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Mappings of ANR's Whose Images are ANR's

Jung-In Kang Choi
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Jung-In Kang Choi entitled "Mappings of ANR's Whose Images are ANR's." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

John J. Walsh, Major Professor

We have read this dissertation and recommend its acceptance:

Robert J. Daverman, Kwangh Jeon

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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MAPPINGS OF ANR'S WHOSE IMAGES ARE ANR'S

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Jung-In Kang Choi

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ABSTRACT

Recently, R.J. Daverman and J.J. Walsh modified an example due to J. Taylor to obtain an example of a cell-like map from a compactum with non-trivial shape onto the Hilbert cube Q such that the non-degeneracy set is contained in the countable union of finite dimensional closed subsets of Q. Previously, G. Kozlowski proved that a cell-like map $f:X' \to X$ from a compact ANR X' onto a metric space X is a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of finite dimensional closed subsets of X such that the non-degeneracy set is contained in $\bigcup_{n=1}^{\infty} B_n$ and $\{f^{-1}(B_n)\}_{n=1}^{\infty}$ forms a pairwise disjoint null-sequence. Here we raise two questions, which we show are equivalent. First: Is a cell-like map $f:X' \to X$ a hereditary shape equivalence if the non-degeneracy set is contained in the countable union of pairwise disjoint finite dimensional closed subsets of X? Second: Is a cell-like map $f:X' \to X$ a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of finite dimensional closed subsets of X such that $\bigcup_{n \neq m} (B_n \cap B_m)$ has a strong transfinite dimension and the non-degeneracy set is contained in $\bigcup_{n=1}^{\infty} B_n$? Even though we are not able to answer these questions, we give affirmative answers to the questions for special cases, and, furthermore, we are able to extend the aforementioned result of Kozlowski's. Also, we attempt to extend certain analyses of cell-like maps, which are proper by definition, to (non-proper) UV^{∞}-maps. We prove that for a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X the following are equivalent:

(1) X is an ANR ; (2) f is a hereditary homotopy equivalence; (3) f is a hereditary shape equivalence; (4) f is a fine homotopy equivalence. Since UV^{∞}-maps are generally not onto, the notion of a "hereditary shape equivalence" is a variation of that for cell-like maps, though it agrees with, say, Kozlowski's for cell-like maps.

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CHAPTER I

INTRODUCTION

It is known that a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X is a hereditary shape equivalence if and only if f is a fine homotopy equivalence and, furthermore, if and only if X is an ANR ; see [An] and [K2] . Furthermore, if the dimension of X' is finite, then X is an ANR if and only if X is finite dimensional and, furthermore, if and only if the dimension of X is less than or equal to the dimension of X'; see [AP], [K1], and [La] .

It is quite natural to expect cell-like maps to be homotopy equivalences. Even the expectation that cell-like maps are fine homotopy equivalences is reasonable. For the most part, cell-like maps behave in the expected fashion by being fine homotopy equivalences in the setting of ANR's and by being hereditary shape equivalences in the setting of general metric spaces; see [An], [K2], [MR], and [Sh].

J. Taylor [Ta] gave an example of a cell-like map F from an infinite dimensional compactum T with non-trivial shape onto the Hilbert cube Q , which is not a hereditary shape equivalence. For further examples of cell-like maps that are not hereditary shape equivalences see [DW], [KMW], and [Mi] .

Now a principal unanswered problem about cell-like maps, which is known as the "dimension raising cell-like map question", takes the following form. Is there a cell-like map $f: X' \to X$ from a finite dimensional ANR X' to a metric space X such that any one of the following five equivalent conditions holds?

(1) f is not a hereditary shape equivalence; (2) f is not a fine homotopy equivalence; (3) X is not an ANR ; (4) X is infinite dimensional; (5) the dimension of X is bigger than the dimension of X' .

R.J. Daverman and J.J. Walsh [DW] modified the example due to J. Taylor [Ta] to obtain an example of a cell-like map F from a compactum T with non-trivial shape onto the Hilbert cube Q satisfying that the non-degeneracy set, that is, the set $\{y \in Q \mid F^{-1}(y) \neq \text{point}\}$, is contained in the countable union of finite dimensional closed subsets of Q . On the other hand G. Kozlowski [K2] proved that a cell-like map $f:X' \to X$ from a compact ANR X' onto a metric space X is a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint finite dimensional closed subsets of X such that the non-degeneracy set is contained in $\bigcup_{n=1}^{\infty} B_n$ and $\{f^{-1}(B_n)\}_{n=1}^{\infty}$ forms a null-sequence, that is, diameter of $f^{-1}(B_n) \to 0$ as $n \to \infty$.

Here arises a pertinent question.

QUESTION A. Is a cell-like map a hereditary shape equivalence if the non-degeneracy set is contained in the countable union of pair-wise disjoint finite dimensional closed subsets?

We may even expect a better result.

QUESTION B. Is a cell-like map $f:X' \to X$ a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of finite dimensional closed subsets of X such that the non-degeneracy set is contained in $\bigcup_{n=1}^{\infty} B_n$ and $\bigcup_{n \neq m}^{\infty} (B_n \cap B_m)$ has a strong transfinite dimension?

We find an affirmative answer to Question A combined with a theorem of F.D. Ancel's [An] gives an affirmative answer to Question B.

Unfortunately we are unable to answer these questions. In Chapter II we give some affirmative answers for special cases of the questions, and we extend the aforementioned theorem of G. Kozlowski's [K2] . Two results are the following.

THEOREM. If $f:X' \to X$ is a cell-like map between compact metrizable spaces such that there exist a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X and an integer k satisfying, for each $n = 1,2,\ldots,$

(1) $f|f^{-1}(B_n): f^{-1}(B_n) \to B_n$ is a hereditary shape equivalence,

(2) the non-degeneracy set is contained in $\bigcup_{i=1}^{\infty} B_i$,

and

(3) for each neighborhood U of B_n , there exists a neighborhood V of B_n satisfying $\overline{V} \subset U$ and $\dim[\partial V \cap (\bigcup_{i=1}^{\infty} B_i)] \leq k$,

then f is a hereditary shape equivalence.

THEOREM. If $f:X' \to X$ is a cell-like map from an ANR X' onto a metric space X such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying, for each $n = 1,2,\ldots,$

(1) $f|f^{-1}(B_n): f^{-1}(B_n) \to B_n$ is a hereditary shape equivalence,

(2) the non-degeneracy set is contained in $\bigcup_{i=1}^{\infty} B_i$,

and

(3) for each neighborhood U of B_n , there exists a neighborhood V of B_n satisfying $\overline{V} \subset U$ and ∂V intersects at most finitely many of $\{B_i\}_{i=1}^{\infty}$,

then f is a hereditary shape equivalence.

In Chapter III, we extend analyses of cell-like maps, which are proper maps by definition, to (non-proper) UV^{∞}-maps. Even though UV^{∞}-

maps are required to be neither surjective nor proper as we see in the example of the inclusion of the open interval $(0, 1)$ into $[0, 1]$, which is a hereditary shape equivalence and a fine homotopy equivalence, we expect UV^{∞}-maps to behave in the same way as cell-like maps.

G. Kozlowski [K2] defined the notion of a hereditary shape equivalence for proper maps between metric spaces; namely, f is a hereditary shape equivalence provided $f|f^{-1}(A) \to A$ is a shape equivalence for each closed subset A of X. Necessarily, such maps are surjective. It follows from Kozlowski's work [K2] that a map $f:X' \to X$ is a hereditary shape equivalence if and only if $f|f^{-1}(U): f^{-1}(U) \to U$ is a shape equivalence for any open subset U of X. The latter equivalent condition turns out to be an adequate definition of a hereditary shape equivalence for non-proper maps and fits well with the notion of a UV^{∞}-map. (We shall give a formal definition later.)

In F.D. Ancel's work [An], we find that for a proper map $f:X' \to X$ from an ANR X' onto a metric space X and a closed subset A of X $f|f^{-1}(A): f^{-1}(A) \to A$ is a hereditary shape equivalence if and only if $f_A:X' \to X' \underset{f}{\cup} A$ is a fine homotopy equivalence, where $X' \underset{f}{\cup} A$ is the adjunction space defined as follows. As a set $X' \underset{f}{\cup} A = (X' - f^{-1}(A)) \cup A$ and the topology on $X' \underset{f}{\cup} A$ is generated by the open sets in $X' - f^{-1}(A)$, together with sets of the form $f^{-1}(U - A) \cup (U \cap A)$ for open subsets U of X, and $f_A(x) = x$ for $x \in X' - f^{-1}(A)$ and $f_A(x) = f(x)$ for $x \in f^{-1}(A)$. Hence we adapt the last equivalent condition to define the notion that a map $f:X' \to X$ between metric spaces is a fine homotopy equivalence over a closed subset A of X.

We generalize a theorem of G. Kozlowski's [K2] stating that a cell-like map $f:X' \to X$ from a compact ANR X' onto a metric space X is a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of X satisfying: the non-degeneracy set is contained in $\bigcup_{n=1}^{\infty} B_n$, $f|f^{-1}(B_n): f^{-1}(B_n) \to B_n$ is a hereditary shape equivalence for each B_n, and $\{f^{-1}(B_n)\}_{n=1}^{\infty}$ forms a null-sequence, to a theorem on a UV^{∞}-map as follows.

<u>THEOREM</u>. If $f:X' \to X$ is a UV^{∞}-map from an ANR X' to a metric space X such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying, for each $n = 1,2,\ldots,$

(1) f is a fine homotopy equivalence over B_n,

(2) for each neighborhood U of B_n, there exists a neighborhood V of B_n with $\overline{V} \subset U$ and $\partial V \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$,

and

(3) the non-degeneracy set is contained in $\bigcup_{i=1}^{\infty} B_i$,

then f is a hereditary shape equivalence.

G. Kozlowski [K2] proved that a cell-like map $f:X' \to X$ from an ANR X' to a metric space X is a hereditary shape equivalence if and only if X is an ANR. We extend this result as follows.

<u>THEOREM</u>. A UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a hereditary shape equivalence if and only if X is an ANR.

We find that the next theorem provides important equivalences among the classical notions.

<u>THEOREM</u>. For a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X the following are equivalent:

(1) X is an ANR ;

(2) f is a hereditary homotopy equivalence;

(3) f is a hereditary shape equivalence;

(4) f is a fine homotopy equivalence.

First notice that (2) easily implies (3). For a cell-like map f F.D. Ancel [An] proved equivalences of (1), (3), and (4) and G. Kozlowski [K2] proved equivalences of (1) and (3). W.E. Haver [Ha] proved equivalences of (1) and (4) for a cell-like map betwen separable metric spaces. For (non-proper) UV^{∞}-maps Kozlowski [K2] proved (4) implies (1) and (1) implies (2). We establish the remaining implications.

Finally, we generalize a theorem of G. Kozlowski's [K2] that states a cell-like map $f:X' \to X$ is a hereditary shape equivalence if $X = \bigcup_{i=1}^{\infty} X_i$ for a closed subset X_i's of X satisfying $f|f^{-1}(X_i): f^{-1}(X_i) \to X_i$ is a hereditary shape equivalence for each X_i to a theorem for a (non-proper) UV^{∞}-map with the same hypotheses.

DEFINITIONS AND NOTATION

CONVENTION. All spaces are metrizable and all metrics are denoted by d .

A map is a continuous function.

A map $f:X \to Y$ is a UV^{∞}-map provided, for each $x \in X$ and each neighborhood U of x , there exists a neighborhood V of x such that $V \subset U$ and the inclusion $f^{-1}(V) \to f^{-1}(U)$ is null-homotopic.

A map $f:X \to Y$ is proper provided $f^{-1}(C)$ is compact for any compact subset C of Y . It is well-known that f is proper if and only if f is closed and each point-inverse is compact.

A <u>cell-like map</u> is a proper map with each point-inverse having trivial shape.

A map $f:X \to Y$ is a <u>hereditary homotopy equivalence</u> provided, for each open subset U of Y, $f|f^{-1}(U): f^{-1}(U) \to U$ is a homotopy equivalence.

An ANR is understood to be a metric space that is an absolute neighborhood retract for the class of metric spaces.

A map $f:X \to Y$ is a <u>shape equivalence</u> provided for each ANR P $f^{\#}: [Y:P] \to [X:P]$ is a bijection of sets, where each of $[Y:P]$ and $[X:P]$ denotes the set of homotopy classes of maps and $f^{\#}([\alpha]) = [\alpha \circ f]$ for a class $[\alpha] \in [Y:P]$.

Recall that a map $f:X \to Y$ is a <u>hereditary shape equivalence</u> provided, for each open subset U of Y, $f|f^{-1}(U):f^{-1}(U) \to U$ is a shape equivalence. We already mentioned that a proper map $f:X \to Y$ is a hereditary shape equivalence if and only if $f|f^{-1}(A): f^{-1}(A) \to A$ is a shape equivalence for any closed subset A of X.

For a fixed map $f:X' \to X$ we introduce the following notation:

$S' = f^{-1}(S)$ for any subset S of X.

$\mathcal{U}' = \{f^{-1}(U) | U \in \mathcal{U}\}$ for any family $\mathcal{U}$ of subsets of X.

I denotes the unit interval $[0, 1]$.

A homotopy $H: X \times I \to Y$ is <u>limited by a family</u> $\mathcal{U}$ of subsets of Y provided $\{H(\{x\} \times I) | x \in X\}$ refines $\mathcal{U}$.

A map $f:X' \to X$ is a <u>fine homotopy equivalence</u> provided, for any open cover $\mathcal{U}$, there exists a map $g:X \to X'$ such that $g \circ f$ is homotopic to identity on X' with the homotopy limited by $\mathcal{U}'$ and $f \circ g$ is homotopic to identity on X with the homotopy limited by $\mathcal{U}$.

Recall that a map $f:X' \to X$ is a <u>fine homotopy equivalence over a closed subset</u> A of X provided $f_A:X' \to X' \underset{f}{\cup} A$ is a fine homotopy equivalence, where $X' \underset{f}{\cup} A$ is the adjunction space defined as follows. As a set $X' \underset{f}{\cup} A = (X' - f^{-1}(A)) \cup A$ and the topology on $X' \underset{f}{\cup} A$ is generated by the open sets in $X' - f^{-1}(A)$ together with sets of the form $f^{-1}(U - A) \cup (U \cap A)$ for open subsets U of X, and $f_A(x) = x$ for $x \in X' - f^{-1}(A)$ and $f_A(x) = f(x)$ for $x \in f^{-1}(A)$. We already mentioned that for a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X and a closed subset A of X, f is a fine homotopy equivalence over A if and only if $f|f^{-1}(A): f^{-1}(A) \to A$ is a hereditary shape equivalence.

For a map $f:X' \to X$ the <u>non-degeneracy set</u>, denoted by N_f, is the subset $\{x \in X \mid \text{either } f^{-1}(x) = \phi, \text{ or } f^{-1}(x) \text{ contains at least two points, or } f^{-1}(x) = \text{point but } \{f^{-1}(B) \mid B \in \mathcal{B}\} \text{ is not a neighborhood basis for } f^{-1}(x) \text{ where } \mathcal{B} \text{ is a neighborhood basis for } x\}$. Notice that for a cell-like map $f:X' \to X$ $N_f = \{x \in X \mid f^{-1}(x) \neq \text{point}\}$.

A neighborhood always refers to an open set.

For a space X and subsets $B \subset A$ of X we introduce the following notation:

$\overline{B}$ = the closure of B in X.

$\overline{B}$ (in A) = the closure of B in the subspace A.

$\text{int}(A)$ = the interior of A in X.

<u>open in A</u> means open in the subspace A.

∂B = the boundary of B in X.

∂B (in A) = the boundary of B in the subspace A.

Finally we state here a modified <u>homotopy extension property</u> of ANR .

<u>HOMOTOPY EXTENSION PROPERTY OF ANR</u> . Suppose A is a closed subset of a metric space X and $\mathcal{U}$ is an open cover of an ANR Y . If $H: A \times I \cup X \times \{0\} \to Y$ is a map such that $\{H(\{x\} \times I) | x \in A\}$ refines $\mathcal{U}$, then there exists an extension $\overline{H}: X \times I \to Y$ of H such that $\overline{H}$ is limited by $\mathcal{U}$.

For standard definitions and notation see [Du], [Hu], and [HW].

CHAPTER II

CELL-LIKE MAPS

Throughout this chapter $f:X' \to X$ denotes a cell-like map unless specified otherwise.

Recall that, for a cell-like map $f:X' \to X$, the non-degeneracy set N_f equals the set $\{x \in X | f^{-1}(x) \text{ is not a point}\}$.

G. Kozlowski [K2] proved the following theorem.

THEOREM. Suppose $f:X' \to X$ is a cell-like map from a compact ANR X' onto a metric space X such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) $f|B_n':B_n' \to B_n$ is a hereditary shape equivalence for each B_n ,

and

(3) $\{B_n\}_{n=1}^{\infty}$ is pairwise disjoint and, for each $\varepsilon > 0$, at most finite members of $\{B_n'\}_{n=1}^{\infty}$ have diameter $> \varepsilon$,

then f is a hereditary shape equivalence.

In section 2.1 we generalize the aforementioned theorem of G. Kozlowski as follows.

THEOREM 1. Suppose $f:X' \to X$ is a cell-like map between metric spaces such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) $f|B_n':B_n' \to B_n$ is a hereditary shape equivalence for each B_n ,

and

(3) for each B_n and a neighborhood U of B_n, there exists a neighborhood V of B_n such that $\overline{V} \subset U$ and $\partial V \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$, then f is a hereditary shape equivalence.

Theorem 1 combined with a theorem of L. Tumarkin's [Na] gives a "better" theorem as follows.

<u>THEOREM 2</u>. Suppose $f:X' \to X$ is a cell-like map between compact metrizable spaces such that there exist a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X and an integer k satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) $f|B_n':B_n' \to B_n$ is a hereditary shape equivalence for each B_n, and

(3) for each B_n and a neighborhood U of B_n, there exists a neighborhood V of B_n satisfying $\overline{V} \subset U$ and $\dim(\partial V \cap (\bigcup_{i=1}^{\infty} B_i)) \leq k$, then f is a hereditary shape equivalence.

The same analysis as in the proof of Theorem 2 gives the equivalences of two questions A and B raised in the introduction.

In section 2.2, we prove Theorem 3 which also extends Theorem 1. But Theorem 3 is quite different from Theorem 2. Theorem 3 replaces the hypothesis (3) in Theorem 1 with the following: for each B_n and a neighborhood U of B_n, there exists a neighborhood V of B_n satisfying $\overline{V} \subset U$ and ∂V intersects at most finitely many of $\{B_i\}_{i=1}^{\infty}$.

Let us introduce definitions and notation which are used throughout Chapter II.

For a subset V of $X \times Y$ we adopt the following:

$\operatorname{dom} V = \{x \in X \mid (x,y) \in V$ for some $y \in Y\}$.

$V|K = \{(x,y) \in V \mid x \in K\}$ for a subset K of X.

$V|x = V|\{x\}$ for $x \in X$.

$V(x) = \{y \in Y \mid (x,y) \in V\}$ for $x \in X$.

