To the Graduate Council:

I am submitting herewith a thesis written by Brenda Kay Harrington entitled "Qualitative and Quantitative Properties of Eigenvalue Problems with Eigenparameter in the Boundary Conditions." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Don Hinton

Don B. Hinton, Major Professor

We have read this thesis
and recommend its acceptance:

Charles E Clark

Suzanne Lenhart

Accepted for the Council:

C W Minkel

Vice Provost
and Dean of The Graduate School

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirments for a Master's degree at The University of Tennessee, Knoxville, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, by the Head of Interlibrary Services when, in the opinion of either, the proposed use of the material is for scholary purposes. Any copying or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature Brenda Kay Harrington

Date May 5, 1988

QUALITATIVE AND QUANTITATIVE PROPERTIES OF EIGENVALUE PROBLEMS WITH EIGENPARAMETER IN THE BOUNDARY CONDITIONS

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Brenda Kay Harrington

June 1988

To my father Dr. Clifton C. Thompson

ACKNOWLEDGMENTS

I would like to thank Dr. Charles E. Clark and Dr. Suzanne Lenhart for serving on my committee. My major professor Dr. Don B. Hinton deserves special thanks for chairing my committee and without whose help this thesis would not have been possible. I also wish to thank my husband Eric for his patience and support.

ABSTRACT

We have examined the general differential equation

$$y'' + q(x)y = -\lambda y$$

with boundary conditions

$$(\cos\gamma)y(L) - (\sin\gamma)y(L) = 0$$

and

$$-[\beta_1 y(0) + \beta_2 y'(0)] = \lambda[\beta_1' y(0) + \beta_2' y'(0)].$$

The qualitative study of this problem involves the investigation of the number of zeros for each eigenfunction and the separation of the zeros of the eigenfunctions.

The quantitative study involves numerical solutions of the above differential equation. The IMSL computer programs ZPOLR and EQRT2S were used in this study and can be found in the program library at The University of Tennessee. We used ZPOLR to solve for the eigenvalues of a tridiagonal symmetric matrix and EQRT2S to solve for the roots of a polynomial. Other numerical methods included Newton's method for solving differential equations and an algorithm to solve for the angle θ using a bisection approach.

The qualitative findings show that the zeros of the eigenfunctions separate with a minor discrepancy when a certain angle crosses a multiple of π. The nth eigenfunction has n zeros until this angle crosses a multiple of π and then the nth eigenfunction has (n-1) zeros.

The qualitative findings show that the finite difference and bisection methods work well in solving for the eigenvalues. The series method, while solving for the first few zeros successfully, also generates negative and extraneous roots and is therefore considered to be unreliable.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION AND QUALITATIVE THEORY

We consider here the eigenvalue problem

(1.1) $y'' + q(t)y = -\lambda y$

with the boundary conditions

(1.2) $(\cos\gamma)y(L) - (\sin\gamma)y'(L) = 0$

and

(1.3) $-[\beta_1 y(0) + \beta_2 y'(0)] = \lambda[\beta_1' y(0) + \beta_2' y'(0)]$,

where q is a continuous real valued function defined on [0,L] and where β_1, β_1', β_2, β_2' are real valued and satisfy the relationship

(1.4) $\beta_1'\beta_2 > \beta_1\beta_2'$.

Condition (1.4) is necessary to guarantee that the problem is self-adjoint [4].

This differential equation is unique since it contains an eigenvalue parameter-dependent boundary condition. There are several different application problems that can be modeled with second and fourth order systems which have eigenvalue parameter-dependent boundary conditions. An example of a reduced second order problem describes the vibrations of a revolving beam with a mass on the tip [5]. There are also fourth order examples involving "transverse vibrations of a uniform rotating beam, elastically restrained at one end and carrying a tip mass at its free end" [1]. The significance of this problem arises in "the design of helicopter blades" [1]. Another fourth order problem involves "vibrations of a vertical pendulum consisting of a bob suspended from a wire..." [2].

We will take a look at a well-known result in differential equations involving the separation of zeros of eigenfunctions and the number of zeros for each eigenfunction. However, we will add the complication that the boundary conditions will be eigenvalue parameter-dependent. First, we will examine the changes the parameter dependency will have on the theoretical concepts. Second, we will look at parameter-dependent differential equations that can also be solved analytically. Then we will look at several different numerical approaches for solving differential equations which are parameter-dependent. The numerical approaches will include finite difference, series, and angle bisection methods.

The differential equation (1.1) with boundary conditions (1.2) and (1.3) can be transformed into polar coordinates. First we let

$$x = r\cos\theta$$

and

$$y = r\sin\theta.$$

Then we find that

$$x' = r'\cos\theta - r\theta'\sin\theta$$

and

$$y' = r'\sin\theta + r\theta'\cos\theta.$$

By letting $y' = x$ we find that $y'' = x'$. By substitution we see that

$$y' = x = r\cos\theta$$

and

$$x' = -(q(t) + \lambda)y = -(q(t) + \lambda)r\sin\theta.$$

Further substitution leads us to the following equations:

$$(1.5) \qquad r'\sin\theta + r\theta'\cos\theta = r\cos\theta \text{ and}$$

$$(1.6) \qquad r'\cos\theta - r\theta'\sin\theta = -(q(t) + \lambda)r\sin\theta.$$

Next, we multiply equation (1.5) by $\cos\theta$ and equation (1.6) by $-\sin\theta$ to obtain,

$$\cos\theta[r'\sin\theta + r\theta'\cos\theta] = \cos\theta[r\cos\theta]$$

and

$$-\sin\theta[r'\cos\theta - r\theta'\sin\theta] = -\sin\theta[-(q(t) + \lambda)r\sin\theta].$$

Adding the above two equations and solving for θ' we find that

$$\theta' = \cos^2\theta + (q(t) + \lambda)\sin^2\theta.$$

Now, we multiply equation (1.5) by $\sin\theta$ and equation (1.6) by $\cos\theta$ to get

$$\sin\theta[r'\sin\theta + r\theta'\cos\theta] = \sin\theta[r\cos\theta]$$

and

$$\cos\theta[r'\cos\theta - r\theta'\sin\theta] = \cos\theta[-(q(t) + \lambda)r\sin\theta].$$

Adding the above two equations and solving for r' we find that

$$r' = r(\sin\theta\cos\theta)[1 - q(t) - \lambda].$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. We can define $r(0,\lambda) = 1$, so that with $\alpha(\lambda) = \theta(0,\lambda)$,

$$y(0,\lambda) = \sin\alpha(\lambda)$$

and

$$y'(0,\lambda) = \cos\alpha(\lambda).$$

Substituting the above formulas for y and y' into boundary condition (1.3) yields that with $\alpha = \alpha(\lambda)$,

$$-[\beta_1\sin\alpha + \beta_2\cos\alpha] = \lambda[\beta_1'\sin\alpha + \beta_2'\cos\alpha].$$

Solving this equation for α gives

$$\tan\alpha = \sin\alpha/\cos\alpha = -(\beta_2 + \beta_2'\lambda)/(\beta_1 + \beta_1'\lambda).$$

Boundary condition (1.2) can be solved for

$$y(L)/y'(L) = \sin\gamma/\cos\gamma = \tan\gamma.$$

Therefore $\theta(L,\lambda) = \gamma(\mathrm{mod}\ \pi)$. Thus λ is an eigenvalue and the eigenvalue exists when $\theta(L,\lambda) = \gamma(\mathrm{mod}\ \pi)$ if $\theta(0,\lambda)$ is chosen so that

$$\tan\theta(0,\lambda) = -(\beta_2 + \beta_2'\lambda)/(\beta_1 + \beta_1'\lambda).$$

We define below a continuous function $\alpha(\lambda)$ such that

(1.7) $$\tan\alpha(\lambda) = -(\beta_2 + \beta_2'\lambda)/(\beta_1 + \beta_1'\lambda).$$

We also define the function

$$g(\lambda) = -(\beta_2 + \beta_2'\lambda)/(\beta_1 + \beta_1'\lambda)$$

and hence,

$$g'(\lambda) = (\beta_1'\beta_2 - \beta_1\beta_2')/(\beta_1 + \lambda\beta_1')^2.$$

Because $\beta_1'\beta_2 - \beta_1\beta_2' > 0$ from (1.4), it is clear that $g'(\lambda) > 0$ for $\lambda \neq -\beta_1/\beta_1'$. Therefore,

$$g(\lambda) = \tan\alpha(\lambda) = -(\beta_2 + \beta_2'\lambda)/(\beta_1 + \beta_1'\lambda),$$

$$g'(\lambda) = [\sec^2\alpha(\lambda)]\alpha'(\lambda),$$

and

$$g'(\lambda) = (-\beta_2'\beta_1 + \beta_1'\beta_2)/(\beta_1 + \beta_1'\lambda)^2.$$

Since $\alpha'(\lambda) = (-\beta_2'\beta_1 + \beta_1'\beta_2)/(\beta_1 + \beta_1'\lambda)^2\sec^2\alpha(\lambda)$ we make the important conclusion that $\alpha'(\lambda) > 0$ for $\lambda \neq -\beta_1/\beta_1'$. To see that $\alpha'(-\beta_1/\beta_1') > 0$ we use

$$\cot\alpha(\lambda) = -(\beta_1 + \beta_1'\lambda)/(\beta_2 + \beta_2'\lambda)$$

and differentiate. This gives

$$[-\csc^2\alpha(\lambda)]\alpha'(\lambda) = [-\beta_1'(\beta_2 + \beta_2'\lambda) + \beta_2'(\beta_1 + \beta_1'\lambda)]/(\beta_2 + \beta_2'\lambda)^2.$$

The above equation reduces to

$$\alpha'(\lambda) = (\beta_1'\beta_2 - \beta_2'\beta_1)/[\csc^2\alpha(\lambda)](\beta_2 + \beta_2'\lambda)^2.$$

From (1.4) we see that $\alpha'(-\beta_1/\beta_1') > 0$ for $\beta_1' \neq 0$.

We have in Figure 1.1 and Figure 1.2 the two possible graphs of $g(\lambda)$.

In the first case we have that $\beta_1' = 0$ which is shown in Figure 1.1. In the second case we have that $\beta_1' \neq 0$ which is shown in Figure 1.2.

We choose a branch of arctan so that the function α is continuous, $\alpha'(\lambda) > 0$, α has limits at $\pm\infty$ and $\alpha(\infty) - \alpha(-\infty) = \pi$. Examining the function $g(\lambda)$ as $\lambda \to -\infty$, we see that there are four cases to consider. In the graphs below, the quantities $-\beta_1/\beta_1'$ and $-\beta_2/\beta_2'$ may be either positive or negative. In the first case we have that

(A) $\qquad g(-\infty) = -\beta_2'/\beta_1' > 0$ if $\beta_1' \neq 0$ and $\beta_2'/\beta_1' < 0$.

Figure 1.3 is a graph of case (A).

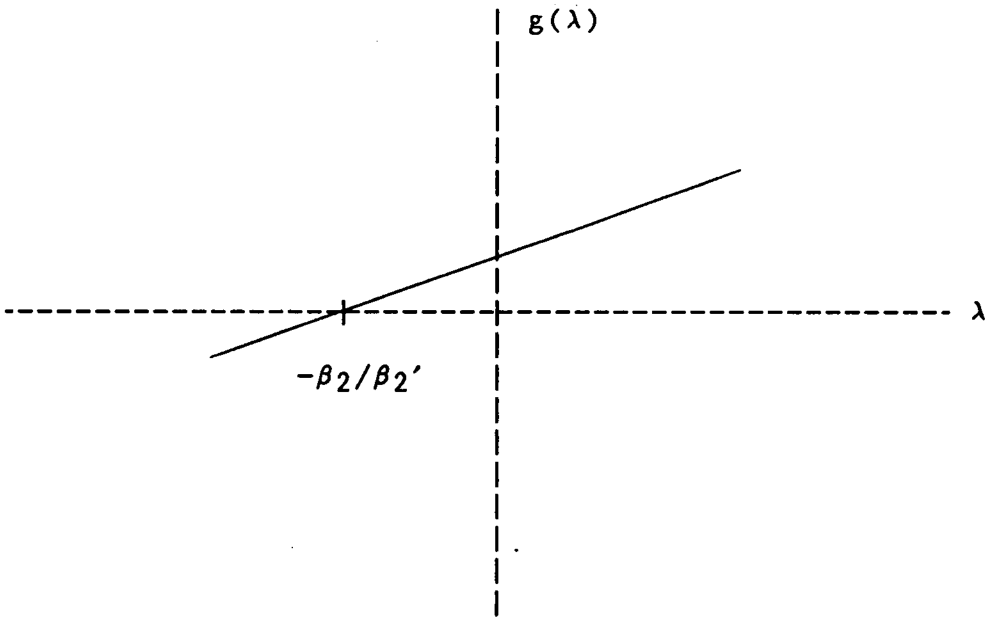

Figure 1.1. Graph of $g(\lambda)$ with $\beta_1' = 0$.

