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N-Person Cooperative Game Theory Solutions, Coalitions, and Applications

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To the Graduate Council:

I am submitting herewith a thesis written by Ragan Nicole Brackin entitled "N-Person Cooperative Game Theory Solutions, Coalitions, and Applications." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Yueh-er Kuo, Major Professor

We have read this thesis and recommend its acceptance:

Suzanne Lenhart, G. Sammuel Jordan

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Dean of Graduate Studies

(Original signatures are on file in the Graduate Student Services Office.)

N-Person Cooperative Game Theory Solutions, Coalitions, and Applications

A Thesis
Presented for the
Masters of Science Degree
The University of Tennessee, Knoxville

Ragan Nicole Brackin

May 2002

Dedication

This thesis is dedicated to my amazing parents

Dennis Ray Brackin and JoAnn Brackin

for all the love and support they have given me

and to my beloved siblings

Christopher Adam Brackin and Lori Beth Brackin

whose words of encouragement will remain

cherished treasures.

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I must give thanks to the Lord Jesus Christ through whom all things are possible. He has blessed me with parents and siblings who have supported me with prayers, love and encouragement.

Abstract

This study explores the topic of N-person cooperative game theory. The following paper begins with an introduction to the basic definitions and theorems of game theory. These definitions and theorems are then used to introduce various solution methods and methods of coalition formation. These results are then applied to the airport game, to the supplier-firm-buyer game, and to evolutionary games.

Contents

Chapter 1

Definitions and Notation

Introduction

Chapter 1 will provide basic definitions and notation necessary to discuss the two major issues at the center of n-person cooperative game theory: coalition formation and distribution of wealth gained through cooperation. The reader will be introduced to the characteristic equation and the properties it may possess. We will define a $(0, 1)$-normalized game and prove that every essential cooperative game is strategically equivalent to one and only one game in this form. The payoff vector is also discussed in this chapter as well as the properties it may hold. The main references of chapter 1 are Colman [2] and [1a], Driessen [3], Forgó-Szép-Szidarovszky [4] , Jianhua [7], Luce-Raiffa [10], Owen [12], Rapoport [13], Thomas [18], and Vorobév [20].

An n-person cooperative game involves n players, where $n > 2$, and agreements regarding the correlation of strategies between two or more players are permitted.

The most concise representation for cooperative games is the characteristic function form.

Definition 1.1 Let $N = \{1, ..., n\}$ represent the n players. If $v(S)$ is a real-valued function defined on the set of all subsets S of N, satisfying

$$v(\emptyset) = 0, \qquad (1.1)$$

$$v(N) \geq \sum_{i=1}^{n} v(\{i\}), \qquad (1.2)$$

we say that $\Gamma \equiv [N, v]$ is an n**-person cooperative game**, and $v(S)$ is said to be the **characteristic function** of the game Γ.

Definition 1.2 Any non-empty subset S of the set N ($S \subset N$) is a possible **coalition** .

Suppose $S \subset N$ is a coalition, then $v(S)$ represents the maximum utility (for simplicity, money) S can get without correlating strategies with the other $N \setminus S$ players.

Definition 1.3 A coalition which contains all n players is called the **grand coalition**, and v(N) represents the maximum expected payoff of this coalition.

There are several properties which can be used to classify an n-person cooperative game. For example, games can be divided into categories such as constant-sum or non-constant-sum, and essential or inessential.

Definition 1.4 A game is said to be **constant-sum** if

$$v(S) + v(N \setminus S) = v(N)$$

for every $S \subset N$.

Definition 1.5 If the sum of expected payoffs, given by $v(S)$, in a game Γ is different depending on which players form a coalition, then game is called a **non-constant-sum game**.

Suppose S and T are disjoint coalitions, then clearly if they form a coalition, they should receive at least as large a payoff as they would by remaining separate.

Theorem 1.1 *Jianhua [7]* Let $\Gamma \equiv [N, v]$ be an n-person cooperative game. Then

$$v(S \cup T) \geq v(S) + v(T) \qquad (1.3)$$

for all $S, T \subseteq N$, $S \cap T = \emptyset$.

Proof: The coalition S can guarantee obtaining an amount $v(S)$ and no more. Similarly, the coalition T can guarantee obtaining the amount $v(T)$ and no more. Hence, the coalition $S \cup T$ can obtain the amount $v(S) + v(T)$ even if S and T fail to cooperate with each other. Since the maximum the coalition $S \cup T$ can obtain under any condition is $v(S \cup T)$, this implies

$$v(S \cup T) \geq v(S) + v(T).$$

Definition 1.6 Inequality (1.3) is called the **superadditive** property of the characteristic function v. If equality holds in (1.3), that is for all $S, T \subseteq N, S \cap T = \emptyset$, we have

$$v(S \cup T) = v(S) + v(T),$$

we say v is **additive** .

The superadditive property of the characteristic function implies that the "union" of all players (i.e. the grand coalition) (and unions of smaller coalitions into larger ones) is profitable because the corresponding payoff is increased. If the additive property holds in the characteristic function, then the players gain nothing by forming coalitions in the given game.

Definition 1.7 An n-person cooperative game with additive characteristic function is called an **inessential** game. All other cooperative games are **essential**.

Inessential games are trivial, because players have nothing to gain by forming a coalition. This can be seen in Theorem 1.2.

Theorem 1.2 *Jianhua [7]* In order that a characteristic function be additive, it is necessary and sufficient that the equality

$$\sum_{i=1}^{n} v(\{i\}) = v(N) \qquad (1.4)$$

be satisfied.

Proof: Necessity of equality (1.4) is directly implied by the additivity assumption. For sufficiency of the condition, assume that (1.4) holds. Let $S, T \subseteq N, S \cap T = \emptyset$. Using the superadditivity of v successively, we have

$$v(N) = \sum_{i=1}^{n} v(\{i\})$$

$$= \sum_{i \in S} v(\{i\}) + \sum_{i \in T} v(\{i\}) + \sum_{i \in N \setminus S \cup T} v(\{i\})$$

$$\leq v(S) + v(T) + v(N \setminus S \cup T)$$

$$\leq v(S \cup T) + v(N \setminus S \cup T)$$

$$\leq v(N)$$

Hence $v(S) + v(T) = v(S \cup T)$.

Since there are many types of cooperative games, it is beneficial to classify them in a manner such that games with similar basic properties belong to the same class. To do this, we must determine what makes two characteristic functions strategically equivalent.

Definition 1.8 Two characteristic functions, v and v' associated with N, are **strategically equivalent** if a positive number k and arbitrary real numbers c_i $(i \in N)$ exist such that for any coalition $S \subset N$, the equality

$$v'(S) = kv(S) + \sum_{i \in S} c_i$$

is satisfied.

It is this property that enables us to normalize a cooperative game.

Definition 1.9 A game $\Gamma = \{N, v\}$ is said to be $(0, 1) -$ **normalized** if

$$v(\{i\}) = 0 \qquad \text{for all} \qquad \text{i} \in \text{N},$$

$$v(N) = 1.$$

It is clear that all $(0, 1)$-normalized games are essential.

Theorem 1.3 *Forgó-Szép-Szidarovszky [4]* Among games strategically equivalent to an essential game $\Gamma = [N, v]$, there is exactly one $(0,1)$-normalized game.

Proof: Taking $\Gamma' \equiv [N, v']$ with $v'(S) = cv(S) - c\sum_{i \in S} v(\{i\})$ and $c^{-1} = v(N) - \sum_{i=1}^{n} v(\{i\})$ shows that there is a (0,1)-normalized game strategically equivalent to $\Gamma \equiv [N, v]$. To show there is only one such game, we need to have a solution to the following system of equations

$$kv(\{i\}) + c_i = 0, \qquad \text{for all} \qquad \text{i} \in \text{N} \qquad (1.5)$$

$$kv(N) + \sum_{i=1}^{n} c_i = 1.$$

for unknowns $k, c_1, ..., c_n$.

Since Γ is essential, $v(N) - \sum_{i=1}^{n} v(\{i\}) > 0$. By elementary algebra, we can find the unique solution of (1.5):

$$k = \frac{1}{v(N) - \sum_{i=1}^{n} v(\{i\})} > 0$$

$$c_i = -kv(\{i\}),$$

which is the game with the desired property.

Every player of a cooperative game has a right to receive his share of the total payoff $v(N)$ available. Let us assume that the distribution of utilities available to the set of players in N is such that each player $i \in N$ receives the amount x_i. The assignment of utilities can be represented by the payoff vector $x = (x_1, ..., x_n)$. These assignments cannot be arbitrary; they must be governed by certain restrictions.

Definition 1.10 A payoff vector x is called an **imputation** if it satisfies the following conditions:

$$x_i \geq v(\{i\}), \qquad \text{for all} \qquad \text{i} = 1, ..., \text{n}, \qquad (1.6)$$

$$\sum_{i=1}^{n} x_i = v(N). \qquad (1.7).$$

If relation (1.6) is not true, then player i will certainly refuse to accept the distribution since he or she is guaranteed the amount $v(\{i\})$ without forming any coalitions. Thus, relation (1.6) is known as the condition of **individual rationality** .

Relation (1.7) is the condition of **group rationality,** also known as the condition of **Pareto optimality** . This condition is necessary, for if

$$v(N) > \sum_{i=1}^{n} x_i,$$

then the players can form the grand coalition N to obtain the total payoff $v(N)$, in which case each player i could receive an additional amount besides

x_i. This distribution scheme would never be accepted by the players. On the other hand,

$$v(N) < \sum_{i=1}^{n} x_i$$

is impossible, since $v(N)$ by definition of the grand coalition is the greatest amount possible and the total distribution cannot exceed the total income.

Theorem 1.4 *Vorobév [20]* An inessential game has only one imputation. This imputation is

$$x = (v(\{1\}), ..., v(\{n\})).$$

Proof: Let $x = (x_1, ..., x_n)$ be an imputation of an inessential game $\Gamma = [N, v]$. Assume that

$$x_i > v(\{i\})$$

for some i. Then by (1.6)

$$\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} v(\{i\}).$$

Now the left-hand side of this inequality equals $v(N)$ by (1.7). By the additivity of the characteristic function, the right-hand side of the inequality also equals $v(N)$. This implies that $v(N) > v(N)$, which is impossible. Thus, for every i we have

$$x_i = v(\{i\}).$$

This proves the game has a unique imputation

$$x = (v(\{1\}), ..., v(\{n\})).$$

As will be seen in the following chapter, imputation domination plays a key role in several solution methods.

