Some Results About Reproducing Kernel Hilbert Spaces of Certain Structure

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To all the people I will ever love.

Abstract

The theory of reproducing kernel Hilbert spaces has been crucial to the development of many of the most significant modern ideas behind functional analysis. In particular, there are two classes of reproducing kernel Hilbert spaces that have seen plenty of interest: that of complete Nevanlinna-Pick spaces and de Branges-Rovnyak spaces.

In this dissertation, we prove some results involving each type of reproducing kernel Hilbert space separately as well as one result regarding their potential overlap. It turns out that a de Branges-Rovnyak space is also of complete Nevanlinna-Pick type as long as there exists a multiplier satisfying a certain identity.

Further, we extend work of Theodor Kaluza to give sufficient conditions for having the Nevanlinna-Pick property in terms of the coefficients of a reproducing kernel's power series. Lastly, we characterize the multiplicative shifts on certain de Branges-Rovnyak spaces by highlighting four specific properties.

Table of Contents

Chapter 1

Introduction

1.1 History

There are two major topics covered in this manuscript: specifically, two types of spaces of analytic functions that are known, respectively, as complete Nevanlinna-Pick spaces and de Branges-Rovnyak spaces. Since there is far too much relevant history to recap in a short section like this, we will focus mainly on the contributions made by the four mathematicians whose last names appear in the names of the type of spaces: Rolf Nevanlinna, Georg Pick, Louis de Branges, and James Rovnyak.

Let us start with the former pair. In 1916 and 1919, respectively, Pick (see [22]) and Nevanlinna (see [20]) established the following theorem.

Theorem 1.1.1: (Nevanlinna-Pick) *Given $n \in \mathbb{N}$ and $\{z_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ in the unit disk $\mathbb{D}$, the following are equivalent:*

(A) there exists analytic $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ with $\phi(z_i) = \lambda_i$ for all $1 \leq i \leq n$.

(B) the matrix $M \in \mathbb{C}^{n\times n}$, defined component-wise by $M_{i,j} := \dfrac{1-\overline{\lambda_j}\lambda_i}{1-\overline{z_j}z_i}$ for $1 \leq i, j \leq n$, is positive definite.

That is, Pick and Nevanlinna proved that a solution to the interpolation problem described in condition A exists if and only if the relatively-easy-to-check condition B holds. Several decades later, Donald Sarason revitalized the subject by giving a new approach to the theorem as well as the construction of the solution ϕ (see [25] and [17]). In the 1980's, Jim Agler—motivated by Sarason's viewpoint—sought to generalize the interpolation problem (see [1]). What if we replace the initial data $\{z_i\}_{i=1}^n$ with $\{x_i\}_{i=1}^n$ in some more abstract set X? Or, more dramatically, what if we replaced the target data $\{\lambda_i\}_{i=1}^n$ with complex-valued matrices $\{W_i\}_{i=1}^n$ of some fixed size? In making these changes, one inherently alters the domain and range of a possible solution ϕ; for example, if $X \subset \mathbb{C}^d$ for $d \in \mathbb{N}$, then ϕ would have to be a function of d variables, as opposed to only one as in the Nevanlinna-Pick theorem.

Under such abstractions, complete Nevanlinna-Pick spaces are ones in which a suitable analogue of Theorem 1.1.1 holds. We will introduce the formal definition at the start of Chapter 2, but we mention now that the work of Jim Agler, John E. McCarthy, Scott McCullough, and Peter Quiggin will allow us to work with a much more favorable equivalence in place of the rather technical definition. With that, many results have been determined to hold for complete Nevanlinna-Pick spaces and we will discuss many of these advantages in Chapter 2 as well, if only briefly.

We would also be remiss not to mention the mathematician Theodor Kaluza, who established a lemma in 1926 (see [14]) that would later prove to be incredibly relevant to these spaces. In fact, the entirety of Chapter 3 will be spent generalizing this one result of Kaluza.

Next, let us reflect on the origins of de Branges-Rovnyak spaces. After Béla Sz.-Nagy and Ciprian Foias had proven their famous dilation theorem (see [26]), de Branges and Rovnyak devised a model to construct a particular type of subspace of the Hardy space H^2 in order to study the invariant subspace problem. In fact, in 1964, de Branges had claimed to have solved the invariant subspace problem using their model, and while this turned out to be an incorrect proof, the spaces used therein have nonetheless become credited to de Branges and Rovnyak.

Not all of the associated history is this unfortunate, as de Branges was able to use the spaces successfully in 1984 to prove the Bieberbach conjecture (see [9]). That being said, the modern view of de Branges-Rovnyak spaces was not developed until the 1990's when, among other contributions, Sarason's book *Sub-Hardy Hilbert Spaces in the Unit Disk* was published in 1994. This drove the theory forward and led to a central dichotomy of de Branges-Rovnyak spaces: the extreme versus the non-extreme case. More recently, Michael Jury and Rob Martin have developed additional (Clark) theory for multivariable analogues of the original de Branges-Rovnyak models (see [13]) and have also investigated a similar type of dichotomy (quasi-extreme versus non-quasi-extreme). Inspired by their work as well as that of Ball and Bolotnikov (see [5]), this dissertation seeks to generalize some previously known results to the setting of several complex variables.

While we have said nothing about how de Branges-Rovnyak spaces are defined/constructed and will defer this until Chapter 2, we state now that de Branges-Rovnyak spaces are reproducing kernel Hilbert spaces parameterized by a function which we usually call b. This manuscript will focus on de Branges-Rovnyak spaces that are much more generalized than their original formulation by considering b other than ones who have domain and range both $\mathbb{D}$, but we will mention that this original case has been shown to have applications in mathematical physics.

1.2 Overview

In Section 1.3, we cover all the background needed for the following three chapters. While we strive for this to be a self-contained document, we will assume the reader knows the basics of inner products, norms, Hilbert spaces, and bounded operators/their adjoints. Specifically, we must develop a certain baseline theory of what we call *reproducing kernel Hilbert spaces* so that we may define two particular classes of these types of spaces, each receiving their own introductory section in Chapter 2: complete Nevanlinna-Pick spaces in Section 2.1 and de Branges-Rovnyak spaces in Section 2.2.

With the basics of those spaces established, we will take the remainder of Chapter 2 to focus on the potential overlap of these two types; we will state and prove the following result in Section 2.3 (see Theorem 2.3.2).

Theorem 1.2.1: *Let $d \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty\}$. For a nonconstant $b \in \mathcal{M}_1(H_d^2 \otimes \mathcal{D}, H_d^2)$ with dim($\mathcal{D}$) $= r$ and $b(0) = 0$, the de Branges-Rovnyak kernel $k_\lambda(z) = \dfrac{1 - b(z)b(\lambda)^*}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}$ has the complete Nevanlinna-Pick property if and only if there exists $\psi \in \mathcal{M}_1(H_r^2 \otimes \mathcal{D}, H_r^2 \otimes \mathbb{C}^d)$ such that $M_z(\psi \circ b)(z) = b(z)$.*

Here, H_d^2 is the Drury-Arveson space, the space of multivariable power series who are square summable under appropriate weights (see Section 1.3 for the specifics). Further, the notation $\mathcal{M}_1$ refers to the set of contractive multipliers on the given spaces and will be covered in our upcoming preliminary section. Further, M_z refers to the row operator of multiplication by the independent variable $z = (z_1, \ ..., \ z_d)$.

While Theorem 1.2.1 assumes we start with a de Branges-Rovnyak space and characterizes whether or not the space is *also* of complete Nevanlinna-Pick type, we will be able to quickly alter our viewpoint into a similar corollary at the very end of Chapter 2 that interchanges the role of the spaces, instead assuming we are given a complete Nevanlinna-Pick space at the start.

As mentioned in the history section, Sections 3.1 and 3.2 will focus on two ways we can generalize a lemma of Theodor Kaluza regarding coefficients of power series of one complex variable. While Kaluza's result was not explicitly used in the environment of complete Nevanlinna-Pick spaces when it was first published, a strong connection from the lemma to this topic was established once interest in these types of spaces arose. In attempting to develop multivariable analogues of Kaluza's result, our two generalized versions will only involve power series of d variables (and their coefficients) in order to properly match Kaluza's original lemma. We remark now that the multi-index notation discussed in Section 1.3 is imperative to statements about these series.

In particular, we will need to define what it means for a sequence indexed by multi-indices to be non-decreasing. First, for α, β in $\mathbb{Z}^d$, we can induce a partial order by defining $\beta \leq \alpha$ to mean $\beta_i \leq \alpha_i$ for $1 \leq i \leq d$. With that, we say a sequence $\{S_\alpha\}_{\alpha \in \mathbb{Z}^d}$ is non-decreasing if and only if $S_\alpha \leq S_\beta$ for $\alpha \leq \beta$.

The following is the first generalization of Kaluza's result (see Lemma 3.1.2).

Lemma 1.2.2 (Kaluza-Type Lemma): *Suppose* $\{C_\alpha\}_{\alpha\in\mathbb{Z}^d}$ *satisfies* $C_{\boldsymbol{0}} = 1, C_\alpha > 0$ *for all other* α*, and* $C_\gamma = 0$ *for any* γ *such that* $\gamma \not\geq \boldsymbol{0}$*. Define* $\{R_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ *by* $R_{\boldsymbol{0}} := 0$ *and*

$$R_\alpha := \frac{C_\alpha}{\displaystyle\sum_{k=1}^{d} C_{\alpha-e_k}}, \quad \alpha \neq \mathbf{0},$$

where the e_k *are the canonical basis multi-indices.*

If $\{R_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ *is non-decreasing and bounded above by 1, then* $f(x) := \displaystyle\sum_{\alpha\in\mathbb{N}_0^d} C_\alpha x^\alpha$ *converges on* $\{x = (x_1,\ \ldots,\ x_d) \mid \|x\|_1 := |x_1| + |x_2| + \ldots + |x_d| < 1\}$ *and* $g(x) := 1 - \dfrac{1}{f(x)} = \displaystyle\sum_{\gamma\in\mathbb{N}_0^d} q_\gamma x^\gamma$ *for some non-negative* $\{q_\gamma\}_{\gamma\in\mathbb{N}_0^d}$.

Before proving the above result, we will spend some time in Section 3.1 connecting this type of result to complete Nevanlinna-Pick spaces, specifically noting how the lemma can be applied to give a sufficient condition for a reproducing kernel k_λ to be of complete Nevanlinna-Pick type and why the function $1 - \dfrac{1}{k_\lambda(z)}$ is relevant to this. Before moving on to Section 3.2, we connect the above lemma back to de Branges-Rovnyak spaces with a corollary showing that if k_λ is of complete Nevanlinna-Pick type by means of Lemma 1.2.2, it must also be a de Branges-Rovnyak kernel of a specific form.

Our second generalization of Kaluza's lemma involves word notation (covered in Section 1.3) and, being somewhat of a technical stepping stone to a more direct result, we omit it here in order to highlight the following corollary instead (see Corollary 3.2.2).

Corollary 1.2.3: *Suppose* $\{C_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ *are such that* $C_{\mathbf{0}} = 1$ *and* $C_\alpha > 0$ *for all other* α*. If* $C_\alpha \leq \dfrac{|\alpha|!}{\alpha!}$,

$$\frac{(C_{\alpha+e_i})^2}{C_\alpha C_{\alpha+2e_i}} \leq \frac{(|\alpha|+1)(\alpha_i+2)}{(|\alpha|+2)(\alpha_i+1)} \text{ for all } 1 \leq i \leq d,$$

$$\text{and } \frac{C_{\alpha+e_i} C_{\alpha+e_j}}{C_\alpha C_{\alpha+e_i+e_j}} \leq \frac{|\alpha|+1}{|\alpha|+2} \text{ for all } 1 \leq i, j \leq d,\ i \neq j,$$

then $f(x) := \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha x^\alpha$ *converges on* $\{x = (x_1,\ ...,\ x_d) \mid \|x\|_1 := |x_1| + |x_2| + ... + |x_d| < 1\}$ *and* $g(x) := 1 - \frac{1}{f(x)} = \sum_{\alpha \neq 0} q_\alpha x^\alpha$ *where each* $q_\alpha \geq 0$.

To close out Chapter 3, we discuss a class of examples of spaces for which Corollary 1.2.3 is applicable. Along the way, we will prove by example that Lemma 1.2.2 and Corollary 1.2.3 are independent generalizations of one another.

In Chapter 4, the longest chapter of the manuscript, we leave complete Nevanlinna-Pick spaces behind and instead focus solely on de Branges-Rovnyak spaces. We start by settling some rather specific operator theory preliminaries in Section 4.1. In Section 4.2, we address an operator-based formulation of the reproducing kernel Hilbert space theory of Section 1.3. Similarly, Section 4.3 re-introduces de Branges-Rovnyak spaces in a similar, more general setting than was done in Section 2.2.

In Section 4.4, we turn our attention to $(M_z, H(B))$ on a de Branges-Rovnyak space where this operator is bounded. With a series of propositions, we state and prove several properties of this operator based entirely on just a few starting assumptions. Finally, Section 4.5 closes out the dissertation by characterizing which abstract row operators $(T, \mathcal{H})$ are unitarily equivalent to multiplication by z on a de Branges-Rovnyak space, as the following result indicates (see Theorem 4.5.1).

Theorem 1.2.4: *Let* $T = (T_1,\ ...,\ T_d) : \mathcal{H}^{(d)} \to \mathcal{H}$ *be a bounded row operator on a separable Hilbert space* $\mathcal{H}$*, and let* $m \in \mathbb{N}$. T *is unitarily equivalent to* $(M_z, H(B))$ *for some* $B \in \mathcal{S}(\mathcal{D}, \mathcal{E})$ *and Hilbert spaces* $\mathcal{D}$ *and* $\mathcal{E}$ *such that* $B(0) = 0$*, dim(*$\mathcal{E}$*)* $= m$*, and that* M_z *is a bounded row operator if and only if* T *satisfies the following four conditions:*

(i) $T_i T_j = T_j T_i$ *for all* $1 \leq i, j \leq d$

(ii) dim([Ran $(T - \boldsymbol{\lambda})]^\perp) = m$ *for every* $\lambda \in \mathbb{B}_d$

(iii) $\|T\boldsymbol{x}\|_{\mathcal{H}} \geq \|\boldsymbol{x}\|_{\mathcal{H}^d}$ *for all* $\boldsymbol{x} \in$ *(Ker* $T)^\perp$

(iv) $\bigcap_{n \geq 0} \sum_{|w|=n} T_w \mathcal{H} = \{0\}$

Here, for $\lambda \in \mathbb{B}_d$, $\boldsymbol{\lambda}$ is a row operator which acts on $\mathbf{x} \in \mathcal{H}^d$ by scaling each component and adding as follows:

$$\boldsymbol{\lambda}\mathbf{x} := \sum_{i=1}^{d} \lambda_i x_i.$$

1.3 Reproducing Kernel Hilbert Spaces

One of the most famous results regarding Hilbert spaces is the Riesz Representation Theorem, which states that for every bounded linear functional ψ on a Hilbert space $\mathcal{H}$, there exists a unique $y_\psi \in \mathcal{H}$ such that $\psi(x) = \langle x, y_\psi \rangle_{\mathcal{H}}$ [8].

What does this theorem look like when we impose some extra assumptions? Specifically, let us assume that (1) $\mathcal{H} \subset \{f : X \to \mathbb{C}\}$ for some set X (that is, $\mathcal{H}$ is a *Hilbert function space*) and (2) the point evaluation functional E_λ given by $E_\lambda(f) = f(\lambda)$ is bounded on $\mathcal{H}$ for every $\lambda \in X$.

In this case, for every fixed $\lambda \in X$, the Riesz Representation Theorem gives a unique element $k_\lambda \in \mathcal{H}$ such that $E_\lambda(f) = f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$. We call k_λ the *reproducing kernel* at λ as its defining property reproduces the value of f at λ.

Therefore, the formal definition of a *reproducing kernel Hilbert space*, hereby referred to as an RKHS, is a Hilbert space $\mathcal{H} \subset \{f : X \to \mathbb{C}\}$ on which all point evaluation functionals are bounded. This is all that is needed to guarantee the existence of $\{k_\lambda\}_{\lambda \in X}$, the set of *kernel functions*.

To continue the general theory of RKHS's, we require a definition. Recall that $\mathcal{B}(\mathcal{D}, \mathcal{E})$ refers to the set of bounded operators from a Hilbert space $\mathcal{D}$ to another one $\mathcal{E}$, and that $\mathcal{B}(\mathcal{E})$ abbreviates $\mathcal{B}(\mathcal{E}, \mathcal{E})$.

Definition 1.3.1: For a set X and a Hilbert space $\mathcal{E}$, we call $K : X \times X \to \mathcal{B}(\mathcal{E})$ *positive definite* and write $K(z, \lambda) \succeq 0$ if and only if for all $n \in \mathbb{N}$, $\{\lambda^i\}_{i=1}^n \subset X$, and $\{a_i\}_{i=1}^n \subset \mathcal{E}$,

$$\sum_{i,j=1}^{n} \left\langle K(\lambda^i, \lambda^j) a_j, a_i \right\rangle_{\mathcal{E}} \geq 0.$$

Note that when $\mathcal{E} = \mathbb{C}$, this definition is the same as requiring that the matrix $K = [k_{i,j}]_{1 \leq i,j \leq n}$ given by $k_{i,j} = K(\lambda^i, \lambda^j)$ is positive definite (in the sense of square matrices) for all $n \in \mathbb{N}$ and $\{\lambda^i\}_{i=1}^n \subset X$.

Let us take the time to discuss some associated facts to Definition 1.3.1. First, we can define an ordering on the set $\{K : X \times X \to \mathcal{B}(\mathcal{E})\}$ by $A \succeq B$ if and only if $A - B \succeq 0$. We point out specifically that this ordering is transitive: $A \succeq B$ and $B \succeq C$ indeed implies $A \succeq C$.

Next, for a second Hilbert space $\mathcal{F}$ and $H : X \to \mathcal{B}(\mathcal{F}, \mathcal{E})$, it can readily be seen from the definition that $H(z)H(\lambda)^* \succeq 0$. More powerful and interesting is a converse-of-sorts to the previous fact: if $K(z, \lambda) \succeq 0$, then there exists an auxiliary Hilbert space $\mathcal{F}$ and $H : X \to \mathcal{B}(\mathcal{F}, \mathcal{E})$ such that we can factor $K(z, \lambda) = H(z)H(\lambda)^*$ (see [21]).

Together, both of the above lead to a quick proof of the following useful proposition.

Proposition 1.3.2: *Given a positive definite $K : X \times X \to \mathcal{B}(\mathcal{E})$ and $g : X \to \mathcal{B}(\mathcal{E}, \mathcal{D})$, $L : X \times X \to \mathcal{B}(\mathcal{D})$ defined by $L(z, \lambda) := g(z)K(z, \lambda)g(\lambda)^*$ is positive definite.*

In Section 4.2, we will examine the more general situation where each $k_\lambda(z)$ is an operator on $\mathbb{E}$ (hereby referred to as the *operator-case*), but for Chapter 1, let us consider the particular case that we take $\mathcal{E} = \mathbb{C}$. This is the case where $k_\lambda(z)$ is a scalar valued function for all λ, and in this setting, we obtain a one-to-one correspondence between positive definite functions and RKHS's, where the underlying set X is the same between the two. Let us now illustrate the specifics of this correspondence.

Given an RKHS $\mathcal{H}$ and therefore a set of kernel functions $\{k_\lambda\}_{\lambda \in X}$, we define $K(z, \lambda) := k_\lambda(z) = \langle k_\lambda, k_z \rangle_{\mathcal{H}}$. The positive definiteness is then near immediate as all we need to check that the quantity

$$\sum_{i,j=1}^n \overline{a_i} a_j K(\lambda^i, \lambda^j) = \sum_{i,j=1}^n \langle a_j k_{\lambda^j}, a_i k_{\lambda^i} \rangle_{\mathcal{H}} = \left\| \sum_{i=1}^n a_i k_{\lambda^i} \right\|_{\mathcal{H}}^2$$

is non-negative for any $n \in \mathbb{N}$, $\{a_i\} \subset \mathbb{C}$, and $\{\lambda^i\} \subset X$. Of course, as a norm, this is clearly the case.

Conversely, given a positive definite K, define functions $\{k_\lambda\}_{\lambda \in X}$ by $k_\lambda(z) := K(z, \lambda)$ and $\mathcal{H}$ as the completion of

$$D := \left\{ \sum_{i=1}^{n} c_i k_{\lambda^i} \mid n \in \mathbb{N},\ \{c_i\}_{i=1}^n \subset \mathbb{C},\ \{\lambda^i\}_{i=1}^n \subset X \right\}$$

in an appropriate Hilbert space norm. We outsource the remaining details that $\mathcal{H}$ is indeed a Hilbert space to [21], wherein the author defines $\langle k_\lambda, k_z \rangle_{\mathcal{H}} := k_\lambda(z)$ and then extends linearly to all of D. With this definition, the desired reproducing property clearly holds for any $f \in D$ and this can then be extended to all of $\mathcal{H}$. Also, it is now clear that $\overline{k_z(\lambda)} = k_\lambda(z)$ for fixed z and λ in X.

To refer to the unique RKHS obtained from a positive definite K, we use the notation $H(K)$, and we comment that by the above, the span of kernel functions is always dense in $H(K)$. While this last fact will prove incredibly useful in many occasions, one might ask for a more specific crtierion as to when an given element $f : X \to \mathbb{C}$ is in such a space $H(K)$? One possible answer comes in the form of the following proposition.

Proposition 1.3.3: *Let $f : X \to \mathbb{C}$ and $H(K)$ be given. $f \in H(K)$ if and only if there exists $c > 0$ such that $c^2 k_\lambda(z) - f(z)\overline{f(\lambda)} \succeq 0$.*

Moreover, in this case, $\|f\|_{H(K)}$ is the least c that satisfies the above.

In Chapter 4, we will state the operator-case version of this result (see Proposition 4.2.1), so we will defer the proof until then. Related to the previous proposition is a fact that characterizes when one RKHS is contained in another.

Proposition 1.3.4: [21] *Given $H(K^1)$ and $H(K^2)$, $H(K^1) \subseteq H(K^2)$ if and only if there exists $c > 0$ such that $c^2 k^2_\lambda(z) - k^1_\lambda(z) \succeq 0$.*

Moreover, in this case, the least c that satisfies the above is the operator norm of the inclusion map $j : H(K^1) \to H(K^2)$.

With so many results relating to the notion of positive definiteness, next we discuss the following way of obtaining positive definite objects, to which we credit [2].

Proposition 1.3.5: *If $u : X \to \mathcal{B}(\mathcal{D}, \mathbb{C})$ with $\|u(z)\|_{op} < 1$ for all $z \in X$, then*

$$K(z, \lambda) := \frac{1}{1 - u(z)u(\lambda)^*} \succeq 0.$$

We comment that in the case that $\mathcal{D}$ has dimension $a \in \mathbb{N} \cup \{\infty\}$, we have $u(z) = (u_1(z),\ ...,\ u_a(z))$ and subsequently

$$u(z)u(\lambda)^* = \langle u(z), u(\lambda)\rangle_{\mathcal{D}} = \sum_{j=1}^{a} u_j(z)\overline{u_j(\lambda)},$$

We mention now that with the above assumptions on u, we also have $\|u\|_\infty := \sup_{z \in X} \|u(z)\|_{\mathrm{op}} \leq 1$.

While there is more to be said about RKHS's in general, let us now take a detour in using the previous fact to provide a particular example of an RKHS whose importance cannot be overstated. First, take

$$X = \mathbb{B}_d := \{(z_1,\ ...,\ z_d) \mid \|z\|_2^2 := |z_1|^2 + ... + |z_d|^2 < 1\}$$

for some fixed $d \in \mathbb{N} \cup \{\infty\}$, the parameter indicating the number of variables the functions in our RKHS's have. In fact, this set (the d-unit ball) will serve as our X for the entirety of the manuscript.

Next, consider $u(z) := \mathbf{z} \in \mathcal{B}(\mathbb{C}^d, \mathbb{C})$ defined by $\mathbf{z}\{a_i\}_{i=1}^d = \sum_{i=1}^{d} z_i a_i$. With $d = 1$, this is simply multiplication by z, and it can still be thought of as multiplication in general if we consider $\mathbf{z}$ as a row operator acting on columns. By the Cauchy-Schwarz inequality, $|\mathbf{z}\{a_i\}_{i=1}^d| \leq \|z\|_2 \|a\|_2 < \|a\|_2$ whenever $z \in \mathbb{B}_d$. Thus, $u(z) = \mathbf{z}$ satisfies $\|u(z)\|_{\mathrm{op}} < 1$ for all $z \in \mathbb{B}_d$.

One can quickly compute that $\mathbf{z}^* c = \{\overline{z_i} c\}_{i=1}^d$ and subsequently that $\mathbf{z}\boldsymbol{\lambda}^* = \langle z, \lambda\rangle_{\mathbb{C}^d}$ for any z and $\lambda \in \mathbb{B}_d$. Proposition 1.3.5, which motivated all this set-up, now gives that $S(z, \lambda) := \dfrac{1}{1 - \langle z, \lambda\rangle_{\mathbb{C}^d}}$ is positive definite. Thus, we obtain an associated RKHS $H_d^2 := H(S)$

that we call the *Drury-Arveson space*. To explain some of the notation, we call the kernel $S(z, \lambda) = s_\lambda(z)$ the *Szegő kernel* and we use the notation H^2_d as we will soon see that the Drury-Arveson space is a multivariable generalization of the Hardy space $H^2 = H^2_1$. To get there, we need to introduce notation that will be used often throughout this dissertation.

Like the Drury-Arveson space exemplifies, the RKHS's we consider will consist of functions of d variables, for fixed $d \in \mathbb{N}$. Since we are interested in power series representations of such functions, we will have a strong need for what is known as *multi-index notation*. We call α a *multi-index* of size d if and only if $\alpha = (\alpha_1, \ ..., \ \alpha_d)$ for $\alpha_i \in \mathbb{N}_0$ for $1 \leq i \leq d$. Going forward, we use the notation $\mathbb{N}_0^d$ to refer to the set of multi-indices of the same size d. Each multi-index α has associated quantities $|\alpha| := \displaystyle\sum_{i=1}^{d} \alpha_i$, which we refer to as either the *order* or *length* of the multi-index, as well as a *multinomial weight* $\dfrac{|\alpha|!}{\alpha!}$. Here, $\alpha! := \displaystyle\prod_{i=1}^{d} \alpha_i!$, and we note that the zero multi-index $\mathbf{0}$ has order equal to zero but weight equal to one. The only other particular multi-indices we mention now are the set of canonical basis elements for $\mathbb{C}^d : \{e^k, \ 1 \leq k \leq d \mid e^k_i = \delta_{i,k}\}$. For the sake of brevity in the future, we will simply refer to this set as the *canonical basis multi-indices*.

For $z = (z_1, \ ..., \ z_d)$ and $\alpha \in \mathbb{N}_0^d$, we use the notation z^α to refer to $\displaystyle\prod_{i=1}^{d} z_i^{\alpha_i}$, and it is implicitly understood when we use this notation that $z_i z_j = z_j z_i$ for all i and j. What if we wanted to write a similar string of products of components of $X = (X_1, \ ..., \ X_d)$ where $X_i X_j$ and $X_j X_i$ may not be the same (e.g. if the $\{X_i\}$ were square matrices)? Instead of using multi-indices, this task requires what we call *word notation*. We call w a *word* (made from the d-alphabet) and write $w \in \mathbb{F}_d$ if and only if w is a string of countably many letters $w_1 w_2 ... w_n$ where $w_j \in \{1 \leq i \leq d\}$ for $1 \leq j \leq n$, with $n =: |w|$ being what we call the *length* of the word (i.e. the number of letters it has). Then, the notation X_w refers precisely to the noncommutative product $X_{w_1} X_{w_2} ... X_{w_n}$. We will also make occasional use of the empty word $\emptyset$, which acts somewhat as a multiplicative identity where multiplication is appendation of a word either on the left or right.

Before we return to H^2_d, let us attempt to bridge the divide between multi-indices and words. We first introduce the *letter-counting map* $\alpha : \mathbb{F}_d \to \mathbb{N}_0^d$ by the following: given any

$w \in \mathbb{F}_d$, let $\alpha(w) \in \mathbb{N}_0^d$ be the multi-index with ith component

$$\alpha(w)_i := \#\{1 \leq j \leq n \mid w_j = i\}.$$

As an example, if $d = 4$ and $w = 132232$, then $\alpha(w) = (1, 3, 2, 0)$. Another example is $\alpha(\emptyset) = \mathbf{0}$, no matter the value of d. It is a known combinatorial fact (an immediate consequence of the multinomial theorem) that

$$\#\{w \in \mathbb{F}_d \mid \alpha(w) = \alpha\} = \frac{|\alpha|!}{\alpha!},$$

and we will make use of this in the sequel. We remark now that for d commuting variables $(z_1, ..., z_d)$, we have that $z_w = z^{\alpha(w)}$ using both sets of our new notation.

Returning to the Szegő kernel, we compute, for any $n \geq 0$,

$$\langle z, \lambda \rangle_{\mathbb{C}^d}^n = \left(\sum_{i=1}^d z_i \overline{\lambda_i} \right)^n = \sum_{\substack{w \in \mathbb{F}_d \\ |w| = n}} z_w \overline{\lambda_w} = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = n}} \sum_{\substack{w \in \mathbb{F}_d \\ \alpha(w) = \alpha}} z^{\alpha(w)} \overline{\lambda^{\alpha(w)}} = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = n}} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{\lambda^\alpha}$$

and subsequently by a geometric series expansion

$$s_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}} = \sum_{n=0}^\infty \langle z, \lambda \rangle_{\mathbb{C}^d}^n = \sum_{\alpha \in \mathbb{N}_0^d} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{\lambda^\alpha}.$$

Thus, each s_λ is analytic by nature of having a power series representation. We now claim that $H_d^2 \subseteq \{f \text{ analytic on } \mathbb{B}_d\}$. Since any $g \in D$ is analytic and any $f \in H_d^2$ has $\{g_n\}_{n \geq 1} \subset D$ converging to it by the density, all we need to do is establish that said convergence is locally uniform. By Montel's theorem, it suffices to show that $N := \{g \in D \mid \|g\|_{H_d^2} < 1\}$ is locally uniformly bounded and subsequently that N is a normal family of functions. Fixing a compact set $C \subset \mathbb{B}_d$ such that we have a real number M satisfying $\|z\|_2 \leq M < 1$ for all $z \in C$, we then have

$$\|s_z\|_{H_d^2}^2 = \langle s_z, s_z \rangle_{H_d^2} = s_z(z) = \frac{1}{1 - \|z\|_2^2} \leq \frac{1}{1 - M^2}, \quad z \in C$$

and this leads to

$$|g(z)| = |\langle g, s_z\rangle_{H^2_d}| \leq \|g\|_{H^2_d}\|s_z\|_{H^2_d} \leq \sqrt{\frac{1}{1-M^2}}, \quad z \in C \text{ and } g \in N.$$

Thus, the convergence in question is locally uniform, as desired.

Therefore, for any $H(K)$ such that $H(K) \subseteq H^2_d$, each k_λ is analytic and therefore has a power series representation. Towards the start of Chapter 3, we will discuss an alternate assumption on $H(K)$ that guarantees a stronger condition: namely that the coefficients of k_λ's power series do not depend on $\lambda \in \mathbb{B}_d$. The following proposition gives a quick way to check positive-definiteness for power series of that type.

Proposition 1.3.6: *If* $K(z,\lambda) = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha z^\alpha \overline{\lambda^\alpha}$ *for* $\{C_\alpha\} \subset [0,\infty)$*, then* $K(z,\lambda) \succeq 0$.

Proof: We simply use the definition: for any $n \in \mathbb{N}, \{a_i\}_{i=1}^n \subset \mathbb{C}$, and $\{\lambda^i\}_{i=1}^n \subset \mathbb{B}_d$, one can show that

$$\sum_{i,j=1}^n \overline{a_i} a_j K(\lambda^i, \lambda^j) = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha \left| \sum_{i=1}^n \overline{a_i} \left(\lambda^i\right)^\alpha \right|^2,$$

which is of course non-negative if each C_α is.

While we are discussing power series, we remark that two absolutely convergent series $\sum_{\alpha \in \mathbb{N}_0^d} A_\alpha z^\alpha$ and $\sum_{\alpha \in \mathbb{N}_0^d} B_\alpha z^\alpha$ are identically equal on $\mathbb{B}_d$ if and only if $A_\alpha = B_\alpha$ for all $\alpha \in \mathbb{N}_0^d$.