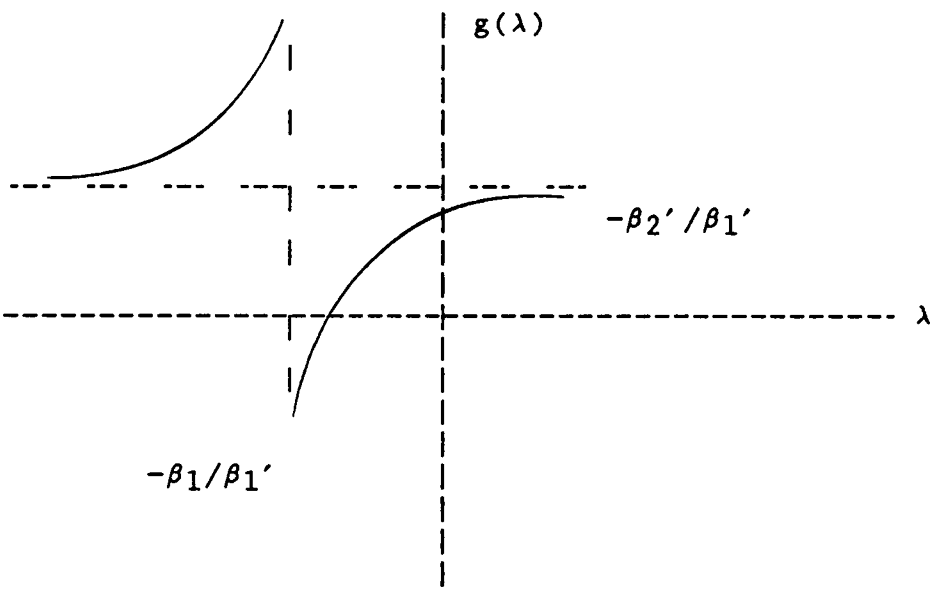

Figure 1.2. Graph of $g(\lambda)$ with $\beta_1' \neq 0$.

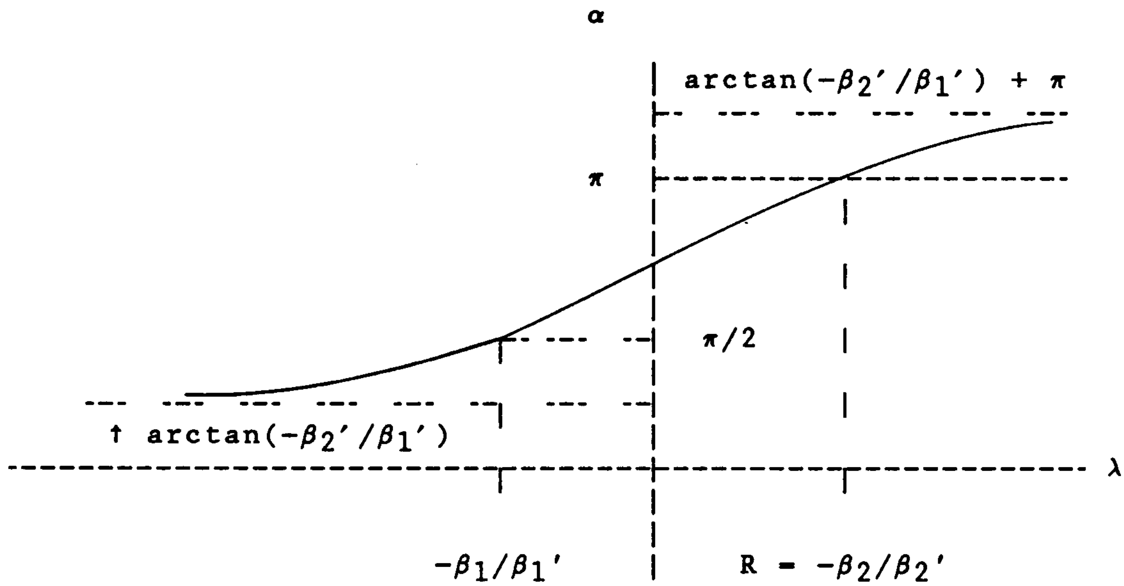

Figure 1.3. Graph of $\alpha(\lambda)$ when $g(-\infty) = -\beta_2'/\beta_1' > 0$ if $-\beta_1' \neq 0$ and $\beta_2'/\beta_1' < 0$.

In the second case we have that

(B) $g(-\infty) = 0$ if $\beta_1' \neq 0$ and $\beta_2' = 0$.

Figure 1.4 is a graph of case (B).

In the next case we have that

(C) $g(-\infty) = -\beta_2'/\beta_1' < 0$ if $-\beta_2'/\beta_1' < 0$ and $\beta_1' \neq 0$.

Figure 1.5 is a graph of case (C).

In the last case we have that

(D) $g(-\infty) = -\infty$ if $\beta_1' = 0$.

Figure 1.6 is a graph of case (D).

We need to define R so that $\alpha(R)$ is equal to π in case (A) and equal to 0 in cases (C) and (D). Therefore we have that $\tan\alpha(\lambda) = 0$ when $R = -\beta_2/\beta_2'$.

The function $\theta(x,\lambda)$ has the following properties:

(1) When $\theta(x,\lambda) = k\pi$, then we have $\partial\theta/\partial x(x,\lambda) > 0$.

Figure 1.7 shows the graph of property (1).

(2) If $\lambda_1 < \lambda_2$, then we have $\alpha(\lambda_1) < \alpha(\lambda_2)$ and $\theta(x,\alpha(\lambda_1)) < \theta(x,\alpha(\lambda_2))$ on $0 \le x \le L$. Figure 1.8 shows a graph of property (2).

(3) When λ tends to negative infinity, then we have in case (A) or (B),

$$\lim_{\lambda \to -\infty} \theta(L,\alpha(\lambda)) = 0.$$

Property (3) is shown in Figure 1.9.

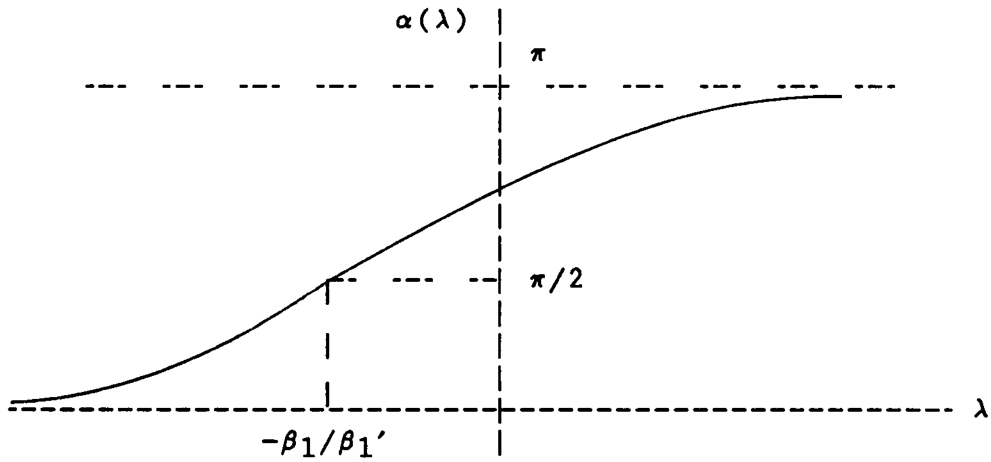

Figure 1.4. Graph of $\alpha(\lambda)$ when $g(-\infty) = 0$ if $\beta_1' \neq 0$ and $\beta_2' = 0$.

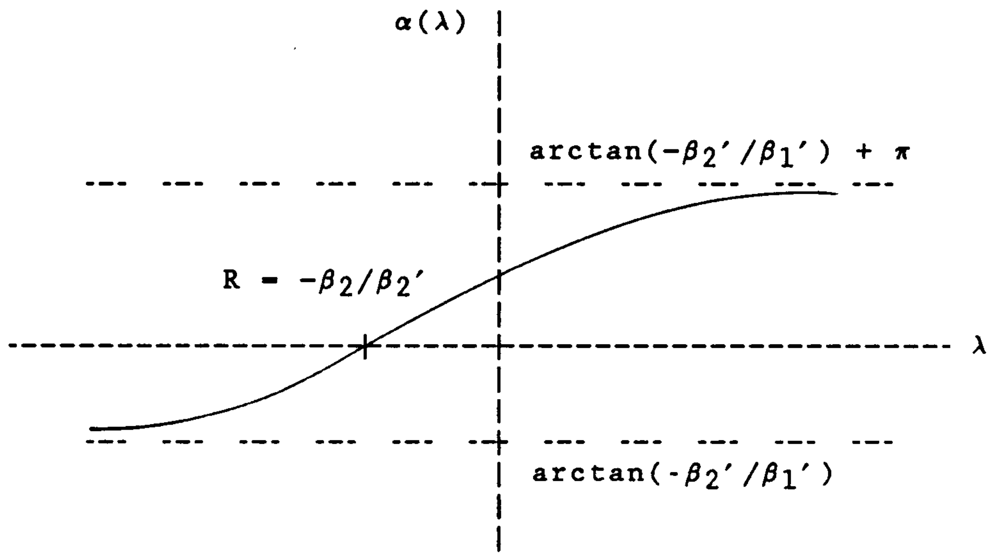

Figure 1.5. Graph of $\alpha(\lambda)$ when $g(-\infty) = -\beta_2'/\beta_1' < 0$ if $-\beta_2'/\beta_1' < 0$ and $\beta_1' \neq 0$.

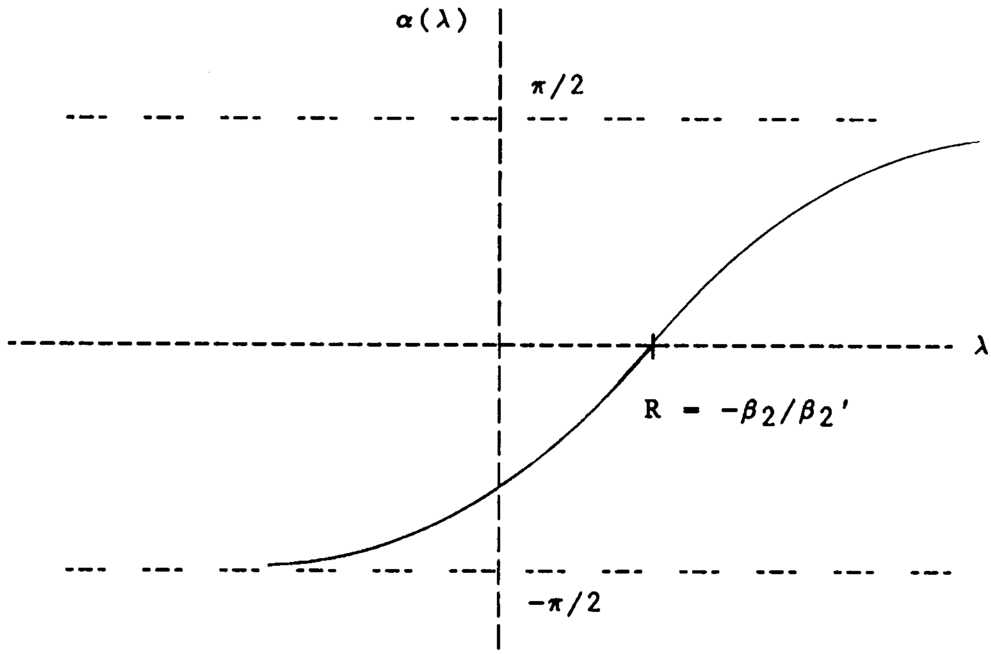

Figure 1.6. Graph of $\alpha(\lambda)$ when $g(-\infty) = -\infty$ if $\beta_1' = 0$.