Definition 1.11 Given v, we say imputation x **dominates** imputation y through coalition S (notation $x >_S y$) if

$$x_i > y_i, \qquad i \in S \qquad (1.8)$$

$$\sum_{i \in S} x_i \leq v(S). \qquad (1.9)$$

Condition (1.8) states that all the members of coalition S prefer imputation x to imputation y, since each player i, a member of S, receives a larger payoff via imputation x. Relation (1.9) says that coalition S was guaranteed to obtain what imputation x give them.

Chapter 2

Solution Methods

Introduction

This chapter supplies the reader with information about several solution concepts used in cooperative game theory. By a solution, we mean how payoffs will be disbursed among players. Some of the earliest defined concepts are that of the core, the strong ϵ-core, the least core, and the von Neumann and Morgenstern solution. More recent solution theories include the Shapley value and the nucleolus. References in this chapter are taken from Driessen [3], Forgó-Szép-Szidarovszk [4], Jianhua [7], Owen [12], Thomas [18], and Vorobév [20].

2.1 The Core

The solution concept known as the core of an n-person cooperative game was first defined by Gillies [6]. In Chapter 1, we discussed both group and individual rationality, thus defining an imputation. Placing a further collective rationality stipulation on all possible coalitions of players produces the solution concept known as the core.

Definition 2.1 The set of all undominated imputations for a game Γ is called the **core** . The core of a game Γ is denoted by $C(\Gamma)$.

Theorem 2.1 *Thomas [18]* In order that imputation x belong to the core of a cooperative game with the characteristic function v, it is necessary and sufficient that the inequality

$$v(S) \leq \sum_{i \in S} x_i \qquad \text{for all S} \subset \text{N} \qquad (2.1)$$

be satisfied.

Proof: Necessity. It is sufficient to consider games in (0,1)-normalized form. Let

$$v(S) > \sum_{i \in S} x_i > 0$$

for the imputation x and for some coalition S. Observe that the coalition S must consist of more than one player, otherwise that last inequality violates the individual rationality of imputation x. For a similar reason, S should be different from N. We now have

$$\sum_{i \notin S} x_i = v(N) - \sum_{i \in S} x_i \geq v(S) - \sum_{i \in S} x_i > 0.$$

Now choose ϵ such that

$$0 < \epsilon < \frac{1}{|S|} \left(v(S) - \sum_{i \in S} x_i \right),$$

and construct the vector $y = (y_1, ... y_n)$ by setting

$$y_i = \begin{cases} x_i + \epsilon, & \text{if} \quad \text{i} \in \text{S}, \\ \frac{1}{|N \setminus S|} (\sum_{i \notin S} x_i - |S|\epsilon), & \text{if} \quad \text{i} \notin \text{S}. \end{cases}$$

Direct verification shows that y is an imputation and moreover $y >_s x$. Therefore, x does not belong to the core and the necessity of (2.1) is verified. Sufficiency. Assuming that imputation x is dominated by imputation y, we have for some coalition S

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq v(S),$$

which violates (2.1) and proves the sufficiency of the condition.

The previous theorem implies that for an imputation to belong to the core of a given cooperative game, the components of the imputation must satisfy a finite system of linear inequalities. The most likely case in which a core exists is that of an inessential game. Finding a solution using the concept of the core is not so easily done for essential games. In fact, this is impossible for the constant-sum case as proved by the following theorem.

Theorem 2.2 *Owen [12]* Let $\Gamma \equiv [N, v]$ be an essential constant-sum n-person cooperative game. The core of the game is the null set.

Proof: Assume the core is not empty, and let $x \in C(\Gamma)$. Then by definition, for every i we have

$$v(\{i\}) \leq x_i,$$

$$v(N \setminus \{i\}) \leq \sum_{j \in N \setminus \{i\}} x_j.$$

Now sum the two inequalities. Since the game is constant sum, the sum of the two left-hand terms is $v(N)$. On the other hand by the definition of an imputation, the two right-hand terms also add to $v(N)$. Hence equality holds in the first inequality for all i, i.e.,

$$v(\{i\}) = x_i, \qquad i = 1, ..., n. \qquad (2.2)$$

It follows from (2.2) that the game is inessential. Therefore, $C(\Gamma) = \emptyset$.

In Examples [2.1] and [2.2] of Chapter 2 the graphs are created using barycentric coordinates. In general, points $X = (x_1,\ x_2,\ x_3)$ satisfying

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad (1)$$

$$x_1 + x_2 + x_3 = 1, \qquad (2)$$

may be represented graphically using barycentric coordinates. Let 123 be an equilateral triangle with unit perpendicular bisectors and with its vertices labeled 1, 2, and 3. For every point X in this closed triangle, let $x_1,\ x_2,\ x_3$ be the distance from X to the sides of the triangle opposite the vertices 1, 2, 3, respectively. Then $x_1,\ x_2,\ x_3$ satisfy the conditions (1) and (2); see Figure (2.1). They are called the barycentric coordinates of X.

The barycentric coordinates of the vertices 1, 2, and 3 are (1,0,0), (0,1,0) and (0,0,1), respectively. The equations of the three sides 23, 31, and 12 of the triangle are

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,$$

respectively.

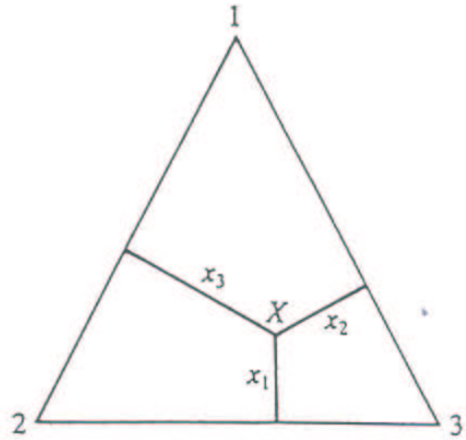

Figure 2.1: *Graph of* $X = (x_1,\ x_2,\ x_3)$

Consider the values of the characteristic function v of a general 3-person cooperative game in (0,1)-normalization

$$v(i) = 0, \qquad i = 1,\ 2,\ 3,$$

$$v(N) = 1, \qquad v(1,2) = c_3, \qquad v(1,2) = c_2, \qquad v(2,3) = c_1,$$

where $c_1,\ c_2,\ c_3$, are constants in the closed interval [0,1]. Domination of imputations is impossible with respect to either a one-person coalition or the grand coalition N. Consider domination with respect to the coalition {1,2}:

$$y \succ_{\{1,2\}} x,$$

where x and y are imputations in X. By definition

$$v(1,2) = c_3 \geq y_1 + y_2,$$

$$y_1 > x_1, \qquad y_2 > x_2.$$

Since y is an imputation, we know $y_1 + y_2 + y_3 = 1$ using substitution, we get

$$y_3 \geq 1 - c_3.$$

This inequality means that point y should lie on or to the right of the line $y_3 = 1 - c_3$. Similarly, $y_2 \geq 1 - c_2$ means that point y should lie on or to the left of $y_2 = 1 - c_2$ and $y_1 \geq 1 - c_1$ means that y should lie on or above the line $y_1 = 1 - c_1$.

Example [2.1] Jianhua[7] Suppose the values of the characteristic function of a three-person cooperative game Γ are

$$v(\{i\}) = 0, \qquad i = 1, 2, 3,$$

$$v(\{1,2\}) = \frac{2}{3}, \qquad v(\{1,3\}) = \frac{7}{12}, \qquad v(\{2,3\}) = \frac{1}{2},$$

$$v(\{1,2,3\}) = 1.$$

By definition of the core, the imputation $x = (x_1, x_2, x_3) \in C(\Gamma)$ if and only if

$$v(\{1\}) = 0 \leq x_1, \qquad v(\{2\}) = 0 \leq x_2,$$

$$v(\{3\}) = 0 \leq x_3, \qquad v(\{1,2\}) = \frac{2}{3} \leq x_1 + x_2,$$

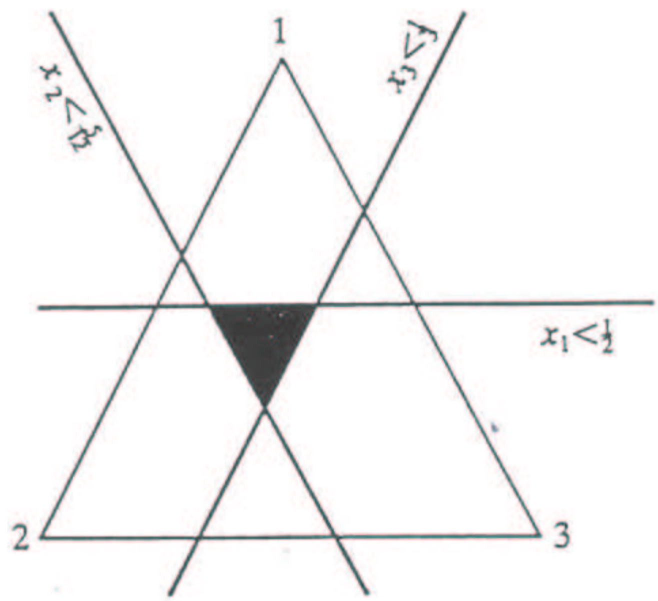

Figure 2.2: *Graphical representation of core in Ex [2.1]*

$$v(\{1,3\}) = \frac{7}{12} \leq x_1 + x_3, \qquad v(\{2,3\}) = \frac{1}{2} \leq x_2 + x_3.$$

By means of the condition of group rationality (1.7), the last three of the above inequalities reduce to

$$x_3 \leq \frac{1}{3}, \qquad x_2 \leq \frac{5}{12}, \qquad x_1 \leq \frac{1}{2}.$$

Therefore, the core of Γ is the shaded area in figure (2.2). It is a triangle including its sides.