Using the previous fact and the derived series form of $s_\lambda(z)$, we can show that the set of monomials $\{z^\beta\}_{\beta \in \mathbb{N}_0^d}$ lie in H^2_d. To do so, fix a β and take $c := \sqrt{\frac{\beta!}{|\beta|!}}$. Then

$$c^2 s_\lambda(z) = z^\beta \overline{\lambda^\beta} + \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ \alpha \neq \beta}} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{\lambda^\alpha}$$

and therefore the power series of $\ell_\lambda(z) := c^2 s_\lambda(z) - z^\beta \overline{\lambda^\beta}$ has positive coefficients and subsequently this shows that $\ell_\lambda(z)$ is positive definite by Proposition 1.3.6. Then, by

Proposition 1.3.3, $z^\beta \in H^2_d = H(S)$ for arbitrary β, with $\|z^\beta\|^2_{H^2_d} \leq \dfrac{\beta!}{|\beta|!}$. We next show that this inequality is in fact sharp via the claim: $\left\langle z^\beta, z^\gamma \right\rangle_{H^2_d} = \delta_{\beta,\gamma}\dfrac{\beta!}{|\beta|!}$ for any $\beta, \gamma \in \mathbb{N}^d_0$.

To prove this, allow λ and μ to be arbitrary elements of $\mathbb{B}_d$. We compute

$$\langle s_\lambda(\cdot), s_\mu(\cdot)\rangle_{H^2_d} = \left\langle \sum_{\gamma\in\mathbb{N}^d_0} \frac{|\gamma|!}{\gamma!} z^\gamma \overline{\lambda^\gamma}, \sum_{\beta\in\mathbb{N}^d_0} \frac{|\beta|!}{\beta!} z^\beta \overline{\mu^\beta} \right\rangle_{H^2_d}$$

$$= \sum_{\beta\in\mathbb{N}^d_0} \sum_{\gamma\in\mathbb{N}^d_0} \frac{|\beta|!|\gamma|!}{\beta!\gamma!} \overline{\lambda^\gamma} \mu^\beta \left\langle z^\gamma, z^\beta \right\rangle_{H^2_d} = \sum_{\beta\in\mathbb{N}^d_0} \left(\frac{|\beta|!}{\beta!} \sum_{\gamma\in\mathbb{N}^d_0} \frac{|\gamma|!}{\gamma!} \overline{\lambda^\gamma} \left\langle z^\gamma, z^\beta \right\rangle_{H^2_d} \right) \mu^\beta.$$

On the other hand, we have by the reproducing kernel property,

$$\langle s_\lambda(\cdot), s_\mu(\cdot)\rangle_{H^2_d} = s_\lambda(\mu) = \sum_{\beta\in\mathbb{N}^d_0} \frac{|\beta|!}{\beta!} \overline{\lambda^\beta} \mu^\beta.$$

Thus, as power series in the variable μ, these two series are identically equal; therefore, in equating the β coefficient of each and taking complex conjugates, we have for any λ:

$$\frac{|\beta|!}{\beta!} \lambda^\beta = \frac{|\beta|!}{\beta!} \sum_{\gamma\in\mathbb{N}^d_0} \frac{|\gamma|!}{\gamma!} \left\langle z^\beta, z^\gamma \right\rangle_{H^2_d} \lambda^\gamma.$$

Again, as two series identically equal in the variable λ, equating the γ coefficient of each reveals

$$\delta_{\beta,\gamma} = \frac{|\gamma|!}{\gamma!} \left\langle z^\beta, z^\gamma \right\rangle_{H^2_d}.$$

Rearranging this slightly, we have proved the claim.

We are finally ready to connect the Drury-Arveson space (when $d = 1$) to the Hardy space by the following claim:

$$H^2_d = \left\{ f(z) = \sum_{\alpha\in\mathbb{N}^d_0} A_\alpha z^\alpha \;\middle|\; \sum_{\alpha\in\mathbb{B}_d} |A_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty \right\}.$$

If so, then for $d = 1$, we see $H^2_1 = H^2$: the set of power series on $\mathbb{D}$ with square summable coefficients (using the fact that α and $|\alpha|$ have no distinction when $d = 1$).

We have already noted that $f \in H^2_d$ reveals that f has an absolutely convergent power series representation $\sum_{\alpha \in \mathbb{N}_0^d} A_\alpha z^\alpha$. To prove the desired set equality, we compute

$$\|f\|^2_{H^2_d} = \langle f, f \rangle_{H^2_d} = \left\langle \sum_{\alpha \in \mathbb{N}_0^d} A_\alpha z^\alpha, \sum_{\beta \in \mathbb{N}_0^d} A_\beta z^\beta \right\rangle_{H^2_d}$$

$$= \sum_{\alpha \in \mathbb{N}_0^d} \sum_{\beta \in \mathbb{N}_0^d} A_\alpha \overline{A_\beta} \left\langle z^\alpha, z^\beta \right\rangle_{H^2_d} = \sum_{\alpha \in \mathbb{N}_0^d} \sum_{\beta \in \mathbb{N}_0^d} A_\alpha \overline{A_\beta} \delta_{\alpha,\beta} \frac{\alpha!}{|\alpha|!} = \sum_{\alpha \in \mathbb{N}_0^d} |A_\alpha|^2 \frac{\alpha!}{|\alpha|!}.$$

Therefore, this last expression is finite if and only if the norm of f in H^2_d is, and this proves our above claim.

Let $\mathcal{H}$ and $\mathcal{D}$ be Hilbert spaces. We can construct the *tensor space* $\mathcal{H} \otimes \mathcal{D}$ by defining the inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{H} \otimes \mathcal{D}} := \langle x_1, x_2 \rangle_{\mathcal{H}} \langle y_1, y_2 \rangle_{\mathcal{D}}$$

for any $x_1, x_2 \in \mathcal{H}$ and $y_1, y_2 \in \mathcal{D}$. It is clear from this definition that tensoring $\mathcal{H}$ with $\mathbb{C}$ does nothing, revealing $\mathcal{H} \otimes \mathbb{C} = \mathcal{H}$.

Now suppose that $\mathcal{H}$ consists of functions (e.g. that $\mathcal{H}$ is an RKHS). Whenever we encounter the tensor notation $\mathcal{H} \otimes \mathcal{D}$, we should think of elements in this space as $\mathcal{D}$-valued functions in $\mathcal{H}$. As a primary example, for any $n \in \mathbb{N}$,

$$\mathcal{H} \otimes \mathbb{C}^n \cong \bigoplus_{i=1}^n \mathcal{H} = \{\text{columns consisting of } \{f_i\}_{i=1}^n \subset \mathcal{H}\}.$$

In the future, we will use $\mathcal{H}^{(n)}$ to abbreviate this space.

Our final topic of this introductory chapter relates to *mulipliers* on a Hilbert space. For Hilbert function spaces $\mathcal{H}$ and $\mathcal{K}$ (on $\mathbb{B}_d$) and auxiliary Hilbert spaces $\mathcal{D}$ and $\mathcal{E}$, define

$$\mathcal{M}(\mathcal{H} \otimes \mathcal{D}, \mathcal{K} \otimes \mathcal{E}) := \{\phi : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathcal{E}) \mid \phi f \in \mathcal{K} \otimes \mathcal{E} \ \ \forall f \in \mathcal{H} \otimes \mathcal{D}\}$$

We adopt the conventions to not write the corresponding tensor product if either $\mathcal{D}$ or $\mathcal{E}$ is $\mathbb{C}$ and that $\mathcal{M}(\mathcal{H} \otimes \mathcal{D})$ is a notational shortening of $\mathcal{M}(\mathcal{H} \otimes \mathcal{D}, \mathcal{H} \otimes \mathcal{D})$.

To continue our discussion, we require the following fundamental theorem of functional analysis.

Theorem 1.3.7 (Closed Graph): [8] *An operator $T : \mathcal{H} \to \mathcal{K}$ is bounded if and only if its graph*

$$\{(x, Tx) \mid x \in \mathcal{H}\}$$

is closed in $\mathcal{H} \times \mathcal{K}$.

Using the Closed Graph Theorem, one can show that each multiplier ϕ defines a bounded operator M_ϕ by $M_\phi f := \phi f$. From there, we can define $\|\phi\|_{\mathcal{M}(\mathcal{H}\otimes\mathcal{D},\mathcal{K}\otimes\mathcal{E})} := \|M_\phi\|_{\text{op}}$.

As one might expect, we are especially interested in the case where $\mathcal{H}$ and $\mathcal{K}$ are both RKHS's of functions defined on the same set: say, $\mathbb{B}_d$. If so, then for a scalar multiplier ϕ, we can determine how the adjoint M_ϕ^* acts on kernel functions. The following proposition will be proven in Chapter 4 when we encounter the operator-case version of it, as was the same for Proposition 1.3.3, but for now consider K^1 and K^2 scalar valued kernels on $\mathbb{B}_d$.

Proposition 1.3.8: *Let $\phi \in \mathcal{M}(H(K^1), H(K^2))$. For any $\lambda \in \mathbb{B}_d$, we have $M_\phi^* k_\lambda^2 = \overline{\phi(\lambda)} k_\lambda^1$.*

We will need a slightly more general version of the above result, one that holds when ϕ is a row multiplier: that is, when $\phi \in \mathcal{M}(H(K^1) \otimes \mathcal{D}, H(K^2))$ for some $\mathcal{D}$ with dimension $r \in \mathbb{N} \cup \{\infty\}$. However, this turns out to be readily available from Proposition 1.3.8. For a fixed $1 \leq i \leq r$, one can show that $\phi_i \in \mathcal{M}(H(K^1), H(K^2))$ by having M_ϕ act on a column with an arbitrary $f \in H(K^1)$ in the ith component and 0's in all the other ones. Therefore, $M_{\phi_i}^* k_\lambda^2 = \overline{\phi_i(\lambda)} k_\lambda^1$ for all components $1 \leq i \leq r$ by the above proposition; we can write this succinctly as $M_\phi^* k_\lambda^2 = \phi(\lambda)^* k_\lambda^1$.

In what follows, a *normalized* kernel simply refers to the fact that $k_0 \equiv 1$. Among other advantages for an RKHS, this guarantees that any constant function is in $H(K)$, as $1 = k_0 \in H(K)$ and thus can be scaled by any element of $\mathbb{C}$. We will near exclusively work with normalized kernels going forward.

As a corollary to (the row multiplier version of) Proposition 1.3.8, we can prove that for any $\phi \in \mathcal{M}(H(K) \otimes \mathcal{D}, H(K))$ such that $\mathcal{D}$ has dimension $r \in \mathbb{N} \cup \{\infty\}$ and that k is normalized, the following inequality holds: $\|\phi\|_\infty \leq \|\phi\|_{\mathcal{M}(H(K)\otimes\mathcal{D},H(K))}$.

To prove this, first let us compute:

$$\|\phi\|_\infty = \sup_{\lambda \in \mathbb{B}_d} \|\phi(\lambda)\|_{\text{op}} = \sup_{\lambda \in \mathbb{B}_d} \|\phi(\lambda)^*\|_{\text{op}} = \sup_{\lambda \in \mathbb{B}_d} \sup_{\substack{y \in \mathbb{C} \\ y \neq 0}} \frac{\|\phi(\lambda)^* y\|_{\mathcal{D}}}{|y|}$$

$$= \sup_{\lambda \in \mathbb{B}_d} \sup_{\substack{y \in \mathbb{C} \\ y \neq 0}} \frac{\sqrt{\sum_{i=1}^{r} |\phi_i(\lambda)|^2 |y|^2}}{|y|} = \sup_{\lambda \in \mathbb{B}_d} \sqrt{\sum_{i=1}^{r} |\phi_i(\lambda)|^2}.$$

Then, we note that for any fixed $\lambda \in \mathbb{B}_d$, we have $M_\phi^* k_\lambda = \phi(\lambda)^* k_\lambda$ by the (more general) proposition and subsequently

$$\sum_{i=1}^{r} |\phi_i(\lambda)|^2 \|k_\lambda\|^2_{H(K)} = \|\phi(\lambda)^* k_\lambda\|^2_{H(K)\otimes\mathcal{D}}$$

$$= \|M_\phi^* k_\lambda\|^2_{H(K)\otimes\mathcal{D}} \leq \|M_\phi^*\|^2_{\text{op}} \|k_\lambda\|^2_{H(K)} = \|M_\phi\|^2_{\text{op}} \|k_\lambda\|^2_{H(K)}.$$

Next, it is the normalization assumption that allows us to now divide by $\|k_\lambda\|^2_{H(K)}$, as $k_\lambda(0) = \overline{k_0(\lambda)} = 1$; therefore, k_λ is not identically 0 and $\|k_\lambda\|_{H(K)} \neq 0$. As mentioned, we can now divide the previous inequality by $\|k_\lambda\|^2_{H(K)}$ and take square roots of both sides to see

$$\|\phi\|_\infty = \sup_{\lambda \in \mathbb{B}_d} \sqrt{\sum_{i=1}^{r} |\phi_i(\lambda)|^2} \leq \|M_\phi\|_{\text{op}} = \|\phi\|_{\mathcal{M}(H(K))},$$

as desired.

In the case that $d = 1$, $H(K) = H^2$, and $\mathcal{D} = \mathbb{C}$, the previous inequality becomes an equality. This is because $\|\phi\|_\infty = \sup_{\lambda \in \mathbb{D}} |\phi(\lambda)|$ precisely coincides with what we call the H^∞ norm and it is well-known that $\mathcal{M}(H^2) = H^\infty$.

The previous proposition has an additional consequence in the form of the following lemma, which will have one particular use in the next chapter.

Lemma 1.3.9: *Suppose $\phi \in \mathcal{M}(H(K^1), H(K^2))$ is non-vanishing on $\mathbb{B}_d$. If $M_\phi M_\phi^* = I_{H(K^2)}$, then M_ϕ is unitary.*

Proof: As we have assumed M_ϕ^* is isometric, it suffices to show M_ϕ^* is a surjective operator (see [8]). Let D_i refer to the set of linear combinations of kernel functions in $H(K^i)$ for $i = 1$ and 2, which we recall are dense in their respective spaces. If $g = \sum_{i=1}^{n} c_i k^1_{\lambda^i}$ is in D_1, then $h := \sum_{i=1}^{n} \frac{c_i}{\phi(\lambda^i)} k^2_{\lambda^i}$ is well-defined since ϕ is assumed non-vanishing; further, we have found $h \in D_2$ such that $M_\phi^* h = g$ for arbitrary $g \in D_1$.

Let $f \in H(K^1)$ and $\{g_n\}_{n\in\mathbb{N}} \subset D_1$ such that $g_n \to f$ in $H(K^1)$. For any $n \in \mathbb{N}$, obtain $h_n \in D_2$ such that $M_\phi^* h_n = g_n$ and further note that $M_\phi g_n = M_\phi M_\phi^* h_n = h_n$ by assumption. Since M_ϕ is bounded and $g_n \to f$, we have $h_n = M_\phi g_n \to M_\phi f =: h \in H(K^2)$. Concluding, we use the boundedness of M_ϕ^* to see $g_n = M_\phi^* h_n \to M_\phi^* h$, but since $g_n \to f$, it must be that $M_\phi^* h = f$ and therefore M_ϕ^* is surjective.

Our final result of the chapter characterizes which $\phi : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathcal{E})$ are multipliers from some RKHS $H(K)$ to itself.

Proposition 1.3.10: [21] *$\phi \in \mathcal{M}(H(K) \otimes \mathcal{D}, H(K) \otimes \mathcal{E})$ if and only if there exists $c > 0$ such that*

$$(c^2 I_{\mathcal{E}} - \phi(z)\phi(\lambda)^*) k_\lambda(z) \succeq 0.$$

Moreover, in this case, $\|\phi\|_{\mathcal{M}(H(K)\otimes\mathcal{D}, H(K)\otimes\mathcal{E})}$ is the least possible c that can satisfy the above.

For $c > 0$, we introduce the notation

$$\mathcal{M}_c(H(K) \otimes \mathcal{D}, H(K) \otimes \mathcal{E}) := \{\phi \in \mathcal{M}(H(K) \otimes \mathcal{D}, H(K) \otimes \mathcal{E}) \mid \|\phi\|_{\mathcal{M}(H(K)\otimes\mathcal{D}, H(K)\otimes\mathcal{E})} \leq c\},$$

noting that the previous proposition gives us a precise criterion to check for when an element is in this set.

Chapter 2

Two Particular Classes of Reproducing Kernel Hilbert Spaces

2.1 Complete Nevanlinna Pick Spaces

In this chapter, we will introduce two particular classes of RKHS's, each getting its own section. We call $k_\lambda(z)$ a Complete Nevanlinna Pick (hereby abbreviated as CNP) kernel on a set X if and only if for all $s, t, n \in \mathbb{N}$, $\{W_i\}_{i=1}^n \subset \mathbb{C}^{s\times t}$, and $\{x_i\}_{i=1}^n \subset X$, the following two conditions are equivalent:

(A) there exists $\phi \in \mathcal{M}_1(H(K) \otimes \mathbb{C}^t, H(K) \otimes \mathbb{C}^s)$ with $\phi(x_i) = W_i$ for all $1 \le i \le n$.

(B) the matrix $M \in \mathbb{C}^{nt\times nt}$, defined by block matrices $M_{i,j} := (I_t - W_i^* W_j)k_{x_j}(x_i)$ for $1 \le i, j \le n$, is positive definite.

By Proposition 1.3.10, one can see that condition A always implies condition B. It is precisely when condition B implies condition A that we call the relevant kernel CNP.

When k is a CNP kernel, we say that the associated RKHS $H(K)$ has the *CNP property.* As we see, the definition entirely relates to when an interpolation problem has a contractive solution. Somewhat fortunately for us, however, we do not need to work with this rather technical definition, given the below theorem, which is credited to the work of

Scott McCullough (see [18]) and Peter Quiggin (see [23]) as well as that of Jim Agler and John E. McCarthy (see [2]).

Theorem 2.1.1 (McCullough-Quiggin-Agler-McCarthy): *Let $k_\lambda(z)$ be a normalized reproducing kernel. $k_\lambda(z)$ is CNP if and only if, for some Hilbert space $\mathcal{D}$ of finite or countable dimension, $k_\lambda(z)$ is of the form $\dfrac{1}{1-u(z)u(\lambda)^*}$ for some $u : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathbb{C})$ with $\|u(z)\|_{op} < 1$ for all $z \in \mathbb{B}_d$.*

That is, the sufficient condition to have a reproducing kernel in Proposition 1.3.5 actually characterizes the CNP kernels. Therefore, as we discussed in Section 1.3, the Szegő kernel is CNP and therefore H^2_d has the CNP property.

Since any positive definite object can be factored into $u(z)u(\lambda)^*$ for some u, the theorem gives us the most convenient equivalence to having a (normalized) CNP kernel:

$$k_\lambda(z) \text{ is CNP if and only if } 1 - \frac{1}{k_\lambda(z)} \succeq 0.$$

This fact will be incredibly useful in proving the main result of Section 2.3.

Let us now take some time to briefly discuss some advantages (outside of solving interpolation problems) to having a space with the CNP property. For each of the four statements below, assume k_λ is CNP.

I) $\{k_\lambda\}_{\lambda \in \mathbb{B}_d} \subset \mathcal{M}(H(K))$. That is, every kernel function is a multiplier (see [11]).

II) For each $f \in H(K)$, we may decompose f into what are known as subinner and free-outer factors. We defer these definitions to [4] but it is entirely motivated by the classic inner-outer factorization in the Hardy space (which is clearly a CNP space, recalling $H^2 = H^2_1$).

III) Also motivated by known results in the Hardy space, we have a Beurling-type theorem: for brevity, we just say that theorems of this type allow for any multiplier-invariant subspace of $H(K)$ to be identified with the range of an operator satisfying certain properties (see [19]).

IV) Most recently, Michael Hartz showed in 2020 [12] that every CNP space satisfies the column-row property (with constant 1), which itself leads to advantages outside of the scope of this text.

There are two additional theorems that apply to CNP spaces, but we will save those for Section 2.3 since that is when we will use them. To close out this section, let us show how any CNP space may be isometrically embedded into the Drury-Arveson space of an appropriate number of variables. In doing so, this will explain why many refer to the Drury-Arveson space as the universal CNP space.

Before that, we introduce one more (to-be-relevant) definition: we say a closed subspace $A \subseteq \mathcal{H}$ is $*$-*invariant* if and only if for all $\phi \in \mathcal{M}(\mathcal{H})$, we have $M_\phi^* A \subseteq A$.

Now, we illustrate the aforementioned embedding. Suppose that $k_\lambda(z)$ is CNP and subsequently that it has the form $\dfrac{1}{1 - u(z)u(\lambda)^*}$ for some $u : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathbb{C})$ such that $\mathcal{D}$ has dimension $a \in \mathbb{N} \cup \{\infty\}$ and $\|u(z)\|_{\text{op}} < 1$ for all $z \in \mathbb{B}_d$.

Let $s_w(\mu)$ refer to the (Szegő) reproducing kernel of H_a^2,

$$s_w(\mu) = \frac{1}{1 - \langle \mu, w \rangle_{\mathcal{D}}},$$

and note that $k_\lambda(z) = s_{u(\lambda)}(u(z))$; through this, we wish to view $H(K)$ as a subspace of H_a^2, which is done as follows. Next, let $C : H(K) \to H_a^2$ be the operator defined by $Ck_\lambda = s_{u(\lambda)}$, which by the previous observation can easily be seen to be norm-preserving from one dense set to another (the linear spans of the respective kernel functions) and thus isometric.

Thus, C provides an isometric embedding of $H(K)$ into H_a^2 by identifying $H(K)$ with the range of C, which we now claim is $*$-invariant. This follows from the action of adjoints of multipliers on kernel functions; for any $\phi \in M(H_a^2)$, we have

$$M_\phi^* C k_\lambda = M_\phi^* s_{u(\lambda)} = \overline{(\phi \circ u)(\lambda)} s_{u(\lambda)} = C \overline{(\phi \circ u)(\lambda)} k_\lambda$$

establishing the $*$-invariance.

We also record now that for $f \in H^2_a, C^* f \in H(K)$, and thus

$$(C^* f)(\lambda) = \langle C^* f, k_\lambda \rangle_{H(K)} = \langle f, C k_\lambda \rangle_{H^2_a} = \left\langle f, s_{u(\lambda)} \right\rangle_{H^2_a} = (f \circ u)(\lambda)$$

for any $\lambda \in \mathbb{B}_d$, revealing $C^* f = f \circ u$.

2.2 de Branges-Rovnyak Spaces

Next, it is time to introduce this chapter's second class of RKHS's. We call a RKHS $H(K)$ on $\mathbb{B}_d$ a *de Branges-Rovnyak Space* if and only if its reproducing kernel has the form

$$k_\lambda(z) = \frac{1 - b(z) b(\lambda)^*}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}$$

for some $b(z) = (b_1(z),\ ...,\ b_r(z))$ and $r \in \mathbb{N} \cup \{\infty\}$. In this case, we typically write $H(b)$ in place of $H(K)$.

From Proposition 1.3.10, such a row function b satisfies $k_\lambda(z) \succeq 0$ if and only if b is in $\mathcal{M}_1(H^2_d \otimes \mathcal{D}, H^2_d)$ where $\mathcal{D}$ has dimension r, revealing the set of b which parameterize de Branges-Rovnyak spaces of the type introduced above.

We can also compute

$$s_\lambda(z) - k_\lambda(z) = \frac{b(z) b(\lambda)^*}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}} = b(z) s_\lambda(z) b(\lambda)^*,$$

which is positive definite by Proposition 1.3.2. Then, by Proposition 1.3.4, we have that $H(b) \subseteq H^2_d$, and we remark now that the Drury-Arveson space is itself a de Branges-Rovnyak space as H^2_d and $H(b)$ with $b \equiv 0$ have the same reproducing kernel.

From the norm inequality towards the end of Section 1.3, we must have that $\|b\|_\infty \leq \|b\|_{\mathcal{M}_1(H^2_d \otimes \mathcal{D}, H^2_d)} \leq 1$ and therefore $\|b(z)^*\|_{\text{op}} = \|b(z)\|_{\text{op}} \leq \|b\|_\infty \leq 1$ for all $z \in \mathbb{B}_d$. Therefore,

$$\sum_{i=1}^{r} |b_i(z)|^2 |w|^2 = \|b(z)^* w\|^2_{\mathcal{D}} \leq |w|^2$$

for any $w \in \mathbb{C}$, and dividing by (nonzero) $|w|^2$ reveals that $\sum_{i=1}^{r} |b_i(z)|^2 \leq 1$ for all z. We will prove this inequality is in fact strict as long as we rule out that $H(b)$ equals $\{0\}$, as we now show that equality for any fixed λ in the above demands that $H(b)$ be only $\{0\}$.

To that end, assume we have $\mu \in \mathbb{B}_d$ such that $1 - \sum_{i=1}^{r} |b_i(\mu)|^2 = 0$. Since $k_\lambda(z)$ is positive definite, for any fixed $\lambda \in \mathbb{B}_d$, it must be that the matrix

$$\begin{bmatrix} 0 & 1 - \sum_{i=1}^{r} b_i(\mu)\overline{b_i(\lambda)} \\ 1 - \sum_{i=1}^{r} b_i(\lambda)\overline{b_i(\mu)} & 1 - \sum_{i=1}^{r} |b_i(\lambda)|^2 \end{bmatrix}$$

is positive definite by the remark we made immediately after Definition 1.3.1.

If so, it must be that its determinant

$$-\left(1 - \sum_{i=1}^{r} b_i(\mu)\overline{b_i(\lambda)}\right)\left(1 - \sum_{i=1}^{r} b_i(\lambda)\overline{b_i(\mu)}\right) = -\left|1 - \sum_{i=1}^{r} b_i(\mu)\overline{b_i(\lambda)}\right|^2 \geq 0.$$

Therefore, we see $1 - \sum_{i=1}^{r} b_i(\mu)\overline{b_i(\lambda)} = 0$, but then the Cauchy-Schwarz inequality gives

$$1 = \left|\sum_{i=1}^{r} b_i(\mu)\overline{b_i(\lambda)}\right| \leq \sqrt{\sum_{i=1}^{r} |b_i(\mu)|^2}\sqrt{\sum_{i=1}^{r} |b_i(\lambda)|^2} = \sqrt{\sum_{i=1}^{r} |b_i(\lambda)|^2} \leq 1,$$

revealing that $\sum_{i=1}^{r} |b_i(\lambda)|^2$ equals 1 for any $\lambda \in \mathbb{B}_d$. From here, we have

$$\|k_\lambda\|_{H(b)}^2 = k_\lambda(\lambda) = \frac{1 - \sum_{i=1}^{r} |b_i(\lambda)|^2}{1 - \|\lambda\|_{\mathbb{C}^d}^2} = 0.$$

With this, every k_λ is identically 0 and subsequently $H(b) = \{0\}$ in the case where there is *any* $\mu \in \mathbb{B}_d$ satisfying the previous inequality sharply. Therefore, as long as $H(b)$ is nonzero, we have strictness in the aforementioned inequality.

In other words, unless $H(b) = \{0\}$, $b(z) \in \mathbb{B}_r$ for all $z \in \mathbb{B}_d$ and we use this to abbreviate $b : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathbb{C})$ as $b : \mathbb{B}_d \to \mathbb{B}_r$.

One particular aspect of de Branges-Rovnyak spaces corresponding to $b : \mathbb{B}_d \to \mathbb{B}_r$ that we should settle is that, in the sense of the upcoming proposition, it is always legal to assume that $b(0) = 0$. The benefit of this is that we can always assume our reproducing kernel is normalized since $b(0) = 0$ gives $k_0 \equiv 1$. In doing so, this eliminates the possibility that $H(b) = \{0\}$.

Proposition 2.2.1: *Let $d \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty\}$ and suppose we have a nonzero $H(b)$ corresponding to $b : \mathbb{B}_d \to \mathbb{B}_r$. $H(b)$ is isomorphic to $H(\hat{b})$ for some $\hat{b} : \mathbb{B}_d \to \mathbb{B}_r$ with $\hat{b}(0) = 0$.*

The proposition's proof will require the following lemma from Rudin regarding automorphisms of the unit ball $\mathbb{B}_r$. Going forward, replace $\mathbb{C}^r$ with ℓ^2 in the case that $r = \infty$. We have tried to circumvent the need for this "split" notation until this moment, but now it is time to formally abbreviate the infinite case as such in order to eliminate the clutter of having to write two entirely different statements.

Lemma 2.2.2: [24] *Let $r \in \mathbb{N} \cup \{\infty\}$ and $\alpha \in \mathbb{B}_r$, $\alpha \neq 0$. Define P_α, Q_α, and ψ_α, all of which are maps from $\mathbb{B}_r$ into $\mathbb{C}^r$, by:*

$$P_\alpha w := \frac{\langle w, \alpha \rangle_{\mathbb{C}^r}}{\|\alpha\|^2_{\mathbb{C}^r}} \alpha; \qquad Q_\alpha w := w - P_\alpha w;$$

$$\psi_\alpha(w) := \frac{\alpha - P_\alpha w - \left(1 - \|\alpha\|^2_{\mathbb{C}^r}\right)^{1/2} Q_\alpha w}{1 - \langle w, \alpha \rangle_{\mathbb{C}^d}}$$

for any $w \in \mathbb{B}_r$.

ψ_α satisfies the following identity for all $w, \mu \in \mathbb{B}_r$:

$$1 - \langle \psi_\alpha(w), \psi_\alpha(\mu) \rangle_{\mathbb{C}^r} = \frac{\left(1 - \|\alpha\|^2_{\mathbb{C}^r}\right)\left(1 - \langle w, \mu \rangle_{\mathbb{C}^r}\right)}{\left(1 - \langle w, \alpha \rangle_{\mathbb{C}^r}\right)\left(1 - \langle \alpha, \mu \rangle_{\mathbb{C}^r}\right)}$$

and subsequently ψ_α maps $\mathbb{B}_r$ into (onto, in fact) $\mathbb{B}_r$.

We note that, in the case that $r = 1$, $P_\alpha w$ identically equals w and therefore we obtain the more familiar disc automorphism $\psi_\alpha(w) = \dfrac{\alpha - w}{1 - \overline{w}\alpha}$.

Proof of Proposition 2.2.1: If $b(0) = 0$, there is nothing to prove. If not, then we apply the lemma with $\alpha := b(0) \neq 0$, recalling our previous observation which confirms that (since $H(K) \neq \{0\}$) $b(0) \in \mathbb{B}_r$.

Define $\hat{b} : \mathbb{B}_d \to \mathbb{B}_r$ by $\hat{b} := \psi_\alpha \circ b$. Looking at the definition of ψ_α from the lemma, it is clear from $P_\alpha \alpha = \alpha$ that $\psi_\alpha(\alpha) = 0$, and therefore $\hat{b}(0) = 0$. Let us now verify that $\hat{b}$ indeed induces a de Branges-Rovnyak space (i.e. that $k^{\hat{b}}$ is positive definite). Using the identity in the lemma, we replace w with $b(z)$ and μ with $b(\lambda)$ to get:

$$1 - \left\langle \hat{b}(z), \hat{b}(\lambda) \right\rangle_{\mathbb{C}^r} = \frac{(1 - \|\alpha\|^2_{\mathbb{C}^r})\,(1 - \langle b(z), b(\lambda)\rangle_{\mathbb{C}^r})}{(1 - \langle b(z), \alpha\rangle_{\mathbb{C}^r})\,(1 - \langle \alpha, b(\lambda)\rangle_{\mathbb{C}^r})}.$$

Dividing both sides of the above by $1 - \langle z, \lambda\rangle_{\mathbb{C}^d}$, we see that for $f(z) := \dfrac{(1 - \|\alpha\|^2_{\mathbb{C}^r})^{1/2}}{1 - \langle b(z), \alpha\rangle_{\mathbb{C}^r}}$, we have:

$$k^{\hat{b}}_\lambda(z) = f(z)\overline{f(\lambda)}k^b_\lambda(z)$$

for all $z, \lambda \in \mathbb{B}_d$. This reveals two things: the former is that $k^{\hat{b}}$ is positive definite by Proposition 1.3.2 and the latter is that, by Proposition 1.3.10, $f \in \mathcal{M}_1(H(b), H(\hat{b}))$ since trivially $0 = k^{\hat{b}}_\lambda(z) - f(z)\overline{f(\lambda)}k^b_\lambda(z) \succeq 0$.