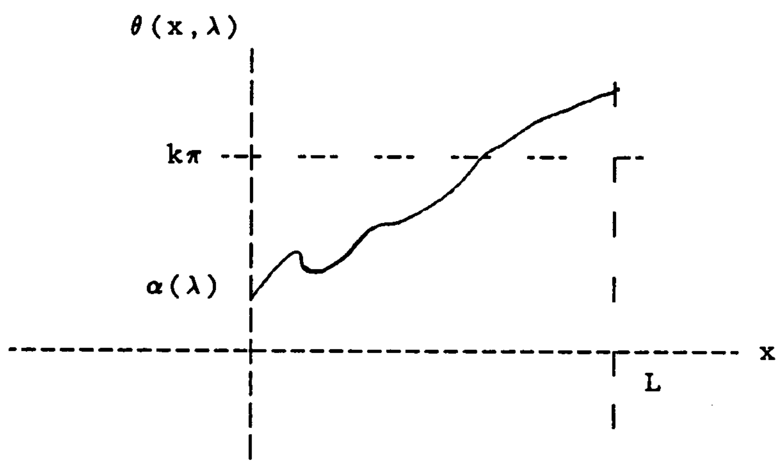

Figure 1.7. Graph of $\theta(x,\lambda)$ crossing $k\pi$ with $\partial\theta/\partial x(x,\lambda) > 0$.

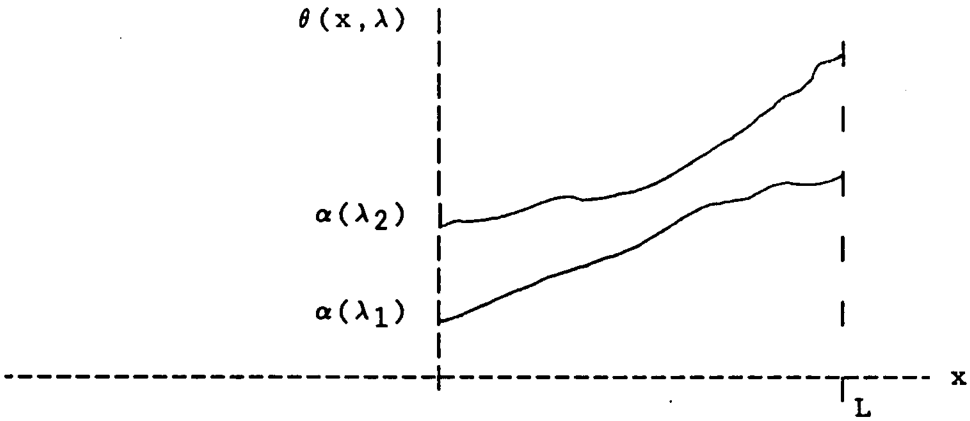

Figure 1.8. Graph of $\theta(x,\lambda)$ on $0 \le x \le L$ when $\lambda_1 < \lambda_2$.

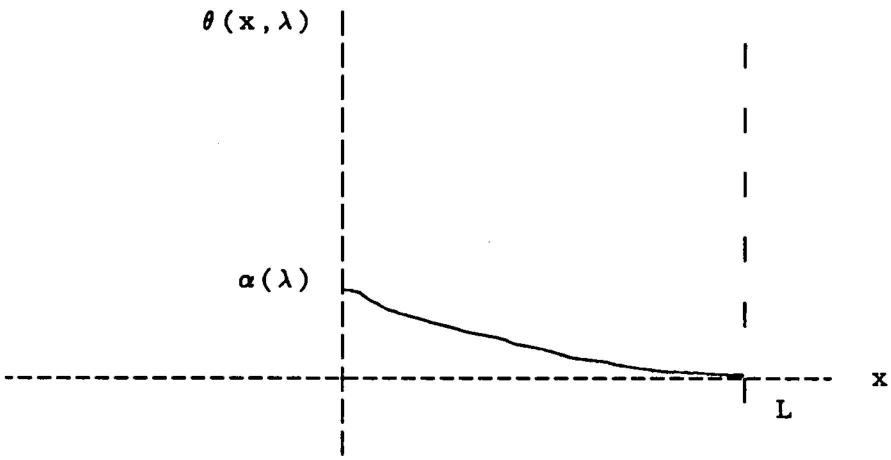

Figure 1.9. Graph of $\theta(L,\alpha(\lambda)) \to 0$ when $\lambda \to -\infty$.

Property (1) follows from the differential equation for θ' (or see (1.9) below). Property (2) follows from theorem 1 [6, p. 144] below.

Theorem 1. Let $k_i \in C^1[0,L]$ and $g_i \in C[0,L]$ for $i = 1,2$ with $0 < k_2 \leq k_1$ and $g_1 \leq g_2$. Let ϕ_i be a solution of $(k_i y')' + g_i y = 0$ and let r_i and θ_i satisfy the corresponding problem in polar coordinates, i.e.,

$$r_i' = [1/k_i(x) - g_i(x)]r_i \cos\theta_i \sin\theta_i,$$

$$\theta_i' = g_i(x)\sin^2\theta_i + \cos^2\theta_i/k_i(x).$$

If $\theta_1(0) \leq \theta_2(0)$, then $\theta_1(x) \leq \theta_2(x)$ for all $x \in J = [0,L]$. If in addition $g_2 > g_1$ on J, then $\theta_1(x) < \theta_2(x)$ for all $x \in (0,L]$.

Property (3) will be proved in Theorem 4.

For completeness we will state the classical separation theorem and proof for zeros of eigenfunctions [6, p. 145].

The following are equations referenced by Theorem 3.

(1.8) $(k(x)y')' - g(x)y = -\lambda\rho(x)y, \quad 0 \leq x \leq L,$

(1.9) $r' = [1/k(x) - g(x)]r\cos\theta\sin\theta,$

and

$$\theta' = (-g(x) + \lambda\rho)\sin^2\theta + \cos^2\theta/k(x),$$

(1.10) $\cos\alpha y(0) - \sin\alpha k(0)y'(0) = 0, \quad 0 \leq \alpha < \pi,$

(1.11) $\cos\gamma y(L) - k(L)\sin\gamma y'(L) = 0, \quad 0 < \gamma \leq \pi,$

(1.12) $(k(x)y')' + g(x)y = 0.$

The functions k, g, and ρ are assumed to be real continuous functions with k and ρ positive.

Theorem 3 also makes reference to Theorem 2 which states:

Theorem 2. Let ϕ_1 be a nonconstant solution of (1.12) on $J = (0,L)$ with consecutive zeros at points x_1 and x_2 of J and $x_1 < x_2$. If ϕ_2 is a second solution of (1.12) on J, then either $\phi_2 = c\phi_1$ for some constant c or else ϕ_2 has one and only one zero in the interval (x_1,x_2) [6, p. 127].

Theorem 3. Equation (1.8) with boundary conditions (1.10) and (1.11) has an infinite number of eigenvalues $\{\lambda_m\}$ which satisfy $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$, and $\lambda_m \to \infty$ as $m \to \infty$. Each eigenvalue λ_m is simple. The corresponding eigenfunction ϕ_m has exactly m zeros in the interval $0 < x < L$. The zeros of ϕ_m separate those of ϕ_{m+1} (i.e., the zeros of ϕ_m lie between the zeros of ϕ_{m+1}).

Proof: Let $\phi(x,\lambda)$ be the unique solution of (1.8) which satisfies

$$\phi(0,\lambda) = \sin\alpha, \qquad k(0)\phi'(0,\lambda) = \cos\alpha.$$

Then ϕ satisfies (1.10). Let $r(x,\lambda)$ and $\theta(x,\lambda)$ be the corresponding polar form of the solution $\phi(x,\lambda)$. The initial conditions are then transformed to $\theta(0,\lambda) = \alpha$, $r(0,\lambda) = 1$. Eigenvalues are those values λ for which $\phi(x,\lambda)$ satisfies (1.11), that is, those values λ for which $\theta(L,\lambda) = \gamma + m\pi$ for some integer m.

By Theorem 1 it follows that for any $x \in [0,L]$, $\theta(x,\lambda)$ is monotone increasing in λ. Note that $\theta(x,\lambda) = 0$ modulo π if and only if $\phi(x,\lambda)$ is zero. From (1.9) it is clear that $\theta' = 1/k > 0$ at a zero of ϕ and hence $\theta(x,\lambda)$ is strictly increasing in a neighborhood of a zero of ϕ.

We claim that for any fixed constant c, $0 < c \le L$, we have

$$\lim_{\lambda \to \infty} \theta(c,\lambda) = \infty \quad \text{and} \quad \lim_{\lambda \to -\infty} \theta(c,\lambda) = 0.$$

To prove the first of these statements, note that $\theta(0,\lambda) = \alpha \ge 0$ and that $\theta' > 0$ if $\theta = 0$. Hence $\theta(x,\lambda) \ge 0$ for all x and λ. Fix $c_0 \in (0,c)$. We shall show that $\theta(c,\lambda) - \theta(c_0,\lambda) \to \infty$ as $\lambda \to \infty$. This will suffice.

Pick constants K, R, and G such that $k(x) \le K$, $\rho(x) \ge R > 0$, and $g(x) \ge -G$. Consider the equation

(1.13) $$Ky'' + (\lambda R - G)y = 0 \qquad (\lambda > 0)$$

with $y(c_0) = \phi(c_0,\lambda)$. $Ky'(c_0) = k(c_0)\phi'(c_0,\lambda)$. If $\psi(x,\lambda)$ is the polar angle of the solution of (1.13), then by Theorem 1 and the choice of K, R, and G, it follows that $\theta(x,\lambda) \ge \psi(x,\lambda)$ for $c_0 < x \le c$. Since $\theta(c_0,\lambda) = \psi(c_0,\lambda)$, this gives

$$\theta(c,\lambda) - \theta(c_0,\lambda) \ge \psi(c,\lambda) - \psi(c_0,\lambda).$$

The successive zeros of (1.13) are easily computed. They occur at intervals $T(\lambda) = \pi[K(\lambda R - G)^{-1}]^{1/2}$. Since $T(\lambda) \to 0$ as $\lambda \to \infty$, then for any integer $j > 1$, ψ will have j zeros between c_0 and c for λ large enough, for example, for $(c - c_0) \ge T(\lambda)j$. Then $\psi(c,\lambda) - \psi(c_0,\lambda) \ge j\pi$. Since j is arbitary, it follows that $\theta(c,\lambda) - \theta(c_0,\lambda) \to \infty$ as $\lambda \to \infty$.

To prove that $\theta(c,\lambda) \to 0$ as $\lambda \to -\infty$, first fix $\epsilon > 0$. We may, without loss of generality, choose ϵ so small that $\pi - \epsilon > \alpha \ge 0$. Choose K, R, and G so that $0 < K \le k(x)$, $0 < R \le \rho(x)$, and $G \ge |g(x)|$. If $\lambda < 0$, $\alpha > \epsilon$ and $\epsilon \le \theta \le \pi - \epsilon$, then (1.9) gives

$$\theta'(x,\lambda) \le G + \lambda R\sin^2\epsilon + 1/K \le -(\alpha - \epsilon)/(c - 0) < 0$$

as soon as $\lambda < \{(\alpha - \epsilon)/(0 - c) - G - 1/K\}(R\sin^2\epsilon)^{-1} < 0$. Since $\theta(0,\lambda) = \alpha > \epsilon$, then for $-\lambda$ sufficiently large,

$$\theta(x,\lambda) \le \alpha - [(\alpha - \epsilon)/(c - 0)](x - 0),$$

for as long as $\theta(x,\lambda) \ge \epsilon$. Let $x = c$ to see that $\theta(x,\lambda)$ must go below ϵ by the time $x = c$. If θ starts less than ϵ or becomes less than ϵ, then $\theta'(t,\lambda) < 0$ at $\theta = \epsilon$ for

$$\lambda < \{(\alpha - \epsilon)/(0 - c) - G - 1/K\}(R\sin^2\epsilon)^{-1}$$

guarantees that it will remain less than ϵ.

