ϕ_t and ϕ_θ , etc, we mean the t and θ derivatives, respectively, of this function. By ϕ_κ and ϕ_λ , etc, we mean the derivatives of the two-variable function $\phi(\kappa, \lambda)$ with respect to its first and second arguments, respectively.

3.1.3 $O(n)$ -invariance and curvature flows

We assume in this section that $\phi \in C^\infty(\Gamma_+)$ is an admissible (but not necessarily non-degenerate) speed function. The following three lemmas (Lemmas (3.1.1), 3.1.2 and 3.1.3) and their corollaries (Corollaries 3.1.1, 3.1.2 and 3.1.3) apply to all $O(n)$ -invariant solutions to the f -flow (by admissible speeds $f = \phi$).

Lemma 3.1.1. *Along any $O(n)$ -invariant solution to the flow (1.1),*

$$\begin{aligned} \kappa_t &= \kappa^2 \phi_\kappa \kappa_{\theta\theta} + \kappa^2 (\phi_{\kappa\kappa} \kappa_\theta^2 + 2\phi_{\kappa\lambda} \kappa_\theta \lambda_\theta + \phi_{\lambda\lambda} \lambda_\theta^2) \\ &\quad + \kappa \phi_\lambda (-\kappa_\theta \lambda_\theta - \lambda \tan \theta (\kappa_\theta - 2\lambda_\theta)) + \kappa (\phi_\kappa \kappa^2 + \phi_\lambda \lambda^2), \end{aligned} \quad (3.4)$$

$$\lambda_t = \kappa^2 \phi_\kappa \lambda_{\theta\theta} - \lambda (\phi + \phi_\kappa \kappa) \tan \theta \lambda_\theta + \lambda (\phi_\kappa \kappa^2 + \phi_\lambda \lambda^2), \quad (3.5)$$

$$\phi_t = \phi_\kappa \kappa^2 \phi_{\theta\theta} - \phi_\lambda \lambda^2 \tan \theta \phi_\theta + \phi (\phi_\kappa \kappa^2 + \phi_\lambda \lambda^2), \quad (3.6)$$

and

$$-\frac{dA}{dt} = \int_0^{2\pi} \phi \, d\theta. \quad (3.7)$$

Proof. With respect to the Gauss map parametrization, the support function $\sigma(\cdot, t) : S^n \rightarrow \mathbb{R}$ of the solution evolves as

$$\partial_t \sigma = -\phi$$

and we can express the principal curvatures κ, λ as

$$\begin{aligned}\kappa^{-1} &= \sigma_{\theta\theta} + \sigma \\ \lambda^{-1} &= \sigma - \sigma_{\theta} \tan \theta.\end{aligned}$$

Differentiating with respect to t gives

$$\begin{aligned}\kappa_t &= \kappa^2(\phi_{\theta\theta} + \phi), \\ \lambda_t &= \lambda^2(\phi - \phi_{\theta} \tan \theta).\end{aligned}$$

Formulae (3.4), (3.5) are obtained now by using the chain rule and relating $\lambda_{\theta\theta}$ to lower order terms by differentiating (3.3).

The evolution equation for ϕ follows from those for κ and λ since

$$\phi_t = \phi_{\kappa} \kappa_t + \phi_{\lambda} \lambda_t.$$

The identity (3.7) is just the usual first variation of enclosed area. □

Lemma 3.1.2. *The ratio $u := \kappa/\lambda$ evolves according to*

$$\begin{aligned}u_t - \kappa^2 \phi_{\kappa} u_{\theta\theta} - 2\phi_{\kappa} \kappa^2 (\lambda_{\theta}/\lambda) u_{\theta} \\ = \lambda^{-1} D^2 \phi((\kappa_{\theta}, \lambda_{\theta}), (\kappa_{\theta}, \lambda_{\theta})) - \phi_{\lambda} \lambda^2 \tan \theta u_{\theta} - 2\phi \tan^2 \theta (\kappa - \lambda).\end{aligned}\tag{3.8}$$

Proof. This follows by direct calculation. □

Lemma 3.1.3. *At a critical point of u , $D^2 \phi((\kappa_{\theta}, \lambda_{\theta}), (\kappa_{\theta}, \lambda_{\theta})) = 0$.*

Proof. At a critical point of u , we have

$$\begin{aligned}u_{\theta} &= 0 \\ \implies \lambda \kappa_{\theta} - \kappa \lambda_{\theta} &= 0 \\ \implies \kappa_{\theta}/\lambda_{\theta} &= \kappa/\lambda,\end{aligned}$$

which means that $(\kappa_\theta, \lambda_\theta)$ is a multiple of (κ, λ) . Now since ϕ is 1-homogeneous, we have that $D^2\phi((\kappa, \lambda), (\kappa, \lambda)) = 0$ and the claim follows. \square

Corollary 3.1.1. *The inequality $\kappa \geq \lambda$ is preserved.*

Proof. Suppose the solution is defined on a time interval $[0, T)$ with $\kappa \geq \lambda$ at the initial time $t = 0$. Given $\varepsilon > 0$, consider the function $u^\varepsilon := u - 1 + \varepsilon e^t$, where $u := \kappa/\lambda$. By hypothesis, $u^\varepsilon(\cdot, 0) \geq \varepsilon > 0$. We claim that u_ε remains positive for all positive times. Suppose, to the contrary, that there is some $t_0 > 0$ and some $\theta_0 \in S^1$ such that $u^\varepsilon(\theta_0, t_0) = 0$ but $\min_{S^1} u^\varepsilon > 0$ for all $t < t_0$. In particular,

$$0 \geq u_t^\varepsilon, \quad 0 \geq -u_{\theta\theta}^\varepsilon, \quad \text{and} \quad 0 = u_\theta^\varepsilon = u_\theta,$$

at (θ_0, t_0) , and hence, by (3.8) and Lemma 3.1.3,

$$\begin{aligned} 0 &\geq u_t^\varepsilon - \kappa^2 \phi_\kappa u_{\theta\theta}^\varepsilon - 2\phi_\kappa \kappa^2 (\lambda_\theta/\lambda) u_\theta^\varepsilon \\ &= \varepsilon e^{t_0} + 2\varepsilon e^{t_0} \phi \lambda \tan^2 \theta \\ &> 0 \end{aligned}$$

at (θ_0, t_0) , yielding a contradiction. So u^ε remains positive and hence, taking $\varepsilon \rightarrow 0$, we conclude that $u - 1$ remains non-negative. \square

Corollary 3.1.2. *$\lambda(\cdot, t)$ is nondecreasing on $[\frac{\pi}{2}, \pi]$ for each t .*

Proof. This is a combination of (3.3) and Corollary 3.1.1 \square

Corollary 3.1.3. *The inequality $\kappa/\lambda \leq C$ is preserved along $O(n)$ -invariant solutions to (1.1) which satisfy $\kappa \geq \lambda$.*

Proof. Since the inequality $\kappa \geq \lambda$ ensures that the reaction term in (3.8) has the correct sign for preserving upper bounds, the claim follows from the maximum principle in a similar manner to Corollary 3.1.1. \square

A well-known argument originally due to Chou (formerly Tso, see [Tso \(1985\)](#)) provides an estimate for the speed for as long as the inradius remains bounded from below. (This estimate applies to all solutions to (1.1) by admissible speeds, not just $O(n)$ -invariant ones).

Proposition 3.2. *If the support function $\sigma : S^n \times [0, t_0] \rightarrow \mathbb{R}$ of a solution $\{\Sigma_t\}_{t \in [0, t_0]}$ to (1.1) satisfies*

$$2r \leq \sigma \leq R$$

for all $t \in [0, t_0]$, then

$$\phi \leq \frac{R}{r} \max \left\{ 2\phi_1 r^{-1}, \max_{M_0} \phi \right\} \quad (3.9)$$

and

$$\phi \leq C \left(2r^{-1} + t^{-\frac{1}{2}} \right), \quad (3.10)$$

where $C = C(n, \phi_1, \frac{R}{r})$ and we recall that $\phi_1 := \phi(1, \dots, 1)$.

Proof. If we define the function $\phi_* : \Gamma_+ \rightarrow \mathbb{R}$ by

$$\phi_*(r) := -\phi(r^{-1}),$$

then, with respect to the Gauss map parametrization,

$$\partial_t \sigma = -\phi = \phi_*(\rho_1, \dots, \rho_n),$$

where $\rho_i := \kappa_i^{-1}$ are the principal radii. Note that, under the Gauss map parametrization, the principal radii are the eigenvalues of the tensor

$$A := \bar{\nabla}^2 \sigma + \sigma \bar{g},$$

where \bar{g} and $\bar{\nabla}$ are the standard metric and connection on S^n .

Consider the function $v := \frac{\phi}{\sigma - r}$. If we denote by $\dot{\phi}^{ij}$ and $\dot{\phi}^i$ the derivatives of ϕ with respect to Π_{ij} and κ_i , respectively, and by $\dot{\phi}_*^{ij}$ and $\dot{\phi}_*^i$ the derivatives of ϕ_* with

respect to A_{ij} and ρ_i , respectively, then, at a new interior maximum of v , we find that

$$\begin{aligned}
0 &\leq (\partial_t - \dot{\phi}_*^{ij} \bar{\nabla}_i \bar{\nabla}_j) \frac{\phi}{\sigma - r} \\
&= \frac{\phi}{\sigma - r} \left(\frac{(\partial_t - \dot{\phi}_*^{ij} \bar{\nabla}_i \bar{\nabla}_j) \phi}{\phi} - \frac{(\partial_t - \dot{\phi}_*^{ij} \bar{\nabla}_i \bar{\nabla}_j) \sigma}{\sigma - r} \right) + 2 \dot{\phi}_*^{ij} \bar{\nabla}_i \frac{\phi}{\sigma - r} \bar{\nabla}_j \sigma \\
&= \frac{\phi}{\sigma - r} \left(\text{tr}(\dot{\phi}_*) + \frac{\phi + \dot{\phi}_*(A) - \sigma \text{tr}(\dot{\phi}_*)}{\sigma - r} \right) \\
&= \frac{\phi}{\sigma - r} \left(\dot{\phi}^i \kappa_i^2 + \frac{2\phi - \dot{\phi}^i \kappa_i^2 \sigma}{\sigma - r} \right).
\end{aligned}$$

That is,

$$\frac{\dot{\phi}^i \kappa_i^2}{\phi^2} \frac{\phi}{\sigma - r} \leq \frac{2r^{-1}}{\sigma - r} \leq 2r^{-2}.$$

Since ϕ is inverse-concave, the first claim now follows from (Andrews et al., 2013, Lemma 5).

To obtain the second claim, consider instead the ratio $\frac{t\phi}{\sigma-r}$. □

Combining the above results we are able to conclude that uniformly convex $O(n)$ -invariant hypersurfaces contract to round points under (1.1).

Proposition 3.3. *Let Σ be an $O(n)$ -invariant bounded convex hypersurface of \mathbb{R}^{n+1} . If $\kappa \geq \lambda$ on Σ , then (for any admissible speed ϕ) the solution to (1.1) starting from Σ contracts to a round point in finite time.*

Proof. Since the profile curve $\gamma : S^1 \times [0, T) \rightarrow \mathbb{R}^2$ of the solution satisfies the equation

$$\langle \gamma_t(\theta, t), \nu(\theta, t) \rangle = -f(\theta, t, \kappa(\theta, t)),$$

where

$$f(\theta, t, k) := \phi(k, \lambda(\theta, t)),$$

the support function satisfies an inhomogeneous parabolic equation with smooth coefficients. By Corollary 3.1.3, this equation remains uniformly parabolic, so Proposition 3.2 and estimates for inhomogeneous uniformly parabolic equations in one space variable (Lieberman, 1996, §XI.6) provide *a priori* estimates in $C^{k,\alpha}$ for every k , depending only on $k, f, \max_{S^1 \times \{0\}} \kappa/\lambda$ and r_0 , on any time interval $[0, t_0]$ on which σ is bounded from below by r_0 . Moreover, by Corollary 3.1.3 and Andrews' lemma (Andrews, 1994a, Theorem 5.1), the ratio of maximum to minimum width of the profile curve remains uniformly bounded throughout the evolution. A standard blow-up argument (see Andrews (1994a)) in conjunction with (3.8) now implies the claim. \square

We note that Proposition 3.3 follows from a more general theorem of McCoy–Mofarreh–Wheeler (McCoy et al., 2015, Theorem 6.1) (cf. (Risa and Sinestrari, 2022, Theorem 1)). It may also be seen as a consequence of a result of Andrews and McCoy (Andrews and McCoy (2016) (cf. Andrews (2010))).

3.2 Existence

We shall construct our ancient pancake solutions by taking the limit of a sequence of old-but-not-ancient solutions. These suitable old-but-not-ancient solutions are obtained by evolving rotated timeslices of the Angenent oval.

3.2.1 The approximating solutions

Let $\gamma : S^1 \times (-\infty, 0) \rightarrow \mathbb{R}^2$ be the turning angle parametrization of the Angenent oval and set $\Gamma_t := \gamma(S^1, t)$. For each $R < 0$, we define Σ^R to be the hypersurface obtained by revolving Γ_{-R} about the x -axis, i.e.

$$\Sigma^R := \{x_R(\theta)e_1 + y_R(\theta)\varphi : \theta \in S^1, \varphi \in S^{n-1} \subset \{e_1\}^\perp\},$$

where x_R, y_R are the coordinates of Γ_{-R} , i.e., $\gamma(\theta, -R) = (x_R(\theta), y_R(\theta))$. Let $F_0^R : S^n \rightarrow \mathbb{R}^{n+1}$ be the Gauss map parametrization of Σ_R .

Consider the maximal $O(n)$ -invariant solution $F_R : S^n \times [-T_R, 0) \rightarrow \mathbb{R}^{n+1}$ to the ϕ -flow with initial data $F_R(\cdot, 0) = F_0^R(\cdot)$. Since Σ^R satisfies $\kappa \geq \lambda$ (see the proof of (Bourni et al., 2021b, Lemma 4.1)), Corollary 3.1.1 ensures that this inequality continues to hold for $t > -T_R$. Proposition 3.3 then guarantees that the timeslices $\Sigma_t^R := F_R(S^n, t)$ shrink to a round point as $t \rightarrow 0$.

Define $\Sigma_t^R := F_R(S^n, t)$ and $\Gamma_t^R := \Sigma_t^R \cap \mathbb{E}^2$. Let $\gamma_R = \gamma_R(\theta, t)$ be the turning angle parametrization of Γ_t^R . By rotating as necessary, we may assume that the unit tangent vector field τ_R satisfies $\tau_R(0) = e_1$. Due to uniqueness of solutions and the symmetry of the initial data, the reflection symmetry about the two axes of \mathbb{E}^2 are preserved under the flow. Consequently, the points $\gamma_R(\frac{\pi}{2}, t), \gamma_R(\pi, t)$ are the unique points of Γ_t^R that lie on the positive axes. Their distances from the origin, $h_R(t) := x(\gamma_R(\frac{\pi}{2}, t))$ and $\ell_R(t) := y(\gamma_R(\pi, t))$, are referred to as the horizontal and vertical displacements of Γ_t^R and they play an important role in our analysis.

We prove the existence of an ancient pancake solution by considering the family of flows $\{F_R\}_{R>0}$ and taking a (subsequential) limit as $R \rightarrow \infty$. For this, we need estimates on the displacements $|F_R|$ and the second fundamental form II that are uniform for sufficiently large R , so that the Arzelà-Ascoli theorem gives us the existence of a convergent subsequence.

3.2.2 Displacement and curvature estimates

For estimates on $|F_R|$, we take advantage of the fact that the flow preserves convexity. Thus it will suffice to have estimates on the horizontal and vertical displacements $h_R(t) := x(\gamma_R(\frac{\pi}{2}, t)) = \langle \gamma_R(\frac{\pi}{2}, t), e_1 \rangle$ and $\ell_R(t) := y(\gamma_R(\pi, t)) = \langle \gamma_R(\pi, t), e_2 \rangle$.

Lemma 3.3.1. $\ell_R(t) \geq h_R(t)$

Proof. Due to Corollary 3.1.1,

$$\begin{aligned}
-\ell'_R(t) &= \phi(\kappa(\pi, t), \lambda(\pi, t)) \\
&\geq \phi(\lambda(\pi, t), \lambda(\pi, t)) \\
&\geq \phi(\lambda(\frac{\pi}{2}, t), \lambda(\frac{\pi}{2}, t)) \\
&= \phi(\kappa(\frac{\pi}{2}, t), \lambda(\frac{\pi}{2}, t)) \\
&= -h'_R(t).
\end{aligned}$$

The claim follows by integrating the above inequality from t to 0, at which time we know that $\ell(0) = h(0) = 0$ due to Proposition 3.3. \square

Define the constant $\phi_1 := \phi(1, 1)$.

Lemma 3.3.2. $-\frac{\pi}{2}t \leq h_R(t)\ell_R(t) \leq -\pi\phi_1 t$.

Proof. Due to Corollary 3.1.1, we have that

$$\kappa = \phi(\kappa, 0) \leq \phi(\kappa, \lambda) \leq \phi(\kappa, \kappa) = \kappa\phi_1. \quad (3.11)$$

Integrating (3.7) and using the above estimate, we get

$$-2\pi t \leq A_R(t) \leq -2\pi\phi_1 t. \quad (3.12)$$

Lastly, by convexity of our solution, we may estimate the area from the inside and outside by the area of quadrilaterals through the points of maximal horizontal and vertical displacements to get

$$2h_R(t)\ell_R(t) \leq A_R(t) \leq 4h_R(t)\ell_R(t). \quad (3.13)$$

The desired estimate for $h_R\ell_R$ now follows. \square

Lemma 3.3.3. $\frac{R}{2\phi_1}(1 - e^{-R}) \leq T_R \leq (R + \log 2)$.

Proof. By estimates in (3.2) for Angenent ovals,

$$\begin{aligned} \frac{\pi}{2} (1 - e^{-R}) &\leq h_R(-T_R) \leq \frac{\pi}{2} \\ R &\leq \ell_R(-T_R) \leq R + \log 2 \end{aligned} \tag{3.14}$$

Now multiplying these inequalities together and using Lemma 3.3.2 yields the desired inequality. \square

Let $\phi^R(\cdot, t) = \phi(\kappa_R(\cdot, t), \lambda_R(\cdot, t))$ be the curvature function evaluated on the hypersurface Σ_t^R whose principal curvatures are denoted $\kappa_R(\cdot, t), \lambda_R(\cdot, t)$. Due to its $O(n)$ -symmetry, we can regard $\kappa_R, \lambda_R, \phi^R$ as functions on the profile curve parametrized by turning angle.

Lemma 3.3.4. $\phi_{\min}^R = \phi^R(\frac{\pi}{2}, \cdot) \leq \frac{2h_R}{h_R^2 + \ell_R^2} \phi_1$.

Proof. A circle \mathcal{C} centered on the x -axis and passing through $\gamma_R(\frac{\pi}{2}, t)$ and $\gamma_R(\pi, t)$ has radius $r = \frac{\ell_R^2 + h_R^2}{2h_R}$ and center $(-(r - h_R), 0)$, and hence any circle with radius $\rho < \frac{\ell_R^2 + h_R^2}{2h_R}$ and center $(-(\rho - h_R), 0)$ is tangent to Γ_t at $\gamma(\frac{\pi}{2}, t)$ and the point $\gamma_R(\pi, t)$ lies to its outside. Due to the curvature of Γ_t being minimized at $\gamma_R(\frac{\pi}{2}, t)$, it then follows that $\kappa_R(\frac{\pi}{2}, t) \leq \frac{2h_R}{\ell_R^2 + h_R^2}$ (See (Bourni et al., 2021b, Claim 4.4.1).) This yields

$$\phi^R\left(\frac{\pi}{2}, t\right) = \phi(\kappa_R(\frac{\pi}{2}, t), \lambda_R(\frac{\pi}{2}, t)) = \phi(\kappa_R(\frac{\pi}{2}, t), \kappa_R(\frac{\pi}{2}, t)) = \phi_1 \kappa_R(\frac{\pi}{2}, t),$$

from which the claim follows. \square

Lemma 3.3.5. $h_R(t) \geq \frac{\pi}{2}(1 - e^{-R}) \exp\left(\frac{2\phi_1}{t}\right)$

Proof. Since $h_R \leq \frac{\pi}{2}$, Lemmas 3.3.2 and 3.3.4 yield

$$-h'_R(t) = \phi^R\left(\frac{\pi}{2}, t\right) \leq \frac{2\phi_1 h_R(t)}{\ell(t)^2} \leq \frac{2\phi_1 h_R(t)}{(-t)^2}.$$

Integrating from T_R to $-t$, we obtain

$$h_R(t) \geq h_R(T_R) \exp\left(\frac{2\phi_1}{T_R} + \frac{2\phi_1}{t}\right).$$

The claim now follows from the estimate (3.2). \square

Lemma 3.3.6. $\ell_R(t) \leq \frac{-2\phi_1 t}{1-e^{-R}} \exp\left(\frac{2\phi_1}{-t}\right)$.

Proof. This follows by combining the estimates of Lemmas 3.3.2 and 3.3.5. \square

3.2.3 The existence argument

Combining the above estimates, we may now extract a limit along some sequence $R_j \rightarrow -\infty$.