2.2 The Strong ϵ-Core and The Least Core

Since the core is often empty, the strong ϵ-core was introduced.

Definition 2.2 An n-dimensional payoff vector which does not satisfy the condition of individual rationality

$$x_i \geq v(i), \qquad \text{for all i} = 1, ..., \text{n}$$

but does satisfy the condition of group rationality

$$x(N) = v(N)$$

is called a **pre-imputation**.

Note that $\sum_{i \in N} x_i = x(N)$ and X^* is the set of all pre-imputations.

Definition 2.3 Let $\Gamma \equiv [N, v]$ be an n-person cooperative game. For every pre-imputation $x \in X^*$ and every coalition $S \subseteq N$, define the excess of S on x by

$$e(S, x) = v(S) - x(S).$$

The excess represents the difference between $v(S)$ and the sum of the payoffs that the distribution scheme x offers to the members of S, if the coalition is formed. If $e(S, x)$ is positive, then there is a gain in the total payoff to the members of S with respect to x. Clearly, if $e(S, x)$ is negative, there is a loss to coalition S compared to the payoff offered to them in vector x.

Definition 2.4 Let $\Gamma \equiv [N, v]$ be an n-person cooperative game, and let ϵ be a real number. The set of pre-imputations

$$C_\epsilon(\Gamma) = \{\mathrm{x} : \mathrm{x} \in \mathrm{X}^*; \mathrm{e}(\mathrm{S}, \mathrm{x}) \leq \epsilon, \text{ for all } \mathrm{S} \subset \mathrm{N}, \mathrm{S} \neq \emptyset, \mathrm{N}\}$$

is called the **strong ϵ-core** of Γ.

By placing the following stipulation on the strong ϵ-core of an n-person cooperative game one can reduce the number of possible solutions and form the least core.

Definition 2.5 Let $\Gamma \equiv [N, v]$ be an n-person cooperative game. If ϵ_0 is the smallest ϵ for which $C_\epsilon(\Gamma) \neq \emptyset$, then $C_{\epsilon_0}(\Gamma)$ is called the **least core** of Γ and is denoted by$LC(\Gamma)$.

To clarify the relationship between the core, strong ϵ-cores and the least core, the following example is given.

Example [2.2] Jianhua [7] Let us begin by simplifying our notations as follows:

$$v(1) = v(\{1\}), \qquad x(1) = x_1 = x(\{1\}),$$

$$e(1) = e(\{1\}, x), \qquad v(13) = v(\{1, 3\}),$$

etc.

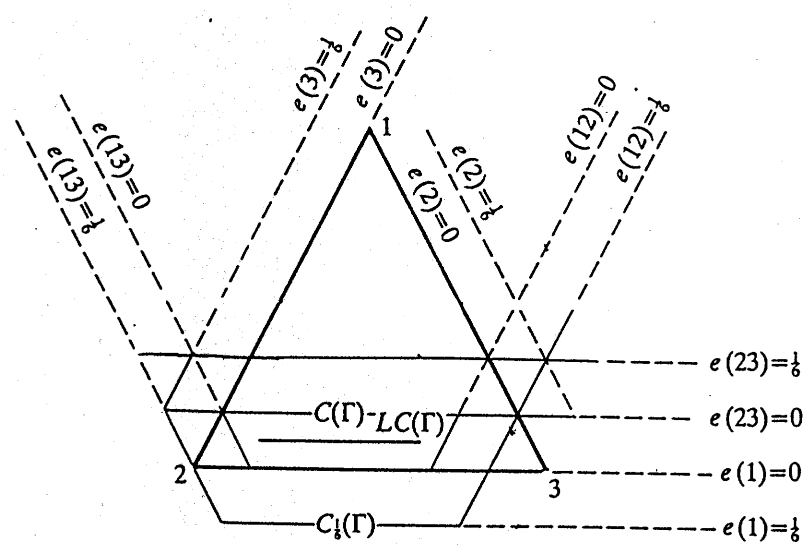

Figure 2.3: *Graphical representation of Ex [2.2]*

The characteristic function v of a three-person cooperative game Γ has the following values:

$$v(i) = 0, \qquad i = 1, 2, 3,$$

$$v(12) = \frac{1}{3}, \qquad v(13) = \frac{1}{6}, \qquad v(23) = \frac{5}{6},$$

$$v(123) = 1.$$

Figure (2.2) shows the core $C(\Gamma) = C_0(\Gamma)$ and the strong ϵ-cores for $\epsilon = \frac{1}{6}$. $C(\Gamma)$ is a quadrangle and $C_{\frac{1}{6}}$ is a pentagon. The least core is $C_{\frac{-1}{12}}(\Gamma)$. It is the line segment parallel to the side 23 of the imputation triangle.

2.3 The von Neumann and Morgenstern Solution

We now introduce another solution-finding method. The von Neumann and Morgenstern solution was developed in 1944 and is also known as the stable set solution. This solution set is partly based on a relationship between imputations called domination.

Definition 2.6 Let X be the set of all imputations of an n-person cooperative game Γ. If $V \subseteq X$ is a set of imputations satisfying the conditions:

1. for any $x \in V$ and $y \in V$,$x \not> y$

2. if $w \notin V$, then there exists $z \in V$ such that $z > w$,

then the set V is called the **von Neumann-Morgenstern solution** of Γ.

If property (1) holds then S is said to be "internally stable", and if property (2) holds S is said to be "externally stable."

Now that we have a method for determining whether an imputation is part of the solution set, how useful is this in finding a solution for a large class of games?

Theorem 2.3 *Forgó-Szép-Szidarovszk [4]* Every superadditive, essential, three person game has at least one von Nuemann and Morgenstern solution.

Proof: Due to the length of the proof it will not be included in this text. The proof can be found in an **Introduction to the Theory of Games**[4].

It has been proved that von Nuemann and Morgenstern solutions do exist in four-person games, however there has been no success in either proving or disproving that all five- to nine-person games have a von Neumann and Morgenstern solution. A 10-person game without a stable set has been constructed. There has been no success in finding a stable set solution in the case of constant-sum games.

Despite all this uncertainty there are a few special n-person games for which stable sets always exist.

Definition 2.7 A superadditive $(0,1)$-normalized game is said to be simple if for each $S \subset N$, either v(0)=0 or v(S)=1.

Theorem 2.4 *Forgó-Szép-Szidarovszky [4]* Let $\Gamma = [N, v]$ be a simple game and let S be a minimal winning coalition (such that $v(S) = 1$, but $v(T) = 0$ for all $T \subset S, T \neq S$). Let V_S be the set of all $x \in I$ (I is the set of imputations) such that $x_i = 0$ for all $i \notin S$. Then V_S is a stable set.

Proof: If $S = N$, then $V_S = I$ is a trivial stable set. So we assume $S \neq N$.

To prove internal stability suppose that $x, y \in V_S$. $x >_M y$ is only possible for a coalition $M \subset S$, if $x_i > 0$, $i \in M$ and $\sum_{i \in M} x_i \leq v(M)$. Since S is a minimal winning coalition, $S = M$. Since $\sum_{l \in S} x_l = \sum_{x \in S} y_l = 1$ if $x_j > y_j$ for some j, then there is a $k \neq j$ such that $x_k < y_k$, $k \in S$, making domination impossible. (Note that $|S| > 1$.)

For proving external stability take $y \notin V_S$, $y_k > 0$ and thus $\delta = 1 - \sum_{i \in S} y_i > 0$. Define $z = (z_1, ..., z_n)$ as

$$z_i = \begin{cases} y_i + \frac{\delta}{|N \backslash S|} & \text{if i} \in \text{S}, \\ 0 & \text{if i} \notin \text{S}. \end{cases}$$

It is elementary to see that $z \in V_S$, and $z >_S y$.

2.4 The Nucleolus

The solution known as the nucleolus has two very useful properties:

(a) every game has one and only one nucleolus, and

(b) if the core exists, the nucleolus is part of it.

The basic idea is to make the most unhappy coalition under the given imputation happier than the most unhappy coalition under any other coalition. The question is then how do we know which coalition is the most unhappy. A coalition's unhappiness is determined by looking at what the coalition expected to get given the characteristic function versus what payoff was actually received, i.e., $v(S) - x(S)$. The larger this number is the more unhappy the coalition is with imputation x.

Given an n-person game Γ and payoff $x = (x_1, ...x_n)$, define the 2^n-vector $\theta(x)$ as the vector whose components are the excesses of the 2^n subsets $S \subset N$, arranged in decreasing order, i.e.,

$$\theta_k(x) = e(S_k, x),$$

where $S_1, S_2, ..., S_{2^n}$ are the subsets of N. These values should be arranged by

$$e(S_k, x) \geq e(S_{k+1}, x).$$

Example [2.3] Owen [12]In a three-person game Γ, where $v(S) = 1$ if S has two or three players and $v(S) = 0$ otherwise, the payoff vector (0.2, 0.4, 0.3) gives us the excesses displayed in Table 2.1. Thus, $\theta(x) = (0.5, 0.4, 0.3, 0.1, 0, -0.2, -0.3, -0, 4)$. Suppose we have a payoff vector $y = (0.3, 0.5, 0.2)$; then $\theta(y) = (0.5, 0.3, 0.2, 0, 0, -0.2, -0.3, -0.5)$.

Table 2.1 Excesses found for Ex [2.3]

S	$v(S) - x(S) = e(S, x)$
$\emptyset$	0 - 0 = 0
{1}	0 - 0.2 = - 0.2
{2}	0 - 0.4 = - 0.4
{3}	0 - 0.3 = - 0.3
{1,2}	1 - 0.6 = 0.4
{1,3}	1 - 0.5 = 0.5
{2,3}	1 - 0.7 = 0.3
N	1 - 0.9=0.1

Definition 2.8 *The* **nucleolus** of an n-person cooperative game Γ is the set $N(\Gamma)$ of imputations $x \in X$ such that $\theta(x)$ of is minimal in the lexicographical ordering.