M_f will provide the desired isomorphism. By Proposition 1.3.8, we have $M_f^* k^{\hat{b}}_\lambda(z) = \overline{f(\lambda)}k^b_\lambda(z)$ and subsequently $M_f M_f^* k^{\hat{b}}_\lambda(z) = k^{\hat{b}}_\lambda(z)$.

Therefore, we can use a usual density argument to get $M_f M_f^* = I_{H(\hat{b})}$. Since f is clearly non-vanishing by its definition, we can employ Lemma 1.3.9 to reveal that M_f is unitary, as desired.

Consider the row operator tuple of multiplication by the independent variable, $M_z = [M_{z_1}, \ ..., \ M_{z_d}]$, which acts on an $f = [f_1, \ ..., \ f_d]^T \in H(b)^d$ by $M_z f = \sum_{i=1}^{d} z_i f_i$. There is a central dichotomy in de Branges-Rovnyak spaces depending on whether this operator is bounded or unbounded. In the case of $d = 1$ and a single function $b(z)$ (as in the original de Branges-Rovnyak spaces), the bounded case was known as the *non-extreme* situation and unbounded the *extreme* situation, as it turns out this entirely depends on whether b is an extreme point of the unit ball of H^∞ or not (see [16]).

In considering row functions b of several variables, we lose this connection and choose to no longer involve the word "extreme" for this reason. Regardless, the following proposition characterizes the bounded case for any d and r, which is all this manuscript will focus on going forward.

Proposition 2.2.3: *Let $b : \mathbb{B}_d \to \mathbb{B}_r$. $(M_z, H(b))$ is a bounded row operator if and only if $b(z)y \in H(b)$ for every $y \in \mathbb{C}^r$.*

Moreover, in this case, the operator norm of $(M_z, H(b))$ and that of the inclusion map $j : \{b(z)y \mid y \in \mathbb{C}^r\} \to H(b)$ are related by $\|M_z\|^2 = \|j\|^2 + 1$.

In Chapter 4, we will re-introduce de Branges-Rovnyak spaces in the case where k^b is an operator, and we will prove the operator-case version of the above proposition at that time (see Proposition 4.3.1).

To close out this section, let us observe a consequence of being in the case where $(M_z, H(b))$ is bounded: for every canonical basis vector e^i in $\mathbb{C}^r$, $1 \leq i \leq r$, the proposition tells us that in this case $b_i(z) = b(z)e^i \in H(b) \subseteq H^2_d$ and therefore every component function b_i must be analytic. We will use this fact in the sequel.

2.3 Regarding the Overlap

Now that we have spent some time developing the basics of two particular types of RKHS's, one may wonder what spaces besides the Drury-Arveson space are both a CNP space *and* a de Branges-Rovnyak space. In 2020, Cheng Chu proved the following result, addressing this question in the case of $H(b)$ for $b : \mathbb{D} \to \mathbb{D}$.

Theorem 2.3.1: [7] *For a nonconstant* $b : \mathbb{D} \to \mathbb{D}$ *with* $b(0) = 0$ *and* $\|b\|_\infty \leq 1$*, the de Branges-Rovnyak kernel* $k_\lambda(z) = \dfrac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}$ *has the complete Nevanlinna-Pick property if and only if there is a holomorphic* $h : b(\mathbb{D}) \to \mathbb{D}$ *such that* $(h \circ b)(z) = z$ *on* $\mathbb{D}$ *and the function* $\psi(z) := \dfrac{z}{h(z)}$ *extends from* $b(\mathbb{D})$ *to a holomorphic* $\psi : \mathbb{D} \to \overline{\mathbb{D}}$.

For $d \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty\}$, we now seek to generalize the previous theorem to one applicable to a nonconstant $b : \mathbb{B}_d \to \mathbb{B}_r$ [i.e. b is the row $b(z) = (b_1(z),\ b_2(z),\ ...,\ b_r(z))$] with $b(0) = 0$ and $\|b\|_{\mathcal{M}(H^2_d \otimes \mathcal{D}, H^2_d)} \leq 1$ where $\mathcal{D}$ has dimension r.

Theorem 2.3.2: *Let* $d \in \mathbb{N}$ *and* $r \in \mathbb{N} \cup \{\infty\}$*. For a nonconstant* $b \in \mathcal{M}_1(H^2_d \otimes \mathcal{D}, H^2_d)$ *with dim(*$\mathcal{D}$*)* $= r$ *and* $b(0) = 0$*, the de Branges-Rovnyak kernel* $k_\lambda(z) = \dfrac{1 - b(z)b(\lambda)^*}{1 - \langle z, \lambda\rangle_{\mathbb{C}^d}}$ *has the complete Nevanlinna-Pick property if and only if there exists* $\psi \in \mathcal{M}_1(H^2_r \otimes \mathcal{D}, H^2_r \otimes \mathbb{C}^d)$ *such that* $M_z(\psi \circ b)(z) = b(z)$.

Before proceeding towards the proof, let us note the details of how, when $d = r = 1$, the right hand side condition of the above is indeed a generalization of the right hand side condition from Chu's original statement.

Assuming the existence of h and ψ from Chu's result, we have $|\psi(z)| \leq 1$ for all $z \in \mathbb{D}$, or in other words $\|\psi\|_{M(H^2)} = \|\psi\|_{H^\infty} \leq 1$. Also, $(\psi \circ b)(z) = \dfrac{b(z)}{h(b(z))} = \dfrac{b(z)}{z}$. Noting that since $d = 1$, M_z can easily be rearranged by (scalar) multiplication/division, we have the condition in Theorem 2.1.2 (with the same ψ).

On the other hand, we claim that the right hand side implication of Theorem 2.3.2, again in the case that $d = 1 = r$, guarantees that b is injective and thus $b : \mathbb{D} \to b(\mathbb{D})$ has an inverse h. The following more general lemma, which relaxes r, shows this is true.

Lemma 2.3.3: *Let* $r \in \mathbb{N} \cup \{\infty\}$*. Suppose we have a de Branges-Rovnyak space* $H(b)$ *corresponding to nonzero* $b : \mathbb{D} \to \mathbb{B}_r$ *with* $b(0) = 0$*. If there exists* $\psi : \mathbb{B}_r \to \mathbb{B}_r$ *such that* $z(\psi \circ b)(z) = b(z)$*, then* $\psi(0) \neq 0$ *and* b *is injective.*

Proof: First let us assume that no b_i is identically 0; this is legal because any $b_j \equiv 0$

have no affect on $b(z)b(\lambda)^* = \sum_{i=1}^{r} b_i(z)\overline{b_i(\lambda)}$ and subsequently no affect on $H(b)$ either as the previous expression is all that uniquely determines k_λ.

Next, for any $1 \leq i \leq r$, recall that we know b_i is analytic on $\mathbb{D}$. Thus, since $b_i(0) = 0$, there must be some $k_i \in \mathbb{N}$ (i.e. the multiplicity at 0) and analytic function c_i with $c_i(0) \neq 0$ such that $b_i(z) = z^{k_i}c_i(z)$ for all $z \in \mathbb{D}$. Let $k := \min_{1 \leq i \leq r} k_i$ and next let j be, say, the smallest index such that $k_j = k$. If we define $d_i(z) := c_i(z)z^{k_i - k}$, then $\{d_i\}_{i=1}^{r}$ form a set of analytic functions (noting that $k_i - k \geq 0$) such that $b_i(z) = z^k d_i(z)$ for all i with $d_j(0) = c_j(0) \neq 0$. If we allow d to refer to $(d_1, \ ..., \ d_r)$, then we have $b(z) = z^k d(z)$.

By the hypothesis, we have that $z\psi_i(b(z)) = b_i(z)$ for all $1 \leq i \leq r$. Taking $i = j$ and employing the above paragraph, we can write this as

$$z\psi_j(z^k d(z)) = z^k d_j(z)$$

and observe from the right-hand side that this function must have multiplicity k at 0.

Now, if $\psi_j(0) = 0$, then $\psi_j(z) = zf(z)$ for some function f. Hence, $z\psi_j(z^k d(z)) = z^{k+1}g(z)$ for $g(z) := d(z)f(z^k d(z))$, and subsequently we observe that $z\psi_j(z^k d_j(z))$ has multiplicity at least $k+1$ at 0. This contradicts the earlier remark that the same function has multiplicity k at 0. Therefore, it must be that $\psi_j(0) \neq 0$ and ultimately that $\psi(0) \neq 0$, as desired.

Continuing the proof, we next address the injectivity. Assume $b(z) = 0$ for some $z \in \mathbb{D}$. If so, then $z\psi_j(0) = b_j(z) = 0$ and therefore z must be 0 since $\psi_j(0)$ cannot be.

Next, assume for $z, \lambda \in \mathbb{D}$ that $b(z) = b(\lambda)$ and is nonzero (or else $z = \lambda = 0$). Bringing in ψ, we have

$$z\psi_i(b(z)) = b_i(z) = b_i(\lambda) = \lambda\psi_i(b(\lambda)) \text{ for all } 1 \leq i \leq r.$$

However, since $z\psi(b(z)) = b(z) \neq 0$, $\psi(b(z))$ cannot be either and therefore $\psi_\ell(b(z)) \neq 0$ for some $1 \leq \ell \leq r$. This leads to $z\psi_\ell(b(z)) = \lambda\psi_\ell(b(\lambda)) = \lambda\psi_\ell(b(z))$ and dividing by $\psi_\ell(b(z))$, we conclude that $z = \lambda$. Having shown that b is injective, we have finished the proof.

Before we prove Theorem 2.3.2, there are some preliminaries to address. Recall that we showed in the previous section that, since our assumption of $b(0) = 0$ guarantees $H(b)$ is nonzero, $\|b(z)\|_{\text{op}} < 1$ for all $z \in \mathbb{B}_d$.. Because of this, the function $L(z, \lambda) = \ell_\lambda(z) := \dfrac{1}{1 - \langle b(z), b(\lambda)\rangle_{\ell^2}}$ is a CNP kernel and we denote $H(L)$ for the associated CNP space. Therefore, we have the advantage of having an auxiliary CNP kernel to which we can apply known results. In particular, we need two known theorems that apply to CNP spaces; the first theorem is due to Robert Leech.

Theorem 2.3.4 (Leech): [15] *Suppose $H(K)$ is CNP and corresponds to a scalar-valued kernel function $k_\lambda(z)$. If, for auxiliary Hilbert spaces $\mathcal{D}$ and $\mathcal{E}$, $g : \mathbb{B}_d \to \mathcal{B}(\mathcal{E}, \mathbb{C})$ and $h : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathbb{C})$ satisfy*

$$[g(z)g(\lambda)^* - h(z)h(\lambda)^*]k_\lambda(z) \succeq 0,$$

then $\exists\ S \in \mathcal{M}_1(H(K) \otimes \mathcal{D}, H(K) \otimes \mathcal{E})$ such that $h(z) = g(z)S(z)$.

The second theorem is a consequence of the Commutant Lifting Theorem and is due to Ball, Trent, and Vinnikov. The proof makes use of the fact that the embedding of any CNP space in the Drury-Arveson space is $*$-invariant, as we observed earlier.

Theorem 2.3.5 (Ball-Trent-Vinnikov): [6] *Suppose $H(K)$ is CNP and corresponds to a scalar-valued kernel function $k_\lambda(z) = \dfrac{1}{1 - u(z)u(\lambda)^*}$ for some $u(z) = (u_1(z),\ \ldots,\ u_a(z))$ and $a \in \mathbb{N} \cup \{\infty\}$. Let $S : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathcal{E})$. The following are equivalent:*

I) $S \in \mathcal{M}_1(H(K) \otimes \mathcal{D}, H(K) \otimes \mathcal{E})$

II) there exists $\psi \in \mathcal{M}_1(H^2_a \otimes \mathcal{D}, H^2_a \otimes \mathcal{E})$ such that $S = \psi \circ u$.

Proof of Theorem 2.3.2: Assuming the CNP property and noting that $b(0) = 0$ means that k_λ is normalized, we have

$$0 \preceq 1 - \frac{1}{k_\lambda(z)} = \frac{\langle z, \lambda\rangle_{\mathbb{C}^d} - b(z)b(\lambda)^*}{1 - \langle b(z), b(\lambda)\rangle_{\ell^2}} = \left[\langle z, \lambda\rangle_{\mathbb{C}^d} - b(z)b(\lambda)^*\right]\ell_\lambda(z).$$

Define $g : \mathbb{B}_d \to \mathcal{B}(\mathbb{C}^d, \mathbb{C})$ by $g(z)x = \sum_{i=1}^{d} z_i x_i$ for any $z \in \mathbb{B}_d$ and $x \in \mathbb{C}^d$. Noting that $g(z)g(\lambda)^* = \langle z, \lambda \rangle_{\mathbb{C}^d}$ and that $H(L)$ is CNP,

$$[\langle z, \lambda \rangle_{\mathbb{C}^d} - b(z)b(\lambda)^*] \, \ell_\lambda(z) \succeq 0$$

is precisely the hypothesis for Leech's theorem with $h = b : \mathbb{B}_d \to \mathcal{B}(\mathbb{C}^r, \mathbb{C})$. Therefore, we obtain $S \in \mathcal{M}_1(H(L) \otimes \mathbb{C}^r, H(L) \otimes \mathbb{C}^d)$ such that $b(z) = g(z)S(z) = M_z S(z)$, where we choose to write $g(z) = M_z$ now that we are acting on a function space and no longer considering $g(\lambda)$ as well.

Next, we apply the Ball-Trent-Vinnikov theorem to get property II as an equivalence to property I with the above S. That is, there exists $\psi \in \mathcal{M}_1(H_r^2 \otimes \mathbb{C}^r, H_r^2 \otimes \mathbb{C}^d)$ such that $S = \psi \circ u$. Combining this with the result of Leech's theorem, we have $M_z(\psi \circ b)(z) = b(z)$, as desired.

The converse can be proved more directly. Having $\psi \in \mathcal{M}_1(H_r^2 \otimes \mathbb{C}^r, H_r^2 \otimes \mathbb{C}^d)$ translates by Proposition 1.3.10 to

$$[I_{\mathbb{C}^d} - \psi(w)\psi(\mu)^*] \; s_\mu(w) \succeq 0,$$

that is, as a function on $\mathbb{B}_r \times \mathbb{B}_r$. In order to invoke our other assumption, we replace w with $b(z)$ as well as μ with $b(\lambda)$ to get

$$[I_{\mathbb{C}^d} - (\psi \circ b)(z)(\psi \circ b)(\lambda)^*] \; \ell_\lambda(z) \succeq 0$$

as a function on $\mathbb{B}_d \times \mathbb{B}_d$, where we recall that $s_{b(\lambda)}(b(z)) = \ell_\lambda(z)$. With the same g as in the proof of the forward direction, we can apply $g(z)$ on the left and $g(\lambda)^*$ on the right to the above and use Proposition 1.3.2 to maintain the positive definiteness:

$$0 \preceq g(z)[I_{\mathbb{C}^d} - (\psi \circ b)(z)(\psi \circ b)(\lambda)^*]g(\lambda)^* \ell_\lambda(z)$$

$$= [g(z)g(\lambda)^* - g(z)(\psi \circ b)(z)[g(\lambda)(\psi \circ b)(\lambda)]^*] \, \ell_\lambda(z).$$

Invoking the fact that $b(z) = M_z(\psi \circ b)(z) = g(z)(\psi \circ b)(z)$, we arrive at

$$[\langle z, \lambda\rangle_{\mathbb{C}^d} - b(z)b(\lambda)^*]\, \ell_\lambda(z) \succeq 0.$$

From here, the calculations from the former part of the proof can be worked backwards to get the desired CNP inequality.

Now, consider the separate question: which CNP kernels have the form of a de Branges-Rovnyak kernel? This turns out to be fairly simple to answer due to the symmetry present in the following equivalent expressions:

$$\frac{1}{1 - u(z)u(\lambda)^*} = \frac{1 - b(z)b(\lambda)^*}{1 - \langle z, \lambda\rangle} \iff \frac{1}{1 - b(z)b(\lambda)^*} = \frac{1 - u(z)u(\lambda)^*}{1 - \langle z, \lambda\rangle}.$$

From here, the chapter's main result therefore answers this alternate question for free, which we include as the following corollary.

Corollary 2.3.6: *Let* $k_\lambda(z) = \dfrac{1}{1 - u(z)u(\lambda)^*}$ *be a CNP kernel for some nonconstant* $u \in \mathcal{M}_1(H^2_d \otimes \mathcal{D}, H^2_d)$ *with dim(*$\mathcal{D}$*)* $= a \in \mathbb{N} \cup \{\infty\}$ *and* $u(0) = 0$*. Then,* $H(K) = H(b)$ *for some* b *if and only if there exists* $\psi \in \mathcal{M}_1(H^2_a \otimes \mathcal{D}, H^2_a \otimes \mathbb{C}^d)$ *such that* $M_z(\psi \circ u)(z) = u(z)$*.*

If $d = 1$, then by pulling from Lemma 2.3.3, the u must be injective if a CNP k_λ is *also* a de Branges-Rovnyak kernel. Therefore, any CNP kernel where the associated u is *not* injective cannot be of de Branges-Rovnyak type.

Let us close out Chapter 2 in the same way we will conclude every following chapter: discussing further directions applicable to the chapter's concepts. In the context of the main result, Theorem 2.3.2, there is one central question: how would one solve for, or otherwise obtain, the multiplier ψ? This may be (at least in general) a very difficult question, but one can approach it with a slightly easier question: is the multiplier ψ uniquely determined by the b?

Assuming that $r = 1$, we would have to investigate $\psi = (\psi_1, ..., \psi_d)^T$ satisfying $M_z\psi(b(z)) = b(z)$. If we write $\psi_i(w) = \sum_{k=0}^{\infty} a_{ik}w^k$ for $1 \leq i \leq d$, then the previous question becomes: can the set of $\{a_{ik}\}$ be determined uniquely only from information about the b? For now, this question remains open.

Chapter 3

Multivariable Generalizations of Kaluza's Lemma

3.1 A Multivariable Kaluza-Type Lemma

In 1926, Theodor Kaluza stated and proved the following fact about power series of *one* complex variable:

Lemma 3.1.1 (Kaluza's Lemma): [14] *Suppose* $\{C_n\}_{n\geq 0}$ *are such that* $C_0 = 1$ *and* $C_n > 0$ *for* $n \geq 1$*, and define an auxiliary sequence* $\{R_n\}_{n\geq 0}$ *by* $R_0 := 0$ *and* $R_n := \dfrac{C_n}{C_{n-1}}$ *for* $n \geq 1$*. If* R_n *is a non-decreasing sequence bounded above by 1, then* $f(x) := \displaystyle\sum_{n=0}^{\infty} C_n x^n$ *converges for* $x \in \mathbb{D}$ *and* $g(x) := 1 - \dfrac{1}{f(x)} = \displaystyle\sum_{n=0}^{\infty} q_n x^n$ *for some non-negative* $\{q_n\}_{n\geq 0}$.

As we will soon state and prove a multivariable version whose proof matches the one written by Kaluza in the case of the dimension d equaling 1, there is no need to re-record Kaluza's original proof. However, we remark here that it's the boundedness by 1 of R_n that makes the $\{q_n\}$ exist and the non-decreasing property of R_n that makes the $\{q_n\}$ non-negative. The same idea will appear in the proof of our generalized version.

Before we get there, let us discuss the usefulness of such a lemma. Despite simply being a statement about a new power series obtained from a prexisting one, it has found significant practicality in the realm of reproducing kernels. However, we will need an assumption on $H(K)$ such that $k_\lambda(z)$ has a power series representation of the following type:

$$k_\lambda(z) = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha \left(\overline{\lambda} z\right)^\alpha .$$

That is, if the coefficients of the power series for each $k_\lambda(z)$, say C_α^λ, are of the type $C_\alpha^\lambda = C_\alpha \overline{\lambda}^\alpha$ for $\{C_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ that do not depend on λ. Of course, we also require each C_α to be non-negative for k_λ to give a legitimate RKHS.

One assumption that will guarantee this is the following: suppose $\{z^\alpha\}_{\alpha \in \mathbb{N}_0^d}$ forms an orthogonal basis for $H(K)$. If so, then for $C_\alpha := (\|z^\alpha\|^2_{H(K)})^{-1}$, the functions $\{g_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ given by $g_\alpha(z) := \sqrt{C_\alpha} z^\alpha$ form an orthonormal basis for $H(K)$. To use this, we employ a well-known fact regarding RKHS's that we chose to omit in Section 1.3: given an orthonormal basis $\{f_n\}_{n \geq 1}$ for $H(K)$, one can obtain $k_\lambda(z)$ through the sum $\sum_{n=1}^{\infty} f_n(z)\overline{f_n(\lambda)}$ (see [21]). Thus, in our case, we have

$$k_\lambda(z) = \sum_{\alpha \in \mathbb{N}_0^d} g_\alpha(z)\overline{g_\alpha(\lambda)} = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha \left(\overline{\lambda} z\right)^\alpha,$$

as was claimed.

Let us now recap some observations from Sections 1.3 and 2.1. First recall from Section 2.1 that we call a normalized kernel $k_\lambda(z)$ (for z, λ in $\mathbb{B}_d$) a Complete Nevanlinna-Pick kernel (or CNP for short) if and only if it is of the form $k_\lambda(z) = \dfrac{1}{1 - u(z)u(\lambda)^*}$ for some $u(z) = (u_1(z),\ \ldots,\ u_a(z))$ and $a \in \mathbb{N} \cup \{\infty\}$ such that $\|u(z)\|_{\text{op}} < 1$ for all $z \in \mathbb{B}_d$.

Recall that hereby we are tacitly assuming that the monomials form an orthogonal basis for $H(K)$. In this case, we have that k_λ is CNP if and only if the auxiliary function $u_\lambda(z) := 1 - \dfrac{1}{k_\lambda(z)}$ is positive definite, for it can then be factored into $u_\lambda(z) = u(z)u(\lambda)^*$ for some u as in the definition, and solving for $k_\lambda(z)$ reveals the desired form. Going one step further, a power series is positive definite if and only if all of its coefficients are non-negative, and since the set of coefficients $\{C_\alpha\}$ do not depend on λ, investigating if such a $k_\lambda(z)$ is CNP reduces to seeing whether $1 - \dfrac{1}{f(x)}$ is a power series with non-negative coefficients, where $f(x) = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha x^\alpha$.

In the case of $d = 1$, the preceding paragraph should be enough to make clear why Kaluza's Lemma gives a sufficient condition for $k_\lambda(z)$ to be CNP with the above correspondence of $k \leftrightarrow f$; for $d > 1$, there is however one caveat to this correspondence. By replacing $\overline{\lambda} z$ with x when moving from k to f, we have altered the domain of absolute convergence/analyticity. That is, if $x_i := \overline{\lambda_i} z_i$ for λ and z in $\mathbb{B}_d$, then $\|x\|_1 := |x_1| + |x_2| + ... + |x_d| \leq \|\lambda\|_2 \|z\|_2 < 1$ by the Cauchy-Schwarz inequality. Also, if we allow the notation $|x|$ to refer to the tuple $(|x_1|, \ ..., \ |x_d|)$, we have that $f(|x|) = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha \left|\overline{\lambda} z\right|^\alpha$.

The comments above thus reveal that to correspond with $k_\lambda(z)$ being absolute convergent for any λ and z in $\mathbb{B}_d$, the appropriate condition on f is that $f(|x|) < \infty$ for any tuple x such that $\|x\|_1 < 1$.

As was previously mentioned, our main goal is to come up with the correct way to generalize the lemma to work for any d. However, we would be remiss not to mention that there is a certain class of power series indexed over $\mathbb{N}_0^d$ for which the original single-variable lemma can be applied for any value of d. We call k_λ *unitarily invariant* if and only if its power series has the form $k_\lambda(z) = \sum_{n=0}^{\infty} d_n \langle z, \lambda \rangle^n$ for some $\{d_n\} \subset (0, \infty]$; the name comes from the fact that $k_{U\lambda}(Uz) = k_\lambda(z)$ for any unitary U acting on $\mathbb{B}_d$.

If k_λ is of this type, then the previous correspondence can simply be $k_\lambda(z) \leftrightarrow f(w) = \sum_{n=0}^{\infty} d_n w^n$ where $w \in \mathbb{D}$ since $| \langle z, \lambda \rangle | < 1$ again by Cauchy-Schwarz if z, λ lie in $\mathbb{B}_d$. One way to think about this is that for a unitary invariant kernel, we get a "free extension" from 1 to d as far as using the original lemma goes. Of course, it is natural to want to come up with a new sufficient condition for a kernel to be CNP which can be used on a far less restrictive set of multi-variable kernels.

To that end, we now begin to set up the generalized lemma. For α, β in $\mathbb{Z}^d$, we can induce a partial order by defining $\beta \leq \alpha$ to mean $\beta_i \leq \alpha_i$ for $1 \leq i \leq d$. As an example, to say $\alpha \in \mathbb{N}_0^d$ exactly means $\alpha \geq \mathbf{0}$ using this ordering. We say a sequence $\{S_\alpha\}_{\alpha \in \mathbb{Z}^d}$ is non-decreasing if and only if $S_\alpha \leq S_\beta$ for $\alpha \leq \beta$.

Now suppose $f(z) = \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha z^\alpha$ satisfies $C_{\mathbf{0}} = 1$ and $C_\alpha > 0$ for all other α. For ease of notation/well-definedness in the sequel, we extend $\{C_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ to $\{C_\alpha\}_{\alpha \in \mathbb{Z}^d}$ by setting $C_\gamma = 0$ for any γ such that $\gamma \not\geq \mathbf{0}$ (i.e. γ that have at least one negative component).

Next, we define an auxillary sequence $\{R_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ by $R_{\mathbf{0}} := 0$ and

$$R_\alpha := \frac{C_\alpha}{\sum_{k=1}^{d} C_{\alpha - e_k}}, \quad \alpha \neq \mathbf{0},$$

where the e_k are the canonical basis multi-indices. We note that there are (infinitely) many $\alpha \geq \mathbf{0}$ that result in $\alpha - e_k \not\geq \mathbf{0}$ for at least one k, and therefore our previous extension comes into play to ensure the above denominator is well-defined; further, demanding that $\alpha \neq \mathbf{0}$ ensures the denominator is strictly positive, as at least one k will result in $\alpha - e_k \geq \mathbf{0}$.

Lemma 3.1.2 (Kaluza-Type Lemma): *Suppose $\{C_\alpha\}_{\alpha \in \mathbb{Z}^d}$ satisfies $C_{\mathbf{0}} = 1, C_\alpha > 0$ for all other α, and $C_\gamma = 0$ for any γ such that $\gamma \not\geq \mathbf{0}$. Define $\{R_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ by $R_{\mathbf{0}} := 0$ and*

$$R_\alpha := \frac{C_\alpha}{\sum_{k=1}^{d} C_{\alpha - e_k}}, \quad \alpha \neq \mathbf{0},$$

where the e_k are the canonical basis multi-indices.

If $\{R_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ is non-decreasing and bounded above by 1, then $f(x) := \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha x^\alpha$ converges on $\{x = (x_1,\ ...,\ x_d) \mid \|x\|_1 := |x_1| + |x_2| + ... + |x_d| < 1\}$ and $g(x) := 1 - \frac{1}{f(x)} = \sum_{\gamma \in \mathbb{N}_0^d} q_\gamma x^\gamma$ for some non-negative $\{q_\gamma\}_{\gamma \in \mathbb{N}_0^d}$.

Before we prove this, let us take a moment to address how the original lemma and this new one lead to the same condition to check when applied in the case that $k_\lambda(z)$ is unitarily invariant.

That is, suppose

$$k_\lambda(z) = \sum_{n=0}^{\infty} d_n \langle z, \lambda\rangle^n = \sum_{n=0}^{\infty} d_n \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} \left(\overline{\lambda} z\right)^\alpha = \sum_{\alpha\in\mathbb{N}_0^d} d_{|\alpha|} \frac{|\alpha|!}{\alpha!} \left(\overline{\lambda} z\right)^\alpha,$$

where we have expanded the series in order to see that $C_\alpha = d_{|\alpha|}\dfrac{|\alpha|!}{\alpha!}$. We comment now that this means that if $\alpha! C_\alpha$ does not depend only on $|\alpha|$, then k_λ is NOT unitarily invariant.

To show the desired consistency, we need the following: for any $\gamma \geq \mathbf{0}$, using that $\dfrac{1}{(\gamma - e_j)!} = \dfrac{\gamma_j}{\gamma!}$ as long as $\gamma_j > 0$,

$$\sum_{\substack{j=1 \\ \gamma_j>0}}^{d} \frac{|\gamma - e_j|!}{(\gamma - e_j)!} = \sum_{\substack{j=1 \\ \gamma_j>0}}^{d} \frac{(|\gamma|-1)!\gamma_j}{\gamma!} = \frac{(|\gamma|-1)!}{\gamma!} \sum_{\substack{j=1 \\ \gamma_j>0}}^{d} \gamma_j = \frac{|\gamma|!}{\gamma!}.$$

With this, we have:

$$\sum_{k=1}^{d} C_{\alpha-e_k} = \sum_{\substack{k=1 \\ \alpha_k>0}}^{d} d_{|\alpha-e_k|} \frac{|\alpha - e_k|!}{(\alpha - e_k)!} = d_{|\alpha|-1} \frac{|\alpha|!}{\alpha!},$$

which makes $R_\alpha = \dfrac{C_\alpha}{\sum_{k=1}^{d} C_{\alpha-e_k}} = \dfrac{d_{|\alpha|}}{d_{|\alpha|-1}}$. Thus, if we use $\hat{R}_n$ to refer to $\dfrac{d_n}{d_{n-1}}$ as in the original lemma, it is immediate that $R_\alpha = \hat{R}_{|\alpha|}$ and so both the conditions of non-decreasing and bounded by 1 are consistent between the old and new lemmas.

Proof of Lemma 3.1.2: Since the two assumptions on $\{R_\alpha\}$ are used in distinct ways, we break the proof into two steps.

Step 1: Use $\{R_\alpha\}$ bounded above by 1 to argue that f converges on

$$\{x = (x_1,\ ...,\ x_d) \mid \|x\|_1 := |x_1| + |x_2| + ... + |x_d| < 1\},$$

and subsequently that the $\{q_\gamma\}$ exist.

First note that $R_\beta \leq 1$ is equivalent to $C_\beta \leq \sum_{j=1}^{d} C_{\beta - e_j}$ as long as $|\beta| \geq 1$. For any α with $|\alpha| \geq 1$, we claim that

$$C_\alpha \leq \sum_{\substack{k=(k_1,\ \ldots,\ k_{|\alpha|})\\ 1 \leq k_i \leq d}} C_{\alpha - S^k} \quad \text{where} \quad S^k := \sum_{i=1}^{|\alpha|} e_{k_i}.$$

Noting that the former inequality is equivalent to the latter when $|\alpha| = 1$, we have shown the base case for an inductive approach of establishing the claim. With that in mind, we now fix an $N \geq 1$ and an α with $|\alpha| = N+1$, assuming the above holds for any multi-indices up to and including order N. Since not only is $|\alpha| \geq 1$ but also $|\alpha - e_j| = |\alpha| - 1 = N$ for all j, we have by the base case inequality as well as the inductive hypothesis:

$$C_\alpha \leq \sum_{j=1}^{d} C_{\alpha - e_j} \leq \sum_{j=1}^{d} \sum_{\substack{k=(k_1,\ \ldots,\ k_{|\alpha|-1})\\ 1 \leq k_i \leq d}} C_{\alpha - (e_j + S^k)} = \sum_{\substack{k=(k_1,\ \ldots,\ k_{|\alpha|})\\ 1 \leq k_i \leq d}} C_{\alpha - S^k}.$$

With that established, we now argue that this sum bounding C_α precisely equals $\dfrac{|\alpha|!}{\alpha!}$. To do this, we first mention that by the additive property of the order $|\cdot|$, it is clear that $|S^k| = |\alpha|$ for any k in the sum. Suppose we have a k such that $\alpha \neq S^k$: if $d = 1$, this is an immediate contradiction of the previous observation; if $d \geq 2$, the only possibility is that $\alpha \neq S^k$ has at least one positive component and one negative component, but the latter of these invokes our extension and thus $C_{\alpha - S^k} = 0$ for these k.

This is to say, that the only k in the sum besides those such that $C_{\alpha - S^k} = C_{\mathbf{0}} = 1$ result in vanishing summands. All that remains in this part of the argument is to comment that, for a fixed α with $|\alpha| = \alpha_1 + \ldots + \alpha_d$, the number of $k = (k_1,\ \ldots,\ k_{|\alpha|})$ satisfying

$$\#\{1 \leq j \leq |\alpha| \mid k_j = i\} = \alpha_i \quad \forall\ 1 \leq i \leq d$$

is precisely $\dfrac{|\alpha|!}{\alpha!}$. In fact, this was the exact same combinatorial idea employed in Section 1.3, just without explicitly referring to noncommutative/word notation.