With these preliminaries completed, we now proceed to the main argument. Recall that θ is jointly continuous in x and λ by Theorem 7.2 [6, p. 70]. Since $0 < \gamma \le \pi$, since $\theta(L,\lambda) \to 0$ as $\lambda \to -\infty$, and since $\theta(x,\lambda)$ is monotone increasing to $+\infty$ with λ, then there is a unique $\lambda = \lambda_0$ at which $\theta(L,\lambda) = \gamma$. Notice that $0 \le \alpha = \theta(0,\lambda_0)$ and $\gamma = \theta(L,\lambda_0) \le \pi$ while $\theta(x,\lambda_0)$ is increasing in a neighborhood of $\theta = 0$ and $\theta = \pi$. Hence θ must satisfy

$$0 < \theta(x,\lambda_0) < \pi \quad \text{when } 0 < x < L.$$

Thus, $\phi_0(x) = \phi(x,\lambda_0)$ is not zero on $0 < x < L$.

Let λ increase from λ_0 to the unique λ_1 where $\theta(L,\lambda_1) = \gamma + \pi$. Since $\alpha = \theta(0,\lambda_1) < \pi < \theta(L,\lambda_1) = \gamma + \pi$ and since $\theta'(x_1,\lambda_1) > 0$ at any point x_1 where $\theta(x_1,\lambda_1) = \pi$, then $\phi_1(x) = \phi(x,\lambda_1)$ will have exactly one zero in $0 < x < L$. We continue in this manner to obtain λ_m where $\theta(L,\lambda_m) = \gamma + m\pi$ and $\phi_m(x) = \phi(x,\lambda_m)$ will have exactly m zeros in $0 < x < L$. That the zeros of ϕ_m and ϕ_{m+1} interlace follows from Theorem

2, the non-crossing of θ's, and monotone behavior with respect to λ. When $\lambda^{\#} > \lambda^{*}$ we know that $\theta(x,\lambda^{\#})$ stays above $\theta(x,\lambda^{*})$. The crossing of π move towards the left as we can see in Figure 1.10.

If we have two consecutive eigenfunctions where $\lambda_k < \lambda_{k+1}$ for m = 1, 2, ... then the crossing of $\theta(x,\lambda_k)$ at $m\pi$ falls between the crossing of $\theta(x,\lambda_{k+1})$ at $m\pi$ and the crossing of $\theta(x,\lambda_{k+1})$ at $(m+1)\pi$. This is true due to the following argument. We know that ϕ_{m+1} has (m+1) zeros and ϕ_m has m zeros. Theorem 2 forces the first zero of ϕ_{m+1} to be before the first zero of ϕ_m and the last zero of ϕ_{m+1} to be after the last zero of ϕ_m. Theorem 1 combined with Theorem 2 forces ϕ_{m+1} to have one and only one zero in each open interval between the zeros of ϕ_m. Another situation we must consider is the possibility that a zero of ϕ_{m+1} could equal a zero of ϕ_m. Figure 1.11 shows this possibility.

However, if we consider the number of zeros ϕ_{m+1} must have there are (m-1) open intervals in which ϕ_{m+1} is required to have only one zero, ϕ_{m+1} must have one zero in the interval before the first zero of ϕ_m, and ϕ_{m+1} must have one zero in the interval after the last zero of ϕ_m. Adding these zeros together we see that we have (m+1) zeros for ϕ_{m+1}. If the situation pictured in Figure 1.11 existed for just one zero of ϕ_m and ϕ_{m+1} then we would have an extra zero, which is not possible. Therefore, the situation pictured in Figure 1.11 is not possible.

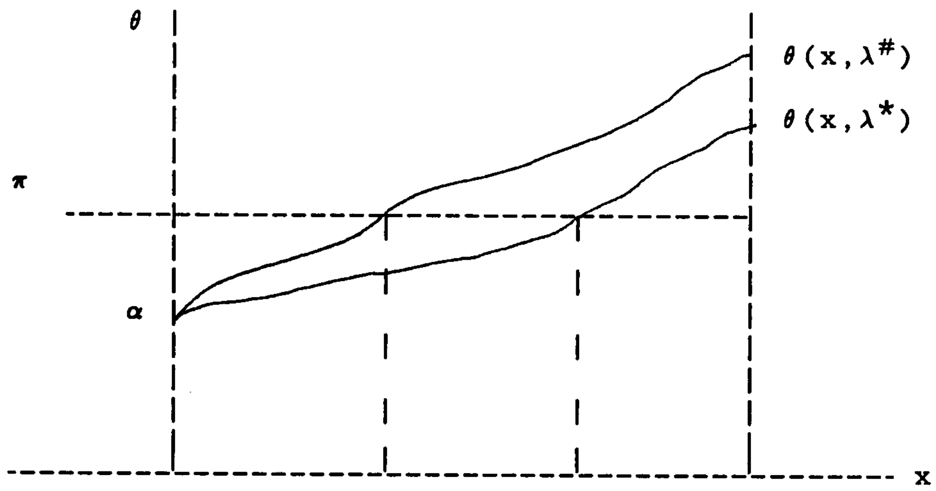

Figure 1.10. Graph of a zero moving towards the left when $\lambda^{\#} > \lambda^{*}$.

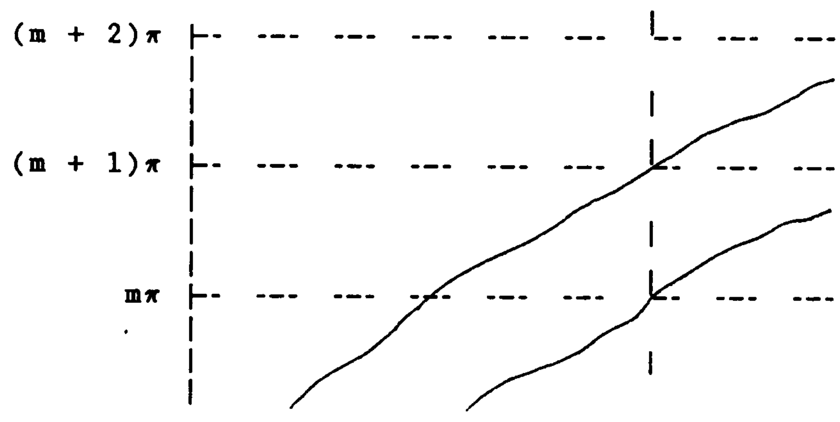

Figure 1.11. Graph where a zero of ϕ_{m+1} is equal to a zero of ϕ_m.

Now we shall look at the same proof and notice the difference that parameter-dependent boundary conditions will make in the previous theorem.

Theorem 4. The differential equation (1.1) with boundary conditions (1.2) and (1.3) has an infinite number of eigenvalues $\{\lambda_m\}$ which satisfy $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$, and $\lambda_m \to \infty$ as $m \to \infty$. All of the eigenvalues are simple. For cases (A), (C), and (D) there exists an integer $r \geq 1$ where if $m < r$, then the corresponding eigenfunction ϕ_m has exactly m zeros in the interval $0 < x < L$. However, if $m \geq r$, then the corresponding eigenfunction ϕ_m has exactly (m-1) zeros in the interval $0 < x < L$. Moreover, the zeros of ϕ_m in (0,L) separate those of ϕ_{m+1} in (0,L) with one exception. The zeros of ϕ_{r-1} are separated by the zeros of ϕ_r except for the first zeros of ϕ_{r-1}. This is shown in Figure 1.12.

For case (B) the number and separation of zeros is exactly the same as in Theorem 3.

Proof: We will prove case (A). Cases (B), (C) and (D) have similar proofs. For each real λ, let $\phi(x,\lambda)$ be the solution of (1.1) such that $\phi(0,\lambda) = \sin\alpha(\lambda)$ and $\phi'(0,\lambda) = \cos\alpha(\lambda)$. Then ϕ satisfies (1.3). Let $r(x,\lambda)$ and $\theta(x,\lambda)$ be the corresponding polar form of the solution $\phi(x,\lambda)$. The initial conditions are then transformed to $\theta(0,\lambda) = \alpha(\lambda)$ and $r(0,\lambda) = 1$. Eigenvalues are those values of λ for which $\phi(x,\lambda)$ satisfies $\cos\gamma y(L) - \sin\gamma y'(L) = 0$, that is, those values λ for which $\theta(L,\lambda) = \gamma + m\pi$ for some integer m.

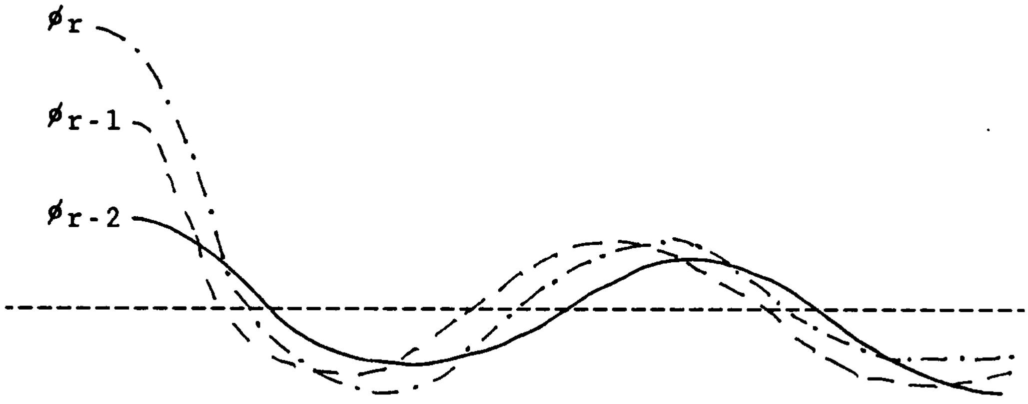

Figure 1.12. Graph of the separation of zeros for ϕ_{r-2}, ϕ_{r-1}, and ϕ_r.

By Theorem 1 it follows that for any $x \in [0,L]$, $\theta(x,\lambda)$ is monotone increasing in λ. Note that $\theta(x,\lambda) = 0$ modulo π if and only if $\phi(x,\lambda)$ is 0. From (1.9) we see that $\theta' = 1 > 0$ at a zero of ϕ and hence $\theta(x,\lambda)$ is strictly increasing in a neighborhood of a zero. We can conclude that $\lambda_0 < R$ because R occurs when the left-hand endpoint hits π. However, $\theta' > 0$ in a neighborhood of π and the first eigenvalue will occur when the right-hand side hits γ. These facts will not allow $\lambda_0 \geq R$ since this would require that the slope of the curve be negative at a multiple of π as shown in Figure 1.13 and Figure 1.14.

For $\lambda_2 > \lambda_1$ we have $\theta(0,\lambda_1) = \alpha(\lambda_1)$ and $\theta(0,\lambda_2) = \alpha(\lambda_2)$ and $\alpha(\lambda_1) < \alpha(\lambda_2)$ since α is increasing. Thus θ_1 and θ_2 can't intersect and θ_i satisfies: $\theta' = [\lambda + q(x)]\sin^2\theta + \cos^2\theta$ (as a function of x) as seen in Figure 1.15.

We would like to show that for any fixed constant c, $0 < c \leq L$, we have

$$\lim_{\lambda \to \infty} \theta(c,\lambda) = \infty \quad \text{and} \quad \lim_{\lambda \to -\infty} \theta(c,\lambda) = 0.$$

To prove the first of these statements note that $\theta(0,\lambda) = \alpha(\lambda) > 0$ and that $\theta' > 0$ if $\theta = 0$. Hence $\theta(x,\lambda) \geq 0$ for all x and λ.

Pick K, G, and R such that $k(x) \leq K$, $G \leq g(x)$ and $1 \geq R > 0$. Compare (for $\lambda > 0$) $(ky')' + (g + \lambda)y = 0$ with $(Ky')' + (G + \lambda R)y = 0$ and with polar angle θ_0 for $(Ky')' + (G + \lambda R)y = 0$ with $\theta_0(0,\lambda) = \alpha(\lambda)$. By Theorem 1, $\theta(x,\lambda) \geq \theta_0(x,\lambda)$ on $[0,L]$.

