# Modular forms, partitions, and $q$-series

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# Abstract

This dissertation explores results pertaining to partition theory and its $q$-series identities using techniques from modular forms. In particular, Chapter 2 proves congruences for $k$-colored generalized Frobenius partitions originally defined by George Andrews in 1984. These congruences generalize parity results for the partition function and generalized Frobenius partitions to weakly holomorphic modular forms of a certain type. Chapter 3 gives results regarding the Andrews-Bressoud identities, a generalization of the famed Rogers-Ramanujan partition identities. When viewed as $q$-series, these series can be connected to irreducible characters from vertex operator algebra theory. Then, when combined with standard tools from modular forms, we see that the Wronskians of these series satisfy nice modularity properties.

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# List of Symbols

| | |
|---|---|
| $p(n)$ | partition function, i.e., the number of partitions of $n$; see p. 1 |
| $c\phi_k(n)$ | $k$-colored generalized Frobenius partitions, i.e. number of such partitions of $n$; see p. 7 |
| $(a)_n$, $(a)_\infty$ | $q$-Pochhammer symbol; see p. 9, 37 |
| $\mathbb{H}$ | complex upper half-plane; see p. 12 |
| $\mathrm{SL}_2(\mathbb{Z})$ | $2 \times 2$ matrices with integer entries of determinant 1, i.e., $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$; see p. 11 |
| $\Gamma_0(N)$ | congruence subgroup where $c \equiv 0 \pmod{N}$; see p. 12 |
| $\Gamma_1(N)$ | congruence subgroup where $c \equiv 0 \pmod{N}$ and $a, d \equiv 1 \pmod{N}$; see p. 12 |
| $\Gamma$ | congruence subgroup; see p. 12 |
| $\mathcal{M}_k(\Gamma, \chi)$ | space of modular forms of weight $k$ for the congruence subgroup $\Gamma$ with character $\chi$; see p. 12 |
| $\mathcal{S}_k(\Gamma, \chi)$ | space of cusp forms of weight $k$ for the congruence subgroup $\Gamma$ with character $\chi$; see p. 12 |
| $E_k(\tau)$ | Eisenstein series of weight $k$; see p. 14 |
| $\Delta(\tau)$ | Delta-function; see p. 15 |
| $j(\tau)$ | modular $j$-function; see p. 15 |
| $\eta(\tau)$ | Dedekind eta-function; see p. 4, 16 |
| $\mathcal{M}_k^!(\Gamma, \chi)$ | space of weakly holomorphic modular forms of weight $k$ for the congruence subgroup $\Gamma$ with character $\chi$; see p. 16 |
| $B_{k,j}(q)$ | Andrews-Bressoud series; see p. 35 |

| | |
|---|---|
| $W(f_1, \ldots, f_k)$ | Wronskian determinant; see p. 35 |
| $\mathcal{W}_k(q)$ | Wronskian determinant of the Andrews-Bressoud series; see p. 36 |
| $\widetilde{\mathcal{W}}_k(q)$ | Wronskian determinant of the derivatives of the Andrews-Bressoud series; see p. 36 |
| $\mathcal{F}_k(\tau)$ | quotient of the Wronskians $\widetilde{\mathcal{W}}_k(q)$ and $\mathcal{W}_k(q)$; see p. 36 |

# Chapter 1

# Introduction

## 1.1 Partitions

The concept of a partition is easily understood, though beneath the surface, its generating function exhibits many surprising properties that have long been studied by numerous mathematicians including Srinivasa Ramanujan, G.H. Hardy, and George Andrews. A partition is a combinatorial object by definition, but those who study them and their properties use techniques from a wide variety of areas including combinatorics, number theory, and representation theory. Finding new depths of this beauty continues today, and the topics discussed in this dissertation all stem from partition-related problems.

To begin, let us first formally define a partition.

**Definition 1.1.** *A **partition** of a positive integer $n$ is a non-increasing sequence $\lambda_1, \lambda_2, \ldots, \lambda_k$, where $k \in \mathbb{N}$, $\lambda_i \in \mathbb{N}$ for $1 \leq i \leq k$, and $n = \sum_{i=1}^{k} \lambda_i$. We call the $\lambda_i$ the **parts** of the partition and $k$ its **length**.*
*The **partition function** $p(n)$ is the number of partitions of $n$.*

For example, the partitions of 4 are

$$4 \quad 3,1 \quad 2,2 \quad 2,1,1 \quad 1,1,1,1, \tag{1.1}$$

**Table 1.1:** Partition function values

| $n$ | $p(n)$ | $n$ | $p(n)$ | $n$ | $p(n)$ | $n$ | $p(n)$ | $n$ | $p(n)$ | $n$ | $p(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 5 | 7 | 10 | 42 | 15 | 176 | 20 | 627 | 25 | 1958 |
| 1 | 1 | 6 | 11 | 11 | 56 | 16 | 231 | 21 | 792 | 26 | 2436 |
| 2 | 2 | 7 | 15 | 12 | 77 | 17 | 297 | 22 | 1002 | 27 | 3010 |
| 3 | 3 | 8 | 22 | 13 | 101 | 18 | 385 | 23 | 1255 | 28 | 3718 |
| 4 | 5 | 9 | 30 | 14 | 135 | 19 | 490 | 24 | 1575 | 29 | 4565 |

and, thus, $p(4) = 5$. Similarly, the partitions of 5 are

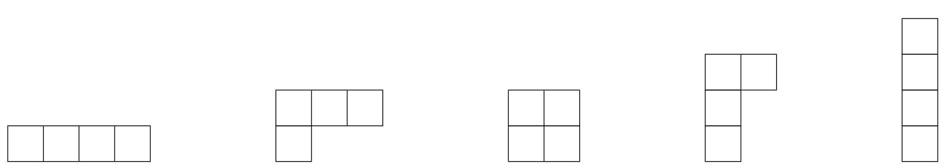
$$5 \quad 4,1 \quad 3,2 \quad 3,1,1 \quad 2,2,1 \quad 2,1,1,1 \quad 1,1,1,1,1, \tag{1.2}$$

which gives that $p(5) = 7$. By convention, we define $p(0) = 1$. The first 30 values of the partition function are included in Table 1.1.

Another way to view a partition is as its Ferrers diagram, which is also called the graphical representation of a partition [4].

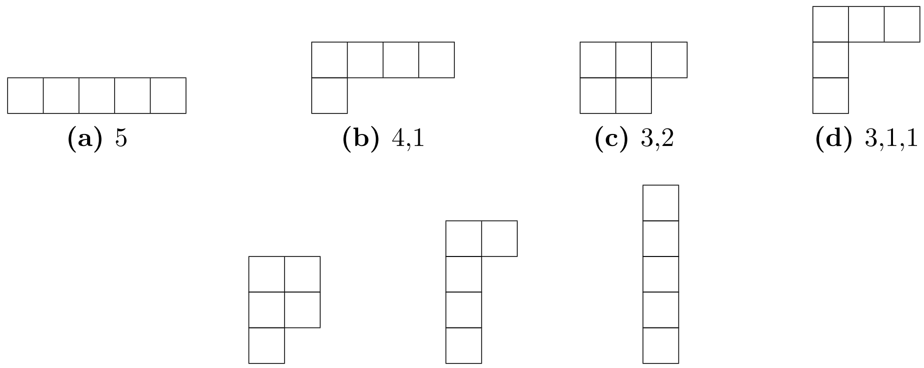
**Definition 1.2.** *For a partition $\lambda_1, \ldots, \lambda_k$ of the integer $n$, the associated **Ferrers diagram** $\lambda$ has $k$ rows, where the $i^{\text{th}}$ row consists of $\lambda_i$ boxes.*

*For each box $s$ in a Ferrers diagram, we define the **arm length** $a_\lambda(s)$ to be the number of boxes to the right of $s$ in the diagram $\lambda$ and the **leg length** $l_\lambda(s)$ to be the number of boxes below $s$ in the diagram $\lambda$.*



**(a)** 4 **(b)** 3,1 **(c)** 2,2 **(d)** 2,1,1 **(e)** 1,1,1,1

**Figure 1.1:** Ferrers diagrams for the partitions of 4



**(a)** 5 **(b)** 4,1 **(c)** 3,2 **(d)** 3,1,1

**(e)** 2,2,1 **(f)** 2,1,1,1 **(g)** 1,1,1,1,1

**Figure 1.2:** Ferrers diagrams for the partitions of 5

For example, the Ferrers diagrams for the partitions of 4 from Equation (1.1) are in Figure 1.1, and the Ferrers diagrams for all of the partitions of 5 from Equation (1.2) are in Figure 1.2. Let $\lambda$ be the diagram in Figure 1.2e. Let $s$ be box in the upper left corner of this diagram; then $a_\lambda(s) = 1$ and $l_\lambda(s) = 2$. Similarly, for the box $t$ in the first row and second column of the same diagram, $a_\lambda(t) = 0$ and $l_\lambda(t) = 1$. This notion of arms and legs will be helpful for us following Definition 1.3.

Now that we have defined a partition and $p(n)$, it is natural to wonder whether or not there exists a general formula for calculating values of $p(n)$ quickly. In 1917, Hardy and Ramanujan [16] made progress toward such a formula in obtaining the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

Rademacher improved upon their result and completed this endeavor in 1937 [40], and the resulting formula for $p(n)$ is

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[ \frac{d}{dx} \left( \frac{\sinh\left( \left(\frac{\pi}{k}\right) \left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{\frac{1}{2}} \right)}{\left(x - \frac{1}{24}\right)^{\frac{1}{2}}} \right) \right]_{x=n}, \tag{1.3}$$

where

$$A_k(n) = \sum_{\substack{h \pmod k \\ \gcd(h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}}$$

with $\omega_{h,k}$ a certain $24k^{\text{th}}$ root of unity (see Section 5.2 of [4] for precise definition). The proof of this formula relies on computing contour integrals, and in order to do so, Rademacher related $p(n)$ to the Dedekind eta-function

$$\eta(\tau) \coloneqq q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.4}$$

where $q \coloneqq e^{2\pi i \tau}$ and $\tau \in \mathbb{H}$, the complex upper half-plane, and employed its known transformation properties under the action of matrices in $\mathrm{SL}_2(\mathbb{Z})$. The process of proving this formula developed what is now known as the Circle Method of computing contour integrals, which is still frequently used today, particularly in the area of analytic number theory.

While this formula proves very useful for quick computations of $p(n)$ for large $n$, it is cumbersome for analyzing the general behavior of $p(n)$ beyond these direct computations. Instead, to study properties of the partition function, we consider its values for all $n$ at once via its generating function, as is common with many combinatorial objects. The generating function for $p(n)$ is the following formal power series in the variable $q$:

$$\sum_{n=0}^{\infty} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots .$$

Later we will experience the usefulness of this generating function and see explicitly its connection to the Dedekind eta function defined in Equation (1.4).

### 1.1.1 Congruences

Table 1.1 is organized in such a way to lead us to our first partition property: congruences. While the partition function is a combinatorial object and grows very rapidly, we still find patterns amongst its values. We notice that all values in the last row of Table 1.1 are multiples of 5. While we only see 6 of these values, this pattern continues indefinitely and was proven by Srinivasa Ramanujan in 1919 [4]. Moreover, Ramanujan observed the following three congruences of the partition function:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.5}$$

In fact, any congruence of the form

$$p(\ell n+\delta) \equiv 0 \pmod{\ell},$$

where $\ell$ is prime and $0 \leq \delta < \ell$, is called a Ramanujan congruence, and in 2003 Ahlgren and Boylan [1] proved that these are the only three Ramanujan congruences for the partition function.