Suppose we are given two vectors

$$\alpha = (\alpha_1, ..., \alpha_q) \text{ and } \beta = (\beta_1, ..., \beta_{\mathrm{q}}).$$

Then α is lexicographically smaller than β if there exists some integer k, $1 \leq k \leq q$, such that

$$\alpha_\ell = \beta_\ell \qquad \text{for } 1 \leq \ell < \text{k},$$

$$\alpha_k < \beta_k.$$

The notation $\alpha <_L \beta$ shall be used for this relation, and $\alpha \leq_L \beta$ if either $\alpha <_L \beta$ or $\alpha = \beta$. In the previous example, we find that $\theta(y) <_L \theta(x)$, because $\theta_1(x) = \theta_1(y)$ but $\theta_2(y) < \theta_2(x)$.

Theorem 2.5 *Jianhua [7]* Let $\Gamma \equiv [N, v]$ be an n-person cooperative game. Then the nucleolus $N(\Gamma)$ of the game is non-empty.

Proof: This proof shall be omitted, however it can be found in **The Theory of Games** [7].

Theorem 2.6 *Forgó-Szép-Szidarovszk [4]* Let $N(\Gamma)$ be the nucleolus of an n-person cooperative game. Then $|N(\Gamma)| = 1$, i.e., $N(\Gamma)$ consists of one element.

Proof: Due to the length of this proof it shall be included in the text. The reader can find this proof in **Introduction to the Theory of Games** [4]. The nucleolus is called a one-point solution due to Theorem 2.6.

2.5 The Shapley Value

One of the most famous one-point solution concepts is the Shapley value. This value represents the payoff a given player will expect to get before play begins. Shapley [16] developed three axioms, which he felt $\phi_i(v)$, player i's expectation in a game with characteristic function v, should satisfy. Before stating these axioms, the following definitions will be introduced.

Definition 2.9 A **carrier** for a game v is a coalition T such that, for any $S, v(S) = v((S \cap T))$.

Definition 2.10 Let Γ be an n-person game, and let π be any permutation of the set N. Then by πv we mean the game u such that, for any $S = \{i_1, ..., i_S\}$,

$$u(\{\pi(i_1), \pi(i_2), ..., \pi(i_S)\}) = v(S).$$

By the **value** of the game Γ, we mean an n-vector,$\phi[v]$ satisfying,

1. If S is any carrier of Γ, then

$$\sum_S \phi_i[v] = v(S).$$

2. For any permutation π, and $i \in N$,

$$\phi_{\pi(i)}[\pi v] = \phi_i[v].$$

3. If u and v are any games,

$$\phi_i[u+v] = \phi_i[u] + \phi_i[v].$$

These are Shapley's axioms. They are sufficient to determine a value ϕ uniquely for all games.

Theorem 2.7 *Owen [12]* There is a unique function ϕ, defined on all games, satisfying Axioms 1-3 namely:

$$\phi_i(v) = \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S - \{i\}) \qquad (2.8)$$

where $s = |S|$ (the number of members of S).

Proof: This proof shall be ommited. It can be in several text including **Game Theory** [12]. This function ϕ_i is called the Shapely value.

Now we will use the Shapely value to determine the disbursement of payoffs for the three-person cooperative game used as an example in the least core section of this chapter.

Example [2.4] The game had the following values:

$$v(i) = 0 \qquad i = 1, 2, 3,$$

$$v(12) = \frac{1}{3}, \qquad v(13) = \frac{1}{6}, \qquad v(23) = \frac{5}{6},$$

$$v(123) = 1.$$

To find $\phi_1(v)$, we sum over the coalitions $S = \{1\}, \{12\}, \{13\}, \{123\}$ and substituting into (2.8) gives:

$$\begin{aligned}\phi_1(v) &= \frac{0!2!}{3!}(0-0) + \frac{1!1!}{3!}\left(\frac{1}{3}-0\right) + \frac{1!1!}{3!}\left(\frac{5}{6}-0\right) + \frac{2!0!}{3!}\left(1-\frac{1}{6}\right) \\ &= 0 + \frac{1}{18} + \frac{1}{36} + \frac{1}{18} \\ &= \frac{5}{36}.\end{aligned}$$

For $\phi_2(v)$ we sum over $S = \{2\}, \{12\}, \{23\} and \{123\}$ so :

$$\begin{aligned}\phi_2(v) &= \frac{0!2!}{3!}(0-0) + \frac{1!1!}{3!}\left(\frac{1}{3}-0\right) + \frac{1!1!}{3!}\left(\frac{5}{6}-0\right) + \frac{2!0!}{3!}\left(1-\frac{1}{6}\right) \\ &= 0 + \frac{1}{18} + \frac{5}{36} + \frac{5}{18} \\ &= \frac{17}{37}.\end{aligned}$$

For $\phi_3(v)$ we sum over $S = \{3\}, \{13\}, \{23\} and \{123\}$ so :

$$\phi_3(v) = \frac{0!2!}{3!}(0-0) + \frac{1!1!}{3!}\left(\frac{1}{6} - 0\right) + \frac{1!1!}{3!}\left(\frac{5}{6} - 0\right) + \frac{2!0!}{3!}\left(1 - \frac{1}{3}\right)$$

$$= 0 + \frac{1}{36} - \frac{5}{36} - \frac{2}{9}$$

$$= \frac{14}{36}.$$

Thus, the Shapley value of this game is

$$\phi(\Gamma) = \left(\frac{5}{36}, \frac{17}{36}, \frac{14}{36}\right).$$

Chapter 3

Coalition Formation

Introduction

Thus far we have addressed the question of how payoffs will be divided among players of an n-person cooperative game. In this chapter, we ask which particular coalitions are likely to form. There are three such theories that will be discussed. Komorita's [8] equal excess model is one of the few theories designed to deal with both the coalition formation and payoff division. The minimal winning coalition strategy, developed by Riker, [14] is based on the idea that if a coalition is large enough to win then, it should avoid adding new members, since payoffs must be shared amongst these new members. Gamson's [5] minimum resource theory assumes that players in the winning coalition should demand payoffs corresponding to their voting strength in the coalition. The main references used in Chapter 3 are Gamson [5], Komorita

[8], and Riker [14].

3.1 Equal Excess Model

There are two basic principles that make up the equal excess model. An individual's "bargaining strength" is one such principle. Bargaining strength in negotiations is based on the alternative coalitions the player can form.

The second underlying principle assumes that members of a potential coalition are most likely to agree on a division based on sharing equally the excess of what can be gained by the coalition, relative to the total outcomes if each chose his best alternative.

We shall assume that individuals are motivated to maximize expected payoffs, which is typically a goal in most theories of coalition formation. Unlike other theories, however, we shall assume social psychological motives also play a role in coalition formation. Furthermore, we shall assume that the value of each coalition represents transferable utilities. The following three-person game will be used to illustrate the model:

Example [3.1] Komorita [8] Here and elsewhere in this chapter we will denote the players by letters rather than integers. Let

$$v(A) = v(B) = v(C) = 0; \qquad v(ABC) = 0;$$

$$v(AB) = 6; \qquad v(AC) = 5; \qquad v(BC) = 4.$$

Players must determine their initial demands in negotiations. An individual's preferences and expectations during various stages are specified in Assumptions (1) and (2). We will illustrate the assumptions afterward with this example.

Assumption (1) Prenegotiations are driven by an individual's desire to form the coalition that maximizes **initial expectation**, given by:

$$E^0_{iS} = \frac{v(S)}{s} \qquad (3.1)$$

where E^0_{iS} denotes the initial expectation of individuals i in coalition S. Note that s represents the number of players in coalition S.

It is impossible for any player to obtain a payoff unless a coalition with exactly two members is formed. In the above example, initial expectations of the three possible coalitions and their payoffs defined by (3.1) are 3 each for A and B in the AB coalition; 2.5 each for A and C in the AC coalition; and 2 each for the B and C in the coalition BC. It is obvious that by Assumption (1) A and B will initiate negotiations. The dividing of rewards is negotiated based on a player's expectations in other possible coalitions. It goes without saying that the better the quality of a player's alternatives, in comparison to the alternatives of the other players, the greater his or her bargaining strength. Due to different expectations of players, one or more players may concede to reach an agreement. These concessions will likely be made in random stages of negotiations. Assumption (2) is designed to account for these concessions.

Assumption (2) Since expectations of players change when a coalition forms after a round of negotiations, the most likely coalition to form is the one in which all players' expectations are jointly maximized. The most probable distribution of payoffs is represented by the **equal excess norm**, as follows:

$$E_{iS}^{r} = max_{S \neq T} E_{iT}^{r-1} + \frac{(v(S) - \sum max E_{jT}^{r-1})}{s} \qquad (3.2)$$

where E_{iS}^{r} represents the expectation of individual i in coalition S on round r; max E_{iT}^{r-1} denotes individual i's maximum expectation in alternative coalitions on round $r-1$; and the summation is over the members of coalition S.

Definition 3.1 A sequential process in which each person makes an offer or counteroffer, and each player accepts or rejects offers, which are received, determines a **round** of the game.

Assumption (2) implies that if $v(S)$ exceeds the sum of players' expectations, the excess will be divided equally among the players of the coalition.

Returning to the example, if player A and B negotiate in the first round, player A would have a maximum expectation of 2.5 via coalition AC, while B's maximum expectation would only be 2.0 from the BC coalition. Applying (3.2) we have:

$$E^1_{A(AB)} = 2.5 + \frac{[6.0 - (2.5 + 2.0)]}{2} = 3.25$$
$$E^1_{B(AB)} = 2.0 + \frac{[6.0 - (2.5 + 2.0)]}{2} = 2.75$$

Given the model, we conclude that if the AB coalition forms on the first interchange of offers, the payoff would be divided as $3.25 - 2.75$ for players A and B, respectively.

Based on the iterations of (3.2), expectation predictions of this model can be made for later rounds. A few of these predictions are listed in Table 3.1. Analyzing the predictions, we see that player A's expectations increase as the number of rounds increases regardless of the coalition in which A resides. However, player C's expectations decrease in both the AC and BC coalition. As rounds progress, player B increases expectation if coalition BC is formed, but decreases expectation when coalition AB is formed. Also note that at the limit, the predicted shares of the players equals their maximum expectations in alternative coalitions. If players reach an agreement at the limiting value, the above fact implies that no player will be tempted to leave the coalition.