Theorem 3.4. *Given any non-degenerate admissible speed ϕ , there is a sequence of approximating solutions which converges locally uniformly in the smooth topology to a smooth, compact, convex, locally uniformly convex, $O(1) \times O(n)$ -invariant ancient solution to the ϕ -flow which lies in the slab $\Omega := \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} : |x| < \frac{\pi}{2}\}$ and in no smaller slab.*

Proof. We need uniform-in- R estimates for F_R and its derivatives on compact subsets of time. A bound for $|F_R|$ follows from Lemma 3.3.6. A bound for the curvature follows from Proposition 3.2 due to the inradius bound (which is a consequence of Lemmas 3.3.1 and 3.3.5 and convexity) and the circumradius bound (which is a consequence of Lemma 3.3.6). Since ϕ_κ is bounded uniformly from above and below, higher order estimates then follow from estimates for inhomogeneous uniformly parabolic equations in one space variable as in Proposition 3.3. Well-known arguments then provide uniform-in- R estimates for F_R in C^k for all k , and also a lower bound for DF_R .

Thus, by the Arzelà–Ascoli theorem, there exists a sequence of flows F_{R_j} , with $R_j \rightarrow -\infty$, converging locally uniformly in the smooth topology to a smooth limit flow. The limit is certainly weakly convex since this is the case for the approximating solutions. Moreover, the upper bound for ℓ_R ensures that the limit is compact (and in particular does not degenerate into two parallel hyperplanes). Strict convexity then follows from applying the strong maximum principle to (3.4) (under slightly stronger

conditions on the speed ϕ , we could also apply (Lynch, 2022, Proposition A.2)). The limit solution inherits rotational and reflection symmetries from the approximating solutions. Convexity and the lower bound for h_R ensure that the solution lies in no smaller slab. \square

3.3 Unique asymptotics and reflection symmetry

We wish to show in Section 3.5 that the solution constructed above is unique in the class of $O(n)$ -invariant convex ancient solutions in the slab $[-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}^n$. We will use Alexandrov's moving plane method to show this. But to be able to do this, we need to know the asymptotics of any such solution; that is the subject of this section.

Consider any convex ancient solution $F : S^n \times (-\infty, 0) \rightarrow \mathbb{R}^{n+1}$ of ϕ -flow that is $O(n)$ -invariant with respect to some $e_1 \in S^n$ and lies in the slab $\Omega := \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} : |x| < \frac{\pi}{2}\}$ and in no smaller slab. Define $\Sigma_t = \Sigma_t^n := F(S^n, t)$. Given some unit vector $e_2 \perp e_1$, define the plane $\mathbb{E}^2 := \text{span}\{e_1, e_2\}$, and parametrize the profile curve $\Gamma_t := \mathbb{E}^2 \cap \Sigma_t$ with respect to turning angle by a curve $\gamma : S^1 \times (-\infty, 0) \rightarrow \mathbb{E}^2$. Denote, as usual, the unit tangent and unit normal by τ, ν respectively and the curvature and rotational curvature by $\kappa(\cdot, t), \lambda(\cdot, t)$ respectively. We continue to use $\phi(\cdot, t) := \phi(\kappa(\cdot, t), \lambda(\cdot, t))$.

Finally, for each $t < 0$ we define the vertical displacement to be

$$\ell(t) := \max_{\theta \in S^1} |\langle \gamma(\theta, t), e_2 \rangle|.$$

Due to convexity and rotational symmetry,

$$\ell(t) = \langle \gamma(\pi, t), e_2 \rangle = -\langle \gamma(0, t), e_2 \rangle.$$

Note that we do not know *a priori* that the curve is reflection-symmetric about the y -axis.

3.3.1 Unique asymptotics

The following two lemmas describe the asymptotic geometry Σ_t as $t \rightarrow \infty$. The statement and proof do not differ from (Bourni et al., 2021b, Lemmas 5.1 and 5.2) in any significant way, since our nonlinearity poses no additional challenges: the proof only depends on convexity, the strong maximum principle, and the Harnack inequality, all of which are available in our nonlinear setting. Hence, we omit the proofs here and refer to the reader to Bourni et al. (2021b) for the proofs.

Lemma 3.4.1. *For any sequence of times $t_i \rightarrow -\infty$ the sequence of flows $\Sigma_t^i := \Sigma_{t+t_i}$ defined on $(-\infty, -t_i)$ converge locally uniformly in the smooth topology to the stationary ϕ -flow defined by $\partial\Omega$.*

Now recall that we have defined a normalization on our speed such that $\phi(1, 0) = 1$. Define α by

$$\alpha^{-1} := \lim_{t \rightarrow -\infty} \phi(\pi, t) \quad (3.15)$$

This limit is well-defined as ϕ is nondecreasing in t due to the differential Harnack inequality (see Andrews (1994c)). Let P be the inverse of the Gauss map, i.e. for any unit vector v , $P(v, t)$ is the point on Σ_t with normal v .

Lemma 3.4.2. *For any sequence of times $t_i \rightarrow -\infty$ and any unit vector $v \perp e_1$ the sequence of flows $\Sigma_t^i := \Sigma_{t+t_i} - P(v, t_i)$ converges locally uniformly in the smooth topology to the scaled Grim hyperplane $\alpha G_{\alpha^{-2t}}^n$ where G_t^n defined by*

$$G_t^n := \{\theta e_1 + (t - \log \cos \theta)v\}, t \in (-\infty, \infty) \quad (3.16)$$

is the standard Grim hyperplane that translates in direction v with unit speed.

Due to containment of the flow in the strip, it is clear that $\alpha \leq 1$. Now we show that $\alpha = 1$. We obtain this from purely geometrical considerations by showing that if $\alpha < 1$, then the area $A(t)$ contained inside Γ_t decreases too quickly (cf. Bourni et al. (2020)).

Lemma 3.4.3. $\ell(t) \geq \alpha^{-1}t$.

Proof. It is a simple consequence of the fact that $-\ell'(t) = \phi(\pi, t) \geq \alpha^{-1}$ due to the Harnack inequality. \square

Lemma 3.4.4. $\lambda(\cdot, t) \leq \kappa(\cdot, t)$.

Proof. By rotational symmetry, we have at the ‘poles’ $\theta_j = \pm\frac{\pi}{2}$ that $\lambda(\frac{\pi}{2}, t) = \kappa(\frac{\pi}{2}, t)$ and $\lambda(-\frac{\pi}{2}, t) = \kappa(-\frac{\pi}{2}, t)$, and by Lemma 3.4.2,

$$\lim_{t \rightarrow -\infty} \frac{\kappa}{\lambda}(\theta, t) = \infty$$

for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The claim now follows from Lemmas 3.1.2 and 3.1.1 and the symmetry of γ . \square

Lemma 3.4.5. $\lim_{t \rightarrow -\infty} A'(t) = -2\pi$.

Proof. We have

$$\begin{aligned} |A' + 2\pi| &= \left| -\int_{\Gamma_t} \phi ds + \int_{\Gamma_t} \kappa ds \right| \\ &= \left| \int_{S^1} (\phi(\kappa, \lambda) - \kappa) \frac{d\theta}{\kappa} \right| \\ &= \left| \int_{S^1} (\phi(1, \lambda/\kappa) - 1) d\theta \right| \end{aligned}$$

By Lemma 3.4.4, $0 \leq \phi(1, \lambda/\kappa) - 1 \leq \phi(1, 1) - 1$ and $\lim_{t \rightarrow -\infty} \lambda/\kappa = 0$ a.e. on S^1 , so that the integrand tends to zero a.e. and the claim follows by Lebesgue dominated convergence. \square

Corollary 3.4.1. *The width of the limiting Grim hyperplane is maximal, i.e. $\alpha = 1$.*

Proof. We claim that for any $\varepsilon > 0$, there exists $t_\varepsilon < 0$ such that for any $t < t_\varepsilon$,

$$A(t) \leq -(2\pi + \varepsilon)t$$

Indeed, by Lemma 3.4.5, given any $\varepsilon > 0$ there exists $t'_\varepsilon < 0$ such that for all $t < t'_\varepsilon$

$$\begin{aligned} A(t) &\leq A(t_\varepsilon) + (2\pi + \frac{\varepsilon}{2})(t'_\varepsilon - t) \\ &= - (2\pi + \frac{\varepsilon}{2})t + \frac{A(t'_\varepsilon) + (2\pi + \frac{\varepsilon}{2})t'_\varepsilon}{t}. \end{aligned}$$

so that we may set $t_\varepsilon \doteq \min\{t'_\varepsilon, -\frac{2}{\varepsilon}(A(t'_\varepsilon) + (2\pi + \frac{\varepsilon}{2})t'_\varepsilon)\}$.

We bound the area from below by an enclosed trapezoid that we describe below. Let $C(t), D(t) \in \Gamma_t$ such that $y(C(t)) = y(D(t)) = 0$. Without loss of generality, assume that $x(C(t)) > x(D(t))$. Let $p(t) = \gamma(0, t)$ be the “tip” of Γ_t . Now for any $\delta \in (0, 1)$ we can find $t_\delta < 0$ such that for all $t < t_\delta$,

$$\pi - \delta \leq x(C(t)) - x(D(t)) \leq \pi.$$

Since the tip region converges locally uniformly to the scaled Grim Reaper αG , one can, (by taking t_δ more negative if need be), also find points $p^\pm(t)$ on Γ_t and a constant C_δ such that

$$y(p^-(t)) = y(p^+(t)),$$

$$\pi\alpha - \delta \leq x(p^+(t)) - x(p^-(t)) \leq \pi\alpha,$$

and

$$0 \leq y(p(t)) - y(p^\pm(t)) = \ell(t) - y(p^\pm(t)) \leq C_\delta.$$

Now since the trapezoid formed by $C(t), D(t), p^\pm(t)$ is contained inside Γ_t (by convexity), we estimate $\ell(t)$ using Lemma 3.4.3 to get

$$(\pi\alpha - \delta + \pi - \delta)(\alpha^{-1}t - C_\delta) \leq A(t).$$

Combining these upper and lower bounds, one sees that for $t < \min\{t_\varepsilon, t_\delta\}$,

$$[\pi(\alpha^{-1} - 1) - 2\delta\alpha^{-1} - \varepsilon](-t) \leq \pi(1 + \alpha)C_\delta + 2\delta C_\delta$$

By choosing ε, δ small, one can make $[\pi(\alpha^{-1} - 1) - 2\delta\alpha^{-1} - \varepsilon] > 0$, and allowing $t \rightarrow -\infty$ yields a contradiction unless $\alpha = 1$. Thus we conclude that $\alpha = 1$. \square

3.3.2 Reflection symmetry

We now exploit the maximality of the width of the limiting Grim hyperplane to deduce the following reflection symmetry.

Theorem 3.5. *Let $\{\Sigma_t\}_{t \leq 0}$ be a convex, ancient $O(n)$ -symmetric solution to ϕ -flow that lies in the slab $\{|x_1| \leq \frac{\pi}{2}\}$ and in no smaller slab. Then it is necessarily reflection symmetric about the hyperplane $\{x_1 = 0\}$.*

Proof. The argument is a standard application of the Alexandrov reflection principle, which holds for parabolic flows, and hence the same as (Bourni et al., 2021b, Theorem 6.2). \square

3.4 Area and displacement estimates

We continue to study arbitrary convex $O(n)$ -invariant ancient solutions lying in a slab. The aim here is to provide improved estimates for the enclosed area $A(t) = \int_{-t}^0 \int_{\Gamma_\tau} \phi \, ds \, d\tau$ and the vertical displacement $\ell(t)$ for such solutions.

Since ϕ is non-degenerate, we may write

$$\begin{aligned} \phi(\kappa, \lambda) &= \kappa\phi(1, \lambda/\kappa) \\ &= \kappa \left(\phi(1, 0) + \phi_\lambda(1, 0)(\lambda/\kappa) + \phi_{\lambda\lambda}(1, \xi)(\lambda/\kappa)^2 \right) \\ &= \kappa\phi(1, 0) + \lambda\phi_\lambda(0, 1) + \phi_{\lambda\lambda}(1, \xi)\lambda^2/\kappa \\ &\leq \kappa + \lambda\phi_\lambda(0, 1) + C\lambda^2/\kappa, \end{aligned}$$

and hence

$$\kappa + \lambda\phi_\lambda(0, 1) - C\lambda^2/\kappa \leq \phi(\kappa, \lambda) \leq \kappa + \lambda\phi_\lambda(0, 1) + C\lambda^2/\kappa \quad (3.17)$$

where $0 \leq \xi \leq \lambda/\kappa$ is given by the mean value theorem, and $C := \sup_{\xi \in [0,1]} |\phi_{\lambda\lambda}(1, \xi)|$. Thus the area estimate will reduce to estimates of $\int_{\Gamma_t} \lambda ds$ and $\int_{\Gamma_t} (\lambda^2/\kappa) ds$.

Due to the reflection symmetry of such solutions (Theorem 3.5), we have that the horizontal displacement

$$h(t) := \max_{\theta \in S^1} \langle \gamma(\theta, t), e_1 \rangle$$

satisfies

$$h(t) = \langle \gamma(\frac{\pi}{2}, t), e_1 \rangle = -\langle \gamma(-\frac{\pi}{2}, t), e_1 \rangle.$$

Moreover, due to Lemma 3.4.4, we have

$$\kappa(\theta, t) \geq \lambda(\theta, t) \geq \lambda(\frac{\pi}{2}, t) = \kappa(\frac{\pi}{2}, t),$$

which then implies that the minimum value of ϕ occurs at the poles, i.e.

$$\min_{\theta \in S^1} \phi(\theta, t) = \phi(\pm \frac{\pi}{2}, t). \quad (3.18)$$

We obtain the desired estimates by using a graphical representation for the solution. The part of the curve Γ_t with $y \geq 0$ can be written as a graph $(x, u(x, t))$ with $x \in [-h(t), h(t)]$. In this representation,

$$\frac{1}{4} \int_{\Gamma_t} \lambda ds = \int_0^{h(t)} \frac{1}{u(x, t)} dx \quad (3.19)$$

and

$$\frac{1}{4} \int_{\Gamma_t} \frac{\lambda^2}{\kappa} ds = \int_0^{h(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} = \int_0^{h(t)} \frac{|\cos \theta|}{\kappa u(x, t)^2} dx \quad (3.20)$$

coming from the fact that $ds = \frac{dx}{|\cos \theta|}$ and $\lambda = \frac{|\cos \theta|}{u}$.

Moreover, the function u satisfies the graphical ϕ -flow equation

$$\frac{du}{dt} = -\phi \sqrt{1 + u_x^2}.$$

Since ϕ is nondecreasing due to the Harnack inequality, and since the solution converges to the Grim reaper, we have the basic estimate

$$-\frac{du}{dt} \geq \phi(1, 0) = 1.$$

Now, given $x \in [0, \frac{\pi}{2})$, we may denote by $T(x)$ the time when the profile curve passes through the point $(x, 0)$, i.e. $h(T(x)) = x$, and hence $u(x, T(x)) = 0$. Now, integrating the previous inequality from T to t , we get for all $x \in [0, h(t))$ that

$$u(x, t) \geq -t + T(x).$$

Observe that

$$\kappa(\frac{\pi}{2}, t) \leq 2 \frac{h(t) - x}{u(x, t)^2},$$

which when combined with $\phi(\kappa, \lambda) \leq \phi(\kappa, \kappa) = \phi(1, 1)\kappa = \phi_1\kappa$ yields

$$\phi_{\min}(t) \leq 2\phi_1 \frac{h(t) - x}{u(x, t)^2}. \quad (3.21)$$

Putting $x = 0$, we get

$$-\frac{dh}{dt} = \phi_{\min} \leq \frac{2\phi_1 h}{(-t)^2}. \quad (3.22)$$

which, upon integration, gives

$$h(t) \geq \frac{\pi}{2} e^{\frac{2\phi_1}{t}} \geq \frac{\pi}{2} \left(1 - \frac{2\phi_1}{-t}\right). \quad (3.23)$$

Setting $t = T(x)$ now gives

$$u(x, t) \geq -t + T(x) \geq -t - \frac{\phi_1 \pi}{\frac{\pi}{2} - x}. \quad (3.24)$$

Iterating yields the following estimates.

Lemma 3.5.1. *For each $k \in \mathbb{N}$ there exists a constant c_k such that for all $t < 0$ and all $x \in [0, h(t)]$,*

$$(i) \quad \phi_{\min}(t) \leq \frac{\phi_1 \pi c_k}{(-t)^{k+1}}$$

$$(ii) \quad h(t) \leq \frac{\pi}{2} \left(1 - \frac{2\phi_1 c_k}{(-t)^k} \right)$$

$$(iii) \quad u(x, t) \geq -t - \left(\frac{\phi_1 \pi c_k}{\frac{\pi}{2} - x} \right)^{1/k}$$

Proof. In view of the above estimates, we may proceed by induction, as in (Bourni et al., 2021b, Lemma 7.1). \square

This allows us to estimate the following integrals:

Lemma 3.5.2. *For any $\varepsilon > 0$,*

$$(i) \quad \int_{\Gamma_t} \lambda ds \leq \frac{2\pi}{-t} + o\left(\frac{1}{(-t)^{2-\varepsilon}}\right)$$

$$(ii) \quad \int_{\Gamma_t} \frac{\lambda^2}{\kappa} ds \leq O\left(\frac{1}{(-t)^2}\right)$$

Proof. The first estimate may be proved as in (Bourni et al., 2021b, Claim 7.2.1). The second estimate is not required in Bourni et al. (2021b) because the mean curvature $H = \kappa + (n-1)\lambda$ is linear in κ and λ . Nonetheless, a similar idea works. Indeed, following (Bourni et al., 2021b, Claim 7.2.1), we define $c(x, t) = \sqrt{\rho(t)^2 - (x - (h(t) - \rho(t)))^2}$, where $\rho(t) = \frac{(-t)^2}{\pi}$, and set $\underline{x}(t) = \inf\{x \in [\frac{\pi}{4}, h(t)] : u(x, t) = c(x, t)\}$, and split the integral to be estimated as

$$\frac{1}{4} \int_{\Gamma_t} \frac{\lambda^2}{\kappa} ds = \int_0^{h(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} = \int_0^{\underline{x}(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} + \int_{\underline{x}(t)}^{h(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|}.$$

Since $\lambda/\kappa \leq 1$,

$$\int_{\underline{x}(t)}^{h(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} \leq \int_{\underline{x}(t)}^{h(t)} \frac{\lambda}{|\cos \theta|} dx = o(t^{-2}) \quad (3.25)$$

just as in (Bourni et al., 2021b, Claim 7.2.1). The remaining term is estimated as follows. Let $k \in \mathbb{N}$ and choose $x_0(t) = \frac{\pi}{2} - \frac{\pi n c_k}{(-t)^k}$. If $\underline{x}(t) \leq x_0(t)$, then

$$\int_0^{\underline{x}(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} \leq \int_0^{x_0(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} \leq \int_0^{x_0(t)} \frac{1}{u(x, t)^2} dx$$

and if not,

$$\begin{aligned} \int_0^{\underline{x}(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} &\leq \int_0^{x_0(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} + \int_{x_0(t)}^{\underline{x}(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} \\ &\leq \int_0^{x_0(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} + \int_{x_0(t)}^{\underline{x}(t)} \lambda \frac{dx}{|\cos \theta|} \\ &\leq \int_0^{x_0(t)} \frac{1}{u(x, t)^2} dx + \int_{x_0(t)}^{h(t)} \frac{dx}{c(x, t)}. \end{aligned}$$

Either way, Lemma 3.5.1 (cf. (Bourni et al., 2021b, Claim 7.2.1)) yields

$$\begin{aligned} \int_0^{\underline{x}(t)} \frac{\lambda^2}{\kappa} \frac{dx}{|\cos \theta|} &\leq \frac{x_0(t)}{u(x_0(t), t)^2} + \int_{x_0(t)}^{h(t)} \frac{dx}{c(x, t)} \\ &\leq \frac{\pi}{2} \left(\frac{1}{-t} + o((-t)^{-1}) \right)^2 + o((-t)^{-2}) \\ &\leq O((-t)^{-2}). \end{aligned} \tag{3.26}$$

Putting equations (3.25) and (3.26) together yields

$$\frac{1}{4} \int_{\Gamma_t} \frac{\lambda^2}{\kappa} ds \leq O((-t)^{-2})$$

as claimed. □

This leads us to the following area estimate:

Corollary 3.5.1.

$$-t + \phi_\lambda(0, 1) \log(-t) - C \leq \frac{A(t)}{2\pi} \leq -t + \phi_\lambda(0, 1) \log(-t) + C \tag{3.27}$$

Proof. The upper bound follows from integrating $A'(t)$ in ((3.7)), estimating ϕ using the upper bound in equation ((3.17)) and using Lemma 3.5.2. The lower bound will be inferred from the upper bound via an application of Hölder's inequality. Indeed,

$$h^2(t) \leq \int_0^{h(t)} u(x, t) dx \int_0^{h(t)} u(x, t)^{-1} dx = \frac{A(t)}{16} \int_{\Gamma_t} \lambda ds,$$

so that Lemma 3.5.1 gives

$$\int_{\Gamma_t} \lambda ds \geq \frac{2\pi \left(1 - \frac{2\phi_1 c_1}{-t}\right)^2}{-t + \phi_1 \log(-t) + C} \geq \frac{2\pi}{-t} - o\left(\frac{1}{(-t)^{2-\epsilon}}\right).$$

Recalling the lower bound in (3.17) and Lemma 3.5.1 (ii), the desired lower bound for the enclosed area now follows by integrating (3.7). \square

Now the following refinement to the displacement estimate is obtained:

Lemma 3.5.3. *For any $\varepsilon \in (0, 1)$,*

$$\ell(t) \leq -t + o((-t)^\varepsilon). \quad (3.28)$$

Proof. This follows from the same area estimation trick of (Bourni et al., 2021b, Lemma 7.3). \square

Corollary 3.5.2. *For any $\varepsilon \in (0, 1)$*

$$\phi(\theta, t) \geq |\cos \theta| \left(1 + \frac{\phi_\lambda(0, 1)}{-t} - o\left(\frac{1}{(-t)^{2-\varepsilon}}\right)\right) \text{ as } t \rightarrow -\infty \text{ for all } \theta \in S^1,$$

where $\phi_\lambda(0, 1)$ is defined by (3.17).