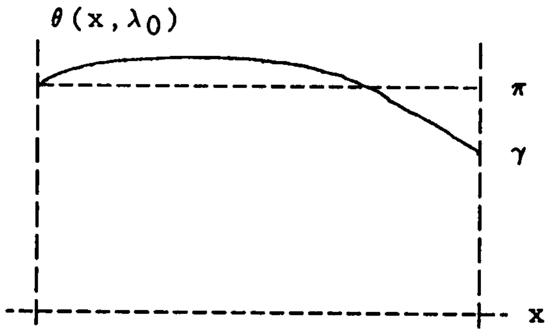

Figure 1.13. Graph of $\theta(x,\lambda_0)$ where $R = \lambda_0$.

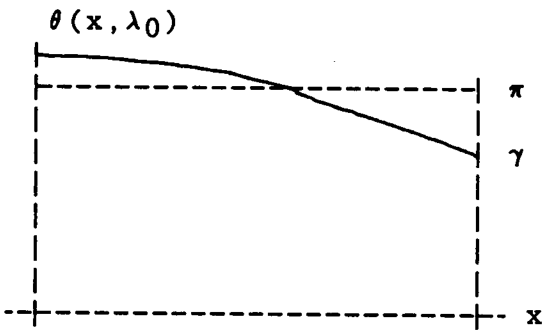

Figure 1.14. Graph of $\theta(x,\lambda_0)$ where $R < \lambda_0$.

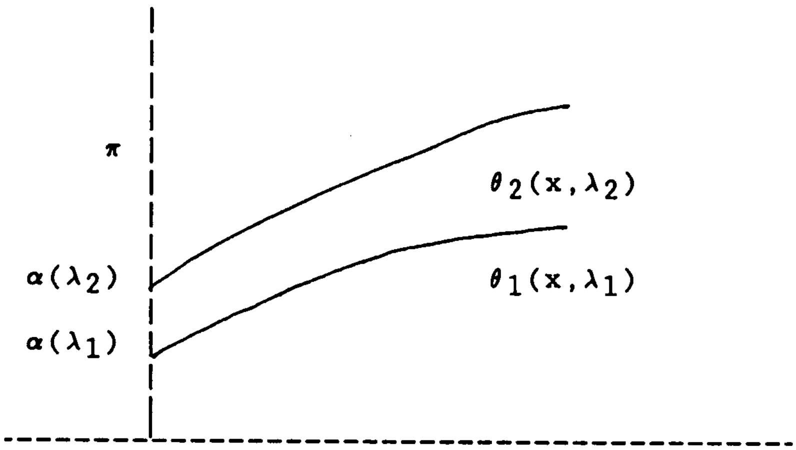

Figure 1.15. Graph of $\theta(x,\lambda_2) > \theta(x,\lambda_1)$ when $\alpha(\lambda_2) > \alpha(\lambda_1)$.

$Ky' + (G + \lambda R)y = 0$ (for $\lambda > 0$) has solutions whose zeros have spacing $\pi[K/(G + \lambda R)]^{1/2} \to 0$ as $\lambda \to \infty$. Thus $\theta_0(L,\lambda) \to \infty$ and $\lambda \to \infty$ and therefore $\theta(L,\lambda) \to \infty$ as $\lambda \to \infty$.

To prove that $\theta(c,\lambda) \to 0$ as $\lambda \to -\infty$, first fix $\epsilon > 0$. We may, without loss of generality, take $\epsilon > 0$, $\lambda^* < 0$ so that $\alpha(-\infty) > \epsilon$ and $\pi - \epsilon > \alpha(\lambda)$ for $\lambda \le \lambda^*$. Choose K, R, and G so that $0 < K \le k(x)$, $0 < R \le \rho(x)$, and $G \ge |g(x)|$. If $\lambda \le \lambda^*$ and $\epsilon \le \theta \le \pi - \epsilon$, then by (1.9),

(1.14)

$$\theta'(x,\lambda) \le G + \lambda R \sin^2\epsilon + 1/K \le -(\alpha(\lambda) - \epsilon)/(c-0) < 0$$

as soon as $\lambda < \{(\alpha(\lambda) - \epsilon)/(0 - c) - G - 1/K\}(R\sin^2\epsilon)^{-1} < 0$. Since $\theta(0,\lambda) = \alpha(\lambda) > \epsilon$, then for $-\lambda$ sufficiently large,

$$\theta(x,\lambda) \le \alpha(\lambda) - [(\alpha(\lambda) - \epsilon)/(c - 0)](x - 0),$$

for as long as $\theta(x,\lambda) \ge \epsilon$. Let $x = c$ to see that $\theta(x,\lambda)$ must go below ϵ by the time $x = c$. If θ starts less than ϵ or becomes less than ϵ, then $\theta'(x,\lambda) < 0$ at $\theta = \epsilon$ by (1.14) if $-\lambda$ is sufficiently large guarantees that it will remain less than ϵ.

With these preliminaries completed, we now proceed to the main argument. Since $0 < \gamma \le \pi$, since $\theta(L,\lambda) \to 0$ as $\lambda \to -\infty$ and since $\theta(x,\lambda)$ is monotone increasing to $+\infty$ with λ, then there is a unique $\lambda = \lambda_0$ at which $\theta(L,\lambda) = \gamma$. Notice that $0 < \alpha(\lambda_0) = \theta(0,\lambda_0)$ and $\gamma = \theta(L,\lambda_0) \le \pi$ while $\theta(x,\lambda_0)$ is increasing in a neighborhood of $\theta = 0$ and $\theta = \pi$. Hence θ must satisfy

$$0 < \theta(x,\lambda_0) < \pi \qquad \text{when } 0 < x < L.$$

Thus, $\phi_0(x) = \phi(x,\lambda_0)$ is not zero on $0 < x < L$.

Let λ increase from λ_0 to the unique λ_1 where $\theta(L,\lambda_1) = \gamma + \pi$. Suppose also $\lambda_1 < R$. Since $\alpha = \theta(0,\lambda_1) < \pi < \theta(L,\lambda_1) = \gamma + \pi$ and since $\theta'(x_1,\lambda_1) > 0$ at any point x_1 where $\theta(x_1,\lambda_1) = \pi$, then $\phi_1(x) = \phi(x,\lambda_1)$ will have exactly one zero in $0 < x < L$. If we look first at case (A) where $\tan\alpha(-\infty) = -\beta_2'/\beta_1' > 0$, we see that if we continue in this manner we can obtain λ_m for $\lambda_m < R$ where $\theta(L,\lambda_m) = \gamma + m\pi$ and $\phi_m(x) = \phi(x,\lambda_m)$. We then see that there are m zeros in $(0,L)$ for the eigenfunction ϕ_m. When $m \geq R$ the $\alpha(\lambda)$ function has passed through π and one zero has been lost. Therefore, we have m-1 zeros for the eigenfunction ϕ_m. In cases (C) and (D) $\alpha(\lambda)$ passes through zero instead of π. Looking at case (B) the function $\alpha(\lambda)$ does not pass through π or 0 and so no zero is lost. That the zeros of ϕ_m and ϕ_{m+1} interlace follows as in the proof of Theorem 3, with the exception of the cases where $\alpha(\lambda)$ crosses π or 0 and a zero is lost. The interlacing is not found on the left-hand side for the first eigenfunction after R, i.e. ϕ_{m+1}. All other eigenfunctions follow the normal interlacing of zeros.

We look at the Rayleigh Quotient now for a problem that we will solve numerically in chapters III, IV, and V. The Rayleigh Quotient can sometimes be used to show under what conditions $\lambda_0 \geq 0$.

We begin with the general equation $\phi'' + q\phi = -\lambda\sigma\phi$ with boundary conditions $\phi(0) = 0$ and $\phi'(L) = \lambda\phi(L)$. Multiplying the above equation through by ϕ we see that

$$\phi''\phi + q\phi^2 = -\lambda\sigma\phi^2.$$

Integrating by parts gives

$$\int_0^L \phi''\phi dx + \int_0^L q\phi^2 dx = -\lambda\int_0^L \sigma\phi^2 dx.$$

Hence,

$$\lambda\int_0^L \sigma\phi^2 dx = -\phi\phi'\Big|_0^L + \int_0^L [(\phi')^2 - q\phi^2]dx$$

and

$$\lambda\int_0^L \sigma\phi^2 dx = -\phi(L)\phi'(L) + \phi(0)\phi'(0) + \int_0^L [(\phi')^2 - q\phi^2]dx.$$

Since $\phi(0) = 0$ and $\phi'(L) = \lambda\phi(L)$,

$$\lambda\int_0^L \sigma\phi^2 dx = -\phi(L)[\lambda\phi(L)] + \int_0^L [(\phi')^2 - q\phi^2]dx,$$

$$\lambda(\phi(L)^2 + \int_0^L \sigma\phi^2 dx) = \int_0^L [(\phi')^2 - q\phi^2]dx$$

and

$$\lambda = [\int_0^L (\phi')^2 - q\phi^2 dx]/[\phi(L)^2 + \int_0^L \sigma\phi^2 dx].$$

Therefore, we can see that $\lambda_0 \geq 0$ if $q(x) \leq 0$. Actually $\lambda_0 > 0$ since $\phi'(x) = 0$ in $[0,L]$ and $\phi(0) = 0$ implies $\phi(x) = 0$ in $[0,L]$.

CHAPTER II

A MODEL PROBLEM

We will look at a problem which consists of a string fixed at one end with an attached mass at the other. The attached mass is allowed to move freely in the vertical direction only as shown in Figure 2.1.

We will basically follow the development found in [3, p. 464]. This system can be analyzed using the one-dimensional partial differential wave equation

$$\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2$$

where $c^2 = T/\rho_0$, the tension T divided by the uniform mass density ρ_0. The boundary condition for the fixed left end at x = 0 is

$$u(0,t) = 0$$

and at the right endpoint with the attached mass m is

$$m\partial^2 u/\partial t^2(L,t) = -T\partial u/\partial x(L,t).$$

The above boundary condition is Newton's law for an attached mass if gravity is ignored.

The initial conditions are

$$u(x,0) = f(x)$$

and

$$\partial u/\partial t(x,0) = g(x).$$

Solving this system using separation of variables we find that with $u(x,t) = \phi(x)h(t)$, then

$$d^2\phi/dx^2 = -\lambda\phi$$

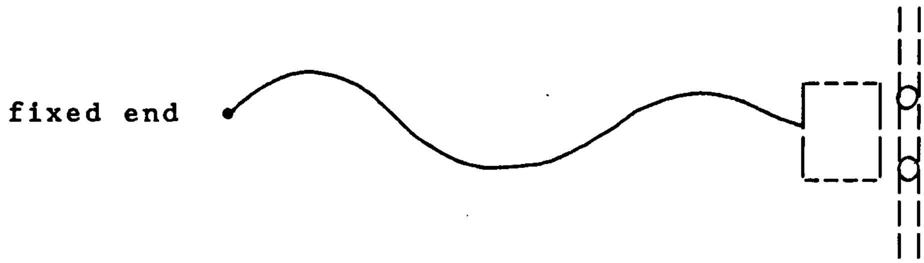

Figure 2.1. Picture of an attached mass.

and

$$d^2h/dt^2 = -\lambda c^2 h.$$

From the boundary condition x = 0 we see that $\phi(0) = 0$. From the boundary condition x = L we have that

$$m\phi(L)d^2h/dt^2 = -Th(t)d\phi/dx(L).$$

When we eliminate the second time derivative from the boundary conditions using

$$d^2h/dt^2 = -\lambda c^2 h,$$

the boundary value problem becomes

$$d^2\phi/dx^2 = -\lambda\phi,$$

$$\phi(0) = 0,$$

and

$$(d\phi/dx)(L) - (\lambda c^2 m/T)\phi(L) = 0.$$

The above equations cannot be solved as a Sturm - Liouville problem, because the boundary condition at x = L is eigenvalue dependent. Oscillating solutions occur only when $\lambda > 0$ and we have already shown by the Rayleigh Quotient that $\lambda_0 > 0$.

We will analyze the case where $\lambda > 0$. The general solution of

$$d^2\phi/dx^2 = -\lambda\phi$$

is

$$\phi = c_1\cos\lambda^{1/2}x + c_2\sin\lambda^{1/2}x.$$

The general solution above must satisfy $\phi(0) = 0$. We therefore have that $c_1 = 0$ and hence $\phi = c_2\sin\lambda^{1/2}x$.