Given a player's desire to maximize expectation, we can deduce from Table 1 that if coalitions form in an early round of play, coalition AB will most likely form. However as the number of rounds increases, the likelihood of coalition BC forming increases. In theory, all three coalitions are equally likely to form if there are an indefinite number of rounds.

A certain degree of indeterminacy is found in this model since the number of rounds necessary for an agreement to be reached is unknown. This number depends partially on the competitiveness of the players. Assuming the players are highly competitive, it may take many rounds of negotiations to reach an agreement. In this case, the asymptotic expectations probably yield the best estimate of payoffs. Now suppose the players are not very competitive. In this case only a few rounds of play will be required for an agreement to be made and expectations found for earlier rounds are likely to yield the best estimate of payoffs.

Table 3.1 Predictions for Ex [3.1]

Coalition	**Expectations over Rounds**				
~	0	1	2	5	Asymptote
AB	3.0-3.0	3.25-2.75	3.37-2.62	3.49-2.51	3.50-2.50
AC	2.5-2.5	3.00-2.00	3.25-1.75	3.48-1.52	3.50-1.50
BC	2.0-2.0	2.25-1.75	2.38-1.62	2.48-1.52	2.50-1.50

3.2 Minimal Winning Coalition Strategy

The focus of this theory is how a players "influence of position" affects payoff distribution.

Definition 3.2 A **decision** is a selection among alternatives.

A decision made by the players will lead to a particular payoff. Often a player may be capable of influencing decisions more than deemed possible given his "resources". This is due to his ability to influence other players with his position.

Definition 3.3 A **resource** is a weight associated with each player in a way that some critical quantity of these weights is necessary for a decision to be made.

Definition 3.4 Any group or individual who carries out the same coalition strategy until a decision is made is called a **social unit**.

A coalition is a social unit from its conception until a decision is made.

Definition 3.5 Any coalition capable of controlling a decision through sufficient resources is a **winning coalition**.

Definition 3.6 The **decision point** is the minimum proportion of resources necessary to control the decision.

Definition 3.7 Losing coalitions which have the power to prevent the other coalitions from winning are called **blocking coalitions**.

According to Gamson[5], a **full-fledged coalition situation** is one in which the following conditions are present:

1. There is a decision to be made and there are more than two social units attempting to maximize their shares of the payoff.

2. No single alternative will maximize the payoff of all participants.

3. No participant has dictatorial powers.

4. No participant has veto power.

Conditions one and two deem that every player has some bearing on the game's outcome. Condition three says no one player initially has the resources to control the decision by himself or herself. In condition four we find that no member is included in every winning coalition. This game may be classified as essential if conditions one, two, and three hold. Note that essential is defined differently here than from Chapter 1. These conditions also prevent the occurrence of the trivial solution. Blocking coalitions are prevented by condition four.

The model requires information about several matters the to predict coalition formation. These are identified and discussed next.

1. "The initial distribution of resources." We must know which resources are of value for a particular decision. Then for some starting point an inventory must be taken of each player's resources.

2. "The payoff for each coalition." The characteristic function defined in chapter one is determined by computing payoffs to any subset of players and an assumption is made that the remaining players will form a single coalition. This assumption reduces every game to a two-player game. The minimal winning coalition strategy does not make such an assumption. We must know every alternative coalition, including partitions into more than two classes of players. This implies that a particular subset may receive one payoff when the complementary set is partitioned in one manner and an entirely different payoff if the complementary set is partitioned in a different way.

   If would seem that the function which is required to supply all such partitions would be more complicated than the characteristic function. Since the Gamson theory yields only one winning coalition and the payoff to all other players, which are not members of the winning coalition, is zero, we need only know the possible winning coalitions.

The probability of achieving future rewards is reflected in the payoff since it includes anticipation of future events.

Definition 3.8 The **expected value** or payoff for a coalition is found by multiplying the total payoff of that coalition by the probability of the coalition's achieving that payoff.

3. "Non-utilitarian strategy preferences." Each player must make a list, which ranks every other player in the order he or she would be inclined to join them in a coalition. Ties are allowed and ranks should be assigned with no regard to the other player's control or resources.

4. "The effective decision point." Often the rules of the game assign a specific amount of resources necessary to control the decision. This amount is known as the formal decision point. This amount, however, may be more than sufficient to gain control of the decision. If control of the decision is possible with fewer resources than specified in the

rules of the game, then that smaller amount of resources is an effective decision point.

Definition 3.9 A winning coalition that is no longer winning if one player leaves the coalition is called a **minimal winning coalition**.

We use the term minimal here to show that this is the smallest number of players whose total resources are large enough to control the decision.

Definition 3.10 The minimal winning coalition with total resources closest to the decision point is called the **cheapest winning coalition** .

Definition 3.11 For an empirical value K in a given coalition situation, a set of payoffs, of which the lowest is no more than K percent less than the highest, is defined as a **payoff class**.

The theory applies to full-fledged coalition situations in which we assume the following to be true:

Assumption (1) The same information about initial distribution of resources and the payoffs to any coalition are known by all the players. This assumption does not imply that the players have perfect information.

Assumption (2) Payoffs in the same payoff class are indistinguishable by players.

Assumption (3) A rank ordering of non-utilitarian coalition preferences for joining with other players is done by every player. By non-utilitarian we mean an ordering not based resources.

The general hypothesis of the theory states that any player will expect others to demand from a coalition, a share of the payoff proportional to the amount of resources that they contribute to a coalition.

It is possible for a player A to know a personal payoff value from any possible coalition formation. To calculate a personal payoff player A multiplies his proportion of the resources in the coalition by the total payoff of the coalition. These values can then be assigned to payoff classes which A will prefer the highest. The coalition whose members have the highest mean rank on player A's scale of non-utilitarian preferences will be A's preference within a payoff class. When the total payoff to the winning coalition is constant, player A will seek to maximize his payoff by maximizing his shares, i.e., he will favor the cheapest winning coalition.

Example [3.2] Gamson [5] Suppose player A has 30 percent of the resources, B has 19 percent, C has 30 percent, and D has 21 percent where the decision point is 51 percent. Player A should consider the minimal winning coalitions of AC and AD. Since both player A and player C own 30 percent of the resources, they would half the payoff. However if player A joined player D, he would expect about $\frac{3}{5}$ of the payoff. If the payoff for each coalition is different, then A's payoff could possibly be higher in coalition AC than in coalition AD. In accordance with the minimal winning coalition theory, if the payoffs are in the same payoff class, player A would choose to join the one in which he ranks higher on non-utilitarian strategy preference.

A coalition will form if and only if there are reciprocal strategy choices between two players. For example, in a three person game, if player A desires to form a coalition with player B, and player B prefers either coalition AB or BC, and player C wants to form a coalition with player A, only A and B have reciprocal strategy choices. Therefore, the theory predicts coalition AB will form.

3.3 The Minimal Resource Theory

Before a discussion of Riker's strategy for coalition building can begin, a brief dynamic model should be introduced. Players are often referred to as the decision-making body, N. This model operates under the rules of an n-person zero-sum game with "side payments" allowed.

Definition 3.12 A player's **weight** refers to his influence, power, or significance in relation to other players.

A coalition with weight m, where $m > \frac{1}{2}\sum_{i=1}^{n} w_i$ where w_i is the weight of a member, i, can act for or impose its will on all players.

Coalition building begins when a leader allies himself with other players on a particular issue for decision.

Definition 3.13 A member who manages the growth of a coalition is known as the **leader**.

Definition 3.14 To form a coalition, a leader must lure **followers**, which are members of the body who join the association the leader forms.

We will now make a distinction between a coalition and a proto-coalition.

Definition 3.15 The word **coalition** will describe the end product of the decision-making process.

Definition 3.16 The association (leading to a coalition) a follower joins is a **proto-coalition**.

More precisely a proto-coalition is any subset of N, when N is partitioned into three or more disjoint subsets such that no subset has the weight m.

Definition 3.17 The acts of joining or resigning from a proto-coalition is defined as a **move** .

Definition 3.18 If two or more members or proto-coalitions agree upon simultaneous actions, we refer to them as **simultaneous moves** .

When moves occur, the proto-coalition will change size. After a move the internal structure of the body changes and a player's strategy will also change.

Definition 3.19 A **stage** is defined as the interrelationship of proto-coalitions just before (or after) a move is made.

In stage one, there are n single-member proto-coalitions. The second stage will have one two-player proto-coalition and $(n-2)$ single player proto-coalitions. The final stage, r, is reached when there exists a winning coalition or two blocking coalitions. Players are not aware when they are in the $(r-1)^{th}$ stage.

Definition 3.20 A **side payment** is the gift of anything of value in order to lure followers into a proto-coalition.

Leaders make use of side-payments to attract followers into proto-coalitions. The leader may possess the side payment before his offer, or he may be making the offer in hopes of paying the follower from the payoff received by the winning coalition.

Since side payments are made with items of value, they obviously affect the coalition-building process. They will limit the number of players a leader asks to become followers. This fact in turn will cause opposition, since some players being left out will form another proto-coalition.

A player's quest to belong to the winning coalition in the r^{th} stage of the game will guide his moves in stage $(r-1)$, as well as in previous stages. Leaders of proto-coalitions must know how alternative outcomes in the r^{th} stage are affected by moves made in all previous stages to insure they will be members of the winning coalition.

Definition 3.21 The last member to join a minimal winning coalition is said to occupy the **pivotal** position.

In order to determine his options at a given stage (other than the initial), a leader needs to know his chance of winning given the immediate state of arrangements into proto-coalitions. At this point he is not concerned with his chance of pivoting. There are three general considerations that may affect a leader's chance of winning. One such consideration is how much dependence upon unique events occurring during the course of play a winning coalition may have. Traditions may also play a major role in a leader's chance of forming a winning coalition. It may depend, in part, on general considerations inherent in the game model, such as the size principle.

The size principle limits in some way the range of possible outcomes of a game and may place drastic restrictions on coalitions that are likely to form. Any proto-coalition in the j^{th} stage that can form a minimal winning coalition, while others cannot, has a strategic advantage. To assure situations arise in which such advantages exist the leader of a proto-coalition needs specifications of the situations. Riker [14]formalizes these situations as:

1. uniquely preferable winning coalitions
2. uniquely favorable proto-coalitions
3. uniquely essential proto-coalitions
4. unique coalitions
5. strategically weak proto-coalitions.