Noting then that the left-hand number/cardinality above is S_i^k, all of our work in aggregate establishes

$$C_\alpha \leq \sum_{\substack{k=(k_1,\ \ldots,\ k_{|\alpha|}) \\ 1\leq k_i \leq d}} C_{\alpha - S^k} = \#\{k = (k_1,\ \ldots,\ k_{|\alpha|}) \mid \alpha = S^k\} = \frac{|\alpha|!}{\alpha!}$$

for nonzero α. Since $C_{\mathbf{0}} = 1$, this is also true (with equality) when $\alpha = \mathbf{0}$ as well.

From here, if we take an $x \in \mathbb{C}^d$ and M satisfying $\|x\|_1 \leq M < 1$, we have

$$f(|x|) = \sum_{\alpha\in\mathbb{N}_0^d} C_\alpha |x|^\alpha \leq \sum_{\alpha\in\mathbb{N}_0^d} \frac{|\alpha|!}{\alpha!}|x|^\alpha = \frac{1}{1-\|x\|_1^2} \leq \frac{1}{1-M^2} < \infty.$$

Therefore, the power series of f is uniformly convergent on $\{x = (x_1,\ \ldots,\ x_d) \mid \|x\|_1 \leq M\}$ for any $M < 1$, proving our convergence claim. Then, since $f(\mathbf{0}) = C_{\mathbf{0}} = 1$, g is thus analytic in a neighborhood of $\mathbf{0}$ and subsequently the $\{q_\gamma\}_{\gamma\in\mathbb{N}_0^d}$ exist, as desired.

Before we move to Step 2, we desire an identity linking the C_α and q_γ. Note that $f(x) - 1 = f(x)g(x)$ and in series notation this gives

$$\sum_{\alpha\neq\mathbf{0}} C_\alpha x^\alpha = \sum_{\alpha\in\mathbb{N}_0^d}\sum_{\gamma\in\mathbb{N}_0^d} C_\alpha q_\gamma x^{\alpha+\gamma}.$$

We rearrange the right hand side to obtain

$$\sum_{\alpha\neq\mathbf{0}} C_\alpha x^\alpha = \sum_{\alpha\neq\mathbf{0}} \left(\sum_{\mathbf{0}\leq\beta\leq\alpha} q_\beta C_{\alpha-\beta}\right) x^\alpha$$

The reason we can exclude $\alpha = \mathbf{0}$ in the right hand sum is that the corresponding term vanishes since $C_{\mathbf{0}} = f(\mathbf{0}) = 1$ implies that $q_{\mathbf{0}} = g(\mathbf{0}) = 0$. Comparing coefficients, we are ready to write what we will hereby refer to as the main identity:

$$C_\alpha = \sum_{\mathbf{0}\leq\beta\leq\alpha} q_\beta C_{\alpha-\beta} = q_\alpha + \sum_{\substack{\mathbf{0}\leq\beta\leq\alpha \\ \beta\neq\alpha}} q_\beta C_{\alpha-\beta}, \quad \alpha \neq \mathbf{0}$$

Step 2: Use $\{R_\alpha\}$ non-decreasing to show that all $q_\gamma \geq 0$.

We will proceed by induction on the length of the multi-index $|\alpha| := \alpha_1 + ... + \alpha_d$. As a base case, $|\alpha| = 0$ means $\alpha = \mathbf{0}$ and we have already noted $q_{\mathbf{0}} = 0$. Also, the main identity reveals $0 < C_{e_k} = q_{e_k}$ for any k, covering all d possibilities of $|\alpha| = 1$.

Moving to the inductive step, we fix an $N \geq 1$ and an α with $|\alpha| = N + 1$, assuming that the $\{q_\gamma\}_{|\gamma| \leq N}$ are all non-negative. All we must show is that $q_\alpha \geq 0$. Define, for every $1 \leq j \leq d$, the scalars $p_j := \dfrac{C_{\alpha - e_j}}{\sum_{k=1}^{d} C_{\alpha - e_k}}$ alongside the index set $\mathcal{J} := \{1 \leq j \leq d \mid C_{\alpha - e_j} \neq 0\}$. Two important properties that we will make use of with these are that $\sum_{j \in \mathcal{J}} p_j = 1$, and that $\dfrac{C_\alpha p_j}{C_{\alpha - e_j}} = R_\alpha$ for every $j \in \mathcal{J}$. Also, $j \in \mathcal{J}$ if and only if $\alpha - e_j \geq \mathbf{0}$ (by our extension, nonzero coefficients only occur when the subscript is in $\mathbb{N}_0^d$), and $|\alpha| \geq 2$ ensures that $\alpha - e_j \neq \mathbf{0}$ for these j. Thus, we can use the main identity $\#\mathcal{J}$ times (resulting in the bracketed expression below being 0 for every j) to obtain

$$0 = \sum_{j \in \mathcal{J}} \frac{C_\alpha p_j}{C_{\alpha - e_j}} \left[C_{\alpha - e_j} - \sum_{\mathbf{0} \leq \beta \leq \alpha - e_j} q_\beta C_{\alpha - e_j - \beta} \right] = C_\alpha - R_\alpha \sum_{j \in \mathcal{J}} \sum_{\mathbf{0} \leq \beta \leq \alpha - e_j} q_\beta C_{\alpha - e_j - \beta}.$$

Since $j \notin \mathcal{J}$ if and only if $\alpha - e_j \not\geq \mathbf{0}$, the inner sum is empty for those such j. Therefore, we choose to write the outside sum over all j, rewriting the above as:

$$C_\alpha = R_\alpha \sum_{j=1}^{d} \sum_{\mathbf{0} \leq \beta \leq \alpha - e_j} q_\beta C_{\alpha - e_j - \beta} = \sum_{\substack{\mathbf{0} \leq \beta \leq \alpha \\ \beta \neq \alpha}} q_\beta R_\alpha \sum_{\substack{j=1 \\ \beta_j < \alpha_j}}^{d} C_{\alpha - e_j - \beta},$$

where we have interchanged the order of the two sums by noting that $\mathbf{0} \leq \beta \leq \alpha - e_j$ if and only if $\mathbf{0} \leq \beta \leq \alpha$ and $\beta_j < \alpha_j$ (which of course excludes $\beta = \alpha$). However, despite the qualifier $\beta_j < \alpha_j$ being a part of the inner sum, it is fully legal for us to not include it; this is because the only other option of $\beta_j = \alpha_j$ again invokes our extension to set $C_\gamma = 0$ for any multi-index γ with at least one negative component.

Next, using the main identity once more as well as substituting in the previous equality, we get

$$q_\alpha = C_\alpha - \sum_{\substack{\mathbf{0}\leq\beta\leq\alpha \\ \beta\neq\alpha}} q_\beta C_{\alpha-\beta} = \sum_{\substack{\mathbf{0}\leq\beta\leq\alpha \\ \beta\neq\alpha}} q_\beta \left[R_\alpha \left(\sum_{j=1}^{d} C_{\alpha-e_j-\beta} \right) - C_{\alpha-\beta} \right]$$

$$= \sum_{\substack{\mathbf{0}\leq\beta\leq\alpha \\ \beta\neq\alpha}} q_\beta \sum_{j=1}^{d} C_{\alpha-e_j-\beta}(R_\alpha - R_{\alpha-\beta}).$$

The factor in parentheses is non-negative due to our assumption of R being non-decreasing, as $\mathbf{0} \leq \alpha - \beta \leq \alpha$ for all β in the sum; as for the factors being multiplied on the outside, the sum of C's are positive by our starting assumption, and every q in the above is assumed to be non-negative by the inductive hypothesis: if $\mathbf{0} \leq \beta \leq \alpha$ but $\beta \neq \alpha$ and $|\alpha| = N+1$, then it must be that $|\beta| \leq N$. All of this combined shows $q_\alpha \geq 0$, which finishes the proof.

Next, we present the following corollary, which reveals that any k_λ that is CNP by means of Lemma 3.1.2 is in fact a de Branges-Rovnyak kernel as well.

Corollary 3.1.3: *Suppose we have a reproducing kernel* $k_\lambda(z) = \sum_{\alpha\in\mathbb{N}_0^d} C_\alpha \left(\overline{\lambda} z\right)^\alpha$ *for some* $\{C_\alpha\}$ *satisfying the assumptions of Lemma 3.1.2. Then*

$$k_\lambda(z) = k_\lambda^b(z) := \frac{1 - b(z)b(\lambda)^*}{1 - \langle z, \lambda \rangle}$$

for some $b(z) = \left(\sqrt{b_\alpha} z^\alpha\right)_{\alpha\in\mathbb{N}_0^d}$ *with non-negative* $\{b_\alpha\}$.

Further, $\{C_\alpha\}$ *and* $\{b_\alpha\}$ *are related by*

$$C_\alpha = \frac{|\alpha|!}{\alpha!} - \sum_{\mathbf{0}\leq\beta\leq\alpha} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} b_\beta.$$

Proof: Since $k_\lambda(\mathbf{0}) = C_\mathbf{0} = 1$ and $k_\lambda^b(\mathbf{0}) = 1 - b_\mathbf{0}$, we see right away that it must be that we take $b_\mathbf{0} := 0$ or else the corollary can't possibly be true.

We take $b_\alpha := \dfrac{C_\alpha}{R_\alpha}(1-R_\alpha)$ for nonzero $\alpha \in \mathbb{N}_0^d$. Since C_α are strictly positive, the assumption that $0 < R_\alpha \leq 1$ gives that $b_\alpha \geq 0$. The rest of the proof is in the form of two distinct computations that collectively establish that $k_\lambda = k_\lambda^b$.

Invoking the expansion of the Drury-Arveson kernel $\dfrac{1}{1-\langle z,\lambda\rangle} = \sum_{\gamma\geq\mathbf{0}} \dfrac{|\gamma|!}{\gamma!}\left(\overline{\lambda}z\right)^\gamma$, we have

$$k_\lambda^b(z) = \frac{1-b(z)b(\lambda)^*}{1-\langle z,\lambda\rangle} = \sum_{\gamma\geq\mathbf{0}} \frac{|\gamma|!}{\gamma!}\left(\overline{\lambda}z\right)^\gamma \left(1-\sum_{\alpha\geq\mathbf{0}} b_\alpha\left(\overline{\lambda}z\right)^\alpha\right)$$

$$= \sum_{\gamma\geq\mathbf{0}} \frac{|\gamma|!}{\gamma!}\left(\overline{\lambda}z\right)^\gamma - \sum_{\gamma\geq\mathbf{0}}\sum_{\alpha\geq\mathbf{0}} \frac{|\gamma|!}{\gamma!} b_\alpha\left(\overline{\lambda}z\right)^{(\alpha+\gamma)}$$

Working on the sum being subtracted, we have:

$$\sum_{\alpha\geq\mathbf{0}}\sum_{\gamma\geq\mathbf{0}} \frac{|\gamma|!}{\gamma!} b_\alpha\left(\overline{\lambda}z\right)^{(\alpha+\gamma)} = \sum_{\alpha\geq\mathbf{0}}\sum_{\beta\geq\alpha} \frac{|\beta-\alpha|!}{(\beta-\alpha)!} b_\alpha\left(\overline{\lambda}z\right)^\beta = \sum_{\beta\geq\mathbf{0}}\left(\sum_{\mathbf{0}\leq\alpha\leq\beta} \frac{|\beta-\alpha|!}{(\beta-\alpha)!} b_\alpha\right)\left(\overline{\lambda}z\right)^\beta$$

Taken together, these expressions show that the α coefficient (for $\alpha \neq \mathbf{0}$) in the power series for k_λ^b is

$$\frac{|\alpha|!}{\alpha!} - \sum_{\mathbf{0}\leq\beta\leq\alpha} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} b_\beta,$$

which holds outside the context of this corollary. The second of our two calculations is to show, from our particular definition of b_α, that this expression equals C_α.

To that effect, we invoke the form of R_α to see $b_\alpha = \sum_{j=1}^{d} C_{\alpha-e_j} - C_\alpha$, which we immediately turn around into $C_\alpha = \sum_{\substack{j=1\\ \alpha_j>0}}^{d} C_{\alpha-e_j} - b_\alpha$; it will soon be clear why we choose at this step to explicitly remove the vanishing terms from the sum.

Like we have already done many times, the remaining calculation takes an inductive approach on the order of $|\alpha|$. Considering the base case of $|\alpha| = 1$ (i.e. $\alpha = e_k$ for some k), we have from the formula for C_α that $C_{e_k} = 1 - b_{e_k}$, and this matches the desired expression since the only β to show up in the latter sum in this case is $\alpha = e_k$ itself.

Recall that for any $\gamma \in \mathbb{N}_0^d$,

$$\sum_{\substack{j=1 \\ \gamma_j > 0}}^{d} \frac{|\gamma - e_j|!}{(\gamma - e_j)!} = \frac{|\gamma|!}{\gamma!},$$

an identity we will need for the inductive step. We now fix an α of length $N+1$ and assume the desired equality holds for any β of order $1 \leq |\beta| \leq N$. Applying this to the $\alpha - e_j$ for the j that show up in the following sum, we have:

$$C_\alpha = \sum_{\substack{j=1 \\ \alpha_j > 0}}^{d} C_{\alpha - e_j} - b_\alpha = \sum_{\substack{j=1 \\ \alpha_j > 0}}^{d} \left(\frac{|\alpha - e_j|!}{(\alpha - e_j)!} - \sum_{\mathbf{0} \leq \beta \leq \alpha - e_j} \frac{|\alpha - e_j - \beta|!}{(\alpha - e_j - \beta)!} b_\beta \right) - b_\alpha$$

$$= \frac{|\alpha|!}{\alpha!} - b_\alpha - \sum_{j=1}^{d} \sum_{\mathbf{0} \leq \beta \leq \alpha - e_j} \frac{|\alpha - e_j - \beta|!}{(\alpha - e_j - \beta)!} b_\beta,$$

where we have removed the $\alpha_j > 0$ qualifier since the inside sum over β is entirely empty in the only other case that $\alpha_j = 0$. Continuing with the last term in the above, we again switch the order of the sums as we did in the proof of the lemma to see it equals:

$$\sum_{\substack{\mathbf{0} \leq \beta \leq \alpha \\ \beta \neq \alpha}} \sum_{\substack{j=1 \\ \beta_j < \alpha_j}}^{d} \frac{|\alpha - e_j - \beta|!}{(\alpha - e_j - \beta)!} b_\beta = \sum_{\substack{\mathbf{0} \leq \beta \leq \alpha \\ \beta \neq \alpha}} b_\beta \sum_{\substack{j=1 \\ (\alpha - \beta)_j > 0}}^{d} \frac{|\alpha - \beta - e_j|!}{(\alpha - \beta - e_j)!} = \sum_{\substack{\mathbf{0} \leq \beta \leq \alpha \\ \beta \neq \alpha}} b_\beta \frac{|\alpha - \beta|!}{(\alpha - \beta)!}.$$

Going back to C_α and adding in the b_α term so that $\beta = \alpha$ indeed shows up in the sum, we have arrived at the destination, establishing

$$C_\alpha = \frac{|\alpha|!}{\alpha!} - \sum_{\mathbf{0} \leq \beta \leq \alpha} \frac{|\alpha - \beta|!}{(\alpha - \beta)!} b_\beta$$

by way of induction. As mentioned, this shows $k_\lambda = k_\lambda^b$ and completes the proof of the corollary.

Now, we approach the ideas featured in Corollary 3.1.3 from a different perspective: suppose we start with $b(z) = \left(\sqrt{b_\alpha} z^\alpha\right)_{\alpha \in \mathbb{N}_0^d}$ for non-negative $\{b_\alpha\}$, with $b_{\mathbf{0}} = 0$ specifically, and construct with it a de-Branges kernel k^b. As seen in the previous proof, we know the expression for each of the k^b power series coordinates C_α. However, our existing framework requires $C_\alpha > 0$. If k^b is positive definite, then its power series coefficients must all be non-negative, so we only need to rule out that any are equal to 0.

We now show that in the case that $(M_z, H(b))$ is bounded, every C_α is strictly positive. Recalling from the start of the chapter that $C_\alpha := (\|z^\alpha\|^2_{H(b)})^{-1}$, we will do this by arguing that the norm appearing in the preceding denominator is finite for all α. Proceeding inductively, the base case is clear since our kernel is normalized so that $\|z^{\mathbf{0}}\|_{H(b)} = \|1\|_{H(b)} = \|k^b_0\|_{H(b)} = |k^b_0(0)| = 1$. Next, assume we have z^α such that $|\alpha| \geq 1$ and $\|z^\beta\|_{H(b)}$ finite for any β of smaller length than α. Since $|\alpha| \geq 1$, there must be at least one component $\alpha_i \neq 0$. If so, then define $f(z)$ to be the column with 0's in every component except the ith and $z^{\alpha - e_i}$ in the ith component. With this, we have

$$\|z^\alpha\|_{H(b)} = \|M_z f\|_{H(b)} \leq \|M_z\|_{\text{op}} \|f\|_{H(b)^d} = \|M_z\|_{\text{op}} \|z^{\alpha - e_i}\|_{H(b)},$$

which is finite by our inductive hypothesis. Therefore, no C_α can be zero and we are ready to continue our discussion now that this has been settled.

One advantage of the $b_\alpha \leftrightarrow C_\alpha$ construction given by $b_\alpha = \dfrac{C_\alpha}{R_\alpha}(1 - R_\alpha)$ is as follows. Suppose we have a k^b just as above for which there is some $M \in \mathbb{N}$ such that $b_\alpha = 0$ for all α with $|\alpha| > M$. Next, say that we wish to check, by Lemma 3.1.2, whether k^b is a CNP kernel. Having each $b_\alpha \geq 0$ means that 1 is the upper bound on the R_α, and from there knowing that $b_\alpha = 0$ if and only if $R_\alpha = 1$ means that we only need to check the non-decreasing condition for the set of α up to and including order M.

To illustrate with one class of examples, take $M = 2$. We only need to make sure that $R_\alpha \leq R_\beta$ when $|\alpha| = 1$ and $|\beta| = 2$. Succinctly, this is demanding

$$R_{e_i} \leq \min\{R_{2e_i},\ R_{e_i + e_j}\} \ \text{ for all } 1 \leq i, j \leq d,\ i \neq j.$$

Examining the definitions, we compute, for fixed i and j, $i \neq j$:

$$R_{e_i} = \frac{C_{e_i}}{C_{\mathbf{0}}} = 1 - b_{e_i}; \quad R_{2e_i} = \frac{C_{2e_i}}{C_{e_i}} = \frac{1 - b_{e_i} - b_{2e_i}}{1 - b_{e_i}};$$

$$R_{e_i+e_j} = \frac{C_{e_i+e_j}}{C_{e_i}+C_{e_j}} = \frac{2-b_{e_i}-b_{e_j}-b_{e_i+e_j}}{2-b_{e_i}-b_{e_j}}.$$

From here, it is only algebra to see that

$$R_{e_i} \leq R_{2e_i} \Longleftrightarrow b_{2e_i} \leq b_{e_i}(1-b_{e_i})$$

$$R_{e_i} \leq R_{e_i+e_j} \Longleftrightarrow b_{e_i+e_j} \leq b_{e_i}(2-b_{e_i}-b_{e_j})$$

Thus, if the $\{b_\alpha\}$ satisfy the above right-hand side inequalities for all i and j, $i \neq j$, as well as the rest of the assumptions, then k^b is a CNP kernel by Lemma 3.1.2.

Before we move on, allow us to make this example class more specific by taking $d = 2$, for then we only have five parameters affecting k^b: $b_{1,0},\ b_{0,1},\ b_{2,0},\ b_{0,2}$, and $b_{1,1}$. Besides needing to all be non-negative, the previous set of inequalities reduces to:

$$b_{2,0} \leq b_{1,0}(1-b_{1,0}); \quad b_{0,2} \leq b_{0,2}(1-b_{0,2}); \quad b_{1,1} \leq \min\{b_{1,0},\ b_{0,1}\}(2-b_{1,0}-b_{0,1})$$

To write one particular example of a set of values satisfying these, we can take $b_{1,0} = b_{0,1} = b_{1,1} = \frac{1}{4}$ and $b_{2,0} = b_{0,2} = \frac{1}{3}$, which corresponds to

$$b(z) = \left(\frac{1}{2}z_1,\ \frac{1}{2}z_2,\ \frac{1}{\sqrt{3}}z_1^2,\ \frac{1}{2}z_1z_2,\ \frac{1}{\sqrt{3}}z_2^2\right).$$

One minor notational note to address with this is that we can always replace $b(z) = \left(\sqrt{b_\alpha}z^\alpha\right)_{\alpha\in\mathbb{N}_0^d}$ with $b(z) = \left(\sqrt{b_\alpha}z^\alpha\right)_{\substack{\alpha\in\mathbb{N}_0^d \\ b_\alpha\neq 0}}$ as the expression $b(z)b(\lambda)^*$ is all that affects the form of k^b.

This last example provides one more important purpose. From the formulas, $C_{2,0} = 1-b_{1,0}-b_{2,0} = \frac{5}{12}$ and $C_{1,1} = 2-b_{1,0}-b_{0,1}-b_{1,1} = \frac{5}{4}$; taken together, this is enough to show that k^b is not unitarily invariant: $2C_{2,0} \neq C_{1,1}$ and therefore $\alpha! C_\alpha$ does not depend only on $|\alpha|$. This motivates our generalization, as the original Kaluza's lemma cannot be applied to show that k^b is a CNP kernel, despite the fact that it indeed is by Lemma 3.1.2.

3.2 An Alternate Generalization

We now aim for a separate generalization of the lemma, one that still gives the same conclusion but with an alternate set of assumptions on (only) the coefficients of a given power series. A major difference present is that what follows uses word notation, the non-commutative analogue of multi-index notation.

Theorem 3.2.1: *Suppose* $\{C_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ *are such that* $C_{\mathbf{0}} = 1$ *and* $C_\alpha > 0$ *for all other* α*. If each* C_α *can be decomposed as* $C_\alpha = \sum_{\alpha(v)=\alpha} D_v$ *for some* $\{D_v\}_{v \in \mathbb{F}_d} \subset (0,1]$ *that satisfy* $D_{iw} D_{wj} \leq D_w D_{iwj}$ *for all letters* $1 \leq i, j \leq d$ *and words* w*, then* $f(x) := \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha x^\alpha$ *converges on* $\{x = (x_1, \ ..., \ x_d) \mid \|x\|_1 := |x_1| + |x_2| + ... + |x_d| < 1\}$ *and* $g(x) := 1 - \dfrac{1}{f(x)} = \sum_{\alpha \neq 0} q_\alpha x^\alpha$*, where* $\{q_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ *are such that* $q_\alpha = \sum_{\alpha(v)=\alpha} \hat{q}_v$ *for some non-negative* $\{\hat{q}_v\}_{v \in \mathbb{F}_d}$*.*

Clearly, each $q_\alpha \geq 0$ in the above, and thus this theorem could be applied to a $k_\lambda(z)$ precisely the same as in Section 3.1. The decomposition of the q_α into the $\hat{q}_v$ is merely an extra that comes out of the proof.

If $d = 1$, then the only allowable decomposition in Theorem 3.2.1 is $D_v = C_\alpha = C_n > 0$, as there is no difference between words and multi-indices in this case. The central inequality in the above then becomes $C_{n+1}^2 \leq C_n C_{n+2}$ which can immediately be rearranged into $R_{n+1} \leq R_{n+2}$ for $R_n := \dfrac{C_n}{C_{n-1}}$. Therefore, this again reduces to the original Kaluza's lemma when we have only one variable.

Proof of Theorem 3.2.1:

Step 1: Use $D_v \leq 1$ to argue that f converges on

$$\{x = (x_1, \ ..., \ x_d) \mid \|x\|_1 := |x_1| + |x_2| + ... + |x_d| < 1\},$$

and subsequently that the $\{q_\alpha\}$ exist.

This step is incredibly similar to the first step in the proof of Lemma 3.1.2. Defining $N_\alpha := \#\{v \mid \alpha(v) = \alpha\} = \dfrac{|\alpha|!}{\alpha!}$, it is immediate from the assumed decomposition that $D_v \leq 1$ for all v implies $C_\alpha \leq N_\alpha$ for all α. This inequality was precisely what we used to argue the convergence claim as well as the existence of the $\{q_\alpha\}$ in the proof of Lemma 3.1.2, again also using the assumption that $f(\mathbf{0}) = 1$.

We will make use of the same central identity from Lemma 3.1.2 as well, which can now be obtained. Repeated here for convenience, we have:

$$q_\alpha = C_\alpha - \sum_{\substack{\mathbf{0} \leq \beta \leq \alpha \\ \beta \neq \alpha}} q_\beta C_{\alpha-\beta}, \quad \alpha \neq \mathbf{0}$$

Step 2: Define $\{\hat{q}_v\}$ inductively through $\hat{q}_\emptyset := 0$ and the equation

$$\hat{q}_v := D_v - \sum_{\substack{uw=v \\ w \neq \emptyset}} \hat{q}_u D_w$$

for $v \neq \emptyset$, and then show by induction that every $\hat{q}_v \geq 0$ using the $D_{iw}D_{wj} \leq D_w D_{iwj}$ assumption.

Let us start with some alternate notation to be used in the definition of the $\hat{q}_v$. For a fixed word v and $1 \leq i \leq |v| - 1$, define $\underline{v}^i = v_1 ... v_i$ and $\overline{v}^i = v_{i+1} ... v_{|v|}$ as well as $\underline{v}^{|v|} := v =: \overline{v}^0$ and $\overline{v}^{|v|} := \emptyset =: \underline{v}^0$. Noting that $\underline{v}^i \overline{v}^i = v$ for any $0 \leq i \leq |v|$, we have the following, written two different ways:

$$\hat{q}_v = D_v - \sum_{i=1}^{|v|-1} \hat{q}_{\underline{v}^i} D_{\overline{v}^i}; \qquad D_v = \sum_{i=1}^{|v|} \hat{q}_{\underline{v}^i} D_{\overline{v}^i}$$

for any $v \neq \emptyset$. Next, since none of the D_v are zero, we can define the auxiliary sequence $R_v := \dfrac{D_v}{D_{\underline{v}^{|v|-1}}}$, where we comment that the denominator is $D_\emptyset = C_{\mathbf{0}} = 1$ in the case that v is only a single letter.

We can already see in the denominator of R_v that the notation here starts to get a tad hairy, so for convenience in the rest of the proof of this step, fix v and introduce $x := \underline{v}^{|v|-1}$. This has easily verified properties $|x| = |v| - 1$ and $\underline{x}^i = \underline{v}^i$ for $1 \leq i \leq |x|$. Less immediate due to admittedly absurd notation yet still true is

$$\overline{x}^i = \underline{\left(\overline{v}^i\right)}^{|\overline{v}^i|-1} \text{ for } 1 \leq i \leq |x|$$

since carefully parsing through the definitions used above reveals both equal $v_{i+1}...v_{|x|}$ for a fixed $1 \leq i \leq |x| - 1$ and both equal $\emptyset$ for $i = |x|$. The use for these properties is that we can now write $R_{\overline{v}^i} = \dfrac{D_{\overline{v}^i}}{D_{\overline{x}^i}}$ as well as

$$D_v = R_v D_x = R_v \sum_{i=1}^{|x|} \hat{q}_{\underline{x}^i} D_{\overline{x}^i} = \sum_{i=1}^{|v|-1} \hat{q}_{\underline{v}^i} D_{\overline{x}^i} R_v$$

and this, more importantly, leads to

$$\hat{q}_v = D_v - \sum_{i=1}^{|v|-1} \hat{q}_{\underline{v}^i} D_{\overline{v}^i} = \sum_{i=1}^{|v|-1} \hat{q}_{\underline{v}^i} D_{\overline{x}^i} R_v - \sum_{i=1}^{|v|-1} \hat{q}_{\underline{v}^i} D_{\overline{x}^i} R_{\overline{v}^i} = \sum_{i=1}^{|v|-1} \hat{q}_{\underline{v}^i} D_{\overline{x}^i} (R_v - R_{\overline{v}^i}).$$

Here is where we use the (rewritten) assumption that $\dfrac{D_{wj}}{D_w} \leq \dfrac{D_{iwj}}{D_{iw}}$, again noting that no denominators will be zero, even in the case of empty subscripts. However, we seek a form of this where the letter i in the right-hand side is extended to be any word u, which we obtain as follows: if u has length m, then we apply the assumption m times (working *backwards* through the letters of u)

$$\frac{D_{wj}}{D_w} \leq \frac{D_{u_m wj}}{D_{u_m w}} \leq \frac{D_{u_{m-1} u_m wj}}{D_{u_{m-1} u_m w}} \leq ... \leq \frac{D_{uwj}}{D_{uw}},$$

as desired. Now, returning to our expression of $\hat{q}_v$ and the associated notation, we apply the obtained inequality with $u = \underline{v}^i$, $w = \overline{x}^i$ and $j = v_{|v|}$ for every $1 \leq i \leq |v| - 1$ to arrive at $R_v - R_{\overline{v}^i} \geq 0$.

All this preparation leads to a quick proof by induction on $|v|$ that the $\hat{q}_v$ are all non-negative: the base cases are taken care by $\hat{q}_\emptyset = 0$ and $\hat{q}_k = D_k$ for any letter k. As for the inductive step, fix a v of length $N+1$ and assume $\hat{q}_u \geq 0$ for any $0 \leq |u| \leq N$. In particular, $\hat{q}_{\underline{v}^i} \geq 0$ for $1 \leq i \leq |v| - 1$. Since the $\{D_w\}$ are all strictly positive, an examination of the above expression for $\hat{q}_v$ reveals it indeed must be non-negative. This completes this step of the proof.

Step 3: Show that $q_\alpha = \sum_{\alpha(v)=\alpha} \hat{q}_v$.

As usual, we will proceed by induction on $|\alpha|$, but some preliminaries are required. First, define the following sets for a fixed nonzero α:

$$G_\beta := \{(u,w) \mid \alpha(u) = \beta,\ \alpha(w) = \alpha - \beta\},\ \text{for } \mathbf{0} \le \beta \le \alpha,\ \beta \neq \alpha.$$

$$H_v := \{(u,w) \mid w \neq \emptyset,\ uw = v\},\ \text{for } v \text{ such that } \alpha(v) = \alpha.$$

Some remarks: first, it is clear that any two distinct G_β must be disjoint, and the same is true for any two distinct H_v; second, we can use the additive identity $\alpha(uw) = \alpha(u) + \alpha(w)$ to argue that

$$\bigcup_{\substack{\mathbf{0}\le\beta\le\alpha \\ \beta\neq\alpha}} G_\beta = \bigcup_{\alpha(v)=\alpha} H_v$$

To see this, we observe (1) that if $(u,w) \in G_\beta$ for an appropriate β, then $\alpha(uw) = \alpha$ by the aforementioned identity and then trivially $(u,w) \in H_{uw}$, and (2) that if $(u,w) \in H_v$ for an appropriate v, then $w \neq \emptyset$ ensures $\alpha(u) \neq \alpha$ and we have that $(u,w) \in G_{\alpha(u)}$ (again using the identity).

With this set identity in tow, we use it to compute:

$$\sum_{\alpha(v)=\alpha} \hat{q}_v = \sum_{\alpha(v)=\alpha} D_v - \sum_{\alpha(v)=\alpha} \sum_{\substack{uw=v \\ w\neq\emptyset}} \hat{q}_u D_w = C_\alpha - \sum_{\substack{\mathbf{0}\le\beta\le\alpha \\ \beta\neq\alpha}} \sum_{\substack{\alpha(u)=\beta \\ \alpha(w)=\alpha-\beta}} \hat{q}_u D_w =$$

$$C_\alpha - \sum_{\substack{\mathbf{0}\le\beta\le\alpha \\ \beta\neq\alpha}} \left(\sum_{\alpha(u)=\beta} \hat{q}_u\right)\left(\sum_{\alpha(w)=\alpha-\beta} D_w\right) = C_\alpha - \sum_{\substack{\mathbf{0}\le\beta\le\alpha \\ \beta\neq\alpha}} C_{\alpha-\beta} \sum_{\alpha(u)=\beta} \hat{q}_u$$

This is precisely the set-up we need to proceed with the inductive step of the argument, noting that we have base cases for the claim $q_{\mathbf{0}} = 0 = \hat{q}_\emptyset$ and $q_{e_k} = C_{e_k} = D_k = \hat{q}_k$ for $1 \le k \le d$. If α is indeed fixed and we assume the claim is true for any $\mathbf{0} \le \beta \le \alpha,\ \beta \neq \alpha$, we are in the immediate position to apply this to where the above string of equalities left off to get

$$\sum_{\alpha(v)=\alpha} \hat{q}_v = C_\alpha - \sum_{\substack{\mathbf{0}\le\beta\le\alpha \\ \beta\neq\alpha}} C_{\alpha-\beta} q_\beta = q_\alpha$$

from the identity obtained at the end of Step 1. The proof is now complete.