Suppose U and V are subsets of $X \times Y$ and $g:\operatorname{Dom}V \to Y$ is a function such that the relation $g \subset U$. A slice-contraction of V onto g in U is a homotopy $\phi:V \times I \to U$ such that ϕ_0 = inclusion $V \to U$, for each $x \in X$ $\phi((V|x) \times I) \subset U|x$, and $\phi_1(V|x) = g|x$. If there is a slice-contraction of V onto g in U, we say that V slice-contracts or is slice-contractible onto g in U.

Suppose R is a relation from a space X to a space Y. Then $R:X \to Y$ is slice-trivial in $X \times Y$ provided R is continuous with compact point images and if each neighborhood U of R in $X \times Y$ contains a neighborhood V of R in $X \times Y$ such that V slice-contracts in U.

Throughout chapter II we use G. Kozlowski's definition of a hereditary shape equivalence defined for cell-like maps as follows. A cell-like map $f:X' \to X$ is a hereditary shape equivalence provided for any closed subset A of X $f|A'$ induces a bijection between $[A:P]$ and $[A':P]$ for any ANR P.

To prove theorems in this chapter we freely use results of F.D. Ancel's [An] and G. Kozlowski's [K2].

2.1. MAPS DETERMINED ON SEQUENCES WITH BOUNDED DIMENSIONAL PAIRWISE INTERSECTIONS

First we prove the following theorem which generalizes a result of G. Kozlowski's [K2] .

THEOREM 1. If $f: X' \to X$ is a cell-like map between metric spaces such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) for each B_n $f|B_n': B_n' \to B_n$ is a hereditary shape equivalence, and

(3) for each B_n and a neighborhood U of B_n , there exists a neighborhood V of B_n such that $\overline{V} \subset U$ and $\partial V \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$,

then f is a hereditary shape equivalence.

PROOF. Since any metric space can be embedded as a closed subset of an ANR , we may assume that X' is an ANR . According to G. Kozlowski [K2], f is a hereditary shape equivalence if, for any open cover $\mathcal{M}$ of X , there exists a map $g: X \to X'$ such that $g \circ f$ is homotopic to the identity map on X' with the homotopy limited by $\mathcal{M}'$.

Let $\mathcal{M}$ be an open cover of X . By Ancel's enlargement lemma [An, A.8], there exists a neighborhood U of the relation f^{-1} in $X \times X'$ such that $\{U(x) | x \in X\}$ refines $\mathcal{M}'$.

According to Ancel [An, Theorem 4.5], $f^{-1}|B_n$ is slice-trivial in $X \times X'$ for each $n = 1,2,\ldots$.

For a subset S of a space Y and $\varepsilon > 0$ we denote $N(S; \varepsilon) = \{y \in Y | d(y, S) < \varepsilon\}$.

By induction we will prove that, for each $n = 1, 2, \ldots$, there exist a (possibly empty) open subset M_n of X, a map $g_n:\overline{M}_n \to X'$, a homotopy $H^n:(\overline{M}_n)' \times I \to X'$ from H_0^n = inclusion $(\overline{M}_n)' \to X'$ to $H_1^n = g_n \circ f|(\overline{M}_n)'$, and an open cover L_n of X such that

(a) $\{\overline{M}_n\}_{i=1}^{\infty}$ consists of pairwise disjoint sets, $\bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} M_i$, and $\partial M_n \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$,

(b) mesh $(L_n) < \frac{1}{n}$ and L_n refines M,

(c) $g_n = f^{-1}$ on ∂M_n, and

(d) $H^n(x, t) = x$ for each $(x, t) \in (\partial M_n)' \times I$ and the homotopy H^n is limited by L_n'.

We will show the first two steps of induction.

Choose an open cover L_1 of X such that mesh $L_1 < 1$ and L_1 refines M. Using the enlargement lemma, choose neighborhoods $U_{1,1}$, $U_{1,2}$, $U_{1,3}$ and $U_{1,4}$ of $f^{-1}|B_1$ in $X \times X'$ and, by the hypothesis (3), a neighborhood M_1 of B_1 such that

(1.a) $U_{1,i}$ slice-contracts onto $g_{1,(i-1)}$ in $U_{1,(i-1)}$ by $\phi_{1,i}$ $(i = 2, 3, 4)$, $\partial M \cap (\bigcup_{n=1}^{\infty} B_n) = \phi$, $\overline{M} \subset \text{Dom } U_{1,4}$, and $\{U_{1,1}(x) | x \in X\}$ refines L_1'.

Now we will find $g_1:\overline{M}_1 \to X'$ satisfying $g_1 = f^{-1}$ on ∂M_1. By Ancel's neighborhood extension property [An. 2.4], $f^{-1}|\partial M_1$ extends to a map g_1' from a neighborhood V_1 of ∂M_1 (in $\overline{M}_1$) into X' such that $g_1'(x) \in U_{1,4}(x)$ for each $x \in X$. Find an open set W_1 (in $\overline{M}_1$) such that $\partial M_1 \subset W_1 \subset \overline{W}_1$ (in $\overline{M}_1$) $\subset V_1 \subset \overline{M}_1$, and find a map $\lambda:\overline{M}_1 \to [0, 1]$ satisfying $\lambda(\partial M_1) = \{0\}$ and $\lambda(\overline{M}_1 - W_1) = \{1\}$. Then we can define $g_1:\overline{M}_1 \to X'$ as $g_1(x) = g_{1,3}(x)$ if $x \notin W_1$ and $g_1(x) = \pi \circ \phi_{1,4}(x, g_1'(x), \lambda(x))$ if $x \in \overline{W}_1$ (in $\overline{M}_1$), where π is

a projection from $X \times X'$ onto X'. Then we easily see $g_1 \subset U_{1,3}$ and $g_1 = f^{-1}$ on ∂M_1.

We will find a homotopy H^1 from $H_0^1 =$ inclusion $(\overline{M}_1)' \to X'$ to $g_1 \circ f|(M_1)'$ satisfying $H^1(\{x\} \times I) \subset U_{1,1}(f(x))$ for each $x \in (\overline{M}_1)'$ and $H^1(x, t) = x$ for each $(x, t) \in (\partial M_1)' \times I$. Since X' is an ANR, there exists a neighborhood C of $(\partial M_1)'$ (in $(\overline{M}_1)'$) and a homotopy $G': C \times I \to X'$ from $G_0' =$ inclusion $C \to X'$ to $G_1' = g_1 \circ f|C$ such that $G'(x, t) = x$ for any $(x, t) \in (\partial M_1)' \times I$ and $G'(\{x\} \times I) \subset U_{1,3}(f(x))$. Let $P = \{C \times I \times I\} \cup \{(\overline{M}_1)' \times [(\{0, 1\} \times I) \cup (I \times \{1\})]\}$, and find a map $\Lambda: (\overline{M}_1)' \times I \to P$ satisfying $\Lambda(x, s) = (x, s, 0)$ for any $(x, s) \in [(\partial M_1)' \times I] \cup [(\overline{M}_1)' \times \{0, 1\}]$ and Λ preserves the first coordinates. Define $\psi = P \to X'$ as follows.

$$\psi(x, s, t) = \begin{cases} \pi \circ \phi_{1,3}(f(x), G'(x, s), t) & \text{on } C \times I \times I \\ \pi \circ \phi_{1,3}(f(x), x, t) & \text{on } (\overline{M}_1)' \times \{0\} \times I \\ \pi \circ \phi_{1,3}(f(x), g_1 \circ f(x), t) & \text{on } (\overline{M}_1)' \times \{1\} \times I \\ g_{1,2} \circ f(x) & \text{on } (\overline{M}_1)' \times I \times \{1\} . \end{cases}$$

Finally we define $H^1: (\overline{M}_1)' \times I \to X'$ as $H^1 = \psi \circ \Lambda$. We can easily check $H^1(\{x\} \times I) \subset U_{1,2}(f(x))$ for each $x \in (\overline{M}_1)'$, hence $H^1(\{x\} \times I)|x \in (\overline{M}_1)'\}$ refines L_1', $H_0^1 =$ inclusion $(\overline{M}_1)' \to X'$, $H_1^1 = g_1 \circ f|(\overline{M}_1)'$, and $H^1(x, t) = x$ for any $(x, t) \in (\partial M)' \times I$.

Now we will show the second step of induction.

Choose an open refinement L_2 of M with mesh $< \frac{1}{2}$. If $B_2 \subset M_1$, let $M_2 = \phi$, $g_2 = \phi$, and $H^2 = \phi$. Suppose B_2 is not contained in M_1. Then we can easily show that for any neighborhood

U of $B_2 - M_1$ there exists a neighborhood V of $B_2 - M_1$ such that $\partial V \cap (\bigcup_{n=1}^{\infty} B_n) = \phi$ and $\overline{V} \subset U$. Also it follows from results of Ancel's and Kozlowski's that $f^{-1}|B_2 - M_1$ is slice-trivial in $X \times X'$. Therefore, by the same argument as in the first step of induction, we obtain a neighborhood M_2 of $B_2 - M_1$, and a map $g_2:\overline{M}_2 \to X'$, and a homotopy $H^2:(\overline{M}_2)' \times I \to X'$ from $H_0^2 =$ inclusion $(\overline{M}_2)' \to X'$ to $H_1^2 = g_2 \circ f|(\overline{M}_2)'$ satisfying that $\overline{M}_1 \cap \overline{M}_2 = \phi$ (Note that $\overline{M}_1 \cap (B_2 - M_1) = \phi$), $B_1 \cup B_2 \subset M_1 \cup M_2$, $\partial M_2 \cap (\bigcup_{n=1}^{\infty} B_n) = \phi$, $g_2 = f^{-1}$ on ∂M_2, $H^2(x, t) = x$ for any $(x, t) \in (\partial M_2)' \times I$, and $\{H^2(\{x\} \times I) | x \in (\overline{M}_2)'\}$ refines $\mathcal{L}_2'$.

Now by induction we claim that for each $n = 1, 2, \ldots,$ there exist an open set M_n of X, a map $g_n:\overline{M}_n \to X'$, a homotopy $H^n:(\overline{M}_n)' \times I \to X'$ from $H_0^n =$ inclusion $(\overline{M}_n)' \to X'$ to $H_1^n = g_n \circ f|(\overline{M}_n)'$, and an open refinement $\mathcal{L}_n$ of $\mathcal{M}$ with mesh $< \frac{1}{n}$ satisfying (a), (b), (c), and (d).

Now define $g:X \to X'$ as $g(x) = g_n(x)$ for $x \in M_n$ $(n = 1, 2, \ldots)$ and $g(x) = f^{-1}(x)$ otherwise, and define $H:X' \times I \to X'$ as $H(x, t) = H^n(x, t)$ for $(x, t) \in (M_n)' \times I$ $(n = 1, 2, \ldots)$ and $H(x, t) = x$ otherwise. Then it can be shown that g and H are continuous [In fact we give a proof in chapter III where the hypotheses are weaker]. Furthermore H is a homotopy between identity on X' and $g \circ f$ limited by $\mathcal{M}'$. Hence f is a hereditary shape equivalence.

A sequence of subsets of a space is a <u>null-sequence</u> provided for any $\varepsilon > 0$ only finitely many members of the sequence have diameter $\geq \varepsilon$.

As a corollary of Theorem 1 we have the following.

COROLLARY 1.1. (G. Kozlowski [K2]) If $f:X' \to X$ is a cell-like map between metrizable spaces such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of X satisfying $N_f \subset \bigcup_{n=1}^{\infty} B_n$, $\{B_n\}_{n=1}^{\infty}$ forms a null-sequence, and $f|B_n'$ is a hereditary shape equivalence for each B_n, then f is a hereditary shape equivalence.

PROOF. It follows from the following lemma.

LEMMA 2.1.1. Suppose $f:X' \to X$ is a map and there is a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X such that $\{B_n'\}_{n=1}^{\infty}$ forms a pairwise disjoint null-sequence. Then for any $n = 1, 2, \ldots$ and any neighborhood U of B_n there exists a neighborhood W of B_n such that $\overline{W} \subset U$ and $\partial W \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$.

PROOF. Consider the decomposition $G = \{B_n' | n = 1, 2, \ldots\} \cup \{\{x\} | x \in X' - \bigcup_{n=1}^{\infty} B_n'\}$ of X' and the decomposition space X'/G. Let $\pi: X' \to X'/G$ be the decomposition map.

We can easily show π is a closed map since $\{B_n'\}_{n=1}^{\infty}$ forms a null-sequence. Hence X'/G is a normal space. Now using Urysohn's functions we can prove the lemma.

Also by the same analysis as in the proof of Corollary 1.1, we have the following.

COROLLARY 1.2. If $f:X' \to X$ is a cell-like map between metrizable spaces such that there is a sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of X satisfying $N_f \subset \bigcup_{n=1}^{\infty} B_n$, $\{B_n\}_{n=1}^{\infty}$ forms a null-sequence, and $f|B_n'$ is a hereditary shape equivalence for each B_n, then f is a hereditary shape equivalence.

REMARK 1. Even though we state Corollary 1.1 and Corollary 1.2 for null-sequences, we immediately see "better results" as follows. A

cell-like map $f:X' \to X$ between metrizable spaces is a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of X such that $N_f \subset \bigcup_{n=1}^{\infty} B_n$, $f|B_n'$ is a hereditary shape equivalence for each B_n, and either of the following two conditions is satisfied:

(1) the decomposition map $\pi':X' \to X'/G'$ is closed, where $G' = \{B_n | n \in N\} \cup \{\{x\} | x \in X' - \bigcup_{n=1}^{\infty} B_n'\}$.

(2) the decomposition map $\pi:X \to X/G$ is closed, where $G = \{B_n | n \in N\} \cup \{\{x\} | x \in X - \bigcup_{n=1}^{\infty} B_n\}$.

REMARK 2. Even though G. Kozlowski states his theorem for the pairwise disjoint null-sequence $\{B_n'\}_{n=1}^{\infty}$, his proof gives a "better theorem" as follows.

THEOREM (G. Kozlowski [K2]). If $f:X' \to X$ is a cell-like map from a compact ANR X' onto a metric space X and there is a sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of X such that $N_f \subset \bigcup_{n=1}^{\infty} B_n$, for each $x_n \in B_n$ $\{f^{-1}(x_n)\}_{n=1}^{\infty}$ forms a null sequence, and each $f|B_n'$ is a hereditary shape equivalence, then f is a hereditary shape equivalence. But his "better theorem" doesn't follow as a corollary of Theorem 1 as his original theorem does.

Now we present Theorem 2 which generalizes Theorem 1 with hypotheses "close to being minimal". We prove Theorem 2 as a corollary of Theorem 1.

THEOREM 2. If $f:X' \to X$ is a cell-like map between compact metrizable spaces such that there are a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X and an integer k such that, for each $n = 1,2,\ldots,$

(1) $N_f \subset \bigcup_{i=1}^{\infty} B_i$,

(2) $f|B'_n$ is a hereditary shape equivalence, and

(3) for each neighborhood U of B_n, there exists a neighborhood V of B_n such that $\overline{V} \subset U$ and $\dim(\partial V \cap (\bigcup_{i=1}^{\infty} B_i)) \leq k$, then f is a hereditary shape equivalence.

<u>PROOF</u>. For each B_n and each integer $m > 0$, choose a neighborhood $V_{n,m}$ of B_n such that $\dim(\partial V_{n,m} \cap (\bigcup_{i=1}^{\infty} B_i)) \leq k$ and $\overline{V}_{n,m} \subset N(B_n; \frac{1}{m}) = \{x \in X | d(B_n, x) < \frac{1}{m}\}$.

Let $K = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} [\partial V_{n,m} \cap (\bigcup_{i=1}^{\infty} B_i)]$. Notice that $\dim K \leq k$. Recall a theorem of L. Tumarkin's which states that a finite dimensional subset of a metric space is contained in a G_δ-subset of the same dimension [Na. Theorem II.10, page 32]. Now choose a subset L of X with the dimension $\leq k$ such that $X - L = \bigcup_{i=1}^{\infty} F_i$ for closed subsets F_i's of X and $K \subset L$. By a theorem of F.D. Ancel's [An. Theorem 5.1], to show f is a hereditary shape equivalence, it is enough to show that $f|F'_n : F'_n \to F_n$ is a hereditary shape equivalence for each $n = 1, 2, \ldots$.

We will show that for each $i = 1, 2, \ldots$, $f|F'_i : F'_i \to F_i$ and the sequence $\{B_n \cap F_i\}_{n=1}^{\infty}$ of closed subsets of F_i satisfy the hypotheses in Theorem 1. The hypothesis (1) is certainly satisfied, and the hypothesis (2) is satisfied by the definition of a hereditary shape equivalence. Suppose U is a neighborhood of $B_n \cap F_i$ (in F_i) and W is an open subset of X such that $U = W \cap F_i$. Then $(B_n - W) \cap F_i = \phi$, hence we can find a neighborhood V of $B_n - W$ (in X) such that $\overline{V} \cap F_i = \phi$. Then certainly we have $B_n \subset W \cup V$. Using compactness of X we get a neighborhood $V_{n,m}$ of B_n such that $\overline{V}_{n,m} \subset W \cup V$. Hence $B_n \cap F_i \subset V_{n,m} \cap F_i \subset (W \cup V) \cap F_i = W \cap F_i = U$.

Furthermore, $\partial(V_{n,m} \cap F_i)$ (in F_i) $= \partial V_{n,m} \cap F_i$. Therefore $[\partial(V_{n,m} \cap F_i)$ (in $F_i)] \cap [\bigcup_{j=1}^{\infty}(B_j \cap F_i)] = [\partial V_{n,m} \cap F_i] \cap [\bigcup_{j=1}^{\infty} B_j] \subset K \cap F_i \subset L \cap F_i = \phi$. Therefore the hypothesis (3) in Theorem 1 is satisfied. Hence each $f|F_i':F_i' \to F_i$ is a hereditary shape equivalence, and hence f is a hereditary shape equivalence.

Early in the introduction we raised two questions.

Question A. Is a cell-like map a hereditary shape equivalence if the non-degeneracy set is contained in the countable union of pairwise disjoint finite dimensional closed sets?

Question B. Is a cell-like map $f:X' \to X$ a hereditary shape equivalence if there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of finite dimensional closed subsets of X such that the non-degeneracy set is contained in $\bigcup_{n=1}^{\infty} B_n$ and $\bigcup_{n\neq m}(B_n \cap B_m)$ has a strong transfinite dimension? Here we present the proof that Question A and Question B are equivalent. We prove it using the same analysis as in the proof of Theorem 2. Before we give the proof, we recall the definition of a strong transfinite dimension. The definition of a strong transfinite dimension, Ind, is given inductively. (See [Na]). Ind $X = -1$ provided X is the empty space. Ind $X \leq \alpha$ for an ordinal α provided for each closed subset A of X and a neighborhood U of A there exists a neighborhood V of A such that $V \subset U$ and $\text{Ind}(\partial V) \leq \beta$ for some ordinal $\beta < \alpha$. The space X is said to have a strong transfinite dimension provided there is an ordinal α such that Ind $X \leq \alpha$. Also we recall a theorem of F.D. Ancel's [An] which states that if a subset S of a metric space X has a strong transfinite dimension, then S is contained in a countable dimensional G_δ-subset of X . Certainly an

affirmative answer to Question B provides an affirmative answer to Question A. Assume an affirmative answer to Question A. Using the aforementioned theorem of Ancel's, find a countable dimensional subset K of X such that $X - K = \bigcup_{i=1}^{\infty} F_i$ for some closed subsets F_i's of X. According to a theorem of Ancel's [An. Theorem 5.1], to claim f is a hereditary shape equivalence, it is enough to show that $f|F_i':F_i' \to F_i$ is a hereditary shape equivalence for each $i = 1, 2, \ldots$. But for each F_i the sequence $\{B_n \cap F_i\}_{n=1}^{\infty}$ and the map $f|F_i':F_i' \to F_i$ satisfy all the hypotheses in Question A. Therefore an affirmative answer to Question A gives an affirmative answer to Question B.