Proof. By symmetry, it suffices to prove the claim for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (note that for $\theta = \pm\frac{\pi}{2}$ the claim holds trivially). Consider the function $w : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-t, 0) \rightarrow \mathbb{R}$ defined by $w(\theta, t) := f(t) \cos \theta$, where the function $f : (-\infty, 0) \rightarrow \mathbb{R}_+$ will be

determined momentarily. Observe that

$$w_t = \phi_\kappa \kappa^2 w_{\theta\theta} + |\mathbb{II}|_\phi^2 w + \left(\frac{f'}{f} - \phi_\lambda \lambda^2 \right) w,$$

where $|\mathbb{II}|_\phi^2 := \phi_\kappa \kappa^2 + \phi_\lambda \lambda^2$. Recalling Lemma (3.1.1), we compute

$$\frac{w}{\phi} (\partial_t - \phi_\kappa \kappa^2 \partial_\theta^2) \frac{\phi}{w} = 2\phi_\kappa \kappa^2 \frac{w}{\phi} \left(\frac{\phi}{w} \right)_\theta \frac{w_\theta}{w} - \phi_\lambda \lambda^2 \tan \theta \frac{\phi_\theta}{\phi} - \left(\frac{f'}{f} - \phi_\lambda \lambda^2 \right).$$

Rewriting

$$\frac{\phi_\theta}{\phi} = \frac{w}{\phi} \left(\frac{\phi}{w} \right)_\theta - \frac{w_\theta}{w} \quad \text{and} \quad \frac{w_\theta}{w} = -\tan \theta$$

we obtain

$$\begin{aligned} \frac{w}{\phi} (\partial_t - \phi_\kappa \kappa^2 \partial_\theta^2) \frac{\phi}{w} + \tan \theta \frac{w}{\phi} \left(\frac{\phi}{w} \right)_\theta (\phi_\lambda \lambda^2 + 2\phi_\kappa \kappa^2) \\ = \phi_\lambda \sec^2 \theta \lambda^2 - \frac{f'}{f} \\ = \frac{\phi_\lambda}{y^2} - \frac{f'}{f} \\ \geq \frac{\phi_\lambda(0,1)}{y^2} - C \frac{\lambda}{\kappa y^2} - \frac{f'}{f} \\ \geq \frac{\phi_\lambda}{\ell^2} - \frac{C}{\ell^3} - \frac{f'}{f} \end{aligned} \tag{3.29}$$

since $\kappa \geq C\phi \geq \cos \theta$.

Now fix any $\varepsilon \in (0, 1)$. By Lemma 3.5.3, there is some $C_\varepsilon < \infty$ such that

$$\frac{1}{\ell(t)^2} \geq \frac{1}{(-t)^2} - \frac{C_\varepsilon}{(-t)^{3-\varepsilon}}$$

for all $t \in (-\infty, -1]$, say. Thus, if we set

$$f(t) := \exp \left(\left[\frac{\phi_\lambda(0,1)}{-t} - \frac{\Lambda}{(2-\varepsilon)(-t)^{2-\varepsilon}} \right] \right),$$

then we may choose Λ so large that

$$\frac{f'(t)}{f(t)} = \left[\frac{\phi_\lambda(0, 1)}{(-t)^2} - \frac{\Lambda}{(-t)^{3-\varepsilon}} \right] \leq \frac{\phi_\lambda(0, 1)}{\ell(t)^2} - \frac{C}{\ell^3}$$

for all $t < -1$. So the maximum principle yields

$$\min_{S^1 \times \{t\}} \frac{\phi}{w} \geq \min_{S^1 \times \{t_0\}} \frac{\phi}{w} \quad \text{for all } -1 > t > t_0.$$

But the right hand side approaches 1 as $t \rightarrow -\infty$. The claim follows by estimating $\exp(\zeta) \geq 1 + \zeta$. \square

Integrating the lower speed bound yields a displacement estimate.

Lemma 3.5.4. *The limit*

$$C := \lim_{t \rightarrow -\infty} (\ell(t) + t - \phi_\lambda(0, 1) \log(-t))$$

exists (in the extended real line $\mathbb{R} \cup \{\infty\}$).

Proof. Given any $\varepsilon \in (0, 1)$ set $f(t) := \frac{C}{1-\varepsilon} \frac{1}{(-t)^{1-\varepsilon}}$ for some $C \in \mathbb{R}$. By Corollary 3.5.2, we can choose C so that

$$\frac{d}{dt} (\ell + t - \phi_\lambda(0, 1) \log(-t) - f) \leq 0.$$

The claim follows because $\lim_{t \rightarrow -\infty} f = 0$. \square

3.5 Uniqueness

In this section, we will show that the pancake solutions constructed in Section 4 are unique. We will do so in several steps. The essential idea, as in [Bourni et al. \(2021b\)](#), is to exploit the constructed solution as a barrier using the Alexandrov reflection

principle. First, we show that on the constructed solution the constant C defined by

$$C := \lim_{t \rightarrow -\infty} (\ell(t) + t - \phi_\lambda(0, 1) \log(-t))$$

is finite. We will use methods developed in Section 3.4 to obtain area and displacement estimates, which will allow us to prove this. We require the following lemma.

Lemma 3.5.5. *On the approximating solutions, $\phi_R(\theta, t) \geq |\cos \theta|$.*

Proof. First observe that $\phi_R(\theta, -T_R) \geq |\cos \theta|$ by construction, and the inequality is always (trivially) satisfied at the poles $\theta = \pm \frac{\pi}{2}$. So we need to show that it continues to hold in $(-\frac{\pi}{2}, \frac{\pi}{2})$ (by reflection symmetry, it will hold on the other half as well). To that end, consider the function $w = \phi / \cos \theta$. Suppose that w develops a new interior minimum at some (θ_0, t_0) . At this point, we have

$$\phi_\theta = -\phi \tan \theta_0$$

and hence

$$0 \geq w_t - \kappa^2 \phi_k w_{\theta\theta} = \phi_\lambda \lambda^2 \phi \sec^3 \theta_0 > 0.$$

This leads to a contradiction and the claim follows. \square

Equations (3.2) now allow us to proceed as in §3.4 to obtain the area estimate

$$-t + \phi_\lambda(0, 1) \log(-t) + C \geq \frac{A(t)}{2\pi} \geq -t + \phi_\lambda(0, 1) \log(-t) - C \quad (3.30)$$

for all $t \in [-T_R, 1)$. Bounding the area of $\Gamma_R(t)$ by a rectangle of height $2\ell_R(t)$ and width π gives the length estimate

$$\ell_R(t) \geq -t + \phi_\lambda(0, 1) \log(-t) - C.$$

Now we show that $\ell_R(t)$ is bounded similarly from above (uniformly with respect to R). Proceeding in the same way as Lemma 3.5.3, we obtain that for any $\varepsilon \in (0, 1)$,

$$\ell(t) \leq -t + C_\varepsilon(-t)^\varepsilon,$$

where C_ε is independent of R . Now, proceeding as in Bourni et al. (2021b), we may choose $C < \infty$ such that

$$\frac{d}{dt}(\ell_R(t) + t - \phi_\lambda(0, 1) \log(-t) - f) \leq 0,$$

where $f := \frac{C}{1-\varepsilon} \frac{1}{(-t)^{1-\varepsilon}}$. Now integration yields

$$\ell_R(t) + t - \phi_\lambda(0, 1) \log(-t) \leq \ell_R(-T_R) - T_R - \phi_\lambda(0, 1) \log(T_R) + f(t) - f(-T_R).$$

Now, since

$$\ell_R(-T_R) \leq T_R + \phi_\lambda(0, 1) \log(-T_R) + C + \log 2$$

and $\lim_{t \rightarrow -\infty} f(t) = 0$, we let $R \rightarrow \infty$ to see that C is finite.

Now we sketch the proof of uniqueness (cf. (Bourni et al., 2021b, §8)). Suppose that $\gamma, \gamma' : S^1 \times (-\infty, 0) \rightarrow \mathbb{R}^2$ are the turning angle parametrizations of the profile curves Γ_t, Γ'_t respectively of the solution constructed in Section 3, and an arbitrary $O(n)$ -invariant compact convex ancient solution to the ϕ -flow that lies in the same slab and no smaller slab. Let Ω_t, Ω'_t be the open sets contained inside the profile curves. Since both solutions contract to the origin at $t = 0$, they must intersect for all previous times due to the avoidance principle. Now consider the two constants

$$C := \lim_{t \rightarrow -\infty} (\ell(t) + t - \phi_\lambda(0, 1) \log(-t))$$

$$C' := \lim_{t \rightarrow -\infty} (\ell'(t) + t - \phi_\lambda(0, 1) \log(-t)),$$

where $\ell(t), \ell'(t)$ are the respective vertical displacements of γ, γ' . We have just shown that C is finite, and that $C' \in \mathbb{R} \cup \{\infty\}$. If $C \neq C'$, then it follows from the strong maximum principle, just as in (Bourni et al., 2021b, Proof of Theorem 1.2), that $\Omega'_t \subset \Omega_t$ or $\Omega_t \subset \Omega'_t$, and hence Γ_t, Γ'_t never intersect. Thus $C = C'$. This implies, by the same reasoning, that $\Omega'_t \subset \Omega_{t+\tau}$ for every $\tau > 0$. Taking $\tau \rightarrow 0$ shows that $\bar{\Omega}'_t \subset \bar{\Omega}_t$. Similarly, $\bar{\Omega}_t \subset \bar{\Omega}'_t$. Thus the solutions coincide.

Chapter 4

Further Results on Translating Solutions

In this chapter, we return to the study of translating solutions to a speed function f , or f -translators. Building on top of Chapter 2, we obtain the asymptotic expansion for the analogue of the bowl-soliton for a large ‘nondegenerate’ class of fully nonlinear curvature flows. We use this to show the uniqueness of these bowl-type solitons in their asymptotic class. We also give examples to illustrate the situation for ‘degenerate’ speeds and how different they can be. Finally, we show how to construct ‘wing-like’ solitons for these flows, which are complete, connected translators that are not graphical, entire or convex. We also obtain asymptotic expansions for them to show the variety of solutions that one can obtain depending on the choice of speed function.

It is worth mentioning that f -translators have been widely studied when $f = H$ (see for instance [Hoffman et al. \(2021\)](#) for a complete survey about H -translators, model of singularities, and minimal surfaces theory.) Another relevant work is [Urbas \(1998c\)](#), which discusses existence and properties of $\sqrt[\alpha]{S_n}$ -translators for $\alpha > 0$ (more commonly known as the powers of Gauss curvature.)

We have seen in Chapter 2 that when f is a 1-homogeneous speed satisfying $f(0, 1, \dots, 1) > 0$ (i.e. nondegenerate in the sense of Chapter 2), the “bowl”-type

soliton is always entire and behaves like a paraboloid at infinity, i.e:

$$\frac{|x|^2}{2f(0, 1, \dots, 1)} + o(|x|^2), \text{ as } |x| \rightarrow \infty.$$

It is an interesting question what the lower order terms are. When f is the mean curvature, [Clutterbuck et al. \(2007\)](#) showed that the bowl soliton is smoothly asymptotic to

$$\frac{|x|^2}{2(n-1)} - \ln(|x|) + O(|x|^{-1}), \text{ as } |x| \rightarrow \infty. \quad (4.1)$$

We extend the above result to a large class of speeds which we define below. For convenience, we denote $\mathbf{e} := (1, \dots, 1) \in \mathbb{R}^{n-1}$, the $(n-1)$ -tuple of 1's.

We consider speed functions with the following properties:

- a. $\Gamma \subset \mathbb{R}^n$ is an open symmetric cone that contains the positive cone $\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$.
- b. f is positive and symmetric, i.e.: $f(\sigma(\lambda_1), \dots, \sigma(\lambda_n)) = f(\lambda_1, \dots, \lambda_n)$ for every permutation $\sigma \in S_n$.
- c. f is strictly increasing in each variable, i.e.: $\frac{\partial f}{\partial \lambda_i}(\lambda) > 0$ holds for every $\lambda \in \Gamma$ and $i = 1, \dots, n$.
- d. f is 1-homogeneous, i.e.: $f(c\lambda) = cf(\lambda)$ for every $c > 0$.
- e. f vanishes at boundary of Γ , i.e.: there exist a continuous function $\tilde{f} : \bar{\Gamma} \rightarrow \mathbb{R}$ such that $\tilde{f}|_{\Gamma} = f$ and $\tilde{f}|_{\partial\Gamma} = 0$.

Remark. *In this chapter, we revert to using f to denote the speed function. However, the principal curvatures are hitherto referred to as λ_i 's. Note that the speeds described above are simply the admissible speeds defined in [Definition 1](#) that are additionally 1-homogeneous.*

Theorem 4.1. *Assume that $f : \Gamma \rightarrow [0, \infty)$ satisfies properties [a-d](#) and is nondegenerate. Then, the entire “bowl”-type solution is smoothly asymptotic to*

$$\frac{|x|^2}{2f(0, \mathbf{e})} - \frac{\partial f}{\partial \lambda_1} \Big|_{(0, \mathbf{e})} \ln(|x|) + O(|x|^{-1}), \text{ as } |x| \rightarrow \infty.$$

The term “smoothly asymptotic” is defined in [Definition 8](#). Furthermore, by applying the same techniques employed in [Martín et al. \(2015\)](#), we show that that the bowl-type soliton is essentially unique in the asymptotic class of $O(|x|^2)$ solutions via the following theorem:

Theorem 4.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a strictly convex complete f -translator with a single end smoothly asymptotic to the “bowl”-type solution. Then, if f satisfies properties [a-e](#) and is nondegenerate, Σ is the “bowl”-type solution up to vertical translations.*

To show that [Theorem 4.1](#) does not apply to all speed functions, we discuss the degenerate speed function $f = \sqrt[n]{S_n}$ (the n^{th} root of the Gauss curvature) and show that the bowl soliton does not have quadratic asymptotics in any dimension $n \geq 2$.

Theorem 4.3. *Let $r := |x|$ be the Euclidean norm of an n -tuple. The “bowl”-type solution for the speed function $f = \sqrt[n]{S_n}$ is smoothly asymptotic to*

$$\begin{cases} \int_0^r e^{\frac{s^2}{2}} ds + O\left(\int_0^r \sqrt{e^{\frac{s^2}{2}} - 1} ds\right), & \text{for } n = 2, \\ \frac{r^4}{12} + O(r), & \text{for } n = 3, \\ \frac{(n-2)^{\frac{n-1}{n-2}}}{2(n-1)^{\frac{1}{n-2}}} r^{\frac{2(n-1)}{n-2}} + O\left(r^{\frac{2}{n-2}}\right), & \text{for } n \geq 4 \end{cases}, \text{ as } r \rightarrow \infty.$$

Then, we discuss a special kind of translator known as the *wing-like* solution first studied in the context of mean curvature flows in [Clutterbuck et al. \(2007\)](#).

Theorem 4.4. *For every $R > 0$, there exist a non-convex complete rotationally symmetric f -translator W_R with respect to x_{n+1} -axis $f = \sqrt[k]{S_k}$ and $f = \frac{S_k}{S_{k-1}}$ such that:*

1. For $f = \sqrt[k]{S_k}$, we distinguish even and odd cases:

(a) When k is even: $W_R \setminus B_{R_1}(0)$ with $R_1 > R$ possesses two graphical branches

$W_R^+, W_R^- : \mathbb{R}^n \setminus B_R(0) \rightarrow \mathbb{R}$ smoothly asymptotic to

$$W_R^\pm(x) = \pm \left(\frac{|x|^2}{2f(0, \mathbf{e})} - \frac{\partial f}{\partial \lambda_1} \Big|_{(0, \mathbf{e})} \ln(|x|) + O(|x|^{-1}) + C^\pm \right), \text{ as } |x| \rightarrow \infty.$$

(b) When k is odd: W_R possesses a \mathbb{S}^{n-1} boundary component and $W_R \setminus B_{R_1}(0)$ with $R_1 > R$ is given by a vertical graph smoothly asymptotic to

$$W_R(x) = \frac{|x|^2}{2f(0, \mathbf{e})} - \frac{\partial f}{\partial \lambda_1} \Big|_{(0, \mathbf{e})} \ln(|x|) + O(|x|^{-1}) + C, \text{ as } |x| \rightarrow \infty.$$

2. For $f = \frac{S_k}{S_{k-1}}$ we have that $W_R \setminus B_{R_1}(0)$ with $R_1 > R$ posses two graphical branches W_R^+, W_R^- such that

$$W_R^+(x) = \frac{|x|^2}{2f(0, \mathbf{e})} - \frac{\partial f}{\partial \lambda_1} \Big|_{(0, \mathbf{e})} \ln(|x|) + O(|x|^{-1}) + C^+, \text{ as } |x| \rightarrow \infty,$$

$$\lim_{|x| \rightarrow \infty} |\nabla W_R^-(x)| = 0.$$

Finally, we provide some applications of the above theorems. The following is an asymptotic growth estimate:

Theorem 4.5. *Let $\Sigma = \{(x, u(x)) : x \in \mathbb{R}^n\}$ be an entire convex translating solution of (1.1) for some nondegenerate f . Assume further, that there exist $a, b, C_1, C_2, R > 0$ such that*

$$C_1|x|^a \leq u(x) \leq C_2|x|^b, \text{ for } |x| \geq R, \quad (4.2)$$

then, $a \leq 2 \leq b$. In addition, if $a = b = 2$, then $u(x)$ agrees with the “bowl”-type solution up to vertical translations.

The organization of this chapter is as follows: In Section 4.1, we discuss the preliminaries of the differential geometry of axially symmetric translators and the ODE theory needed for the analysis of the ODE that the translator solves. In Section 4.2, we derive the asymptotic expansion up to $o(|x|^{-2})$ of the slope field of the translator. We use this estimate in Section 4.3 to prove the uniqueness result in Theorem 4.2. Section 4.4 discusses the degenerate example $\sqrt[n]{S_n}$. Section 4.5 concerns the discussion of winglike translators. Finally, in Section 4.6 we give a proof of Theorems 4.5.

4.1 Preliminaries

4.1.1 The rotational translator ODE

Since the general case for $\alpha > 0$ has been described in Chapter 2 Section 2.1, we will directly recall the relevant equations when $\alpha = 1$. For a real-valued C^2 -function u of a single real variable we consider Σ to be the graph of $y = u(r)$, where $r = |x|$ and $x \in \mathbb{R}^n$. For it to be a translator, it must satisfy the ODE

$$f\left(\frac{u''}{(1+u'^2)^{3/2}}, \frac{u'}{r\sqrt{1+u'^2}}\right) = \frac{1}{\sqrt{1+u'^2}}.$$

Remark. *Note that we are abusing notation by using $f(x, y)$ to mean $f(x, \mathbf{y})$. In the context of axially symmetric solutions, there are only two distinct principal curvatures, and hence we choose this more economical notation.*

With $v = u'$, and with the 1-homogeneity of f we may write it as

$$f\left(\frac{v'}{1+v^2}, \frac{v}{r}\right) = 1. \tag{4.3}$$

Recall the function g defined implicitly by

$$f(g(y, z), y) = z. \quad (4.4)$$

Note that in Eq. (4.3), we have $z = 1$, and in this case, we will suppress the second argument and refer to $g(y, 1)$ as simply $g(y)$. We will return to using $g(y, z)$ in Section 4.5 when the second argument is allowed to be different from 1.

Then in canonical form, v solves the initial value problem

$$\begin{cases} v'(r) = (1 + v^2(r)) g\left(\frac{v(r)}{r}\right), r \geq 0, \\ v(0) = 0. \end{cases}, \quad (4.5)$$

Example 1. *In addition to the examples of the first chapter, we give a couple more examples of admissible speeds f and the corresponding g so that we may recall them for future use when we consider winglike solutions of these speeds.*

1. *The k th-root of the symmetric elemental polynomials, $\sqrt[k]{S_k}$:*

$$f(x, y) = \sqrt[k]{\binom{n-1}{k} y^k + \binom{n-1}{k-1} x y^{k-1}} \text{ and } g(y) = \binom{n-1}{k-1}^{-1} y^{1-k} - \frac{n-k}{k} y.$$

2. *The quotients of the symmetric elemental polynomials, $Q_{k+1,k} = \frac{S_{k+1}}{S_k}$:*

$$f(x, y) = \frac{\binom{n-1}{k} x y^k + \binom{n-1}{k+1} y^{k+1}}{\binom{n-1}{k-1} x y^{k-1} + \binom{n-1}{k} y^k} \text{ and } g(y) = \frac{n-k}{k+1} y \frac{(k+1) - (n-k-1)y}{(n-k)y - k}.$$

4.1.2 Differentiability properties of nondegenerate speeds

We always assume without loss of generality that $f(0, \mathbf{e}) = 1$ whenever f is nondegenerate. In addition, since $f(x, y)$ is 1-homogeneous, we have the Euler identity

$$f(x, y) = f_x x + f_y y.$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. We note that the partial derivatives f_x and f_y are 0-homogeneous functions, and by our normalization, we have

$$f_y(0, 1) = 1 \tag{4.6}$$

Now, we outline an estimation trick that we will use repeatedly in this chapter. Due to 1-homogeneity of f , we have $f(x, y) = yf(x/y, 1)$ and hence:

- When f is C^1 , using the mean value theorem, we may write

$$f(x, y) = y(1 + f_x(\xi, 1))\frac{x}{y} = y + f_x(\xi, 1)x \tag{4.7}$$

for some $0 \leq \xi \leq \frac{x}{y}$.