The boundary condition at x = L implies that

$$\lambda^{1/2}\cos(\lambda^{1/2}L) = \lambda(c^2 m/T)\sin(\lambda^{1/2}L)$$

or

$$\tan(\lambda^{1/2}L) = (TL/c^2m)(1/\lambda^{1/2}L).$$

When the above equation is solved graphically we see that there are an infinite number of eigenvalues which occur at the intersections, as shown in Figure 2.2.

The intersections give the eigenvalues. These intersections can be solved by using Newton's method. The first 10 eigenvalues are listed in Table 2.1. For our computations we take $L = 1$ and $T/c^2m = 1$.

In the differential equation solved in [3] the parameter-dependency is found in the terminal boundary condition. In the theoretical development we notice that the parameter-dependency is found in the initial boundary condition. This discrepancy causes no major difficulties since $\phi(x)$ could be defined to be $\phi(x) = y(L - x)$. This reverses the problem so that $\phi(0) = y(L)$, $\phi(L) = y(0)$, $\phi'(0) = -y'(L)$ and $\phi'(L) = -y'(0)$. We will place the eigenparameter with the most convenient boundary condition for the numerical solutions of the differential equations.

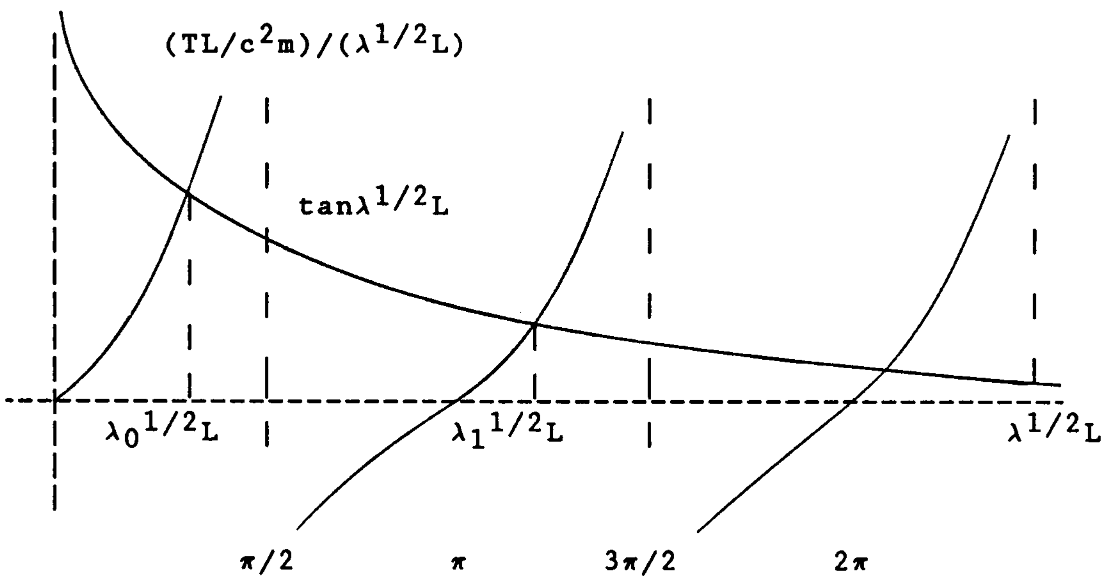

Figure 2.2. Graph of $\tan(\lambda^{1/2}L)$ - $(TL/c^2m)(1/\lambda^{1/2}L)$.

Table 2.1. Eigenvalues found by Newton's method.

Eigenvalues	Newton's Method
λ_0	.74017
λ_1	11.735
λ_2	41.438
λ_3	90.808
λ_4	159.90
λ_5	248.73
λ_6	357.30
λ_7	485.61
λ_8	633.65
λ_9	801.44

CHAPTER III

FINITE DIFFERENCE METHOD

We will examine solving parameter-dependent boundary condition problems using the method of finite differences. If we divide the interval [0,L] into n equal parts we may approximate $y'(x)$ and $y''(x)$. $y'(x)$ is approximated by, with $h = L/n$,

$$(y(x+h) - y(x))/h$$

and $y''(x)$ is approximated by

$$[(y(x+h) - y(x))/h - (y(x) - y(x-h))/h]/h,$$

which reduces to

$$(y(x+h) - 2y(x) + y(x-h))/h^2.$$

Using the principles of finite differences we will be able to solve the test problem

$$\phi'' = -\lambda\phi$$

with the boundary conditions

$$\phi(0) = 0 \text{ and } \phi'(1) = \lambda\phi(1).$$

Using $\phi_i = \phi(ih)$ and the above approximations, we obtain that

$$\begin{cases} \phi_0 = 0 \\ (\phi_{i+1} - 2\phi_i + \phi_{i-1})/h^2 = -\lambda\phi_i \qquad \text{where } i = 1,\ldots,n. \end{cases}$$

From the boundary condition $\phi'(1) = \lambda\phi(1)$ we see that

$$(\phi_{n+1} - \phi_n)/h = \lambda\phi_n.$$

Solving for ϕ_{n+1} we see that

$$\phi_{n+1} = (\lambda h + 1)\phi_n.$$

This relationship allows us to set up a recursion formula for the finite differences for $i = 1,\ldots,n-1$. However, the last interval is defined using information in the $(n + 1)$st interval. A simple substitution can be made for the case $i = n$ so that no term involving the $(n + 1)$st interval exists. The system generally can be transformed into a matrix equation with the following form

$$A\phi = -\mu B\phi,$$

where A and B are real symmetric matrices with $B > 0$. Setting

$$\phi = B^{-1/2}\xi,$$

we have

$$B^{-1/2}AB^{-1/2}\xi = -\mu\xi$$

or

$$A^*\xi = -\mu\xi$$

where

$$A^* = B^{-1/2}AB^{-1/2}.$$

It turns out that A^* is a tridiagonal symmetric matrix and the eigenvalues can be found by solving for the solutions of this system.

In the case at hand we have

$$\phi_{n+1} - 2\phi_n + \phi_{n-1} = -\lambda h^2\phi_n,$$

$$(\lambda h + 1)\phi_n - 2\phi_n + \phi_{n-1} = -\lambda h^2\phi_n, \text{ and}$$

$$-\phi_n + \phi_{n-1} = -\lambda(h^2 + h)\ \phi_n.$$

Therefore the system becomes

$$\begin{cases} \phi_0 = 0 \\ \phi_{i+1} - 2\phi_i + \phi_{i-1} = -\lambda h^2 \phi_i, \quad i = 1, \ldots, n-1 \\ -\phi_n + \phi_{n-1} = -\lambda(h^2 + h)\phi_n. \end{cases}$$

We can transform the above system into the matrix system

$$A\phi = C\phi$$

where A and C are n X n matrices. This system is

$$\begin{bmatrix} -2 & 1 & 0 & & & 0 \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \end{bmatrix} =$$

$$\begin{bmatrix} -\lambda h^2 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & -\lambda h^2 & \\ 0 & & & -\lambda(h^2 + h) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \end{bmatrix}.$$

Setting $\mu = \lambda h^2$, in the above system we find that

$$A\phi = \begin{bmatrix} -\mu & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & -\mu & \\ 0 & & & -\mu(1 + 1/h) \end{bmatrix} \phi .$$

Then $A\phi = -\mu B\phi$ where,

$$B = \begin{bmatrix} 1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & 1 & \\ 0 & & & 1 + 1/h \end{bmatrix}$$

and

$$B^{-1/2} = \begin{bmatrix} 1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & 1 & \\ 0 & & & (1 + 1/h)^{-1/2} \end{bmatrix} .$$

From the substitutions above, we have that

$$A^* = B^{-1/2} A B^{-1/2}$$

which is given by

$$A^* = \begin{bmatrix} -2 & 1 & 0 & & & & 0 \\ & 1 & -2 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & (1 + 1/h)^{-1/2} & \\ 0 & & & & (1+1/h)^{-1/2} & & -(1 + 1/h)^{-1} \end{bmatrix}.$$

We solved this system using ZPOLR, an IMSL program designed to find the eigenvalues of a tridiagonal symmetric matrix. The solutions are found in Table 3.1.

We can also solve

$$\phi'' = -\lambda\phi$$

using different boundary conditions. The possible boundary conditions at $x = 0$ are

(3.1) $\phi(0) = 0$

and

(3.2) $\phi'(0) = k\phi(0)$.

The possible boundary conditions at $x = L$ are

(3.3) $\phi'(L) = \lambda\phi(L)$

and

(3.4) $-\beta_1\phi(L) + \beta_2\phi'(L) = \lambda[\beta_1'\phi(L) - \beta_2'\phi'(L)]$,

where $\beta_1'\beta_2 > \beta_1\beta_2'$. We have already examined cases (3.1) and (3.3).

The finite difference equation for the differential equation is

Table 3.1. Eigenvalues of $\phi'' = -\lambda\phi$ found by the finite difference method.

Eigenvalues	n=4	n=8	n=16	n=32	n=64
λ_0	.67404	.70672	.72325	.73275	.74085
λ_1	10.782	11.435	11.632	11.696	11.719
λ_2	32.862	39.062	40.802	41.266	41.388
λ_3	54.882	80.302	88.046	90.098	90.627
λ_4		128.93	151.61	157.78	159.37
λ_5		177.56	229.05	243.67	247.46
λ_6		218.78	317.40	346.95	354.69
λ_7		246.33	413.27	466.62	480.80
λ_8			512.97	601.52	625.49
λ_9			612.67	750.37	788.41

(3.5) $$(\phi_{i+1} - 2\phi_i + \phi_{i-1})/h^2 = -\lambda\phi_i$$

for $i = 0, 1, 2, \ldots$.

However, the above equation involves intervals outside the domain for the first and last division. When $i = 0$ (3.5) becomes

(3.6) $$\phi_1 - 2\phi_0 + \phi_{-1} = -\lambda h^2\phi_0$$

and we have from boundary condition (3.2)

(3.7) $$(\phi_0 - \phi_{-1})/h = k\phi_0.$$

(3.7) contains a term involving the 0th interval. Solving (3.7) for ϕ_{-1} we find that

$$\phi_{-1} = (1-kh)\phi_0.$$

Substituting this information into (3.6) we have that

$$\phi_1 - 2\phi_0 + (1 - kh)\phi 0 = -\lambda h^2\phi_0,$$

which no longer has a ϕ_{-1} term. The above equation reduces to

$$\phi_1 + (-1 - kh)\phi_0 = -\lambda h^2\phi_0.$$

Hence the matrix system

$$A\phi = C\phi$$

is

$$\begin{bmatrix} -1-kh & 1 & 0 & 0 & 0 & \cdots & \\ 1 & -2 & 1 & 0 & 0 & \cdots & \\ 0 & & 1 & -2 & 1 & 0 & \cdots \\ & & & \cdot & & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} -\lambda h^2 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

When $i = n$ (3.5) becomes

(3.8) $\phi_{n+1} - 2\phi_n + \phi_{n-1} = -\lambda\phi_n h^2$

and we have from boundary condition (3.4) that

(3.9) $-\beta_1\phi_n + \beta_2(\phi_{n+1} - \phi_n)/h =$

$$\lambda[\beta_1'\phi_n - \beta_2'(\phi_{n+1} - \phi_n)/h].$$

However (3.9) contains a term involving the (n+1)st interval. Solving (3.8) for ϕ_{n+1} we find that

$$\phi_{n+1} = [1 + h((\beta_1 + \lambda\beta_1')/(\beta_2 + \lambda\beta_2'))]\phi_n.$$

Substituting the above equation into (3.7) we have

$$\phi_n[h((\beta_1 + \lambda\beta_1')/(\beta_2 + \lambda\beta_2')) + 1] - 2\phi_n + \phi_{n-1} = -\lambda h^2\phi_n$$

which reduces to

$$\beta_2\phi_{n-1} + (-\beta_2 + h\beta_1)\phi_n =$$

$$\lambda[-\beta_2'\phi_{n-1} + (\beta_2' - \beta_1'h - h^2\beta_2)\phi_n] - \lambda^2h^2\beta_2'\phi_n.$$

However unless $\beta_2' = 0$ the equation is quadratic in λ. If $\beta_2' \neq 0$, to keep the problem linear and symmetric an additional variable $z = a\phi_n + b\phi_{n+1}$ is required. Now that we have investigated the different but similar boundary conditions we will limit our discussion to boundary conditions (3.1) and (3.3).