2.2. MAPS DETERMINED ON SEQUENCES WITH EMPTY INFINITE INTERSECTIONS

In this section we have a theorem which generalizes Theorem 1. But the theorem is quite different from Theorem 2.

THEOREM 3. If $f:X' \to X$ is a cell-like map from an ANR X' onto a metric space X such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) $f|B_n':B_n' \to B_n$ is a hereditary shape equivalence for each $n = 1, 2, \ldots$, and

(3) for each B_n and a neighborhood U of B_n, there exists a neighborhood V of B_n such that $\overline{V} \subset U$ and ∂V intersects at most finitely many of $\{B_i\}_{i=1}^{\infty}$,

then f is a hereditary shape equivalence.

PROOF. As in the proof of Theorem 1, it is enough to show that for any open cover $\mathcal{M}$ of X there exists a map $g:X \to X'$ such that $g \circ f$

is homotopic to the identity on X' with the homotopy limited by $\mathcal{M}'$. We will show it in 5 steps.

Let $\mathcal{M}$ be an open cover of X. As in the proof of Theorem 1, find a neighborhood U_0 of f^{-1} in $X \times X'$ such that $\{U_0(x) | x \in X\}$ refines $\mathcal{M}'$.

Step 1. It easily follows from a lemma of F.D. Ancel's [An. 3.5] that for each $n = 1, 2, \ldots$ there exist an open cover $\mathcal{L}_n$ of X, a neighborhood U_n of f^{-1} in $X \times X'$, and a neighborhood M_n of B_n such that

(1) mesh $\mathcal{L}_n < \frac{1}{n}$ and $\mathcal{L}_n'$ refines $\{U_{n-1}(x) | x \in X\}$,

(2) $U_n \subset U_{n-1}$ and $\{U_n(x) | x \in X\}$ refines $\mathcal{L}_n'$, and

(3) $U_n | M_n$ slice-contracts in U_{n-1}.

Step 2. For each $n = 1, 2, \ldots$ we will find open subsets K_n and H_n of X and a slice-contraction χ_n such that

(2.1) $\overline{K}_n \subset H_n \subset \overline{H}_n \subset M_n$, $\bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} K_i$, $\overline{K}_n$ intersects at most finitely many of $\{\overline{H}_i\}_{i=1}^{\infty}$ and

(2.2) a slice-contraction $\chi_n : [\bigcup_{i=1}^{n} (U_i | \overline{K}_i)] \times I \to \bigcup_{i=1}^{n} (U_{i-1} | \overline{K}_i)$ such that $\chi_n = \chi_{n-1}$ on $(U_i | \overline{K}_i) \times I$ if $B_n \cap \partial K_i = \phi$ for $i < n$.

We will show it by induction.

For $B_1 \subset M_1$, we simply choose open subsets K_1 and H_1 such that $B_1 \subset K_1 \subset \overline{K}_1 \subset H_1 \subset \overline{H}_1 \subset M_1$ and ∂K_1 intersects at most finitely many of $\{B_i\}_{i=1}^{\infty}$. Certainly there is a slice-contraction $\chi_1 : (U_1 | \overline{K}_1) \times I \to U_0 | \overline{K}_1$.

Let $k > 0$, and assume that for each $n = 1, 2, \ldots, k$ there exist open subsets H_n and K_n of X and a map χ_n such that

(2.1)' $\overline{K}_n \subset H_n \subset \overline{H}_n \subset M_n$, $\bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} K_i$, ∂K_n intersects at most finitely many of $\{B_i\}_{i=1}^{\infty}$, and $\overline{H}_n \cap \overline{K}_i = \phi$ if $B_n \cap \partial K_i = \phi$ for $i < n$ and

(2.2)' $\chi_n:[\bigcup_{i=1}^{n} (U_i|\overline{K}_i)] \times I \to \bigcup_{i=1}^{n} (U_{i-1}|\overline{K}_i)$ is so that $\chi_n = \chi_{n-1}$ on $(U_i|\overline{K}_i) \times I$ if $B_n \cap \partial K_i = \phi$ for $i < n$.

Consider $B_{k+1} \subset M_{k+1}$. If $B_{k+1} \subset \bigcup_{i=1}^{k} K_i$, simply we choose $K_{k+1} = H_{k+1} = \phi$ and $\chi_{k+1} = \chi_k$. Otherwise, first we choose open subsets H_{k+1} and K_{k+1} of X such that $\overline{K}_{k+1} \subset H_{k+1} \subset \overline{H}_{k+1} \subset M_{k+1}$, $\bigcup_{i=1}^{k+1} B_i \subset \bigcup_{i=1}^{k+1} K_i$, $\overline{H}_{k+1} \cap \overline{K}_i = \phi$ if $\partial K_i \cap B_{k+1} = \phi$ for $i < k+1$, and ∂K_{k+1} intersects at most finitely many of $\{B_i\}_{i=1}^{\infty}$, and then we define $\chi_{k+1}:[\bigcup_{i=1}^{k+1} (U_i|\overline{K}_i)] \times I \to \bigcup_{i=1}^{k+1} (U_{i-1}|\overline{K}_i)$ as follows.

$$\chi_{k+1}(x, y, t) = \begin{cases} \chi_k(x, y, t) & \text{on } \{\bigcup_{i=1}^{k+1} U_i|(\overline{K}_i - \overline{H}_{k+1})\} \times I \\ \chi_k(\psi(x, y, \lambda(x)\cdot t), \mu(x)\cdot t) & \\ \qquad \text{on } \{\bigcup_{i=1}^{k+1} U_i|[\overline{K}_i \cap (\overline{H}_{k+1} - K_{k+1})]\} \times I & \\ \psi(x, y, t) & \text{on } \{U_{k+1}|\overline{K}_{k+1}\} \times I , \end{cases}$$

where ψ is a slice-contraction of $U_{k+1}|\overline{H}_{k+1}$ in $U_k|\overline{H}_{k+1}$ and $\mu, \lambda:X \to [0, 1]$ are maps such that $\lambda(X - H_{k+1}) = \{0\}$, $\mu(\overline{K}_{k+1}) = \{0\}$, and $\mu^{-1}(1) \cup \lambda^{-1}(1) = X$. Then we easily see that for each $n = 1, 2, \ldots, k+1$ (2.1)' and (2.2)' are satisfied by $\{K_n\}_{n=1}^{k+1}$, $\{H_n\}_{n=1}^{k+1}$, and $\{\chi_n\}_{n=1}^{k+1}$. Now we can easily claim (2.1) and (2.2) .

Step 3. For each $n = 1, 2, \ldots$, we define a slice-contraction $\phi_n:[\bigcup_{i=1}^{n} (U_i|\overline{K}_i)] \times I \to \bigcup_{i=1}^{n} (U_{i-1}|\overline{K}_i)$ as $\phi_n(x, y, t) = \chi_{n(0)}(x, y, t)$ where $n(0)$ is the smallest integer satisfying $\overline{K}_i \cap \overline{H}_m = \phi$ for any $i \leq n$ and $m > n(0)$. Using (2.1) and (2.2) we can show that each

ϕ_n is well-defined and continuous. Furthermore, $\phi_n = \phi_{n-1}$ on $\bigcup_{i=1}^{n-1}(U_i|\overline{K}_i) \times I$ for each $n = 1, 2, \ldots$.

Step 4. Let $K = \bigcup_{i=1}^{\infty} K_i$ and $U = \bigcup_{i=1}^{\infty}(U_{i-1}|\overline{K}_i)$. By induction we will show that for each $n = 1, 2, \ldots$, there exist a map $g_n : \bigcup_{i=1}^{n} K_i \to X'$ and a homotopy $F^n : (\bigcup_{i=1}^{n} \overline{K}_i)' \times I \to X'$ such that

$$(4.1) \quad g_n = g_{n-1} \text{ on } \bigcup_{i=1}^{n-1} \overline{K}_i \text{ and } g_n = f^{-1} \text{ on } (\bigcup_{i=1}^{n} \overline{K}_i) - K$$

and

$$(4.2) \quad F_0^n = \text{incusion } (\bigcup_{i=1}^{n} \overline{K}_i)' \to X' \ , \ F_1^n = g_n \circ f|(\bigcup_{i=1}^{n} \overline{K}_i)' \ ,$$

$$F^n(x, t) = x \text{ for each } (x, t) \in (\bigcup_{i=1}^{n} \overline{K}_i - K)' \times I \ ,$$

$$F^n = F^{n-1} \text{ on } (\bigcup_{i=1}^{n-1} \overline{K}_i)' \times I \ , \text{ and } F^n(\{x\} \times I) \subset U(f(x))$$

$$\text{for any } x \in (\bigcup_{i=1}^{n} \overline{K}_i)' \ .$$

Let ϕ_n slice-contract $\bigcup_{i=1}^{n} U_i|\overline{K}_i$ onto g_n' in $U = \bigcup_{i=1}^{\infty}(U_{i-1}|\overline{K}_i)$. Then , by a proposition of F.D. Ancel's [An. 2.1. (1)], we find that for each $n = 1, 2, \ldots$ g_n' is a continuous function and $g_n' = g_{n+1}'$ on $\bigcup_{i=1}^{n}(U_i|\overline{K}_i)$.

Let $\pi : X \times X' \to X'$ be the projection map.

Notice that $f^{-1}|(\overline{K}_1 - K) : \overline{K}_1 - K \to X'$ is a continuous map and $G' : (\overline{K}_1 - K) \times I \to X'$ defined by $G'(x, t) = \pi \circ \phi_1(x, f^{-1}(x), t)$ is a homotopy from $G_0' = f^{-1}|(\overline{K}_1 - K)$ to $G_1' = g_1'|(\overline{K}_1 - K)$ such that $G'(\{x\} \times I) \subset U(x)$ for each $x \in \overline{K}_1 - K$. Hence, by a modified homotopy extension property of an ANR X' , we obtain an extension $\overline{G}' : \overline{K}_1 \times I \to X'$ of G' such that $\overline{G}'(\{x\} \times I) \subset U(x)$ for each $x \in \overline{K}_1$ Define $g_1 : \overline{K}_1 \to X'$ as $g_1(x) = \overline{G}'(x, 0)$. Let $P_1 = [(\overline{K}_1 - K)' \times$

$I \times I] \cup [(\overline{K}_1)' \times \{0, 1\} \times I] \cup [(\overline{K}_1)' \times I \times \{1\}]$, and define $\Lambda_1 : P_1 \to X'$ as follows.

$$\Lambda_1(x, s, t) = \begin{cases} \pi o \phi_1(f(x), x, st) & \text{on } (\overline{K}_1 - K)' \times I \times I \\ x & \text{on } (\overline{K}_1)' \times \{0\} \times I \\ \overline{G}'(f(x), t) & \text{on } (\overline{K}_1)' \times \{1\} \times I \\ \pi o \phi_1(f(x), x, s) & \text{on } (\overline{K}_1)' \times I \times \{1\} . \end{cases}$$

Then we can easily check that Λ_1 is a well-defined continuous function. By a modified homotopy extension property of an ANR X' , we get an extension $\overline{\Lambda}_1 : (\overline{K}_1)' \times I \to X'$ of Λ_1 such that $\overline{\Lambda}_1(\{x\} \times I \times I) \subset U(f(x))$ for each $x \in (\overline{K}_1)'$. Now we define $F^1 : (\overline{K}_1)' \times I \to X'$ as $F'(x, s) = \overline{\Lambda}_1(x, s, 0)$. Then $F'(x, 0) = x$ and $F'(x, 1) = g_1 o f(x)$ for any $x \in (\overline{K}_1)'$ and $F'(x, s) = x$ for any $(x, s) \in (\overline{K}_1 - K)' \times I$. Furthermore, $F'(\{x\} \times I) \subset U(f(x))$ for each $x \in (\overline{K}_1)'$.

Now we will show how to get g_2 and F^2 . First we notice that the function $g_1 \cup f^{-1}|(\overline{K}_2 - K) : \overline{K}_1 \cup (\overline{K}_2 - K) \to X'$ defined as g_1 on $\overline{K}_1$ and $f^{-1}|(\overline{K}_2 - K)$ on $\overline{K}_2 - K$ is well-defined and continuous. Define $G^2 : [\overline{K}_1 \cup (\overline{K}_2 - K)] \times I \to X'$ as $G^2 = G^1$ on $\overline{K}_1 \times I$ and $G^2(x, t) = \pi o \phi_2(x, f^{-1}(x), t)$ for $(x, t) \in (\overline{K}_2 - K) \times I$. Then G^2 is a homotopy from $g_1 \cup f^{-1}|(\overline{K}_2 - K)$ to g_2' such that $G^2(\{x\} \times I) \subset U(x)$ for each $x \in \overline{K}_1 \cup (\overline{K}_2 - K)$. Hence, by a modified homotopy extension property of an ANR , there is an extension $\overline{G}^2 : (\overline{K}_1 \cup \overline{K}_2) \times I \to X'$ of G^2 such that $\overline{G}^2(\{x\} \times I) \subset U(x)$ for each $x \in \overline{K}_1 \cup \overline{K}_2$. We define $g_2 : \overline{K}_1 \cup \overline{K}_2 \to X'$ as $g_2(x) = \overline{G}^2(x, 0)$.

Then g_2 is an extension of g_1 and $g_2 = f^{-1}$ on $(\overline{K}_1 \cup \overline{K}_2) - K$. Let $P_2 = \{[\overline{K}_1 \cup (\overline{K}_2 - K)]' \times I \times I\} \cup \{(\overline{K}_1 \cup \overline{K}_2)' \times \{0, 1\} \times I\}$ $\cup \{(\overline{K}_1 \cup \overline{K}_2)' \times I \times \{1\}\}$, and define $\Lambda_2 : P_2 \to X'$ as follows.

$$\Lambda_2(x, s, t) = \begin{cases} \pi \circ \phi_2(f(x), x, st) & \text{on } (\overline{K}_2 - K)' \times I \times I \\ x & \text{on } (\overline{K}_2)' \times \{0\} \times I \\ \overline{G}^2(f(x), t) & \text{on } (\overline{K}_2)' \times \{1\} \times I \\ \pi \circ \phi_2(f(x), x, s) & \text{on } (\overline{K}_2)' \times I \times \{1\} \\ \overline{\Lambda}_1(x, s, t) & \text{on } (\overline{K}_1)' \times I \times I \end{cases}.$$

Then we can easily show that Λ_2 is a well-defined continuous function that extends $\overline{\Lambda}_1$. Using a modified homotopy extension property of an ANR, we obtain an extension $\overline{\Lambda}_2 : (\overline{K}_1 \cup \overline{K}_2)' \times I \times I \to X'$ of Λ_2 such that $\overline{\Lambda}_2(\{x\} \times I \times I) \subset U(f(x))$ for any $x \in (\overline{K}_1 \cup \overline{K}_2)'$. Now we define $F^2 : (\overline{K}_1 \cup \overline{K}_2)' \times I \to X'$ as $F^2(x, s) = \overline{\Lambda}_2(x, s, 0)$. Then we can show that F^2 is an extension of F^1, F^2_0 = inclusion, and $F^2_1 = g_2 \circ f | (\overline{K}_1 \cup \overline{K}_2)'$. Furthermore, $F^2(\{x\} \times I) \subset U(f(x))$ for each $x \in (\overline{K}_1 \cup \overline{K}_2)'$ and $F^2_1(x, t) = x$ for any $(x, t) \in [(\overline{K}_1 \cup \overline{K}_2)' - K'] \times I$.

Now, by induction, we claim that for each $n = 1, 2, \ldots$ there exist a map $g_n : \bigcup_{i=1}^{n} \overline{K}_i \to X'$ and a homotopy $F^n : (\bigcup_{i=1}^{n} \overline{K}_i)' \times I \to X'$ satisfying (4.1) and (4.2).

Step 5. Finally we define $g : X \to X'$ as

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in K_n \text{ for some } n = 1, 2, \ldots \\ f^{-1}(x) & \text{if } x \in X - \bigcup_{i=1}^{\infty} K_i \end{cases},$$

and we define $F : X' \times I \to X'$ as

$$F(x,\ t) = \begin{cases} F^n(x,\ t) \text{ if } (x,\ t) \in (K_n)' \times I \\ \qquad\qquad\qquad \text{for some } n = 1,\ 2,\ \ldots \\ x \qquad \text{if } (x,\ t) \in [X' - (\bigcup_{i=1}^{\infty} K_i)'] \times I\ . \end{cases}$$

We will show g is continuous. First notice that it is enough to show that, for any sequence $\{x_n\}_{n=1}^{\infty}$ in K converging to x in ∂K, $\{g(x_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(x)$. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in K converging to x in ∂K. If there is K_i such that K_i contains infinitely many $\{x_{i(n)}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, then $g(x_{i(n)}) = g_i(x_{i(n)})$ and $\{g_i(x_{i(n)})\}_{n=1}^{\infty}$ converges to $g_i(x) = f^{-1}(x)$. Since each $\overline{K}_i$ intersects at most finitely many of $\{\overline{K}_n\}_{n=1}^{\infty}$ (see (2.1)), there are at most fintely many K_i's which contain infinitely many of $\{x_n\}_{n=1}^{\infty}$. Hence it is enough to show that, for any sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in K_n$ converging to x in ∂K, $\{g(x_n) = g_n(x_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(x)$. Consider a neighborhood $B(f^{-1}(x);\ \varepsilon) = \{y \in X' \mid d(f^{-1}(x),\ y) < \varepsilon\}$ for some $\varepsilon > 0$. Since f is proper, there exists a neighborhood $B(x;\ \delta)$ of x in X for some $\delta > 0$ such that $f^{-1}(B(x;\ \delta)) \subset B(f^{-1}(x);\ \varepsilon)$. Choose an integer n_1 such that $x_n \in B(x;\ \frac{\delta}{2})$ for any $n \geq n_1$, and an integer n_2 such that $n_2 \geq n_1$ and $\frac{1}{n_2} < \frac{\delta}{2}$. Since each $\overline{K}_i$ intersects at most fintely many of $\{\overline{K}_n\}_{n=1}^{\infty}$, we can find an integer n_3 such that $n_3 \geq n_2$ and $\overline{K}_i \cap (\bigcup_{n=n_3}^{\infty} K_n) = \phi$ for any $i \leq n_2$. If $n \geq n_3$, then $x_n \notin \bigcup_{i=1}^{n_2} \overline{K}_i$. Let $n(0)$ be the smallest integer such that $x_n \in \overline{K}_{n(0)}$. Notice that for any $n \geq n_3$ $n(0) \geq n_2 + 1$ and $U(x_n) = U_{n(0)-1}(x_n) \subset f^{-1}(L)$ for some $L \in \mathcal{L}_{n(0)-1}$. Since $f^{-1}(x_n) \cup g(x_n) \subset U(x_n)$, we have $\{x_n,\ f\circ g(x_n)\} \subset L$ and

diam $L < \frac{1}{n(0)-1} \leq \frac{1}{n_2} < \frac{\delta}{2}$ for $n \geq n_3$. Therefore for $n \geq n_3$ $d(f \circ g(x_n), x_n) \leq d(f \circ g(x_n), x_n) + d(x_n, x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Hence for $n \geq n_3$ $f \circ g(x_n) \in B(x; \delta)$ and $g(x_n) \in f^{-1}(B(x; \delta)) \subset B(f^{-1}(x); \varepsilon)$. Therefore g is continuous.