- When f is C^2 , Taylor's remainder theorem yields

$$f(x, y) = y \left(1 + f_x(0, 1)\frac{x}{y} + \frac{1}{2}f_{xx}(\xi, 1)\frac{x^2}{y^2} \right) = y + f_x(0, 1)x + \frac{1}{2y}f_{xx}(\xi, 1)x^2 \tag{4.8}$$

Therefore, by the normalization of f , we will assume in most calculations in this chapter that

$$0 < x \leq y \Leftrightarrow 0 < \xi \leq 1,$$

and consequently, $|f_x(\xi, 1)|, |f_{xx}(\xi, 1)|$ are bounded by compactness of $[0, 1]$ and the continuity of these functions.

4.1.3 Differentiability properties of g

Due to our normalization $f(0, 1) = 1$, we have $g(1) = 0$. Moreover, by the chain rule, we get

$$f_x g_y + f_y = 0.$$

Differentiating once more, we get

$$f_{xx}g_y^2 + 2f_{xy}g_y + f_{yy} + f_x g_{yy} = 0.$$

Next, we note that whenever $f_x \neq 0$ it holds that

$$g_y = -\frac{f_y}{f_x}.$$

In particular, $g_y \leq 0$, indicating that $g(y)$ is decreasing.

Remark. *In particular, when f_{xx}, f_{xy}, f_{yy} are defined at $(x, y) = (g(y), y)$, we have*

$$g_{yy} = -\frac{f_{xx}g_y^2 + 2f_{xy}g_y + f_{yy}}{f_x}.$$

Therefore, it is interesting that g will be convex when f is concave and vice versa. However, since our results do not rely on the convexity properties of f , we won't be using this fact.

4.2 Asymptotics of bowl-type solutions

Recall from the introduction that a bowl-type soliton is a complete strictly convex smooth solution of (4.5) with $u' = v$, which is defined in $[0, R)$ where $R \in \left\{ \frac{1}{f(1,1)}, \infty \right\}$.

We will do so by showing that v satisfies

$$v(r) = r - \frac{c}{r} + o(r^{-2}), \text{ as } r \rightarrow \infty,$$

whence the claim of the theorem follows at once by integration.

The method that we follow is a bootstrapping approach where we progressively refine the asymptotics. For this, we will use the ODE sub- and super-solutions (reference attached in A.1) denoted by $w_{\pm, \varepsilon}(r)$, where the ε will be used in asymptotic little-oh notation.

By definition, w is a sub-solution (resp. super-solution) to Eq. (4.5) if

$$w' \leq (1 + w^2)g(w/r), \quad (\text{ resp. } \geq).$$

However, in some cases it may be more convenient to check the following equivalent condition

$$f\left(\frac{w'}{1 + w^2}, \frac{w}{r}\right) \leq 1, \quad (\geq \text{ resp.}). \quad (4.9)$$

Proposition 4.6. *The functions $w_+(r) = r$ and $w_{-, \varepsilon}(r) = (1 - \varepsilon)r$ satisfy the following properties:*

1. $w_+(r)$ is a super-solution to Eq. (4.5) $r > 0$.
2. For every $\varepsilon \in (0, 1)$, $w_{-, \varepsilon}(r)$ is a sub-solution to Eq. (4.5) for sufficiently large r . Moreover, given any $r_0 > 0$, there exists $r_1 > r_0$ such that $v(r_1) \geq r_1$. Thus, $v \geq (1 - \varepsilon)r$ for sufficiently large r .

Proof. The proofs are all direct computations.

1. Firstly, we note that $f(x, y)$ is increasing in each variable, then it holds

$$f\left(\frac{w'_+}{1 + w_+^2}, \frac{w_+}{r}\right) = f\left(\frac{1}{1 + r^2}, 1\right) \geq f(0, 1) = 1.$$

Therefore, $w_+(r)$ is a super-solution to Eq. (4.5) $r > 0$.

2. Next, by evaluating $w_{-, \varepsilon}$ in (4.3) and taking the limit as $r \rightarrow \infty$, we see that

$$\begin{aligned} \lim_{r \rightarrow \infty} f\left(\frac{w'_{-, \varepsilon}}{1 + w_{-, \varepsilon}^2}, \frac{w_{-, \varepsilon}}{r}\right) &= \lim_{r \rightarrow \infty} f\left(\frac{1 - \varepsilon}{1 + ((1 - \varepsilon)r)^2}, 1 - \varepsilon\right) \\ &= f(0, 1 - \varepsilon) \\ &< f(0, 1) = 1. \end{aligned}$$

Therefore, the claim is true for sufficiently large r .

On the other hand, we will prove the following part by contradiction. Let $r_0 > 0$ and assume that $v(r) < (1 - \epsilon)r$ for all $r > r_0$. Then, since $g(y)$ is decreasing in y , it follows that

$$\begin{aligned} v' &= (1 + v^2)g\left(\frac{v}{r}\right) \\ &> (1 + v^2)g(1 - \epsilon) \\ &= C(1 + v^2) \end{aligned}$$

for $r > r_0$ and some $C > 0$. Consequently, this inequality implies that v blows up at some finite $r_1 > r_0$ contradicting that $v(r)$ exist for all $r \geq 0$.

□

Remark. Proposition 4.6 implies that the solution $v(r) = r + o(r)$ as $r \rightarrow \infty$. Note that this proposition was proved in Chapter 2, but this proof style foreshadows the technique used in the following steps.

Proposition 4.7. The function $w_{-\epsilon}(r) = r - \epsilon$ satisfies:

1. For every $\epsilon \in (0, 1)$, $w_{-\epsilon}(r)$ is a sub-solution to Eq. (4.3) for sufficiently large r .
2. Given any $r_0 > 0$, there exists $r_1 > r_0$ such that $v(r_1) \geq r_1$. Thus, $v \geq r - \epsilon$ for sufficiently large r .

Proof. 1. We will show that $w_{-\epsilon}$ verifies Eq. (4.9). Indeed, by Eq. (4.7), we may write

$$\begin{aligned} f\left(\frac{w'_{-\epsilon}}{1 + w_{-\epsilon}^2}, \frac{w_{-\epsilon}}{r}\right) &= f\left(\frac{1}{1 + (r - \epsilon)^2}, 1 - \frac{\epsilon}{r}\right) \\ &= 1 - \frac{\epsilon}{r} + \frac{f_x(\xi, 1)}{1 + (r - \epsilon)^2}. \end{aligned}$$

Then, since

$$0 \leq \xi \leq \frac{1}{\left(1 - \frac{\varepsilon}{r}\right)(1 + (r - \varepsilon)^2)} < 1$$

for r sufficiently large, we obtain

$$f\left(\frac{w'_{-, \varepsilon}}{1 + w_{-, \varepsilon}^2}, \frac{w_{-, \varepsilon}}{r}\right) = 1 - \frac{\varepsilon}{r} + o(r^{-1}) < 1.$$

for sufficiently large r .

2. Next, by arguing by contradiction, we fix $r_0 > 0$ and assume that $v(r) < r - \varepsilon$ holds for all $r > r_0$ and $\varepsilon \in (0, 1)$.

Then, by part 2 of Proposition 4.6, we have that $v \geq \frac{r}{2}$ for sufficiently large r and

$$\begin{aligned} v' &= (1 + v^2)g\left(\frac{v}{r}\right) \\ &> \left(1 + \frac{r^2}{4}\right)g\left(1 - \frac{\varepsilon}{r}\right). \end{aligned}$$

Finally, recall that $g(1) = 0$. Then, by the mean value theorem, we have

$$v' \geq \left(1 + \frac{r^2}{4}\right)\left(g(1) - g_y(\xi)\frac{\varepsilon}{r}\right),$$

for some $\xi \in \left(1 - \frac{\varepsilon}{r}, 1\right)$. Consequently, since g is decreasing, there is $C > 0$ depending on ε such that

$$v' \geq Cr, \text{ for all } r > r_0.$$

However, this fact contradicts $v(r) < r - \varepsilon$ for all $r > r_0$.

□

Remark. Proposition 4.7 implies that $v(r) = r + o(1)$ as $r \rightarrow \infty$.

Proposition 4.8. *Let $c = f_x(0, 1)$ and consider*

$$w_{+,\varepsilon}(r) = r - \frac{c - \varepsilon}{r} \text{ and } w_{-,\varepsilon}(r) = r - \frac{c + \varepsilon}{r}.$$

1. *For every $\varepsilon \in (0, c)$, $w_{+,\varepsilon}(r)$ is a super-solution to Eq. (4.3) for sufficiently large r . In addition, given any $r_0 > 0$, there exists $r_1 > r_0$ such that $v(r_1) \leq r_1 - \frac{c - \varepsilon}{r_1}$. Thus, $v \leq r - \frac{c - \varepsilon}{r}$ for sufficiently large r .*
2. *For every $\varepsilon > 0$, $w_{-,\varepsilon}(r)$ is a sub-solution to Eq. (4.3) for sufficiently large r . Moreover, given any $r_0 > 0$, there exists $r_1 > r_0$ such that $v(r_1) \geq r_1$. Thus, $v \geq r - \frac{c + \varepsilon}{r}$ for sufficiently large r .*

Proof. 1. Taking sufficiently large r as in part 2 of Proposition 4.7, but using Eq. 4.8 instead, we have

$$\begin{aligned} f\left(\frac{w'_{+,\varepsilon}}{1 + w_{+,\varepsilon}^2}, \frac{w_{+,\varepsilon}}{r}\right) &= f\left(\frac{1 + \frac{c - \varepsilon}{r^2}}{1 + (r - \frac{c - \varepsilon}{r})^2}, 1 - \frac{c - \varepsilon}{r^2}\right) \\ &\geq f\left(\frac{1}{1 + r^2}, 1 - \frac{c - \varepsilon}{r^2}\right) \\ &\geq 1 - \frac{c - \varepsilon}{r^2} + \frac{f_x(0, 1)}{1 + r^2} + \frac{1}{2(1 - \frac{c - \varepsilon}{r^2})} f_{xx}(\xi, 1) \frac{1}{(1 + r^2)^2} \\ &= 1 + \frac{\varepsilon}{r^2} - \frac{c}{r^2(1 + r^2)} + \frac{1}{2(1 - \frac{c - \varepsilon}{r^2})} \frac{f_{xx}(\xi, 1)}{(1 + r^2)^2} \\ &\geq 1 + \frac{\varepsilon}{r^2} - O(r^{-4}) \\ &> 1, \text{ for sufficiently large } r. \end{aligned}$$

For the next part, let $r_0 > 0$ and we assume

$$v(r) > r - \frac{c - \varepsilon}{r}, \text{ for all } r > r_0.$$

Then, by Proposition 4.6, we have

$$\begin{aligned} v' &= (1 + v^2)g\left(\frac{v}{r}\right) \\ &\leq (1 + r^2)g\left(1 - \frac{c - \varepsilon}{r^2}\right), \end{aligned}$$

since g is decreasing.

We note that the second-order expansion of g allows us to write

$$v \leq (1 + r^2) \left(g(1) - g_y(1) \frac{c - \varepsilon}{r^2} + \frac{g_{yy}(\xi) c^2}{2 r^4} \right).$$

Recall (4.6), $c = f_x(0, 1)$ and $g(1) = 0$, then we have $g_y(1) = -c^{-1}$ and

$$\begin{aligned} v' &\leq (1 + r^2) \left(\frac{c - \varepsilon}{cr^2} + \frac{g_{yy}(\xi) c^2}{2 r^4} \right) \\ &= (1 + r^2) \frac{c - \varepsilon}{cr^2} + O(r^{-2}) \\ &\leq \left(1 - \frac{\varepsilon}{c} \right) + O(r^{-2}). \end{aligned}$$

This would imply that v grows no more rapidly than $\left(1 - \frac{\varepsilon}{c}\right)r$, contradicting the assumption that $v(r) > r - \frac{c - \varepsilon}{r}$ for large enough $r > r_0$.

2. Arguing as in the previous part, for sufficiently large r , we have

$$\begin{aligned} f\left(\frac{w'_{-, \varepsilon}}{1 + w_{-, \varepsilon}^2}, \frac{w_{-, \varepsilon}}{r}\right) &= f\left(\frac{1 - \frac{c + \varepsilon}{r^2}}{1 + \left(r - \frac{c + \varepsilon}{r}\right)^2}, 1 - \frac{c + \varepsilon}{r^2}\right) \\ &\leq f\left(\frac{1}{1 + r^2}, 1 - \frac{c - \varepsilon}{r^2}\right) \\ &= 1 - \frac{c + \varepsilon}{r^2} + \frac{f_x(0, 1)}{1 + r^2} + \frac{f_{xx}(\xi, 1)}{2 \left(1 - \frac{c + \varepsilon}{r^2}\right)} \frac{1}{(1 + r^2)^2} \\ &= 1 - \frac{\varepsilon}{r^2} - \frac{c}{r^2(1 + r^2)} + O(r^{-4}) \\ &= 1 - \frac{\varepsilon}{r^2} + O(r^{-4}) \\ &< 1. \end{aligned}$$

For the next part, we fix $r_0 > 0$ and assume $v(r) < r - \frac{c+\epsilon}{r}$ for all $r > r_0$. Then, by part (2) of Proposition 4.7 for sufficiently large r , we have $v \geq r - 1$ and hence,

$$\begin{aligned}
v' &= (1 + v^2)g\left(\frac{v}{r}\right) \\
&\geq (1 + (r - 1)^2)g\left(1 - \frac{c + \epsilon}{r^2}\right) \\
&= (1 + (r - 1)^2)\left(g(1) - g_y(1)\frac{c + \epsilon}{r^2} + \frac{g_{yy}(\xi)}{2}\left(\frac{c + \epsilon}{r^2}\right)^2\right) \\
&\geq (1 + (r - 1)^2)\left(\frac{c + \epsilon}{cr^2} - O(r^{-4})\right) \\
&\geq 1 + \frac{\epsilon}{c} + O(r^{-2}).
\end{aligned}$$

This implies v grows faster than $\left(1 + \frac{\epsilon}{c}\right)r + O(r^{-1})$, contradicting the assumption that $v(r) < r - \frac{c + \epsilon}{r}$ for large enough $r > r_0$. □

Remark. Proposition 4.8 implies $v(r) = r - \frac{c}{r} + o(r^{-1})$ as $r \rightarrow \infty$.

Remark. The results of Proposition 4.8 were obtained independently in Cogo et al. (2023).

Proposition 4.9. Let $c = f_x(0, 1)$ and consider

$$w_{+,\epsilon}(r) = r - \frac{c}{r} + \frac{\epsilon}{r^2} \text{ and } w_{-,\epsilon}(r) = r - \frac{c}{r} - \frac{\epsilon}{r^2}.$$

1. For every $\epsilon \in (0, c)$, $w_{+,\epsilon}(r)$ is a super-solution to Eq. (4.3) for sufficiently large r . In addition, given any $r_0 > 0$, there exists $r_1 > r_0$ such that $v(r_1) \leq r_1 - \frac{c}{r_1} + \frac{\epsilon}{r_1^2}$. Thus, $v \leq r - \frac{c}{r} + \frac{\epsilon}{r^2}$ for sufficiently large r .
2. For every $\epsilon > 0$, $w_{-,\epsilon}(r)$ is a sub-solution to Eq. (4.3) for sufficiently large r . Moreover, given any $r_0 > 0$, there exists $r_1 > r_0$ such that $v(r_1) \geq r_1 - \frac{c}{r_1} - \frac{\epsilon}{r_1^2}$. Thus, $v \geq r - \frac{c}{r} - \frac{\epsilon}{r^2}$ for sufficiently large r .

Proof. 1. Indeed, as in the previous proof for sufficiently large r , we have

$$\begin{aligned} LHS &= f\left(\frac{w'_{+, \varepsilon}}{1 + w_{+, \varepsilon}^2}, \frac{w_{+, \varepsilon}}{r}\right) = f\left(\frac{1 + \frac{c}{r^2} - \frac{2\varepsilon}{r^3}}{1 + \left(r - \frac{c}{r} + \frac{\varepsilon}{r^3}\right)^2}, 1 - \frac{c}{r^2} + \frac{\varepsilon}{r^3}\right) \\ &\geq f\left(\frac{1 + \frac{c}{r^2} - \frac{2\varepsilon}{r^3}}{1 + r^2}, 1 - \frac{c}{r^2} + \frac{\varepsilon}{r^3}\right) \end{aligned}$$

Estimating further we get

$$\begin{aligned} LHS &\geq 1 - \frac{c}{r^2} + \frac{\varepsilon}{r^3} + \frac{c}{1 + r^2} \left(1 + \frac{c}{r^2} - \frac{2\varepsilon}{r^3}\right) \\ &\quad + \frac{f_{xx}(\xi, 1)}{2} \frac{\left(1 + \frac{c}{r^2} - \frac{2\varepsilon}{r^3}\right)^2}{\left(1 + \frac{c}{r^2} - \frac{\varepsilon}{r^3}\right)(1 + r^2)^2} \\ &= 1 + \frac{\varepsilon}{r^3} - O(r^{-4}) \\ &\geq 1. \end{aligned}$$

For the next part, fix $r_0 > 0$ and suppose $v(r) > r - \frac{c}{r} + \frac{\varepsilon}{r^2}$ for all $r > r_0$. Then, by recalling that $g(1) = 0$, $g_y(1) = -c^{-1}$, we obtain

$$\begin{aligned} v' &= (1 + v^2)g\left(\frac{v}{r}\right) \\ &\leq (1 + r^2)g\left(1 - \frac{c}{r^2} + \frac{\varepsilon}{r^3}\right) \\ &\leq (1 + r)^2 \left(g(1) - g_y(1) \left(\frac{c}{r^2} - \frac{\varepsilon}{r^3}\right) + \frac{g_{yy}(\xi)}{2} \left(\frac{c}{r^2} - \frac{\varepsilon}{r^3}\right)^2\right) \\ &\leq 1 - \frac{\varepsilon}{cr} + O(r^{-2}). \end{aligned}$$

However, this contradicts $v(r) > r - \frac{c}{r} + \frac{\varepsilon}{r^2}$ for all $r > r_0$.

2. As in part one for sufficiently large r , we have

$$\begin{aligned} f\left(\frac{w'_{-, \varepsilon}}{1+w_{-, \varepsilon}^2}, \frac{w_{-, \varepsilon}}{r}\right) &= f\left(\frac{1 + \frac{c}{r^2} + \frac{2\varepsilon}{r^3}}{1 + \left(r - \frac{c}{r} - \frac{\varepsilon}{r^2}\right)^2}, 1 - \frac{c}{r^2} - \frac{\varepsilon}{r^3}\right) \\ &\leq 1 - \frac{\varepsilon}{r^3} + O(r^{-4}) \\ &\leq 1. \end{aligned}$$

On the other hand, fix $r_0 > 0$ and we assume that $v(r) < r - \frac{c}{r} - \frac{\varepsilon}{r^2}$ for all $r > r_0$. Then, by Part 2 of Proposition 4.8 for sufficiently large r , we have $v > r - \frac{2c}{r}$ and hence,

$$\begin{aligned} v' &= (1 + v^2) g\left(\frac{v}{r}\right) \\ v' &\geq \left(1 + \left(r - \frac{2c}{r}\right)^2\right) g\left(1 - \frac{c}{r^2} - \frac{\varepsilon}{r^3}\right) \\ &\geq 1 + \frac{\varepsilon}{cr} - O(r^{-2}). \end{aligned}$$

However, this contradicts $v(r) < r - \frac{c}{r} - \frac{\varepsilon}{r^2}$ for large enough $r > r_0$.

□

This completes the proof of Theorem 4.1.

Remark. *It is possible to continue the above bootstrapping process ad infinitum. For instance, when $f = Q_{k+1,k} = \frac{S_{k+1}}{S_k}$, we obtained the following expression for the “bowl”-type solution*

$$v(r) = r - \frac{c}{r} + \frac{n^2(n-k-1)(n-k-4)}{(k+1)^3(n-k)^2} \frac{1}{r^3} + O(r^{-5}).$$

Note that when $k = 0$, $Q_{k+1,k} = H$ and this expression coincide with the one obtained in Clutterbuck et al. (2007).

4.3 Uniqueness of entire solutions

With the fine asymptotic information about the bowl-type solitons that we have obtained so far, we can now demonstrate the uniqueness result for entire strictly convex solutions of (1.2) that are smoothly asymptotic to the “bowl”-type solution for nondegenerate speed f .

To this end, we will recall a tangential principle for f -translators in \mathbb{R}^{n+1} from [Torres-Santaella \(2023\)](#).

Theorem 4.10. *Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ be two complete, embedded, connected f -translators such that*

1. $f : \Gamma \rightarrow (0, \infty)$ satisfies properties [a-d](#).
2. Σ_1 is strictly convex, i.e: the principal curvatures $\lambda \in \Gamma_+$.
3. Σ_2 is convex, i.e: the principal curvatures $\lambda \in \bar{\Gamma}_+$.

Then, the following tangency principles hold:

1. *Assume that there exists an interior point $p \in \Sigma_1 \cap \Sigma_2$ such that the tangent spaces coincide at p . If Σ_1 lies at one side of Σ_2 , then both hypersurfaces coincide.*
2. *Assume that the boundaries $\partial\Sigma_i$ lie in the same hyperplane Π and the intersection of Σ_i with Π is transversal. If Σ_1 lies at one side of Σ_2 and there exist $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that the tangent spaces to Σ_i and $\partial\Sigma_i$ coincide, then both hypersurfaces coincide.*

Proof. See [Torres-Santaella \(2023\)](#) Theorem 1.4 for a proof and an additional remark.