The preceding theorem/proof applies noncommutative notation to functions of commuting variables and entirely forgoes noncommutative function theory, which is outside the scope of this text. Regardless, we now make some observations about how one might attempt to restate the theorem as one applied to $f(X) = \sum_{v\in\mathbb{F}_d} D_v X_v$ where X is now, say, a d-tuple of square matrices of the same size. Step 2 in the above proof essentially proved the main part of a noncommutative version of Kaluza's lemma, that the condition $D_{iw}D_{wj} \le D_w D_{iwj}$ for all words w and letters i and j is enough to guarantee that the set of $\hat{q}_v$ are all non-negative.

However, note that an analogue of the expression $1 - \dfrac{1}{f(x)}$ becomes harder to define in this situation as $f(X)$ is not scalar valued in general. Perhaps the best way to phrase such a result is by saying that, with the inequality hypothesis, any noncommutative function g satisfying $g(X)f(X) = f(X) - I_n = f(X)g(X)$ whenever X is a n by n matrix must have non-negative coefficients in its noncommutative power series representation. Yet, even then, this totally ignores the details of what would it mean for a noncommutative series to converge (e.g. finding a analogous argument to Step 1). All that being said, it may be fairly straightforward to revamp Theorem 3.2.1 into a fully noncommutative version of Kaluza's lemma, but this was not the focus of our current chapter.

Moving on, note that while Theorem 3.2.1 allows for any decomposition of each C_α as the sum of $\{D_v\}_{\alpha(v)=\alpha}$, there is a certain "canonical" way to do so by taking $D_v := \dfrac{C_{\alpha(v)}}{N_{\alpha(v)}}$. Each D_v will be in $(0, 1]$ as long as we again have $C_\alpha \le N_\alpha$, the necessary condition obtained along the way in the proofs of both the lemma and the theorem.

The condition we should investigate further is $D_{iw}D_{wj} \le D_w D_{iwj}$ in this particular case. Substituting in and rearranging the C's to the left-hand side and the N's to the right-hand, this condition is equivalent to

$$\frac{C_{\alpha(iw)}C_{\alpha(wj)}}{C_{\alpha(w)}C_{\alpha(iwj)}} \le \frac{N_{\alpha(iw)}N_{\alpha(wj)}}{N_{\alpha(w)}N_{\alpha(iwj)}}.$$

However, since $\alpha(iw) = \alpha(w) + e_i = \alpha(wi)$ (and therefore $\alpha(iwj) = \alpha(w) + e_i + e_j$), this condition needs not hold for all w, i, and j, but rather for all α, i, and j: in other words, we only need to check the condition for a single element of $\{w \mid \alpha(w) = \alpha\}$ for each α and thus there is no need to involve words moving forward in this part of the discussion.

We will soon state our results as a formal corollary but first let us simplify the right-hand side of the central inequality, making use of the Kronecker delta. We compute

$$N_{\alpha+e_i+e_j} = \frac{|\alpha + e_i + e_j|!}{(\alpha + e_i + e_j)!} = \frac{(|\alpha| + 1)!(|\alpha| + 2)}{(\alpha + e_i)!(\alpha_j + 1 + \delta_{ij})} = N_{\alpha+e_i} \frac{|\alpha| + 2}{\alpha_j + 1 + \delta_{ij}}$$

$$N_{\alpha+e_j} = \frac{|\alpha + e_j|!}{(\alpha + e_j)!} = N_\alpha \frac{|\alpha| + 1}{\alpha_j + 1}$$

Together, we obtain:

$$\frac{N_{\alpha+e_i} N_{\alpha+e_j}}{N_\alpha N_{\alpha+e_i+e_j}} = \frac{(|\alpha| + 1)(\alpha_j + 1 + \delta_{ij})}{(|\alpha| + 2)(\alpha_j + 1)}$$

We now summarize with a corollary which has been proved by the above set of observations.

Corollary 3.2.2: *Suppose* $\{C_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ *are such that* $C_{\mathbf{0}} = 1$ *and* $C_\alpha > 0$ *for all other* α. *If* $C_\alpha \leq \dfrac{|\alpha|!}{\alpha!}$,

$$\frac{(C_{\alpha+e_i})^2}{C_\alpha C_{\alpha+2e_i}} \leq \frac{(|\alpha| + 1)(\alpha_i + 2)}{(|\alpha| + 2)(\alpha_i + 1)} \text{ for all } 1 \leq i \leq d,$$

$$\text{and } \frac{C_{\alpha+e_i} C_{\alpha+e_j}}{C_\alpha C_{\alpha+e_i+e_j}} \leq \frac{|\alpha| + 1}{|\alpha| + 2} \text{ for all } 1 \leq i, j \leq d,\ i \neq j,$$

then $f(x) := \sum_{\alpha \in \mathbb{N}_0^d} C_\alpha x^\alpha$ *converges on* $\{x = (x_1,\ \ldots,\ x_d) \mid \|x\|_1 := |x_1| + |x_2| + \ldots + |x_d| < 1\}$ *and* $g(x) := 1 - \dfrac{1}{f(x)} = \sum_{\alpha \neq 0} q_\alpha x^\alpha$ *where each* $q_\alpha \geq 0$.

Recall earlier in the chapter that

$$b(z) = \left(\frac{1}{2} z_1,\ \frac{1}{2} z_2,\ \frac{1}{\sqrt{3}} z_1^2,\ \frac{1}{2} z_1 z_2,\ \frac{1}{\sqrt{3}} z_2^2 \right)$$

led to a de Branges-Rovnyak kernel k^b whose coefficients $\{C_\alpha\}$ satisfied the conditions of the chapter's first generalization. It turns out they fail the conditions of Corollary 3.2.2. To see this, note that with $\alpha = 0$, we must have

$$\frac{(C_{e_i})^2}{C_{2e_i}} = \frac{(C_{\alpha+e_i})^2}{C_\alpha C_{\alpha+2e_i}} \le \frac{(|\alpha|+1)(\alpha_i+2)}{(|\alpha|+2)(\alpha_i+1)} = 1.$$

Yet, one can check using formulas from earlier in the chapter that $C_{1,0} = 3/4$ and $C_{2,0} = 5/12$ leading to $\frac{(C_{1,0})^2}{C_{2,0}} = 27/20 > 1$. This shows there are CNP kernels for which Lemma 3.1.2 applies but Corollary 3.2.2 does not.

Now, let us construct a class of examples actually applicable to Corollary 3.2.2, starting more general and moving towards the specific. Say that $C_\alpha = \frac{|\alpha|!}{\alpha!}\prod_{i=1}^{d} s^i_{\alpha_i}$ for $\{\{s^i_n\}_{n\ge 0}\}_{i=1}^d$ that satisfy the following for every $1 \le i \le d$: $s^i_0 = 1$ and, for all $n \in \mathbb{N}$, $s^i_n \in (0,1]$ and $(s^i_n)^2 \le s^i_{n-1}s^i_{n+1}$. We observe for any fixed i that these assumptions reflect the ones in Kaluza's original lemma: in particular, the inequality can be rearranged to quickly see that it is equivalent to the condition that $R_n \le R_{n+1}$ for $R_n := \frac{s^i_n}{s^i_{n-1}}$.

We now show that such C_α satisfy the suppositions of the corollary. The immediate ones to check are that $C_{\mathbf{0}} = 1$ and that $0 < C_\alpha \le \frac{|\alpha|!}{\alpha!}$ from the corresponding assumptions on the s^i_n. To help us with the computations required to check the remaining two, we introduce the notation

$$P_i := \prod_{\substack{k=1\\k\ne i}}^{d} s^k_{\alpha_k}; \qquad Q_{i,j} := \prod_{\substack{k=1\\k\ne i,j}}^{d} s^k_{\alpha_k},$$

where the latter is only defined for $i \ne j$. With this, we have for any $1 \le i, j \le d,\ i \ne j$:

$$C_\alpha = \frac{|\alpha|!}{\alpha!}s^i_{\alpha_i}P_i; \qquad C_{\alpha+e_i} = \frac{(|\alpha|+1)!}{\alpha!(\alpha_i+1)}s^i_{\alpha_i+1}P_i; \qquad C_{\alpha+2e_i} = \frac{(|\alpha|+2)!}{\alpha!(\alpha_i+1)(\alpha_i+2)}s^i_{\alpha_i+2}P_i$$

$$C_{\alpha+e_i+e_j} = \frac{(|\alpha|+2)!}{\alpha!(\alpha_i+1)(\alpha_j+1)}s^i_{\alpha_i+1}s^j_{\alpha_j+1}Q_{i,j}.$$

Subsequently, this leads to

$$\frac{(C_{\alpha+e_i})^2}{C_\alpha C_{\alpha+2e_i}} = \left(\frac{(s^i_{\alpha_i+1})^2}{s^i_{\alpha_i}s^i_{\alpha_i+2}}\right)\left(\frac{(|\alpha|+1)(\alpha_i+2)}{(|\alpha|+2)(\alpha_i+1)}\right) \le \frac{(|\alpha|+1)(\alpha_i+2)}{(|\alpha|+2)(\alpha_i+1)},$$

using the assumed inequality for a fixed i. Also, for $i \neq j$, we have

$$\frac{C_{\alpha+e_i}C_{\alpha+e_j}}{C_\alpha C_{\alpha+e_i+e_j}} = \left(\frac{P_j}{s^i_{\alpha_i}Q_{i,j}}\right)\left(\frac{|\alpha|+1}{|\alpha|+2}\right) = \frac{|\alpha|+1}{|\alpha|+2},$$

and we have now shown $\{C_\alpha\}$ satisfies every assumption in the statement of the corollary.

Next, we move toward building a reproducing kernel whose $\{C_\alpha\}$ are of the preceding form. First, if μ is a probability measure on $[0,1]$, it is known that $F(s) := \int_0^1 t^s d\mu(t)$ defines a function on $[0,\infty)$ for which $\ln \circ F$ is convex (see [3], Lemma 5.1). Therefore, for any $n \in \mathbb{N}$, we have

$$\ln\left[F\left(\frac{n-1}{2}+\frac{n+1}{2}\right)\right] \leq \frac{1}{2}\left(\ln[F(n-1)]+\ln[F(n+1)]\right).$$

Rearranging and taking exponentials of both sides, we then arrive at the inequality $[F(n)]^2 \leq F(n-1)F(n+1)$, which will prove useful soon.

Now, suppose on $[0,1]^d$ we have a measure $\mu = \prod_{i=1}^{d}\mu_i$ where each μ_i is a probability measure on $[0,1]$ and that $\int_0^1 t^n d\mu_i \neq 0$ for all $n \geq 1$ and $1 \leq i \leq d$.

For $t = (t_1, \ ..., \ t_d)$, we use this to define

$$k_\lambda(z) := \int_{[0,1]^d} \frac{1}{1-\sum_{i=1}^{d} t_i\overline{\lambda_i}z_i}\, d\mu(t)$$

$$= \int_{[0,1]^d} \sum_{\alpha\in\mathbb{N}_0^d} \frac{|\alpha|!}{\alpha!}(t\overline{\lambda}z)^\alpha d\mu(t) = \sum_{\alpha\in\mathbb{N}_0^d}\left(\frac{|\alpha|!}{\alpha!}\int_{[0,1]^d} t^\alpha d\mu(t)\right)\left(\overline{\lambda}z\right)^\alpha.$$

By expanding it this way and since μ is a product measure, we see that we have $C_\alpha = \frac{|\alpha|!}{\alpha!}\prod_{i=1}^{d}\int_0^1 t_i^{\alpha_i} d\mu_i(t_i)$. Thus, with $s^i_n := \int_0^1 t_i^n d\mu_i(t_i)$, we are in the exact set-up from before, noting that the preceding discussion regarding log-convexity shows that the main inequality is satisfied, where the other properties of $s^i_0 = 1$ and $0 < s^i_n \leq 1$ are more readily verified. All this is to say, any such μ leads to a kernel that is CNP by the corollary. In fact, it was shown all the way back in Kaluza's original paper that these types of moment sequences produced the exact type of inequality relevant to Kaluza's lemma.

By taking $d = 2$ and both μ_1 and μ_2 to be Lebesgue measure on $[0,1]$, we can show the resulting $\{C_\alpha\}$ do not satisfy the conditions of Lemma 3.1.2, namely that the sequence R_α is not non-decreasing as required. Let us write (m,n) for arbitrary $\alpha \in \mathbb{N}_0^2$; we claim that $R_{0,2} > R_{1,2}$. To compute these quantities, we note that $s_n^i = \dfrac{1}{n+1}$ for $i = 1$ and 2, and subsequently that $C_{m,n} = \dfrac{(m+n)!}{(m+1)!(n+1)!}$ (from the preceding paragraph). We use this to calculate $C_{0,1} = C_{1,1} = C_{1,2} = 1/2$ and $C_{1,2} = 1/2$. From there, all we need to do is employ the definition of $R_{m,n}$ to see:

$$R_{0,2} = \frac{C_{0,2}}{C_{0,1}} = 2/3 > 3/5 = \frac{C_{1,2}}{C_{0,2} + C_{1,1}} = R_{1,2}.$$

Coupled with the other non-example discussed right after Corollary 3.2.2, we have shown that our two multivariable generalizations of Kaluza's lemma (Lemma 3.1.2 and Corollary 3.2.2) are independent of one another.

In terms of future directions, this chapter presents quite a few: first, our examples that we used to show independence of the two generalizations ignore a potential connection between Lemma 3.1.2 and Theorem 3.2.1. Is it possible that there is some decomposition/choice of $\{D_v\}$ that would give Lemma 3.1.2 out of applying Theorem 3.2.1? This question is fairly intimidating mainly because that it is difficult to come up with *any* other decomposition of $\{D_v\}$ other than the somewhat canonical one that we chose in order to obtain Corollary 3.2.2. Nonetheless, it may be doable.

Next, note that we did not obtain a Section 3.2 analogue to Corollary 3.1.3, which said that a kernel that is CNP by means of Lemma 3.1.2 must also be of de Branges-Rovnyak type. It seems reasonable to assume that a kernel satisfying Corollary 3.2.2 (or Theorem 3.2.1 for that matter) must be of de Branges-Rovnyak type as well, but we could not obtain an argument similar to that seen in the proof of Corollary 3.1.3.

Finally, we connect these ideas back to those of Section 2.3. At least in the case of Lemma 3.1.2, any kernel satisfying its assumptions must be both CNP and of de Branges-Rovnyak type, which gives us the multiplier ψ from Theorem 2.3.2. It seems likely that we could determine more information about the ψ in this case; as in, one wonders whether one could tie the $\{C_\alpha\}$ to the coefficients of power series representations of the components of ψ.

Chapter 4

Characterization of Forward Shifts on de Branges-Rovnyak Spaces

4.1 Operator Theory Preliminaries

Let $d \in \mathbb{N}$. The main goal of this introductory section is to study the implications of having a row operator $T = (T_1, \ ..., \ T_d)$ (whose component operators act on an arbitrary Hilbert space $\mathcal{H}$) bounded below by 1 on the orthogonal complement of its kernel. That is, $\|Tx\|_{\mathcal{H}} \geq \|x\|_{\mathcal{H}^d}$ for all $x \in (\text{Ker } T)^{\perp}$.

At the outset, we first remark that any such T has closed range. If $\{Tx_n\}$ converges to some y, we may as well assume that each $x_n \perp \text{Ker } T$, for if not $Tx_n = Tz_n$ for $z_n := P_{(\text{Ker } T)^{\perp}} x_n$ anyway. From here, the property easily gives that if $\{Tx_n\}$ is Cauchy, then so is $\{x_n\}$, thus converging to some x. By the boundedness of T, it is immediate that $Tx = y$, concluding the argument.

One naturally might wonder about the converse, whether T having closed range is enough to ensure T is bounded below when restricted to the complement of its kernel (although perhaps with some non-negative constant besides 1).

The answer to this is affirmative: if T has closed range, then $R : (\text{Ker } T)^{\perp} \to \text{Ran } T$ given by $Rx = Tx$ for $x \in (\text{Ker } T)^{\perp}$ is invertible. This is because (1) assuming $Rx = Tx = 0$ for some $x \in (\text{Ker } T)^{\perp}$ reveals that x can only be 0, and (2) that the closure of Ran R equals the closure of Ran T, which is Ran T by our only assumption. Therefore, R is both injective and surjective, and thus invertible. Then, as claimed, we obtain for any $x \in (\text{Ker } T)^\perp$,

$$\|x\|_{\mathcal{H}^d} = \|R^{-1}Rx\|_{\mathcal{H}^d} \leq \|R^{-1}\|_{\text{op}}\|Rx\|_{\mathcal{H}} = c\|Rx\|_{\mathcal{H}},$$

where $c := \|R^{-1}\|_{\text{op}} \geq 0$.

We will make use of the fact that this property is invariant under adjoints, which we will now prove as a lemma. In the below, S could be a row operator *or* a column operator with the same proof, so for that reason we omit subscripts from the norms.

Lemma 4.1.1: *If $\|Sx\| \geq \|x\|$ for all $x \in$ (Ker $S)^\perp$, then $\|S^*y\| \geq \|y\|$ for all $y \in$ (Ker $S^*)^\perp$.*

Proof: If $y = 0$, the claim is immediate. Let $y \in (\text{Ker } S^*)^\perp = \text{Ran } S$, and suppose $y = Sx$ for some nonzero $x \in (\text{Ker } S)^\perp$, for otherwise $y = SP_{(\text{Ker } S)^\perp}x$ anyway. All we need to do is compute:

$$\|S^*y\| = \sup_{\|z\|=1} \langle S^*y, z\rangle \geq \left\langle S^*y, \left(\frac{x}{\|x\|}\right)\right\rangle = \frac{\|Sx\|^2}{\|x\|} \geq \|Sx\| = \|y\|,$$

as desired.

Lemma 4.1.1 combined with our earlier observation that an operator has closed range if and only if it is bounded below on the complement of the kernel of its adjoint allows for an immediate proof of the Closed Range Theorem for operator tuples $T = (T_1, \ldots, T_d)$, which simply states that Ran T is closed if and only if Ran T^* is.

Since T and T^* have closed range, we can decompose $\mathcal{H}^{(d)} = \text{Ran } T^* \oplus \text{Ker } T$ as well as $\mathcal{H} = \text{Ran } T \oplus \text{Ker } T^*$ and subsequently express T as a block matrix with respect to these decompositions. Specfically, since $T\big|_{\text{Ker } T}$ and $P_{\text{Ker } T}T^*$ equal 0 in $\mathcal{H}$ and $\mathcal{H}^d$ respectively, we have $T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ with respect to $\text{Ran } T^* \oplus \text{Ker } T \to \text{Ran } T \oplus \text{Ker } T^*$, where $A := P_{\text{Ran } T}T\big|_{\text{Ran } T^*}$. As a result, we have $T^* = \begin{bmatrix} A^* & 0 \\ 0 & 0 \end{bmatrix}$ with respect to $\text{Ran } T \oplus \text{Ker } T^* \to$

Ran $T^*\oplus$Ker T as well as $T^*T = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}$ with respect to Ran $T^*\oplus$Ker $T \to$ Ran $T^*\oplus$Ker T. Here, U : Ran $T^* \to$ Ran T^* has the form $U = A^*A = P_{\text{Ran } T^*}T^*T\big|_{\text{Ran } T^*}$.

We claim that U is invertible, which we will prove by establishing that it is bounded below (on its entire domain) and that it has dense range. For the former, our two main tools are

$$\|TT^*y\|_{\mathcal{H}} \geq \|T^*y\|_{\mathcal{H}^d} \; \forall y \in \mathcal{H} \;\text{ and }\; \|T^*Tx\|_{\mathcal{H}^d} \geq \|Tx\|_{\mathcal{H}} \; \forall x \in \mathcal{H}^d$$

since our central assumption gives us both that T is bounded below by 1 on Ran T^* and that T^* is bounded below by 1 on Ran T. Then:

$$\|UT^*y\|_{\mathcal{H}^d} = \|T^*TT^*y\|_{\mathcal{H}^d} \geq \|TT^*y\|_{\mathcal{H}} \geq \|T^*y\|_{\mathcal{H}^d}$$

and U is bounded below on its domain, as desired.

As for its (closed) range, we clearly have Ran U = Ran T^*T = Ran T^*, with the last equality again using the fact that T^* has closed range. For if so, we can write any $y \in \mathcal{H}$ as $y = P_{\text{Ran } T}y + P_{\text{Ker}T^*}y$, leading to $T^*y = T^*P_{\text{Ran } T}y = T^*Tz$ for some $z \in \mathcal{H}^d$. This establishes Ran $T^* \subseteq$ Ran T^*T, with the opposite containment being immediate. Therefore, the range of U is dense.

With U invertible, we can define $S := \begin{bmatrix} U^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with respect to $\mathcal{H}^d$ = Ran $T^* \oplus$ Ker T into itself. Similarly, we now define the column operator $L := ST^* = \begin{bmatrix} U^{-1}A^* & 0 \\ 0 & 0 \end{bmatrix}$ with respect to Ran $T \oplus$ Ker $T^* \to$ Ran $T^* \oplus$ Ker T, which we call the *Cauchy dual* of T^*.

Before moving to a proposition getting at the main properties of L, we first show that Ker T^* = Ker L and Ran T^* = Ran L.

For the former, it is immediate that Ker $T^* \subseteq$ Ker L. Let us now compute the action of L on an arbitrary $y \in \mathcal{H}$:

$$Ly = U^{-1}A^*P_{\text{Ran } T}y = U^{-1}P_{\text{Ran } T^*}T^*\big|_{\text{Ran } T}P_{\text{Ran } T}y = U^{-1}T^*P_{\text{Ran } T}y.$$

Therefore, using the invertibility of U^{-1}, $Ly = 0$ means that $T^* P_{\text{Ran } T} y = 0$. Then, $P_{\text{Ran } T} y \in \text{Ker } T^* \cap \text{Ran } T = \{0\}$ so it must be that $y \perp \text{Ran } T = (\text{Ker } T^*)^\perp$. We have thus shown $\text{Ker } L \subseteq \text{Ker } T^*$, proving our first claim regarding set equality.

The previous action of L immediately reveals that $\text{Ran } L \subseteq \text{Ran } U^{-1} = \text{Ran } T^*$, so we only need to show the opposite containment. To get that, we observe

$$LT = ST^*T = \begin{bmatrix} I_{\text{Ran } T^*} & 0 \\ 0 & 0 \end{bmatrix} = T^*TS = T^*L^*.$$

This tells us that $L^*x = 0$ results in $P_{\text{Ran } T^*} x = T^*L^*x = 0$ and thus $x \in (\text{Ran } T^*)^\perp = \text{Ker } T$. It is then enough to take orthogonal complements of both sides of the now-proven fact $\text{Ker } L^* \subseteq \text{Ker } T$ to prove the second claim as well.

Proposition 4.1.2: *$LT = P_{\text{Ran } T^*}$ and $TL = P_{\text{Ran } T}$.*

Proof: Since we've previously shown $LT = \begin{bmatrix} I_{\text{Ran } T^*} & 0 \\ 0 & 0 \end{bmatrix}$, there is nothing left to prove for the first statement, and we only need to show $TL = P_{\text{Ran } T}$ to finish the proof.

To start, we observe that S is self-adjoint since U and therefore U^{-1} is, which gives us $TS = L^*$. Also, by applying T^* on the right to $\begin{bmatrix} I_{\text{Ran } T^*} & 0 \\ 0 & 0 \end{bmatrix} = T^*L^*$, we obtain the identity $T^* = T^*L^*T^*$, or equivalently $T = TLT$ by taking adjoints.

With these, we directly compute that

$$(TL)^* = L^*T^* = TST^* = TL \quad \text{and} \quad (TL)^2 = TLTL = TL.$$

Since TL clearly maps $\mathcal{H}$ into $\text{Ran } T$, all that is left to show to establish $TL = P_{\text{Ran } T}$ is that $y - TLy \perp \text{Ran } T$ for every $y \in \mathcal{H}$. For $x \in \mathcal{H}^d$, we have:

$$\langle y - TLy, Tx \rangle = \langle y, Tx \rangle - \langle y, L^*T^*Tx \rangle = \langle y, (T - L^*T^*T)x \rangle = 0$$

due to the previously established fact $T = TLT = (TL)^*T = L^*T^*T$.

The final proposition of this section will require some new notation, which will be used throughout the entire chapter. For any Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and any $\lambda \in \mathbb{B}_d$, we denote $\boldsymbol{\lambda}$ to refer to the operator in $\mathcal{B}(\mathcal{H}^d, \mathcal{H})$, which acts on an $x \in \mathcal{H}^d$ by $\boldsymbol{\lambda}x = \sum_{i=1}^{d} \lambda_i x_i$. The notation serves to make use of how rows act on columns in order to not have to write the summation every time.

We now wish to make several observations involving this notation in order to make the future computations involving it go more smoothly.

First, it can be checked that for any $y \in \mathcal{H}, \boldsymbol{\lambda}^* y = [\overline{\lambda_1} y,\ ...,\ \overline{\lambda_d} y]^T$, and this immediately leads to the identity $\boldsymbol{\mu}\boldsymbol{\lambda}^* = \langle \mu, \lambda \rangle_{\mathbb{C}^d} I_{\mathcal{H}}$ for any $\mu, \lambda \in \mathbb{B}_d$. If $\mathcal{H}$ is a function space over d complex variables $(z_1,\ ...,\ z_d)$, we can write this identity as $M_z \boldsymbol{\lambda}^* = \langle z, \lambda \rangle_{\mathbb{C}^d} I_{\mathcal{H}}$.

Another quick idea we can establish is the fact that, by definition, $\boldsymbol{\lambda}$ commutes with any singleton operator O applied on the left in the following sense: $O\boldsymbol{\lambda} = \boldsymbol{\lambda} O^{(d)}$, where the notation $O^{(d)}$ refers to the diagonal matrix with constant operator entry O (otherwise the dimensions do not align in the preceding equality).

Finally, for any column A, we have for $y \in \mathcal{H}$,

$$(\boldsymbol{\lambda}A)^* y = A^* \boldsymbol{\lambda}^* y = \sum_{i=1}^{d} A_i^* \overline{\lambda_i} y = \sum_{i=1}^{d} \overline{\lambda_i} A_i^* y = \overline{\boldsymbol{\lambda}} A^* y,$$

and thus $(\boldsymbol{\lambda}A)^* = \overline{\boldsymbol{\lambda}} A^*$.

Proposition 4.1.3: *L is a contraction, and subsequently $(I_{\mathcal{H}} - \boldsymbol{\lambda}L)^{-1}$ and $(I_{\mathcal{H}} - \overline{\boldsymbol{\lambda}}L^*)^{-1}$ are well-defined for any $\lambda \in \mathbb{B}_d$.*

Proof: First note that $P_{\text{Ker } T^*} = I - P_{\text{Ran } T} = I - TL$, where $I = I_{\mathcal{H}}$ unless otherwise noted.

For arbitrary $x = P_{\text{Ker } T^*} x + TLx \in \mathcal{H}$, we have $\|x\|^2 \geq \|TLx\|^2 \geq \|Lx\|^2$ since we know T expands the norm on $(\text{Ker } T)^\perp = \text{Ran } L$. Thus L is a (column) contraction, as claimed.

As for the invertibility of $I - \boldsymbol{\lambda} L$ for $\lambda \in \mathbb{B}_d$, we have for any $x \in \mathcal{H}$, $\|\boldsymbol{\lambda} L x\| =$

$$\left\| \sum_{i=1}^{d} \lambda_i L_i x \right\|^2 \leq \left(\sum_{i=1}^{d} |\lambda_i| \|L_i x\| \right)^2 \leq \sum_{i=1}^{d} |\lambda_i|^2 \sum_{i=1}^{d} \|L_i x\|^2 \leq \|\lambda\|_{\mathbb{C}^d}^2 \|x\|^2 < \|x\|^2,$$

where the second estimate in the above is precisely the Cauchy-Schwarz inequality. Therefore, we have established $\|\boldsymbol{\lambda} L\|_{\text{op}} < 1$, a well-known criterion to ensure $(I - \boldsymbol{\lambda} L)^{-1}$ exists.

Concluding, we immediately have $(I_{\mathcal{H}} - \overline{\boldsymbol{\lambda}} L^*) = (I_{\mathcal{H}} - \boldsymbol{\lambda} L)^*$, so this operator must be invertible as well.

Before we move to the next section, we invoke that perhaps the most significant identity involving this T-dependent column operator L is that $TL = I_{\mathcal{H}} - P_{\text{Ker } T^*}$, as was established at the start of the above proof.

4.2 Facts about Operator Valued Kernels

So far, all of our reproducing kernels have been scalar valued; it is now time to introduce more general formulations of RKHS's, ones in which the reproducing kernel itself is a bounded operator.

To wit, suppose we have an RKHS $H(K)$ such that, for all $\lambda \in \mathbb{B}_d$, k_λ is $\mathcal{B}(\mathcal{E})$-valued for some underlying Hilbert space $\mathcal{E}$. Then, it must be that $H(K) \subseteq \{f : \mathbb{B}_d \to \mathcal{E}\}$. The reproducing property of the kernel is now:

$$\langle f(\lambda), x \rangle_{\mathcal{E}} = \langle f, k_\lambda(\cdot) x \rangle_{H(K)} \quad \forall f \in H(K), \lambda \in \mathbb{B}_d, \text{and } x \in \mathcal{E}.$$

While we will not completely redo the theory in this more advanced setting, one property we will mention at the outset is that the set

$$D := \left\{ \sum_{i=1}^{n} k_{\lambda^i}(z) x_i \mid n \in \mathbb{N}, \{\lambda^i\}_{i=1}^n \subset \mathbb{B}_d, \{x_i\}_{i=1}^n \subset \mathcal{E} \right\}$$

is dense in $H(K)$, just as we might expect to carry over from the scalar case.

Using this density, we can show that $f_n \to f$ weakly in $H(K)$ if and only if $f_n(\lambda) \to f(\lambda)$ weakly in $\mathcal{E}$ for any fixed $\lambda \in \mathbb{B}_d$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{H(K)} < \infty$. Assuming the former convergence, we have

$$\langle f_n(\lambda), x\rangle_{\mathcal{E}} = \langle f_n, k_\lambda(\cdot)x\rangle_{H(K)} \to \langle f, k_\lambda(\cdot)x\rangle_{H(K)} = \langle f(\lambda), x\rangle_{\mathcal{E}},$$

showing the desired convergence in $\mathcal{E}$ since x was arbitrary. Conversely, assuming the pointwise weak convergence in $\mathcal{E}$, we can show that for any $g(z) = \sum_{i=1}^{m} k_{\lambda^i}(z)x_i \in D$, we have $\langle f_n, g\rangle_{\mathcal{H}(\mathcal{K})} =$

$$\sum_{i=1}^{m} \langle f_n, k_{\lambda^i}(\cdot)x_i\rangle_{\mathcal{H}(\mathcal{K})} = \sum_{i=1}^{m} \left\langle f_n(\lambda^i), x_i\right\rangle_{\mathcal{E}} \to \sum_{i=1}^{m} \left\langle f(\lambda^i), x_i\right\rangle_{\mathcal{E}} = \langle f, g\rangle_{\mathcal{H}(\mathcal{K})}.$$

Finally, the assumed norm boundedness and the density of D in $H(K)$ allows for a routine argument that $f_n \to f$ weakly in $H(K)$.

The following is an analogue of Proposition 1.3.3 (Aronszajn's classical result) for operator-valued reproducing kernels.

Proposition 4.2.1: *Suppose for some Hilbert space $\mathcal{D}$, we have $F : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathcal{E})$, and consider the space*

$$\mathcal{H} := \{F(\cdot)y \mid y \in \mathcal{D}\} \subset \{f : \mathbb{B}_d \to \mathcal{E}\}.$$

$\mathcal{H} \subseteq H(K)$ if and only if there exists $c > 0$ such that $c^2 k_\lambda(z) - F(z)F(\lambda)^* \succeq 0$, as an operator on $\mathbb{B}_d \times \mathbb{B}_d$.