The continuity of F can be checked by the same analysis that established the continuity of g.

Finally we can claim that F is a homotopy from the identity on X' to $g \circ f$ limited by M' by noticing that $F(\{x\} \times I) \subset U(f(x)) \subset U_0(f(x))$ and $\{U_0(x) | x \in X\}$ refines M'. Therefore f is a hereditary shape equivalence.

CHAPTER III

UV^∞-MAPS

In this chapter we generalize some theorems for cell-like maps to theorems for (non-proper) UV^∞-maps.

Throughout this chapter $f:X' \to X$ is a map between metric spaces unless it is specified otherwise.

In section 3.1 we recall two definitions. First we give the definition of f being a hereditary shape equivalence that generalizes G. Kozlowski's notion of a hereditary shape equivalence for a proper map. Then we have the definition of f being a fine homotopy equivalence over a closed subset A of X which generalizes Kozlowski's notion of $f|A'$ being a hereditary shape equivalence for a proper map.

In section 3.2 we generalize Theorem 1 for a UV^∞-map by adopting the above definitions and by modifying the proof of Theorem 1.

In section 3.3, we have Theorem 5 which extends a result of G. Kozlowski's [K2] . The proof is given there for maps between separable metric spaces, while the proof in the case of maps between (non-separable) metric spaces is given in the appendix.

As an important corollary of Theorem 5, we get the following.

<u>THEOREM 6</u>. For a UV^∞-map $f:X' \to X$ from an ANR X' to a metric space X the following are equivalent:

(1) X is an ANR .

(2) f is a hereditary homotopy equivalence.

(3) f is a hereditary shape equivalence.

(4) f is a fine homotopy equivalence.

Finally in section 3.4 we improve a theorem of G. Kozlowski's [K2] to a theorem for a (non-proper) UV^{∞}-map.

Before we precede further, let us recall and introduce definitions and notation.

$S' = f^{-1}(S)$ for the map $f: X' \to X$ and a subset S of X.

For a map $g: X \to Y$ between metric spaces and a closed subset A of Y, the <u>adjunction space</u> $X \underset{g}{\cup} A$ <u>obtained from</u> g <u>and</u> A is the space defined as follows. As a set $X \underset{g}{\cup} A = (X - g^{-1}(A)) \cup A$, and the topology on $X \underset{g}{\cup} A$ is generated by the open sets in $X - g^{-1}(A)$ together with sets of the form $g^{-1}(U - A) \cup (U \cap A)$ for open subsets U of X.

$M(f)$ = the adjunction space $X' \times [0, 1] \underset{(f,id)}{\cup} X \times \{1\}$ obtained from $(f, id): X' \times I \to X \times I$ and $X \times \{1\}$, where $(f, id)(x, t) = (f(x), t)$ for any $(x, t) \in X' \times I$.

$DM(f)$ = the adjunction space $X' \times [-1, 1] \underset{(f,id)}{\cup} X \times \{\pm 1\}$ obtained from $(f, id): X' \times [-1, 1] \to X \times [-1, 1]$ and $X \times \{\pm 1\}$, where $(f, id)(x, t) = (f(x), t)$ for any $(x, t) \in X' \times [-1, 1]$.

$f': DM(f) \to X \times [-1, 1]$ denotes the "natural map" defined as $f'(x, t) = (f(x), t)$ for $(x, t) \in X' \times (-1, 1)$ and $f'(x, t) = (x, t)$ for $(x, t) \in X \times \{\pm 1\}$.

$f_A: X' \to X' \underset{f}{\cup} A$ and $\overline{f}_A: X' \underset{f}{\cup} A \to X$ are the "natural maps".

$f'_A: DM(f_A) \to (X' \underset{f}{\cup} A) \times [-1, 1]$ is the "natural map".

$M(f:B) = \{(x, t) \in M(f) \mid x \in B \text{ or } x \in B'\}$ for a subset B of X.

$DM(f:B) = \{(x, t) \in DM(f) \mid x \in B \text{ or } x \in B'\}$ for a subset B of X.

We shall not distinguish between a subset and an embedding. For example we identify X' with $X' \times \{0\}$ in either $M(f)$ or $DM(f)$ and X with $X \times \{1\}$ in $M(f)$.

$\mathcal{L}' = \{L' | L \in \mathcal{L}\}$, $\overline{\mathcal{L}} = \{\overline{L} | L \in \mathcal{L}\}$, and $\overline{\mathcal{L}'} = \{\overline{L'} | L \in \mathcal{L}\}$ for any family of subsets of X and $f:X' \to X$.

3.1. INTRODUCTION AND PRELIMINARIES

Recall a map $f:X' \to X$ is a UV^{∞}-map provided for each $x \in X$ and each neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and the inclusion $V' \to U'$ is null-homotopic.

As we see in the example of the inclusion $(0, 1) \to [0, 1]$, a UV^{∞}-map does not force surjectivity of the map unlike a cell-like map which forces surjectivity of the map. But we have the following proposition for a UV^{∞}-map.

PROPOSITION 3.1. If $f:X' \to X$ is a UV^{∞}-map, then $f(X')$ is dense in X.

PROOF. Suppose $f(X')$ is not dense in X. Then we can find a non-empty open subset U of X such that $U' = \phi$. Then for any $x \in U$ and any open subset V of X with $x \in V \subset U$ the inclusion $V' = \phi \to U' = \phi$ can not be null-homotopic. Therefore $f(X')$ is dense in X.

Recall a map $f:X' \to X$ between metric spaces is a hereditary shape equivalence provided for each open subset U of X $f|U':U' \to U$ is a shape equivalence. G. Kozlowski [K2] introduced the notion of a hereditary shape equivalence for a proper map between metric spaces; namely, a map $f:X' \to X$ between metric spaces is a hereditary shape

equivalence provided for each closed subset A of X $f|A':A' \to A$ is a shape equivalence. Necessarily, his notion of a hereditary shape equivalence forces hereditary shape equivalent maps to be surjective. Since a UV^∞-map does not force surjectivity of the map, certainly the stereotype of his definition would not work for a UV^∞-map. But Kozlowski [K2] showed that if a cell-like $f:X' \to X$ of an ANR X' is a hereditary shape equivalence, then for any subset S of X $f|S':S' \to S$ is a shape equivalence. Also implicitly he showed that if a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X is such that for any open subset U of X $f|U':U' \to U$ is a shape equivalence, then for any closed subset A of X $f|A':A' \to A$ is a shape equivalence. Therefore we get the following proposition which shows that our definition of a hereditary shape equivalence is a generalized notion of Kozlowski's hereditary shape equivalence.

PROPOSITION 3.2. For a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X the following are equivalent:

(1) f is a hereditary shape equivalence.

(2) for any closed subset A of X, $f|A':A' \to A$ is a shape equivalence.

Recall that a map $f:X' \to X$ between metric spaces is a fine homotopy equivalence over a closed subset A of X provided $f_A:X' \to X' \cup_f A$ is a fine homotopy equivalence. G. Kozlowski [K2] introduced the notion of $f|A':A' \to A$ being a hereditary shape equivalence for a map $f:X' \to X$ between metric spaces and a closed subset A of X. We would like to extend his notion to UV^∞-maps. Though the stereotype definition of Kozlowski's does not work for (non-surjective)

UV^∞-maps, we find an extension of this notion in F.D. Ancel's works [An] . Ancel showed that, for a cell-like map $f:X' \to X$ between metric spaces, the following are equivalent:

(1) f is a hereditary shape equivalence.

(2) for any closed embedding of X' to an ANR X'_+ , the trivial extension $f^+:X'_+ \to X'_+ \cup_f X$ of f is a fine homotopy equivalence; see [An, 4.5].

(3) there exists a closed embedding of X' to an ANR X'_+ such that the trivial extension $f^+: X'_+ \to X'_+ \cup_f X$ is a fine homotopy equivalence.

If we notice that, for a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X and a closed subset A of X , $f_A:X' \to X' \cup_f A$ is the trivial extension of $f|A':A' \to A$ arising from the closed embedding A' into an ANR X' , then immediately we get the following proposition which shows that the notion of a map $f:X' \to X$ between metric spaces being a fine homotopy equivalence over a closed subset A of X is an extended notion of $f|A':A' \to A$ being a hereditary shape equivalence that Kozlowski [K2] introduced.

PROPOSITION 3.3. For a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X and a closed subset A of X the following are equivalent:

(1) f is a fine homotopy equivalence over A .

(2) $f|A':A' \to A$ is a hereditary shape equivalence.

3.2. MAPS DETERMINED ON PAIRWISE DISJOINT SEQUENCES

In this section we generalize Theorem 1 to a theorem for UV^∞-maps.

Recall $B(x; \frac{1}{n})$ denotes the neighborhood $\{y \in X | d(x, y) < \frac{1}{n}\}$ of x for a point x in a metric space X and $n > 0$. Also recall that, for a map $f:X' \to X$ between metric spaces, N_f is the subset $\{x \in X |$ either $f^{-1}(x) = \phi$, or $f^{-1}(x)$ contains at least two points, or $f^{-1}(x) =$ point but $\{f^{-1}(B) | B \in \mathcal{B}\}$ is not a neighborhood basis for $f^{-1}(x)$, where $\mathcal{B}$ is a neighborhood basis for $x\}$.

Immediately we get the following proposition which shows that a UV^{∞}-map behaves "nicely" in the complement of the non-degeneracy set.

PROPOSITION 3.4. For a map $f:X' \to X$ between metric spaces, the restriction $f^{-1}|X - N_f: X - N_f \to X'$ is a continuous function.

The following two are modified homotopy extension properties of ANR's, which can be easily shown using compactness of I and $I \times I$, normality, and the homotopy extension property of ANR's.

PROPOSITION 3.5. If A is a closed subset of a metric space X, Y is an ANR, and $H:A \times I \cup X \times \{0\} \to Y$ is a map such that $\{H(\{x\} \times I) | x \in A\}$ refines an open cover $\mathcal{U}$ of Y, then there exists an extension $\overline{H}:X \times I \to Y$ of H such that $\{\overline{H}(\{x\} \times I) | x \in X\}$ refines $\mathcal{U}$.

PROPOSITION 3.6. If $H:(A \times I \times I) \cup (X \times \{0, 1\} \times I) \cup (X \times I \times \{1\}) \to Y$ is a map to an ANR Y and $\{H(\{x\} \times I \times I) | x \in A\} \cup \{H([\{x\} \times \{0, 1\} \times I] \cup [\{x\} \times I \times \{1\}]) | x \in X\}$ refines an open cover $\mathcal{U}$ of Y, where A is a closed subset of X, then there exists an extension $\overline{H}:X \times I \times I \to Y$ of H such that $\{\overline{H}(\{x\} \times I \times I) | x \in X\}$ refines $\mathcal{U}$.

Before we generalize Theorem 1, we need to present the following three lemmas.

LEMMA 3.2.1. If a map $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence over a closed subset B of

X, then for any open cover L of X, there exist a neighborhood V of B in X and a map $g:V \to X'$ such that $g \circ f|V'$ is homotopic to the inclusion $V' \to X'$ with the homotopy limited by L'.

PROOF. Let L be an open cover of X. Find an open star refinement M of L. Since $f_B:X' \to X' \underset{f}{\cup} B$ is a fine homotopy equivalence, there exists a map $h:X' \underset{f}{\cup} B \to X'$ such that $h \circ f_B$ is homotopic to the identity on X' with the homotopy limited by M'. Consider $h|B:B \to X'$, and regard B as a closed subset of X. Since X' is an ANR, $h|B$ has an extension $g:\overline{W} \to X'$ for some neighborhood W of B in X. Define $H:[(X' \underset{f}{\cup} B) \times \{0\}] \cup [B \times I] \cup [(f_B)^{-1}(\overline{W}) \times \{1\}] \to X'$ as $H(x, 0) = h(x)$ for $(x, 0) \in (X' \underset{f}{\cup} B) \times \{0\}$, $H(x, t) = h(x)$ for $(x, t) \in B \times I$, and $H(x, 1) = g \circ \overline{f}_B(x)$ for $(x, 1) \in (\overline{f}_B)^{-1}(\overline{W}) \times \{1\}$. Then there exist a neighborhood V of B in X and a map $\overline{H}:(\overline{f}_B)^{-1}(V) \times I \to X'$ such that $\overline{H}_0 = h$ on $(\overline{f}_B)^{-1}(V)$, $\overline{H}_1 = g \circ \overline{f}_B$ on $(\overline{f}_B)^{-1}(V)$, and $\overline{H}$ is limited by M'. Hence there is a homotopy from $g \circ f|V'$ to the inclusion $V' \to X'$ limited by L'.

LEMMA 3.2.2. Suppose $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence over a closed subset B of X and B is such that for any neighborhood W of B there exists a neighborhood V of B satisfying $\overline{V} \subset W$ and $\partial V \cap N_f = \phi$.

Then, for any open cover L of X, there exist a neighborhood V of B and a map $g:\overline{V} \to X$ such that $\partial V \cap N_f = \phi$, $g = f^{-1}$ on ∂V, and $g \circ f|(\overline{V})'$ is homotopic (rel. $(\partial V)'$) to the inclusion $(\overline{V})' \to X'$ with the homotopy limited by L'.

PROOF. Let L be an open cover of X, and choose an open star-refinement M of L. Using Lemma 3.2.1, we obtain a neighborhood W

of B and a map $h:W \to X'$ such that $h \circ f|W'$ is homotopic to the inclusion map $W' \to X'$ with the homotopy limited by $\mathcal{M}'$. Choose a neighborhood V of B such that $\overline{V} \subset W$ and $\partial V \cap N_f = \phi$. Let $g' = h|\overline{V}$, and let $G:(\overline{V})' \times I \to X'$ be a homotopy from $G_1 = g' \circ f|(\overline{V})'$ to $G_0 =$ inclusion $(\overline{V})' \to X'$ limited by $\mathcal{M}'$. Regard $f^{-1}|\overline{V}$ as a subset of $X \times X'$. Define $\phi:(f^{-1}|\overline{V}) \times I \to X'$ as $\phi(f(x), x, t) = G(x, t)$, and define $H:[(\partial V) \times I] \cup [\overline{V} \times \{1\}] \to X'$ as $H(x, t) = \phi(x, f^{-1}(x), t)$ for $(x, t) \in \partial V \times I$ and $H(x, t) = g'(x)$ for $(x, t) \in \overline{V} \times \{1\}$. Then by the modified homotopy extension property of ANR, H has an extension $\overline{H}:\overline{V} \times I \to X'$ limited by $\mathcal{M}'$. Define $g:\overline{V} \to X'$ as $g = \overline{H}_0$.

Let $P = [(f^{-1}|\partial V) \times I \times I] \cup [(f^{-1}|\overline{V}) \times \{0, 1\} \times I] \cup [(f^{-1}|\overline{V}) \times I \times \{1\}]$, and define $\psi:P \to X'$ as follows.

$$\psi(f(x), x, s, t) = \begin{cases} \phi(f(x), x, st) & \text{on } (f^{-1}|\partial V) \times I \times I \\ x & \text{on } (f^{-1}|\overline{V}) \times \{0\} \times I \\ \overline{H}(f(x), t) & \text{on } (f^{-1}|\overline{V}) \times \{1\} \times I \\ \phi(f(x), x, s) & \text{on } (f^{-1}|\overline{V}) \times I \times \{1\} \end{cases}$$

Then easily we can check ψ is a well-defined continuous function. Furthermore, we can check that $\{\psi(\{(f(x), x)\} \times I \times I) \mid (f(x), x) \in f^{-1}|\partial V\} \cup \{\psi(\{(f(x), x)\} \times [(\{0, 1\} \times I) \cup (I \times \{1\})]) \mid f(x), x) \in f^{-1}|\overline{V}\}$ refines $\mathcal{L}'$. Hence by the modified homotopy extension property of ANR's, ψ has an extension $\overline{\psi}:(f^{-1}|\overline{V}) \times I \times I \to X'$ such that $\{\psi(\{(f(x), x)\} \times I \times I) \mid (f(x), x) \in f^{-1}|\overline{V}\}$ refines $\mathcal{L}'$. Finally we define a homotopy $K:(\overline{V})' \times I \to X'$ as $K(x, s) = \overline{\psi}(f(x), x, s, 0)$ for each $(x, s) \in (\overline{V})' \times I$. Then

K_0 = inclusion $(\overline{V})' \to X'$, $K_1 = g \circ f|(\overline{V})'$, $K(x, s) = \overline{\psi}(f(x), x, s, 0) = \phi(f(x), x, 0) = G(x, 0) = x$ for each $(x, s) \in (\partial V)' \times I$, and K is limited by L' .

The following lemma is a variation of Lemma 3.2.2, which can be proved by the exact same analysis as Lemma 3.2.2. We state the lemma without a proof.

LEMMA 3.2.3. Suppose $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence over a closed subset A of X and B is a closed subset of A such that for any neighborhood W of B there exists a neighborhood V of B satisfying $\overline{V} \subset W$ and $\partial V \cap N_f = \phi$.

Then, for any open cover L of X , there exist a neighborhood V of B and a map $g:\overline{V} \to X'$ such that $\partial V \cap N_f = \phi$, $g = f^{-1}$ on ∂V , and $g \circ f|(\overline{V})'$ is homotopic (rel. $(\partial V)'$) to the inclusion $(\overline{V})' \to X'$ with the homotopy limited by L' .

Now we are ready to present the improved version of Theorem 1.

THEOREM 4. If $f:X' \to X$ is a UV^∞-map from an ANR X' to a metric space X such that there exists a sequence $\{B_n\}_{n=1}^\infty$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) f is a fine homotopy equivalence over each B_n ,

and

(3) for each B_n and each neighborhood W of B_n , there exists a neighborhood V of B_n such that $\overline{V} \subset W$ and $\partial V \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$,

then f is a fine homotopy equivalence.

PROOF. As in the proof of Theorem 1, we will show that for any open cover L of X there exists a map $g:X \to X'$ such that $g \circ f$ is homotopic to the identity map on X' with the homotopy limited by L' .

Let L be an open cover of X . First, by induction, we will show that, for each $n = 1,2,\ldots$, there exist an open subset V_n of X , an open refinement U_n of L , a map $g_n:\overline{V}_n \to X'$, and a homotopy $H^n:(\overline{V}_n)' \times I \to X'$ such that

(1) U_n has mesh $< \frac{1}{n}$,

(2) $\partial V_n \cap (\bigcup_{i=1}^{\infty} B_i) = \phi$, $\bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} V_i$, and $\{V_i\}_{i=1}^{n}$ is pairwise disjoint,

(3) $g_n = f^{-1}$ on ∂V_n ,

and

(4) H^n is a homotopy from H_0^n = inclusion $(\overline{V}_n)' \to X'$ to $H_1^n = g_n \circ f|(\overline{V}_n)'$ (rel. $(\partial V_n)'$) limited by U_n' .