While these are the only congruences for $p(n)$ of this specific form, many "non-Ramanujan" congruences exist for the partition function. For example, [39] lists several examples including

$$\begin{aligned} p(48037937n+1122838) &\equiv 0 \pmod{17} \\ p(14375n+3474) &\equiv 0 \pmod{23} \\ p(4063467631n+30064597) &\equiv 0 \pmod{31}. \end{aligned}$$

In fact, Ahlgren and Ono [2] proved that for any prime $\ell \geq 5$ and positive integer $m$, there exist infinitely many non-nested arithmetic progressions $\{An+B\}$ such that for every positive integer $n$,

$$p(An+B) \equiv 0 \pmod{\ell^m}.$$

However, if we turn our attention to the primes 2 and 3, the partition function behaves quite differently. In the 1960s, Subbarao conjectured the non-existence of congruences in arithmetic progressions of $p(n)$ modulo 2, which was then proven by Radu in 2012 [41]. The result states that for any arithmetic progression $r \pmod{t}$, there are infinitely many integers $N \equiv r \pmod{t}$ for which $p(N)$ is even, and there are infinitely many integers $M \equiv r$

(mod $t$) for which $p(M)$ is odd. In the same paper [41], Radu proves a conjecture of Ahlgren and Ono regarding the partition function in arithmetic progressions modulo 3, and Theorem 1.1 of [49] (Theorem 2.5 in Chapter 2) implies that in any arithmetic progression there are infinitely many values of the partition function that are multiples of 3, but there still remain open problems in this area.

### 1.1.2 Generalized Frobenius partitions

We mentioned earlier that in proving Equation (1.3), Rademacher made use of connections between $p(n)$ and the Dedekind eta function (Equation (1.4)). In particular, Euler proved that the generating function for $p(n)$ can be rewritten as the infinite product below, which then allows us to explicitly connect $p(n)$ with $\eta(\tau)$

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = q^{\frac{1}{24}}\eta(\tau). \tag{1.6}$$

We can also define variations on the partition function. For example, we could consider the partitions of $n$ where all of the parts are odd or all of the parts are distinct or even where each part has an associated color ("colored partitions"). Of course, we could continue this process nearly indefinitely by imposing restrictions on the number, size, and type of parts of a partition, but one of the most exciting consequences of defining these variations is finding unexpected connections between these new partition functions, called identities.

This alternate presentation of the generating function $p(n)$ as an infinite product (Equation (1.6)) provides for many partition identity proofs. For example, the number of partitions of $n$ into distinct parts is the same as the number of partitions of $n$ into odd parts, which is easily seen through manipulation of the generating functions for the associated partition functions. To see this, first note that the generating functions for these variations on the partition function are as follows:

$$\sum_{n=0}^{\infty} p(n|\text{distinct parts})q^n = \prod_{n=1}^{\infty}(1+q^n)$$
$$\sum_{n=0}^{\infty} p(n|\text{odd parts})q^n = \prod_{n \text{ odd}} \frac{1}{(1-q^n)}.$$

Now, we start with the generating function of partitions into distinct parts and multiply each factor by $\frac{1-q^n}{1-q^n}$ so that the numerators become the difference of two squares, which then cancel to give the generating function for the partitions into odd parts:

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^n) &= (1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)\cdots \\ &= \left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^4}{1-q^2}\right)\left(\frac{1-q^6}{1-q^3}\right)\left(\frac{1-q^8}{1-q^4}\right)\left(\frac{1-q^{10}}{1-q^5}\right)\cdots \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} \\ &= \prod_{n \text{ odd}} \frac{1}{(1-q^n)}.\end{aligned}$$

This identity can also be proven using combinatorial methods, but this technique of $q$-series/generating function manipulation provides a method of proof for some identities where a combinatorial proof does not exist.

In Chapter 2, we prove results regarding a specific variation of the partition function, $k$-colored generalized Frobenius partitions. Generalized Frobenius partitions were defined combinatorially in 1984 by Andrews [5] as in Definition 1.3.

**Definition 1.3.** *A* ***generalized Frobenius partition*** *of an integer $n$ is a two-rowed array of integers*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

*where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$, and*

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

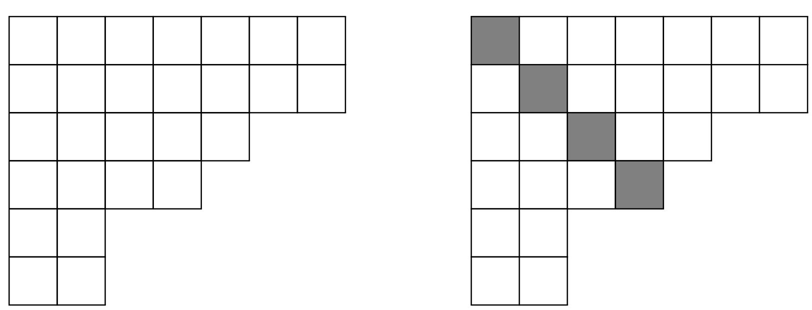
*A $k$-**colored generalized Frobenius partition** of $n$ is a generalized Frobenius partition of $n$ where each $a_i$, $b_i$ is chosen from $k$ copies of the integers; we denote the number of $k$-colored generalized Frobenius partitions of $n$ by $c\phi_k(n)$.*

It is not immediately obvious why these new objects are classified as partitions so let's explore this notation through an example. Here, consider the partition 7, 7, 5, 4, 2, 2 of 27 and its Ferrers diagram (Figure 1.3a). The diagram has 4 boxes on its diagonal (shaded boxes in Figure 1.3b). Then note that the arms of each box on the diagonal form the decreasing sequence

$$a_1 = 6, \qquad a_2 = 5, \qquad a_3 = 2, \qquad a_4 = 0.$$

Similarly, the legs of each box on the diagonal form the decreasing sequence

$$b_1 = 5, \qquad b_2 = 4, \qquad b_3 = 1, \qquad b_4 = 0.$$



**(a)** Ferrers diagram    **(b)** shaded diagonal

**Figure 1.3:** Ferrers diagrams for the partition 7, 7, 5, 4, 2, 2

Now, if $r = 4$ (the length of the diagonal), and we let the $a_i$ form the first row of the array and the $b_i$ form the second row of the array, we can now visualize this original partition of 27 as

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix},$$

which is called Frobenius symbol. Note that in this notation, the parts of the partition must be strictly decreasing and some parts may be 0, $27 = 4 + (6 + 5 + 2 + 0) + (5 + 4 + 1 + 0)$. Thus $c\phi_k(n)$ counts the number of these symbols in $k$ colors.

Similarly to that of $p(n)$, we use the generating function for $c\phi_k(n)$ to study its behavior in a general sense. After defining these objects combinatorially, Andrews [5] gives the generating function for $c\phi_k(n)$:

$$\sum_{n=0}^{\infty} c\phi_k(n)q^n = \frac{\displaystyle\sum_{\vec{m}\in\mathbb{Z}^{k-1}} q^{Q(\vec{m})}}{\displaystyle\prod_{n\geq 1}(1-q^n)^k}, \tag{1.7}$$

where $Q(\vec{m})$ is the quadratic form $Q(\vec{m}) := \sum_{1\leq i<j\leq k-1} m_i m_j$.

Once a partition variation is defined, it is natural to ask the same questions of this new function that we have of $p(n)$. For example, does $c\phi_k(n)$ have any congruences? Does $c\phi_k(n)$ satisfy a Subbarao-type conjecture of its own? These questions are answered in Chapter 2.

### 1.1.3 Infinite families of partition identities

As mentioned previously, many examples of partition identities exist, but some that are of particular interest are those that give rise to infinite families of partition identities. The classical first examples of such identities are the Rogers-Ramanujan identities,

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})} \tag{1.8}$$

$$= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m+2})(1-q^{5m+3})} \tag{1.9}$$

$$= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + \cdots ,$$

where $(a)_n := \prod_{k=0}^{n}(1-aq^k)$ is the $q$-Pochhammer symbol. These identities were originally discovered by Rogers in 1894 but went mostly unnoticed until Ramanujan independently discovered them and communicated his ideas to Hardy around 1913 [44, 4].

Combinatorially, the first Rogers-Ramanujan identity asserts that the partitions of an integer $n$ where the difference between parts is at least 2 (summation side) are in one-to-one correspondence with the partitions of $n$ into parts that are each congruent to 1 or 4 modulo 5 (product side). The second identity gives that the size of the set of partitions of $n$ where each part is larger than one and the difference between parts is at least 2 (summation side) is the same as the size of the set of partitions of $n$ into parts congruent to 2 or 3 modulo 5 (product side). Thus, combined, Equations (1.8) and (1.9) are identities that relate different partition functions based on the modulus 5.

These Rogers-Ramanujan identities led to much more work into partition identities relating an infinite sum to an infinite product and have multiple generalizations. The Andrews-Gordon identities, for example, are an infinite family of identities which generalize the Rogers-Ramanujan identities to all odd moduli (including 5) [14, 4]; thus the first two Andrews-Gordon identities in Equations (1.8) and (1.9). On the even side, we have multiple infinite families of identities: the Göllnitz-Gordon-Andrews identities and the Andrews-Bressoud identities. The Göllnitz-Gordon-Andrews identities generalize the Rogers-Ramanujan identities to moduli that are multiples of 8 [13, 14, 4], and the Andrews-Bressoud identities generalize the Rogers-Ramanujan identities to all even moduli [8]. Each of these generalizations has their own associated partition function interpretation as well. For example, the Andrews-Bressoud identities are

$$\sum_{n\geq 0} b_{k,j}(n)q^n = \frac{\prod_{m\geq 1}(1-q^{2km})(1-q^{2km-k-j+1})(1-q^{2km-k+j-1})}{\prod_{m\geq 1}(1-q^m)}, \tag{1.10}$$

for $2 \leq k \in \mathbb{Z}$ and $j \in \{1, \ldots, k\}$, and combinatorially, they state that the number of partitions of $n$ into parts not congruent to 0 or $\pm(k+j-1)$ modulo $2k$ is the same as the number of partitions $b_{k,i}(n)$, $\pi = (\pi_1, \ldots, \pi_s)$ of $n$ (with $\pi_t \geq \pi_{t+1}$) such that

1. $\pi_t - \pi_{t+k-1} \geq 2$,
2. $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \cdots + \pi_{t+k-2} \equiv j + k \pmod 2$, and
3. at most $k = j$ parts of $\pi$ equal 1. [20]

Much is already known about the Rogers-Ramanujan identities, and then by extension, the Andrews-Gordon identities, but the Andrews-Bressound and Göllnitz-Gordon-Andrews identities are less-studied thus far. Results concerning the Andrews-Gordon identities and Wronskian determinants (see Equation (3.2)) of them [34] inspired work proving similar results about the Andrews-Bressoud identities in Chapter 3 of this dissertation.

While the partition function and its variations are combinatorial objects, many proofs of partition congruences and identities rely on techniques from number theory. In particular, the partition function contains the deep connection (Equation (1.6)) to the Dedekind eta function, which is a basic example of an number theoretic object called a modular form. Modular forms are introduced in the next section, and the proofs in the following subsequent chapters rely on the tools and techniques of the theory of modular forms.

## 1.2 Modular forms

Behind many famous "easy-to-state" results in number theory lie very deep mathematical objects. For example, Fermat's Last Theorem states that there do not exist rational numbers $x, y, z$ such that

$$x^n + y^n = z^n$$

for any integer $n \geq 3$, and its proof depends on number theoretic objects called modular forms [42, 51, 48]. These objects are often studied as infinite series called $q$-series and apply to many areas of mathematics including elliptic curves, representations of numbers as sums of squares, representation theory, and mathematical physics. Two other applications of these objects include partition theory and vertex operator algebras.

Essentially modular forms are complex-valued functions that act symmetrically on a specific subgroup of the special linear group $\mathrm{SL}_2(\mathbb{Z})$, also called the (full) modular group. Many introductory texts exist on this subject that contain more details than are included in this introduction and dissertation (for example, see [6, 22, 39, 12, 35, 9, 21]). Here we include necessary definitions and facts for the results and their proofs in Chapters 2 and 3.

To begin, we note that $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex upper half-plane $\mathbb{H}$ by the transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for any $\tau \in \mathbb{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We also want to recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which take $\tau$ to $\tau + 1$ and $-\frac{1}{\tau}$, respectively. There are special subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with similar behavior; we define these next.