Moreover, in this case, the least c that satisfies the above is the operator norm of the inclusion map $j : \mathcal{H} \to H(K)$.

Before we start a formal proof, let us first make several preliminary comments, which will start to shed some light on how the proof should proceed. First, we wish to endow $\mathcal{H}$

with an inner product, making it into a Hilbert space. To do so, first define

$$\mathcal{D}_0 := \bigvee_{z \in \mathbb{B}_d} \text{Ran } F(z)^* \subseteq \mathcal{D},$$

which is clearly a closed subspace.

Next, define $\langle F(z)y_1, F(z)y_2 \rangle_{\mathcal{H}} := \langle P_{\mathcal{D}_0} y_1, P_{\mathcal{D}_0} y_2 \rangle_{\mathcal{D}}$, where we note right away the property that $\|F(z)y\|_{\mathcal{H}} = \|P_{\mathcal{D}_0} y\|_{\mathcal{D}} \leq \|y\|_{\mathcal{D}}$. One next notices that $F(z)y = 0$ for all $z \in \mathbb{B}_d$ if and only if $y \in \bigcap_{z \in \mathbb{B}_d} \text{Ker } F(z) = \mathcal{D}_0^{\perp}$, which happens if and only if $P_{\mathcal{D}_0} y = 0$, as it should be in order to ensure our norm definition is well-defined.

To see that the norm is complete, take a Cauchy sequence $\{F(z)y_n\}$ in $\mathcal{H}$; as always, we can remove any zeros from the sequence, which by the previous argument is akin to assuming that $\{y_n\}$ is in $\mathcal{D}_0$, and by definition we see:

$$\|F(z)y_n - F(z)y_m\|_{\mathcal{H}} = \|F(z)(y_n - y_m)\|_{\mathcal{H}} = \|P_{\mathcal{D}_0}(y_n - y_m)\|_{\mathcal{D}} = \|y_n - y_m\|_{\mathcal{D}},$$

which clearly reveals that $\{y_n\}$ is Cauchy in $\mathcal{D}$. Since $\mathcal{D}$ is a Hilbert space, y_n converges to some $y \in \mathcal{D}_0$ (again using that $\mathcal{D}_0$ is closed). Then, we have that $\|F(z)y_n - F(z)y\|_{\mathcal{H}} = \|y_n - y\|_{\mathcal{D}}$ and clearly $F(z)y_n$ converges to $F(z)y$ in $\mathcal{H}$.

Thus, we have shown $\mathcal{H}$ is a Hilbert space; on top of that, we now show $\mathcal{H}$ is an RKHS. For any $\lambda \in \mathbb{B}_d$, we have that the point-evaluation functional $E_\lambda : \mathcal{H} \to \mathcal{E}$ is bounded. To see this, we let $y = P_{\mathcal{D}_0} y + P_{\mathcal{D}_0^{\perp}} y \in \mathcal{D}$, noting that our earlier work shows $F(z) P_{\mathcal{D}_0^{\perp}} y = 0$ (as an element of $\mathcal{H}$). Then, we compute

$$\|E_\lambda[F(z)y]\|_{\mathcal{E}} = \|E_\lambda[F(z)P_{\mathcal{D}_0} y]\|_{\mathcal{E}} = \|F(\lambda) P_{\mathcal{D}_0} y\|_{\mathcal{E}}$$

$$\leq \|F(\lambda)\|_{\text{op}} \|P_{\mathcal{D}_0} y\|_{\mathcal{D}} = \|F(\lambda)\|_{\text{op}} \|F(z)y\|_{\mathcal{H}},$$

establishing the desired boundedness. Therefore, since $\mathcal{H}$ is clearly a function space, we have established $\mathcal{H}$ is an RKHS.

That is, $\mathcal{H}$ is in fact $H(L)$ for some reproducing kernel $\ell_\lambda : \mathbb{B}_d \to \mathcal{B}(\mathcal{E})$, which we claim is given by $\ell_\lambda(z) = F(z)F(\lambda)^*$. To prove this, we compute, for any $x \in \mathcal{E}, y \in \mathcal{D}$, and $\lambda \in \mathbb{B}_d$,

$$\langle F(\cdot)y, F(\cdot)F(\lambda)^*x\rangle_{H(L)} = \langle y, F(\lambda)^*x\rangle_{\mathcal{D}} = \langle F(\lambda)y, x\rangle_{\mathcal{E}} = \langle F(\cdot)y, \ell_\lambda(\cdot)x\rangle_{H(L)},$$

which proves the form of $\ell_\lambda(z)$.

Allow us to now break down exactly what $c^2k_\lambda(z) - F(z)F(\lambda)^* \succeq 0$ (for some $c > 0$) means. This is equivalent, by definition, to requiring for any $n \in \mathbb{N}$, $\{\lambda^i\}_{i=1}^n \subset \mathbb{B}_d$, and $\{x_i\}_{i=1}^n \subset \mathcal{E}$ the following inequality:

$$\sum_{i,j=1}^n \left\langle [c^2 k_{\lambda^j}(\lambda^i) - F(\lambda^i)F(\lambda^j)^*]x_j, x_i \right\rangle_{\mathcal{E}} \geq 0.$$

Since $\langle k_{\lambda^j}(\lambda^i)x_j, x_i\rangle_{\mathcal{E}} = \langle k_{\lambda^j}(\cdot)x_j, k_{\lambda^i}(\cdot)x_i\rangle_{H(K)}$ by the reproducing kernel property, we can rearrange the previous inequality to state it in terms of norms. Precisely, after taking a square root, we arrive at

$$\left\| \sum_{i=1}^n F(\lambda^i)^* x_i \right\|_{\mathcal{D}} \leq c \left\| \sum_{i=1}^n k_{\lambda^i}(z) x_i \right\|_{H(K)}.$$

Lastly, we recall the following theorem from Chapter 1 before starting the proof.

Theorem 4.2.2 (Closed Graph): *An operator $T : \mathcal{H} \to \mathcal{K}$ is bounded if and only if its graph*

$$\{(x, Tx) \mid x \in \mathcal{H}\}$$

is closed (in the product topology) in $\mathcal{H} \times \mathcal{K}$.

Proof of Proposition 4.2.1: First, we assume that $\mathcal{H} \subseteq H(K)$. Let us first use the preceding theorem to show that the inclusion map $j : \mathcal{H} \to H(K)$ is bounded. Assume we have $\{F(z)y_n\}$ in $\mathcal{H}$ and recall that we may assume $\{y_n\} \subset \mathcal{D}_0$ so that the zeros are removed from $\{F(z)y_n\}$. If we assume $F(z)y_n$ converges in $\mathcal{H}$ to some element $F(z)y$ as well as assuming $jF(z)y_n = F(z)y_n$ converges in $H(K)$ to some function $g(z)$, then by the closed graph theorem, all we need to do is show that $g(z) = F(z)y = jF(z)y$ to obtain boundedness of j.

We clearly have that $\{F(z)y_n\}$ is Cauchy in $\mathcal{H}$, and pulling on our previous work when we showed the norm was complete, we know $\{y_n\}$ is Cauchy in $\mathcal{D}$ and thus convergent to some w, which we know is in $\mathcal{D}_0$ by the fact that $\mathcal{D}_0$ is closed. Next, for any λ, since $F(\lambda)$ is bounded, we must have that $F(\lambda)y_n \to F(\lambda)w$ in $\mathcal{E}$. Further, since $\{y_n\}$ must be bounded in $\mathcal{D}$, we have that $\sup_{n\in\mathbb{N}} \|F(z)y_n\|_{H(L)} \leq \sup_{n\in\mathbb{N}} \|y_n\|_{\mathcal{D}} < \infty$.

By our discussion at the start of this section regarding pointwise convergence in any RKHS, we can now invoke that $F(z)y_n \to F(z)w$ in $\mathcal{H} = H(L)$. The uniqueness of the limit in $\mathcal{H}$ gives us that $y = w$ and the uniqueness of the limit in $H(K)$ gives that $g(z) = F(z)w = F(z)y$, as desired. Therefore, $\|j\|_{\text{op}} < \infty$.

Using $\mathcal{H} = H(L)$, we now wish to compute the adjoint $j^* : H(K) \to H(L)$. We compute

$$\langle F(\cdot)y, j^* k_\lambda(\cdot)x\rangle_{H(L)} = \langle F(\cdot)y, k_\lambda(\cdot)x\rangle_{H(K)} = \langle F(\lambda)y, x\rangle_{\mathcal{E}} = \langle F(\cdot)y, \ell_\lambda(\cdot)x\rangle_{H(L)},$$

where the middle equality is only legal since we know $H(L) \subseteq H(K)$. This shows that $j^* k_\lambda(z)x = \ell_\lambda(z)x$ and of course that j^* is also bounded with $\|j\|_{\text{op}} = \|j^*\|_{\text{op}}$.

Therefore, we have for any $n \in \mathbb{N}$, $\{\lambda^i\}_{i=1}^n \subset \mathbb{B}_d$, and $\{x_i\}_{i=1}^n \subset \mathcal{E}$,

$$\left\| j^* \sum_{i=1}^n k_{\lambda^i}(\cdot)x_i \right\|_{H(L)} \leq \|j\|_{\text{op}} \left\| \sum_{i=1}^n k_{\lambda^i}(\cdot)x_i \right\|_{H(K)}$$

However, working only with the first term, we have

$$\left\| j^* \sum_{i=1}^n k_{\lambda^i}(\cdot)x_i \right\|_{H(L)} = \left\| \sum_{i=1}^n \ell_{\lambda^i}(\cdot)x_i \right\|_{H(L)} = \left\| F(\cdot) \sum_{i=1}^n F(\lambda^i)^* x_i \right\|_{H(L)}$$

$$= \left\| P_{\mathcal{D}_0} \sum_{i=1}^n F(\lambda^i)^* x_i \right\|_D = \left\| \sum_{i=1}^n F(\lambda^i)^* x_i \right\|_D,$$

since clearly $\sum_{i=1}^n F(\lambda^i)^* x_i \in \mathcal{D}_0$ by definition. This is to say, pulling from the preliminary calculations before the proof, that $c^2 k_\lambda(z) - F(z)F(\lambda)^* \succeq 0$ with $c = \|j\|_{\text{op}}$.

Conversely, assume we have

$$\left\| \sum_{i=1}^{n} F(\lambda^i)^* x_i \right\|_{\mathcal{D}} \leq c \left\| \sum_{i=1}^{n} k_{\lambda^i}(\cdot) x_i \right\|_{H(K)}$$

for some $c > 0$ and any of the relevant parameters $n, \{\lambda_i\}, \{x_i\}$. The proof of the forward direction reveals (1) that this inequality is the same thing as

$$\left\| T \sum_{i=1}^{n} k_{\lambda^i}(\cdot) x_i \right\|_{H(L)} \leq c \left\| \sum_{i=1}^{n} k_{\lambda^i}(\cdot) x_i \right\|_{H(K)}$$

for the map $T : H(K) \to H(L)$, $Tk_\lambda(z)x = \ell_\lambda(z)$, and that (2) T^* is the inclusion map j. In fact, invoking the fact that linear combinations of reproducing kernels are dense in their corresponding RKHS, the previous inequality is precisely the minimum requirement to say T is bounded with $\|T\|_{\text{op}} \leq c$. And thus j is bounded as well, showing $\mathcal{H} \subset H(K)$, as desired.

As a final note, we comment that both directions of the proof combined show that $\|j\|_{\text{op}}$ is the minimum possible c that can arise in the theorem's right hand side.

One particular observation we wish to record now in the aftermath of the proof is that, the boundedness of the inclusion map gives

$$\|F(\cdot)y\|_{H(K)} \leq \|j\|_{\text{op}} \|F(\cdot)y\|_{H(L)} \leq \|j\|_{\text{op}} \|y\|_{\mathcal{D}}$$

for any $y \in \mathcal{D}$. Quick access to such an inequality will be very useful in the sequel.

The next part of our discussion involves multipliers. In previous chapters, we allowed for multipliers to be of varying size, employing tensor notation. There is no need for this (at least in the scope of this dissertation) in the case of having an operator-valued kernel, so we simply define

$$\mathcal{M}(H(K^1), H(K^2)) := \{\phi : \mathbb{B}_d \to \mathbb{C} \mid \phi f \in H(K^2) \ \ \forall f \in H(K^1)\}$$

as the *multipliers* from $H(K^1)$ to $H(K^2)$.

As we promised in Chapter 1, we owe a proof of Proposition 1.3.8. We are now ready to prove the below retitled proposition in the case of K^1 and K^2 both being $\mathcal{B}(\mathcal{E})$-valued.

Proposition 4.2.3: *Let $\phi \in \mathcal{M}(H(K^1), H(K^2))$. For any $\lambda \in \mathbb{B}_d$, we have $M_\phi^* k_\lambda^2 = \overline{\phi(\lambda)} k_\lambda^1$.*

Proof: For any $f \in H(K^1)$, $\lambda \in \mathbb{B}_d$, and $x \in \mathcal{E}$,

$$\left\langle f, M_\phi^* k_\lambda^2(\cdot)x \right\rangle_{H(K^1)} = \left\langle M_\phi f, k_\lambda^2(\cdot)x \right\rangle_{H(K^2)} = \langle \phi(\lambda) f(\lambda), x \rangle_{\mathcal{E}}$$

$$= \phi(\lambda) \langle f(\lambda), x \rangle_{\mathcal{E}} = \phi(\lambda) \left\langle f, k_\lambda^1(\cdot)x \right\rangle_{H(K^1)} = \left\langle f, \overline{\phi(\lambda)} k_\lambda^1 \right\rangle_{H(K^1)}.$$

Since f was arbitrary, this finishes the proof.

We remark that just like before, we have access to the inequality $\|\phi\|_\infty \leq \|\phi\|_{\mathcal{M}(H(K))}$.

One last general fact that we will need in the next section is the following lemma.

Lemma 4.2.4: *If $k_\lambda(0) = I_{\mathcal{E}}$ for all $\lambda \in \mathbb{B}_d$, then $k_\lambda(z) \succeq 0$ if and only if $k_\lambda(z) - I_{\mathcal{E}} \succeq 0$.*

Proof: Since the symbol $\succeq$ gives a (transitive) order relation, the backward direction is immediate as we have $k_\lambda(z) \succeq I_{\mathcal{E}} \succeq 0$.

For the forward direction, we assume that k_λ is indeed a reproducing kernel and that we therefore may use Proposition 4.2.1 with $\mathcal{D} = \mathcal{E}$ and $F(z) = k_0(z) = I_{\mathcal{E}}$, where we have used the fact that $k_0(z) = k_z(0)^* = I_{\mathcal{E}}$ for all $z \in \mathbb{B}_d$. Since we certainly have $\{k_0(\cdot)x \mid x \in \mathcal{E}\} \subseteq H(K)$, the proposition gives that $\|j\|_{op}^2 k_\lambda(z) - I_{\mathcal{E}} \succeq 0$ where $j : \{k_0(\cdot)x \mid x \in \mathcal{E}\} \to H(K)$ is the inclusion map.

Recalling some from the proof of Proposition 4.2.1, we have that the space $\mathcal{D}_0 = \mathcal{E}$ and therefore the norm we had induced on $\{k_0(\cdot)x \mid x \in \mathcal{E}\}$ is simply $\|k_0(\cdot)x\| := \|x\|_{\mathcal{E}}$. Thus, we check

$$\|jk_0(\cdot)x\|_{H(K)} = \|k_0(\cdot)x\|_{H(K)} = \|k_0(0)x\|_{\mathcal{E}} = \|x\|_{\mathcal{E}} = \|k_0(\cdot)x\|,$$

revealing that $\|j\|_{op} = 1$ and completing the proof.

4.3 de Branges-Rovnyak Spaces: The Operator Case

Recall that every prior de Branges-Rovnyak space we have encountered thus far has corresponded to a function b from $\mathbb{B}_d$ to $\mathbb{B}_r$. It turns out that instead of only considering row functions b, we may parameterize our de Branges-Rovnyak kernels with an operator from one Hilbert space to another. To make this situation more distinct, we will use B to indicate such an object instead of a lowercase letter as before. In this section and the next, we will explore properties of these more advanced de Branges-Rovnyak spaces.

Let $\mathcal{D}, \mathcal{E}$ be any Hilbert spaces, and suppose B is in the Schur class

$$S_d(\mathcal{D}, \mathcal{E}) := \mathcal{M}_1(H_d^2 \otimes \mathcal{D}, H_d^2 \otimes \mathcal{E}),$$

with the extra assumption that $B(0) = 0$.

Therefore, since $H_d^2 = H(S)$ for the Szegő kernel $S(z, \lambda) = s_\lambda(z) = \dfrac{1}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}$, we can use Proposition 1.3.10 to see $(I_\mathcal{E} - B(z)B(\lambda)^*)s_\lambda(z) \succeq 0$. Thus

$$k_\lambda(z) := \frac{I_\mathcal{E} - B(z)B(\lambda)^*}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}$$

defines a reproducing kernel and subsequently an RKHS that we again call $H(B)$ in place of $H(K)$.

We have particular interest in the row operator tuple of multiplication by the independent variable, $M_z = [M_{z_1}, \ ..., \ M_{z_d}]$, which acts on an $f = [f_1, \ ..., \ f_d]^T \in H(B)^d$ by $M_z f = \sum_{i=1}^d z_i f_i$. Let us first observe that in the case that $(M_z, H(B))$ is bounded, it must be that $\|M_z\|_{\text{op}} \geq 1$.

To see this, we note that one can immediately see that with $(M_z, H(B))$ bounded, it must be that each coordinate function z_i is in $\mathcal{M}(H(B))$ for $1 \leq i \leq d$. We then use Proposition 4.2.3 to see $M_{z_i}^* k_\lambda = \overline{\lambda_i} k_\lambda$ for all i. Subsequently, we obtain

$$\|M_z^* k_\lambda\|_{H(B)^d}^2 = \sum_{i=1}^d \|M_{z_i}^* k_\lambda\|_{H(B)}^2 = \sum_{i=1}^d |\lambda_i|^2 \|k_\lambda\|_{H(B)}^2 = \|\lambda\|_2^2 \|k_\lambda\|_{H(B)}^2$$

for any $\lambda \in \mathbb{B}_d$. If we let $g_\lambda := \dfrac{k_\lambda}{\|k_\lambda\|_{H(B)}}$, then $\|g_\lambda\|_{H(B)} = 1$ and $\|M_z^* g_\lambda\|_{H(B)^d} = \|\lambda\|_2$. From there, we see

$$\|M_z\|_{\text{op}} = \|M_z^*\|_{\text{op}} = \sup\{\|M_z^* f\|_{H(B)^d} \mid \|f\|_{H(B)} = 1\} \geq \sup\{\|M_z^* g_\lambda\|_{H(B)^d} \mid \lambda \in \mathbb{B}_d\} = 1,$$

as desired.

The following is an analogue of Proposition 2.2.3 for the operator-valued de Branges-Rovnyak spaces.

Proposition 4.3.1: *If $B(0) = 0$, then $(M_z, H(B))$ is a bounded row operator if and only if $B(z)y \in H(B)$ for every $y \in \mathcal{D}$.*

Moreover, in this case, the operator norms of $(M_z, H(B))$ and the inclusion map $j : \{B(z)y \mid y \in \mathcal{D}\} \to H(B)$ are related by $\|M_z\|^2 = \|j\|^2 + 1$.

Proof: A relevant fact that we will use in both directions of the proof is the reproducing-kernel-based tautology

$$\langle z, \lambda\rangle_{\mathbb{C}^d} k_\lambda(z) + I_{\mathcal{E}} = k_\lambda(z) + B(z)B(\lambda)^*.$$

Pulling on Propositions 1.3.10 and 4.2.1, respectively, the main claim reduces to showing that there exists $c > 0$ such that $(c^2 I_{\mathcal{E}} - \langle z, \lambda\rangle_{\mathbb{C}^d})k_\lambda(z) \succeq 0$ if and only if there exists $d > 0$ such that $d^2 k_\lambda(z) - B(z)B(\lambda)^* \succeq 0$.

First assume that we have $c := \|M_z\|_{\text{op}} > 1$ satisfying the former positive definiteness relation. We will handle the case that $c = 1$ at the end.

Since $r_\lambda(z) := \left(I_{\mathcal{E}} - \dfrac{1}{c^2}\langle z, \lambda\rangle_{\mathbb{C}^d}\right) k_\lambda(z)$ satisfies $r_\lambda(0) = I_{\mathcal{E}}$ and we are assuming $c^2 r_\lambda(z) \succeq 0$ and therefore $r_\lambda(z) \succeq 0$, Lemma 4.2.4 gives us that $r_\lambda(z) - I_{\mathcal{E}} \succeq 0$. Then, since $\dfrac{c^2}{c^2-1} > 0$, we have $\dfrac{c^2}{c^2-1}(r_\lambda(z) - I_{\mathcal{E}}) \succeq 0$. We now compute this object:

$$\frac{c^2}{c^2-1}(r_\lambda(z) - I_{\mathcal{E}}) = \frac{c^2}{c^2-1}(k_\lambda(z) - I_{\mathcal{E}}) - \frac{1}{c^2-1}\langle z, \lambda\rangle k_\lambda(z) =$$

$$\frac{c^2}{c^2-1}(k_\lambda(z)-I_\mathcal{E})-\frac{1}{c^2-1}[k_\lambda(z)+B(z)B(\lambda)^*-I_\mathcal{E}]=k_\lambda(z)-\frac{1}{c^2-1}B(z)B(\lambda)^*-I_\mathcal{E}.$$

Then, by defining $t_\lambda(z) := k_\lambda(z) - \frac{1}{c^2-1}B(z)B(\lambda)^*$, we have that $t_\lambda(0) = I_\mathcal{E}$ (since $B(0) = 0$) and $t_\lambda(z) - I_\mathcal{E} \succeq 0$, therefore $t_\lambda(z) \succeq 0$ by Lemma 4.2.4. Finally, with $d := \sqrt{c^2-1} > 0$, we have $d^2 k_\lambda(z) - B(z)B(\lambda)^* = (c^2-1)t_\lambda(z) \succeq 0$, as desired.

We further note by the second statement in Proposition 4.2.1 that we must have $\|j\|_{\text{op}}^2 \leq d^2 = \|M_z\|_{\text{op}}^2 - 1$.

Conversely, assume $d := \|j\|_{\text{op}} > 0$ is such that $d^2 k_\lambda(z) - B(z)B(\lambda)^* \succeq 0$; again, we postpone the case where $d = 0$ until the end of the proof. By defining $p_\lambda(z) := k_\lambda(z) - \frac{1}{d^2}B(z)B(\lambda)^*$, we have that $p_\lambda(0) = I_\mathcal{E}$ and $p_\lambda(z) \succeq 0$ since we know $d^2 p_\lambda(z)$ is. Therefore, Lemma 4.2.4 give us

$$0 \preceq p_\lambda(z) - I_\mathcal{E} = k_\lambda(z) - I_\mathcal{E} - \frac{1}{d^2}B(z)B(\lambda)^* = k_\lambda(z) - I_\mathcal{E} - \frac{1}{d^2}[\langle z,\lambda\rangle_{\mathbb{C}^d} k_\lambda(z) - (k_\lambda(z) - I_\mathcal{E})]$$

$$= \frac{d^2+1}{d^2}(k_\lambda(z) - I_\mathcal{E}) - \frac{1}{d^2}\langle z,\lambda\rangle_{\mathbb{C}^d} k_\lambda(z)$$

Then, by multiplying by $\frac{d^2}{d^2+1}$, we have

$$0 \preceq k_\lambda(z) - I_\mathcal{E} - \frac{1}{d^2+1}\langle z,\lambda\rangle_{\mathbb{C}^d} k_\lambda(z) = q_\lambda(z) - I_\mathcal{E},$$

where $q_\lambda(z) := k_\lambda(z) - \frac{1}{d^2+1}\langle z,\lambda\rangle_{\mathbb{C}^d} k_\lambda(z)$. Since $q_\lambda(0) = I_\mathcal{E}$, we also obtain $q_\lambda(z) \succeq 0$ by way of Lemma 4.2.4 and subsequently $(d^2+1)q_\lambda(z) = [(d^2+1)I_\mathcal{E} - \langle z,\lambda\rangle_{\mathbb{C}^d}]k_\lambda(z)$ must also be positive-definite. Taking $c := \sqrt{d^2+1} > 0$ finishes proving the biconditional statement.

Similarly to before, we close out the argument by applying the second statement of Proposition 1.3.10 to observe $\|M_z\|_{\text{op}}^2 \leq c^2 = \|j\|_{\text{op}}^2 + 1$. Therefore, the previous inequality reveals $\|j\|_{\text{op}}^2 + 1 = \|M_z\|_{\text{op}}^2$, as desired.

To finish the proof, we must show that $\|M_z\|_{\text{op}} = 1$ if and only if $\|j\|_{\text{op}} = 0$. If the latter is zero, then of course j must be 0 and subsequently $B \equiv 0$. Then, by examing the reproducing kernel, $H(B) = H_d^2$ and is it well-known that the Drury Arveson shift has norm 1.

Conversely, assume $\|M_z\|_{\text{op}} = 1$. In returning to the ideas seen at the very start of the proof, we have that $(I_{\mathcal{E}} - \langle z, \lambda\rangle_{\mathbb{C}^d})k_\lambda(z)$ is positive definite; then, rearranging the tautology between these objects and $B(z)B(\lambda)^*$, we see that $I_{\mathcal{E}} - B(z)B(\lambda)^*$ must be positive definite as well.

We claim this last relation is enough to say $B(z)$ is a constant operator. To see this, employ the definition of positive definiteness (with two points/vectors):

$$0 \leq \sum_{i,j=1}^{2} \left\langle [I_{\mathcal{E}} - B(\lambda^i)B(\lambda^j)^*]a_j, a_i \right\rangle_{\mathcal{E}} = \sum_{i,j=1}^{2} \langle a_j, a_i\rangle_{\mathcal{E}} - \sum_{i,j=1}^{2} \left\langle B(\lambda^j)^* a_j, B(\lambda^i)^* a_i \right\rangle_{\mathcal{E}}$$

$$= \|a_1 + a_2\|_{\mathcal{E}}^2 - \|B(\lambda^1)^* a_1 + B(\lambda^2)^* a_2\|_{\mathcal{D}}^2.$$

Then, by taking $a_2 = -a_1$, we get $0 \leq -\|B(\lambda^1)^* a_1 - B(\lambda^2)^* a_1\|_{\mathcal{D}}^2$. By rearranging this and employing non-negativity of norms, it must be that $B(\lambda^1)^* a_1 = B(\lambda^2)^* a_1$ for all arbitrary $a_1 \in \mathcal{E}$. Therefore, $B(z)^*$ is constant and necessarily $B(z)$ is as well. Finally, since $B(0)$ is assumed to be the zero operator, $B(z)$ must be identically 0. This shows that $\|j\|_{\text{op}} = 0$ as desired.

In the theory of de Branges-Rovnyak spaces (of functions of d variables), we say a column operator X solves the *Gleason problem* in $H(B)$ if and only if

$$\forall f \in H(B), \qquad \sum_{i=1}^{d} z_i (X_i f)(z) := f(z) - f(0).$$

That is, if and only if $M_z X = I_{H(B)} - P_0$ where $P_0 f := f(0)$, the orthogonal projection onto the subspace $\mathcal{M}_0$. In the case that $d = 1$, then X defined by $(Xf)(z) := \dfrac{f(z) - f(0)}{z}$ (what we usually refer to as the *backward shift*) defines a unique solution to the Gleason problem. For higher d, we cannot replicate this and in general the solution is not guaranteed to be unique.

While the Gleason problem can be studied in any $H(B)$ space, we will make the extra assumption that $(M_z, H(B))$ is a bounded (row) operator for some function B in the Schur class $S_d(\mathcal{D}, \mathcal{E})$ satisfying $B(0) = 0$. Here, $\mathcal{D}$ and $\mathcal{E}$ are abstract Hilbert spaces that do not play much of a role in the proof of the following proposition.

Proposition 4.3.2: *If $(M_z, H(B))$ is bounded, the Gleason problem is solvable.*

Proof: At the start, let us write S for the Schur function parameterizing the de Branges-Rovnyak space $H(S)$ so that we can use B for a different object throughout the proof. We remind the reader that, pointwise, $S(z) \in \mathcal{B}(\mathcal{D}, \mathcal{E})$.

Unsurprisingly, the proof will make use of the $H(S)$ reproducing kernel $k_\lambda : \mathbb{B}_d \to \mathcal{B}(\mathcal{E})$ given by $k_\lambda(z) = \dfrac{I_\mathcal{E} - S(z)S(\lambda)^*}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}$ as well as the corresponding tautology

$$\langle z, \lambda \rangle_{\mathbb{C}^d} k_\lambda(z) + I_\mathcal{E} = k_\lambda(z) + S(z)S(\lambda)^*.$$

Since we are assuming $k_\lambda(z)$ is positive definite on $\mathbb{B}_d \times \mathbb{B}_d$, the associated theory gives an auxiliary Hilbert space $\mathcal{F}$ and an operator valued function $H : \mathbb{B}_d \to \mathcal{B}(\mathcal{F}, \mathcal{E})$ such that $k_\lambda(z) = H(z)H(\lambda)^*$ for every $z, \lambda \in \mathbb{B}_d$.

The proof will make heavy use of the preceding $\boldsymbol{\lambda}$ notation, as well as the observations made when this notation was introduced in Section 1. Define

$$\mathcal{N}_1 := \bigvee_{\substack{\lambda \in \mathbb{B}_d \\ x \in \mathcal{E}}} \begin{bmatrix} H(\lambda)^* x \\ S(\lambda)^* x \end{bmatrix} \subset \mathcal{F} \oplus \mathcal{D},$$

$$\mathcal{N}_2 := \bigvee_{\substack{\lambda \in \mathbb{B}_d \\ x \in \mathcal{E}}} \begin{bmatrix} \boldsymbol{\lambda}^* H(\lambda)^* x \\ x \end{bmatrix} \subset \mathcal{F}^d \oplus \mathcal{E},$$

and $U : \mathcal{N}_1 \to \mathcal{N}_2$ by $U \begin{bmatrix} H(\lambda)^* x \\ S(\lambda)^* x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda}^* H(\lambda)^* x \\ x \end{bmatrix}$ and then extending linearly to a dense set in $\mathcal{N}_1$.

We claim that U extends to be isometric on $\mathcal{N}_1$. Fixing μ and λ in $\mathbb{B}_d$ as well as x and y in $\mathcal{E}$, we compute

$$\left\langle U \begin{bmatrix} H(\lambda)^* y \\ S(\lambda)^* y \end{bmatrix}, U \begin{bmatrix} H(\mu)^* x \\ S(\mu)^* x \end{bmatrix} \right\rangle_{\mathcal{F}^d \oplus \mathcal{E}} = \left\langle \begin{bmatrix} \boldsymbol{\lambda}^* H(\lambda)^* y \\ y \end{bmatrix}, \begin{bmatrix} \boldsymbol{\mu}^* H(\mu)^* x \\ x \end{bmatrix} \right\rangle_{\mathcal{F}^d \oplus \mathcal{E}}$$

$$= \langle [H(\mu) \boldsymbol{\mu} \boldsymbol{\lambda}^* H(\lambda)^* + I_{\mathcal{E}}]\, y, x \rangle_{\mathcal{E}} = \langle [\langle \mu, \lambda \rangle_{\mathbb{C}^d}\, k_\lambda(\mu) + I_{\mathcal{E}}]\, y, x \rangle_{\mathcal{E}}$$

$$= \langle [k_\lambda(\mu) + S(\mu) S(\lambda)^*]\, y, x \rangle_{\mathcal{E}} = \left\langle \begin{bmatrix} H(\lambda)^* y \\ S(\lambda)^* y \end{bmatrix}, \begin{bmatrix} H(\mu)^* x \\ S(\mu)^* x \end{bmatrix} \right\rangle_{\mathcal{F} \oplus \mathcal{D}}.$$

With this, it can be easily verified that $\|Un\|_{\mathcal{F}^d \oplus \mathcal{E}} = \|n\|_{\mathcal{F} \oplus \mathcal{D}}$ for any $n \in \mathcal{N}_1$ and thus our claim has been established.

We wish to extend U to be defined on all of $\mathcal{F} \oplus \mathcal{D}$ by setting $Um = 0$ for every $m \in \mathcal{N}_1^\perp$. While U is no longer isometric on the full space, we do obtain that the extended U is contractive.