Notice that immediately we get an open refinement U_n of L satisfying (1) for each $n = 1,2,\ldots$. Using the exact same analysis as in the proof of Theorem 1, Lemma 3.2.2 and Lemma 3.2.3 provide (2), (3), and (4) .

Define $g:X \to X'$ as $g(x) = g_n(x)$ for $x \in V_n$ $(n = 1,2,\ldots)$ and $g(x) = f^{-1}(x)$ for $x \in X - \bigcup_{i=1}^{\infty} V_i$, and define $H:X' \times I \to X'$ as $H(x, t) = H^n(x, t)$ for $(x, t) \in V_n' \times I$ $(n = 1,2,\ldots)$ and $H(x, t) = x$ for $(x, t) \in [X' - \bigcup_{i=1}^{\infty} V_i'] \times I$.

First we will show g is continuous. It is enough to show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\bigcup_{n=1}^{\infty} V_n$ converging to $x \in \partial(\bigcup_{n=1}^{\infty} V_n)$, then $\{g(x_n)\}_{n=1}^{\infty}$ converges to $g(x) = f^{-1}(x)$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\bigcup_{n=1}^{\infty} V_n$. If there exists V_m containing infinitely many of $\{x_n\}_{n=1}^{\infty}$, then $x \in \overline{V}_m$ and each V_n $(\neq V_m)$ intersects at most finitely many of $\{x_n\}_{n=1}^{\infty}$ because $\{\overline{V}_i\}_{i=1}^{\infty}$

is pairwise disjoint. But for any sequence $\{a_n\}_{n=1}^{\infty}$ in V_m converging to a point in ∂V_m, we notice that $\{g_m(a_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(x)$. Hence to claim g is continuous, it is enough to show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\bigcup_{i=1}^{\infty} V_i$ converging to $x \in \partial(\bigcup_{n=1}^{\infty} V_n)$ such that each V_i contains at most finitely many of $\{x_n\}_{n=1}^{\infty}$, then $\{g(x_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(x)$. Reindexing if necessary, it is enough to show that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\bigcup_{n=1}^{\infty} V_n$ converging to $x \in \partial(\bigcup_{n=1}^{\infty} V_n)$ such that $x_n \in V_n$ for each $n = 1,2,\ldots$, then $\{g(x_n) = g_n(x_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(x)$.

Let $\{x_n\}$ be a sequence converging to $x \in \partial(\bigcup_{n=1}^{\infty} V_n)$ such that $x_n \in V_n$ for each $n = 1,2,\ldots$. Suppose W is a neighborhood of $f^{-1}(x)$. Since $x \notin N_f$ there exists an integer n_0 such that $f^{-1}(B(x; \frac{1}{n_0})) \subset W$. Since $\{x_n\}_{n=1}^{\infty}$ converges to x, there exists an integer n_1 such that $\frac{1}{n_1} < \frac{1}{2n_0}$ and $x_n \in B(x; \frac{1}{2n_0})$ for any $n \geq n_1$. Notice that, for any $n = 1,2,\ldots,$ and $y \in f^{-1}(x_n)$, $\{y, g_n(x_n)\}$ is contained in $f^{-1}(U)$ for some $U \in \mathcal{U}_n$, hence $\{x_n, f \circ g(x_n)\}$ is contained in U for some $U \in \mathcal{U}_n$. For $n \geq n_1$, $d(x_n, x) < \frac{1}{2n_0}$ and $\{x_n, f \circ g_n(x_n)\} \subset U$ for some $U \in \mathcal{U}_n$ with diameter $< \frac{1}{n} < \frac{1}{2n_0}$. Hence for $n \geq n_1$, $f \circ g_n(x_n) \in B(x; \frac{1}{n_0})$, and $g_n(x_n) \in f^{-1}(B(x; \frac{1}{n_0})) \subset W$. Therefore $\{g_n(x_n)\}_{n=1}^{\infty}$ converges to $f^{-1}(x)$ if $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \partial(\bigcup_{n=1}^{\infty} V_n)$ such that $x_n \in V_n$ for each $n = 1,2,\ldots,$ and we see that g is continuous.

Continuity of H can be checked by the same analysis that established the continuity of g. Also we easily see that H is a homotopy from $g \circ f$ to identity on X' limited by $\mathcal{L}'$.

As corollaries of Theorem 4, we can improve Corollary 1.1 and Corollary 1.2. The notion of "null-sequence" as "only finitely many members having diameter $> \varepsilon$ for all $\varepsilon > 0$" is too limiting in non-compact, especially non-locally compact spaces. A proper notion is suggested as follows. A sequence $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint closed subsets of a metric space X is a locally <u>null-sequence</u> provided, for any limit point x of $\{B_n\}_{n=1}^{\infty}$ (that is, each neighborhood of x intersects infinitely many of $\{B_n\}_{n=1}^{\infty}$) and a neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and any B_n intersecting V with $x \notin B_n$ is contained in U. Now using the later notion of null-sequence we improve Corollary 1.1. and Corollary 1.2.

<u>COROLLARY 4.1</u>. If $f:X' \to X$ is a UV^{∞}-map from an ANR X' to a metric space X such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) f is a fine homotopy equivalence over each B_n ,

and

(3) $\{f^{-1}(B_n)\}_{n=1}^{\infty}$ forms a pairwise disjoint locally null-sequence,

then f is a fine homotopy equivalence.

<u>COROLLARY 4.2</u>. If $f:X' \to X$ is a UV^{∞}-map from an ANR X' to a metric space X such that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of closed subsets of X satisfying

(1) $N_f \subset \bigcup_{n=1}^{\infty} B_n$,

(2) f is a fine homotopy equivalence over each B_n ,

and

(3) $\{B_n\}_{n=1}^{\infty}$ forms a pairwise disjoint locally null-sequence, then f is a fine homotopy equivalence.

3.3. CRITERIA ENSURING IMAGES OF ANR'S ARE ANR'S

In this section we will extend a theorem of G. Kozlowski's [K2] stating that a cell-like map $f:X' \to X$ from an ANR X' onto a metric space X is a hereditary shape equivalence if and only if X is an ANR to a theorem for a UV^{∞}-map. Due to the complexity of the proof for (non-separable) metric spaces, in section 3.3 we give a proof of the theorem for separable metric spaces and in the appendix we prove the theorem for metric spaces.

THEOREM 5. A UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a hereditary shape equivalence if and only if X is an ANR .

As a corollary of Theorem 5, we get the following important equivalences among the classical notions.

THEOREM 6. For a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X , the following are equivalent:

(1) X is an ANR .

(2) f is a hereditary homotopy equivalence.

(3) f is a hereditary shape equivalence.

(4) f is a fine homotopy equivalence.

Before we prove theorems, we need to present several lemmas.

The next two lemmas show how to construct a continuous function from an adjunction space using a well-defined function. Variations of the construction will be used often in this section and in section 3.4 to obtain continuous functions from adjunction spaces.

LEMMA 3.3.1. If $f:X' \to X$, $h:X' \to Y$, and $g:X \to Y$ are maps such that there is a homotopy $H:X' \times I \to Y$ from $H_0 = h$ to $H_1 = g \circ f$, then $G:M(f) \to Y$ defined by

$$G(x, t) = \begin{cases} H(x, 2t) & \text{on } X' \times [0, \frac{1}{2}] \\ g \circ f(x) & \text{on } X' \times [\frac{1}{2}, 1) \\ g(x) & \text{on } X \times \{1\} \end{cases}$$

is continuous.

PROOF. It is enough to show that if a sequence $\{(x_n, t_n)\}_{n=1}^{\infty}$ in $X' \times [\frac{1}{2}, 1)$ converges to $(x, 1)$ in $X \times \{1\}$, then $\{G(x_n, t_n)\}_{n=1}^{\infty}$ converges to $g(x)$. But it does because the map $C:M(f) \to X \times \{1\}$ collapsing $M(f)$ onto $X \times \{1\}$ is continuous.

LEMMA 3.3.2. If $f:X' \to X$ and $g_1, g_2:X \to Y$ are maps such that $g_1 \circ f$ is homotopic to $g_2 \circ f$, then there is a map $G:DM(f) \to Y$ with $G = g_1$ on $X \times \{-1\}$ and $G = g_2$ on $X \times \{1\}$.

PROOF. It follows immediately by the same analysis as in the proof of Lemma 3.3.1.

Now we present three more lemmas needed later. Lemma 3.3.3 is a basic tool needed in the transfinite induction argument used to prove Lemma 3.3.4, and Lemma 3.3.4 is needed in the proof of Lemma 3.3.5. We prove Theorem 5 using Lemma 3.3.5 as a basic tool for a transfinite induction argument.

Recall we identify X' with $X' \times \{0\}$ in $M(f)$ or in $DM(f)$ and X with $X \times \{1\}$ in $M(f)$.

LEMMA 3.3.3. Suppose a map $f:X' \to X$ is a hereditary shape equivalence. Assume $h:X' \to Y$ is a map to an ANR Y, U and V are

open subsets of X, $H_u:M(f:U) \to Y$ and $H_v:M(f:V) \to Y$ are maps satisfying $H_u = h$ on U', $H_v = h$ on V', $H_v(V \times \{1\}) = \{\text{a point}\}$, and $H_u(M(f:U \cap V)) \cup H_v(M(f:V))$ is contained in an open subset 0 of Y, and A is closed (in $U \cup V$).

Then there exists a map $H:M(f:U \cup V) \to Y$ such that $H = h$ on $(U \cup V)'$, $H = H_u$ on $M(f:(U - V) \cup A)$, and $H(M(f:V)) \subset 0$.

PROOF. Let $W = U \cap V$, $g:W \to 0$ be the map defined as $g(x) = H_u(x, 1)$, $c:W \to 0$ be the constant map defined as $c(x) = H_v(x, 1)$, and $c(W) = \{y\}$. Since f is a hereditary shape equivalence, there exists a homotopy $G:W \times I \to 0$ with $G_0 = g$ and $G_1 = c$. Define $L:W' \times [-1, 2] \to 0$ as follows. Let $q:X' \times I \to M(f)$ be the "natural map". $L(x, -t) = H_v \circ q(x, t)$, $L(x, t) = H_u \circ q(x, t)$, and $L(x, 1 + t) = G(f(x), t)$ for each $(x, t) \in W' \times I$. Then L defines a map $\lambda:W' \to \Omega 0$ to the space of loops in 0 based at y and parameterized by the interval $[-1, 2]$. Since $\Omega 0$ is an ANR, λ has an extension $\Lambda:M(f:W) \to \Omega 0$. Then Ω defines a map $M:W' \times [-1, 2] \times [0, 1] \to 0$ satisfying $M_0 = L$ and $M(W' \times \{-1, 2\} \times [0, 1]) = \{y\}$. Let T be the triangle in the plane whose vertices are $(-1, 0)$, $(1, 0)$, and $(0, 1)$. Find a homeomorphism $\theta:W' \times T \to W' \times [-1, 2] \times [0, 1]$ such that $\theta|W' \times [-1, 1] \times \{0\}$ = inclusion and θ preserves the first coordinate. Let $q_2:W' \times T \to M(\overline{f}:W)$ be the "natural map", where $\overline{f}:DM(f) \to X$ is the "natural map". Then $M \circ \theta \circ q_2^{-1}:M(\overline{f}:W) \to 0$ is a well-defined function. Using a variation of Lemma 3.3.1, we will get a continuous function from $M(f:W)$ to 0 as follows. Let $\alpha_1:[-1, 2] \times [0, 1] \to [-1, 2] \times [0, \frac{3}{4}]$ be the map obtained by projecting $[-1, 2] \times [\frac{3}{4}, 1]$ onto $[-1, 2] \times \{\frac{3}{4}\}$

vertically. Let S denote the hexagon whose vertices are $(-1, 0)$, $(2, 0)$, $(1\frac{1}{2}, \frac{1}{4})$, $(1\frac{1}{2}, \frac{3}{4})$, $(-\frac{1}{2}, \frac{3}{4})$, and $(-\frac{1}{2}, \frac{1}{4})$, and $\alpha_2: [-1, 2] \times [0, \frac{3}{4}] \to S$ be the map obtained by projecting the quadrilateral whose vertices are $(-1, 0)$, $(-\frac{1}{2}, \frac{1}{4})$, $(-\frac{1}{2}, \frac{3}{4})$, and $(-1, \frac{3}{4})$ onto the polygonal arc joining $(-1, 0)$, $(-\frac{1}{2}, \frac{3}{4})$, and $(-\frac{1}{2}, \frac{3}{4})$ horizontally and projecting the quadrilateral whose vertices are $(2, 0)$, $(1\frac{1}{2}, \frac{1}{4})$, $(1\frac{1}{2}, \frac{3}{4})$, and $(2, \frac{3}{4})$ onto the polygonal arc joining $(2, 0)$, $(1\frac{1}{2}, \frac{1}{4})$, and $(1\frac{1}{2}, \frac{3}{4})$ horizontally. Let $\alpha_3: S \to [-\frac{1}{2}, 1\frac{1}{2}] \times [\frac{1}{4}, \frac{3}{4}]$ be the map obtained by projecting the quadrilateral whose vertices are $(-1, 0)$, $(2, 0)$, $(1\frac{1}{2}, \frac{1}{4})$, and $(-\frac{1}{2}, \frac{1}{4})$ onto the line segment joining $(-\frac{1}{2}, \frac{1}{4})$ and $(1\frac{1}{2}, \frac{1}{4})$ radially. Let $\alpha_4: [-\frac{1}{2}, 1\frac{1}{2}] \times [\frac{1}{4}, \frac{3}{4}] \to [-1, 2] \times [0, 1]$ be the homeomorphism satisfying $\alpha_4(-\frac{1}{2}, \frac{1}{4}) = (-1, 0)$, $\alpha_4(-\frac{1}{2}, \frac{3}{4}) = (-1, 1)$, $\alpha_4(1\frac{1}{2}, \frac{1}{4}) = (2, 0)$, and $\alpha_4(1\frac{1}{2}, \frac{3}{4}) = (2, 1)$. Define $\alpha: W' \times [-1, 2] \times [0, 1] \to W' \times [-1, 2] \times [0, 1]$ as $\alpha(w, s, t) = (w, \alpha_4 \circ \alpha_3 \circ \alpha_2 \circ \alpha_1(s, t))$. Then $M \circ \alpha \circ \theta \circ q_2^{-1}$ is continuous. Furthermore, $M \circ \alpha \circ \theta \circ q_2^{-1}$ defines a homotopy from $H_u|M(f:W)$ to $H_v|M(f:W)$ (rel. $W' \times \{0\}$). Then, using normality, we easily get $H: M(f: U \cup V) \to Y$ satisfying $H = h$ on $(U \cup V)'$, $H = H_u$ on $M(f:(U - V) \cup A)$, and $H(M(f:V)) \subset 0$.

LEMMA 3.3.4. Suppose a UV^{∞}-map $f: X' \to X$ is a hereditary shape equivalence. If U and V are open subsets of X, F is closed in $U \cup V$ satisfying $F \subset U$, and $h: X' \to Y$ and $H: M(f:U) \to Y$ are maps to an ANR Y satisfying that $H = h$ on U' and $H(M(f:U \cap V)) \cup h(V')$ is contained in an open subset 0 of Y, then there exists a map $\overline{H}: M(f: U \cup V) \to Y$ satisfying $\overline{H} = h$ on $(U \cup V)'$, $\overline{H} = H$ on $M(f: F \cup (U - V))$, and $\overline{H}(M(f:V)) \subset 0$.

PROOF. For each $x \in V - F$ choose a neighborhood U_x of x such that there exists $H_x: M(f:U_x) \to 0$ satisfying $H_x(U_x \times \{1\}) = \{\text{a point}\}$, $H_x = h$ on $(U_x)'$, and $U_x \subset V - F$. Choose a locally finite refinement $\mathcal{U} = \{U_0, U_\alpha | \alpha \in A'\}$ of the open cover $\{U, U_x | x \in V - F\}$ of $U \cup V$ with $U_0 = U$ and $\{U_\alpha | \alpha \in A'\} < \{U_x | x \in V - F\}$, and choose a precise refinement $\mathcal{V} = \{V_0, V_\alpha | \alpha \in A'\}$ of $\mathcal{U}$ with $\overline{V}_0$ (in $U \cup V) \subset U_0$ and $\overline{V}_\alpha$ (in $U \cup V) \subset U_\alpha$ for each $\alpha \in A'$. Notice that $F \cup (U - V)$ is contained in V_0.

Assume $A = \{0\} \cup A'$ is a well-defined set with "$\leqq$" such that 0 is the first element of A.

Let $P(\alpha)$ denote the following proposition; for each $i \leq j \leq \alpha$ there exist an open set W_i^j and a map $H^j: M(f:\cup W_\beta^j | \beta \leq j\}) \to Y$ satisfying

(1) $\overline{V}_i$ (in $U \cup V) \subset W_i^j \subset U_i$,

(2) $W_i^j \subset W_i^i$,

(3) $W_i^i - \cup\{\overline{U}_\beta \text{ (in } U \cup V) | i < \beta < j\} \subset W_i^j$,

(4) $H^j = H^i$ on $M(f:\cup\{W_\beta^j | \beta \leq i\})$,

(5) $H^j = h$ on $(\cup\{W_\beta^j | \beta \leq j\})'$,

and

(6) $H^j(M(f:(\cup\{W_\beta^j | \beta \leq j\}) \cap V)) \subset 0$.

We will show that $P(\alpha)$ is true for any $\alpha \in A$ using transfinite induction.

Certainly $P(0)$ is true with $W_0^0 = U_0$ and $H^0 = H$.