We will also solve the equation

$$(3.10) \qquad y'' - xy = -\lambda y$$

with the boundary conditions

$$(3.11) \qquad y(0) = 0$$

and

$$(3.12) \qquad y'(1) = \lambda y(1)$$

by using finite difference methods. The above system reduces to

$$\begin{cases} \phi_0 = 0 \\ \phi_{i+1} - (2 + ih^3)\phi_i + \phi_{i-1} = -\lambda h^2 \phi_i, \ i = 1, \ldots, n-1, \\ -(h^2 + 1)\phi_n + \phi_{n-1} = -\lambda(h^2 + h)\phi_n. \end{cases}$$

The matrix system $A\phi = C\phi$ is

$$\begin{bmatrix} -(2+h^3) & 1 & & & 0 \\ 1 & -(2+2h^3) & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & 1 & -(2+(n-1)h^3) & 1 \\ 0 & & & 1 & -(nh^3+1) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \end{bmatrix} = \begin{bmatrix} -\lambda h^2 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & -\lambda h^2 \\ 0 & & & -\lambda(h^2+h) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \phi_n \end{bmatrix}.$$

We can let $\mu = \lambda h^2$ so that the matrix equation becomes

$$A\phi = -\mu B\phi$$

where

$$B = \begin{bmatrix} 1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & 1 & \\ 0 & & & 1 + 1/h \end{bmatrix}.$$

Also

$$B^{-1/2} = \begin{bmatrix} 1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & 1 & \\ 0 & & & & (1 + 1/h)^{-1/2} \end{bmatrix}$$

and

$A^* =$

$$\begin{bmatrix} -(2 + h^3) & 1 & & & & 0 \\ 1 & -(2 + 2h^3) & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -(2 + (n-1)h^3) & (1 + 1/h)^{-1/2} \\ 0 & & & & (1 + 1/h)^{-1/2} & -(h^2 + 1)/(1 + 1/h) \end{bmatrix}.$$

The eigenvalues are listed in Table 3.2 for the different values of n.

Using the finite difference method we will also solve the problem

(3.13) $y'' = -\lambda(1 + x)y$

with the boundary conditions (3.11) and (3.12).

The above system becomes

Table 3.2. Eigenvalues of y" - xy = $-\lambda y$ found by the finite difference method.

Eigenvalues	n=4	n=8	n=16	n=32	n=64
λ_0	.94489	.93950	.93739	.93699	.93592
λ_1	11.215	11.847	12.036	12.096	12.117
λ_2	33.353	39.538	41.272	41.732	41.855
λ_3	55.387	80.793	88.533	90.581	91.108
λ_4		129.43	152.10	158.27	159.86
λ_5		178.06	229.55	244.17	247.95
λ_6		219.28	317.90	347.45	355.18
λ_7		246.83	413.77	467.11	481.29
λ_8			513.47	602.02	625.98
λ_9			613.17	750.87	788.91

$$\begin{cases} \phi_0 = 0 \\ \phi_{i+1} - 2\phi_i + \phi_{i-1} = -\lambda(h^2 + ih^3)\phi_i, \quad i = 1, \; . \; . \; . \; , n - 1, \\ -\phi_n + \phi_{n-1} = -\lambda(2h^2 + h)\phi_n. \end{cases}$$

The matrix equation $A\phi = C\phi$ becomes

$$\begin{bmatrix} -2 & 1 & & & & 0 \\ 1 & -2 & 1 & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ . \\ \phi_n \end{bmatrix} =$$

$$\begin{bmatrix} -\lambda(h^2 + h^3) & & & 0 \\ & . & & \\ & & . & \\ & & & -\lambda(h^2 + (n-1)h^3) \\ 0 & & & -\lambda(2h^2 + h) \end{bmatrix} \begin{bmatrix} \phi_1 \\ o \\ . \\ . \\ \phi_n \end{bmatrix}.$$

Letting $\mu = \lambda h^2$ we have

$$A\phi = \begin{bmatrix} -\mu(1 + h) & & & & 0 \\ & . & & & \\ & & . & & \\ & & & -\mu(1 + (n-1)h) & \\ 0 & & & & -\mu(2 + 1/h) \end{bmatrix} \phi.$$

The system becomes

$$A\phi = -\mu B\phi$$

where

$$B = \begin{bmatrix} 1+h & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & 1+(n-1)h & \\ 0 & & & & 2+1/h \end{bmatrix}.$$

Since

$$B^{-1/2} = \begin{bmatrix} (1+h)^{-1/2} & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & (1+(n-1)h)^{-1/2} & \\ 0 & & & & (2+1/h)^{-1/2} \end{bmatrix}$$

we have

$$A^* = B^{-1/2}AB^{-1/2}$$

which is given by

$$A^* = \{A^*_{i,j}\}$$

where

$$A^*_{i,j} = \begin{cases} 0, & \text{if } (j - i) \geq 2 \text{ or } (j - i) \leq 2, \\ (1 + ih)^{-1/2}(1 + jh)^{-1/2}, & \text{if } |\, j - i \,| = 1, \\ -2(1 + ih)^{-1}, & \text{if } j - i = 0 \text{ , } i \neq n, \\ (2 + 1/h)^{-1}, & \text{if } j = i = n. \end{cases}$$

The eigenvalues are listed in Table 3.3 for the different values of n.

Table 3.3. Eigenvalues of $y'' = -\lambda(1 + x)y$ found by the finite difference method.

Eigenvalues	n=4	n=8	n=16	n=32	n=64
λ_0	.52921	.57126	.59330	.60466	.61149
λ_1	7.5584	8.1950	8.4173	8.4996	8.5315
λ_2	22.041	26.727	28.117	28.509	28.624
λ_3	37.758	54.150	59.845	61.381	61.788
λ_4		86.144	102.45	106.90	108.05
λ_5		117.47	154.23	164.63	167.32
λ_6		145.64	213.11	233.00	239.46
λ_7		182.29	276.67	314.33	324.29
λ_8			342.20	404.80	421.59
λ_9			406.76	504.52	531.13

CHAPTER IV

THE SERIES METHOD

We will examine the test problem using the series method approach. The test problem is $\phi'' = -\lambda\phi$ where $\phi(0) = 0$ and $\phi'(1) = \lambda\phi(1)$. For the series method we have that $\phi(x)$ can be represented by an infinite series in x. We can substitute ϕ and ϕ' into the differential equation and by truncating after n terms we can arrive at a polynomial approximation P. The eigenvalues can be found by solving $\phi(0) = 0$, $\phi'(1) = \lambda\phi(1)$. Using the polynomial approximation $P'(1,\lambda) = \lambda P(1,\lambda)$ or $\lambda P(1,\lambda) - P'(1,\lambda) = 0$, we find the eigenvalues by solving for the roots of this approximation. Therefore we take $\phi(x) = c_0 + c_1x + c_2x^2 + \ldots$ and $\phi'(x) = c_1 + 2c_2x + 3c_3x^2 + \ldots$. Examining the boundary conditions we see that $\phi(0) = 0 = c_0$. When we substitute the infinite series into the differential equation we find that

$[2c_2 + 3(2)c_3x + 4(3)c_4x^2 + \ldots] = -\lambda[c_0 + c_1x + c_2x^2 + \ldots]$.

Since we have a homogeneous problem and if we let $c_1 = 0$ then $c_0 = c_1 = 0$ would imply that $\phi(x) = 0$. To solve for a non-trivial solution we must choose a value of $c_1 \neq 0$. Therefore, without loss of generality we will let $c_1 = 1$ so that the following recursion formula can be formed.

$\phi(0) = c_0 = 0$

$2c_2 = -\lambda c_0$

$3(2)c_3 = -\lambda c_1$

$4(3)c_4 = -\lambda c_2$

.
.
.

$(n + 2)(n + 1)\, c_{n+2} = -\lambda c_n$.

Since $c_0 = 0$, we see that the recursion formula gives $c_{even} = 0$. Further, with $c_1 = 1$,

$c_3 = -\lambda/3(2)$

$c_5 = \lambda^2/5!$

$c_7 = -\lambda^3/7!$

.
.
.

$c_{2k+1} = (-1)^k\lambda^k/(2k + 1)!$.

We used EQRT2S, an IMSL program, to find the roots of the polynomial formed with an n number of non-zero terms. For example, with $n = 6$, we are using the approximation for $\phi(x)$,

$\phi(x) \approx c_1x + c_3x^3 + c_5x^5 + c_7x^7 + c_9x^9 + c_{11}x^{11}$, so that $P(1, \lambda) = c_1 + c_3 + c_5 + c_7 + c_9 + c_{11}$, and $P'(1, \lambda) = c_1 + 3c_3 + 5c_5 + 7c_7 + 9c_9 + 11c_{11}$ with the c_n's given above. The eigenvalues are listed in Table 4.1.

The Series Method is only converging for the first few eigenvalues. Since this method is producing extraneous and negative eigenvalues we cannot depend on its reliability. Therefore, we will not pursue this method with any of the other differential equations.

Table 4.1. Eigenvalues of $\phi'' - -\lambda\phi$ found by the series method.

Eigenvalues	n-6	n-10	n-15
λ_0	.74017	.74017	.74017
λ_1	11.726	11.735	11.735
λ^2	32.292	41.462	41.439
λ_3	32.292	74.399	90.814
λ_4	17.321	78.284	147.86
λ_5	17.321	78.284	160.62
λ_6		58.956	160.62
λ_7		58.956	121.06
λ_8		*	121.06
λ_9		*	157.64

* The remaining eigenvalues are negative and are not included here.

CHAPTER V

BISECTION METHOD

The bisection method will be used to solve our test problem. We will also solve two other eigenvalue problems using the bisection method.

The general solution for r′ and θ' of the second degree differential equation

$$y'' + q(x)y = \Gamma(x)\lambda y$$

can be found by using the same process that was used to find r′ and θ' of equation (1.1). Solving our general problem we see that

$$r' = [\Gamma(x)\lambda - q(x)]r\sin\theta$$

and

$$\theta' = \cos^2\theta - [\Gamma(x)\lambda - q(x)]\sin^2\theta.$$

From Chapter I, the initial boundary condition for θ can be solved for

$$\alpha(\lambda) = \tan^{-1}[(-\beta_2 + \lambda\beta_2')/(\beta_1 + \lambda\beta_1')]$$

and must satisfy the condition that $\beta_1'\beta_2 > \beta_1\beta_2'$. We will choose for our computations the boundary condition $-y'(0) = \lambda y(0)$ so that

$$\beta_1 = 0,\ \beta_1' = 1,\ \beta_2 = 1 \text{ and } \beta_2' = 0.$$

The above conditions allow us to simplify $\alpha(\lambda)$ to

$$\alpha(\lambda) = \tan^{-1}[-1/\lambda].$$

However, we must also add π to $\alpha(\lambda)$ when $\lambda > 0$ so that the angle will occur as in Figure 1.4.

The terminal boundary condition $y(L) = 0$ allows us to solve for the terminal angle. Since we know that

$$\tan\gamma = 0,$$

then

$$\theta(L,\lambda) = 0 \pmod{\pi}.$$

In our particular problem we have $L = 1$ so we can conclude that

$$\theta(1,\lambda) = k\pi.$$

Therefore, the eigenvalues occur when the terminal angle equals multiples of π.