Next, we define the column operators $A : \mathcal{F} \to \mathcal{F}^d$ and $B : \mathcal{D} \to \mathcal{F}^d$ as well as the singleton operators $C : \mathcal{F} \to \mathcal{E}$ and $D : \mathcal{D} \to \mathcal{E}$ by representing U as the block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We observe now that for any $x \in \mathcal{F}$,

$$\|Ax\|_{\mathcal{F}^d}^2 \le \|Ax\|_{\mathcal{F}^d}^2 + \|Cx\|_{\mathcal{E}}^2 = \|U(x \oplus 0)\|_{\mathcal{F}^d \oplus \mathcal{E}}^2 \le \|x \oplus 0\|_{\mathcal{F} \oplus \mathcal{D}}^2 = \|x\|_{\mathcal{F}}^2,$$

and thus A is contractive as well. This reasoning could apply to show that *any* operator arising in a block decomposition of a contractive operator is also contractive; in particular, we will make use of this fact applied to B^* as part of the block decomposition of the contractive U^*. Going one step further, since $B^* = [B_1^*,\ \ldots,\ B_d^*]$, applying B^* to the canonical basis elements of $\mathcal{F}^d$, one can check that B_i^* is contractive for $1 \le i \le d$.

Since U is isometric on $\mathcal{N}_1$, we can determine the actions of A, B, C, and D (rather, the actions of the restrictions of these operators to the appropriate spaces) more explicitly by the fact that

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}^* H(\lambda)^* x \\ x \end{bmatrix} = \begin{bmatrix} H(\lambda)^* x \\ S(\lambda)^* x \end{bmatrix}$$

for all λ and x. This gives the following equalities in terms of operators in $\mathcal{B}(\mathcal{E})$ for fixed λ:

$$A^*\boldsymbol{\lambda}^*H(\lambda)^* + C^* = H(\lambda)^*$$

$$B^*\boldsymbol{\lambda}^*H(\lambda)^* + D^* = S(\lambda)^*.$$

The first thing we can do is plug in $\lambda = 0$ to the latter equality to obtain $D^* = S(0)^* = 0$ and therefore $D = 0$. Next, by taking adjoints and rearranging the former, we obtain $C = H(\lambda)(I_{\mathcal{F}} - \boldsymbol{\lambda}A)$.

Now, since A is a column contraction, this allows for $I_{\mathcal{F}} - \boldsymbol{\lambda}A$ to be invertible for any $\lambda \in \mathbb{B}_d$, as $\|\boldsymbol{\lambda}A\|_{\text{op}}$ can be shown to be less than or equal to 1, just as was done in the proof of Proposition 4.1.3.

Therefore, $H(\lambda) = C(I_{\mathcal{F}} - \boldsymbol{\lambda}A)^{-1}$ and $S(\lambda) = H(\lambda)\boldsymbol{\lambda}B = \boldsymbol{\lambda}H(\lambda)^{(d)}B$. Going forward, we choose to write the latter as $S(z) = M_zH(z)^dB$.

We now claim that for $b_i : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathcal{E})$ given by $b_i(z) := H(z)B_i$, $b_i(z)y$ is in $H(S)$ for all $y \in \mathcal{D}$. If so, then for any $y \in \mathcal{D}$, by taking $b(z)y$ to be the column $H(z)^{(d)}By = [b_1(z)y, \ ..., \ b_d(z)y]^T$, we have found $b(z)y \in H(S)^d$ such that $S(z)y = M_zb(z)y$, which will be the key to defining our Gleason problem solution.

To show this claim, we will use Proposition 4.2.1: that is, fixing $1 \leq i \leq d$, we will show that $b_i(z) = H(z)B_i$ satisfies that $b_i(z)y \in H(S)$ for all $y \in \mathcal{D}$ by showing that

$$k_\lambda(z) - b_i(z)b_i(\lambda)^* = H(z)H(\lambda)^* - H(z)B_iB_i^*H(\lambda)^* = H(z)[I - B_iB_i^*]H(\lambda)^*$$

is positive definite. Since B_i^* being contractive (as our prior remark mentioned) means precisely that $I - B_iB_i^* \succeq 0$, employing Proposition 1.3.2 reveals this is indeed the case. Therefore, the aforementioned proposition can be applied d times with each b_i acting as our operator valued function F.

However, we have one more reason to momentarily keep $1 \leq i \leq d$ fixed. Define the operator $Y_i : H(S) \to H(S)$ on a dense set by $Y_ik_\lambda(z)x = b_i(z)S(\lambda)^*x$ and extending linearly; we claim Y_i extends to be a bounded operator, which we will prove by showing there exists a constant c such that

$$\left\| Y_i \sum_{j=1}^{n} k_{\lambda^j}(\cdot) x_j \right\|_{H(S)} \leq c \left\| \sum_{j=1}^{n} k_{\lambda^j}(\cdot) x_j \right\|_{H(S)},$$

using the usual density argument in $H(S)$. To do this, after fixing a relevant set of parameters, we will simply combine two inequalities to which we readily have access. Since we previously applied Proposition 4.2.1 to $b_i(z)$, we can also recall the comment after the proof of said proposition that tells us there exists a constant $c_1 > 0$ such that

$$\left\| Y_i \sum_{j=1}^{n} k_{\lambda^j}(\cdot) x_j \right\|_{H(S)} = \left\| b_i(\cdot) \sum_{j=1}^{n} S(\lambda^j)^* x_j \right\|_{H(S)} \leq c_1 \left\| \sum_{j=1}^{n} S(\lambda^j)^* x_j \right\|_{\mathcal{D}}.$$

For the second inequality, recall that Proposition 4.3.1 tells us that $(M_z, H(S))$ being bounded (which we are assuming) gives the existence of some constant $c_2 > 0$ such that $c_2^2 k_\lambda(z) - S(z)S(\lambda)^* \succeq 0$. From the discussion in between Proposition 4.2.1 and its proof, recall that we showed this is equivalent to

$$\left\| \sum_{j=1}^{n} S(\lambda^j)^* x_j \right\|_{\mathcal{D}} \leq c_2 \left\| \sum_{j=1}^{n} k_{\lambda^j}(\cdot) x_j \right\|_{H(S)}.$$

From here, it is clear we have shown the desired estimate with $c := c_1 c_2$.

With each Y_i bounded, we can assuredly say that $Y = [Y_1, \ ..., \ Y_d]^T$ defines a bounded operator. We are finally ready to define our Gleason problem solution X, which we remind the reader should be a column operator, by $X := \boldsymbol{\lambda}^* - Y$, clearly bounded as the difference of two bounded columns. In other words, we have

$$X k_\lambda(z) x = \boldsymbol{\lambda}^* k_\lambda(z) x - b(z) S(\lambda)^* x.$$

Finally, we have $M_z X k_\lambda(z) =$

$$M_z \boldsymbol{\lambda}^* k_\lambda(z) - M_z b(z) S(\lambda)^* = \langle z, \lambda \rangle_{\mathbb{C}^d} k_\lambda(z) - S(z)S(\lambda)^* = k_\lambda(z) - I_{\mathcal{E}} = k_\lambda(z) - k_\lambda(0).$$

Thus, this establishes $M_z X = I - P_0$, and therefore X is a Gleason problem solution, as desired.

We close out this section by crediting Ball and Bolotnikov (see [5]) for the ideas seen in the above proof, namely that of the *colligation* U, which allows for a *realization* of S in terms of the component operators of U.

4.4 Properties of $(M_z, H(B))$

In this section, we will continue to study $H(B)$ for some $B \in \mathcal{S}_d(\mathcal{D}, \mathcal{E})$ and Hilbert spaces $\mathcal{D}$ and $\mathcal{E}$. Our starting assumptions are that $B(0) = 0$, $\dim(\mathcal{E}) = m \in \mathbb{N}$, and that M_z is a bounded (row) operator on $H(B)$. Knowing only these, there are many properties to derive regarding M_z, which we will list and prove through a series of propositions.

Many of these properties will pertain to the operator $M_z - \boldsymbol{\lambda}$ and its adjoint, for an arbitrary $\lambda \in \mathbb{B}_d$. Here, $\boldsymbol{\lambda}$ refers to the operator in $\mathcal{B}(H(B)^d, H(B))$, which acts on an $f \in H(B)^d$ by $\boldsymbol{\lambda} f = \sum_{i=1}^{d} \lambda_i f_i$, just like it did in the previous sections.

Proposition 4.4.1: *Consider $\mathcal{M}_\lambda := \{k_\lambda(z)x \mid x \in \mathcal{E}\} \subset H(B)$. The following properties all hold:*

a) $dim(\mathcal{M}_\lambda) = dim(\mathcal{E})$

b) $\mathcal{M}_\lambda \subseteq Ker\ (M_z^* - \overline{\boldsymbol{\lambda}})$

c) $\|M_z^* g\|_{H(B)^d} \geq \|g\|_{H(B)}$ for all $g \perp Ker\ M_z^*$

Proof of a): Recalling that we set $m = \dim(\mathcal{E})$, let us specify a basis for $\mathcal{E}$, say $\{y_i\}_{i=1}^m$. Fix $\lambda \in \mathbb{B}_d$ and consider $\{k_\lambda(z)y_i\}_{i=1}^m$, which we claim is a basis for $\mathcal{M}_\lambda$. From the fact that the y_i span $\mathcal{E}$, it is clear that the $k_\lambda(z)y_i$ span $\mathcal{M}_\lambda$. As for the linear independence, suppose that $\sum_{i=1}^{m} c_i k_\lambda(z) y_i = 0$, as in that this function is 0 for any $z \in \mathbb{B}_d$. Since $B(0) = 0$, $k_\lambda(0) = I_\mathcal{E}$, and therefore plugging in 0 gives $\sum_{i=1}^{m} c_i y_i = 0$; by the linear independence of the

y_i, each c_i must be 0. This establishes our claim and reveals $m = \dim(\mathcal{M}_\lambda)$.

Proof of b): For any $f \in H(B)$, $\lambda \in \mathbb{B}_d$, $x \in \mathcal{E}$, and $1 \le i \le d$, we have by Proposition 4.2.3 that $M_{z_i}^* k_\lambda(z)x = \overline{\lambda_i} k_\lambda(z)x$. This can be written succinctly as $M_z^* k_\lambda(z)x = \overline{\boldsymbol{\lambda}} k_\lambda(z)x$, showing $\mathcal{M}_\lambda \subseteq \text{Ker } (M_z^* - \overline{\boldsymbol{\lambda}})$, as desired.

Proof of c): First, we note that it is straightforward to see that

$$\mathcal{M}_\lambda^\perp = \{f \in H(B) \mid f(\lambda) = 0\},$$

again by using the reproducing kernel to relate the inner products in $\mathcal{E}$ and $H(B)$.

Considering that b), with $\lambda = 0$, shows that $(\text{Ker } M_z^*)^\perp \subseteq \mathcal{M}_0^\perp$, it suffices to show that $\|M_z^* g\|_{H(B)^d} \ge \|g\|_{H(B)}$ for any g such that $g(0) = 0$ (i.e. that $g \in \mathcal{M}_0^\perp$, a potentially larger set than to which the claim applies, although we will establish equality before the end of the section).

We now claim that

$$D_0 := \left\{ \sum_{i=1}^n [k_{w^i}(z) - k_0(z)]x_i \mid n \in \mathbb{N}, \{w^i\}_{i=1}^n \subset \mathbb{B}_d, \{x_i\}_{i=1}^n \subset \mathcal{E} \right\}$$

is a set dense in $\{g \in H(B) \mid g(0) = 0\}$, which D_0 is evidently contained in since $k_\alpha(0) = I_\mathcal{E} = k_\beta(0)$, regardless of α and β. To prove our claim, assume $g(0) = 0$ for some $g \in H(B)$. Since $D = \left\{ \sum_{i=1}^m k_{w^i}(z)x_i \mid m \in \mathbb{N}, \{w^i\}_{i=1}^n \subset \mathbb{B}_d, \{x_i\}_{i=1}^n \subset \mathcal{E} \right\}$ is dense in $H(B)$, there exists some sequence $d_n(z) = \sum_{i=1}^{m_n} k_{w^{i,n}}(z)x_{i,n}$ converging to g. In particular, we have pointwise convergence at $z = 0$, so

$$d_n^0(z) := \sum_{i=1}^{m_n} k_0(z)x_{i,n} = \sum_{i=1}^{m_n} x_{i,n} = \sum_{i=1}^{m_n} k_{w^{i,n}}(0)x_{i,n} \to g(0) = 0.$$

Thus, $\{d_n - d_n^0\}_{n\ge1} \subset D_0$ converges to g, and the desired density has been shown.

By this density, it suffices to assume g is of the form $\sum_{i=1}^{n}(k_{w^i}(z)-k_0(z))x_i$ and remarking that $M_{z_j}^* k_0(z)=0$ using the previously written action of M_z^* on the reproducing kernel, we compute

$$\|M_z^* g\|_{H(B)^d}^2 = \sum_{j=1}^{d} \|M_{z_j}^* g\|_{H(B)}^2 = \sum_{j=1}^{d} \left\langle \sum_{i=1}^{n} \overline{w_j^i} k_{w^i}(z) x_i, \sum_{\ell=1}^{n} \overline{w_j^\ell} k_{w^\ell}(z) x_\ell \right\rangle_{H(B)}$$

$$= \sum_{i,\ell=1}^{n} \left(\sum_{j=1}^{d} w_j^\ell \overline{w_j^i} \right) \langle k_{w^i}(z) x_i, k_{w^\ell}(z) x_\ell \rangle_{H(B)} = \sum_{i,\ell=1}^{n} \left\langle w^\ell, w^i \right\rangle_{\mathbb{C}^d} \left\langle k_{w^i}(w^\ell) x_i, x_\ell \right\rangle_{\mathcal{E}}.$$

Then, since $\langle [k_{w^i}(z)-k_0(z)]x_i, [k_{w^\ell}(z)-k_0(z)]x_\ell \rangle_{H(B)} = \left\langle k_{w^i}(w^\ell)x_i, x_\ell \right\rangle_{\mathcal{E}} - \langle x_i, x_\ell \rangle_{\mathcal{E}}$ for any i and ℓ, we have $\|g\|_{H(B)}^2 = \sum_{i,\ell=1}^{n} \left\langle [k_{w^i}(w^\ell) - I_{\mathcal{E}}]x_i, x_\ell \right\rangle_{\mathcal{E}}$.

Collecting both sets of calculations, we see:

$$\|M_z^* g\|_{H(B)^d}^2 - \|g\|_{H(B)}^2 = \sum_{i,\ell=1}^{n} \left\langle [\left\langle w^\ell, w^i \right\rangle_{\mathbb{C}^d} k_{w^i}(w^\ell) - k_{w^i}(w^\ell) + I_{\mathcal{E}}]x_i, x_\ell \right\rangle_{\mathcal{E}}$$

$$= \sum_{i,\ell=1}^{n} \left\langle B(w^\ell) B(w^i)^* x_i, x_\ell \right\rangle_{\mathcal{E}} = \left\| \sum_{i=1}^{n} B(w^i)^* x_i \right\|_{\mathcal{D}}^2 \geq 0,$$

and thus our claim is established.

By Lemma 4.1.1, we note that c) above holds for M_z in place of M_z^*, and therefore both M_z and M_z^* have closed range. Also, by the discussion in the first section, the Cauchy dual to M_z^* exists, which we will call $\mathscr{L}$ instead of L. We summarize the properties we will presently need as the identity $M_z \mathscr{L} = P_{\text{Ran } M_z} = I_{\mathcal{H}} - P_{\text{Ker } M_z^*}$.

Proposition 4.4.2: *Again, we consider $\mathcal{M}_\lambda = \{k_\lambda(z)x \mid x \in \mathcal{E}\}$ for some fixed $\lambda \in \mathbb{B}_d$ and finite-dimensional $\mathcal{E}$. The following all hold:*

a) $\mathcal{M}_0^\perp \subseteq$ *Ran* M_z

b) $\|(M_z^* - \overline{\boldsymbol{\lambda}})g\|_{H(B)^d} \geq \|(I - \boldsymbol{\lambda} M_z^*)g\|_{H(B)} \quad \forall g \perp$ *Ker* M_z^*

c) $M_z - \boldsymbol{\lambda}$ has closed range

d) $k_\lambda(z) = (I - \overline{\boldsymbol{\lambda}}\mathscr{L}^*)^{-1}k_0(z)$

e) $\mathcal{M}_\lambda = [Ran\ (M_z - \boldsymbol{\lambda})]^\perp$

Proof of a): It is now time to invoke the existence of a Gleason problem solution X, as proven in Proposition 4.3.2. That is, we have a bounded column operator X satisfying $M_z X = I_{H(B)} - P_0$ where $P_0 f := f(0)$, the orthogonal projection onto the subspace $\mathcal{M}_0$. Thus, we immediately have that if $f \in \mathcal{M}_0^\perp$ (i.e. $f(0) = 0$), then $M_z(Xf) = f \in \text{Ran } M_z$.

Before moving to prove *b*), we address that we have now established $\mathcal{M}_0^\perp = (\text{Ker } M_z^*)^\perp$ by Proposition 4.4.1b) with $\lambda = 0$, Proposition 4.4.2a), and the fact that M_z has closed range. This is enough to show that $\mathscr{L}$ is *also* a Gleason problem solution, as we can now establish $P_{\text{Ker } M_z^*} = P_{\mathcal{M}_0} = P_0$. For this last equality, we can see this by first writing $f = P_{\mathcal{M}_0} f + P_{\mathcal{M}_0{}^\perp} f$. Next, we can plug in 0 to see $f(0) = (P_{\mathcal{M}_0} f)(0)$ since we have already established that any $g \in \mathcal{M}_0^\perp$ satisfies $g(0) = 0$. Then, since $\mathcal{M}_0 = \{k_0(z)x \mid x \in \mathcal{E}\}$ consists of constant functions (as $k_0(z) = I_\mathcal{E}$), it must be that $P_{\mathcal{M}_0} f = f(0)$.

Proof of b): The claim is a somewhat auxilary estimate that acts as a λ-dependent generalization of M_z^* being bounded below on the adjoint of its kernel. Taking $\lambda = 0$ returns the already established property, which we will use to prove this one. Computing the square of each norm separately, we have $\|(M_z^* - \overline{\boldsymbol{\lambda}})g\|^2_{H(B)^d} =$

$$\sum_{j=1}^d \|M_{z_j}^* g - \overline{\lambda_j} g\|^2_{H(B)} = \|M_z^* g\|^2_{H(B)^d} - 2\text{Re}\left(\sum_{j=1}^d \lambda_j \left\langle M_{z_j}^* g, g\right\rangle_{H(B)}\right) + |\lambda|^2 \|g\|^2_{H(B)}$$

and $\|(I - \boldsymbol{\lambda} M_z^*)g\|^2_{H(B)} =$

$$\|g - \boldsymbol{\lambda} M_z^* g\|^2_{H(B)} = \|g\|^2_{H(B)} - 2\text{Re}\left(\langle \boldsymbol{\lambda} M_z^* g, g\rangle_{H(B)}\right) + \|\boldsymbol{\lambda} M_z^* g\|^2_{H(B)}$$

$$= \|g\|^2_{H(B)} - 2\text{Re}\left(\sum_{j=1}^d \lambda_j \left\langle M_{z_j}^* g, g\right\rangle_{H(B)}\right) + \|\boldsymbol{\lambda} M_z^* g\|^2_{H(B)}$$

Before we subtract the latter from the former, we comment that the Cauchy-Schwarz inequality reveals:

$$\|\boldsymbol{\lambda} M_z^* g\|_{H(B)}^2 = \left\| \sum_{j=1}^{d} \lambda_j M_{z_j}^* g \right\|_{H(B)}^2 \leq \left(\sum_{j=1}^{d} |\lambda_j|^2 \right) \left(\sum_{j=1}^{d} \|M_{z_j}^* g\|_{H(B)}^2 \right) = |\lambda|^2 \|M_z^* g\|_{H(B)^d}^2$$

Collecting all this, we have $\|(M_z^* - \overline{\boldsymbol{\lambda}})g\|_{H(B)^d}^2 - \|(I - \boldsymbol{\lambda} M_z^*)g\|_{H(B)}^2 =$

$$\|M_z^* g\|_{H(B)^d}^2 - \|g\|_{H(B)}^2 + |\lambda|^2 \|g\|_{H(B)}^2 - \|\boldsymbol{\lambda} M_z^* g\|_{H(B)}^2 \geq (1 - |\lambda|^2)(\|M_z^* g\|_{H(B)^d}^2 - \|g\|_{H(B)}^2),$$

which is non-negative for any $\lambda \in \mathbb{B}_d$ as long as $g \in \text{Ker } M_z^*$ since we can then use Proposition 4.4.1c).

Proof of c): With the previous estimate proven, we are ready now to start a direct proof that Ran $(M_z^* - \overline{\boldsymbol{\lambda}})$ is closed, which by the Closed Range Theorem will confirm c).

To wit, assume $\{f_n\} \subset H(B)$ is such that $(M_z^* - \overline{\boldsymbol{\lambda}})f_n$ converges to some $g = [g_1, \ ..., \ g_d]^T \in H(B)^d$. Define an auxiliary sequence $\{h_n\} \subset H(B)$ by $h_n(z) := f_n(z) - f_n(0)k_\lambda(z)$, noting that $h_n(0) = 0$ for every n and therefore by the comment we made after the proof of Proposition 4.4.2a), we have $\{h_n\} \subset \mathcal{M}_0^\perp = \text{Ker } M_z^*$. Also, since $(M_z^* - \overline{\boldsymbol{\lambda}})k_\lambda(z) = 0$, we have that $(M_z^* - \overline{\boldsymbol{\lambda}})h_n = (M_z^* - \overline{\boldsymbol{\lambda}})f_n$. This is to say that $(M_z^* - \overline{\boldsymbol{\lambda}})h_n$ also converges to g and is therefore a Cauchy sequence.

Now, for any $n, m \in \mathbb{N}_0$, $h_n - h_m \perp \text{Ker } M_z^*$, so our earlier estimate can be applied to see

$$\|(M_z^* - \overline{\boldsymbol{\lambda}})(h_n - h_m)\|_{H(B)^d} \geq \|(I - \boldsymbol{\lambda} M_z^*)(h_n - h_m)\|_{H(B)},$$

and therefore $(I - \boldsymbol{\lambda} M_z^*)h_n$ is *also* a Cauchy sequence. The same sequence must then converge to some function $r \in H(B)$.

Let us now invoke component convergence when considering the convergence to g. Since for any $1 \leq j \leq d$, we have $(M_{z_j}^* - \overline{\lambda_j})h_n$ converging to g_j, we can first multiply by λ_j then sum over j to see that $(\boldsymbol{\lambda} M_z^* - |\lambda|^2 I)h_n$ converges to $s := \boldsymbol{\lambda} g \in H(B)$.

Taking this one step further by adding and subtracting h_n, we have that $(1-|\lambda|^2)h_n - (I - \boldsymbol{\lambda} M_z^*)h_n$ converges to s. Yet, now we can bring in the r and rearrange the convergence to see that $(1-|\lambda|^2)h_n$ converges to $r+s$, which of course reveals that h_n converges to $\dfrac{r+s}{1-|\lambda|^2} =: h \in H(B)$.

Finally, since $M_z^* - \overline{\boldsymbol{\lambda}}$ is continuous, it must be that $g = (M_z^* - \overline{\boldsymbol{\lambda}})h$, as desired.

Proof of d): We first comment that Proposition 4.1.3 establishes the invertibility of the operator in the statement. Next, since $M_z\mathscr{L} = I - P_0$ and $I - P_0$ is self-adjoint, we also have $\mathscr{L}^* M_z^* = I - P_0$. Therefore for any $x \in \mathcal{E}$, we have:

$$k_\lambda(z)x - k_0(z)x = [k_\lambda(z) - k_\lambda(0)]x = \mathscr{L}^* M_z^* k_\lambda(z)x = \overline{\boldsymbol{\lambda}}\mathscr{L}^* k_\lambda(z)x.$$

Rearranging this and solving for $k_0(z)x$, we obtain the desired identity.

Proof of e): By c), it suffices to show that $\mathcal{M}_\lambda = \text{Ker }(M_z^* - \overline{\boldsymbol{\lambda}})$. Even further, by Proposition 4.4.1b), we only need check that $\text{Ker }(M_z^* - \overline{\boldsymbol{\lambda}}) \subseteq \mathcal{M}_\lambda$. Therefore, suppose $f \in H(B)$ satisfies $M_z^* f = \overline{\boldsymbol{\lambda}} f$. By applying $\mathscr{L}^*$ on the left to both sides, we have $\mathscr{L}^* M_z^* f = \overline{\boldsymbol{\lambda}}\mathscr{L}^* f$. Pulling on d) and its proof, this allows us to say $f - f(0) = \overline{\boldsymbol{\lambda}}\mathscr{L}^* f$, and subsequently that $f = (I - \overline{\boldsymbol{\lambda}}\mathscr{L}^*)^{-1} f(0)$. Yet, since $f(0) = k_0(z) f(0)$, we have that $f = (I - \overline{\boldsymbol{\lambda}}\mathscr{L}^*)^{-1} k_0(z) f(0) = k_\lambda(z) f(0) \in \mathcal{M}_\lambda$, as desired.

We note now that Proposition 4.4.1a) and Proposition 4.4.2e) together show that we have $\dim([\text{Ran }(M_z - \boldsymbol{\lambda})]^\perp) = \dim(\mathcal{E}) = m$ for every $\lambda \in \mathbb{B}_d$. This property of $[\text{Ran }(M_z - \boldsymbol{\lambda})]^\perp$ having constant dimension across any λ is referred to in the literature as M_z being a weak dual Cowen-Douglas tuple (of rank m).

The next property involves a definition that requires multi-index notation. For $\{T_i\}_{i=1}^d \subset \mathcal{B}(\mathcal{H})$ that pairwise commute, we can define T^α as $T_1^{\alpha_1} ... T_d^{\alpha_d}$. Then, for a commuting operator tuple $T = (T_1, \ ..., \ T_d)$, we call T *analytic* if and only if

$$\bigcap_{n \geq 0} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = n}} T^\alpha \mathcal{H} = \{0\}.$$

In the future, we will replace the index under the sum with simply $|\alpha| = n$ where it is understood that this refers to α in the set $\mathbb{N}_0^d$ with order n.

Proposition 4.4.3: *$(M_z, H(B))$ is analytic.*

Proof: Since $H(B) \subseteq (H_d^2 \ \otimes \ \mathcal{E}) \cong (H_d^2 \ \otimes \ \mathbb{C}^m)$, we only need to show

$$\bigcap_{n \geq 0} \sum_{|\alpha| = n} M_z^\alpha (H_d^2 \otimes \mathbb{C}^m) = \{0\}$$

to see that the restricted shift $(M_z, H(B))$ is analytic.

First, we show this in the case that $m = 1$. Let us assume for any $n \geq 0$, $f = \sum_{|\alpha| = n} z^\alpha h_\alpha$ for some functions $\{h_\alpha\} \subset H_d^2$. We claim that $\left(\frac{\partial^\gamma}{\partial z^\gamma} f\right)(0) = 0$ for any $\gamma \geq 0$, which would establish $f \equiv 0$. The main tool for proving this comes from observing that for any h, $\alpha \neq \emptyset$, and $1 \leq i \leq d$, $\frac{\partial}{\partial z_i} z^\alpha h = z^\beta g$ for some function g and $|\beta| \in \{|\alpha|, |\alpha| - 1\}$. Fixing γ and letting $k := |\gamma| \geq 0$, we can apply this observation k times sequentially to each summand of $f = \sum_{|\alpha| = k+1} z^\alpha h_\alpha$ and see that $\frac{\partial^\gamma}{\partial z^\gamma} f$ vanishes upon plugging in 0.

For general m, if, for any $n \geq 0$, a column $f = [f^1, \ ..., \ f^m]^T$ can be written as $\sum_{|\alpha| = n} z^\alpha [h_\alpha^1, \ ..., \ h_\alpha^m]^T$ for some $\{h_\alpha^k\} \subset H_d^2$, then applying the previous case to each component function f^k shows $f^k \equiv 0$ for any k and therefore $f \equiv 0$.

Proposition 4.4.4: *If $(M_z, H(B))$ is bounded, then dim$(H(B)) = \infty$.*

Proof: Fix a nonzero $x \in \mathcal{E}$. By the only assumption, we have that $z_1 k_0(z) x = M_z [k_0(z) x, \ 0, \ ..., \ 0]^T \in H(B)$. This serves as a base case for an inductive approach to showing $z_1^n k_0(z)x = M_z[z^{n-1}k_0(z)x,\ 0,\ ...,\ 0]^T \in H(B)$ for any $n \geq 1$, which should be clear enough. We claim that the set $\{z_1^n k_0(z)x\}_{n\geq 1}$ is linearly independent, which will be enough to finish the proof.

Assume $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ are such that $\sum_{n=1}^{\infty} a_n z_1^n k_0(z)x = 0$. Using that $k_0(z) = I_{\mathcal{E}}$, we have for any z,

$$0 = \left\langle \sum_{n=1}^{\infty} a_n z_1^n x, x \right\rangle_{\mathcal{E}} = \sum_{n=1}^{\infty} a_n \|x\|_{\mathcal{E}}^2 z_1^n,$$

and a power series (in this case in the single variable z_1) is identically zero if and only if all its coefficients are, leading to $a_n\|x\|_{\mathcal{E}}^2 = 0$ for all n and therefore every $a_n = 0$, as needed to show the linear independence.

4.5 The Main Result

In this section, our only concern is to state and prove the result below: Theorem 4.5.1.

Theorem 4.5.1: *Let $T = (T_1,\ ...,\ T_d) : \mathcal{H}^{(d)} \to \mathcal{H}$ be a bounded row operator on an infinite dimensional Hilbert space $\mathcal{H}$, and let $m \in \mathbb{N}$. T is unitarily equivalent to $(M_z, H(B))$ for some $B \in \mathcal{S}(\mathcal{D}, \mathcal{E})$ and Hilbert spaces $\mathcal{D}$ and $\mathcal{E}$ such that $B(0) = 0$, dim($\mathcal{E}$) $= m$, and that M_z is a bounded row operator if and only if T satisfies the following four conditions:*

(i) $T_iT_j = T_jT_i$ *for all* $1 \leq i, j \leq d$

(ii) dim([Ran $(T - \boldsymbol{\lambda})]^{\perp}) = m$ *for every* $\lambda \in \mathbb{B}_d$

(iii) $\|T\boldsymbol{x}\|_{\mathcal{H}} \geq \|\boldsymbol{x}\|_{\mathcal{H}^d}$ *for all* $\boldsymbol{x} \in$ *(Ker* $T)^{\perp}$

(iv) $\bigcap_{n\geq 0} \sum_{|w|=n} T_w\mathcal{H} = \{0\}$

Here, $\boldsymbol{\lambda}$ acts exactly the same as it did earlier in this section but instead is in $\mathcal{B}(\mathcal{H}^d, \mathcal{H})$. Also, since (i) allows us to replace words with multi-indices, property (iv) is precisely that T is analytic. $\mathcal{H}$ must be assumed to be infinite dimensional (by means of the separability assumption), otherwise Proposition 4.4.4 would be contradicted since $\mathcal{H} \cong H(B)$ as a consequence of the theorem.

Theorem 4.5.1 is mostly a multivariable generalization of Theorem 4.6 in [16], wherein the same type of characterization as below was done with $d = 1 = m$. However, assumptions (ii) and (iii) are more complex as soon as $d \geq 2$, so we advise slight caution when comparing the two theorems. Specifically, the authors of [16] show that $\dim([\text{Ran } T]^{\perp}) = 1$ is all that is needed to guarantee condition (ii) in the case that $d = 1 = m$, so condition (ii) with only $\lambda = 0$ is all that is assumed in their case. We will show at the end of this chapter with a particular example that this is not necessarily the case for $d = 2$.

Secondly, Theorem 4.6 in [16] has that T is norm-expanding on all of $\mathcal{H}$ instead of only on $(\text{Ker } T)^{\perp}$. This is because, in one variable, it is an immediate necessary condition that Ker $T = \{0\}$ if T is unitarily equivalent to a bounded shift (on any function space). Yet, as soon as $d = 2$, one can show that Ker M_z contains nonzero elements, and the norm-expanding condition must likewise be weakened in the above way.