Let $w \in A$, and assume that $P(\alpha)$ is true for any $\alpha < w$. For each $\alpha < w$ define $L_\alpha = \cap\{W_\alpha^\beta | \alpha \leq \beta < w\}$. We will show that L_α is open for each $\alpha < w$. If w has the immediate predecessor $w - 1$, then $L_\alpha = W_\alpha^{w-1}$ by (2). Suppose w does not have the immediate

predecessor. For $x \in L_\alpha$, using the local finiteness of $\mathcal{U}$, we can find a neighborhood N_x of x with $N_x \subset W_\alpha^\alpha$ such that there exists $\alpha(x)$ $(< w)$ satisfying $N_x \cap \overline{U}_\beta$ (in $U \cup V) = \phi$ for any $\beta \geq \alpha(x)$. Then $N_x \cap W_\alpha^{\alpha(x)} \subset W_\alpha^\beta$ for any $\beta < w$ by (2) and (3), hence $N_x \cap W_\alpha^{\alpha(x)} \subset L_\alpha$. Therefore L_α is a neighborhood of $\overline{V}_\alpha$ (in $U \cup V$) for any $\alpha < w$. Furthermore, $W_\alpha^\beta - \cup\{\overline{U}_i$ (in $U \cup V) | \beta < i < w\} \subset L_\alpha$ for any $\alpha \leq \beta < w$ by (3). For each $\alpha < w$, choose a neighborhood M_α of $\overline{V}_\alpha$ (in $U \cup V$) satisfying $\overline{V}_\alpha$ (in $U \cup V) \subset M_\alpha \subset \overline{M}_\alpha$ (in $U \cup V) \subset L_\alpha$, and define $W_\alpha^w = M_\alpha \cup (L_\alpha - \overline{U}_w$ (in $U \cup V$)) and $W_w^w = U_w$. Notice that there exists a map $G:M(f:\cup\{L_\alpha | \alpha < w\}) \to Y$ defined as $G = H^\alpha$ on $M(f:L_\alpha)$ for each $\alpha < w$ such that $G(M(f:(\cup\{L_\alpha | \alpha < w\}) \cap V)) \subset 0$ and $G = h$ on $(\cup\{L_\alpha | \alpha < w\})'$. Since $U_w \subset U_x$ for some $x \in V - F$, there exists $H_w:M(f:U_w) \to 0$ satisfying $H_w = h$ on $(U_w)'$ and $H_w(U_w \times \{1\}) =$ point. Therefore we can apply Lemma 3.3.3 with open sets $\cup\{L_\alpha | \alpha < w\}$ and U_w, maps G and H_w, and a closed set $\cup\{\overline{M}_\alpha$ (in $U \cup V) | \alpha < w\}$ in $U \cup V$ to obtain a map $\overline{G}:M(f:(\cup\{L_\alpha | \alpha < w\}) \cup U_w) \to Y$ satisfying $\overline{G} = h$ on $(\cup\{L_\alpha | \alpha < w\})' \cup U_w'$, $\overline{G} = G$ on $M(f:(\cup\{L_\alpha | \alpha < w\} - U_w) \cup (\cup\{\overline{M}_\alpha$ (in $U \cup V) | \alpha < w\}))$, and $\overline{G}(M(f:U_w)) \subset 0$. Now we define $H^w:M(f:\cup\{W_\alpha^w | \alpha \leq w\}) \to Y$ as $H^w = \overline{G}$. We will check $P(w)$ is true. Certainly (1), (2), (4), (5), and (6) hold. Since $W_\alpha^\alpha - \cup\{\overline{U}_i$ (in $U \cup V) | \alpha < i \leq w\} \subset L_\alpha - \overline{U}_w$ (in $U \cup V$) $\subset W_\alpha^w$ for each $\alpha < w$, (3) also holds.

Therefore, by transfinite induction, we claim that $P(\alpha)$ is true for any $\alpha \in A$.

Finally, we define $\overline{H}:M(f:U \cup V) \to Y$ as $\overline{H} = H^\alpha$ on $M(F:V_\alpha)$ for each $\alpha \in A$. Then $\overline{H}$ is a well-defined continuous map by (1),

(2), and (4), $\overline{H} = h$ on $(U \cup V)'$ by (5), $\overline{H} = H$ on $M(f:F \cup (U - V))$ since $F \cup (U - V) \subset V_0$, and $\overline{H}(M(f:V)) \subset 0$ by (6).

To prove Lemma 3.3.5, we need to recall standard facts about geometrical nerves and simplicial complexes.

Suppose X is a metric space and $\mathcal{U}$ is a locally finite open cover of X. Let $\mathcal{N}$ be the geometric nerve of $\mathcal{U}$ with the weak topology.

As a convention, we regard each vertex (of $\mathcal{N}$) as the barycenter of itself.

Let B_k be the set of all barycenters of k-dimensional simplices in $\mathcal{N}$ for $k = 0,1,\ldots$. Then we easily see that, for each $k = 0,1,\ldots$, the star of $b \in B_k$ in the second barycentric refinement intersects the star of $c \in B_k$ in the second barycentric refinement if and only if $b = c$. Hence, for each $k = 0,1,\ldots$, there exists a (possibly empty) family $S_k = \{s_b | b \in B_k\}$ of pairwise disjoint closed subsets of $\mathcal{N}$ such that $\mathcal{N} = \bigcup_{k=0}^{\infty} (\cup S_k)$.

Let $\kappa : X \to \mathcal{N}$ be the map such that $\kappa^{-1}(St(u, \mathcal{N})) \subset U$ for each vertex u (of $\mathcal{N}$) corresponding to a member U of $\mathcal{U}$.

For a barycenter b of a simplex σ in $\mathcal{N}$, σ_b denotes σ and $I_b = U_0 \cap \ldots \cap U_m$, where $\{U_0, \ldots, U_m\}$ corresponds to σ_b. Then $\kappa^{-1}(s_b) \subset I_b$ for each $b \in \bigcup_{k=0}^{\infty} B_k$.

Also we can find that, for each $k = 0,1,\ldots$, there exists a (possibly empty) family $L_k = \{l_b | b \in B_k\}$ of pairwise disjoint open subsets of $\mathcal{N}$ such that $s_b \subset l_b$ and $\kappa^{-1}(l_b) \subset I_b$ for each $b \in B_k$.

Let $\mathcal{C}_k = \{\kappa^{-1}(s_b) | b \in B_k\}$ and $\mathcal{N}_k = \{\kappa^{-1}(1_b) | b \in B_k\}$ for $k = 0,1,\ldots$. Then we have the following proposition.

PROPOSITION 3.7. For a metric space X and a locally finite open cover $\mathcal{U}$ of X, there exist (possibly empty) families $\mathcal{C}_k = \{C_b | b \in B_k\}$ and $\mathcal{N}_k = \{N_b | b \in B_k\}$ for $k = 0,1,\ldots$ such that, for each $k = 0, 1,\ldots$,

($*_1$) $C_b \subset N_b$ for any $b \in B_k$, where B_k is the set of all barycenters of k-dimensional simplices in the geometric nerve $\mathcal{N}$ of the open cover $\mathcal{U}$,

($*_2$) $\mathcal{N}_k$ is a family of pairwise disjoint open subsets of X ,

($*_3$) $\mathcal{C}_k$ is a family of pairwise disjoint closed subsets of X satisfying $\bigcup_{k=0}^{\infty} (\cup \mathcal{C}_k) = X$,

($*_4$) $N_b \subset I_b$ for each $b \in \bigcup_{k=0}^{\infty} B_k$,

and

($*_5$) if $N_b \cap N_c \neq \phi$, then either $\sigma_b \subset \sigma_c$ or $\sigma_c \subset \sigma_b$.

The next lemma is the main lemma used in the proof of Theorem 5.

Once again we recall that we do not distinguish subsets from embeddings.

For a map $H:M(f:A) \to X'$, $H|\{0\}$ denotes the restricted map $H|A' \times \{0\}$.

LEMMA 3.3.5. Suppose $\mathcal{U}$ is a locally finite open cover of a metric space X and, for each $k = 0,1,\ldots,$ $\mathcal{C}_k$ and $\mathcal{N}_k$ are families of subsets of X as in the previous proposition satisfying ($*_1$) through ($*_5$) .

Suppose that a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a hereditary shape equivalence. Suppose D is an open subset of X , $H:M(f:D) \to X'$ is a map, and $\mathcal{M}_k = \{M_b | b \in B_k\}$ is a

family of open subsets of X for each $k = 0,1,\ldots$ such that $H|\{0\}$ = inclusion, $C_b \subset M_b \subset N_b$, and $H(M(f:D \cap M_b)) \subset I_b'$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Then, for any open subset E of X intersecting only finitely many $U_0,\ldots, U_n$ of $\mathcal{U}$, there exist a map $\overline{H}:M(f:D \cup E) \to X'$ and a family $\mathcal{L}_k = \{L_b | b \in B_k\}$ of open subsets of X for $k = 0,1,\ldots$ such that $\overline{H}|\{0\}$ = inclusion, $C_b \subset L_b \subset M_b$, $M_b - \overline{E} \subset L_b$, and $\overline{H}(M(f:(D \cup E) \cap L_b)) \subset I_b'$ for each $b \in \bigcup_{k=0}^{\infty} B_k$. Furthermore, for any closed set F in $U \cup V$ such that $F \subset D$, we can choose $\overline{H}$ so that $\overline{H} = H$ on $M(f:F \cup (D - E))$.

PROOF. Let F be a closed set in $U \cup V$ such that $F \subset D$.

Let m be the dimension of the subcomplex K (of the geometric nerve of $\mathcal{U}$) determined by $\{U_0,\ldots, U_n\}$.

Suppose c is a barycenter of an m-dimensional simplex in K satisfying $M_c \cap E \neq \phi$. Then, by applying Lemma 3.3.4, we obtain a map $H_{mc}:M(f:D \cup (M_c \cap E)) \to X'$ such that $H_{mc} = H$ on $M(f:F \cup (D - M_c \cap E))$, $H_{mc}|\{0\}$ = inclusion, and $H_{mc}(M(f:M_c \cap E)) \subset I_c'$. Hence $H_{mc}(M(f:[D \cup (M_c \cap E)] \cap M_b)) \subset I_b'$ for any $M_b \in \bigcup_{k=0}^{\infty} \mathcal{M}_k$ by $(*_5)$. By noticing $(*_2)$ and $H_{mc} = H$ on $M(f:D - (M_c \cap E))$, we claim that there exists a map $H_m:M(f:D \cup [(\cup \mathcal{M}_m) \cap E]) \to X'$ defined as $H_m = H_{mc}$ on $M(f:D \cup (M_c \cap E))$ for $M_c \in \mathcal{M}_m$ such that $H_m = H$ on $M(f:F \cup [D - (\cup \mathcal{M}_m) \cap E])$, $H_m|\{0\}$ = inclusion, and $H_m(M(f:\{D \cup [(\cup \mathcal{M}_m) \cap E]\} \cap M_b)) \subset I_b'$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Now suppose $M_c \in \mathcal{M}_{m-1}$ and $M_c \cap E \neq \phi$. For each M_b in $\mathcal{M}_m$ satisfying $\sigma_c \subset \sigma_b$ and $M_b \cap E \neq \phi$, choose an open subset O_b^c of X

satisfying $C_b \subset O_b^c \subset \overline{O}_b^c \subset M_b$, and let $O^c = \cup\{O_b^c | \sigma_c \subset \sigma_b$ and $M_b \cap E \neq \phi\}$. Now we apply Lemma 3.3.4 to obtain a map $H_{(m-1)c}$: $M(f:D \cup [(\cup \mathcal{M}_m) \cap E] \cup (M_c \cap E)) \to X'$ such that $H_{(m-1)c}|\{0\}$ = inclusion, $H_{(m-1)c} = H_m$ on $M(f:F \cup (\overline{O}^c \cap E) \cup \{[D \cup (\cup \mathcal{M}_m) \cap E] - (M_c - E)\})$, and $H_{(m-1)c}(M(f:M_c \cap E)) \subset I_c'$. Let $L_b^c = O_b^c \cup (M_b - \overline{E})$ for any $b \in \bigcup_{k=0}^{\infty} B_k$ satisfying $\sigma_c \subset \sigma_b$ and $M_b \cap E \neq \phi$. Otherwise, let $L_b^c = M_b$. Then we can show that $H_{(m-1)c}(M(f:\{D \cup [(\cup \mathcal{M}_m) \cup M_c] \cap E\} \cap L_b^c)) \subset I_b'$, $C_b \subset L_b^c \subset M_b$, and $M_b - \overline{E} \subset L_b^c$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Therefore, for each $M_c \in \mathcal{M}_{m-1}$ satisfying $M_c \cap E \neq \phi$, there exist a map $H_{(m-1)c}:M(f:D \cup [(\cup \mathcal{M}_m) \cup M_c] \cap E) \to X'$ and a family $\mathcal{L}_k^c = \{L_b^c | b \in B_k\}$ of open subsets of X for $k = 0,1,...$ such that $H_{(m-1)c}|\{0\}$ = inclusion, $H_{(m-1)c} = H_m$ on $M(f:F \cup \{[D \cup (\cup \mathcal{M}_m \cap E)] - (M_c \cap E)\})$, $H_{(m-1)c}(M(f:\{D \cup [(\cup \mathcal{M}_m) \cup M_c] \cap E\} \cap L_b^c)) \subset I_b'$, $C_b \subset L_b^c \subset M_b$, $M_b - \overline{E} \subset L_b^c$ for any $b \in \bigcup_{k=0}^{\infty} B_k$. Now by noticing $(*_2)$, $H_{(m-1)c} = H_m$ on $M(f:[D \cup (\cup \mathcal{M}_m \cap E)] - (M_c \cap E))$, and that there are at most finitely many $M_c \in \mathcal{M}_{m-1}$ satisfying $M_c \cap E \neq \phi$, we claim that there exist a map $H_{m-1}:M(f:D \cup [(\cup \mathcal{M}_m) \cup (\cup \mathcal{M}_{m-1})] \cap E) \to X'$ and a family $\mathcal{L}_k^{m-1} = \{L_b^{m-1} | b \in B_k\}$ of open subsets of X for each $k = 0,1,...$ such that $H_{m-1}|\{0\}$ = inclusion, $H_{m-1} = H_m$ on $M(f:F \cup \{[D \cup (\cup \mathcal{M}_m \cap E] - (\cup \mathcal{M}_{m-1} \cap E)\})$, $C_b \subset L_b^{m-1} \subset M_b$, $H_{m-1}(M(f:\{D \cup [(\cup \mathcal{M}_m) \cup (\cup \mathcal{M}_{m-1})] \cap E\} \cap L_b^{m-1})) \subset I_b'$, and $M_b - \overline{E} \subset L_b^{m-1}$ for any $b \in \bigcup_{k=0}^{\infty} B_k$. In fact, each L_b^{m-1} can be defined as the finite intersection $\cap\{L_b^c | M_c \in \mathcal{M}_{m-1}$ and $M_c \cap E \neq \phi\}$.

Finally, by finite induction, we claim that there exist a map $\overline{H} = H_0:M(f:D \cup [(\cup \mathcal{M}_m) \cup \ldots \cup (\cup \mathcal{M}_0)] \cap E)$ $(= M(f:D \cup E)) \to X'$ and

a family $\mathcal{L}_k = \{L_b | b \in B_k\}$ of open subsets of X for each $k = 0,1,...$ such that $\overline{H}|\{0\}$ = inclusion, $\overline{H} = H$ on $M(f:F \cup (D - E))$, $C_b \subset L_b \subset M_b$, $M_b - \overline{E} \subset L_b$, $\overline{H}(M(f:(D \cup E) \cap L_b)) \subset I_b'$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Now we are ready to present a criterion ensuring that images of ANR's are ANR's . Here we establish the criterion for maps between separable metric spaces, while the proof in the case of maps between (non-separable) metric spaces is given in the appendix.

THEOREM 5. A UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a hereditary shape equivalence if and only if X is an ANR .

PROOF. It follows from a theorem of G. Kozlowski's [K2] that a UV^{∞}-map between ANR's is a hereditary homotopy equivalence, hence a hereditary shape equivalence. Therefore we need to show only necessity. According to Kozlowski, it is enough to show that, for any open cover $\mathcal{U}$ of X , there exists a map $H:M(f) \to X'$ satisfying $H|\{0\}$ = identity and $\{H(M(f:\{x\})|x \in X\}$ refines $\mathcal{U}'$.

Assume X is a separable metric space.

Let $\mathcal{U}$ be an open cover of X . We may assume that $\mathcal{U}$ is a locally finite open cover of X . Let $\mathcal{C}_k$, $\mathcal{N}_k$, and B_k $(k = 0,1,...)$ be the families as in Proposition 3.7 satisfying $(*_1)$ through $(*_5)$.

Choose a countable open cover $\mathcal{V} = \{V_1, V_2,...\}$ of X such that each $\overline{V}_n$ $(n = 1,2,...)$ intersects only finitely many members of $\mathcal{U}$, and choose a precise open refinement $\mathcal{M} = \{M_1, M_2,...\}$ of $\mathcal{V}$ such that $\overline{M}_n \subset V_n$ for each $n = 1,2,...$. For each $n = 1,2,...$, choose open subsets $W_{n,m}$ $(m = 1,2,...)$ of X such that

$$\overline{M}_n \subset W_{n,m+1} \subset \overline{W}_{n,m+1} \subset W_{n,m} \subset \overline{W}_{n,m} \subset V_n \quad \text{for } m = 1,2,... .$$

We will show, by induction, that, for each $n = 1,2,\ldots$, there exist $H^n:M(f:\bigcup_{m=1}^{n} W_{m,n}) \to X'$ and a family $\mathcal{L}_k^n = \{L_b^n | b \in B_k\}$ $(k = 0,1,\ldots)$ of open subsets of X such that $H^n|\{0\}$ = inclusion, $H^n = H^{n-1}$ on $M(f:\bigcup_{m=1}^{n-1} W_{m,n})$, $H^n(M(f:(\bigcup_{m=1}^{n} W_{m,n}) \cap L_b^n)) \subset I_b'$, and $C_b \subset L_b^n \subset N_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$ and $n = 1,2,\ldots$.

Since V_1 intersects only finitely many members of $\mathcal{U}$, by applying Lemma 3.3.5, we obtain $H':M(f:W_{1,1}) \to X'$ and a family $\mathcal{L}_k' = \{L_b' | b \in B_k\}$ $(k = 0,1,\ldots)$ of open subsets of X such that $H'|\{0\}$ = inclusion, $H'(M(f:W_{1,1} \cap L_b')) \subset I_b'$, and $C_b \subset L_b' \subset N_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Let $i > 0$, and assume that we have a map $H^i:M(f:\bigcup_{m=1}^{i} W_{m,i}) \to X'$ and a family $\mathcal{L}_k^i = \{L_b^i | b \in B_k\}$ $(k = 0,1,\ldots)$ of open subsets of X such that $H^i|\{0\}$ = inclusion, $H^i = H^{i-1}$ on $M(f:\bigcup_{m=1}^{i-1} W_{m,i})$, $H^i(M(f:(\bigcup_{m=1}^{i} W_{m,i}) \cap L_b^i)) \subset I_b'$, and $C_b \subset L_b^i \subset N_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$. Since $W_{i+1,i+1}$ $(\subset V_{i+1})$ intersects only finitely many members of $\mathcal{U}$, we can apply Lemma 3.3.5 to obtain $\overline{H}:M(f:(\bigcup_{m=1}^{i} W_{m,i}) \cup W_{i+1,i+1}) \to X'$ and a family $\mathcal{L}_k^{i+1} = \{L_b^{i+1} | b \in B_k\}$ $(k = 0,1,\ldots)$ of open subsets of X such that $\overline{H}|\{0\}$ = inclusion, $\overline{H} = H^i$ on $M(f:\bigcup_{m=1}^{i} \overline{W}_{m,i+1})$, $\overline{H}(M(f:[(\bigcup_{m=1}^{i} W_{m,i}) \cup W_{i+1,i+1}] \cap L_b^{i+1})) \subset I_b'$, and $C_b \subset L_b^{i+1} \subset N_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$. Define $H^{i+1}:M(f:\bigcup_{m=1}^{i+1} W_{m,i+1}) \to X'$ as $H^{i+1} = \overline{H}|M(f:\bigcup_{m=1}^{i+1} W_{m,i+1})$. Then $H^{i+1} = H^i$ on $M(f:\bigcup_{m=1}^{i} W_{m,i+1})$, $H^{i+1}|\{0\}$ = inclusion, and $H^{i+1}(M(f:(\bigcup_{m=1}^{i+1} W_{m,i+1}) \cap L_b^{i+1})) \subset I_b'$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Therefore, by induction, we claim that, for each $n = 1,2,\dots,$ there exist a map $H^n : M(f : \bigcup_{m=1}^{n} W_{m,n}) \to X'$ and a family $L_k^n = \{L_b^n \mid b \in B_k\}$ $(k = 0,1,\dots)$ of open subsets of X such that $H^n|\{0\} =$ inclusion, $H^n = H^{n-1}$ on $M(f : \bigcup_{m=1}^{n-1} W_{m,n})$, $H^n(M(f:(\bigcup_{m=1}^{n} W_{m,n}) \cap L_b^n)) \subset I_b'$, and $C_b \subset L_b^n \subset N_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Now we can define $H : M(f) \to X'$ as $H = H^n$ on $M(f:M_n)$ $(n = 1,2,\dots)$. Then H is a well-defined continuous function such that $\{H(M(f:\{x\})) \mid x \in X\}$ refines $\mathcal{U}'$ and $H|\{0\} =$ identity.