□

Remark. *We highlight that the hypotheses of the tangency principle can be modified by the following:*

1. $\Gamma \subset \mathbb{R}^n$ is a connected component of $\{\lambda \in \mathbb{R}^n : f(\lambda) > 0\}$ containing Γ_+ and it is a convex cone.
2. $f : \Gamma \rightarrow \mathbb{R}$ is smooth and symmetric, and satisfies properties *c-d*.
3. The principal curvatures of Σ_1 lie in Γ and the principal curvatures of Σ_2 lie in $\partial\Gamma \cap \left\{ \frac{\partial f}{\partial \lambda_i} > 0 \right\}$.

This is because the proof is based on a convex combination argument of the local graphs near a tangency point of the hypersurfaces. In particular, by shrinking the domain if necessary, the principal curvatures of the convex combination is an admissible family for the functional $f(\lambda) - \langle \nu, e_{n+1} \rangle$, where a Hopf maximum principle holds. We refer the reader to Thm. 1.1 in [Fontenele and Silva \(2001\)](#) for details.

Definition 8. Let Σ be an entire translating graph that satisfies Equation (1.1). Then, the end of Σ is **smoothly asymptotic** to the “bowl”-type soliton if Σ can be expressed outside a ball as a vertical graph of a function u_Σ such that

$$u_\Sigma(x) = \frac{|x|^2}{2} - \frac{c}{2} \ln(|x|^2) + O(|x|^{-1}), \text{ as } |x| \rightarrow \infty. \quad (4.10)$$

Note that the normalization $f(0, 1) = 1$ is used and $c := \frac{\partial f}{\partial \lambda_1}(0, 1)$.

Remark. As is noted in [Torres-Santaella \(2023\)](#) (see sec. 4), Equation (1.2) is invariant under rotations about the x_{n+1} -axis in \mathbb{R}^{n+1} . Consequently, under the hypothesis of Theorem 4.2, it is enough to show that Σ is symmetric along the plane $\{x_1 = 0\}$ to obtain that Σ is rotationally symmetric.

We will now use Alexandrov’s moving plane method to establish uniqueness. To this end, we recall the following definitions used in [Martín et al. \(2015\)](#) and [Martínez and Martínez-Triviño \(2022\)](#):

- $\Pi_t := \{x \in \mathbb{R}^{n+1} : \mathbf{p}(x) = t\}$, where $\mathbf{p}(x_1, \dots, x_{n+1}) = x_1$ is the projection onto the first coordinate. In addition, $\Pi := \Pi_0$.

- $Z_t := \{x_{n+1} > t\}$.
- Let $A \subset \mathbb{R}^{n+1}$ be an arbitrary subset. Then we set $A_+(t) := \{x \in A : \mathbf{p}(x) \geq t\}$, $A_-(t) := \{x \in A : \mathbf{p}(x) \leq t\}$ and $\delta_t(A) := A \cap \Pi_t$. Note that $A_+(t)$ and $A_-(t)$ are the right hand side and the left hand side, respectively, of A relative to Π_t (see Definition 9 given below).
- The 1-parameter families of right (respectively, left) reflections of A , respectively, along the hyperplane Π_t are given by

$$A_+^*(t) := \{(2t - x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \dots, x_{n+1}) \in A_+(t)\}$$

$$A_-^*(t) := \{(2t - x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \dots, x_{n+1}) \in A_-(t)\}$$

- We denote by $\pi : \mathbb{R}^{n+1} \rightarrow \Pi_0 \subset \mathbb{R}^{n+1}$ the orthogonal projection given by $\pi(x_1, \dots, x_{n+1}) = (0, x_2, \dots, x_{n+1})$.

Definition 9. Let A, B be two subsets of \mathbb{R}^{n+1} . Then, we say “ A is on the right side of B ” (denoted by $B \leq A$) iff for every $x \in \Pi = \{x_1 = 0\}$ such that

$$\pi^{-1}(\{x\}) \cap A \neq \emptyset \text{ and } \pi^{-1}(\{x\}) \cap B \neq \emptyset,$$

we have that

$$\sup \{\mathbf{p}(p) : p \in \pi^{-1}(\{x\}) \cap B\} \leq \inf \{\mathbf{p}(p) : p \in \pi^{-1}(\{x\}) \cap A\}. \quad (4.11)$$

Remark. The proof of Theorem 4.2 uses the method of moving planes of Alexandrov in the spirit of Schoen (1983), Martín et al. (2015) and Martínez and Martínez-Triviño (2022). This method specifically requires three properties to be applied:

- A family of hyperplanes as translating solutions of Equation (1.2). To accomplish this, we need Property e on f .

- *Tangential principles in the interior and at the boundary, since we will need to understand how the translators intersect tangentially with their reflections along Π_t .*
- *To “start the reflection process”, we need to find $t > 0$ large enough such that the reflection of Σ along Π_t is a graph over Π and $\Sigma_-(t) \leq \Sigma_+^*(t)$. To compare the horizontal distance between Σ_+^* and Σ_- , we will need the asymptotic behavior of the “bowl”-type solution at infinity.*

Next, we will follow the arguments used in [Martínez and Martínez-Triviño \(2022\)](#) Theorem 6, for f -translators which are smoothly asymptotic to the “bowl”-type solution.

Lemma 4.10.1. *There exists $r_0 > R$ such that $\Sigma_+(t)$ is a graph over Π for every $t > r_0$.*

Proof. Firstly, we note that for every $t > R$, $\Sigma_+(t)$ possesses only one unbounded connected component. If this were false, one could choose a compact component $\Sigma' \subset \Sigma_+(t)$ and a t large enough such that $\Sigma \cap \Pi_t = \emptyset$. Then, by translating Π_t until it touches Σ' at a first order contact point*, say $\Sigma' \cap \Pi_{t'} = \{p\}$ for some $0 < t' < t$, Theorem 4.10, applied to Σ' with $\Pi_{t'}$, implies that Σ' is totally geodesic contradicting that Σ' is strictly convex.

Next, by Equation (4.10), we have

$$d_x u_\Sigma(e_1) \geq \left(1 - \frac{C}{|x|^2}\right) \langle x, e_1 \rangle,$$

for some constant $C > 0$ and $|x| \geq R$. Therefore, by choosing $r_0 > R$ large enough such that $1 - \frac{C}{|x|^2} \geq \varepsilon > 0$, it follows that $d_x u_\Sigma(e_1) > 0$ whenever $\langle x, e_1 \rangle \geq r_0$. Finally,

*First, this is possible because the first-order contact point coincides with the point at which the two hypersurfaces first touch. In addition, the first order contact point means that the graph and gradient coincide. In particular, the tangent planes coincide because the unit normal vectors of the two hypersurfaces coincide at this point.

Lemma 4.10.1 follows since Σ is properly embedded and $\Sigma_+(r_1) \cup \pi(\Sigma_+(r_1))$ bounds a domain in \mathbb{R}^{n+1} . \square

From Lemma 4.10.1, it follows that $\Sigma_+^*(t) \cap Z_R$ is a vertical graph of a function satisfying

$$u_\Sigma^*(x) = u_\Sigma(2t - x_1, x_2, \dots, x_n)$$

for any $t > r_0$.

Lemma 4.10.2. *Let $a > 0$. There exists $r_1 > r_0$ such that for $|x| \geq r_1$ and $t > a + x_1$ we have*

$$u_\Sigma^*(x) - u_\Sigma(x) \geq \epsilon,$$

for some $\epsilon > 0$.

Proof. The proof follows the same method as in Martín et al. (2015) Step 3 Theorem A. \square

From Lemma 4.10.2, for every $a > 0$ and $t \geq r_1$ we have that

$$\Sigma_-(t+a) \cap \{x_1 \leq t\} \leq \Sigma_+^*(t+a) \cap \{x_1 \leq t\}.$$

Moreover, since $\Sigma_+(t)$ is a graph over Π for $t \geq r_1$, it follows that

$$\Sigma_-(t+a) \cap \{t \leq x_1 \leq t+a\} \leq \Sigma_+^*(t+a) \cap \{t \leq x_1 \leq t+a\}.$$

In conclusion, by setting

$$\mathcal{A} := \{t \in [0, \infty) : \Sigma_+(t) \text{ is a graph over } \Pi \text{ and } \Sigma_-(t) \leq \Sigma_+^*(t)\},$$

we see that $\mathcal{A} \neq \emptyset$ since $t+a \in \mathcal{A}$ for $t \geq r_1$.

Proof of Theorem 4.2. Firstly, by Remark 4.3 it is only necessary to prove that Σ is symmetrical about the hyperplane $\Pi = \{x_1 = 0\}$. Moreover, by the lemmas 4.10.1

and 4.10.2, the set

$$\mathcal{A} = \{t \in [0, \infty) : \Sigma_+(t) \text{ is a graph over } \Pi \text{ and } \Sigma_-(t) \leq \Sigma_+^*(t)\},$$

is not empty. Then, we apply the same arguments as in Martín et al. (2015) to obtain \mathcal{A} is closed and open subset of $[0, \infty)$. Consequently, $\mathcal{A} = [0, \infty)$, which means $0 \in \mathcal{A}$. In fact, we have obtained $\Sigma_-(0) \leq \Sigma_+^*(0)$, and by analogous arguments, it can be shown that $\Sigma_-^*(0) \leq \Sigma_+(0)$. Note that the combination of these two properties implies that Σ is symmetric with respect the hyperplane Π . \square

4.4 A Degenerate Example: $\sqrt[n]{S_n}$.

As seen in the previous section, the asymptotic behavior of the “bowl”-type solution is that of a paraboloid when f is nondegenerate.

In this section we will show that the situation can be very different for nondegenerate speeds, taking as an example the function $\sqrt[n]{S_n}$, the n^{th} root of the Gauss curvature.

Theorem 4.11. *The slope of the “bowl”-type $\sqrt[n]{S_n}$ -translator is given by*

$$v(r) = \begin{cases} e^{\frac{r^2}{2}} + O\left(\sqrt{e^{\frac{r^2}{2}} - 1}\right), & \text{for } n = 2, \\ \frac{r^3}{3} + O(1), & \text{for } n = 3, \text{ as } r \rightarrow \infty \\ \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + O\left(\frac{1}{r^{\frac{n}{n-2}}}\right), & \text{for } n \geq 4. \end{cases}$$

The proof of this theorem will be split into the following propositions. First, recall that the slope of the “bowl”-type $\sqrt[n]{S_n}$ -translator satisfies the ODE

$$\begin{cases} v' = (1 + v^2) \left(\frac{r}{v}\right)^{n-1}, \\ v(0) = 0. \end{cases} \quad (4.12)$$

We start with the case of $n = 2$.

Proposition 4.12. *For $n = 2$ the smooth solution to (4.12) satisfies*

$$v(r) = e^{\frac{r^2}{2}} + O\left(\sqrt{e^{\frac{r^2}{2}} - 1}\right), \text{ as } r \rightarrow \infty.$$

Proof. Firstly, for $n = 2$, we note that Equation (4.12) has the form

$$v' = (1 + v^2)\frac{r}{v}. \quad (4.13)$$

Therefore, $v = \sqrt{e^{\frac{r^2}{2}} - 1}$ for $r > 0$. Then, Proposition 4.12 holds by the explicit expression of the solution. \square

Proposition 4.13. *For $n = 3$ the smooth solution to (4.12) satisfies*

$$v(r) = \frac{r^3}{3} + O(1), \text{ as } r \rightarrow \infty.$$

Proof. For $n \geq 3$, Equation (4.12) can be solved implicitly by

$$\frac{v^{n-2}}{n-2} - \int_0^v \frac{t^{n-3}}{1+t^2} dt = \frac{r^n}{n}. \quad (4.14)$$

Particularly, for $n = 3$, we have the implicit equation

$$v - \arctan(v) = \frac{r^3}{3}.$$

On the other hand, the function $f(x) = x - \arctan(x)$ has the following asymptotic expansion

$$f(x) = x - \frac{\pi}{2} + O(x^{-1}), \text{ as } x \rightarrow \infty.$$

Then, an easy calculation reveals that $f^{-1}(x)$ has the following asymptotic given by

$$f^{-1}(x) = x + \frac{\pi}{2} - O(x^{-1}), \text{ as } x \rightarrow \infty.$$

Consequently, we obtain that the asymptotic expansion of v is given by

$$v(r) = \frac{r^3}{3} + \frac{\pi}{2} - O(r^{-3}), \text{ as } r \rightarrow \infty.$$

□

Proposition 4.14. *For $n > 3$, the asymptotic behavior of the solution to (4.12) is given by*

$$v(r) = \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + O\left(\frac{1}{r^{\frac{n}{n-2}}}\right), \text{ as } r \rightarrow \infty.$$

Proof. The proof is divided in several steps.

Step 4.4.1. $v(r) \geq \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}}$ for $r > 0$.

Firstly, we are going to show that the function $w(r) = \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}}$ is a sub-solution to Eq. (4.12) for $r \geq R(n) := \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}$. Indeed, we note that

$$\begin{aligned} & w' - (1 + w^2) \left(\frac{r}{w}\right)^{n-1} \\ &= \left(\frac{n}{n-2}\right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-1}} - \left(1 + \left(\frac{n-2}{n}\right)^{\frac{2}{n-2}} r^{\frac{2n}{n-2}}\right) \left(\frac{n-2}{n}\right)^{-\frac{(n-1)}{n-2}} r^{\frac{-2(n-1)}{n-2}} \\ &= -\frac{1}{\left(\frac{n-2}{n}\right)^{\frac{n-1}{n-2}} r^{\frac{2(n-1)}{n-2}}} \\ &< 0. \end{aligned}$$

Since $w(0) = v(0) = 0$, we get $v(r) \geq w(r)$ for all $r \geq R(n)$.

Step 4.4.2. *For $n > 3$, $v(r) = \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + o\left(r^{\frac{n}{n-2}}\right)$ as $r \rightarrow \infty$.*

We argue by contradiction: suppose there exists $\varepsilon > 0$ such that for all $r_0 \gg 1$ it holds

$$(1 + \varepsilon)w(r) \leq v(r), \text{ for all } r \geq r_0.$$

In particular,

$$v' = (1 + v^2) \left(\frac{r}{v}\right)^{n-1} \leq \frac{r^{n-1}w^{-(n-3)}}{(1 + \varepsilon)^{n-1}} = \frac{1}{(1 + \varepsilon)^{n-1}} \left(\frac{n-2}{n}\right)^{-\frac{n-3}{n-2}} r^{\frac{2}{n-2}}.$$

Then, by the previous step, we have

$$\left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} \leq v \leq \frac{\left(\frac{n-2}{n}\right)^{\frac{1}{n-2}}}{(1 + \varepsilon)^{n-1}} r^{\frac{n}{n-2}}.$$

But this is impossible since $\varepsilon > 0$. Consequently, $v(r) = w(r) + o\left(r^{\frac{n}{n-2}}\right)$ for $r \rightarrow \infty$.

Next, since $v(r) = \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + o\left(r^{\frac{n}{n-2}}\right)$ as $r \rightarrow \infty$, we can find a non-negative function $\varphi(r)$ with the following property: for every $C > 0$ and all $r \gg 1$ it holds $|\varphi(r)| \leq Cr^{\frac{n}{n-2}}$ and

$$v(r) = \left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi(r), \text{ for } r \geq r_0.$$

In addition, φ satisfies the following equation

$$\begin{aligned} \varphi' &= v' - \left(\frac{n}{n-2}\right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}} \\ &= \left(1 + \left(\left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi\right)^2\right) \frac{r^{n-1}}{\left(\left(\frac{n-2}{n}\right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi\right)^{n-1}} \\ &\quad - \left(\frac{n}{n-2}\right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}}. \end{aligned}$$

Step 4.4.3. $\varphi \rightarrow 0$ as $r \rightarrow \infty$.

We argue by contradiction: suppose there exist $\varepsilon > 0$ such that for all $r_0 \gg 1$ and every $C > 0$ it holds

$$\varepsilon \leq \varphi(r) \leq Cr^{\frac{n}{n-2}}, \text{ for all } r \geq r_0.$$

In particular, by letting $C \rightarrow 0$ in

$$\begin{aligned} \varphi' &= \left(1 + \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi \right)^2 \right) \frac{r^{n-1}}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi \right)^{n-1}} \\ &\quad - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}} \\ &\leq \left(1 + r^{\frac{2n}{n-2}} \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^2 \right) \frac{r^{n-1}}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varepsilon \right)^{n-1}} \\ &\quad - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}} \\ &= \left(\left(\frac{n}{n-2} \right)^{\frac{n-1}{n-2}} \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^2 - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} \right) r^{\frac{2}{n-2}} \\ &\quad - \frac{(n-1) \left(\frac{n}{n-2} \right)^{\frac{n}{n-2}} \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^2 \varepsilon}{r} + O\left(\frac{1}{r^{\frac{(n-1)}{(n-2)}}} \right), \end{aligned}$$

we observe that

$$\varphi' \leq -(n-1) \left(\frac{n}{n-2} \right) \frac{\varepsilon}{r} + O\left(\frac{1}{r^{\frac{(n-1)}{(n-2)}}} \right) \leq 0.$$

Therefore, φ is strictly decreasing but this contradicts that $\varepsilon \leq \varphi$.

Step 4.4.4. $\varphi \geq \left(\frac{n}{n-2} \right)^{\frac{1}{n-2}} \frac{1}{r^{\frac{n}{n-2}}}$ for large enough r .

Let $r_1 \gg 1$ and $C > 0$ such that

$$0 \leq \varphi(r) \leq Cr^{\frac{n}{n-2}}, \text{ for all } r \geq r_1.$$

Then, we may estimate φ by letting $C \rightarrow 0$ in the following expression:

$$\begin{aligned} \varphi' &= \left(1 + \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi \right)^2 \right) \frac{r^{n-1}}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi \right)^{n-1}} \\ &\quad - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}} \\ &\geq \frac{1 + r^{\frac{2n}{n-2}} \left(\frac{n-2}{n} \right)^{\frac{2}{n-2}}}{r^{\frac{2(n-1)}{n-2}} \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^{n-1}} - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}}. \\ &= \left(\frac{\left(\frac{n-2}{n} \right)^{\frac{2}{n-2}}}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^{n-1}} - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} \right) r^{\frac{2}{n-2}} + \frac{1}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^{n-1} r^{\frac{2(n-1)}{n-2}}}. \end{aligned}$$

Therefore, we have that

$$\varphi' \geq \left(\frac{n}{n-2} \right)^{\frac{n-1}{n-2}} \frac{1}{r^{\frac{2(n-1)}{n-2}}}, \text{ as } r \rightarrow \infty.$$

Then, by integrating from r to ∞ we finally obtain

$$\varphi \geq \left(\frac{n}{n-2} \right)^{\frac{1}{n-2}} \frac{1}{r^{\frac{n}{n-2}}}.$$

Step 4.4.5. $\varphi = O\left(\frac{1}{r^{\frac{n}{n-2}}}\right)$.

We note that for $r_0 \gg 1$ it holds

$$\left(\frac{n}{n-2} \right)^{\frac{1}{n-2}} \frac{1}{r^{\frac{n}{n-2}}} \leq \varphi(r) \leq Cr^{\frac{n}{n-2}}, \text{ for all } r \geq r_0.$$

Next, by letting $C \rightarrow 0$ in

$$\begin{aligned}
\varphi' &= \left(1 + \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi \right)^2 \right) \times \\
&\quad \frac{r^{n-1}}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{n}{n-2}} + \varphi \right)^{n-1}} - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}} \\
&\leq \left(1 + r^{\frac{2n}{n-2}} \left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} + C \right)^2 \right) \times \\
&\quad \frac{r^{\frac{2(n-1)^2}{n-2}}}{\left(\left(\frac{n-2}{n} \right)^{\frac{1}{n-2}} r^{\frac{2n}{n-2}} + \left(\frac{n}{n-2} \right)^{\frac{1}{n-2}} \right)^{n-1}} \\
&\quad - \left(\frac{n}{n-2} \right)^{\frac{n-3}{n-2}} r^{\frac{2}{n-2}},
\end{aligned}$$

we deduce that

$$\varphi' \leq - \frac{(n-2) \left(\frac{n}{n-2} \right)^{\frac{n-1}{n-2}}}{r^{\frac{2(n-1)}{n-2}}} + O\left(\frac{1}{r^{\frac{4(n-1)}{n-2}}} \right).$$

Finally, by integrating from r to ∞ , we obtain

$$\varphi \leq \frac{(n-2) \left(\frac{n}{n-2} \right)^{\frac{1}{n-2}}}{r^{\frac{n}{n-2}}}.$$

□

This concludes the proof of Proposition 4.12.