Discussion of the uniquely preferable winning coalition will be limited (1) to the $(r-1)^{th}$ stage and (2) to those situations in which there are no more than five proto-coalitions.

Example [3.3] Riker [14] Suppose there are five proto-coalitions, $A, B, C, D, andE$, which are disjoint subsets of N. Note that $A \cup B \cup C \cup D \cup E = N$. Then their weights must satisfy the inequality $m > w(A), w(B), w(C), w(D), w(E)$ since A, B, C, D, and E are proto-coalitions. Without loss of generality we may assume $w(A) \geq w(B) \geq w(C) \geq w(D) \geq w(E)$. If only three subsets are nonempty, then we label them A, B, and C and D and E do not occur. Similarly if the set is partitioned into four subsets, then only subset E does not occur. Notation for partitions of N into a different number of subsets is needed. Thus, A^n will be used for the weightiest proto-coalition (or one of the equally weightiest) when the players are partitioned into n proto-coalitions. This would imply that A^1 is equivalent to N. If N is partitioned into two subsets then A^2 and B^2 are the names of the winning and losing coalition (or of two blocking coalitions). Assume that A^2 is the winning coalition. This does not tell us specifically which coalition is formed since it may refer to any coalition for which $m \leq w(A^2) \leq w(N)$. To distinguish amongst the winning coalitions, arrange them in order of increasing weights from $w(A^2) = m$ to $w(A^2) = w(N)$. The minimal winning coalition, for which $w(A^2) = m$, should be given "rank" 1 and written as A^2_1. For $w(A^2) = m + a$, where a is the weight of player i such that $0 < w(i) \leq w(j)$, for all $i, j \in N$ assign

A^2 a rank of 2 and write A^2_2. If $A^2 < m$, i.e. a blocking coalition, assign it rank zero written as A^2_0. B^2 will have ranks which are the negative of A^2. The losing coalitions are written as $B^2_{-1}, B^2_{-2}, ...B^2_{-s}$ and for the blocking coalitions write B^2_0, Assume that $i < j < ... < s$ for $i, j, ..., s$ numbers of ranks.

Two conditions must be placed on the proto-coalitions to insure they are viewed as indivisible units. First, when a follower joins a proto-coalition, he is not allowed to leave. Secondly, a leader must distribute the side payment amount he offered the follower to join the proto-coalition. A lower amount may be given only if the follower agrees.

The symbol α_{X^k} will be used to define the payoff to a particular proto-coalition, X^k, and an imputation will be a set of numbers $(\{\alpha_{A^k}, \alpha_{B^k}, ..., \alpha_{E^k}\})$ such that $\sum_{X^k=A^k}^{E^k} \alpha_{X^k} = 0$. Also note that α_i designates the payoff to player, i. In a zero-sum game that imputation is defined as the set of numbers $(\{\alpha_1, \alpha_2, ..., \alpha_n\})$ such that $\sum_{i-1}^{n} \alpha_i = 0$.

It is time to supply the reader with formal definitions for Gamson's initial expectation and the five situations in which an advantage exists.

Definition 3.22 An **initial expectation**, $E(X^k)$, for a proto-coalition, X^k, is an imputation in the r^{th} stage anticipated by X^k in the $(r-1)^{th}$ stage.

Note that this is not the same definition found for initial expectation in Komorita's equal excess theory.

Definition 3.23 Given a partition in the $(r-1)^{th}$ stage such that $A_i^2, A_j^2, ...$ may exist in the r^{th} stage, then a **uniquely preferable winning coalition** P_i^2, is a coalition such that

1. $v(A_i^2) > v(A_j^2)$ and

2. A_i^2 may have an imputation $(\gamma_{X_k}, \gamma_{Y^k}, ...)$ such that, for X_k and $Y_k \in A_i^2$, $\gamma_{X_k} \geq \alpha_{X^k}$ and $\gamma_{Y_k} \geq \beta_{Y^k}$, where $E(X^k) = (\alpha_{X^k}, \alpha_{Y^k}, ...)$ and $E(Y^k) = (\beta_{X^k}, \beta_{Y^k}, ...)$.

Definition 3.24 A **uniquely favored proto-coalition** , is a proto-coalition, X^k, such that

1. for A_i^2, where $X^k \in A_i^2$, and for A_j^2, where $X^k \notin A_j^2$, then $v(A_i^2) > v(A_j^2)$; and

2. for $X^k, Y^k, ...$ satisfying condition 1, some A_k^2 is possible such that $X^k \in A_k^2$ and $Y^k \notin A_k^2$.

Definition 3.25 A proto-coalition, which is contained in all winning coalitions when no other proto-coalition holds that same property, is called an **uniquely essential proto-coalition** .

Definition 3.26 A **unique coalition** is a coalition that is the only possible winning coalition.

Definition 3.27 A **strategically weak proto-coalition** , X^k, is one that cannot become a part of the most valuable winning coalition due to the given partition in the $(r-1)^{th}$ stage, i.e., $X^k \notin A_i^2$.

Given the above definitions, it is possible to examine behavior in the $(r-1)^{th}$ stage of the model. Our discussion will be limited to one case. If the reader wishes to explore all cases, they can be found in Appendix II of Riker's **Theory of Political Coalitions**[14].

Example [3.4] Riker [14] Partition the set N into three proto-coalitions, A^3, B^3, C^3 in the $(r-1)^{th}$ stage. Assume that $w(A^3) > w(B^3) > w(C^3)$. If $m = \frac{n+1}{2}$ or if $m = \frac{n}{2}+1$, then the possible winning coalitions in the r^{th} stage are $(B^3 \cup C^3), (A^3 \cup C^3)$, and $(A^3 \cup B^3)$. The size principle implies that the values of these coalitions are related thus:

If

$$v((B^3 \cup C^3) = a = -v(A^3)$$

$$v((A^3 \cup C^3) = b = -v(B^3)$$

$$v((A^3 \cup B^3) = c = -v(C^3),$$

then $a > b > c$. We must determine whether a uniquely preferable winning coalition exists. In other words,

1. Is there a coalition which has a greater value than any other?

2. Are the initial expectations of every member of that coalition satisfied?

Since $a > b > c$, the answer to the first question is yes, clearly $v(B^3 \cup C^3)$ has a larger value than any other possible coalition. This leaves us with question two. To answer this question we need to calculate the initial expectation of B^3. If B^3 joins A^3 in a coalition and A^3 receives a payoff of zero from the value of $v((A^3 \cup B^3)$, B^3 would obtain the whole payoff, c. Suppose that B^3 and C^3 form a coalition; the imputation would then be: $\alpha_{A^3} = -a,\ \ \alpha_{Q^3} = c$, $\alpha_{R^3} = (a-c)$, which is written precisely as $(-a, c, a-c)$. Similarly C^3 has an initial expectation of b. The coalition containing B^3 and C^3 can expect $(-a, a-b, b)$. Knowledge of the relative sizes of a, b, c are needed to determine if a uniquely preferable winning coalition exists. The size of a, b, c would be determined by the shape of the curve of the characteristic function. Suppose that $c < a - b$; then $b < a - c$ and, C^3 would receive the expected initial payoff of b and the remaining amount, $a - b$ would be more than enough to pay B^3 its initial expected value. Therefore, if the players in the model are rational, then the uniquely preferable winning coalition ($B^3 \cup C^3$) will form.

Observations made during analysis of the relative position of proto-coalitions when N is partitioned into three, four, and five subsets produce some general strategies of coalition-building.

If coalitions are not equally weighted, it is possible that near the end of the process of coalition-building one or more proto-coalitions will have some unique advantage. The terms "uniquely favored," "uniquely preferable," "uniquely essential," and "unique" have been used to describe these situations. A general theory of strategy for end-play is as follows:

1. Any proto-coalition, which finds itself in an advantageous situation, should exploit the advantage.

2. If a proto-coalition fails to have some sort of advantage, then it should seek to eliminate or diminish the advantage of other proto-coalitions.

It seems that smaller proto-coalitions more often have uniquely advantageous positions than do the larger, weighter ones. Since one coalition or proto-coalition frequently has an advantage, this implies the model has a bias toward a decision and in turn has no kind of equilibrium.

Chapter 4

Applications

Introduction

Chapter 4 provides the reader with three applications of game theory. This chapter includes definitions and examples dealing with airport games, supplier-firm-buyer games, and evolutionary games. Information found in this chapter is based primarily on the research of Colman [1a], Driessen [3], Littlechild and Owen [9] and Stuart [17].

4.1 The Airport Game

One of the most popular applications of game theory is that which determines the landing fees at airports. It is our desire to develop a "fair" allocation of these fees. We will show that the simple rules proposed for calculating airport landing charges are precisely those of the Shapley value for an appropriately defined game.

At any airport there occur two types of expenses:

1. variable operating cost due to landing planes of different types and

2. a fixed capital cost for things such as terminal and runway construction.

The problem is determining the fees different types of aircraft will be charged for landing. It is easy to determine the amount of money each aircraft should be responsible for paying due to operating cost. Since these costs are directly connected with the number of times a plane uses the airport, charges are assigned on a per-landing basis. Fixed capital costs, however, are not so easily distributed among aircraft. Since larger planes require enlarged runways, it would not be appropriate to force smaller planes to share in the expense of maintaining this portion of the runway.

For this reason the following rules, proposed by Baker[1], will be used to allocate the fixed capital costs to aircraft.

1. Divide the cost of catering for the smallest type of aircraft equally among the number of landings of all aircraft.

2. Divide the incremental cost of catering for the second smallest type of aircraft equally among the number of landings of all but the smallest

type of aircraft.

3. Continue this process until the incremental cost of the largest type of aircraft is divided equally among the number of landings made by the largest aircraft type.

The landing charges found as a result of the previous rules is precisely the Shapley value where the characteristic function is a "cost" function and the cost of any subset is equal to the cost of the "largest" player in the subset.