**Definition 1.4.** *Let $0 < N \in \mathbb{Z}$. Define the following* ***congruence subgroups*** *of $\mathrm{SL}_2(\mathbb{Z})$:*

- $$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$
- $$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N} \text{ and } c \equiv 0 \pmod{N} \right\}.$$

*We often denote a general congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by $\Gamma$ and call $N$ the* ***level*** *of $\Gamma$.*

It is straightforward to verify that $\Gamma_0(N)$ and $\Gamma_1(N)$ are subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and that they act on $\mathbb{H}$ in the same way as $\mathrm{SL}_2(\mathbb{Z})$. In addition to acting on $\mathbb{H}$, every congruence subgroup (including $\mathrm{SL}_2(\mathbb{Z})$) acts on $\mathbb{Q} \cup \{\infty\}$ via the same action; we call an equivalence class of a rational number a cusp of that congruence subgroup. For example, the only cusp of $\mathrm{SL}_2(\mathbb{Z})$ is the equivalence class of $\infty$, and $\Gamma_0(2)$ has cusps at 0 and $\infty$.

Now that we have explored this matrix action on $\mathbb{H}$, we can define a modular form and a few of its variations.

**Definition 1.5.** *A function $f$ on $\mathbb{H}$ is a* ***modular form of weight*** *$k \in \mathbb{Z}$* ***for*** *$\mathrm{SL}_2(\mathbb{Z})$ if*

1. *$f$ is holomorphic on $\mathbb{H}$,*
2. *$f$ is holomorphic at infinity, i.e., if $q := e^{2\pi i\tau}$, the Fourier expansion of $f$ is of the form*

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

*called a $q$-series, and*

3. *f satisfies the transformation property*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \tag{1.11}$$

*for all* $\tau \in \mathbb{H}$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

*If in addition to these properties, f vanishes at infinity, then we call f a* ***cusp form of weight*** $k$ ***for*** $\mathrm{SL}_2(\mathbb{Z})$. *If f is a cusp form, then* $a_f(0) = 0$*, i.e., the constant term of its q-series expansion is 0.*

Once we have this definition of a modular form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, we can define the same objects for a particular congruence subgroup instead.

**Definition 1.6.** *Let* $0 \leq k \in \mathbb{Z}$, $\Gamma$ *be a congruence subgroup of level* $N$*, where* $1 \leq N \in \mathbb{Z}$*, and* $\chi$ *be a Dirichlet character modulo* $N$*. We say f is a* ***modular form of weight*** $k$***, character*** $\chi$***, and level*** $N$ *if*

1. *f is holomorphic on* $\mathbb{H}$*,*
2. *f is holomorphic at the cusps of* $\Gamma$*, i.e.,*

$$\chi(d)^{-1}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right),$$

*is holomorphic at infinity for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*, and*

3. *f satisfies the transformation property*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau) \tag{1.12}$$

*for all* $\tau \in \mathbb{H}$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

*A* ***cusp form of weight*** $k$***, character*** $\chi$***, and level*** $N$ *is a modular form of the same weight and level that vanishes at the cusps of* $\Gamma$.

*Note that if* $\Gamma = \Gamma_1(N)$*, then all Dirichlet characters* $\chi$ *modulo* $N$ *satisfy the transformation property given in Equation (1.12) so we omit the character.*

*We denote the space of modular forms (resp. cusp forms) for a congruence subgroup* $\Gamma$ *of a given weight and character as* $\mathcal{M}_k(\Gamma,\chi)$ *(resp.* $\mathcal{S}_k(\Gamma,\chi)$*). We often suppress the character in this notation if the character is trivial, i.e.* $\chi = 1$*, and similarly, we denote the space of modular forms (resp. cusp forms) of weight k for the full modular group by* $\mathcal{M}_k$ *(resp.* $\mathcal{S}_k$*).*

Note that $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma_0(N)$ and acts trivially on $\mathbb{H}$. Suppose $k$ is odd. By the transformation property in Equation (1.11) we have

$$f(\tau) = (-1)^k f(\tau),$$

which implies that $f(\tau) = 0$ for all $\tau$. Thus there are no nonzero modular forms of odd weight for $\Gamma_0(N)$.

One well-known fact about modular forms is that all of the modular forms of a given weight, level, and character $\mathcal{M}_k(\Gamma_0(N), \chi)$ forms a finite-dimensional vector space over $\mathbb{C}$; it follows, then, that the corresponding cusp forms $\mathcal{S}_k(\Gamma_0(N), \chi)$ is a subspace of $\mathcal{M}_k(\Gamma_0(N), \chi)$.

### 1.2.1 Examples of modular forms

Many examples of modular forms can be found throughout the literature and appear in Chapters 2 and 3 of this dissertation. Here, we give several, which are among the first and most notable examples we encounter in an introduction to modular forms as well as useful examples to keep in mind for the remaining chapters.

The first fundamental examples of modular forms are the Eisenstein series. Let $k \geq 2$ be even. The weight $k$ Eisenstein series is

$$E_k(\tau) \coloneqq 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $B_k$ is the $k^{\text{th}}$ Bernoulli number and $\sigma_{k-1}$ is the divisor sum

$$\sigma_{k-1}(n) \coloneqq \sum_{1 \leq d | n} d^{k-1}.$$

For $k \geq 4$, $E_k$ is a modular form of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$; this result is often obtained by analyzing the behavior of the alternate expression for $E_k$

$$2\zeta(k) E_k(\tau) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k},$$

under the action of the generators $S$ and $T$ of $\mathrm{SL}_2(\mathbb{Z})$. In the case of $k = 2$, we have

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

which satisfies $E_2(\tau) = E_2(\tau + 1)$ but

$$\tau^{-2}E_2\left(-\frac{1}{\tau}\right) = E_2(\tau) + \frac{12}{2\pi i\tau},$$

and so it does not satisfy the transformation property for $S \in \mathrm{SL}_2(\mathbb{Z})$ (see Proposition III.2.7 of [22] or Section 1.2 of [12] for proof). However, $E_2$ is still important in the theory of modular forms and is what is called a quasi-modular form, which we will see in Chapter 3.

One last fact about Eisenstein series is that $E_4$ and $E_6$ generate the graded algebra $\bigoplus_{k=0}^{\infty} \mathcal{M}_k$ as an algebra, i.e., $\mathcal{M}_k$ is generated by monomials of the form

$$E_4(\tau)^a E_6(\tau)^b,$$

where $a, b \geq 0$ and $4a + 6b = k$; this is Theorem 1.23 of [39]. From this, it follows that for $k \in \{4, 6, 8, 10, 14\}$, $\mathcal{M}_k = \mathbb{C} \cdot E_k(\tau)$, and since the constant term for $E_k(\tau)$ is 1 for all $k$, $\mathcal{S}_k = 0$ for the same such $k$.

In fact, the first value of $k$ for which $\mathcal{S}_k$ is nonzero is 12, which brings us to our next example. The Delta-function $\Delta(\tau)$ is a cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$; in fact, $\mathcal{S}_{12} = \mathbb{C} \cdot \Delta(\tau)$. We can define $\Delta(\tau)$ in terms of $E_4$ and $E_6$ by

$$\Delta(\tau) \coloneqq \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots .$$

The coefficients of this cusp form are the values of the Ramanujan $\tau$ function [6].

Using the Delta-function, we define the modular $j$-function

$$j(\tau) \coloneqq \frac{E_4(\tau)^3}{\Delta(\tau)} = q^{-1} + 744 + 196884q + 21493769q^2 + 864299979q^3 + \cdots .$$

This function is holomorphic on $\mathbb{H}$, but it has a simple pole at $\infty$ (as seen by the $q^{-1}$ term in its Fourier expansion) and so it is an example of what is called a weakly holomorphic modular form. A weakly holomorphic modular form for a congruence subgroup $\Gamma$ is a modular form that is allowed to have poles at the cusps at $\Gamma$ but is still holomorphic on $\mathbb{H}$ and satisfies the transformation property from Equation 1.11. For $k$, $N$ positive integers and $\chi$ a Dirichlet character modulo $N$, we let $\mathcal{M}^!_{\frac{k}{2}}(\Gamma_0(N), \chi)$ denote the space of weakly holomorphic modular forms of weight $k$ and character $\chi$ for the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$. (The theory of weakly holomorphic modular forms will appear again in Chapter 2.)

Note that since $E_4(\tau)^3$ and $\Delta(\tau)$ are both weight 12, it follows that the weight of the $j$-function is 0; in fact, every weakly holomorphic modular form of weight 0 is a polynomial in the function $j$. For more details on the $j$-function, including its connection to elliptic curves, see [6, 22].

The Dedekind eta-function (from p. 4)

$$\eta(\tau) \coloneqq q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

provides a building-block for creating modular forms as well as the connection between the generating for $p(n)$ and modular forms. In our definition of modular form (Definition 1.5), we require $k \in \mathbb{Z}$, but we can define a modular form of half-integer weight, i.e. of weight $\frac{k}{2}$ for any $k \in \mathbb{Z}$ the same way as long as we adjust the transformation property from Equation (1.11) appropriately. For more background on modular forms of half-integer weight, see [22, 35, 39].

Using this new notion of a modular form of half-integer weight, we have

$$\eta(24\tau) \in \mathcal{S}_{\frac{1}{2}}(\Gamma_0(576), \chi_{12}),$$

where $\chi_{12}$ is the Legendre symbol

$$\chi_{12} \coloneqq \left(\frac{12}{n}\right) = \begin{cases} 1 & \text{if } n \equiv 1, 11 \pmod{12}, \\ -1 & \text{if } n \equiv 5, 7 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we have that $\eta(\tau)^{24} = \Delta(\tau) \in \mathcal{S}_{12}$. This half-integral weight cusp form is well-understood and appears throughout this dissertation. For example, we know exactly when a quotient of eta-functions is a modular form of a specific weight and level [15, 28, 36, 37], which is used in the proofs of Theorems 2.2 and 2.3 in Chapter 2 to connect a specific partition function to modular forms. Then, in Chapter 3, the transformation properties of $\eta(\tau)$ are given and used to prove Theorem 3.1, a result regarding the invariance of the Andrews-Bressoud series (Equation (3.1)).

The last example of modular forms we give in this introduction are theta-functions. The quintessential example of a theta-function is

$$\theta_0(\tau) \coloneqq \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots,$$

which is in the space $\mathcal{M}_{\frac{1}{2}}(\Gamma_0(4))$. From this first example, we can build families of theta-functions, which are also modular forms of half-integral weight (see [39, 35, 18]). The most famous application of theta-functions is to the representation of integers as the sum of squares; for example, the theta-function

$$\theta(\tau) = \sum_{x,y,z,w \in \mathbb{Z}} q^{x^2+y^2+z^2+w^2} \in \mathcal{M}_2(\Gamma_0(4))$$

is used to prove that every nonnegative integer can be written as the sum of 4 squares (for proof, see Problem 2 of §III.5 of [22] or Theorem 5.33 of [21]). A similar infinite sum appears in the generating function for $c\phi_k(n)$ (Equation (1.7)), which is a main focus of Chapter 2. Then Theorem 3.1 in Chapter 3 expresses the Andrews-Bressoud series (Equation (3.1)) as a quotient involving a variation of theta-functions and uses their transformation properties to prove the invariance of these series.

While we have not developed in detail some of the techniques and tools of modular forms used in Chapters 2 and 3 of this dissertation, this section provides the basic definitions, facts, and examples that are most helpful for our purposes.

The remainder of this dissertation is organized as follows: in Chapter 2 we study congruences of coefficients of weakly holomorphic modular forms. In particular, we view specific partition functions $c\phi_k(n)$ and $\overline{c\phi_2}(n)/4$ as these types of modular forms to study their congruences and parity, respectively. Chapter 3 contains modular invariance results about the Andrews-Bressoud series as well as the modular properties of some Wronskian determinants thereof through the use of modular forms and the series' connection to irreducible characters of certain vertex operator algebras.

# Chapter 2

# Congruences for modular forms and $k$-colored generalized Frobenius partitions

A majority of this chapter appears in the previously published paper [19] (full citation below). All results were completed jointly.

Jameson, M. and Wieczorek, M. (2019). Congruences for modular forms and generalized Frobenius partitions. *The Ramanujan Journal*.