The bisection method we will use to solve for the eigenvalues of differential equations can be found using the following process. First, a particular value of λ is chosen. This value of λ is used to calculate the initial angle

$$\alpha(\lambda) = \theta(0,\lambda).$$

Then by using the second order Runga-Kutta method we can solve for the terminal angle $\theta(1,\lambda)$. The first computer program involving this method calculates the terminal angles for every fifth integer value of λ from 0 through 500. Having the list of terminal values allows us to find initial values whose terminal angles have values which are upper and lower bounds for multiples of π. In other words, if λ^*_n and λ^*_{n+1} are to be lower and upper bounds, respectively, for the nth eigenvalue λ_n, then we choose λ^*_n such that $0 < \lambda_{n-1} < \lambda^*_n < \lambda_n$ and λ^*_{n+1} such that $\lambda_n < \lambda^*_{n+1} < \lambda_{n+1}$. The second program for this method uses the upper and lower bounds and then calculates the midpoint λ^x of these values. The midpoint is used to calculate the terminal angle $\theta(1,\lambda^x)$. If the terminal angle is less than the multiple of π that was bounded, then the program replaces the left-hand endpoint with the midpoint. If the terminal value is greater than the multiple of π that was bounded then the program replaces the right-hand endpoint with the midpoint. The midpoint is calculated again using the new left-hand and right-hand endpoints and the process continues until the difference between the right-hand and left-hand endpoints divided by the right-hand endpoint is less than .00005.

The test problem

$$y''(x) = -\lambda y(x), \quad -y'(0) = \lambda y(0), \quad y(1) = 0,$$

has the following relation to our general problem: $q(x) = 0$ and $\Gamma(x) = -1$. Therefore we have that

$$\theta' = \cos^2\theta + \lambda\sin^2\theta.$$

For this test problem we chose three different mesh sizes for the Runga-Kutta method. The mesh size determines the number of intervals between 0 and 1 for which the computer evaluates the right hand side of the differential equation. The three mesh sizes are $n = 16$, $n = 64$ and $n = 128$. The following tables show the terminal angle, the initial lower and upper bounds and the eigenvalue for each different mesh size. As the number of divisions decreases, the number of eigenvalues that can be calculated decreases since the

Runga-Kutta method for solving the differential equation breaks down due to round-off error and the fact that θ has large slope for λ large. Table 5.1 shows the eigenvalues found with n = 16, Table 5.2 with n = 64 and Table 5.3 with n = 128.

We will also look at two other eigenvalue problems but, since the mesh with n = 128 allows us to find several more eigenvalues and since the improvement in the accuracy of the first eigenvalues is not significant, we will limit our examination of the other problems to the case of n = 128.

The next eigenvalue problem we will look at is

$$y'' - xy = -\lambda y, \quad y(0) = 0, \quad y'(1) = \lambda y(1).$$

Letting $\phi(x) = y(1 - x)$ and differentiating we have that $\phi'(x) = -y'(1 - x)$, $\phi''(x) = y''(1 - x)$. Substituting into the above differential equation we have

$$y''(1 - x) - (1 - x)y(1 - x) = -\lambda y(1 - x).$$

Thus the related equation becomes

$$\phi'' - (1 - x)\phi = -\lambda\phi, \quad \phi(1) = 0, \quad -\phi'(0) = \lambda\phi(0).$$

The relation to our general problem is $q(x) = x - 1$ and $\Gamma(x) = -1$. Therefore we have that

$$\theta' = \cos^2\theta + (\lambda + x - 1)\sin^2\theta.$$

Table 5.4 shows the terminal angle, lower bound, upper bound and the eigenvalues for this problem.

Table 5.1. Eigenvalues of y" = $-\lambda y$ using the bisection method with n = 16.

	terminal angle	lower bound	upper bound	eigenvalue
λ_0	3.141593	.1	1.	.74020
λ_1	6.283185	9.	13.	11.746
λ_2	9.424778	30.	42.	34.950

Table 5.2. Eigenvalues of y" = $-\lambda y$ found by the bisection method with n = 64.

	terminal angle	lower bound	upper bound	eigenvalue
λ_0	3.141593	.1	1.	.74015
λ_1	6.283185	9.	13.	11.735
λ_2	9.424778	30.	50.	41.456
λ_3	12.56637	80.	100.	90.974
λ_4	15.70796	140.	200.	141.63

Table 5.3. Eigenvalues of $y'' = -\lambda y$ found by the bisection method with $n = 128$.

	terminal angle	lower bound	upper bound	eigenvalue
λ_0	3.141593	.1	1.	.74017
λ_1	6.283185	9.	13.	11.735
λ_2	9.424778	30.	50.	41.439
λ_3	12.56637	80.	100.	90.852
λ_4	15.70796	140.	170.	160.30
λ_5	18.84956	220.	270.	247.01

Table 5.4. Eigenvalues of $y'' - xy = -\lambda y$ found by the bisection method.

	terminal angle	lower bound	upper bound	eigenvalue
λ_0	3.141593	.1	5.	.93565
λ_1	6.283185	10.	15.	12.131
λ_2	9.424778	40.	50.	41.906
λ_3	12.56637	80.	100.	91.335
λ_4	15.70796	155.	185.	160.81
λ_5	18.84956	225.	265.	247.61

The last eigenvalue problem we will look at is

$$y'' = -\lambda(1 + x)y,\ y(0) = 0,\ y'(1) = \lambda y(1).$$

By making the same substitution as before we see that the related equation becomes

$$\phi'' = -\lambda(2 - x)\phi,\ \phi(1) = 0,\ -\phi'(0) = \lambda\phi(0).$$

The relationship of this problem to the general problem is $q(x) = 0$ and $\Gamma(x) = x - 2$. Therefore, we have that

$$\theta' = \cos^2\theta + (2 - x)\lambda\sin^2\theta.$$

Table 5.5 shows the terminal angle, lower bound, upper bound and the eigenvalues for $y'' = -\lambda(1 + x)y$.

Table 5.5. Eigenvalues of $y'' = -\lambda(1 + x)y$ found by the bisection method.

	terminal angle	lower bound	upper bound	eigenvalue
λ_0	3.141593	.1	5.	.61607
λ_1	6.283185	5.	15.	8.5629
λ_2	9.424778	25.	35.	28.685
λ_3	12.56637	50.	70.	61.985
λ_4	15.70796	100.	125.	108.67

CHAPTER VI

COMPARISON

We will begin with our test problem (2.1) with the boundary conditions (3.11) and (3.12). We will look at this problem solved using Newton's method and finite differences. The finite difference method solutions listed have $n = 64$. The comparison is shown in Table 6.1.

We will look at the above test problem (2.1) that has the boundary conditions (3.11) and (3.12) using Newton's method and the series method. The series method solutions that are listed have $n = 15$. The comparison is shown in Table 6.2.

We will look at test problem (2.1) with boundary conditions (3.11) and (3.12) using Newton's method and the bisection method. The bisection method solutions listed have $n = 128$. The comparison is shown in Table 6.3.

We will also look at the differential equation (3.10) with the boundary conditions (3.11) and (3.12) using the finite difference method and the bisection method. The finite difference method solutions listed have $n = 64$ and the bisection method solutions listed have $n = 128$. The comparison is shown in Table 6.4.

Finally we will look at the differential equation (3.13) with the boundary conditions (3.11) and (3.12) using the finite difference method and the bisection method. The

Table 6.1. Comparison of eigenvalues found by Newton's method and the finite difference method.

	Newton's Method	Finite Difference	Newton - Finite Difference
λ_0	.74017	.74085	-.00068
λ_1	11.735	11.719	.016
λ_2	41.438	41.388	.05
λ_3	90.808	90.627	.181
λ_4	159.90	159.37	.53
λ_5	248.73	247.46	1.27
λ_6	357.30	354.69	2.61
λ_7	485.61	480.80	4.81
λ_8	633.65	625.49	8.16
λ_9	801.44	788.41	13.03

Table 6.2. Comparison of eigenvalues found by Newton's method and the series method.

	Newton's Method	Series Method	Newton-Series
λ_0	.74017	.74017	0.0
λ_1	11.735	11.735	0.0
λ_2	41.438	41.439	-.001
λ_3	90.808	90.814	-.006
λ_4	159.90	147.86	12.04
λ_5	248.73	147.86	12.04
λ_6	357.30	160.62	88.11
λ_7	485.61	121.06	364.55
λ_8	633.65	121.06	512.59
λ_9	801.44	157.64	643.80

Table 6.3. Comparison of eigenvalues found by Newton's method and the bisection method.

	Newton's Method	Bisection Method	Newton-Bisection
λ_0	.74017	.74017	0.0
λ_1	11.735	11.735	0.0
λ_2	41.438	41.439	-.001
λ_3	90.808	90.852	-.044
λ_4	159.90	160.30	-.4
λ_5	248.73	247.01	1.72

Table 6.4. Comparison of eigenvalues of $y'' - xy = -\lambda$ found by the finite difference method and the bisection method.

	Finite Difference	Bisection Method	Finite-Bisection
λ_0	.93592	.93565	.00027
λ_1	12.117	12.131	-.014
λ_2	41.855	41.906	-.051
λ_3	91.108	91.335	-.227
λ_4	159.86	160.81	-.95
λ_5	247.95	247.61	.34

finite difference solutions listed have n = 64 and the bisection method solutions have n = 128. The comparison is shown in Table 6.5.

Table 6.5. Comparison of eigenvalues of $y'' = -\lambda(1 + x)y$ found by the finite difference method and the bisection method.

	Finite Difference	Bisection Method	Finite-Bisection
λ_0	.61149	.61607	-.00458
λ_1	8.5315	8.5629	-.0314
λ_2	28.624	28.685	-.061
λ_3	61.788	61.985	-.197
λ_4	108.05	108.67	-.62

CHAPTER VII

CONCLUSIONS

The major theoretical results show that parameter-dependent boundary conditions basically maintain the separation of zeros and the number of zeros of the eigenfunction. The exception occurs when the initial angle passes through a multiple of π. The next eigenfunction after the initial angle passes a multiple of π loses a zero and all eigenfunctions after that one have one less zero. The next eigenfunction after the initial angle passes a multiple of π also fails to have a zero before the first zero of the previous eigenfunction.

The simple test problem

$$\phi'' = -\lambda\phi$$

with the boundary conditions

$$\phi(0) = 0$$

and

$$\phi'(1) = \lambda\phi(1),$$

was solved using Newton's method. Since Newton's method is very accurate we can assume that the answers given using Newton's method are exact.

Looking at the comparison of the test problem solved with Newton's method and the Finite Difference method we see that the results are fairly good for eigenvalues below 200. Greater accuracy of higher eigenvalues can be achieved by increasing the size of the matrices. However, increasing the matrix size can cause overflow or underflow problems.

Looking at the comparison of the test problem solved with Newton's method and the Series method we see that the series method is only accurate for the first few eigenvalues. Accuracy can be improved by solving for the roots of the polynomial P with a larger number of terms. However, this method gives extraneous roots and negative answers. Because of these difficulties we consider this method to be a poor choice for solving eigenvalue problems.

Looking at the comparison of the test problem with the bisection method we see that we get good results for eigenvalues less than 200. Accuracy can be improved by increasing the number of subdivisions in the interval. The larger the value of λ the more likely the solutions by the Runga-Kutta method are to break down. Care should therefore be taken to reduce the tolerance level for larger values of λ.

The last two eigenvalue problems were solved using the Finite Difference method and the Bisection method. Looking at the comparison of these last two problems we see that the two different methods used give comparable results.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Ahn, Hyun J., "On Random Transverse Vibrations of Rotating Beam with a Tip Mass," Q. J. Mech. Appl. Math. 36 (1983), 97-109.

2. Ahn, Hyun J., "Vibration of a Pendulum Consisting of a Bob Suspended From a Wire," Q. Appl. Math. 39 (1981), 109-117.

3. Haberman, Richard, Elementary Applied Partial Differential Equations, Prentice-Hall, New Jersey, 1983.

4. Hinton, Don B., "An Expansion Theorem for an Eigenvalue Parameter in the Boundary Condition," Q. J. Math. Oxford (2), 30 (1979), 33-42.

5. Jones, Louise H., "The Transverse Vibration of a Rotating Beam with Tip Mass," Q. Appl. Math. 23 (1975), 193-203.

6. Miller, Richard K. and Michel, Anthony N., Ordinary Differential Equations, Academic Press, New York, 1982.

VITA

Brenda Kay Harrington was born in Oxford, Mississippi on February 27, 1963. She attended elementary schools in Memphis, Tennessee. In 1973 she moved to Springfield, Missouri and then graduated from Parkview High School in May of 1981. The following month she entered Southwest Missouri State University and in August 1985 she received a Bachelor of Science degree in Mathematics. In the fall of 1985 she accepted a teaching assistantship at The University of Tennessee, Knoxville and began study toward a Master's degree in Mathematics. This degree was awarded in June 1988.