As an immediate corollary of Theorem 5, we get the following equivalences.

<u>THEOREM 6</u>. For a UV^{∞}-map $f : X' \to X$ from an ANR X' to a metric space X, the following are equivalent:

(1) X is an ANR.

(2) f is a hereditary homotopy equivalence.

(3) f is a hereditary shape equivalence.

(4) f is a fine homotopy equivalence.

<u>PROOF</u>. G. Kozlowski [K2] already showed (1) implies (2) and (4) implies (1). Notice that (2) easily implies (3). Hence we need to show (3) implies (4).

Notice how we proved Theorem 5. We proved that if $f : X' \to X$ is a hereditary shape equivalence from an ANR X' to a separable metric space X, then, for any open cover $\mathcal{U}$ of X, there exists $g : X \to X'$ such that $g \circ f$ is homotopic to identity on X' with the homotopy limited by $\mathcal{U}'$. In fact, we will show the same thing for the case of (non-separable) metric space X. If we have such $g : X \to X'$, we find out that $f \circ g$ and identity on X are $St(St(St\,\mathcal{U}))$-close. Since

we know X is an ANR , using refinements of $\mathcal{U}$, we can easily show that f is a fine homotopy equivalence if a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a hereditary shape equivalence. Hence we have equivalences.

3.4. MAPS OF ANR'S WHOSE IMAGES ARE ANR's

In this section we improve the following theorem of G. Kozlowski's [K2] to a theorem for a UV^{∞}-map.

THEOREM (G. Kozlowski). If $f:X' \to X$ is a cell-like map between metric spaces and $\{X_n\}_{n=1}^{\infty}$ is a sequence of closed subsets of X such that $X = \bigcup_{i=1}^{\infty} X_i$ and $f|X_i':X_i' \to X_i$ is a hereditary shape equivalence for each $i = 1,2,\ldots,$ then f is a hereditary shape equivalence.

We prove Theorem 7 using the criteria developed in section 3.3. Before we prove it we need to present nine lemmas.

Here we recall some notations and introduce some more.

For the map $f:X' \to X$, $f':DM(f) \to X \times [-1, 1]$ is the "natural map" induced by f and identity.

$f_A:X' \to X' \cup_f A$ and $\overline{f}_A:X' \cup_f A \to X$ are the "natural maps", where A is a closed subset of X .

$f_A':DM(f_A) \to (X' \cup_f A) \times [-1, 1]$ is the "natural map".

$f'_{A\times[-1,1]}:DM(f) \to DM(\overline{f}_A)$ is the "natural map".

$f''_{A\times[-1,1]}:DM(f'_{A\times[-1,1]}) \to DM(\overline{f}_A) \times [-1, 1]$ is the "natural map".

LEMMA 3.4.1. If $f:X' \to X$ and $f':DM(f) \to X \times [-1, 1]$ are hereditary homotopy equivalences, A is a closed subset of X , U is a neighborhood of A , and $G:M(f:U) \cup X' \times \{0\} \to Y$, then $G|M(f:A) \cup X' \times \{0\} \to Y$ extends to a map from $M(f)$ to Y .

PROOF. Since f is a hereditary homotopy equivalence, we have a map $H:DM(f:U) \cup M(f) \to Y$ satisfying $H(x,t) = G(x, -t)$ for each $(x,t) \in DM(f:U)$ with $t \leq 0$ and $H(x,0) = G(x,0)$ for any $x \in X'$. Then since f' is a hereditary homotopy equivalence, $H|DM(f:U)$ has an extension $\overline{H}:M(f':U \times [-1,1]) \to Y$. Certainly $\overline{H}$ induces a homotopy (rel. $X' \times \{0\}$) from G to $H|M(f:U) \cup X' \times \{0\}$. Therefore, using normality, we claim that $G|M(f:A) \cup X' \times \{0\}$ extends to a map from $M(f)$ to Y.

LEMMA 3.4.2. Suppose $f_A:X' \to X' \underset{f}{\cup} A$ and $f_A':DM(f_A) \to (X' \underset{f}{\cup} A) \times [-1, 1]$ are hereditary homotopy equivalences. If B is a closed subset of X, U is a neighborhood of B, and $H:X' \times \{0\} \cup M(f:U) \to Y$ is a map to an ANR Y, then there exist a neighborhood W of $A \cup B$ and a map $\overline{H}:X' \times \{0\} \cup M(f:W) \to Y$ such that $H = \overline{H}$ on $M(f:B) \cup X' \times \{0\}$.

PROOF. Let $\overline{f}_{AM}:M(f_A) \to M(f)$ be the "natural map". By Lemma 3.4.1, $H \circ \overline{f}_{AM}|M(f_A:(\overline{f}_A)^{-1}(B)) \cup X' \times \{0\}: M(f_A:(\overline{f}_A)^{-1}(B)) \cup X' \times \{0\} \to Y$ has an extension $F:M(f_A) \to Y$. Define $H': M(f:B \cup A) \cup X' \times \{0\} \to Y$ as follows.

$$H'(x, t) = \begin{cases} F(x, t) & \text{on } M(f:A) \\ H(x, t) & \text{on } M(f:B) \\ H(x, 0) & \text{on } X' \to \{0\}. \end{cases}$$

Let $g:A \cup B \to Y$ be the map defined as $g(x) = H'(x, 1)$. Since Y is an ANR, g has an extension $\overline{g}$ from a neighborhood W_1 of $A \cup B$ to Y. Notice that $\overline{g} \circ \overline{f}_{AM} = F$ on $(\overline{f}_{AM})^{-1} (A \cup B)$. Since Y is an ANR, we can find a neighborhood W of $A \cup B$ and a homotopy

from $\overline{g} \circ \overline{f}_{AM}|(\overline{f}_{AM})^{-1}(W)$ to $F|(\overline{f}_{AM})^{-1}(W)$ (rel. $(\overline{f}_{AM})^{-1}(A \cup B)$) , which induces a homotopy from $[F|X' \times \{0\} \cup M(f:A \cup B)]$ $\cup [\overline{g} \circ \overline{f}_{AM}|(\overline{f}_{AM})^{-1}(W)]$ to $F|X' \times \{0\} \cup M(f_A:A \cup B) \cup (\overline{f}_{AM})^{-1}(W)$. Hence by the homotopy extension property of an ANR Y , $[F|X' \times \{0\} \cup M(f:A \cup B)] \cup (\overline{g} \circ \overline{f}_{AM}|(\overline{f}_{AM})^{-1}(W)]$ has an extension $F':X' \times \{0\} \cup M(f_A:(\overline{f}_{AM})^{-1}(W)) \to Y$. Now, using a variation of Lemma 3.3.1, we can find $H'':X' \times \{0\} \cup M(f:W) \to Y$ such that $H''|M(f:A \cup B) \cup X' \times \{0\}$ is homotopic to H' . Therefore H' has an extension $\overline{H}:M(f:W) \cup X' \times \{0\} \to Y$. Certainly $\overline{H} = H' = H$ on $M(f:B) \cup X' \times \{0\}$.

LEMMA 3.4.3. Suppose $f:X' \to X$ is a map and $\{X_i\}_{i=1}^{\infty}$ is a sequence of closed subsets of X such that $X = \bigcup_{i=1}^{\infty} X_i$. Let $f_i:X' \to X' \cup_f X_i$ denote the map f_{X_i} for $i = 1,2,\ldots$. If f_i and $f_i':DM(f_i) \to (X' \cup_f X_i) \times [-1, 1]$ are hereditary homotopy equivalences, then any map $h:X' \to Y$ to an ANR Y has an extension $H:M(f) \to Y$.

PROOF. First, by induction, we will show that for each $m = 1,2,\ldots$ there exist open subsets V_m and N_m of X and an extension $H^m:M(f:V_m) \cup X' \times \{0\} \to Y$ of h such that $X_m \subset N_m \subset \overline{N}_m \subset V_m$, $X_{m+1} \cup \overline{N}_m \subset N_{m+1} \subset \overline{N}_{m+1} \subset V_{m+1}$, and $H^m = H^{m+1}$ on $M(f:\overline{N}_m) \cup X' \times \{0\}$ $(m = 1,2,\ldots)$.

By Lemma 3.4.2, we have a neighborhood V_1 of X_1 and an extension $H':M(f:V_1) \cup X' \times \{0\} \to Y$ of h . Choose a neighborhood N_1 of X_1 such that $X_1 \subset N_1 \subset \overline{N}_1 \subset V_1$. Let $m > 0$, and assume we have the following:

(1) open subsets $V_1,\ldots, V_m$ of X .

(2) open subsets $N_1, \ldots, N_m$ of X such that $\overline{N}_i \cup X_{i+1} \subset N_{i+1} \subset \overline{N}_{i+1} \subset V_{i+1}$ $(i = 1,2,\ldots, m-1)$.

(3) $H^i : M(f:V_i) \cup X' \times \{0\} \to Y$ such that $H^i = H^{i+1}$ on $M(f:\overline{N}_i) \cup X' \times \{0\}$ $(i = 1,2,\ldots, m-1)$.

We apply Lemma 3.4.2 with $\overline{N}_m \subset V_m$, $H^m : M(f:V_m) \to Y$, f_{m+1}, and f'_{m+1} to obtain a neighborhood V_{m+1} of $\overline{N}_m \cup X_{m+1}$ and $H^{m+1} : M(f:V_{m+1}) \cup X' \times \{0\} \to Y$ such that $H^{m+1} = H^m$ on $M(f:\overline{N}_m) \cup X' \to \{0\}$. Then we choose an open set N_{m+1} such that $\overline{N}_m \cup X_{m+1} \subset N_{m+1} \subset \overline{N}_{m+1} \subset V_{m+1}$.

Now, by induction, we claim that, for each $m = 1,2,\ldots$, there exist open subsets V_m and N_m of X and an extension $H^m : M(f:V_m) \cup X' \times \{0\} \to Y$ of h such that $X_m \subset N_m \subset \overline{N}_m \subset V_m$, $X_{m+1} \cup \overline{N}_m \subset N_{m+1} \subset \overline{N}_{m+1} \subset V_{m+1}$, and $H^m = H^{m+1}$ on $M(f:\overline{N}_m) \cup X' \times \{0\}$ $(m = 1,2,\ldots)$. Then we can define $H : M(f) \to Y$ as $H = H^m$ on $M(f:N_m)$ $(m = 1,2,\ldots)$.

<u>LEMMA 3.4.4</u>. If a UV^∞-map $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence, then $f \times id : X' \times I \to X \times I$ defined by $f \times id(x, t) = (f(x), t)$ is a fine homotopy equivalence.

<u>PROOF</u>. Since $f \times id$ is a UV^∞-map between ANR's, it follows from Theorem 6.

<u>LEMMA 3.4.5</u>. If a UV^∞-map $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence, then, for any closed subset A of X, $f_A : X' \to X' \cup_f A$ and $\overline{f}_A : X' \cup_f A \to X$ are fine homotopy equivalences.

<u>PROOF</u>. By Theorem 6, it is enough to show that, for any open cover $\mathcal{U}$ of $X' \cup_f A$, there exists a map $h : X' \cup_f A \to X'$ such that $h \circ f_A$ is homotopic to identity on X' with the homotopy limited by $\{f_A^{-1}(U) \mid U \in \mathcal{U}\}$.

Let $\mathcal{U}$ be a given open cover of $X' \underset{f}{\cup} A$. Find a family $\mathcal{V}$ of open subsets of X such that $\{(\overline{f}_A)^{-1}(V) \mid V \in \mathcal{V}\}$ refines $\mathcal{U}$ and $A \subset \cup\mathcal{V}$. Choose open subsets W_0, W_1, and W_2 of X such that $A \subset W_0 \subset \overline{W}_0 \subset W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset \cup\mathcal{V}$. For each $x \in X - \overline{W}_2$, choose a neighborhood V_x of x in X such that $V_x \subset X - \overline{W}_2$. Let $\mathcal{V}^T = \mathcal{V} \cup \{V_x \mid x \in X - \overline{W}_2\}$. Since f is a fine homotopy equivalence, there exist a map $g: X \to X'$ and a homotopy $H: X' \times I \to X'$ from H_0 = identity on X' to $H_1 = g \circ f$ limited by $(\mathcal{V}^T)'$. Notice that $H|W_2' \times I$ is limited by $\mathcal{V}'$, hence limited by $\{f_A^{-1}(U) \mid U \in \mathcal{U}\}$. Find a map $\alpha: X' \underset{f}{\cup} A \to [0, 1]$ satisfying $\alpha((\overline{f}_A)^{-1}(\overline{W}_0)) = \{1\}$ and $\alpha((\overline{f}_A)^{-1}(X - W_1)) = \{0\}$. And define $h: X' \underset{f}{\cup} A \to X'$ as follows.

$$h(x) = \begin{cases} f_A^{-1}(x) & \text{on } X' \underset{f}{\cup} A - (\overline{f}_A)^{-1}(\overline{W}_1) \\ H(f_A^{-1}(x), \alpha(x)) & \text{on } (\overline{f}_A)^{-1}(W_2) - A \\ g \circ \overline{f}_A(x) & \text{on } (\overline{f}_A)^{-1}(W_0) . \end{cases}$$

Also define $G: X' \times I \to X'$ as follows.

$$G(x, t) = \begin{cases} x & \text{on } X' - (\overline{W}_1)' \\ H(x, t \cdot \alpha(x)) & \text{on } W_2' - A' \\ H(x, t) & \text{on } W_0' . \end{cases}$$

Then we can easily show that G is a homotopy from $h \circ f_A$ to identity on X' limited by $\{f_A^{-1}(U) \mid U \in \mathcal{U}\}$.

<u>LEMMA 3.4.6</u>. If a UV^∞-map $f: X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence over a closed subset A of X, then, for each open cover $\mathcal{U}$ of $(X' \underset{f}{\cup} A) \times [-1, 1]$, there

exists a map $g:(X' \underset{f}{\cup} A) \times [-1, 1] \to DM(f_A)$ such that $g \circ f_A^{-1}$ is homotopic (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) to identity on $DM(f_A)$ with the homotopy limited by $\{(f_A')^{-1}(U) | U \in \mathcal{U}\}$ and $f_A' \circ g$ is homotopic (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) to identity on $(X' \underset{f}{\cup} A) \times [-1, 1]$ with the homotopy limited by $\mathcal{U}$.

<u>PROOF</u>. By Lemma 3.4.4 and Lemma 3.4.5, $f_A':DM(f_A) \to (X' \underset{f}{\cup} A) \times [-1, 1]$ is a fine homotopy equivalence between ANR's .

First notice that it is enough to show that, for any open cover $\mathcal{U}$ of $(X' \underset{f}{\cup} A) \times [-1, 1]$, there exists a map $g:(X' \underset{f}{\cup} A) \times [-1, 1] \to DM(f_A)$ such that $g \circ f_A'$ is homotopic (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) to identity on $DM(f_A)$ with the homotopy limited by $\{(f_A')^{-1}(U) | U \in \mathcal{U}\}$ (see the proof of Theorem 6).

Let $\mathcal{U}$ be an open cover of $(X' \underset{f}{\cup} A) \times [-1, 1]$. Since f_A' is a fine homotopy equivalence, there exist a map $h:(X' \underset{f}{\cup} A) \times [-1,1] \to DM(f_A)$ and a homotopy $H:DM(f_A) \times I \to DM(f_A)$ from H_0 = identity to $H_1 = h \circ f_A'$ limited by $\{(f_A')^{-1}(U) | U \in \mathcal{U}\}$. Notice that $H_t \circ (f_A')^{-1} | (X' \underset{f}{\cup} A) \times \{\pm 1\}$ $(t \in I)$ defines a homotopy from $(f_A')^{-1} | (X' \underset{f}{\cup} A) \times \{\pm 1\}$ to $h | (X' \underset{f}{\cup} A) \times \{\pm 1\}$ limited by $\{(f_A')^{-1}(U) | U \in \mathcal{U}\}$. Hence by the modified homotopy extension property of an ANR , there is a homotopy $G:(X' \underset{f}{\cup} A) \times [-1, 1] \to DM(f_A)$ limited by $\{(f_A')^{-1}(U) | U \in \mathcal{U}\}$ such that $G_1 = h$ and $G_t = H_t (f_A')^{-1}$ on $(X' \underset{f}{\cup} A) \times \{\pm 1\}$.

Let $P = [(X' \underset{f}{\cup} A) \times \{\pm 1\} \times I \times I] \cup [DM(f_A) \times I \times \{1\}] \cup [DM(f_A) \times \{0, 1\} \times I]$, and define $\lambda:P \to DM(f_A)$ as follows.

$$\lambda(x, s, t) = \begin{cases} x & \text{if } s = 0 \\ G(f'_A(x), t) & \text{if } s = 1 \\ H(x, s) & \text{if } t = 1 \\ H(x, st) & \text{if } (x,s,t) \in (X' \underset{f}{\cup} A) \times \{\pm 1\} \times I \times I . \end{cases}$$

We can easily check λ is a well-defined continuous map. Since $DM(f_A)$ is an ANR, λ extends to a map from $DM(f_A) \times I \times I \to DM(f_A)$. Hence $G_0 \circ f'_A$ is homotopic (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) to identity on $DM(f_A)$. Furthermore, by the modified homotopy extension property of an ANR, we can claim that $G_0 \circ f'_A$ is homotopic (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) to identity on $DM(f_A)$ with the homotopy limited by $\{(f'_A)^{-1}(U) | U \in \mathcal{U}\}$.

As an immediate corollary of Lemma 3.4.6, we have the following lemma.

LEMMA 3.4.7. If a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence over a closed subset A of X, then for any open subset N of $(X' \underset{f}{\cup} A) \times [-1, 1]$ there exists a map $g:N \to (f'_A)^{-1}(N)$ such that $g \circ f'_A | (f'_A)^{-1}(N)$ is homotopic (rel. $[(X' \underset{f}{\cup} A) \times \{\pm 1\}] \cap (f'_A)^{-1}(N))$ to identity on $(f'_A)^{-1}(N)$ and $f'_A \circ g$ is homotopic (rel. $[(X' \underset{f}{\cup} A) \times \{\pm 1\}] \cap N)$ to identity on N.