## 2.1 Motivation and statement of results

In this chapter, we discuss the extent to which the congruence properties of $p(n)$ discussed in Section 1.1.1 carry over to the $k$-colored generalized Frobenius partitions $c\phi_k(n)$ defined by Andrews [5].

To begin, we recall four of the most well-known properties for the partition function from Section 1.1. First, we have the Ramanujan congruences (Equation (1.5)), which assert that for every nonnegative integer $n$, we have

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Second, we know from work of Ahlgren and Boylan [1] that the Ramanujan congruences are the only "simple" congruences, i.e., congruences of the form

$$p(\ell n+\beta) \equiv 0 \pmod{\ell},$$

where $\ell$ is prime and $0 \leq \beta < \ell$. Although this may seem to imply that congruences for $p(n)$ are quite rare, the third well-known property of $p(n)$ is that they occur in great abundance, as long as we relax our restrictions on the shape of the arithmetic progression: for any prime $\ell$ coprime to 6 and positive integer $m$, Ahlgren and Ono [2] proved that there exist infinitely many non-nested arithmetic progressions $\{An+B\}$ such that for every positive integer $n$ we have

$$p(An+B) \equiv 0 \pmod{\ell^m}.$$

However, the behavior of $p(n)$ modulo 2 and 3 is quite different, essentially because 2 and 3 are the primes which divide the level of the associated modular form. This leads us to the fourth well-known fact about the partition function, Subbarao's Conjecture, [41]: for any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $p(M)$ is odd, and there are infinitely many integers $N \equiv r \pmod{t}$ for which $p(N)$ is even.

Now, we turn to the case of $k$-colored generalized Frobenius partitions (see Definition 1.3). It turns out that the generating function (Equation 1.7) is essentially a modular form (given explicitly in Section 2.2). In fact, when $k=1$, we obtain the generating function

$$\sum_{n=0}^{\infty} c\phi_1(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum_{n=0}^{\infty} p(n)q^n,$$

i.e., $c\phi_1(n) = p(n)$, and thus we have another reason to view $c\phi_k(n)$ as a generalization of the partition function. For $k = 2$ we have

$$\sum_{n=0}^{\infty} c\phi_2(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{4n-2})}{(1-q^{2n-1})^4(1-q^{4n})}.$$

In his 1984 work, Andrews found several Ramanujan congruences for $c\phi_k(n)$ where $k > 1$. For example, he proved that when $k$ is prime and $k \nmid n$, we have that

$$c\phi_k(n) \equiv 0 \pmod{k^2},$$

giving at least one Ramanujan congruence for each prime $k$. When $k = 2$, we also have an additional congruence, giving two Ramanujan congruences [5]

$$c\phi_2(2n+1) \equiv 0 \pmod{2}$$
$$c\phi_2(5n+3) \equiv 0 \pmod{5}.$$

In this context, we also have that there are only finitely many Ramanujan congruences: Dewar [11] proved that the congruences above are the only ones for $c\phi_2(n)$. While Dewar's theorem does not apply for all $k$, he notes that his approach of using Tate cycles should apply in more generality.

There are also many other congruences for $k$-colored generalized Frobenius partitions which can be found in the literature. In fact, there are infinitely many such congruences. This follows by applying a theorem of Treneer [49], which gives congruences for all weakly holomorphic modular forms.

**Theorem 2.1.** *Let $m$ and $k$ be positive integers and let $\ell$ be prime with $(\ell, 6k) = 1$. Then there exist infinitely many non-nested arithmetic progressions $\{An + B\}$ such that for every positive integer $n$ we have*

$$c\phi_k(An+B) \equiv 0 \pmod{\ell^m}.$$

**Remark.** By combining this theorem with the Chinese Remainder Theorem, we have that for any integer $M$ coprime to $6k$, there are infinitely many congruences of the form $c\phi_k(An+B) \equiv 0 \pmod{M}$.

Finally, it remains to consider whether there are congruences for $c\phi_k(n)$ modulo $\ell$, where $\ell \mid 6k$. A natural first step is to consider $c\phi_2(n) \pmod{2}$. In fact, this turns out to be a particularly interesting case, since we already know that Ramanujan congruences exist modulo 2, but Theorem 2.1 does not guarantee that there are infinitely many (non-Ramanujan) congruences. However, using the fact that $(1-x^2)^2 \equiv (1-x)^4 \pmod{4}$, we have that

$$\begin{aligned} \sum_{n=0}^{\infty} c\phi_2(n) q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{4n-2})}{(1-q^{2n-1})^4(1-q^{4n})} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^4(1-q^{4n})^2} \\ &\equiv \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})} = \sum_{n=0}^{\infty} p(n/2) q^n \pmod{4}. \end{aligned} \tag{2.1}$$

In other words, the parity of $c\phi_2(n)$ is completely dictated by the parity of the partition function, and so we define

$$\overline{c\phi_2}(n) := c\phi_2(n) - p(n/2)$$

and set out to now study the parity of $\frac{\overline{c\phi_2}(n)}{4}$. This function was previously studied by Kolitsch and others [23, 45], and prior results show that the analogue of Subbarao's conjecture is not true. However, we can prove some results in that direction, in the style of Ono [38].

**Theorem 2.2.** *For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $N \equiv r \pmod{t}$ for which $\frac{\overline{c\phi_2}(N)}{4}$ is even.*

**Theorem 2.3.** *For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $\frac{\overline{c\phi_2}(M)}{4}$ is odd, provided there is one such $M$. Furthermore, if there does exist an $M \equiv r \pmod{t}$ for which $\frac{\overline{c\phi_2}(M)}{4}$ is odd, then the smallest such $M$ is less than $C_{r,t}$ where*

$$C_{r,t} := \frac{2^{18+j} \cdot 3^7 t^6}{d^2} \prod_{p \mid 6t} \left(1 - \frac{1}{p^2}\right) - 2^j,$$

*$d := \gcd(12r-1, t)$, and $j$ is an integer satisfying $2^j > \frac{t}{12}$.*

In fact, Theorem 2.2 and 2.3 (as well as their analogues for the partition function, which were proved by Ono [38]) are special cases of a much more general phenomenon, which applies to the coefficients of any weakly holomorphic modular form with algebraic integer coefficients and any arithmetic progression of modulus $t > 1$. We prove these general results along the way (see Theorems 2.7 and 2.8).

This article is organized as follows: in Section 2.2, we outline the connection between $c\phi_k(n)$ and the theory of modular forms and apply results of Treneer [49] to prove Theorem 2.1, which guarantees congruences modulo $\ell$ for $c\phi_k(n)$ provided $(\ell, 6k) = 1$. In Section 2.3, we step back from partition functions to prove results on the parity of coefficients of weakly holomorphic modular forms in the style of Ono [38]. Lastly, in Section 2.4, we see these results applied to a specific partition function, $\overline{c\phi_2}(n)/4$.

## 2.2 Congruences for $c\phi_k(n)$ when $\gcd(\ell, 6k) = 1$

### 2.2.1 Modular forms and the generating function for $c\phi_k(n)$

In order to prove Theorem 2.1, we first need to study the generating function for $c\phi_k(n)$ and relate it to the theory of modular forms. Recall the generating function for $c\phi_k(n)$ [5, Theorem 5.2] as in Equation (1.7) is

$$\sum_{n=0}^{\infty} c\phi_k(n) q^n = \frac{\sum_{\overrightarrow{m} \in \mathbb{Z}^{k-1}} q^{Q(\overrightarrow{m})}}{\prod_{n=1}^{\infty} (1-q^n)^k},$$

where $Q(\overrightarrow{m})$ is given by

$$Q(\overrightarrow{m}) \coloneqq \sum_{i=1}^{k-1} m_i^2 + \sum_{1 \le i < j \le k-1} m_i m_j.$$

Next, we apply the theory of theta-functions to show that this generating function is essentially a modular form. We use the notation for weakly holomorphic modular forms from p. 16. Note also that a similar lemma appears in [17].

**Lemma 2.4.** *Let $k$ be a positive integer and let $Q(\overrightarrow{m})$ be defined as above.*

*(a) If $k$ is odd, then we have that*

$$\sum_{\overrightarrow{m}\in\mathbb{Z}^{k-1}} q^{Q(\overrightarrow{m})} \in \mathcal{M}_{\frac{k-1}{2}}(\Gamma_0(k), \chi_k), \tag{2.2}$$

*where we set $\chi_k(\bullet) := \left(\frac{(-1)^{(k-1)/2}k}{\bullet}\right)$ for $k$ odd.*

*(b) If $k$ is even, then we have that*

$$\sum_{\overrightarrow{m}\in\mathbb{Z}} q^{Q(\overrightarrow{m})} \in \mathcal{M}^!_{\frac{k-1}{2}}(\Gamma_0(2k), \chi_k), \tag{2.3}$$

*where we set $\chi_k(\bullet) := \left(\frac{2k}{\bullet}\right)$ for $k$ even.*

*Proof.* Set $r := k-1$, $A := \mathbf{I}_{k-1} + \mathbf{1}_{k-1}$ (i.e., the sum of the identity matrix and the all-ones matrix), $h := 0, N := k$, $P(x) := 1$, and $v := 0$. Following the notation of Section 4.9 of [35], we have that

$$\theta(\tau; h, A, N, P) = \sum_{\overrightarrow{m}\in\tau^{k-1}} q^{Q(\overrightarrow{m})}.$$

Part (a) follows immediately from part (3) of Corollary 4.9.5 of [35]. Part (b) follows from Theorem 4.9.3 (and the remark which follows it) of [35], since for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2k)$,

$$\theta(\gamma\tau; h, a, N, P) = \left(\frac{2k}{d}\right)\left(\frac{c}{d}\right)^r \varepsilon_d^{-r}(c\tau+d)^{r/2}\theta(\tau; h, A, N, P),$$

where $\varepsilon_d := \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv 3 \pmod 4. \end{cases}$ ∎

### 2.2.2 Treneer's theorems and the proof of Theorem 2.1

Powerful results of Treneer from 2006 [49] establish congruences for coefficients of certain types of modular forms.

**Theorem 2.5** (Theorem 1.1 of [49])**.** *Suppose that $\ell$ is an odd prime, and that $k$ and $m$ are integers with $k$ odd. Let $N$ be a positive integer with $4 \mid N$ and $(N, \ell) = 1$, and let $\chi$ be a*

*Dirichlet character modulo $N$. Let $K$ be an algebraic number field with ring of integers $\mathcal{O}_K$, and suppose $f(\tau) = \sum a(n)q^n \in \mathcal{M}^!_{\frac{k}{2}}(\Gamma_0(N), \chi) \cap \mathcal{O}_K((q))$. If $m$ is sufficiently large, then for each positive integer $j$, a positive proportion of the primes $P \equiv -1 \pmod{N\ell^j}$ have the property that*

$$a(P^3\ell^m n) \equiv 0 \pmod{\ell^j}$$

*for all $n$ coprime to $P\ell$.*

We can now prove Theorem 2.1, which is essentially a corollary of Theorem 2.5.