Divide the planes into m types ($m \geq 1$). Let N_j denote the set of landings by planes of type j ($j = 1, ...m$) and let there be $n_j > 0$ landings by aircraft of type j. Let $N = \cup_{j=1}^{m} N_j$ denote the set of all landings at the airport. Let C_j be the "cost" associated with providing an adequate runway for planes of type j. Without loss of generality, assume that

$$0 = C_0 < C_1 < C_2 < ... < C_m.$$

Let $S \subset N$, $S \neq \emptyset$. Then the cost $c(S)$ of a runway adequate to receive all landings in S is given by

$$c(S) = max(\{C_j | \; 1 \leq j \leq m, S \cap N_j \neq \emptyset\}) \qquad (4.1)$$

and $c(\emptyset) = 0$.

Theorem 4.1 *Driessen [3]* Let (N, c) be the airport cost game of (4.1) and put $m_j = \sum_{k=j}^{m} |N_k|$ for all $1 \leq j \leq m$. Then

$$\Phi_i(c) = \sum_{k=1}^{j} m_k^{-1}(C_k - C_{k-1})$$

for $i \in N_j$. (Note that $\Phi_i(c)$ is simply the Shapley value defined in chapter 2.)

Proof: The essential idea in the proof is to write the game of (4.1) as the sum of m related cost games for which the Shapley values can easily be determined with the aid of the standard properties for values. Formally, for any type $j = 1, 2, ..., m$, we define the cost game (N, c_j) by

$$c_j(S) = \begin{cases} C_j - C_{j-1} & \text{if } \mathrm{S} \cap \mathrm{M_j} \neq \emptyset \\ 0 & \text{if } \mathrm{S} \cap \mathrm{M_j} = \emptyset \end{cases}$$

where $M_j = \cup_{k=j}^{m} N_k$ represents the set of all landings by planes of type j and larger planes.

1. We assert that $c(S) = \sum_{k=1}^{m} c_k(S)$ for all $S \subset N$ (4.2). Let $S \subset N$, $S \neq \emptyset$. By the definition of the airport game c, there exists a unique $1 \leq j \leq m$ such that $c(S) = C_j$. Then we obtain $S \cap N_j \neq \emptyset$ and $S \cap N_k = \emptyset$ for all $j < k \leq m$. Particularly, $S \cap M_k = \emptyset$ if and only if $j < k \leq m$. Now it follows that

$$\sum_{k=1}^{m} c_k(S) = \sum_{k=1}^{j} (C_k - C_{k-1}) = C_j - C_0 = C_j = c(S).$$

So, (4.2) holds.

2. Let $1 \leq j \leq m$. We assert that any $i \in N - M_j$ is a dummy in the game c_j. Let $i \in N - M_j$ and $S \subset N - \{i\}$. Then we have $c_j(\{i\}) = 0$ and further, $S \cap M_j \neq \emptyset$ if and only if $(S \cup \{i\}) \cap M_j \neq \emptyset$. It follows that $c_j(S \cup \{i\}) - c_j(S) = 0 = c_j(\{i\})$, and thus any $i \in N - M_j$ is a dummy in the game c_j. Now we conclude from the dummy player property of the Shapley value that

$$\Phi_i(c_j) = c_j(\{i\}) = 0 \qquad \text{for all i} \in \text{N} - \text{M}_\text{j}.$$

3. Let $1 \leq j \leq m$. We assert that the players in M_j are symmetric in the game c_j. Let $i_1 \in M_j$, $i_2 \in M_j$ and let $\theta : N \to N$ be a permutation such that $\theta(i_1) = i_2$, $\theta(i_2) = i_1$ and $\theta(i) = i$, for all $i \in N - \{i_1, i_2\}$. For any coalition $S \neq \emptyset$, we have the equivalence $S \cap M_j \neq \emptyset$ if and only if $\theta S \cup M_j \neq \emptyset$ and therefore, $(\theta c_j)(\theta S) = c_j(S) = c_j(\theta S)$. It follows that $\theta c_j = c_j$. Now we conclude from the symmetry property of the Shapley value that

$$\Phi_{i_1}(c_j) = \Phi_{\theta(i_2)}(\theta c_j) = \theta_{i_2}(c_j) \qquad \text{for all } \text{i}_1, \text{i}_2 \in \text{M}_\text{j}.$$

4. Let $1 \leq j \leq m$. Together with parts (2),(3), the efficiency property of the Shapley value yields

$$\Phi_i(c)=\begin{cases} 0 & \text{if i} \in \text{N} - \text{M}_\text{j} \\ m_j^{-1}(C_j - C_{j-1}) & \text{if i} \in \text{n}_\text{j} \end{cases}$$

From this, the formula (4.2) and the additivity property of the Shapley value, it follows that for all $i \in n_j$

$$\Phi_i(c) = \Phi_i(\sum_{k=1}^{m} c_k) = \sum_{k=1}^{m} \Phi_i(c_k) = \sum_{k=1}^{j} m_k^{-1}(C_k - C_{k-1}).$$

Example [**4.1**] Littlechild-Owen [9] For a numerical example we shall look at the total landings at the Birmingham (U.K.) airport during 1968 and 1969. In this example there are a total of 13,572 landings of 11 different types of aircraft. As mentioned earlier, there are two types of expenses. A complete list of notation to be used in Tables [4.1] is the following:

$$n_j = \text{the number of landings by type} - \text{j planes}$$

$$c_j = \text{annual capital cost (runway construction)}$$

$$\ell_j = \text{variable cost per landing}$$

$$a_j = \text{actual landing fee charged}$$

$$\Phi_j = \text{capital charge for a type} - \text{j plane by the Shapley value}$$

$$(j = 1, ..., 11)$$

Table 4.1 Numeric solutions for Ex [4.1]

Aircraft type	j	c_j	n_j	ℓ_j	Φ_j	$\Phi_j + a_j$	a_j
Fokker Friendship 27	1	65889	42	5.23	4.86	10.09	5.80
Viscount 800	2	76725	9555	6.09	5.66	11.75	11.40
Hawker Siddeley Trident	3	95200	288	7.55	10.30	17.85	21.70
Britannia 100	4	97200	303	7.71	10.85	18.56	29.80
Caravelle VLR	5	97436	151	7.73	10.92	18.65	20.30
BAC 11 (500)	6	98142	1315	7.79	11.13	18.92	16.70
Vanguard 953	7	102496	505	8.13	13.40	21.53	26.40
Coment 4B	8	104849	1128	8.32	15.07	23.39	29.40
Britannia 300	9	113322	151	8.99	44.80	53.79	34.70
Convair Corronado	10	115440	112	9.16	60.61	69.77	48.30
Boeing 707	11	117676	22	9.34	162.24	171.58	66.70

4.2 Supplier-Firm-Buyer Games

In this section we shall demonstrate how cooperative game theory can be applied to business strategy. In particular we will examine the Supplier-Firm-Buyer game. This example of unrestricted bargaining can be modeled by either the core of the game or the added-value principle.

Definition 4.1 The **added value** of a coalition $S \subset N$ is defined to be the coalition's marginal contribution: $v(N) - v(N \setminus S)$.

Definition 4.2 An outcome satisfies the **added-value principle** if no player captures more than his or her added value, that is, if

$$\forall i \in N, \qquad x_i \leq v(N) - v(N \setminus \{i\}), \qquad \text{and x(N) = v(N).}$$

Note that if an outcome is in the core of a game it must also satisfy the added value principle. The supplier-firm-buyer game is a "three-sided assignment game" with the following restrictions:

$$a_{ijk} = w_{jk} - c_{ij} \qquad \text{for all i} \in \text{N}_1,\ \text{j} \in \text{N}_2,\ \text{k} \in \text{N}_3. \qquad (4.3)$$

Sets N_1,N_2,and N_3 represent sets of suppliers, firms, and buyers, respectively. The term w_{jk} denotes buyer k's willingness to pay for transacting with firm j, while c_{ij} represents supplier i's opportunity cost for transacting with firm j.

Definition 4.3 A **three-sided assignment game** consists of three disjoint sets, N_1,N_2, and N_3, and a three dimensional assignment matrix, A. The dimensions of the matrix are $n_1 \times n_2 \times n_3$ where $n_i = |N_i|$. The disjoint sets represent the sets of players. A **matching** is a 3-tuple ijk consisting of a player i from set N_1, a player j from set N_2, and a player k from the set N_3. Element a_{ijk} of the matrix A is interpreted as the value that can be created by the matching ijk. We say the matchings $i^a j^a k^a$ and $i^b j^b k^b$ are **distinct** if $i^a \neq i^b$, $j^a \neq j^b$, $k^a \neq k^b$. An **assignment** of size r is a set of r distinct matchings.

There are two principles that govern the construction of a cooperative game from an m-sided assignment game. First, distinct matchings determine value creation. Second, this value is taken to be as large as feasibly possible. We derive equations (4.4), (4.5) and (4.6) from these principles.

Define the player set, N, to be $N_1 \cup N_2 \cup N_3$. The characteristic function v is defined by $v(\emptyset) = 0$, and for all $T \subseteq N$,

$$v(T) = 0 \qquad \text{if } \mathrm{T} \cap \mathrm{N_m} = \emptyset, \mathrm{m} \in \{1, 2, 3\}, \qquad (4.4)$$

$$v(\{i, j, k\}) = a_{ijk} \qquad \forall i \in N_1, j \in N_2, k \in N_3, and \qquad (4.5)$$

$$v(T) = max_{r,AS_r}(a_{i^1j^1k^1} + ... + a_{i^rj^rk^r}), \qquad (4.6)$$

where $r \leq min\{|T \cup N_1|, |T \cup N_2|, |T \cup N_3|\}$ and AS_r is the set of assignments of size r constructed from set T.

Example [4.2] Stuart [17] The game described below has two suppliers, two firms, and only one buyer. Our goal is to determine which firm and supplier will actually sell to the buyer.

$$N_1 = \{s_1, s_2\}, \qquad N_2 = \{f_1, f_2\}, \qquad N_3 = \{b_1\};$$

$$w_{11} = 100, \qquad w_{21} = 150; \qquad c_{11} = c_{12} = c_{21} = c_{12} = 10.$$

We can use equations (4.3)-(4.6) to construct the characteristic function for this example. Since this example has only one buyer, there is only one possible matching. Now equation (4.6) dictates that $v(N)$ is the largest possible matching, namely the buyer with the second firm and either of the two suppliers. This implies $v(N) = 140$.