LEMMA 3.4.8. If a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space X is a fine homotopy equivalence over a closed subset A of X, then $f'_{A\times[-1,1]}:DM(f) \to DM(\overline{f}_A)$ is a hereditary homotopy equivalence.

PROOF. By Lemma 3.4.6, we can choose a map $h:(X' \underset{f}{\cup} A) \times [-1, 1] \to DM(f_A)$, a homotopy (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) $H:DM(f_A) \times I \to DM(f_A)$ from $H_0 =$ identity to $H_1 = h \circ f'_A$, and

a homotopy (rel. $(X' \underset{f}{\cup} A) \times \{\pm 1\}$) $G:(X' \underset{f}{\cup} A) \times [-1, 1] \times I \to (X' \underset{f}{\cup} A) \times [-1, 1]$ from G_0 = identity to $G_1 = f'_A \circ h$.

Let $c_1:DM(f_A) \to DM(f)$ is the map defined as $c_1 = \overline{f}_A$ on $(X' \underset{f}{\cup} A) \times \{\pm 1\}$ and c_1 = identity on $X' \times (-1, 1)$, and $c_2:(X' \underset{f}{\cup} A) \times [-1, 1] \to DM(\overline{f}_A)$ is the map defined as $c_2 = \overline{f}_A$ on $(X' \underset{f}{\cup} A) \times \{\pm 1\}$ and c_2 = identity on $(X' \underset{f}{\cup} A) \times (-1, 1)$.

Define $h':(X' \underset{f}{\cup} A) \times [-1, 1] \to DM(f_A)$, $H':DM(f_A) \times I \to DM(f_A)$, and $G':(X' \underset{f}{\cup} A) \times [-1, 1] \times I \to (X' \underset{f}{\cup} A) \times [-1, 1]$ as follows.

$$h'(x, t) = \begin{cases} h(x, 2t) & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ h(x, 1) & \text{if } \frac{1}{2} \leq t \leq 1 \\ h(x, -1) & \text{if } -1 \leq t \leq -\frac{1}{2} \end{cases} .$$

$$H'(x,s,t) = \begin{cases} (x, s) & \text{if } s = \pm 1 \\ (f_A(x), -1) & \text{if } 0 \leq t \leq \frac{1}{2} \text{ and } -1 < s \leq t - 1 \\ (f_A(x), 1) & \text{if } 0 \leq t \leq \frac{1}{2} \text{ and } 1 - t \leq s < 1 \\ H(x, \frac{s}{1 - t}, 0) & \text{if } 0 \leq t \leq \frac{1}{2} \text{ and } t-1 \leq s \leq 1 - t \\ (f_A(x), -1) & \text{if } \frac{1}{2} \leq t \text{ and } -1 < s \leq -\frac{1}{2} \\ (f_A(x), 1) & \text{if } \frac{1}{2} \leq t \text{ and } \frac{1}{2} \leq s < 1 \\ H(x, 2s, 2t - 1) & \text{if } \frac{1}{2} \leq t \text{ and } -\frac{1}{2} \leq s \leq \frac{1}{2} \end{cases} .$$

$$G'(x,s,t) = \begin{cases} G(x, -1, 0) & \text{if } 0 \le t \le \frac{1}{2} \text{ and } -1 \le s \le t - 1 \\ G(x, 1, 0) & \text{if } 0 \le t \le \frac{1}{2} \text{ and } 1 - t \le s \le 1 \\ G(x, \frac{s}{1 - t}, 0) & \text{if } 0 \le t \le \frac{1}{2} \text{ and } t - 1 \le s \le 1 - t \\ G(x, -1, 2t - 1) & \text{if } \frac{1}{2} \le t \text{ and } -1 \le s \le -\frac{1}{2} \\ G(x, 1, 2t - 1) & \text{if } \frac{1}{2} \le t \text{ and } \frac{1}{2} \le s \le 1 \\ G(x, 2s, 2t - 1) & \text{if } \frac{1}{2} \le t \text{ and } -\frac{1}{2} \le s \le \frac{1}{2} . \end{cases}$$

Then we can easily check that $c_1 \circ H'_t \circ c_1^{-1}$ $(0 \le t \le 1)$ is a homotopy from identity on $DM(f)$ to $c_1 \circ h' \circ c_2^{-1} \circ f'_{A\times[-1,1]}$ and $c_2 \circ G'_t \circ c_2^{-1}$ $(0 \le t \le 1)$ is a homotopy from identity on $DM(\overline{f}_A)$ to $f'_{A\times[-1,1]} \circ c_1 \circ h' \circ c_2^{-1}$. Hence $f'_{A\times[-1,1]}$ is a homotopy equivalence. But if we use Lemma 3.4.7 together with the above analysis, we can easily show that $f'_{A\times[-1,1]}$ is a hereditary homotopy equivalence.

LEMMA 3.4.9. If a UV^{∞}-map $f:X' \to X$ from an ANR X' to a metric space is a fine homotopy equivalence over a closed subset A of X, then $f''_{A\times[-1,1]}:DM(f'_{A\times[-1,1]}) \to DM(\overline{f}_A) \times [-1, 1]$ is a hereditary homotopy equivalence.

PROOF. Since $f'_A:DM(f_A) \to (X' \cup_f A) \times [-1, 1]$ is a UV^{∞}-map between ANR's, $f''_A:DM(f'_A) \to (X' \cup_f A) \times [-1, 1] \times [-1, 1]$ is a UV^{∞}-map between ANR's. Hence f''_A is a fine homotopy equivalence.

By the same argument as in the proof of Lemma 3.4.6, we can prove that, for any open subset N of $(X' \cup_f A) \times [-1, 1] \times [-1, 1]$, there exists a map $g:N \to (f''_A)^{-1}(N)$ such that $g \circ f''_A|(f''_A)^{-1}(N)$ is

homotopic (rel. $[(X' \cup_f A) \times \{\pm 1\} \times I] \cap (f''_A)^{-1}(N))$ to identity on $(f''_A)^{-1}(N)$ and $f''_A \circ g$ is homotopic (rel. $[(X' \cup_f A) \times \{\pm 1\} \times I] \cap N)$ to identity on N. Then, by modifying the map and homotopies as we did in the proof of Lemma 3.4.8, we can show that $f''_{A\times[-1,1]}$: $DM(f'_A) \to DM(\overline{f}_A) \times [-1, 1]$ is a hereditary homotopy equivalence.

Finally, we are ready to improve a theorem of G. Kozlowski's [K2].

<u>THEOREM 7</u>. If $f:X' \to X$ is a UV^∞-map from an ANR X' to a metric space X and $\{X_n\}_{n=1}^\infty$ is a sequence of closed subsets of X such that $\bigcup_{n=1}^\infty X_n = X$ and f is a fine homotopy equivalence over each X_n, then f is a hereditary shape equivalence.

<u>PROOF</u>. First notice that $f_i:X' \to X' \cup_f X_i$ defined by $f_i = f_{X_i}$ and $f'_i:DM(f_i) \to (X' \cup_f X_i) \times [-1, 1]$ are hereditary homotopy equivalences by Lemma 3.4.4 and Lemma 3.4.5. Therefore, for any map $h:X' \to Y$ to an ANR Y, there exists an extension $H:M(f) \to Y$ of h by Lemma 3.4.3. Hence $f^\#:[X:Y] \to [X':Y]$ is onto for any ANR Y.

We will show $f^\#$ is one-to-one.

Consider the map $f':DM(f) \to X \times [-1, 1]$. Certainly $X \times [-1, 1] = \bigcup_{i=1}^\infty (x_i \times [-1, 1])$, $f'_{X_i\times[-1,1]}:DM(f) \to DM(\overline{f}_{X_i})$ is a hereditary homotopy equivalence by Lemma 3.4.8, and $f''_{X_i\times[-1,1]}$: $DM(f'_{X_i\times[-1,1]}) \to DM(\overline{f}_{X_i}) \times [-1, 1]$ is a hereditary homotopy equivalence by Lemma 3.4.9 $(i = 1,2,\ldots)$. Now suppose $g_1, g_2:X \to Y$ are maps to an ANR Y such that $g_1 \circ f$ is homotopic to $g_2 \circ f$. Then we have a map $G:DM(f) \to Y$ satisfying $G(x, -1) = g_1(x)$ and $G(x, 1) =$

$g_2(x)$ for each $x \in X$. Hence we can apply Lemma 3.4.3 with f' , $f'_{X_i \times [-1,1]}$, $f''_{X_i \times [-1,1]}$ $(i = 1,2,\ldots)$, and G to obtain an extension $\overline{G}:DM(f') \to Y$ of G . Certainly $\overline{G}$ induces a homotopy from g_1 to g_2 . Therefore $f^{\#}:[X:Y] \to [X':Y]$ is one-to-one.

Hence f is a shape equivalence.

Since we can apply the above analysis to any open subset U of X , we can claim that f is a hereditary shape equivalence.

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APPENDIX

PROOF of Theorem 5 for (non-separable) metric spaces.

We will show that, for each open cover $\mathcal{U}$ of X, there exists a map $K:M(f) \to X'$ satisfying $K|\{0\}$ = identity and $\{K(M(f:\{x\}))|x \in X\}$ refines $\mathcal{U}'$ as we did to prove the theorem for the case of separable metric spaces.

Let $\mathcal{U}$ be an open cover of X. We may assume that $\mathcal{U}$ is a locally finite open cover of X. Let C_k, $\mathcal{N}_k$, and B_k be the families as in Proposition 3.7 satisfying $(*_1)$ through $(*_5)$.

Choose a locally finite open cover $\mathcal{V} = \{V_\alpha|\alpha \in A\}$ of X such that each $\overline{V}_\alpha$ intersects only finitely many members of $\mathcal{U}$, and choose a precise open refinement $\mathcal{M} = \{M_\alpha|\alpha \in A\}$ such that $\overline{M}_\alpha \subset V_\alpha$ for each $\alpha \in A$.

Assume A is well-ordered with "$\leqq$", and define the property $P(\alpha)$ as follows; for each $\beta \leq \alpha$ and each $\gamma \leq \beta$, there exist an open subset W^β_γ of X, a map $H^\beta:M(f: \cup\{W^\beta_i\ |i \leq \beta\}) \to X'$, and a family $\mathcal{L}^\beta_k = \{\mathcal{L}^\beta_b|b \in B_k\}$ $(k = 0, 1, \ldots)$ of open subsets of X such that

(1) $\overline{M}_\gamma \subset W^\beta_\gamma \subset V_\gamma$, (2) $W^\beta_\gamma \subset W^\gamma_\gamma$,

(3) $W^\gamma_\gamma - \cup\{\overline{V}_i|\gamma < i \leq \beta\} \subset W^\beta_\gamma$,

(4) $H^\beta = H^\gamma$ on $M(f: \cup\{W^\beta_i|i \leq \gamma\})$,

(5) $H^\beta|\{0\}$ = inclusion,

(6) $H^\beta(M(f:[\cup\{W^\beta_i|i \leq \beta\}] \cap L^\beta_b)) \subset I'_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k)$,

(7) $C_b \subset L^\beta_b \subset L^\gamma_b \subset N_b$ for each $b \in \bigcup_{k=0}^{\infty} B_k$,

(8) $L^\gamma_b - \{\cup\overline{V}_i|\gamma < i \leq \beta\} \subset L^\beta_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

We will show that $P(\alpha)$ is true for any $\alpha \in A$ by transfinite induction.

Let 0 be the first element of A. By using Lemma 3.3.5, we certainly get $W_0^0 = V_0$, $H^0 : M(f : W_0^0) \to X'$, and a family $\mathcal{L}_k^0 = \{L_b^0 \mid b \in B_k\}$ $(k = 0, 1, \dots)$ of open subsets of X such that (1) through (8) are hold.

Let $\alpha \in A$, and assume $P(\beta)$ is true for any $\beta < \alpha$. Hence, for each $\gamma \leq \beta < \alpha$, we have an open subset W_γ^β of X, a map $H^\beta : M(f : \cup\{W_i^\beta \mid i \leq \beta\}) \to X'$, and a family $\mathcal{L}_k^\beta = \{L_b^\beta \mid b \in B_k\}$ $(k = 0, 1, \dots)$ of open subsets of X satisfying (1) through (8).

For each $\beta < \alpha$, let $W_\beta = \cap \{W_\beta^i \mid \beta \leq i < \alpha\}$. We will show that each W_β is a neighborhood of $\overline{M}_\beta$. If α has the immediate predecessor $\alpha-1$, then $W_\beta = W_\beta^{\alpha-1}$ by (2), hence W_β is a neighborhood of $\overline{M}_\beta$. Suppose α does not have a immediate predecessor. Then for each $x \in M_\beta$, we can choose a neighborhood G of x in X such that there exists $\alpha(x)$ in A satisfying $\beta < \alpha(x) < \alpha$ and $G \cap \overline{V}_i = \phi$ for any $i \geq \alpha(x)$. Hence $G \cap W_\beta^{\alpha(x)} \subset W_\beta^i$ for any i with $\beta \leq i < \alpha$ by (2) and (3). Hence W_β is a neighborhood of $\overline{M}_\beta$ for any $\beta < \alpha$.

For each $b \in \bigcup_{k=0}^{\infty} B_k$, let $L_b = \cap \{L_b^\beta \mid \beta < \alpha\}$. We will show that each L_b is an open set containing C_b. If α has the immediate predecessor $\alpha-1$, then $L_b = L_b^{\alpha-1}$ by (7), hence L_b is an open set containing C_b. Suppose α has no immediate predecessor. Then for each $x \in L_b$ we can choose a neighborhood G of x such that there exists $\alpha(x)$ in A satisfying $\alpha(x) < \alpha$ and $G \cap \overline{V}_i = \phi$ for any i with $\alpha(x) \leq i$. Then $G \cap L_b^{\alpha(x)} \subset L_b^i$ for any $i < \alpha$ by (7) and (8). Therefore each L_b is an open set containing C_b. Hence

we have a family $\mathcal{L}_k = \{L_b | b \in B_k\}$ $(k = 0, 1, \ldots)$ of open subsets of X satisfying $C_b \subset L_b \subset N_b$ for any $b \in \bigcup_{k=0}^{\infty} B_k$.

Certainly we can define a map $H : M(f: \cup\{W_\beta | \beta < \alpha\}) \to X'$ as $H = H^\beta$ on $M(f:W_\beta)$ using (4). Then it is easy to see that $H(M(f:[\cup\{W_\beta | \beta < \alpha\}] \cap L_b)) \subset I'_b$ for each $b \in \bigcup_{k=0}^{\infty} B_k$.

For each $\beta < \alpha$, choose an open set G_β satisfying $\overline{M}_\beta \subset G_\beta \subset \overline{G}_\beta \subset W_\beta$, and let $W_\beta^\alpha = G_\beta \cup (W_\beta - \overline{V}_\alpha)$.

Now we apply Lemma 3.3.5 with $D = \cup \{W_\beta | \beta < \alpha\}$, $E = V_\alpha$, $H : M(f: \cup \{W_\beta | \beta < \alpha\}) \to X'$, $\mathcal{L}_k (k = 0, 1, \ldots)$, and a closed set $F = [\cup \{\overline{G}_\beta | \beta < \alpha\}] \cup [(\cup \{W_\beta | \beta < \alpha\}) - V_\alpha]$ in $(\cup \{W_\beta | \beta < \alpha\}) \cup V_\alpha$ to obtain $\overline{H} : M(f(\cup\{W_\beta | \beta < \alpha\}) \cup V_\alpha) \to X'$ and a family $\mathcal{L}_k^\alpha = \{L_b^\alpha | b \in B_k\}$ $(k = 0, 1, \ldots)$ of open subsets of X such that $\overline{H} = H$ on $M(f:F)$, $\overline{H}|\{0\}$ = inclusion, $\overline{H}(M(f:[(\cup \{W_\beta | \beta < \alpha\}) \cup V_\alpha] \cap L_b^\alpha)) \subset I'_b$, $C_b \subset L_b \subset L_b^\alpha \subset N_b$, and $L_b - \overline{V}_\alpha \subset L_b^\alpha$ for each $b \in \bigcup_{k=0}^{\infty} B_k$.

Let $W_\alpha^\alpha = V_\alpha$, and define $H^\alpha : M(f: \cup\{W_i^\alpha | i \leq \alpha\}) \to X'$ as $H^\alpha = \overline{H}|M(f: \cup\{W_i^\alpha | i \leq \alpha\})$.

By now we have found that, for each $\beta \leq \alpha$ and each $\gamma \leq \beta$, there exist an open subset W_γ^β of X , a map $H^\beta : M(f: \cup\{W_i^\beta | i \leq \beta\}) \to X'$, and a family $\mathcal{L}_k^\beta = \{L_b^k | b \in B_k\}$ $(k = 0, 1, \ldots)$ of open subsets of X for which (1), (2), (4), (5), (6), and (7) are clearly held. Notice that, for each $\beta < \alpha$ and $\gamma < \beta$, $W_\gamma^\gamma - \cup\{V_i | \gamma < i < \alpha\} \subset W_\gamma^\gamma - \cup\{\overline{V}_i \quad i \leq \beta\} \subset W_\gamma^\beta$, hence $W_\gamma^\gamma - \cup\{\overline{V}_i | \gamma < i < \alpha\} \subset W_\gamma$. Therefore $W_\gamma^\gamma - \cup\{\overline{V}_i | \gamma < i \leq \alpha\} \subset W_\gamma - \overline{V}_\alpha \subset W_\alpha^\alpha$, hence (3) is held. Also, by the same analysis, we can show that $L_b^\gamma - \cup\{\overline{V}_i | \gamma < i \leq \alpha\} \subset L_b^\alpha$ for any $\gamma < \alpha$ and any $b \in \bigcup_{k=0}^{\infty} B_k$, hence (8) is held.

Therefore $P(\alpha)$ is true for any $\alpha \in A$.

Finally we define $K: M(f) \to X'$ as $K = H^{\beta}$ on $M(f:M_{\beta})$. Then we can easily see that K is a well-defined continuous function such that $K|\{0\}$ = identity on X' . If we let $D_b = \cap\{L_b^{\alpha} | \alpha \in A\}$, then certainly $C_b \subset D_b$ and $K(M(f:D_b)) \subset I_b'$ for any $b \in \bigcup_{k=0}^{\infty} B_k$. Since $\cup\{C_b | b \in \bigcup_{k=0}^{\infty} B_k\} = X$ and $\{I_b' | b \in \bigcup_{k=0}^{\infty} B_k\}$ refines $\mathcal{U}'$, $\{K(M(f:\{x\})) | x \in X\}$ refines $\mathcal{U}'$. Therefore X is an ANR .

VITA

Jung-In Kang Choi was born in Junju, Korea, on August 17, 1952. She graduated from Kijun Girl's High School in Junju, Korea, in 1971. She holds a Bachelor of Science degree in Mathematics from the Soongjun University in Taejon (1975) and a Master's degree in Mathematics from the Kyungbook National University (1977). In 1978, she began graduate study at The University of Tennessee, Knoxville, where she was awarded the Doctor of Philosophy degree in Mathematics in June 1986.

The author is married to Kyung-Ju Choi who holds a Doctor of Philosophy degree in Polymer Engineering. They have an eight-year old son, Jin, and a three-year old son, Joon.