**Corollary 2.6.** *Let $m$ and $k$ be positive integers and let $\ell$ be prime with $(\ell, 6k) = 1$. Then if $m$ is sufficiently large, for each positive integer $j$, a positive proportion of the primes $P \equiv -1 \pmod{576k\ell^j}$ have the property that*

$$c\phi_k\left(\frac{P^3\ell^m n + k}{24}\right) \equiv 0 \pmod{\ell^j}$$

*for all $n$ coprime to $P\ell$.*

*Proof.* Let $k$ be a positive integer and define

$$f(\tau) := \frac{\sum_{\vec{m}\in\mathbb{Z}^{k-1}} q^{24Q(\vec{m})}}{\eta(24\tau)^k}, \tag{2.4}$$

where $\eta(\tau)$ is the Dedekind eta-function from Equation (1.2.1). Recall that $\eta(24\tau) \in \mathcal{S}_{1/2}(\Gamma_0(576), \chi_{12})$, where $\chi_{12}(\bullet) := \left(\frac{12}{\bullet}\right)$. Thus by Lemma 2.4, we have that $f(\tau)$ is a weakly holomorphic modular form of level $576k$, weight $-\frac{1}{2}$ and character $\chi_k\chi_{12}$, i.e.

$$f(\tau) \in \mathcal{M}^!_{-\frac{1}{2}}(\Gamma_0(576k), \chi_k\chi_{12}).$$

From equations (2.2.1) and (2.4) it follows that

$$f(\tau) = \sum_{n=-k}^{\infty} c\phi_k\left(\frac{n+k}{24}\right) q^n.$$

Now, let $\ell \geq 5$ be a prime such that $\ell \nmid k$ and $m$ be a positive integer. Theorem 2.5 states that if $m$ is sufficiently large, then for each positive integer $j$, a positive proportion of the primes $P \equiv -1 \pmod{576k\ell^j}$ have the property that

$$c\phi_k\left(\frac{P^3\ell^m n + k}{24}\right) \equiv 0 \pmod{\ell^j}$$

for all $n$ coprime to $P\ell$ as desired. □

Note that for each prime $P$ guaranteed by Corollary 2.6, we may let $n$ vary in an appropriate arithmetic progression (i.e., one which guarantees that $P^3\ell^m n \equiv -k \pmod{24}$ and $\gcd(n, P\ell) = 1$) to obtain Theorem 1.

## 2.3 Parity results for coefficients of modular forms

In this section, we will prove the following general results regarding the parity of coefficients of weakly holomorphic modular forms, which we assume to have algebraic integer coefficients.

**Theorem 2.7.** *Let $N_0, \alpha, \beta, t$ be integers with $N_0, \alpha, t$ positive, and let*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in \mathcal{M}_k^!(\Gamma_0(N_0), \chi),$$

*where $c(n)$ are algebraic integers in some number field. For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $c(M)$ is even.*

**Theorem 2.8.** *Let $N_0, \alpha, \beta, t$ be integers with $N_0, \alpha$ positive, and $t > 1$, and let*

$$\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta} \in \mathcal{M}_k^!(\Gamma_0(N_0), \chi),$$

*where $c(n)$ are algebraic integers in some number field. For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $c(M)$ is odd, provided there is one such $M$.*

*Furthermore, if there does exist an* $M \equiv r \pmod{t}$ *for which* $c(M)$ *is odd, then the smallest such* $M$ *is less than* $C_{r,t}$ *for*

$$C_{r,t} := \frac{2^j \cdot 12 + k}{12\alpha} \left[\frac{N\alpha^2 t^2}{d}\right]^2 \prod_{p \mid N\alpha t} \left(1 - \frac{1}{p^2}\right) - 2^j,$$

*where* $N := \operatorname{lcm}(\alpha t, N_0)$*,* $d := \gcd(\alpha r + \beta, t)$*, and* $j$ *is a sufficiently large integer (as in Proposition 2.9).*

### 2.3.1 Proof of Theorem 2.7

First, we adapt the methods of [38] to prove Theorem 2.7.

**Proposition 2.9.** *Let* $N_0, \alpha, \beta, t$ *be integers with* $N_0, \alpha, t$ *positive, and let*

$$\sum_{n=0}^{\infty} c(n) q^{\alpha n + \beta} \in \mathcal{M}_k^!(\Gamma_0(N_0), \chi),$$

*where* $c(n)$ *are algebraic integers in some number field. Then for sufficiently large* $j$*, we have that*

$$f_t(\tau) := \Delta^{2^j}(\alpha t \tau) \sum_{n=0}^{\infty} c(n) q^{\alpha n + \beta}$$

*is a cusp form in* $\mathcal{S}_{12 \cdot 2^j + k}(\Gamma_0(N), \chi)$*, where* $N := \operatorname{lcm}(\alpha t, N_0)$*. Moreover, the Fourier expansion of* $f_t(\tau)$ *modulo 2 can be factored as:*

$$f_t(\tau) \equiv \left(\sum_{n=0}^{\infty} q^{\alpha \cdot 2^j t (2n+1)^2}\right) \left(\sum_{n=0}^{\infty} c(n) q^{\alpha n + \beta}\right) \pmod{2}.$$

*Proof.* Note that $\sum_{n=0}^{\infty} c(n) q^{\alpha n + \beta}$ is a weakly holomorphic modular form of level $N$ and $\Delta^j(\alpha t \tau)$ is a cusp form of level $N$. Thus by choosing $j$ sufficiently large (to ensure vanishing at cusps), it follows that $f_t(\tau) \in \mathcal{S}_{12 \cdot 2^j + k}(\Gamma_0(N), \chi)$. This proves the first statement of the theorem.

The second statement of the theorem follows from the well-known fact that (see, for example, [38])

$$\Delta(\tau) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod{2},$$

together with substitution and the Freshman Binomial Theorem. □

By combining Proposition 2.9 with the following result of Serre [46] concerning the divisibility of coefficients of modular forms, we will be able to prove Theorem 2.7.

**Corollary 2.10** (Serre [46]). *Let $f(\tau)$ be a holomorphic modular form of positive integer weight $k$ on some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n,$$

*where a(n) are algebraic integers in some number field. If $m$ is a positive integer, then*

$$a(n) \equiv 0 \pmod{m}$$

*for almost all $n$ in any given fixed arithmetic progression $r \pmod{t}$.*

*Proof of Theorem 2.7.* By Proposition 2.9, we have a cusp form $f_t(\tau) = \sum_{n=0}^{\infty} a_t(n) q^{\alpha n+\beta}$ such that

$$\sum_{n=0}^{\infty} a_t(n) q^{\alpha n+\beta} \equiv \left( \sum_{n=0}^{\infty} q^{\alpha \cdot 2^j t(2n+1)^2} \right) \left( \sum_{n=0}^{\infty} c(n) q^{\alpha n+\beta} \right) \pmod{2}.$$

Thus we have

$$\sum_{n=0}^{\infty} a_t(n) q^n \equiv \left( \sum_{n=0}^{\infty} q^{2^j t(2n+1)^2} \right) \left( \sum_{n=0}^{\infty} c(n) q^n \right) \pmod{2} \qquad (2.5)$$

and, by Corollary 2.10, almost all of the coefficients $a_t(n)$ are even.

Now, assume for contradiction that there are finitely many integers $M \equiv r \pmod{t}$ for which $c(M)$ is even, i.e., that there exists some $n_0$ such that $c(M)$ is odd for all $M \geq n_0$ with $M \equiv r \pmod{t}$.

For $n \geq n_0$ with $n \equiv r \pmod{t}$ and $\kappa \equiv 1 \pmod{4}$ such that $\kappa > \frac{n}{2^{j+2}t} - 1$, we now compare the coefficient of $q^{2^j t\kappa^2+n}$ on each side of equation (2.5) to obtain

$$a_t(2^j t\kappa^2 + n) \equiv \sum_{i\geq 1 \text{ odd}} c(2^j t(\kappa^2 - i^2) + n) \pmod{2}. \tag{2.6}$$

In order to simplify the right side of equation (2.6), note that for odd $i \leq \kappa$, we have that $2^j t(\kappa^2 - i^2) + n \geq n \geq n_0$ and $2^j t(\kappa^2 - i^2) + n \equiv n \equiv r \pmod{t}$, so our assumption above guarantees that the summand $c(2^j t(\kappa^2 - i^2) + n)$ is odd. On the other hand, for odd $i > \kappa$, we use the fact that $\kappa + 1 > \frac{n}{2^{j+2}t}$ to obtain

$$2^j t(\kappa^2 - i^2) + n \leq 2^j t(\kappa^2 - (\kappa+2)^2) + n = -2^{j+2}t(\kappa+1) + n < 0,$$

and thus $c(2^j t(\kappa^2 - i^2) + n) = 0$. Hence, for such $n, \kappa$, equation (2.6) can be rewritten as

$$a_t(2^j t\kappa^2 + n) \equiv \sum_{1\leq i\leq \kappa \text{ odd}} 1 = \frac{\kappa+1}{2} \equiv 1 \pmod{2}.$$

Thus for $\kappa$ sufficiently large such that $\kappa \equiv 1 \pmod{4}$, the above argument guarantees that for all $M \equiv r \pmod{t}$ in the interval

$$\left[2^j t\kappa^2 + n_0, 2^j t(\kappa+2)^2 + r - t\right],$$

$a_t(M)$ is odd. Note that for distinct values of such $\kappa$, the above intervals are disjoint and the number of such $M$'s in each associated interval is $2^{j+2}(\kappa+1) + \frac{r-n_0}{t}$, where we are assuming without loss of generality that $n_0 \equiv r \pmod{t}$.

Taking into account these intervals for all such values of $\kappa$, we see that if there are only finitely many positive integers $M$ for which $a_t(M)$ is even, then a positive proportion of all $M \equiv r \pmod{t}$ have $a_t(M)$ odd, contradicting Corollary 2.10. $\square$

### 2.3.2 Proof of Theorem 2.8

Now, we prove Theorem 2.8 using similar methods to those found in [38]. First, we state the following lemmas.

**Lemma 2.11** (Lemma 2 of [38]). *Let $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ be a modular form in $\mathcal{M}_k(\Gamma_0(N), \chi)$ and let $d := \gcd(r,t)$. If $0 \le r < t$, then*

$$f_{r,t}(\tau) = \sum_{n \equiv r \pmod{t}} a(n)q^n$$

*is the Fourier expansion of a modular form in $\mathcal{M}_k(\Gamma_1(Nt^2/d))$.*

**Lemma 2.12** (Lemma 1 of [38]). *Let $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ where the coefficients $a(n)$ are algebraic integers in some number field. Let $s$ and $w$ be positive integers and $b_1, b_2, \ldots, b_s$ distinct non-zero integers. If $m$ is a positive integer and*

$$f(\tau) \equiv \sum_{1 \le i \le s} \sum_{n=0}^{\infty} a_i(n) q^{w(2n+1)^2 + b_i} \pmod{m}$$

*where $a_i(n) \not\equiv 0 \pmod{m}$ for all $n \ge 0$, then $f(\tau)$ is not in $\mathcal{M}_k(\Gamma_1(N))$ for any pair of positive integers $k$ and $N$.*

Finally, we need the following well-known theorem of Sturm. Here, for a $q$-series $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ and positive integer $m$, we let $\mathrm{Ord}_m(f)$ denote the smallest integer $n$ such that $a(n) \not\equiv 0 \pmod{m}$, and if no such $n$ exists, we say that $\mathrm{Ord}_m(f) = \infty$.

**Lemma 2.13** ([47]). *Let $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{M}_k(\Gamma_1(N))$ for some positive integer $N$ with algebraic integer Fourier coefficients from a fixed number field. If $m$ is a positive integer and*

$$\mathrm{Ord}_m(f) > \frac{k}{12} N^2 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right),$$

*then $\mathrm{Ord}_m(f) = \infty$, i.e. $a(n) \equiv 0 \pmod{m}$ for all $n$.*

*Proof of Theorem 2.8.* By Proposition 2.9, we have a cusp form $f_t(\tau) = \sum_{n=0}^{\infty} a_t(n)q^{\alpha n+\beta}$ such that

$$\sum_{n=0}^{\infty} a_t(n)q^{\alpha n+\beta} \equiv \left(\sum_{n=0}^{\infty} q^{2^j\alpha t(2n+1)^2}\right)\left(\sum_{n=0}^{\infty} c(n)q^{\alpha n+\beta}\right) \pmod 2.$$

Then by Lemma 2.11, we have that

$$f_{\alpha r+\beta,\alpha t}(\tau) := \sum_{\alpha n+\beta\equiv\alpha r+\beta \pmod{\alpha t}} a_t(n)q^{\alpha n+\beta} \in \mathcal{S}_{2^j\cdot 12+k}\left(\Gamma_1\left(\frac{N\alpha^2t^2}{d}\right)\right)$$

and, moreover,

$$f_{\alpha r+\beta,\alpha t}(\tau) = \sum_{n\equiv r \pmod t} a_t(n)q^{\alpha n+\beta} \equiv \left(\sum_{n=0}^{\infty} q^{2^j\alpha t(2n+1)^2}\right)\left(\sum_{n\equiv r \pmod t} c(n)q^{\alpha n+\beta}\right) \pmod 2.$$

If $c(M)$ is odd for at least one $M \equiv r \pmod t$ but for only finitely many, then this factorization modulo 2 contradicts Lemma 2.12. This proves the first statement of the theorem.