Remark. We note that the proof of the uniqueness theorem presented in section 4.3 also applies for entire strictly convex solutions smoothly asymptotic to the “bowl”-type translators of the $\sqrt[n]{S_n}$ -flow. To see this, we choose $R > 0$. Then, the entire

$\sqrt[n]{S_n}$ -translator can be written as a vertical graph $u_\Sigma : \mathbb{R}^n \setminus B_R(0) \rightarrow \mathbb{R}$ such that

$$u_\Sigma = h_1(r^2) + O(h_2(r^2)), \text{ as } r \rightarrow \infty,$$

where $|x| = r$ and

$$h_1(r^2) = \begin{cases} \int e^{\frac{r^2}{2}} dr, & \text{for } n = 2, \\ \frac{1}{12}r^4, & \text{for } n = 3, \\ \frac{(n-2)^{\frac{n-1}{n-2}}}{2(n-1)n^{\frac{1}{n-2}}} r^{\frac{2(n-1)}{n-2}}, & \text{for } n \geq 4. \end{cases} \quad h_2(r^2) = \begin{cases} \int \sqrt{e^{\frac{r^2}{2}} - 1} dr, & \text{for } n = 2, \\ r, & \text{for } n = 3, \\ \frac{1}{r^{\frac{n-2}{2}}}, & \text{for } n \geq 4. \end{cases}$$

Then, Lemma 4.10.1 holds for $f = \sqrt[n]{S_n}$, since

$$d_x u_\Sigma(e_1) \geq (h'_1(|x|^2) - Ch_2(|x|^2)) \langle x, e_1 \rangle,$$

and

$$0 < h'_1(|x|^2) - Ch'_2(|x|^2) = \begin{cases} e^{\frac{|x|^2}{2}} - C\sqrt{e^{\frac{|x|^2}{2}} - 1}, & \text{for } n = 2 \\ \frac{|x|^3}{3} - C, & \text{for } n = 3, \\ \frac{(n-1)}{(n-2)^{\frac{n-3}{n-2}}} |x|^{\frac{2}{n-2}} - \frac{C}{|x|^{\frac{n}{n-2}}}, & \text{for } n \geq 4. \end{cases} \quad (4.15)$$

for large enough $r \geq R$ and some positive constants C .

In addition, we have that Lemma 4.10.2 also holds, since by the mean value theorem, it follows that

$$u_\Sigma^*(x) - u_\Sigma(x) = 2(t - x_1) \frac{\partial u_\Sigma}{\partial x_1}(\xi, x_2, \dots, x_n),$$

for some $\xi \in (x_1, 2t - x_1)$ with $|x| > r_0$.

Then, by writing $(\xi, x') = (\xi, x_2, \dots, x_n)$, the asymptotic expression of u_Σ gives us

$$\begin{aligned} 2(t - x_1) \frac{\partial u_\Sigma}{\partial x_1}(\xi, x_2, \dots, x_n) &\geq 4(t - x_1) \xi (h'_1(|(\xi, x')|^2) - Ch'_2(|(\xi, x')|^2)) \\ &\geq 4ar_0(h_1(|(\xi, x')|^2)' - h'_2(|(\xi, x')|^2)). \end{aligned}$$

Finally, by choosing $r_1 \geq r_0$ as in (4.15), the difference $h_1(|(\xi, x')|^2)' - h'_2(|(\xi, x')|^2)$ is uniformly bounded from below. Consequently, the proof of Theorem 4.2 holds.

4.5 Wing-like translators

An exotic translating solution to the mean curvature is the *wing-like* translator discussed in Clutterbuck et al. (2007). This solution is connected but not graphical, but is the union of two graphs each defined on the complement of a ball $\mathbb{R}^n - B_R$. The upper and lower halves are each asymptotic to paraboloids.

In this section, we will discuss the existence and the asymptotic behavior of *wing-like* translators for *nondegenerate* speeds (see Definition 5). In particular, we will see that the upper and lower branches need not always have the same asymptotics.

More precisely, a *wing-like* solution is a non-convex rotationally symmetric hypersurface in \mathbb{R}^{n+1} with compact boundary (possibly empty) along the x_{n+1} -axis which is at distance $R > 0$ from the origin that satisfies Eq. (1.2).

Furthermore, by removing a large enough ball from a *wing-like* solution we have at possible two branches given by vertical graphs:

- We are going to show that the upper branch is always asymptotic to the “bowl”-type solution.
- Meanwhile, the asymptotic geometry of the lower branch is different for different speed functions f . In particular, we will show that for the function $f = \sqrt[k]{S_k}$ with k even the lower branch is asymptotic to symmetric reflection of “bowl” type solution, for k odd, the lower possesses a boundary component,

and for the function $f = Q_{k+1,k}$, the lower branch whose gradient vanishes at infinity.

Remark. We note that the ODE satisfying a rotationally symmetric $Q_{k+1,k}$ -translator has as solution $v = 0$, but recall that the horizontal hyperplanes are not vertical translators and that $Q_{k+1,k}$ is not well defined when all principal curvatures are 0.

Remark. It is an open problem to characterize the dichotomy of the lower branch of wing-like solutions for any nondegenerate speed f .

Now we will follow the construction given in [Clutterbuck et al. \(2007\)](#). Firstly, we will start with the a small non-convex portion of the *wing-like* solution.

Step 4.5.1. *Construction of a small portion of the wing-like f -translator as a graph over the x_{n+1} -axis.*

At the point where the tangent space is not orthogonal to e_{n+1} , the translator can be represented locally as a graph of a function $r : (a, b) \rightarrow (0, \infty)$ over the x_{n+1} -axis by

$$\bigcup_{x_{n+1}} r(x_{n+1}) \cdot \mathbb{S}^{n-1} \times \{x_{n+1}\}.$$

Note that $r(x_{n+1})$ represents the radius of the hypersurface given its last coordinate.

Next, we are going to find the ODE the r satisfies. Firstly, at points where the tangent space is not parallel to e_{n+1} , the *wing-like f -translator* can be described as a graph of a rotationally symmetric function $u(r)$ over the hyperplane $\{x_{n+1} = 0\}$.

Note that by construction $u \circ r(x_{n+1}) = x_{n+1}$, and by the chain rule, we have $u'(r) = \frac{1}{r'}$ and $u''(r) = -\frac{r''}{(r')^3}$.

Consequently, since a *wing-like* solution is not convex, Eq. (1.2) has the form (4.4). Then, since u satisfies equation (4.5), we obtain

$$r'' = -(1 + (r')^2)g(r^{-1}, r'),$$

where $g(r^{-1}, r') = r'g((rr')^{-1}, 1)$ since $g(y, z)$ is 1-homogeneous. We noted that in previous sections we denote $g(y)$ instead of $g(y, 1)$ to economize notation.

In addition, at points of the *wing-like* f -translator where the tangent space is vertical, we may argue in the same way as before by considering the branches separately.

Finally, we claim that there exists $\varepsilon > 0$ and a strictly convex solution to the problem

$$\begin{cases} r'' = -(1 + (r')^2)g(r^{-1}, r'), & x_{n+1} \in (h_0 - \varepsilon, h_0 + \varepsilon), \\ r(h_0) = R, & r'(h_0) = 0. \end{cases}, \quad (4.16)$$

where $h_0 \in \mathbb{R}$ and $R > 0$. Note that a different choice of h_0 corresponds to a vertical translation of the *wing-like* f -translator. Therefore, we will assume that $h_0 = 0$.

Indeed, since Eq. (4.16) is not degenerate and the right-hand side is at least of class \mathcal{C}^2 , the classical theory of ODEs holds. This means that there exist $\varepsilon > 0$ and a solution to Eq. (4.16) in $(-\varepsilon, \varepsilon)$.

Then, by shrinking $\varepsilon > 0$ if necessary, we may assume that this solution is strictly convex. This is due to continuity and the fact that

$$r''(0) = -g(R^{-1}, 0) > 0.$$

To see this, we note that by taking derivatives with respect to z in Eq.(4.4), we have $f_x g_z = 1$. Then, since f is strictly increasing, $g_z(y, z) > 0$. Therefore, by recalling that

$$g(1, 1) = 0 \text{ and } g(R^{-1}, 0) < 0$$

the claim holds. This completes the construction of the small portion of the “wing”-like f -translator.

Remark. In the particular case of $f = \sqrt[k]{S_k}$, the ODE that r satisfies is

$$r'' = -(1 + (r')^2) \left(\frac{(r')^k r^{k-1}}{\binom{n-1}{k-1}} - \frac{(n-k)}{kr} \right).$$

Therefore, when k is an even number, we have that $\tilde{r} = r(-x_{n+1})$ is also a solution to Eq. (4.16).

Proposition 4.15. The principal curvatures of the upper half small portion of the wing-like f -translator belong into the cone

$$\{f \text{ satisfies properties } a - e \text{ and } f(0, 1) > 0\}.$$

Proof. Firstly, we note that Eq. (2.1) implies that

$$\lambda_1 = \frac{-r''}{(1 + r'^2)^{3/2}} \text{ and } \lambda_i = \frac{1}{r\sqrt{1 + r'^2}}.$$

are the principal curvatures of the small portion of the *wing-like* f -translator.

Then, by differentiating them

$$\lambda_1' = \frac{-r'''}{(1 + r'^2)^{3/2}} + \frac{3(r'')^2 r'}{(1 + r'^2)^{5/2}} \text{ and } \lambda_i' = \frac{-r'}{r^2 \sqrt{1 + r'^2}} - \frac{r' r''}{r(1 + r'^2)^{3/2}}$$

Next, we will be evaluate the principal curvatures at $u = 0^\dagger$. Differentiating r'' , we have

$$r''' = -2r' r'' g - (1 + r'^2) g_y \frac{r'}{r^2} - (1 + r'^2) g_z r'',$$

where g, g_y, g_z are all evaluated at the point (r^{-1}, r') . Therefore, we obtain

$$r'''(0) = -g_z (R^{-1}, 0) r''(0) = g_z (R^{-1}, 0) g (R^{-1}, 0).$$

[†] $u > 0$ is the upper part of the winglike solution and $u < 0$ is the lower.

Finally, by recalling $r(0) = R$, $r'(0) = 0$ and $r''(0) = -g(R^{-1}, 0) < 0$, we see that

$$\lambda_1'(0) = -g_z(R^{-1}, 0) g(R^{-1}, 0) > 0, \text{ and } \lambda_i'(0) = 0.$$

Therefore, since in the upper half of the small portion $\{u > 0\}$, r is strictly decreasing. We obtain that $\lambda_i' > 0$ for all $i = 1, \dots, n$ finalizing the proposition. \square

Step 4.5.2. *Construction of the upper branch of the wing-like f -translator.*

We have a solution on some interval $u \in [0, \varepsilon)$. To continue the solution, we may revert to the standard representation

$$\begin{cases} v' = (1 + v^2)g\left(\frac{v}{r}\right) \\ v(r_0) = v_0 \end{cases}$$

where we may choose $r_0 \in (R, R + \varepsilon)$ arbitrarily. Then, since $v \rightarrow \infty$ as $r \rightarrow R^+$, we may assume $\frac{v_0}{r_0} \geq 1$, by choosing r_0 sufficiently close to R . By standard ODE theory we have existence and uniqueness as long as the right hand side is well-defined.

If $\frac{v_0}{r_0} > 1$, the right hand side of the ODE is initially negative, hence the function decreases until $g\left(\frac{v}{r}\right) = 0$ (i.e.: $\frac{v}{r} = 1$). So we only need to discuss the case $\frac{v_0}{r_0} = 1$. If this is the case, then the function $v_+ = r$ is a supersolution to the ODE, and $v_- = \frac{r}{f(1, 1)}$ is a subsolution, as can be seen in [Rengaswami \(2021\)](#) Section 7.1. The proof that $\frac{v}{r} \rightarrow 1$ is just a special case of Proposition 2 of [Rengaswami \(2021\)](#).

On the other hand, for the lower branch of the *wing-like f -translator*, we have a solution on some interval $(-\varepsilon, 0]$. As with the upper branch, we may revert to the

standard representation

$$\begin{cases} v' = (1 + v^2)g\left(\frac{v}{r}\right), \\ v(r_0) = v_0. \end{cases},$$

where $r_0 \in (R, R + \varepsilon)$. Then, since $v \rightarrow -\infty$ as $r \rightarrow R^+$, we may assume $\frac{v_0}{r_0} \leq -1$, by choosing r_0 sufficiently close to R . By standard ODE theory we have existence and uniqueness as long as the right hand side is well-defined.

Step 4.5.3. *Construction and asymptotic behavior of the lower branch of the wing-like $\sqrt[k]{S_k}$ -translator for k even.*

Firstly, by construction the principal curvatures of the small portion of W_R are given by

$$\lambda_1 = \frac{g(r^{-1}, r')}{\sqrt{1 + (r')^2}} = \frac{\frac{(r')^k r^{k-1}}{\binom{n-1}{k-1}} - \frac{(n-k)}{kr}}{\sqrt{1 + (r')^2}} \text{ and } \lambda_i = \frac{1}{r\sqrt{1 + (r')^2}}.$$

Then, since $\lambda \in \Gamma_k$ if, and only if,

$$\binom{n-1}{l} \lambda_i^l + \binom{n-1}{l-1} \lambda_1 \lambda_i^{l-1} > 0, \text{ for } l = 1, \dots, k,$$

or equivalently, we have

$$0 < \frac{\binom{n-1}{l-1}}{r^l (1 + (r')^2)^{\frac{l}{2}}} \left(r g(r^{-1}, r') + \frac{n-l}{l} \right) \Leftrightarrow 0 < \frac{(r'r)^k}{\binom{n-1}{k-1}} + \frac{n(k-l)}{kl}.$$

Consequently, since r' only vanishes at the origin, it follows that the principal curvatures of the small portion satisfy $\lambda(0) \in \partial\Gamma_k \cap \Gamma_{k-1}$, or equivalently

$$\lambda(0) \in \partial\Gamma_k \cap \left\{ \frac{\partial \sqrt[k]{S_k}}{\partial \lambda_i} > 0 \right\},$$

and $\lambda(p) \in \Gamma_k$ for $p \in W_R \setminus \{0\}$.

Next, the ODE for $f = \sqrt[k]{S_k}$ is given by

$$\begin{cases} v' = (1 + v^2) \left(\frac{1}{\binom{n-1}{k-1}} \left(\frac{r}{v}\right)^{k-1} - \frac{(n-k)v}{kr} \right), & r \geq r_0, \\ v(r_0) = v_0. \end{cases} \quad (4.17)$$

We note that when k is an even number, then the function $-v$, where v denotes the upper half solution, is also a solution to (4.17)[‡].

In particular, the right hand side of Eq. (4.17) is always negative and finite. This means the solution exists for all $r \geq r_0$, and the asymptotic expression of v for k even is given by

$$v(r) = -\frac{r}{\sqrt[k]{\binom{n-1}{k}}} + \frac{\binom{n-1}{k} \binom{n-1}{k-1}}{k \sqrt[k]{\binom{n-1}{k}}} \frac{1}{r} + O(|x|^{-2}), \text{ as } r \rightarrow \infty.$$

Consequently, the principal curvatures of W_R^- do not belong to Γ_k for any k as $r \rightarrow \infty$.

Remark. *It is important to note that when k is odd, the right hand side of Eq. (4.17) has a singularity when $v = 0$. This is because the translator equation 4.3 can never be satisfied by any axially symmetric graph that possesses a point where $u' = 0$ and $r > 0$. Thus, if $u' \rightarrow 0$ at some point $r_1 > r_0$, u' can never be extended past r_1 , i.e. such graph will possess a boundary S^{n-1} .*

Step 4.5.4. *Construction and asymptotic behavior of the lower branch of the wing-like $Q_{k+1,k}$ -translator.*

Firstly, the ODE that r satisfies is

$$r'' = -(1 + (r')^2) \frac{(n-k)}{(k+1)r} \left(\frac{(k+1)r'r - (n-k-1)}{(n-k) - krr'} \right).$$

[‡]Recall Remark 4.5 and the construction of the *wing-like* solution is by extending the small piece r as the initial data of Eq. (4.17) in terms of u with $v' = u$.

Therefore, the principal curvatures of the small portion of W_R are given by

$$\lambda_1 = \frac{g(r^{-1}, r')}{\sqrt{1 + (r')^2}} = \frac{(n-k)}{(k+1)} \frac{(k+1)r'r - (n-k-1)}{(n-k) - krr'} \text{ and } \lambda_i = \frac{1}{r\sqrt{1 + (r')^2}}.$$

Then, we will have that $\lambda \in \Gamma_{k+1}$ if, and only if,

$$0 < \frac{(n-k)}{k+1} \frac{(k+1)rr' - (n-k-1)}{(n-k) - krr'} + \frac{n-l}{l}, \quad l = 1, \dots, k+1.$$

Consequently, since

$$0 < \frac{n-l}{l} - \frac{n-k-1}{k+1} \Leftrightarrow 0 < k+1-l,$$

we obtain that $\lambda(0) \in \partial\Gamma_{k+1} \cap \Gamma_k$, or equivalently

$$\lambda(0) \in \partial\Gamma_{k+1} \cap \left\{ \frac{\partial Q_{k+1,k}}{\partial \lambda_i} > 0 \right\},$$

and $\lambda(p) \in \Gamma_k$ for $p \in W_R \setminus \{0\}$.

Next, the ODE for $f = Q_{k+1,k}$ is given by

$$\begin{cases} v' = \frac{n-k}{k+1} (1+v^2) \frac{v}{r} \frac{(k+1) - (n-k-1)\frac{v}{r}}{(n-k)\frac{v}{r} - k}, \quad r \geq r_0, \\ v(r_0) = v_0. \end{cases}, \quad (4.18)$$

Firstly, since $v \rightarrow -\infty$ as $r \rightarrow R^+$, we may take r_0 to be sufficiently close to R so that $v_0 < 0$. Note that the right hand side of Eq. (4.18) is positive when $v < 0$. Thus the solution is increasing as long as this is the case. On the other hand, $v_+ = 0$ is a solution to the Eq. (4.18), therefore v and v_+ cannot coincide which means that v remains negative for all $r \geq r_0$.

Now we show that $v \rightarrow 0$ as $r \rightarrow \infty$. Firstly, since v is increasing and bounded above by 0, $L := \lim_{r \rightarrow \infty} v$ exists in $[v_0, 0]$. If $L \neq 0$, then there exists $\varepsilon > 0$ such that

$$\frac{v}{r} < \frac{-\varepsilon}{r}.$$

Then, since $k < n - 1$ and the function $g(y) = \frac{n-k}{k+1}y \frac{(k+1) - (n-k-1)y}{(n-k)y - k}$ is decreasing, we have

$$v' \geq (1+v^2)g\left(\frac{-\varepsilon}{r}\right) \geq (1+v^2)\frac{(n-k)\varepsilon}{kr}.$$

Therefore, $\tan\left(\frac{(n-k)\varepsilon}{k} \ln(r)\right) = O(v)$, but this contradicts that v is bounded.

Remark. We remark that the asymptotic behavior of the lower branch of the $Q_{k+1,k}$ -translator cannot decay to 0 faster than $\frac{-1}{r^{\frac{n-k}{k}(1+\varepsilon)}}$ for every $\varepsilon > 0$ and $1 \leq k$.

To see this, let $C, \varepsilon > 0$ and consider $w_{C,\varepsilon}(r) = -Cr^{-(1+\varepsilon)a}$ where $a = \frac{n-k}{k}$. Then, $w_{C,\varepsilon}$ is a super-solution to Eq. (4.18) with $v(r_0) < 0$. In fact,

$$w'(r) = \frac{(1+\varepsilon)aC}{r^{(1+\varepsilon)a+1}}$$

and the RHS of Eq. (4.18) is

$$\begin{aligned} & aC(1+w^2)\frac{1}{r^{(1+\varepsilon)a+1}} \frac{(k+1)r^{(1+\varepsilon)a+1} + (n-k-1)}{(n-k) + kr^{(1+\varepsilon)a+1}} \\ & \leq aC(1+w^2)\frac{1}{r^{(1+\varepsilon)a+1}} \\ & \leq w', \end{aligned}$$

for $r > r_0$, where r_0 is sufficiently large. Then, for each $\varepsilon > 0$, we may fix $r_1 > r_0$ and then choose C small enough so that $v(r_1) < w_{C,\varepsilon}(r_1)$. Consequently, we have that $w_{C,\varepsilon} > v$ for all $r \geq r_1$.

Remark. By the above remark, we have that the asymptotic behavior of the principal curvatures of the lower branch at infinity behave as $O(-r^{-a-1})$, where $a = \frac{n-k}{k}$. Then, it follows that $\lambda \in \Gamma_{k+1}$ if, and only if,

$$0 < \binom{n-1}{l} \frac{(-1)^{l-1} n(k-l)}{r^{l(a+1)} kl}, \text{ holds for } l = 1, \dots, k+1.$$

Therefore, $\lambda \notin \Gamma_{k+1}$ for any k as $r \rightarrow \infty$ of the lower branch.

Step 4.5.5. The complete construction of the wing-like solution.

This is simply done by extending the small piece $r : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ in the two branches by plugging the initial conditions

$$\begin{cases} u'(r(-\varepsilon)) = \frac{1}{r'(-\varepsilon)} \\ u(r(-\varepsilon)) = -\varepsilon. \end{cases} \quad \begin{cases} u'(r(\varepsilon)) = \frac{1}{r'(\varepsilon)} \\ u(r(\varepsilon)) = \varepsilon. \end{cases}$$

in Equations (4.17) and (4.18), respectively.

4.6 Application: Growth estimate

In this section we will prove the theorem 4.5, which we repeat here for the convenience of the reader.

Theorem 4.16. Let $\Sigma = \{(x, u(x)) : x \in \mathbb{R}^n\}$ be an entire convex f -translator for some nondegenerate speed f . Assume further, that there exist $a, b, C_1, C_2, R > 0$ such that

$$C_1|x|^a \leq u(c) \leq C_2|x|^b, \text{ for } |x| \geq R,$$

then, $a \leq 2 \leq b$.

In addition, if $a = b = 2$, then $u(x)$ agrees with the “bowl”-type solution up to vertical translations.