One of the most important tools in the (backward direction of the) forthcoming proof is the L that was introduced (as the Cauchy dual of T^*) in Section 4.1, as a specific consequence of property (ii). As a quick remark, let us address that in [16], the L is defined more directly by $L := (T^*T)^{-1}T^*$. To see that this is still true with our slightly different assumptions in the case that $d = 1$, we simply use the theorem to say that, under conditions (i) through (iv), T is unitarily equivalent to the operator M_z, which is of course injective by being a (singleton) multiplication operator. Therefore, Ker $T^*T =$ Ker $T = \{0\}$ and by examining the construction of L back in Section 4.1, we see that the form of L greatly reduces to $(T^*T)^{-1}T^*$. However, it seems likely that a direct proof could be obtained: that is, without invoking any equivalence to an operator on a function space, one could presumably show that conditions (i) through (iv) *must* guarantee that Ker $T = \{0\}$. We will leave this conjecture open.

One particular aspect of the proof of Theorem 4.5.1 requires the introduction of notation, a definition, and a lemma. We will begin the proof once these have been established.

Notation: For two closed subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of a Hilbert space $\mathcal{H}$, we write $\mathcal{H} = \mathcal{M}_1 \dot{+} \mathcal{M}_2$ to indicate an *algebraic direct sum*, with the plus symbol having the precise meaning that (1) any element in $\mathcal{H}$ has a decomposition into a sum of two elements $x_1 + x_2$ where $x_i \in \mathcal{M}_i$

and that (2) $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$ (i.e. such a decomposition is unique).

Nothing about the above involves any notion of orthogonality, which is why we avoid using the symbol $\oplus$. *Idempotent* operators (Q that satisfy $Q^2 = Q$ but are not necessarily self-adjoint) are to the above type of algebraic direct sum as projection operators are to what we normally refer to as a direct sum of two subspaces; that is, given an idempotent Q, we immediately obtain an algebraic direct sum with $\mathcal{M}_1 = \text{Ran } Q$ and $\mathcal{M}_2 = \text{Ran } (I - Q)$. At this point, it should be noted that $Q^2 = Q$ is enough to quarantee that Q has closed range, as one can readily check.

Conversely, every algebraic direct sum is associated with a unique idempotent Q that *gives* the decomposition in the sense that for $x \in \mathcal{H}$, $x = Qx + (I - Q)x$ is the unique decomposition.

One last fact we will need is that given such an algebraic direct sum, it is still true that $\dim(\mathcal{M}_2) = \dim(\mathcal{M}_1^\perp)$. To see this, we consider the orthogonal projection $P = \text{Proj}_{\mathcal{M}_1^\perp}$; since clearly $P\mathcal{M}_1 = 0$, we have that $P\mathcal{M}_2 = \text{Ran } P = \mathcal{M}_1^\perp$ and it is well-known that $\dim(P\mathcal{M}_2) \leq \dim(\mathcal{M}_2)$.

To get the opposite inequality, we will show $\mathcal{H} = \mathcal{M}_1^\perp + \mathcal{M}_2^\perp$, since then we can apply the inequality we just derived to obtain $\dim((\mathcal{M}_2^\perp)^\perp) \leq \dim(\mathcal{M}_1^\perp)$. Noting that the $\mathcal{M}_i$ are assumed to be closed, this will establish the equality we need.

First note that the empty intersection of $\mathcal{M}_1$ and $\mathcal{M}_2$ is equivalent to saying $\mathcal{H} = \mathcal{M}_1^\perp \vee \mathcal{M}_2^\perp$. Yet, by considering the quotient map $\mathcal{Q} : \mathcal{H} \to \mathcal{H}/\mathcal{M}_1^\perp$ and the fact that $\text{span}(\mathcal{M}_1^\perp, \mathcal{M}_2^\perp) = \mathcal{Q}^{-1}([\mathcal{M}_2^\perp])$, we see this set must be closed and the symbol $\vee$ may be replaced with $+$, as desired.

Up until now, we have avoided using $\boldsymbol{z}$ (in place of $\boldsymbol{\lambda}$ or $\boldsymbol{\mu}$) when referring to the row operator defined in the first section of the chapter. In what follows, we can no longer afford to do this, as we want to use this notation while simultaneously allowing z to refer to the independent variable on the de Branges-Rovnyak space that we will construct.

Definition: We call a row T of operators on $\mathcal{B}(\mathcal{H})$ *regular* if and only if $\text{Ran } (T - \mathbf{z})$ is closed and $\mathcal{H} = \text{Ran } (T - \mathbf{z}) + \text{Ker } T^*$ for all $z \in \mathbb{B}_d$.

While other authors may refer to the preceding as *regularity at 0* or otherwise replace $\mathbb{B}_d$ with an arbitrary ϵ-ball, we (for now) have no need for anything besides the above definition of simply *regular*.

It is now an opportune time to relax our usage of simply P for the projection onto Ker T^* as we will soon require other projections and we also choose to use P for something else entirely in the following lemma, for which we are indebted to Jörg Eschmeier and Sebastian Langendörfer (see [10]).

Lemma 4.5.2: *Suppose $T = (T_1, \ ..., \ T_d) : \mathcal{H}^{(d)} \to \mathcal{H}$ satisfies property (iii) of Theorem 4.5.1. Define a map $P : \mathbb{B}_d \to \mathcal{B}(\mathcal{H})$, given by $P(z) = (T - \boldsymbol{z})L(I - \boldsymbol{z}L)^{-1}$. For any fixed z, $P(z)$ has the following properties:*

a) $P(z)$ is idempotent.

b) $[I - P(z)]x = P_{Ker\ T^*}(I - \boldsymbol{z}L)^{-1}x$ for all $x \in \mathcal{H}$.

c) $\mathcal{H} = Ran\ P(z) + Ker\ T^*$.

If T is regular, the following is additionally true:

d) $Ran\ P(z) = Ran\ (T - \boldsymbol{z})$.

a′) $P(z)$ is the idempotent associated with $\mathcal{H} = Ran\ (T - \boldsymbol{z}) + Ker\ T^*$.

Proof of a): We start by recalling that, due to property (iii), we have a column L satisfying that LT is a projection onto Ran L = Ran T^*, and subsequently $I - LT = P_{\text{Ker } T}$. Using these facts and the well-established identity $(I - \mathbf{z}L)^{-1} = I + (I - \mathbf{z}L)^{-1}\mathbf{z}L$:

$$L(I - \mathbf{z}L)^{-1}(T - \mathbf{z}) = L(I + (I - \mathbf{z}L)^{-1}\mathbf{z}L)T - L(I - \mathbf{z}L)^{-1}\mathbf{z} =$$

$$LT - L(I - \mathbf{z}L)^{-1}\mathbf{z}(I - LT) = P_{\text{Ran } T^*} - L(I - \mathbf{z}L)^{-1}\mathbf{z}P_{\text{Ker } T}.$$

Invoking the projections as the right-most operators in each term is the essential step, since we know $P_{\text{Ran } T^*} L = L$ and $P_{\text{Ker } T} L = 0$ and this leads to

$$P(z)^2 = (T - \mathbf{z}) \left[L(I - \mathbf{z}L)^{-1}(T - \mathbf{z})L \right] (I - \mathbf{z}L)^{-1} = P(z)$$

because the entire bracketed operator is simply L by the above, establishing a).

Proof of b): Next, we directly compute (using a slightly altered version of the identity for $(I - \mathbf{z}L)^{-1}$ that we used just above):

$$I - P(z) = I - (T - \mathbf{z})L(I - \mathbf{z}L)^{-1} = I - TL(I - \mathbf{z}L)^{-1} + \mathbf{z}L(I - \mathbf{z}L)^{-1}$$

$$= (I - \mathbf{z}L)^{-1} - TL(I - \mathbf{z}L)^{-1} = (I - TL)(I - \mathbf{z}L)^{-1} = P_{\text{Ker } T^*}(I - \mathbf{z}L)^{-1}.$$

Applying the above operators to $x \in \mathcal{H}$, we have clearly established b), although we note here that this separately shows Ran $[I - P(z)] =$ Ran $P_{\text{Ker } T^*}(I - \mathbf{z}L)^{-1} =$ Ran $P_{\text{Ker } T^*} =$ Ker T^* where the second-to-last equality comes out of the fact that clearly $(I - \mathbf{z}L)^{-1}$ is onto.

Proof of c): The decomposition in c) is a fairly simple consequence of a) and b) combined: a) gives that $\mathcal{H} =$ Ran $P(z) +$ Ran $[I - P(z)]$ and, as mentioned just above, we have Ran $[I - P(z)] =$ Ker T^* from the proof of b).

Proof of d): It is clear that Ran $P(z) \subseteq$ Ran$(T - \mathbf{z})$ by definition. To get the opposite containment, we need to assume that T is regular, which we do from here on. In words, the key idea is that we now have two algebraic direct sums to work with, where one summand space is the same and there is a known containment between the summand spaces that are *not* (yet established to be) the same. More formally, the argument goes like this: assume $x \in$ Ran$(T - \mathbf{z})$. By $\mathcal{H} =$ Ran $P(z) +$ Ran $[I - P(z)]$, x has unique decomposition into $x = P(z)x + [I - P(z)]x$. However, since x and $P(z)x$ are in Ran $(T - \mathbf{z})$, so must be the element $y := [I - P(z)]x$, and we know that Ran $(T - \mathbf{z}) \cap$ Ran $[I - P(z)] =$ Ran $(T - \mathbf{z}) \cap$ Ker $T^* = \{0\}$. Therefore $y = 0$ and $x = P(z)x \in$ Ran $P(z)$, as desired.

Proof of a'): As the notation suggests, this is a strengthening of a) in the case that T is regular. Let us suppose that $x = x_1 + x_2$ where $x_1 \in \text{Ran } (T - \mathbf{z})$ and $x_2 \in \text{Ker } T^*$. By d), $x_1 = P(z)y_1$ for some y_1, and since $\text{Ker } T^* = \text{Ran } [I - P(z)]$, we also have an y_2 such that $x_2 = [I - P(z)]y_2$. Then, since $P(z)[I - P(z)] = 0$, we get $P(z)x = P(z)x_1 + P(z)x_2 = P(z)^2 y_1 + 0 = P(z)y_1 = x_1$. With this, the proof is finished, since it is now immediate that $x_2 = x - x_1 = [I - P(z)]x$.

We are now fully equipped to prove the main result of the chapter.

Proof of Theorem 4.5.1: First, we assume the unitary equivalence. For conditions (i) through (iv), (i) is clear for M_z since all multipliers commute by definition, and the remaining three were derived in the previous section: (ii) comes from the remark after Proposition 4.4.2, (iii) is established by Proposition 4.4.1c) and the lemma following it, and (iv) is entirely Proposition 4.4.3.

Therefore, all that must be done in this direction of the proof is to show that each condition is preserved under unitary equivalence. To that end, we now specify a unitary operator $U : \mathcal{H} \to H(B)$ such that $UT_i = M_{z_i}U$ for $1 \leq i \leq d$. Alternatively, we can write this as $UT = M_z U^{(d)}$, where we recall that the notation $A^{(d)}$ refers to a diagonal matrix with constant operator entry A.

From writing $T_i = U^* M_{z_i} U$, one immediately sees from $UU^* = I_{H(B)}$ that (i) holds for T as well.

For (ii), let us first notate the distinction of the two relevant λ-dependent operators by letting $\widehat{\boldsymbol{\lambda}}$ be the operator in $\mathcal{B}(H(B)^d, H(B))$, where it can be easily established based on the definitions that $\widehat{\boldsymbol{\lambda}} U^{(d)} = U\boldsymbol{\lambda}$. With that, it is a routine exercise to show that if $\{g_i\}_{i=1}^m$ form a basis for Ker $\left(M_z^* - \overline{\widehat{\boldsymbol{\lambda}}}\right)$, then $\{U^* g_i\}_{i=1}^m$ form a basis for Ker $(T^* - \overline{\boldsymbol{\lambda}})$. Now bringing in the fact that unitary equivalence preserves range closure, we have established (ii) for T.

To now show (iii) holds for T, we first claim $U^{(d)}(\text{Ker } T)^\perp \subseteq (\text{Ker } M_z)^\perp$. Assume that $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}^{(d)}} = 0$ whenever $T\mathbf{y} = 0$, and that $\mathbf{f}$ is such that $M_z \mathbf{f} = 0$. Then $\left\langle U^{(d)}\mathbf{x}, \mathbf{f} \right\rangle_{H(b)^{(d)}} = \left\langle \mathbf{x}, (U^*)^{(d)}\mathbf{f} \right\rangle_{\mathcal{H}^d} = 0$ since $T(U^*)^{(d)}\mathbf{f} = U^* M_z \mathbf{f} = 0$, and our claim is established.

With that, for $\mathbf{x} \in (\text{Ker } T)^{\perp}$,

$$\|T\mathbf{x}\|_{\mathcal{H}} = \|U^* M_z U^{(d)}\mathbf{x}\|_{\mathcal{H}} = \|M_z U^{(d)}\mathbf{x}\|_{H(b)} \geq \|U^{(d)}\mathbf{x}\|_{H(b)^{(d)}} = \|\mathbf{x}\|_{\mathcal{H}^{(d)}},$$

where our previous claim was needed to justify the inequality.

Finally, we address (iv) for T. Assume x is such that for every fixed n, $x = \sum_{|\alpha|=n} T^{\alpha} y_{\alpha}$ for some $\{y_{\alpha}\} \subset \mathcal{H}$. Since U^* is surjective, there is an f_{α} such that $y_{\alpha} = U^* f_{\alpha}$ for every α that appears in the sum. We then have

$$Ux = \sum_{|\alpha|=n} UT^{\alpha}U^* f_{\alpha} = \sum_{|\alpha|=n} M_z^{\alpha} f_{\alpha}$$

and by (iv) for M_z, Ux must be 0 and thus x must be 0, as desired.

This finishes the forward direction of the theorem's proof.

Conversely, assume properties (i) through (iv) hold for T. From the first section and its propositions, we know property (iii) gives us a contractive column operator L satisfying $LT = P_{\text{Ran } T^*}$ and $TL = P_{\text{Ran } T}$, as well as allowing for the well-definedness of $(I - \boldsymbol{z}L)^{-1}$ and $(I - \overline{\boldsymbol{\lambda}}L^*)^{-1}$ for any $z, \lambda \in \mathbb{B}_d$. We reserve I for $I_{\mathcal{H}}$; any other identity operators will be specifically denoted.

We now begin a fairly lengthy construction of the required objects. Since $\mathcal{H}$ is infinite dimensional, there exists an abstract unitary row operator $U : \mathcal{H}^d \to \mathcal{H}$. Also, L being contractive leads to the operator $D := (I_{\mathcal{H}^d} - LL^*)^{1/2}$ being well-defined, and we are now ready to define the spaces $\mathcal{D} := \overline{\text{Ran } UD}$ and $\mathcal{E} := \text{Ker } T^*$, which as a reminder serve as the domain and codomain respectively for the Schur class of which our soon-to-be defined B will be a member. We note now that by taking $\lambda = 0$ in property (ii), we can see that $m = \dim([\text{Ran } T]^{\perp}) = \dim(\mathcal{E})$, as we require. Lastly, the construction is also in need of an auxiliary column $C := DU^*\big|_{\mathcal{D}}$.

Beyond the above, our construction has three main pieces. In the below, we reserve the notation P for the operator $P_{\text{Ker } T^*} : \mathcal{H} \to \text{Ker } T^*$ which has as its adjoint the inclusion map $\iota : \text{Ker } T^* \to \mathcal{H}$.

$$B : \mathbb{B}_d \to \mathcal{B}(\mathcal{D}, \mathcal{E}) \qquad B(z) := P(I - \mathbf{z}L)^{-1}\mathbf{z}C$$

$$V : \mathcal{H} \to H(B) \qquad Vx(z) := P(I - \mathbf{z}L)^{-1}x$$

$$k_\lambda(z) = P(I - \mathbf{z}L)^{-1}(I - \overline{\boldsymbol{\lambda}}L^*)^{-1}\iota$$

V will provide the desired the unitary equivalence, but we first show consistency in the above (i.e. verifying that the above $k_\lambda(z)$ is indeed the reproducing kernel corresponding to the de-Branges kernel formed from our B). We will do this by showing the identity $B(z)B(\lambda)^* = \langle z, \lambda \rangle k_\lambda(z) - k_\lambda(z) + I_{\mathcal{E}}$. Another important goal this computation will accomplish is showing that our B is indeed in its appropriate Schur class, since the corresponding $k_\lambda(z)$ is clearly positive definite as it can be written as $H(z)H(\lambda)^*$ for $H(z) := P(I - \mathbf{z}L)^{-1}$.

Before we proceed with this task, now is perhaps the best time to address that at first it may seem odd for an abstract unitary U, which is in no way guaranteed to be unique, to play a role in a very particular construction; however, as we will see in the immediate sequel, $B(z)B(\lambda)^*$ will result in the same operator for any U, and this is what uniquely determines the kernel/space.

A few preliminary calculations are required, listed in the order they will be used. First, we compute $CC^* = DU^*UD = D^2 = I_{\mathcal{H}^d} - LL^*$ which tells us $C_iC_j^* = \delta_{ij}I - L_iL_j^*$ for any $1 \leq i, j \leq d$.

Next, we list the easily verified general identities $(I - R)^{-1}R = (I - R)^{-1} - I$ and $S(I - S)^{-1} = (I - S)^{-1} - I$ for any operators R, S such that $(I - R)$ and $(I - S)$ are invertible.

Lastly, we note that from the fact that Ker $T^* =$ Ker L, $L\iota$ and PL^* are both 0. Then, from the geometric series expansions we also obtain $(I - \mathbf{z}L)^{-1}\iota = I_{\mathcal{E}}$ and $P(I - \overline{\boldsymbol{\lambda}}L^*)^{-1} = I_{\mathcal{E}}$ (i.e. only the 0th summand does not vanish.)

We compute $B(z)B(\lambda)^* =$

$$P(I - \mathbf{z}L)^{-1} \sum_{i,j=1}^{d} z_i\overline{\lambda_j}C_iC_j^*(I - \overline{\boldsymbol{\lambda}}L^*)^{-1}P^* = P(I - \mathbf{z}L)^{-1}\left[\langle z, \lambda \rangle - \mathbf{z}L\overline{\boldsymbol{\lambda}}L^*\right](I - \overline{\boldsymbol{\lambda}}L^*)^{-1}\iota$$

$$= \langle z, \lambda \rangle k_\lambda(z) - P(I - \mathbf{z}L)^{-1}\mathbf{z}L\overline{\boldsymbol{\lambda}}L^*(I - \overline{\boldsymbol{\lambda}}L^*)^{-1}\iota$$

We now use the two matrix identities with $R = \mathbf{z}L$ and $S = \overline{\boldsymbol{\lambda}}L^*$ to continue computing

$$B(z)B(\lambda)^* = \langle z, \lambda \rangle \, k_\lambda(z) - P[(I - \mathbf{z}L)^{-1} - I][(I - \overline{\boldsymbol{\lambda}}L^*)^{-1} - I]\iota$$

$$= \langle z, \lambda \rangle \, k_\lambda(z) - P[(I - \mathbf{z}L)^{-1}(I - \overline{\boldsymbol{\lambda}}L^*)^{-1} - (I - \mathbf{z}L)^{-1} - (I - \overline{\boldsymbol{\lambda}}L^*)^{-1} + I]\iota$$

Finally, we use the previously mentioned L-vanishing action of P and ι (as well as the fact that $P\iota = I_{\mathcal{E}}$) when expanding the bracketed expression to obtain

$$B(z)B(\lambda)^* = \langle z, \lambda \rangle \, k_\lambda(z) - k_\lambda(z) + 2I_{\mathcal{E}} - P\iota = \langle z, \lambda \rangle \, k_\lambda(z) - k_\lambda(z) + I_{\mathcal{E}},$$

as desired.

Now to address the unitary equivalence, we first establish that V is unitary by showing that it is injective, has dense range, and is isometric.

To show that V is injective, we require the following identity: for any $n \geq 1$,

$$I - \sum_{|w|=n} T_w L_w = \sum_{k=0}^{n-1} \sum_{|v|=k} T_v P L_v$$

which can be established by substituting $I - TL = I - \sum_{i=1}^{d} T_i L_i$ in for P on the right hand side and writing $T_v T_i = T_i T_v$ by assumption (i) (i.e. that $T_w = T^\alpha$ when $\alpha(w) = \alpha$).

We next identify the kernel of V by computing

$$Vx(z) = P(I - \mathbf{z}L)^{-1}x = P \sum_{n=0}^{\infty} \left(\sum_{i=1}^{d} z_i L_i \right)^n x = \sum_{n=0}^{\infty} P \sum_{|w|=n} z_w L_w x$$

$$= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} P \sum_{\alpha(w)=\alpha} L_w x z^\alpha = \sum_{\alpha \in \mathbb{N}_0^d} \left(P \sum_{\alpha(w)=\alpha} L_w x \right) z^\alpha$$

which leads to the conclusion Ker $V = \bigcap_{\alpha \in \mathbb{N}_0^d} [\text{Ker} \left(P \sum_{\alpha(w)=\alpha} L_w \right)]$, since a power series is identically 0 if and only if all of its coefficients are 0.

Therefore, if $x \in \text{Ker } V$, we use the same previous identity to get $x =$

$$\sum_{k=0}^{n-1} \sum_{|v|=k} T_v P L_v x + \sum_{|w|=n} T_w L_w x = \sum_{k=0}^{n-1} \sum_{|\alpha|=k} T^\alpha P \sum_{\alpha(w)=\alpha} L_w x + \sum_{|w|=n} T_w L_w x = \sum_{|w|=n} T_w L_w x.$$

Since this holds for every n, (iv) reveals that $x = 0$ and thus V is injective.

Next, given an arbitrary linear combination of kernel functions $\sum_{j=1}^{N} k_{\lambda^j}(z) x_j$ for some $N, \{\lambda^j\} \subset \mathbb{B}_d$, and $\{x_j\} \subset \text{Ker } T^*$, we can define

$$x = \sum_{j=1}^{N} \left(I - \sum_{i=1}^{d} \overline{\lambda_i^j} L_i^* \right)^{-1} \iota x_j$$

to obtain $Vx(z) = \sum_{j=1}^{N} k_{\lambda^j}(z) x_j$. Since functions of this form are dense in $H(B)$, V has dense range.

Lastly, to see that V is isometric, we note from V having kernel equal to $\{0\}$ that

$$\mathcal{H} = (\text{Ker } V)^\perp = \left(\bigcap_{\lambda \in \mathbb{B}_d} \text{Ker } P(I - \boldsymbol{\lambda} L)^{-1} \right)^\perp = \bigvee_{\lambda \in \mathbb{B}_d} \text{Ran } (I - \overline{\boldsymbol{\lambda}} L^*)^{-1} \iota,$$

where we note that $\text{Ran } (I - \overline{\boldsymbol{\lambda}} L^*)^{-1} \iota$ is closed for any $\lambda \in \mathbb{B}_d$.

Therefore, elements of the form $x = \sum_{j=1}^{N} \left(I - \sum_{i=1}^{d} \overline{\lambda_i^j} L_i^* \right)^{-1} \iota x_j$ for some $N, \{\lambda^j\} \subset \mathbb{B}_d$, and $\{x_j\} \subset \text{Ker } T^*$ form a set that is dense in $\mathcal{H}$. We previously noted that for such x, $Vx(z) = \sum_{j=1}^{N} k_{\lambda^j}(z) x_j$, and we compute

$$\|Vx\|^2_{\mathcal{H}(\mathcal{B})} = \sum_{j,\ell=1}^{N} \langle k_{\lambda^j}(z) x_j, k_{\lambda^\ell}(z) x_\ell \rangle_{\mathcal{H}(\mathcal{B})} = \sum_{j,\ell=1}^{N} \left\langle k_{\lambda^j}(\lambda^\ell) x_j, x_\ell \right\rangle_{\mathcal{H}} =$$

$$\sum_{j,\ell=1}^{N} \left\langle P \left(I - \sum_{i=1}^{d} \lambda_i^\ell L_i \right)^{-1} \left(I - \sum_{i=1}^{d} \overline{\lambda_i^j} L_i^* \right)^{-1} \iota x_j, x_\ell \right\rangle_{\mathcal{H}} =$$

$$\sum_{j,\ell=1}^{N}\left\langle \left(I-\sum_{i=1}^{d}\overline{\lambda_i^j}L_i^*\right)^{-1}\iota x_j, \left(I-\sum_{i=1}^{d}\overline{\lambda_i^\ell}L_i^*\right)^{-1}\iota x_\ell\right\rangle_{\mathcal{H}} = \|x\|_{\mathcal{H}}^2$$

By the aforementioned density, this establishes that V is isometric and ultimately unitary.

All that remains is to confirm the intertwining identity $M_z V^{(d)} = VT$, which is where we invoke Lemma 4.5.2. However, in order to use Lemma 4.5.2a′), we need to know that our tuple T is indeed regular. Property (ii) plus Lemma 4.5.2c) will give us this, as we will now show, using the following fact from linear algebra: if $A \subseteq B$ and $\dim(A) = \dim(B) < \infty$, then $A = B$.

Considering Lemma 4.5.2c), all we need to show for regularity is that for a fixed z, that Ran $(T - \mathbf{z})$ is closed and that it equals Ran $P(z)$. Recalling that Ran $P(z)$ is contained in Ran $(T - \mathbf{z})$, we will apply the prior fact with $A :=$ Ker $(T^* - \overline{\mathbf{z}})$, which has dimension m by (ii), and $B :=$ Ran $[P(z)]^\perp$, which has dimension m by the decomposition in Lemma 4.5.2c) and the fact we proved earlier that the notion of codimension for algebraic direct sums works just as it does for (orthogonal) direct sums, noting that Ker T^* has dimension m by (ii) as well. This tells us that $A^\perp = B^\perp$, and we note the latter equals Ran $P(z)$ by our earlier comment that idempotents have closed range. Then, since Ran $P(z) \subseteq$ Ran $(T - \mathbf{z}) \subseteq \overline{\text{Ran } (T - \mathbf{z})} =$ Ran $P(z)$, we clearly have that $T - \mathbf{z}$ has closed range.

Let us finally confirm the intertwining relations. Combining lemmas 4.5.2b) and 4.5.2a′), let us state what we need only in terms of V: for a fixed $z \in \mathbb{B}_d$ and $y \in \mathcal{H}$, the lemma tells us that if $\hat{y} \in$ Ker T^* and $y - \hat{y} \in$ Ran $(T - \mathbf{z})$, it must be that $\hat{y} = Vy(z)$.

Fix $x \in \mathcal{H}, z \in \mathbb{B}_d$, and $1 \le i \le d$. Since $Vx(z) \in$ Ker T^*, we can clearly scale it by z_i and have it remain in Ker T^*. Also, note that $x - Vx(z) \in$ Ran $(T - \mathbf{z})$ means, of course, that there are $\{x_j\}_{j=1}^d$ such that $x - Vx(z) = \sum_{j=1}^{d}(T_j - z_j)x_j$; noting that $T_i(T_j - z_j) = (T_j - z_j)T_i$ for any j, it should be clear that we also have that $T_i[x - Vx(z)] \in$ Ran $(T - \mathbf{z})$. Using this, we simply add and subtract $T_iVx(z)$ in the following to get:

$$T_ix - z_iVx(z) = T_i[x - Vx(z)] + (T_i - z_i)Vx(z) \in \text{Ran } (T - \mathbf{z}).$$

Applying the restatement of the lemma with $y = T_ix$ and $\hat{y} = z_iVx(z)$, we arrive at $z_iVx(z) = V[T_ix](z)$. Unfixing z and x, we have finally established the desired relation:

$M_{z_i}V = VT_i$. With this, the proof of the theorem is now complete. $\diamond$

As a following remark, observe that the proof reveals that the space $\mathcal{D}$ in the theorem has dimension equal to that of Ran $(I_{\mathcal{H}^d} - LL^*)^{1/2}$. While this may not be finite, it decides how many columns a matrix representation of $B(z)$ would have, where the number of rows is $\dim(\mathcal{E}) = m < \infty$.

Let us now produce an example motivating the need for all four conditions in Theorem 4.5.1 instead of a strict subset of them. Suppose S in $\mathcal{B}(\mathcal{H})$ satisfies the conditions of Theorem 4.5.1 (in the case that $d = 1$), and consider wanting to check if Theorem 4.5.1 (in the case that $d = 2$) applies to $T := (S, 0)$.

(i) is trvially satisifed since $S0 = 0 = 0S$ and (iv) comes quickly as well since

$$\bigcap_{n\geq 0} \sum_{\substack{\alpha\in\mathbb{N}_0^2 \\ |\alpha|=n}} T^\alpha\mathcal{H} = \bigcap_{n\geq 0} S^n\mathcal{H} = \{0\},$$

if S is analytic.

For (iii), we first note that $(\text{Ker } T)^\perp = (\text{Ker } S \oplus \mathcal{H})^\perp = (\text{Ker } S)^\perp \oplus \{0\}$. Therefore, if $x = [x_1,\ 0]^T$ is in $(\text{Ker } T)^\perp$, then $Tx = Sx_1$ for x_1 in $(\text{Ker } S)^\perp$. With that, it is clear that S norm-expanding on the orthogonal complement of its kernel implies that T has the same property.

However, it turns out that (ii) is not satisfied. For $\lambda = (\lambda_1, \lambda_2)$, $T - \boldsymbol{\lambda} = (S - \lambda_1, -\lambda_2)$ and subsequently $[\text{Ran } (T - \boldsymbol{\lambda})]^\perp = [\text{Ran } (S - \lambda_1)]^\perp \oplus [\text{Ran } (-\lambda_2 I_{\mathcal{H}})]^\perp$. Then, since $[\text{Ran } (-\lambda_2 I_{\mathcal{H}})]^\perp = \left\{ \begin{array}{ll} \{0\} & \lambda_2 \neq 0 \\ \mathcal{H} & \lambda_2 = 0 \end{array} \right\}$, there is no way $[\text{Ran } (T - \boldsymbol{\lambda})]^\perp$ can satisfy having constant dimension across all possible $\lambda \in \mathbb{B}_2$.

Of course, for this example, T having 0 in its second component reveals T could never be equivalent to a de Branges-Rovnyak shift, showing that all four conditions are required in Theorem 4.5.1. In fact, condition (ii) with $\lambda = 0$ is enough for the theorem in the single-variable case, but as soon as $d \geq 2$ we need to add the Cowen-Douglas condition: that the dimension in question is constant across all $\lambda \in \mathbb{B}_d$.

Moving on to potential future directions, it would be ideal to examine some examples of 2-tuples that satisfy all four conditions of Theorem 4.5.1 and can thus be modeled as de Branges-Rovnyak multiplicative shifts. In [16], the authors mention that the shift on what are known as *weighted Dirichlet spaces* satisfies the desired properties when $d = 1$; however, one can show that these types of spaces are also de Branges-Rovnyak spaces by a direct computation, so a more abstract T for which to apply the theorem would reveal considerable more information and subsequently motivate the use of the theorem. However, none of these examples seem readily available since the analyticity condition must be satisfied.

Therefore, one might ask for a weaker theorem revealing exactly which operators T satisfy only conditions (i), (ii), and (iii) of Theorem 4.5.1. Without condition (iv), the unitary map V in the proof of Theorem 4.5.1 no longer has zero kernel; yet, the work of Eschmeier and Langendorfer (see [10], Theorem 10) has shown that under weaker conditions, we can still obtain a unitary equivalence from such an operator tuple T to a direct sum of a multiplicative shift and a type of operator tuple known as a *spherical co-isometry*. That being said, there is no assumption of expansiveness in their aforementioned Theorem 10, leaving the de Branges-Rovnyak setting of this type of result stilll not formalized. Before moving on, we remark that Eschmeier and Langendorfer's result is entirely related to the classical Wold decomposition in the case of having only one operator.

Next, one wonders if the $B(0) = 0$ assumption can be removed (not just for Theorem 4.5.1 but for several of the results in this chapter). Recall that Proposition 2.2.1 stated that in the case where $B = b$ was a row function, we were always allowed to assume $b(0) = 0$ for if not, $H(b)$ was unitary equivalent to $H(\hat{b})$ with $\hat{b}(0) = 0$. Yet, that argument relied entirely on ball automorphisms which would not be (immediately) applicable for an arbitrary Schur function B. Therefore, a much more general analogue of Proposition 2.2.1 remains open for now.

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Vita

Jesse Sautel was born in Louisville, Kentucky in June 1994. He developed an interest in taking math courses toward the end of high school and decided to pursue a Bachelor's degree in Mathematics at Murray State University in the western part of his home state. After graduating there in 2016, he decided to go for a Master's degree in math at the University of Tennessee but made a last minute change to go for a Doctorate instead. Once he graduates from UT, he plans to move back to Lousiville after ten long years away from home.