To prove the second statement of the theorem, suppose that $c(M)$ is even for all $M \equiv r \pmod t$ where $0 \leq M \leq C_{r,t}$. It follows that $c(M)$ is even for all $M \equiv r \pmod t$ where

$$r \leq M \leq \frac{2^j\cdot 12+k}{12\alpha}\left[\frac{N\alpha^2t^2}{d}\right]^2 \prod_{p\mid N\alpha t}\left(1-\frac{1}{p^2}\right) - 2^jt + r.$$

Thus the first odd term of the factor $\sum_{n\equiv r \pmod t} c(n)q^{\alpha n+\beta}$ has exponent at least

$$\frac{2^j\cdot 12+k}{12}\left[\frac{N\alpha^2t^2}{d}\right]^2 \prod_{p\mid N\alpha t}\left(1-\frac{1}{p^2}\right) - 2^j\alpha t + \alpha(t+r) + \beta.$$

Noting that the first odd term of the factor $\sum_{n=0}^{\infty} q^{2^j\alpha t(2n+1)^2}$ has exponent $2^j\alpha t$, we find that

$$\begin{aligned}\mathrm{Ord}_2(f_{\alpha r+\beta,\alpha t}(\tau)) &\geq \frac{2^j\cdot 12+k}{12}\left[\frac{N\alpha^2t^2}{d}\right]^2\prod_{p\mid N\alpha t}\left(1-\frac{1}{p^2}\right)+\alpha(t+r)+\beta\\ &> \frac{2^j\cdot 12+k}{12}\left[\frac{N\alpha^2t^2}{d}\right]^2\prod_{p\mid N\alpha^2t^2/d}\left(1-\frac{1}{p^2}\right)\end{aligned}$$

By Lemma 2.13, this implies that $f_{\alpha r+\beta,\alpha t}(\tau)\equiv 0 \pmod 2$, and thus $c(M)$ is even for all $M\equiv r \pmod t$. □

## 2.4 The parity of $\frac{\overline{c\phi_2}(n)}{4}$

Recall that we defined $\overline{c\phi_2}(n) := c\phi_2(n) - p(n)$. In fact, this is a special case of a function $\overline{c\phi_k}(n)$, which was defined combinatorially by Kolitsch [23, 24], who also found congruences for this function. For example, he proved the following generalization of Equation (2.1):

$$\overline{c\phi_k}(n)\equiv 0 \pmod{k^2}.$$

This was strengthened by Sellers [45], who proved congruences modulo higher powers of 2 and 3. For instance, we have that

$$\overline{c\phi_2}(2n)\equiv 0 \pmod 8.$$

This fact as well as results of Cui et al. [10] provide counterexamples to the analogue of Subbarao's conjecture for $\overline{c\phi_2}(n)/4$. However, in this section, we use Theorems 2.7 and 2.8 to prove Theorems 2.2 and 2.3, which are analogous to theorems of Ono that give strong results on the parity of $p(n)$.

*Proof of Theorems 2.2 and 2.3.* In [45], Sellers proved that the generating function for $\overline{c\phi_2}(n)$ is

$$\sum_{n=0}^{\infty}\overline{c\phi_2}(n)q^n = 4q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{(1-q^n)^2(1-q^{8n})}.$$

Using the fact that $\frac{\eta(\tau)^2}{\eta(2\tau)} \equiv 1 \pmod 2$, we then have

$$\sum_{n=0}^{\infty} \frac{\overline{c\phi_2}(n)}{4} q^{12n-1} = \frac{\eta(192\tau)^2}{\eta(12\tau)^2\eta(96\tau)} \equiv \frac{\eta(192\tau)^2\eta(\tau)^2}{\eta(12\tau)^2\eta(96\tau)\eta(2\tau)} \pmod 2.$$

Using standard results regarding the modularity properties of eta-quotients [15, 36, 37], one can check that $\frac{\eta(192\tau)^2\eta(\tau)^2}{\eta(12\tau)^2\eta(96\tau)\eta(2\tau)} \in \mathcal{M}_0^!(\Gamma_0(576), (\frac{12}{\bullet}))$.

Then by Proposition 2.9 (together with standard results on the order of vanishing of eta-quotients at cusps [28]) it follows that for any integer $j$ with $2^j > \frac{t}{12}$ we have that

$$f_t(\tau) := \frac{\eta(192\tau)^2\eta(\tau)^2}{\eta(12\tau)^2\eta(96\tau)\eta(2\tau)} \Delta^{2^j}(12t\tau) \in \mathcal{S}_{12\cdot 2^j}\left(\Gamma_0(576t), \left(\frac{12t^2}{\bullet}\right)\right).$$

Finally, our desired results follow immediately by applying Theorems 2.7 and 2.8 to $\sum \frac{\overline{c\phi_2}(n)}{4} q^{12n-1}$. □

## 2.5 Future directions

In the process of studying congruences for generalized Frobenius partitions, we used the computer algebra system Sage to investigate values of $\overline{c\phi_2}(n)/4$ in specific arithmetic progressions. For some of these progressions, the code calculated the parity of millions of coefficients of the generating functions without finding a single odd value. However, these computations alone are not enough to prove or disprove the existence of an odd coefficient given the size of the bound $C_{r,t}$ from Theorem 2.3. It remains open to explore these progressions further whether computationally or by reducing the bound $C_{r,t}$ to classify these progressions in terms of Theorems 2.2 and 2.3.

Furthermore, this project only explored the values of $k$ for which $\gcd(6k, \ell) = 1$ and $k = 2$. At this time, the literature does not provide results regarding $c\phi_k(n)$ modulo 3 or, for $k \geq 3$, $c\phi_k(n)$ modulo $k$.

# Chapter 3

# Andrews-Bressoud series and Wronskians

Most of this chapter appears in the arXiv preprint of the paper [50] (full citation below).
Wieczorek, M. (2020). Andrews-Bressoud series and Wronskians.

## 3.1 Motivation and statement of results

Recall that the generalization of the Rogers-Ramanujan identities (Equations (1.8) and (1.9)) to all odd moduli is given by the Andrews-Gordon identities. In their 2008 paper [34], Milas, Mortenson, and Ono examine the modularity of the associate $q$-series and the Wronskian determinants thereof. Similarly, the Rogers-Ramanujan identities can be generalized to all even moduli through the Andrews-Bressoud series as given in Equation (3.1):

$$\sum_{n\geq 0} b_{k,j}(n)q^n = \frac{\prod_{m\geq 1}(1-q^{2km})(1-q^{2km-k-j+1})(1-q^{2km-k+j-1})}{\prod_{m\geq 1}(1-q^m)},$$

where $2 \leq k \in \mathbb{Z}$ and $j \in \{1, \ldots, k\}$. This chapter seeks to prove results on the Andrews-Bressoud series similar to some of those found in [34] about the Andrews-Gordon series.

First, we must note that the product side of the Andrews-Bressoud identities (Equation (3.1)) resembles a generalized eta-quotient apart from a power of $q$. To this end, we define

$$a_{k,j} \coloneqq \frac{6j^2 - 12j - k + 6}{24k}$$

and consider the functions

$$B_{k,j}(q) \coloneqq q^{a_{k,j}} \sum_{n \geq 0} b_{k,j}(n) q^n = q^{a_{k,j}} \prod_{m \geq 1} \frac{(1 - q^{2km})(1 - q^{2km-k-j+1})(1 - q^{2km-k+j-1})}{(1 - q^m)}. \quad (3.1)$$

Then each $B_{k,j}$ is a generalized eta-quotient [43, 29]. For the remainder of this chapter, the term "Andrews-Bressoud series" refers to the functions $B_{k,j}$.

The modularity of the Rogers-Ramanujan series is well-known; in fact, $q^{-\frac{1}{60}}G(q)$ and $q^{\frac{11}{60}}H(q)$ are modular forms of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ [3]. In the generalized case of the Andrews-Gordon identities, the series correspond to irreducible characters of rational vertex operator algebras (see [31, 30, 34]), which follows from the fact that the space generated by these series are invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ [52]. The Andrews-Bressoud series do not correspond to such objects and are not invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$; however, in Section 3.3 we prove that they still possess modularity properties as follows:

**Theorem 3.1.** *For a fixed $k \geq 2$, the space generated by $B_{1,k}(q), \ldots, B_{k,k}(q)$ is $\Gamma_0(2)$-invariant.*

We explore the modularity properties of the Andrews-Bressoud series further by taking Wronskian determinants of the series. We define the Wronskian determinant of $q$-series $f_1, \ldots, f_k$ to be

$$W(f_1, \ldots, f_k) \coloneqq \begin{vmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \cdots & f_k^{(k-1)} \end{vmatrix}, \quad (3.2)$$

where differentiation is defined as

$$\left( \sum a(n) q^n \right)' \coloneqq \sum n a(n) q^n,$$

which is the same as $\frac{1}{2\pi i} \cdot \frac{d}{d\tau}$ when $q := e^{2\pi i\tau}$. Furthermore, we define two Wronskian determinants specific to the Andrews-Bressoud series

$$\mathcal{W}_k(q) := \alpha(k) \cdot \begin{vmatrix} B_{1,k} & B_{2,k} & \cdots & B_{k,k} \\ B'_{1,k} & B'_{2,k} & \cdots & B'_{k,k} \\ \vdots & \vdots & \ddots & \vdots \\ B^{(k-1)}_{1,k} & B^{(k-1)}_{2,k} & \cdots & B^{(k-1)}_{k,k} \end{vmatrix} \tag{3.3}$$

and

$$\widetilde{\mathcal{W}}_k(q) := \beta(k) \cdot \begin{vmatrix} B'_{1,k} & B'_{2,k} & \cdots & B'_{k,k} \\ B''_{1,k} & B''_{2,k} & \cdots & B''_{k,k} \\ \vdots & \vdots & \ddots & \vdots \\ B^{(k)}_{1,k} & B^{(k)}_{2,k} & \cdots & B^{(k)}_{k,k} \end{vmatrix} \tag{3.4}$$

as well as the function

$$\mathcal{F}_k(\tau) := \frac{\widetilde{\mathcal{W}}_k(q)}{\mathcal{W}_k(q)}. \tag{3.5}$$

Here, $\alpha(k)$ and $\beta(k)$ are chosen so that the $q$-expansions of $\mathcal{W}_k(q)$ and $\widetilde{\mathcal{W}}_k(q)$, respectively, have leading coefficient 1.

In [32], Milas studies the irreducible characters of certain vertex operator algebras and gives results regarding the Wronskians of these $q$-series. In particular, he expresses the Wronskians of these characters as certain eta-quotients. The Andrews-Bressoud series are factors of these irreducible characters (see Lemma 3.4), and we can then use Milas's results to write the Wronskian $\mathcal{W}_k$ as an eta-quotient as well.

**Theorem 3.2.** *The Wronskian $\mathcal{W}_k$ formed from the Andrews-Bressoud series is*

$$\mathcal{W}_k(q) = \frac{\eta(\tau)^{2k^2-1}}{\eta(2\tau)^{2k-1}},$$

*where $\eta(\tau)$ is the Dedekind eta function (see Equation (1.2.1)).*

Then, combining this result about $\mathcal{W}_k$, facts about Wronskians and differential equations, and Theorem 3.1, we find that the modularity properties of the function $\mathcal{F}_k$ are "nicer" than that of the original Andrews-Bressoud series $B_{k,j}$ in that they are modular forms. The following result parallels Theorem 1.2 of Milas, Mortenson, and Ono about the Andrews-Gordon series [34].

**Theorem 3.3.** *The function $\mathcal{F}_k(q)$ is a modular form of weight $2k$ for $\Gamma_0(2)$.*

This chapter is organized as follows: in Section 3.2 we connect the theory of vertex operator algebras with the Andrews-Bressoud series and provide the proof of Theorem 3.2, and Section 3.3 contains results regarding the differential equation satisfied by the Andrews-Bressoud series as well as the proofs of Theorems 3.1 and 3.3.