The guaranteed minimum for a given player, ℓ, can be calculated by $max\{0, v(N) - \sum_{m \in N \setminus \{\ell\}} (v(N) - v(N \setminus \{m\})\}$. Since, by the added value principle, no player can receive more than his added value, the above calculation holds. Note that if every player receives his added value and if there is still some remaining value, then player ℓ gains that value. Table [4.2] provides the added value analysis of the game.

The opportunity cost for each supplier to provide resources is \$10 regardless of which firm is used. The added value can be interpreted in terms of competition. Since there is only one supplier needed and all suppliers are identical, the added value for each supplier is \$0. The buyer is willing to pay \$100 for the first firm's product and willing to pay \$150 for the second firm's product. Thus, the first firm has an added value of zero, while the second firm has an added value of \$50. Competition between the suppliers and partial competition between the firms guarantee \$90 to the buyer, and the remaining \$50 is divided between the second firm and the buyer.

Table 4.2 Solutions for Ex [4.2]

Player ℓ	$v(N)$	$v(N \setminus \{\ell\})$	Added Value	Guaranteed Minimum
Buyer	140	0	140	90
Firm 1	140	140	0	0
Firm 2	140	90	50	0
Supplier i	140	140	0	0

4.3 Evolutionary Games

This section provides information about one of the most unusual applications of game theory. It is in the area of biological evolution. Characteristics that affect the reproducing capacity of the individuals in a population are not easily analyzed. The evolution of such characteristics is dependent on interaction among individuals.

Mating behavior of animals, more specifically the male individual's need to establish dominance, will be the focus of this section. The males challenge each other, but often the battle is one of endurance rather than a battle to death. It would be unusual, from a biological view point, that animals would seek to do what is best for the group, i.e. not killing one of its own.

These conflicts are modeled as games by Maynard-Smith [11]. In this model it is thought each individual chooses from among a set of strategies. Suppose the first individual chooses strategy, x, and his opponent uses strategy, y; the resulting payoff to the first individual is $E(x, y)$. To biologists payoffs are related to the fitness of the individual. Genetics prompt an individual to choose strategy x and this individual will pass this gene to his offspring. Since this individual survives he is able to pass this gene. Individuals whose strategies have a high payoff will produce more offspring than those with less successful gene types and thus will eventually dominate the population.

Definition 4.4 Let X be the set of strategies in an evolutionary game. A strategy $x^* \in X$ is called an **evolutionary stable strategy** (ESS) if for every $y \in X$, $y \neq x^*$,and for $\bar{x} = (1-\epsilon)x^* + \epsilon y$, then $e(x^*, \bar{x}) > e(y, \bar{x})$, for sufficiently small $\epsilon > 0$.

Note that y may be thought of as a mutation which affects ϵ of the population.

Definition 4.5 A strategy $x^* \in X$ is an ESS if for every $y \in X, y \neq x^*$,

$$e(x^*, x^*) \geq e(y, x^*),$$

and, moreover, if $e(x^*, x^*) = e(y, x^*)$, then

$$e(x^*, y) > e(y, y). \qquad (4.7)$$

Before examining a certain multi-person game model, we must first look at the two-person model. Suppose each individual has to choose between two pure fighting strategies. The Hawk (H) strategy involves intensifying the fight until the opponent is injured and must withdraw. Basically this means the Hawk always attacks. The Dove (D) strategy involves conventional fighting. The Dove will retreat before injury occurs if the opponent escalates fighting. In other words, the Dove always runs.

The payoff, involved when a Hawk is in battle with a Dove is written as $E(H, D)$. The payoff is a measure of the increase or decrease in the individual's expected lifetime reproductive success in comparison to some baseline measure. This payoff measure is often referred to as the change in Darwinian fitness of the individual and is dependent upon three factors. According to Maynard-Smith and Price these factors are as follows:

1. **The advantage of winning:** the resources over which the contest takes place is assumed to be worth V (for victory) units of Darwinian fitness to the winner.

2. **The disadvantage of being injured:** injury alters an animal's fitness by $-W$ (for wounds) units.

3. **The time and energy wasted in a long contest:** the alteration of the fitness of each contestant by $-T$ (for time) units in conventional

fights.

Using the components described above, we can determine payoffs for every possible combination of strategies. Matrix 4.1 gives a summary of the payoffs for the Hawk-Dove game.

$$\begin{array}{ccc} & Hawk & Dove \\ Hawk & \frac{V-W}{2} & V \\ Dove & 0 & \frac{V}{2}-T \end{array}$$

Matrix 4.1

Since there are equal chances of success and injury when the two similar strategies compete, we must divide the payoff by two. Now we shall analyze examples of two different cases.

Example [4.3] Thomas [18] In this case the advantage of winning outweighs any disadvantage due to injury; i.e., $V \geq D$. Put $V = 4, W = 2$. Then we get the following matrix:

$$\begin{array}{ccc} & Hawk & Dove \\ Hawk & 1 & 4 \\ Dove & 0 & 2 \end{array}.$$

Matrix 4.2

Look at the pure Hawk strategy, $x = (1, 0)$, and take any other strategy $y = (y, 1 - y),\ y \neq 1$:

$$e(x, x) = 1 \geq y = e(y, x),$$

and so Hawk is an ESS.

Example [4.4] Thomas [18] In this scenario injury outweighs the advantage of winning; i.e., $V \leq W$. Put $V = 2, W = 4$. Thus we get the following matrix:

$$\begin{array}{ccc} & Hawk & Dove \\ Hawk & -1 & 2 \\ Dove & 0 & 1 \end{array}.$$

Matrix 4.3

In this game strategy $x = (\frac{1}{2}, \frac{1}{2})$ is an ESS, as shown below. Let $y = (y, 1 - y)\ y \neq \frac{1}{2}$; then

$$e(x, x) = \frac{1}{2} = e(y, x).$$

and by condition (4.7)

$$e(y, y) - = 1 - 2y^2, \qquad e(x, y) = \frac{3}{2} - 3y.$$

Now $(\frac{3}{2} - 2y) - (1 - 2y^2) = 2(y - \frac{1}{2})^2 > 0$, so since $y \neq \frac{1}{2}$, we have

$$e(x, y) - e(y, y) > 0,$$

which proves x is an ESS.

Let us now turn our attention to a multi-person game model. Colman [2] suggests constructing a compound multi-person game in which the players compete in a series of two-person contests, which follow the payoff structure found in Matrix 4.1.

The expected payoff will be a linear function of the proportion of other members of the population playing Hawk or Dove. This is due to the fact that animals have to adopt either a Hawk or a Dove strategy. It is not important that each animal plays every other animal, but only that animals are involved in the same number of competitions. Our focus is on the expected payoffs per contest since our only concern is the relative fitness of the genotypes.

If members of the population play the two-person game given by Matrix 4.1, then payoffs in each contest are based on two factors. The first is the individual's own strategy, and the second, is the proportion of other animals in the population adopting the Hawk strategy. Let k represent that proportion, where $0 \leq k \leq 1$; then it follows that the proportion of Dove opponents is $1 - k$. Suppose the individual adopts the Hawk strategy; then the expected payoff $E(H)$ is $E(H) = k\frac{V-W}{2} + (1-k)V$ and if the individual adopts the Dove strategy then,the expected payoff $E(D)$ is $E(D) = (1-k)(\frac{V}{2} - T.)$

Suppose for illustrative purposes that $V = 10$, $W = 20$, $T = 3$; then Figure (4.1) depicts the resultant compound game. From this figure we can conclude there is no pure ESS. If the population contains mostly Doves (k is small) then, the expected payoff E(H) to a Hawk is higher than the expected payoff E(D) to a Dove. If this is true, the population of Hawks will increase by natural selection shifting the outcome toward the equilibrium. At equilibrium the payoff to the Hawk and Dove are equal and the proportion of population of Hawks to Doves will remain stable.

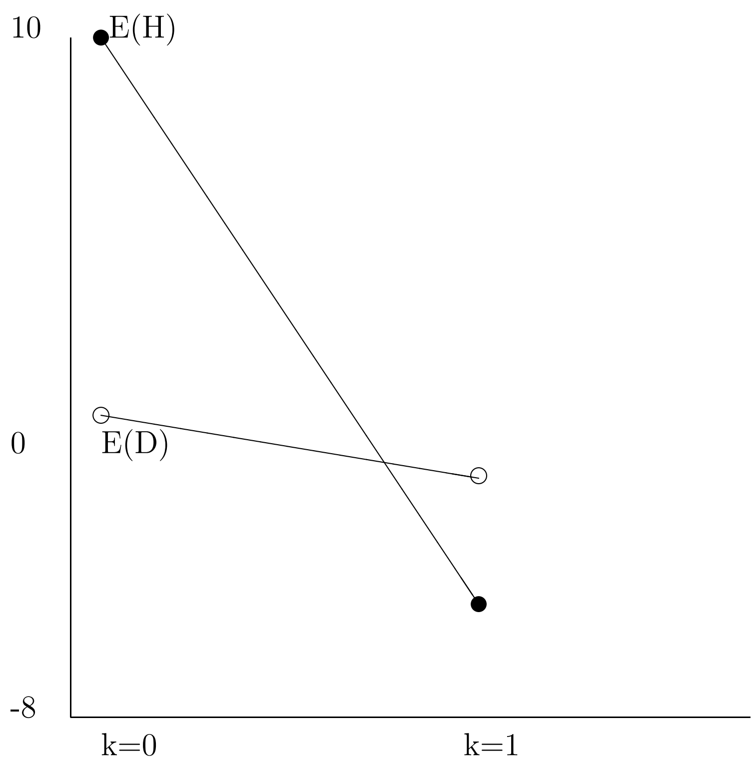

Figure (4.1) Resultant of the Hawk verses Dove Game

Suppose the population contains mostly Hawks (k is large). In this case the payoff E(D) to Doves is higher. Therefore the Hawk population will decrease until equilibrium is reached. This implies that no matter what proportions of Hawks and Doves initially exit, the population will evolve to the ESS equilibrium.

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