Proof. Let assume first that $a > 2$, and let P be the “bowl”-type f -translator in \mathbb{R}^{n+1} . Recall that P is an entire strictly convex rotationally symmetric graph smoothly asymptotic to

$$\frac{|x|^2}{f(0, 1)} - \frac{\partial f}{\partial \lambda_1} \Big|_{\lambda=(0,1)} \ln(|x|) + O(|x|^{-1}).$$

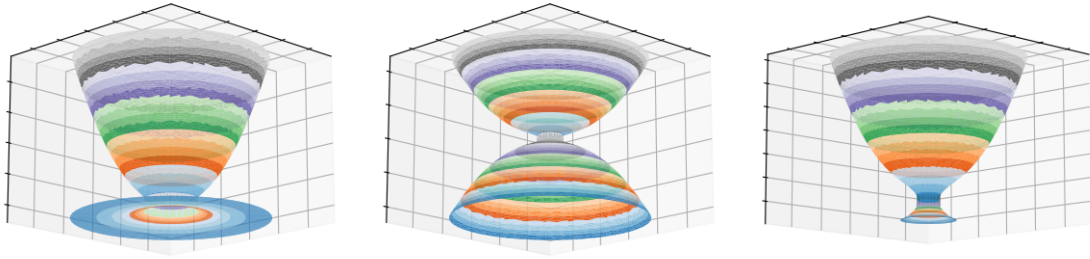
Next, by translating P suitably over Σ , we can find a $t_0 > 0$ such that $P - te_{n+1}$ lies strictly below Σ for $t \geq t_0$. Note that this is possible since $a > 2$. Now, we may translate $P - te_{n+1}$ upward until touches Σ for the first time. Finally, by the interior tangential principle Theorem 4.10, we obtain $\Sigma = P$, but this contradicts $a > 2$.

The case $b < 2$ is analogous, the only change is to place the “bowl”-type soliton above Σ and move it down until it touches Σ .

Finally, when $a = b = 2$, then Σ is smoothly asymptotic to P , and by Theorem 4.2, $\Sigma = P$ up to a vertical translation. \square

Remark. *Theorem 4.5 also holds when the hypotheses of f are changed by the one given in Remark 4.3. This means also holds for entire f -translators such that the principal curvatures of the graph belongs to Γ , and Γ is a convex cone of $\{f > 0\}$ that contains the point $(1, \mathbf{e})$.*

Appendix

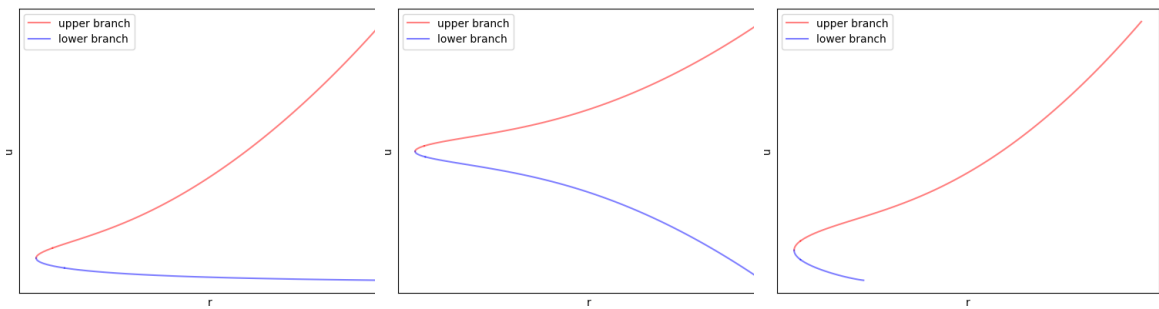


(a) S_k/S_{k-1}

(b) $\sqrt[k]{S_k}$ for k even

(c) $\sqrt[k]{S_k}$ for k odd

Figure 4.1: Winglike translators for various speed functions



(a) $Q_{k+1,k}$

(b) $\sqrt[k]{S_k}$ for k even

(c) $\sqrt[k]{S_k}$ for k odd

Figure 4.2: Profile curves for winglike translators for various speed functions

Chapter 5

Graph Ricci Curvatures and Community Structure

The connection between curvature and topology is a fundamental question in Riemannian geometry. Notable examples include the Gauss-Bonnet theorem, which relates the curvature of a surface to its Euler characteristic, and Myer's theorem, which bounds the diameter of a manifold in terms of its curvature (see [Lee \(2018\)](#) for more details). Remarkably, successful definitions of curvature have been extended to graphs ([Forman \(2003\)](#), [Ollivier \(2009\)](#), [Lin et al. \(2011\)](#), [Devriendt and Lambiotte \(2022\)](#)), generalizing the curvature notions on Riemannian manifolds and establishing analogous connections between curvature and topology.

A crucial topological question in the study of complex networks is their community structure (See [Fortunato \(2010\)](#), [Fortunato and Hric \(2016\)](#) for a survey on community detection). This involves clustering nodes such that many edges connect nodes within the same cluster, while few edges connect nodes between different clusters. These clusters, referred to as communities, are defined based on the application at hand. This task is significant across various fields including computer science, biology, chemistry, logistics, where graphs are commonly used to model real-world systems. Consequently, numerous methods based on diverse theories are available, such as

partitioning algorithms and spectral methods (Fortunato (2010) contains a survey of commonly used algorithms.)

A recent approach to community detection draws inspiration from the geometric notion of curvature. We use the definition of graph Ricci curvature given by Ollivier Ollivier (2009) using optimal transport theory. The essential idea is to compare the distance $d(x, y)$ between two vertices x and y to the distance between the neighbors of x and y (defined in terms of optimal transport). If the latter distance is smaller, the edge is positively curved; if greater, it is negatively curved. Negatively curved edges act as “bottlenecks,” indicating that to move from the neighbors of x to those of y , one must pass through the edge xy .

A basic question regarding curvature and community structure is the following: it is observed that a single edge connecting two disjoint communities will have negative curvature, whereas a complete graph formed by all possible intercommunity edges will have positively curved edges. (This is discussed in Section 5.1.) Therefore, the critical question is: what is the maximum number of intercommunity edges such that we can guarantee each one is negatively curved? More broadly, when examining two particular communities in a graph, what can we infer about the curvature of intercommunity edges? We aim to address this question (and its generalization) quantitatively, as provided by the main theorem.

Theorem 5.1. *Suppose \mathcal{G} is a graph comprised of several communities. Let C_i, C_j be distinguished communities in \mathcal{G} whose sizes are m and n . Let k be the total number of edges that are either intercommunity edges between C_i and C_j , or from any other community to C_i or C_j .*

Then, if

$$k \leq \frac{-m + \sqrt{m^2 + 4(m-1)(2n-1)}}{2},$$

we have $\kappa(e) \leq 0$ for every intercommunity edge e between C_i and C_j .

In particular, if $k \leq \min_l |C_l| - 1$ where C_l are the various communities, the same conclusion holds.

We have the following when the community sizes are all the same:

Corollary 5.1.1. *Let \mathcal{G} be a graph as in previous theorem. Suppose in addition that all communities have the same size n . Then if $k \leq n - 1$, we have $\kappa(e) \leq 0$ for every intercommunity edge e between C_1 and C_2 .*

In the case where the graph only has two communities, we have the following:

Corollary 5.1.2. *Suppose \mathcal{G} is a graph comprised of only two communities C_1, C_2 whose sizes are m and n . Let k be the total number of intercommunity edges between C_1 and C_2 .*

Then, if

$$k \leq \frac{-m + \sqrt{m^2 + 4(m-1)(2n-1)}}{2},$$

we have $\kappa(e) \leq 0$ for every intercommunity edge e between C_1 and C_2 .

If $m = n$, the same conclusion holds for $k \leq n - 1$.

This chapter is organized as follows. In Section 5.1, we define some terms relating to community structure, define the Ollivier-Ricci curvature of an edge of a graph, and give some examples. In Section 5.2, we give a key example that simultaneously motivates the above claims and highlights the computational techniques used to prove the main theorem. In Section 5.3, we provide a proof of the main theorem and the corollaries. In Section 5.4, we show that the bound prescribed in the theorem is sharp, but at the same time we show experimental results that show that we have a lot of latitude even if we go beyond the theoretical limit prescribed in the theorem.

5.1 Preliminaries

5.1.1 Definitions

Let $\mathcal{G} = (\mathcal{V}, \mathbb{E})$ be a graph.

1. A *community* of \mathcal{G} is a maximally connected subgraph of \mathcal{G} .

2. An edge whose endpoints lie in different communities is called an *intercommunity edge*.
3. An edge whose endpoints both lie inside the same community is called an *intracommunity edge*.

We remark that this definition of community is only one of several possible definitions, but this simplifies our mathematical analysis.

5.1.2 The Ollivier-Ricci Curvature

Ollivier [Ollivier \(2009\)](#) defined Ricci curvatures on general Markov chains on metric spaces, with random walks as the building blocks. The following is a simplified exposition for graphs. A *random walk* is a family of probability distributions $\{m_x\}_{x \in \mathcal{G}}$. For any $\alpha \in [0, 1]$, the α -*lazy random walk* is one where the probability measures at each node are given by

$$m_x(z) = m_x^\alpha(z) = \begin{cases} \alpha, & z = x \\ \frac{1-\alpha}{d_x}, & (zx) \in \mathbb{E} \\ 0, & \text{otherwise} \end{cases}$$

Finally, the α -*Ollivier-Ricci curvature* of an edge $e = (xy)$ is defined as

$$\kappa(e) = \kappa^\alpha(e) := 1 - \frac{W(m_x, m_y)}{d(x, y)}. \quad (5.1)$$

where $d(x, y)$ is simply the graph distance between x, y and $W(m_x, m_y)$ is the 1-Wasserstein transportation distance between the probability distributions m_x, m_y (See [Appendix B](#) for a quick review of optimal transportation on graphs.)

We only deal with combinatorial graphs in this volume, and therefore, all edges have length 1. Thus the formula reduces to

$$\kappa(e) = 1 - W(m_x, m_y). \quad (5.2)$$

Example 2. *The intuition behind the Ollivier-Ricci curvature is captured by the following examples:*

- a. *An edge of the lattice \mathbf{Z}^n has curvature $\kappa = 0$. So does an edge of the cycle C_n for $n \geq 6$.*
- b. *An edge of the complete graph K_n has curvature $\kappa = \frac{n(1-\alpha)}{n-1} > 0$.*
- c. *Imagine a ‘dumbbell’ graph comprised of two communities of size m, n , joined by a single intercommunity edge. This edge has curvature $\kappa = 2(1 - \alpha) \left(\frac{1}{m} + \frac{1}{n} - 1\right)$, which is negative for all $m, n \geq 3$.*

Remark. *The claims in the above example can be proved by following the general strategy described here.*

1. *To establish an upper bound on the curvature, we need a lower bound on $W(m_x, m_y)$. This uses (16) and requires prescribing an explicit potential function.*
2. *To establish a lower bound on the curvature, we need an upper bound on $W(m_x, m_y)$. This uses (15) and requires prescribing an explicit transference plan.*
3. *If we can show that the curvature is bounded above and below by the same constant C , then the curvature equals C .*

We will apply this strategy to an important nontrivial example in the next section.

5.2 A Zero-Curvature Example

To see what motivates our claims, we start with the case that the two communities have the same size $n \geq 3$. Consider the number of intercommunity edges below which it is guaranteed that every intercommunity edge is nonpositively curved. We show an explicit construction which suggests that this number can be no more than $n - 1$. (A proof of the sufficiency of this claim is given by the main theorem of this chapter. Sharpness is given by Proposition 5.3.)

Theorem 5.2. *For two communities of size n each, there is a configuration of $n - 1$ edges between the communities such that each intercommunity edge has zero curvature.*

Proof. We give an explicit construction. Let A, B be complete graphs on the vertices $\{a_0, \dots, a_{n-1}\}, \{b_0, \dots, b_{n-1}\}$ respectively. Add inter-community edges $a_i b_i, i = 0, \dots, n - 2$ (See Figure 5.1 for an illustration.) We claim that every edge $a_i b_i$ has zero curvature. Indeed by symmetry, we only need to show it for one of them, say $a_0 b_0$.

To get an upper bound on the curvature, we obtain a lower bound on $W(m_{a_0}, m_{b_0})$. We use f as defined in the table 5.1. Applying the equation (16), an explicit calculation shows that $W(m_{a_0}, m_{b_0}) \geq 1$, which implies $\kappa(a_0 b_0) \leq 0$.

To get a lower bound on curvature, we obtain an upper bound on $W(m_{a_0}, m_{b_0})$. We use the following transference plan:

$$\pi_{a_i b_i} = \begin{cases} \alpha - \frac{1-\alpha}{n}, & i = 0 \\ \frac{1-\alpha}{n}, & i = 1, \dots, n - 1 \end{cases}$$

together with $\pi_{b_0 b_0} = \frac{1-\alpha}{n}$ and 0 between any other pair of vertices not specified previously. Note that the associated distances are

$$d(a_i, b_i) = \begin{cases} 1, & i = 0, \dots, n - 2 \\ 3, & i = n - 1 \end{cases}$$

Applying equation (15), it follows that $W(m_{a_0}, m_{b_0}) \leq 1$ and hence $\kappa(a_0 b_0) \geq 0$. Thus we conclude that

$$\kappa(a_0 b_0) = 0.$$

□

Remark. *The configuration in the previous example is not unique. Indeed, it can be shown using the same technique that the intercommunity edges in Figure 5.2 have curvature 0.*

5.3 Proof of Main Theorem

Let \mathcal{G} be graph, and C_1, C_2 be two communities in \mathcal{G} . Suppose $e = xy \in \mathbb{E}$, $x \in C_1, y \in C_2$. We obtain an upper bound on $\kappa(e)$ by getting a lower bound on $W(m_x, m_y)$. To do so, we need a potential function. The key step is to partition the graph into several subsets on which f will be constant.

Let $C_3 = \mathcal{G} - C_1 - C_2$, which could be a union of several communities. We partition C_1 in five subsets $\{x\}, A, B, C, J, C_2$ in $\{y\}, D, E, F, K$ and C_3 in G, H, I, L, M , which we define in the following way:

- $B = \{z \in C_1 - x : yz \in \mathbb{E}\}$
- $D = \{w \in C_2 - y : xw \in \mathbb{E}\}$
- $C = \{z \in C_1 - x - B : zw \in \mathbb{E} \text{ for some } w \in C_2\}$
- $E = \{z \in C_2 - y - D : zw \in \mathbb{E} \text{ for some } w \in C_2\}$
- $J = \{z \in C_1 - x - B - C : d(z, w) = 2, \text{ for some } w \in C_2\}$
- $K = \{w \in C_2 - y - B - E : d(z, w) = 2, \text{ for some } z \in C_1\}$
- $A = C_1 - x - B - C - J$
- $F = C_2 - y - D - E - K$
- $G = \{v \in \mathcal{G} - C_1 - C_2 : xv, yv \in \mathbb{E}\}$
- $H = \{v \in \mathcal{G} - C_1 - C_2 - G : xv \in \mathbb{E}\}$
- $I = \{v \in \mathcal{G} - C_1 - C_2 - G : yv \in \mathbb{E}\}$

- $L = \{v \in \mathcal{G} - C_1 - C_2 - G - H - I : zv \in \mathbb{E} \text{ for some } z \in J\} = \{v \in \mathcal{G} - C_1 - C_2 : wv \in \mathbb{E} \text{ for some } w \in K\}$
- $M = \mathcal{G} - C_1 - C_2 - G - H - I - L$

This configuration, along with the values of the function f are illustrated in Figure 5.3. The number shown inside each “node” (or subset of \mathcal{G} to be precise) is the value of the function f . Note that we have not added all possible edges: for instance, there could be edges between A, M . The important thing is that A, J do not have neighbours in C_2 , F, K do not have neighbours in C_1 . This ensures that the function in Table 5.2 is 1-Lipschitz.

In Table 5.2 we depict the measures m_x, m_y together with the value of f . We let $\beta := 1 - \alpha$.

In the following we abuse notation by conflating S with its cardinality $|S|$ where S could be any of the sets A, \dots, M . Let $W = W(m_x, m_y)$. Using (16) we have the following bound:

$$W \geq \alpha + (1 - \alpha) \left[\frac{-1 - C - 2 - 2D - G}{d_x} + \frac{-1 + B + 2D + 2E + 2K + 3F + G + I}{d_y} \right] \quad (5.3)$$

We also have the following constraint equations from counting nodes and edges:

$$\begin{aligned} A + B + C + J + 1 &= n, \\ D + E + F + K + 1 &= m, \\ n + D + G + H &= d_x, \\ m + B + G + I &= d_y. \end{aligned} \quad (5.4)$$

Equations (5.3) and (5.4) together give

$$W \geq \alpha + (1 - \alpha) \left[\frac{n - 1 + A + J + G + H}{d_x} + \frac{m - 1 + F}{d_y} - 1 \right]. \quad (5.5)$$

Note that the lack of symmetry between x and y in the above expression is due to the lack of symmetry in the way we defined f .

Let k_1 be the number of intercommunity edges between C_1 and C_2 , and k_2 be the number of edges from C_1 or C_2 to the rest of \mathcal{G} . Then,

$$\begin{aligned} k_1 &\geq 1 + B + D + \max\{C, E\}, \\ k_2 &\geq 2G + H + I + \max\{J, L\} + \max\{K, L\}. \end{aligned} \tag{5.6}$$

Defining $k = k_1 + k_2$, we have

$$k \geq 1 + B + D + \max\{C, E\} + 2G + H + I + \max\{J, L\} + \max\{K, L\} \tag{5.7}$$

which, together with the constraint equations (5.4), yields

$$\begin{aligned} k + A &\geq n + D + 2G + H + I + \max\{K, L\} \\ &= d_x + G + I + \max\{K, L\} \\ &\geq d_x \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} k + F &\geq m + B + 2G + H + I + \max\{J, L\} \\ &= d_y + G + I + \max\{J, L\} \\ &\geq d_y. \end{aligned} \tag{5.9}$$

Now we need to find an optimal k such that

$$\frac{n - 1 + A + J + G + H}{d_x} + \frac{m - 1 + F}{d_y} - 1 \geq \frac{k + A + J + G + H}{d_x} + \frac{k + F}{d_y} - 1,$$

which reduces to

$$\begin{aligned}
& \frac{n-1}{d_x} + \frac{m-1}{d_y} \geq \frac{k}{d_x} + \frac{k}{d_y} \\
& \iff (m-1)d_x + (n-1)d_y \geq k(d_x + d_y) \\
& \iff (m-n)d_x + (n-1)(d_x + d_y) \geq k(d_x + d_y) \\
& \iff (m-n)d_x \geq (k-n+1)(d_x + d_y) \\
& \iff \frac{d_x}{d_x + d_y} \geq \frac{k-n+1}{m-n}
\end{aligned}$$

Since $d_x \geq n$ and $d_x + d_y \leq n + m + k - 1$, a sufficient condition for the above to be true is

$$\frac{n}{n + m + k - 1} \geq \frac{k - n + 1}{m - n},$$

which results in the quadratic inequality

$$k^2 + mk - (m-1)(2n-1) \leq 0.$$

Theorem 5.1 now follows directly from this.

Proof of Corollary 1:

By letting $m = n$, we get

$$\begin{aligned}
k & \leq \frac{-n + \sqrt{n^2 + 4(n-1)(2n-1)}}{2} \\
& = \frac{-n + \sqrt{n^2 + 4(2n^2 - 3n + 1)}}{2} \\
& = \frac{-n + \sqrt{9n^2 - 12n + 4}}{2} \\
& = \frac{-n + 3n - 2}{2} \\
& = n - 1. \quad \square
\end{aligned}$$

Proof of Corollary 2:

In the proof of main theorem, we may assume that $G, H, L, I, M = \emptyset$. Consequently, $J, K = \emptyset$ as well, and $k_2 = 0$, and the claim follows immediately. \square

5.4 Empirical Results

In the previous section, we gave theoretical bounds on the number of intercommunity edges that guarantee the negativity of curvatures. This bound is sharp due to the following proposition:

Proposition 5.3. *For communities of size (n, n) , there is a configuration of n edges between the communities such that each intercommunity edge has positive curvature.*

Proof. Consider the product graph of the complete graph K_n with the graph consisting of only two points joined by an edge. See Figure 5.4 for an illustration with $n = 4$.

We can show that edge a_0b_0 (and hence every intercommunity edge) has positive curvature. The distributions m_{a_0}, m_{b_0} are given in Table 5.3

Consider the transference plan

$$\pi_{a_i b_j} = \begin{cases} \alpha - \frac{1-\alpha}{n}, & i = j = 0 \\ \frac{1-\alpha}{n}, & i = j \neq 0 \\ 0, & i \neq j \end{cases} \quad (5.10)$$

This plan has a cost of

$$\sum_{i,j} \pi_{a_i b_j} d(a_i, b_j) = \alpha + \frac{(n-2)(1-\alpha)}{n} = 1 - \frac{2(1-\alpha)}{n}$$

which is an upper bound on $W(m_{a_0}, m_{b_0})$, and hence

$$\kappa(a_0 b_0) \geq \frac{2(1-\alpha)}{n} > 0.$$

□

Even though it is theoretically possible for all edges to be positively curved when we have $k = n$ intercommunity edges, the point of the remainder of this section is to share experimental findings that indicate how unlikely such a situation is. For the sake of simplicity, we generate two-community graphs with randomly chosen intercommunity edges and observe the empirical proportion of nonpositively curved edges.