## 3.2 Connections to vertex operator algebras and $\mathcal{W}_k$

### 3.2.1 Connections to vertex operator algebras

The Rogers-Ramanujan series as well as its generalizations are connected to the area of vertex operator algebras (see [25, 26, 27, 7, 30]). The Andrews-Bressoud series are the irreducible characters of a specific vertex operator algebra (see Lemma 3.4). In order to see this, we first explore the work of Milas [32]. In Lemma 7.1 of this paper he proves that the irreducible characters of the $N = 1$ superconformal minimal model in the Ramond sector can be written as (correcting a minor typographical error and setting $j := k' + 1$)

$$\mathrm{ch}_{k,j}(q) := q^{\frac{(j-1)^2}{4k}} \cdot \frac{(-q)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{kn^2+(j-1)n}, \tag{3.6}$$

where $j \in \{1, \ldots, k\}$ and $(a)_\infty := \prod_{n\geq 0}(1 - aq^n)$.

**Lemma 3.4.** *The irreducible characters in equation (3.6) can be written as $\mathrm{ch}_{k,j}(q) = \frac{\eta(2\tau)}{\eta(\tau)} \cdot B_{k,j}(q)$ for even $k \geq 2$ and $j \in \{1, \ldots, k\}$.*

*Proof.* First note that $\frac{(j-1)^2}{4k} = a_{k,j} + \frac{1}{24}$. Recall Jacobi's Triple Product Identity (see, for example, Theorem 2.8 of [4])

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = \prod_{n\geq 0}(1 - q^{2n+1})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}).$$

Thus

$$\begin{aligned}\frac{\eta(2\tau)}{\eta(\tau)} \cdot B_{k,j}(q) &= q^{\frac{1}{24}} \cdot (-q)_\infty \cdot q^{a_{k,j}} \prod_{m\geq 1} \frac{(1-q^{2km})(1-q^{2km-k-j+1})(1-q^{2km-k+j-1})}{(1-q^m)} \\ &= q^{\frac{(j-1)^2}{4k}} \cdot \frac{(-q)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-q^{j-1})^n (q^k)^{n^2} \\ &= q^{\frac{(j-1)^2}{4k}} \cdot \frac{(-q)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-1)^n q^{kn^2+(j-1)n} \\ &= \mathrm{ch}_{k,j}(q),\end{aligned}$$

as desired. □

### 3.2.2 Andrews-Bressoud series and differential equations

From Theorem 5.1 (ii) in [32], the irreducible characters of the $N = 1$ superconformal minimal model in the Ramond sector discussed above form a fundamental system of solutions for the homogeneous linear $k^{\text{th}}$-order differential equation

$$\left(q\frac{d}{dq}\right)^k y - \{2k(k-1)E_2(\tau) + (-k+1)E_{2,1}(\tau)\}\left(q\frac{d}{dq}\right)^{k-1} y + \cdots + F_0(\tau)y = 0, \quad (3.7)$$

where $F_n \in \mathbb{Q}[E_{2\ell}, E_{2\ell,1}]$, the Eisenstein series $E_{2\ell}(\tau)$ and $E_{2\ell,1}(\tau)$ are defined via

$$\begin{aligned} E_{2\ell}(\tau) &:= \frac{B_{2k}}{4k} - \sum_{n\geq 1} \frac{n^{2k-1}q^n}{1-q^n} \\ E_{2\ell,1}(\tau) &:= \frac{B_{2k}}{4k} + \sum_{n\geq 1} \frac{n^{2k-1}q^n}{1+q^n}, \end{aligned}$$

and $B_{2k}$ are the Bernoulli numbers. In the proof of Proposition 3.5 we use the fact that $E_{2,1}(\tau)$ is the logarithmic derivative of $\mathfrak{f}_1(\tau)$, where $\mathfrak{f}_1$ is the Weber modular function defined by

$$\mathfrak{f}_1(\tau) := \frac{\eta(2\tau)}{\eta(\tau)}.$$

Furthermore, it is worth noting that $E_2(\tau)$ is a quasi-modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$, and $E_{2,1}(\tau)$ is a modular form of weight 2 for $\Gamma_0(2)$.

Using Proposition 3.4 and the technique of Lemma 6.2 in [31], we manipulate this differential equation into one with a fundamental system given by the functions $B_{k,j}(q)$.

**Proposition 3.5.** *After the substitution* $\tilde{y}(\tau) = \frac{y(\tau)}{\mathfrak{f}_1(\tau)}$*, the homogeneous differential equation* (3.7) *becomes*

$$\left(q\frac{d}{dq}\right)^k \tilde{y} - \{2k(k-1)E_2(\tau) + (-2k+1)E_{2,1}(\tau)\}\left(q\frac{d}{dq}\right)^{k-1}\tilde{y} + \cdots + P_0(\tau)\tilde{y} = 0, \quad (3.8)$$

*which has a fundamental system of solutions formed by* $B_{k,j}(q)$ *for* $j \in \{1,\ldots,k\}$*. where* $P_n(\tau)$ *are quasi-modular forms for* $\Gamma_0(2)$*.*

*Proof.* We know from Lemma 3.4 that each of these irreducible characters $\mathrm{ch}_{k,j}(q)$ can be written as $\mathfrak{f}_1(\tau)\cdot B_{k,j}(q)$ and so we make the substitution $\tilde{y}(\tau) = \frac{y(\tau)}{\mathfrak{f}_1(\tau)}$ and use the differential equation (3.7) as our starting point. Taking the logarithmic derivative of $\tilde{y}(\tau)$, we find

$$\begin{aligned}\left(q\frac{d}{dq}\right)\tilde{y}(\tau) &= \tilde{y}(\tau)\left[\frac{\left(q\frac{d}{dq}\right)y(\tau)}{y(\tau)} - E_{2,1}(\tau)\right] \\ &= \frac{1}{\mathfrak{f}_1(\tau)}\left(q\frac{d}{dq}\right)y(\tau) - \tilde{y}(\tau)E_{2,1}(\tau),\end{aligned}$$

which becomes

$$\frac{1}{\mathfrak{f}_1(\tau)}\left(q\frac{d}{dq}\right)y(\tau) = \left[\left(q\frac{d}{dq}\right) + E_{2,1}(\tau)\right]\tilde{y}(\tau).$$

Then, by induction on $r$,

$$\frac{1}{\mathfrak{f}_1(\tau)}\left(q\frac{d}{dq}\right)^r y(\tau) = \left[\left(q\frac{d}{dq}\right) + E_{2,1}(\tau)\right]^r \tilde{y}(\tau).$$

If we now apply the Leibniz rule (i.e., generalized product rule), we find

$$\frac{1}{\mathfrak{f}_1(\tau)}\left(q\frac{d}{dq}\right)^r y(\tau) = \left(q\frac{d}{dq}\right)^r \tilde{y}(\tau) + rE_{2,1}(\tau)\left(q\frac{d}{dq}\right)^{r-1}\tilde{y}(\tau) + \cdots, \quad (3.9)$$

where the dots denote terms with lower order derivatives of $\tilde{y}(\tau)$. The proof now follows after we multiply (3.7) by $\frac{1}{\mathfrak{f}_1(\tau)}$ and apply (3.9) for $r = 1,\ldots,k$. □

### 3.2.3 Proof of Main Theorem 3.2

By Theorem 0.3 of [32], the Wronskian formed using $\mathrm{ch}_{k,j}$ is

$$W(\mathrm{ch}_{k,1}(q), \ldots, \mathrm{ch}_{k,k}(q)) = \frac{\eta(\tau)^{2k(k-1)}}{\mathfrak{f}_1(\tau)^{k-1}}.$$

Again from Lemma 3.4, we know that $\mathrm{ch}_{k,j}(q) = \frac{\eta(2\tau)}{\eta(\tau)} \cdot B_{k,j}(q)$. Then, using the well-known fact that

$$W(f \cdot f_1, \ldots, f \cdot f_k) = f^k \cdot W(f_1, \ldots, f_k),$$

it follows that

$$\begin{aligned}\frac{\eta(\tau)^{2k(k-1)}}{\mathfrak{f}_1(\tau)^{k-1}} &= W\big(\mathfrak{f}_1(\tau) \cdot B_{k,1}(q), \ldots, \mathfrak{f}_1(\tau) \cdot B_{k,k}(q)\big) \\ &= \mathfrak{f}_1(\tau)^k \cdot \mathcal{W}_k(q).\end{aligned}$$

Therefore,

$$\mathcal{W}_k(q) = \frac{\eta(\tau)^{2k^2-1}}{\eta(2\tau)^{2k-1}},$$

as desired.

## 3.3 Modularity of $\mathcal{F}_k$

In order to prove Main Theorem 3.3, we first need to prove Theorem 3.1. The proof of Theorem 3.1 requires that we view the Andrews-Bressoud series $B_{k,j}(q)$ as a linear combination of theta-functions divided by $\eta(\tau)$. We recall a particular definition from Chapter 10 of [18]: for $r = 1$, the symmetric, positive definite matrix $A = (16k)$, the spherical function $P(m) = 1$, and $N = 16k$, we define the congruent theta-functions for $h \in \mathbb{Z}$,

$$\Theta(\tau; h) := \sum_{m \equiv h \pmod{16k}} q^{\frac{m^2}{32k}}, \tag{3.10}$$

in order to rewrite our $B_{k,j}(q)$.

We also need the transformation properties of $\Theta(\tau;h)$ and $\eta(\tau)$ when we apply the generators of $\Gamma_0(2)$. Note that $\Gamma_0(2)$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad M = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = -ST^2S, \tag{3.11}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then Propositions 10.3 and 10.4 of [18] give the following transformation properties for our theta-functions:

$$\Theta(T\tau;h) = q^{\frac{h^2}{32k}} \cdot \Theta(\tau;h) \tag{3.12}$$

$$\Theta(M\tau;h) = \frac{\sqrt{1-2\tau}}{8k} \sum_{\ell\in\mathcal{H}} \left( \sum_{h'\in\mathcal{H}} e\left( \frac{h'(h'+h+\ell)}{8k} \right) \right) \Theta(\tau;\ell). \tag{3.13}$$

Then we use the transformation properties of $\eta(\tau)$ from [6] to obtain

$$\eta(T\tau) = q^{\frac{1}{24}} \cdot \eta(\tau) \tag{3.14}$$

$$\eta(M\tau) = \frac{\sqrt{1-2\tau}}{q^{\frac{1}{12}}} \cdot \eta(\tau). \tag{3.15}$$

*Proof of Theorem 3.1.* First we give an alternate representation of the character $\mathrm{ch}_{k,j}$ from Proposition 2.1 of [32],

$$\mathrm{ch}_{k,j}(q) = q^{\frac{(j-1)^2}{4k}} \cdot \frac{(-q)_\infty}{(q)_\infty} \cdot \sum_{n\in\mathbb{Z}} \left( q^{2n(2kn+j-1)} - q^{(2n+1)(2kn+k-j+1)} \right).$$

Then using this, we rewrite the $B_{k,j}$ in terms of the theta-functions from Equation (3.10) and the Dedekind eta-function:

$$B_{k,j}(q) = \frac{\Theta(\tau;2(j-1)) - \Theta(\tau;4k-2(j-1))}{\eta(\tau)}.$$

Using the transformation properties for $\eta(\tau)$ and $\Theta(\tau;h)$ in Equations (3.12) - (3.15), we find that

$$B_{k,j}(Tq) = \gamma(k,j) \cdot B_{k,j}(q),$$

where $\gamma(k,j) \in \mathbb{C}$ constant, and

$$B_{k,j}(Mq) = \sum_{j=1}^{k} \delta(k,j) \cdot B_{k,j}(q),$$

where $\delta(k,j) \in \mathbb{C}$ constant. □

*Proof of Main Theorem 3.3.* First, well-known facts about Wronskians give that $\mathcal{F}_k(q)$ is the constant term of (3.8), which implies that $\mathcal{F}_k(q)$ is holomorphic on the upper half-plane and at the cusps of $\Gamma_0(2)$. Thus we need only show the transformation property holds.

Consider $A$ to be the linear transformation matrix given by the action of $T$ on the vector space spanned by $\{B_{k,j}\}$; we know this linear transformation is invertible so $\det(A) \neq 0$. Then using Lemma 1.3 (a) of [33], we have

$$\alpha(k) \cdot W(A \cdot B_{k,1}(q), \ldots, A \cdot B_{k,k}(q)) = \det(A) \cdot \mathcal{W}_k(q),$$

and

$$\beta(k) \cdot W(A \cdot B'_{k,1}(q), \ldots, A \cdot B'_{k,k}(q)) = \det(A) \cdot \widetilde{\mathcal{W}}_k(q),$$

which implies

$$\mathcal{F}_k(Tq) = \frac{\det(A) \cdot \widetilde{\mathcal{W}}_k(q)}{\det(A) \cdot \mathcal{W}_k(q)} = \mathcal{F}_k(q).$$

Now, if $C$ is the linear transformation matrix given by the action of $M$ on the vector space spanned by $\{B_{k,j}\}$ as described in Lemma 3.1, $\det(C) \neq 0$ as the linear transformation is invertible. Again, using Lemma 1.3 (a) of [33],

$$\alpha(k) \cdot W(C \cdot B_{k,1}(q), \ldots, C \cdot B_{k,k}(q)) = \det(C) \cdot \mathcal{W}_k(q),$$

and

$$\beta(k) \cdot W(C \cdot B'_{k,1}(q), \ldots, C \cdot B'_{k,k}(q)) = (-2\tau + 1)^{2k} \cdot \det(C) \cdot \widetilde{\mathcal{W}}_k(q).$$

Thus

$$\mathcal{F}_k(Mq) = \frac{(-2\tau + 1)^{2k} \cdot \det(C) \cdot \widetilde{\mathcal{W}}_k(q)}{\det(C) \cdot \mathcal{W}_k(q)} = (-2\tau + 1)^{2k} \cdot \mathcal{F}_k(q).$$

Therefore, $\mathcal{F}_k(q)$ is a modular form of weight $2k$ for $\Gamma_0(2)$. □

## 3.4 Future directions

While this chapter contains many results for the Andrews-Bressoud series parallel to those of the Andrews-Gordon series in [34], Milas, Mortenson, and Ono were able to characterize when the quotient of Wronskians for the Andrews-Gordon series is identically zero. This has not yet been determined for the Andrews-Bressoud series.

Unlike the proof of Theorem 3.3, in the proof of Theorem 1.2 of [34], the authors rely more heavily upon techniques from the theory of vertex operator algebras and manipulation of the differential equation from the corresponding Wronskian determinant (their Theorem 2.1). The proof presented here for Theorem 3.3 contains methods involving modular forms instead, but it is believed that it is also possible to prove this result in the same style as Milas, Mortenson, and Ono.

In addition, there exist other generalizations of the Rogers-Ramanujan identities, including the Göllnitz-Gordon-Andrews identities. While it is believed that these identities/series and Wronskians thereof also behave similarly to the Andrews-Gordon and Andrews-Bressoud series, proofs of such conjectures do not yet exist. In this particular case, the connection between the theory of vertex operator algebras and these $q$-series has not yet been explicitly reported in the literature, and so we believe that in order to prove results about the Göllnitz-Gordon-Andrews series, we must either prove this connection or find different methods to approach these problems.

# Bibliography

[1] Ahlgren, S. and Boylan, M. (2003). Arithmetic properties of the partition function. *Invent. Math.*, 153(3):487–502.

[2] Ahlgren, S. and Ono, K. (2001). Congruence properties for the partition function. *Proc. Natl. Acad. Sci. USA*, 98(23):12882–12884.

[3] Andrews, G. E. (1974). An analytic generalization of the Rogers-Ramanujan identities for odd moduli. *Proc. Nat. Acad. Sci. U.S.A.*, 71:4082–4085.

[4] Andrews, G. E. (1976). *The theory of partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam. Encyclopedia of Mathematics and its Applications, Vol. 2.

[5] Andrews, G. E. (1984). Generalized Frobenius partitions. *Mem. Amer. Math. Soc.*, 49(301):iv+44.

[6] Apostol, T. M. (1990). *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

[7] Berkovich, A. and McCoy, B. M. (1997). Generalizations of the Andrews-Bressoud identities for the $N = 1$ superconformal model $\mathrm{SM}(2, 4\nu)$. *Math. Comput. Modelling*, 26(8-10):37–49. Combinatorics and physics (Marseilles, 1995).

[8] Bressoud, D. M. (1979). A generalization of the Rogers-Ramanujan identities for all moduli. *J. Combin. Theory Ser. A*, 27(1):64–68.

[9] Bruinier, J. H., van der Geer, G., Harder, G., and Zagier, D. (2008). *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[10] Cui, S.-P., Gu, N. S. S., and Huang, A. X. (2016). Congruence properties for a certain kind of partition functions. *Adv. Math.*, 290:739–772.

[11] Dewar, M. (2011). Non-existence of Ramanujan congruences in modular forms of level four. *Canad. J. Math.*, 63(6):1284–1306.

[12] Diamond, F. and Shurman, J. (2005). *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[13] Göllnitz, H. (1967). Partitionen mit Differenzenbedingungen. *J. Reine Angew. Math.*, 225:154–190.

[14] Gordon, B. (1961). A combinatorial generalization of the Rogers-Ramanujan identities. *Amer. J. Math.*, 83:393–399.

[15] Gordon, B. and Hughes, K. (1993). Multiplicative properties of $\eta$-products. II. In *A tribute to Emil Grosswald: number theory and related analysis*, volume 143 of *Contemp. Math.*, pages 415–430. Amer. Math. Soc., Providence, RI.

[16] Hardy, G. H. and Ramanujan, S. (2000). Asymptotic formulæin combinatory analysis [Proc. London Math. Soc. (2) **17** (1918), 75–115]. In *Collected papers of Srinivasa Ramanujan*, pages 276–309. AMS Chelsea Publ., Providence, RI.

[17] Huat Chan, H., Wang, L., and Yang, Y. (2017). Modular Forms and $k$-colored Generalized Frobenius Partitions. *ArXiv e-prints*.

[18] Iwaniec, H. (1997). *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

[19] Jameson, M. and Wieczorek, M. (2019). Congruences for modular forms and generalized Frobenius partitions. *The Ramanujan Journal*.

[20] Kanade, S., Lepowsky, J., Russell, M. C., and Sills, A. V. (2017). Ghost series and a motivated proof of the Andrews-Bressoud identities. *J. Combin. Theory Ser. A*, 146:33–62.

[21] Kilford, L. J. P. (2015). *Modular forms*. Imperial College Press, London, second edition. A classical and computational introduction.

[22] Koblitz, N. (1993). *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

[23] Kolitsch, L. W. (1989). A relationship between certain colored generalized Frobenius partitions and ordinary partitions. *J. Number Theory*, 33(2):220–223.

[24] Kolitsch, L. W. (1991). $M$-order generalized Frobenius partitions with $M$ colors. *J. Number Theory*, 39(3):279–284.

[25] Lepowsky, J. and Milne, S. (1978). Lie algebraic approaches to classical partition identities. *Adv. in Math.*, 29(1):15–59.

[26] Lepowsky, J. and Wilson, R. L. (1981). The Rogers-Ramanujan identities: Lie theoretic interpretation and proof. *Proc. Nat. Acad. Sci. U.S.A.*, 78(2, part 1):699–701.

[27] Lepowsky, J. and Wilson, R. L. (1982). A Lie theoretic interpretation and proof of the Rogers-Ramanujan identities. *Adv. in Math.*, 45(1):21–72.

[28] Ligozat, G. (1975). *Courbes modulaires de genre* 1. Société Mathématique de France, Paris. Bull. Soc. Math. France, Mém. 43, Supplément au Bull. Soc. Math. France Tome 103, no. 3.

[29] Löbrich, S. (2017). Linear incongruences for generalized eta-quotients. *Res. Number Theory*, 3:Art. 18, 8.

[30] Milas, A. (2004a). Ramanujan's "Lost Notebook" and the Virasoro algebra. *Comm. Math. Phys.*, 251(3):567–588.

[31] Milas, A. (2004b). Virasoro algebra, Dedekind $\eta$-function, and specialized Macdonald identities. *Transform. Groups*, 9(3):273–288.

[32] Milas, A. (2007). Characters, supercharacters and Weber modular functions. *J. Reine Angew. Math.*, 608:35–64.

[33] Milas, A. (2010). On certain automorphic forms associated to rational vertex operator algebras. In *Moonshine: the first quarter century and beyond*, volume 372 of *London Math. Soc. Lecture Note Ser.*, pages 330–357. Cambridge Univ. Press, Cambridge.

[34] Milas, A., Mortenson, E., and Ono, K. (2008). Number theoretic properties of Wronskians of Andrews-Gordon series. *Int. J. Number Theory*, 4(2):323–337.

[35] Miyake, T. (2006). *Modular forms.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition. Translated from the 1976 Japanese original by Yoshitaka Maeda.

[36] Newman, M. (1957). Construction and application of a class of modular functions. *Proc. London. Math. Soc. (3)*, 7:334–350.

[37] Newman, M. (1959). Construction and application of a class of modular functions. II. *Proc. London Math. Soc. (3)*, 9:373–387.

[38] Ono, K. (1996). Parity of the partition function in arithmetic progressions. *J. Reine Angew. Math.*, 472:1–15.

[39] Ono, K. (2004). *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI.

[40] Rademacher, H. (1937). On the Partition Function p(n). *Proc. London Math. Soc. (2)*, 43(4):241–254.

[41] Radu, C.-S. (2012). A proof of Subbarao's conjecture. *J. Reine Angew. Math.*, 672:161–175.

[42] Ribet, K. A. (1990). From the Taniyama-Shimura conjecture to Fermat's last theorem. *Ann. Fac. Sci. Toulouse Math. (5)*, 11(1):116–139.

[43] Robins, S. (1994). Generalized Dedekind $\eta$-products. In *The Rademacher legacy to mathematics (University Park, PA, 1992)*, volume 166 of *Contemp. Math.*, pages 119–128. Amer. Math. Soc., Providence, RI.

[44] Rogers, L. J. (1893/94). Second Memoir on the Expansion of certain Infinite Products. *Proc. Lond. Math. Soc.*, 25:318–343.

[45] Sellers, J. (1994). New congruences for generalized Frobenius partitions with two or three colors. *Discrete Math.*, 131(1-3):367–373.

[46] Serre, J.-P. (1974). Divisibilité des coefficients des formes modulaires de poids entier. *C. R. Acad. Sci. Paris Sér. A*, 279:679–682.

[47] Sturm, J. (1987). On the congruence of modular forms. In *Number theory (New York, 1984–1985)*, volume 1240 of *Lecture Notes in Math.*, pages 275–280. Springer, Berlin.

[48] Taylor, R. and Wiles, A. (1995). Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572.

[49] Treneer, S. (2006). Congruences for the coefficients of weakly holomorphic modular forms. *Proc. London Math. Soc. (3)*, 93(2):304–324.

[50] Wieczorek, M. (2020). Andrews-Bressoud series and Wronskians.

[51] Wiles, A. (1995). Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551.

[52] Zhu, Y. (1996). Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302.

# Vita

Margaret was born in Peru, Illinois to parents Pat and Dan. She is the final of three children, including her older siblings Chris and Liz. While fifth grade long division brought much frustration, Margaret eventually developed a love for the logic behind mathematics and a passion for solving the puzzles it presents.

She earned a Bachelor of Arts degree in mathematics and French from North Central College in Naperville, Illinois in 2014 before continuing on to graduate studies at the University of Tennessee. In Knoxville, Margaret explored a variety of areas of mathematics while fulfilling requirements for her Masters of Science degree, which she achieved in 2017. She then settled into the area of number theory for her doctoral research under the supervision of Dr. Marie Jameson.

Margaret is looking forward to all the adventures which await her following the receipt of her doctorate in May 2020.