Figure 5.5 shows the result of one such experiment. Here, we choose the community sizes $|C_1| = |C_2| = 128$. We experiment with $k = 128, 256, 384$, and 512 intercommunity edges. For each k , we generate 100 graphs where k intercommunity edges are chosen at random. In each of those random graphs, we compute the proportion $p_{\leq 0}^k$ of nonpositively curved edges. Finally, we plot $p_{\leq 0}^k$ versus its frequency of occurrence.

One notices in Figure 5.5 that for $k = 128$ and 256, almost all edges were negatively curved in every one of the 100 randomly generated graphs. When $k = 384$, most of the edges are negatively curved. But when $k = 512$, the proportion of negatively curved edges is small.

In fact, we find something similar for different sizes. In Figure 5.6, we examine community sizes $n = 16, 32, 64, 128, 256$, and 512. For each n , we generate graphs with number of intercommunity edges $k = n, 2n, 3n, 4n$ and empirically find the proportion of intercommunity edges that have nonpositive curvature. For each choice of (n, k) , we generate 100 graphs at random. We plot the empirical proportion of nonpositive edges with error bars one standard deviation wide (over the 100 runs.) What we find is that for $n \geq 32$, even when we have $2n$ intercommunity edges, almost all of them are negatively curved. When $n = 512$, even when $k = 4n$ we have almost all edges negatively curved.

5.5 Conclusion

In summary, we have examined the relation between curvature and community structure from a theoretical point of view. In particular, we have sought to understand how the sizes of communities and the number of intercommunity edges affects the curvature of those intercommunity edges. More specifically, we have given sufficient conditions for intercommunity edges to be negatively curved. In addition, we show these requirements are sharp, in the sense that there are counterexamples as soon as we cross the cutoff. We have achieved this by exploiting two equivalent definitions of the Wasserstein distance between probability distributions, which give us concrete computational tools for proving curvature estimates.

In the experimental section, we explored how likely it is to find positively curved edges when the number of intercommunity edges k exceeds the theoretical bound. We found that as the community sizes become large, k can get much larger than the theoretical bound while most of the intercommunity edges have negative curvature. A likely explanation for this is in the estimate (5.5). Here we see that the Wasserstein distance is large when the “unmatched” sections A, F are large and the node degrees d_x, d_y are small. When we randomly sample intercommunity edges from the list of all possible intercommunity edges, we are less likely to sample an edge configuration where degrees are very large and the unmatched sections are very small, at least when the number of intercommunity edges is very small.

This analysis raises several interesting questions. For instance, it would be very interesting to study the curvature of intracommunity edges and obtain criteria that ensure that they are positively curved. In effect, this would provide a theoretical underpinning for curvature-based community detection via edge deletion. Another interesting direction is the analysis of curvature distribution as a function of the number of intercommunity edges. For instance, for a fixed k , let I be a random sample of k intercommunity edges from the list of all possible ones. Now we can generate a graph with I as the set of intercommunity edges. Let $\kappa_{max}(I)$ be the

maximum curvature among edges in I , which we may regard as a random variable. What is the probability distribution of κ_{max} ? How is it affected by k ? Another interesting question is one that concerns “surgery”, or edge deletion. How does the deletion of a subset of edges of a graph affect the curvature of the remaining edges? Questions of this nature provide ideal circumstances for the synthesis of tools from differential geometry, graph theory, statistics, and programming.

Appendix

Figures

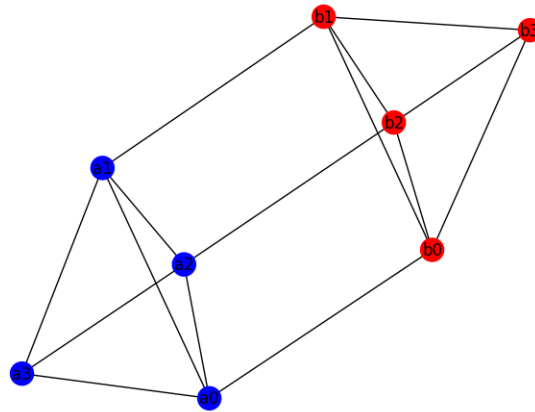


Figure 5.1: A configuration with zero curvature on all intercommunity edges

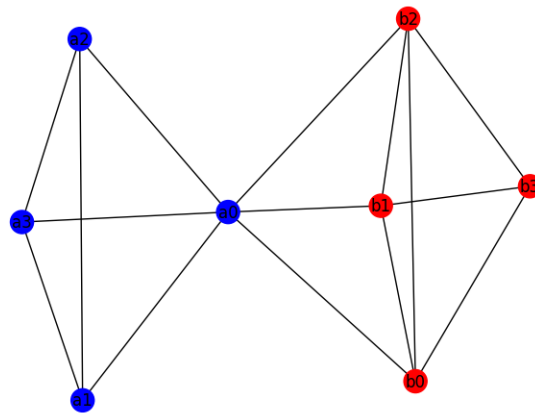


Figure 5.2: Another configuration with zero curvature on intercommunity edges

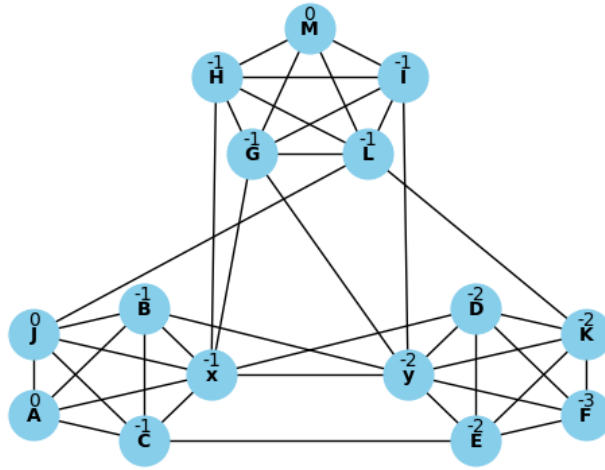


Figure 5.3: Three communities, with a potential function

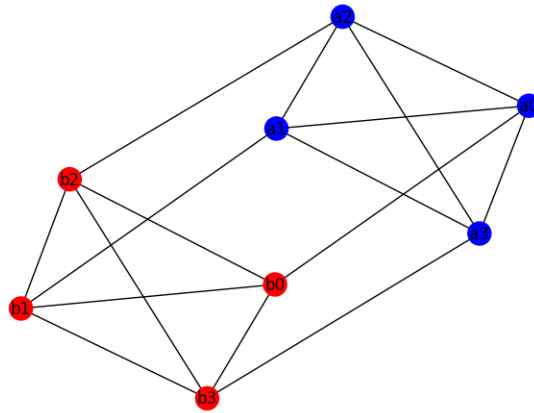


Figure 5.4: Two communities with all n intercommunity edges positively curved

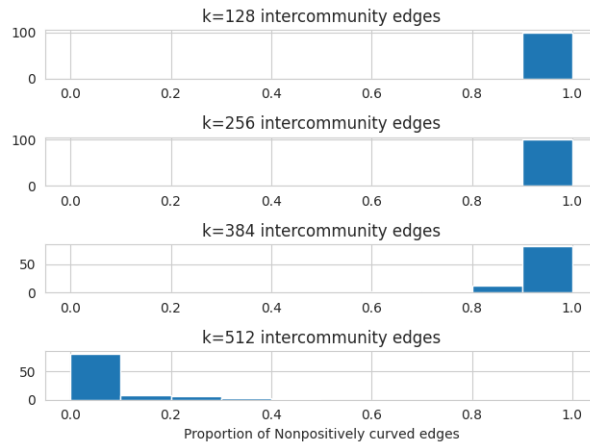


Figure 5.5: Distribution of Proportion of Nonpositively Curved Intercommunity Edges

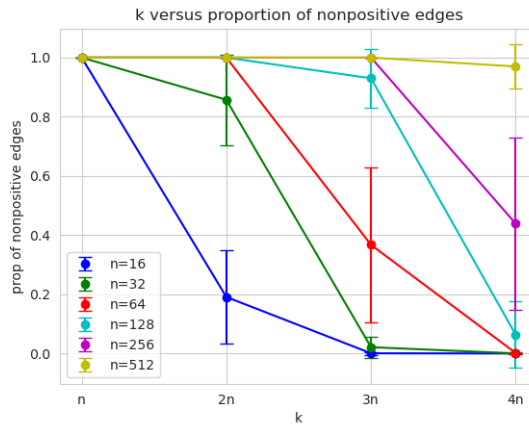


Figure 5.6: Proportion of Negative Edges for different k and n

Tables

Table 5.1: A potential function for curvature lower upper bounds

-	a_{n-1}	a_1, \dots, a_{n-2}	a_0	b_0	b_1, \dots, b_{n-2}	b_{n-1}
m_{a_0}	$\frac{1-\alpha}{n}$	$\frac{1-\alpha}{n}$	α	$\frac{1-\alpha}{n}$	0	0
m_{b_0}	0	0	$\frac{1-\alpha}{n}$	α	$\frac{1-\alpha}{n}$	$\frac{1-\alpha}{n}$
f	0	-1	-1	-2	-2	-3

Table 5.2: Probabilities and potential associated with curvature of edge xy for multiple community case

-	J	A	B	C	x	y	D	E	F	K	G	H	M	I	L
m_x	$\frac{\beta}{d_x}$	$\frac{\beta}{d_x}$	$\frac{\beta}{d_x}$	$\frac{\beta}{d_x}$	α	$\frac{\beta}{d_x}$	$\frac{\beta}{d_x}$	0	0	0	$\frac{\beta}{d_x}$	$\frac{\beta}{d_x}$	0	0	0
m_y	0	0	$\frac{\beta}{d_y}$	0	$\frac{\beta}{d_y}$	α	$\frac{\beta}{d_y}$	$\frac{\beta}{d_y}$	$\frac{\beta}{d_y}$	$\frac{\beta}{d_y}$	$\frac{\beta}{d_y}$	0	0	$\frac{\beta}{d_y}$	0
f	0	0	-1	-1	-1	-2	-2	-2	-3	-2	-1	-1	0	-1	-1

Table 5.3: Probability distributions for positive curvature example

-	a_0	a_1, \dots, a_n	b_0	b_1, \dots, b_n
m_{a_0}	α	$\frac{1-\alpha}{n}$	$\frac{1-\alpha}{n}$	0
m_{b_0}	$\frac{1-\alpha}{n}$	0	α	$\frac{1-\alpha}{n}$

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Appendices

A Differential Equations Theory

A.1 Ordinary Differential Equations

ODE initial value problems

This subsection summarizes the requisite theory that we need to perform abstract ODE analysis. References to all definitions and theorems in this subsection can be found in Teschl's standard text [Teschl \(2012\)](#). The form of ODE most amenable to analysis is

$$x' = f(t, x), \tag{11}$$

where $x(t) : I \rightarrow \mathbb{R}^n$, I is an interval, $f : I \times U \rightarrow \mathbb{R}^n$ is some known function, $U \subset \mathbb{R}^n$ and $x' := dx/dt$. For us, $n = 1$ will suffice.

A differentiable function $x_+(t)$ satisfying

$$x'_+(t) > f(t, x_+(t)) \text{ (respectively, } x'_-(t) < f(t, x_-(t)) \text{)}$$

is called a *supersolution* (respectively, *subsolution*) to (11).

The following is a comparison principle for ODEs:

Proposition A.0.1. *Let $x_+(t)$, $x_-(t)$ be super, sub solutions to (11) on $[t_0, T)$ respectively. For every solution $x(t)$ on $[t_0, T)$ we have*

$$x(t) < x_+(t), t \in [t_0, T) \text{ provided } x(t_0) \leq x_+(t_0)$$

respectively

$$x_-(t) < x(t), t \in [t_0, T) \text{ provided } x_-(t_0) \leq x(t_0).$$

Remark. *If one replaces strong inequality by weak inequality in the definitions of sub and supersolutions, one gets weak inequalities instead of strong ones in the above lemma.*

A problem of the form

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{12}$$

is called an *initial value problem*, or *IVP*.

We recall the fundamental local existence-uniqueness and extensibility results.

Theorem A.0.2. *Suppose $f \in C(U, \mathbb{R}^n)$ where U is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, then there exists a unique local solution $\bar{x}(t) \in C(I)$ of the IVP (12), where I is some interval around t_0 .*

Moreover, if $f \in C^k(U, \mathbb{R}^n)$, then $\bar{x}(t) \in C^{k+1}(I)$.

Theorem A.0.3. *Let $\phi(t)$ be a solution of the IVP (12) defined on the interval (t_-, t_+) . Suppose there is a compact subset $[t_0, t_+] \times C \subset U$ such that $\phi(t_m) \in C$ for some sequence $t_m \in [t_0, t_+)$ converging to t_+ . Then there exists an extension to the interval $(t_-, t_+ + \varepsilon)$ for some $\varepsilon > 0$.*

In particular, if there is such a compact set C for every $t_+ > 0$ (C might depend on t_+), then the solution exists for all $t > t_0$.

The analogous statement holds for an extension to $(t_- - \varepsilon, t_+)$.

Remark. The form $[t_0, t_+] \times C$ for the compact set can be relaxed. We can simply require some compact set $K \subset U$, such that the projection of K onto the t -coordinate contains $[t_0, t_+]$, and that $(t_m, \phi(t_m)) \in K$.

The above theorem is an extensibility result. For instance, it guarantees that if you can solve your ODE locally at any $t = t_0$, and have super and subsolutions that exist for all $t > t_0$, then your solution extends to all “future times” $t > t_0$.

The following theorem provides estimates for the difference between two solutions of an ODE. This is a special case of (Teschl, 2012, Theorem 2.8).

Theorem A.0.4. Suppose $f \in C(U, \mathbb{R}^n)$, $f = f(t, x)$ is Lipschitz continuous (with Lipschitz constant L) in the second argument, uniformly with respect to the first. If $x(t), y(t)$ are solutions of the respective IVPs

$$x'(t) = f(t, x), \quad x(t_0) = x_0$$

$$y'(t) = f(t, y), \quad y(t_0) = y_0,$$

then,

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{L|t-t_0|}$$

for as long as both $x(t), y(t)$ are defined.

We also state a basic lemma which will be used to determine whether a given ODE blows up or not.

Proposition A.0.5. Consider the problem

$$x' = x^\theta$$

$$x(t_0) = x_0 > 0.$$

Its solution is

$$x(t) = \begin{cases} x_0 + \log\left(\frac{t}{t_0}\right) & \text{when } \theta = 1 \\ (x_0^{1-\theta} + (1-\theta)(t-t_0))^{\frac{1}{1-\theta}} & \text{when } \theta \neq 1. \end{cases}$$

The solution exists for all $t > t_0$ if $\theta \leq 1$. It tends to infinity at $t = t_0 + x_0^{1-\theta}/(\theta - 1)$ if $\theta > 1$.

A.2 A PDE regularity lemma

The following result provides higher regularity of C^2 -solutions to elliptic PDE. It is a consequence of Schauder's estimate (see (Gilbarg and Trudinger, 2001, Lemma 17.16)).

Proposition A.0.6. *Suppose that $u \in C^2(\Omega)$ satisfies*

$$F(\cdot, u, Du, D^2u) = 0 \text{ in } \Omega,$$

where $F : \Gamma \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n} \rightarrow \mathbb{R}$ is monotone increasing with respect to the matrix variable. If $F \in C^{k,\alpha}(\Gamma)$ for some $k \geq 1$ and $0 < \alpha < 1$, then $u \in C^{k+2,\alpha}(\Omega)$. In particular, if F is smooth, then so is u .

A.3 Level sets of the expression $\frac{w}{r(1+w^2)^\beta}$

Let $m > 0$ be a constant, and let $\frac{w}{m(1+w^2)^\beta} = r$. Then

$$\frac{dr}{dw} = \frac{1}{m} \frac{1 + (1 - 2\beta)w^2}{(1 + w^2)^{1+\beta}} \quad (13)$$

which means r behaves very differently for $\beta > 1/2$ and $\beta < 1/2$. When $\beta > 1/2$, r is increasing for small values of w and decreasing for large values, which means r is not invertible. But if $\beta < 1/2$, then $dr/dw > 0$, meaning r is increasing with respect

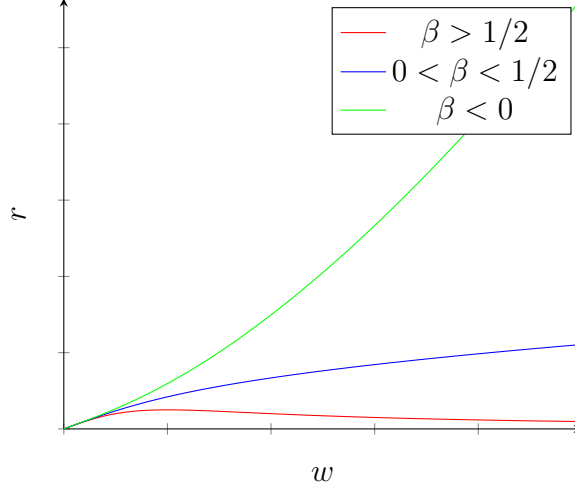


Figure 7: The graph of $\frac{w}{m(1+w^2)^\beta} = r$

to w and hence, so is the inverse function which defines w as a function of r . Figure 7 illustrates this contrast. We consider only the latter case, since this corresponds to $\alpha > 0$.

We say $f(t) \sim g(t)$ as $t \rightarrow \infty$ if $f(t)/g(t) \rightarrow C$ for some constant $C \neq 0$. Then in our level set equation, $r(w) \sim \frac{w^{1-2\beta}}{m} = \frac{w^{1/\alpha}}{m}$. Therefore,

$$w(r) \sim (mr)^\alpha.$$

And by equation (13),

$$\frac{dw}{dr}(r) \sim w^{2\beta} \sim r^{2\alpha\beta} = r^{\alpha-1},$$

so that

$$\frac{dw}{dr} \sim r^{\alpha-1}. \tag{14}$$

For subsequent reference, we collect the following facts about the relation

$$\frac{w}{m(1+w^2)^\beta} = r.$$

The hypothesis $\beta < 1/2$ is used for these lemmas.

Proposition A.0.7. *For constant $m > 0$, r is increasing with respect to w .*

Proposition A.0.8. *For a fixed $r > 0$, if we regard m as a function of w , it is increasing with respect to w .*

Proof. This is due to the symmetry of r and m . □

A.4 The domain of $g(\cdot, 1)$

Note that the values of a homogeneous function f of n variables are completely determined by the values of f on S^{n-1} . Thus, our admissible speeds are completely determined in the positive cone Γ_+^n by their values on $S^{n-1} \cap \Gamma_+^n$, the points on the unit sphere that have all positive coordinates. Further, since in our setting all inputs of f but the first are equal, it suffices to consider points in $S^{n-1} \cap \Gamma_+^n$ of the form $(x, y\mathbf{e})$, that is, $(x, y) \in \Gamma_+^2$ and $x^2 + (n-1)y^2 = 1$. So we may describe y as a function of x : $y(x) = \sqrt{\frac{1-x^2}{n-1}}$. Now we can describe the level set $f(x, y\mathbf{e}) = 1$ as

$$E = \left\{ \frac{1}{f(x, y(x)\mathbf{e})^{1/\alpha}} (x, y(x)) : 0 < x < 1 \right\} \subset \Gamma_+^2.$$

From this point of view, we see that E is a connected set, as it is the continuous image of an interval. From another point of view, E can be identified with the graph of the function $g(\cdot, 1)$: it is the set of points $(g(y, 1), y)$ such that y is in the domain $D \subset \mathbb{R}$ of $g(\cdot, 1)$. D is the projection of E onto the y -coordinate, and hence it is connected. Now, due to the implicit function theorem, the domain of $g(\cdot, 1)$ is an open interval, as we already know it is connected. Since $\lim_{x \rightarrow 0} y(x) = 1/\sqrt{n-1}$, we see that, in the degenerate case, D is unbounded above. In the nondegenerate case, we see that D is bounded above; here we find that $\sup D = 1/f(0, \mathbf{e})^{1/\alpha}$, and we can extend g continuously by defining $g(\sup D, 1) = 0$.

B Optimal Transportation on Graphs

Let μ, ν be probability measures on a graph, and let the vertices be enumerated $1, \dots, n$. A *transference plan* $\pi = (\pi_{ij})$ is an $n \times n$ matrix with nonnegative entries such that $\sum_j \pi_{ij} = \mu(i)$, and $\sum_i \pi_{ij} = \nu(j)$. More concisely, π is a joint probability distribution on $\mathcal{V} \times \mathcal{V}$ whose marginals are μ and ν . We define Π to be the set of all transference plans between μ and ν .

Closeness between two probability distributions can be measured via the *1-Wasserstein distance* (or *earthmover's distance*) which is defined as follows:

$$W(\mu, \nu) = W_1(\mu, \nu) := \inf_{\pi \in \Pi} \sum_{i,j} \pi_{ij} d(i, j) \quad (15)$$

where $d(i, j)$ is the graph distance between vertices i, j .

Proposition B.0.1. *Under the metric W defined above, the set of all probability measures on \mathcal{V} is a metric space.*

Note that computing W amounts to solving a linear programming problem. Therefore, by duality, we have an equivalent formulation for W via a maximization problem. To present this formulation, we first define a *1-Lipschitz* function to be a function f such that $|f(x) - f(y)| \leq d(x, y)$. On (combinatorial) graphs, this is equivalent to insisting that the values of f on adjacent nodes do not differ by more than 1.

Proposition B.0.2. *Let \mathcal{F} be the set of 1-Lipschitz functions on G . Then,*

$$W(\mu, \nu) = \sup_{f \in \mathcal{F}} \left\{ \sum_{z \in G} f(z) (\mu(z) - \nu(z)) \right\} \quad (16)$$

A function f that achieves this supremum is called a *Kantorovich potential*.

Vita

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