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# Spectral Theory of Self-Adjoint Ordinary Differential Operators

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To the Graduate Council:

I am submitting herewith a dissertation written by Charles C. Oehring entitled "Spectral Theory of Self-Adjoint Ordinary Differential Operators." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

F. A. Ficken, Major Professor

We have read this dissertation and recommend its acceptance:

Leo Simons, Walter Snyder, O. M. Harrod, D. D. Lillian

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

November 24, 1958

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Dale Nantling
Dean of the Graduate School

# SPECTRAL THEORY OF SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

A THESIS

Submitted to
The Graduate Council
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
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## INTRODUCTION

Many of the properties of the ordinary Fourier series expansion of a given function are shared by the orthogonal expansions in terms of eigenfunctions of a second order ordinary differential operator. Let $p = p(x)$ and $q = q(x)$ be real-valued functions such that $p$, $p'$, and $q$ are continuous, and $p(x) > 0$, on a finite interval $a \leq x \leq b$. Let $\lambda$ be a complex parameter. The classical Sturm-Liouville theory [9, section 27; 4, Chapter 7; 21, Chapter 1][1] is concerned with solutions of the differential equation

$$-(py')' + qy = \lambda y ,$$

which satisfy certain real boundary conditions whose form need not be given here. These solutions, the so-called eigenfunctions, exist only for certain values of $\lambda$, the corresponding eigenvalues. The Sturm-Liouville theory states that the eigenvalues constitute a countable set of real numbers which cluster only at $+\infty$. The corresponding eigenfunctions constitute an orthogonal system on $[a,b]$ which is complete in $L^2(a,b)$. Thus the Parseval relation is also valid.

If now the possibilities $a = -\infty$ and $b = +\infty$ are allowed, or if the restriction $p(x) > 0$ is required merely for $a < x < b$, the complexity of the situation increases. In these cases the problem is called singular, in contrast to the regular case considered above.

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[1]Numbers in brackets refer to the bibliography at the end of this paper.

To obtain satisfactory analogues of the completeness and Parseval theorems, it now becomes necessary to replace the series expansion of an arbitrary function in $L^2(a,b)$ by an expansion in terms of a Stieltjes integral. The singular case has been treated exhaustively by Weyl [23;24], by Stone [20] who uses the general theory of unbounded symmetric operators, by Titchmarsh [21] who uses function theoretic methods, by Kodaira [10] who combines and simplifies the ideas of Weyl and Stone, by Yosida [25], and by Levinson [13;14] who obtains results as limiting cases of theorems valid for compact subintervals of $(a,b)$.

The basic facts in the regular case extend verbatim to a formally self-adjoint differential operator $L$ (see Chapter I) of arbitrary order $n$, with complex coefficients defined on a compact interval, provided the coefficient of the $n^{th}$ order derivative does not vanish on that interval [4; Chapter 7]. Proofs in this case require nothing more than the Hilbert-Schmidt theory of integral equations. When the problem is singular, general results have been obtained only recently. Glazman [5] has generalized several important results of Weyl and Stone concerning the nature of the boundary conditions when the coefficients are real. Kodaira [11] has also discussed boundary conditions in the case of real coefficients and proved the analogues of the completeness and Parseval relations. Coddington [2] and Levinson [15] have considered these questions when the coefficients are complex and have obtained the expansion and Parseval theorems in two important cases. In these cases they also prove the inverse transform theorem which is the analogue of the Riesz-Fischer theorem in the theory of ordinary Fourier series.

One of the main results of the present paper is a proof, in Chap-

ter II, of the expansion, Parseval, and inverse transform theorems, by use of the theory of generalized direct sums of Hilbert spaces, which theory is due to von Neumann [22]. These theorems are most elegantly stated in a form given them by Kodaira [10]. Let $(\rho_{jk})$ be an Hermitian, non-decreasing n by n matrix whose elements are Lebesgue-Stieltjes measures on the real $\lambda$ axis, $\Lambda$ (see Chapter II). Let $\chi$ and $\omega$ be n-vector valued functions of $\lambda$ with $i^{th}$ component $\chi_i(\lambda)$ and $\omega_i(\lambda)$ respectively. If we introduce the inner product

$$(\omega, \chi) = \int_\Lambda \sum_{j,k=1}^{n} \omega_j(\lambda) \overline{\chi_k(\lambda)} d\rho_{jk}(\lambda) ,$$

it is easy to see that the set $\Omega$ of those $\omega$ for which $(\omega, \omega) < \infty$ becomes a Hilbert space in this inner product. The expansion, Parseval, and inverse transform theorems may then be stated as the following theorem, which will be proved in Chapter II.

If L is a formally self-adjoint differential operator of order n , and if H is a self-adjoint extension of L in the Hilbert space $L^2(a,b)$ , then corresponding to each system of linearly independent solutions $s_j(x, \lambda)$ $j = 1, \ldots, n$, of $Ly = \lambda y$ on $(a,b)$ , there exists a spectral matrix $(\rho_{jk})$ , with the properties described above, such that the associated Hilbert space $\Omega$ is unitarily equivalent to $L^2(a,b)$ . Thus if $u \in L^2(a,b)$ and $\omega$ is its image in $\Omega$ , the Parseval equality

$$\|u\| = \|\omega\|$$

is valid. The fact that to each $\omega$ in $\Omega$ corresponds a $u \in L^2(a,b)$ is the content of the inverse transform theorem. Finally the expansion theorem, the analogue of the completeness theorem in the regular case, states that $u$ and $\omega$ are related by the specific formulas

$$\omega_j(\lambda) = \int_a^b \overline{s_j(x,\lambda)}\, u(x)\, dx \qquad \text{and}$$

$$u(x) = \int_\Lambda \sum_{j,k=1}^{n} s_j(x,\lambda)\, \omega_k(\lambda)\, d\rho_{kj}(\lambda) ,$$

where the first integral converges in the norm of $\Omega$ and the second in the norm of $L^2(a,b)$ .

Chapter I contains definitions and facts pertinent to later chapters.

In Chapter III a generalization, with a different proof, is given of a theorem of Coddington [2, Theorem 3] which characterizes the self-adjoint extensions of L , when such extensions exist, in terms of certain boundary conditions.

It is shown in Chapter IV that a theorem of Hartman and Wintner [7] concerning the eigenvalues of L in one of the cases considered by Weyl is actually valid in a much more general setting.

Since the printing of the text of this paper the author has noted the appearance in the Canadian Journal of Mathematics, Vol. X, No. 3 (1958), pp. 431-446, of a paper by F. Brauer in which the eigenfunction expansion problem is treated using the method of generalized direct sums of Hilbert spaces. Brauer's approach to the problem is very similar to the one used in the present paper; and in fact he proves the analogues of the main results of Theorem 2.64 for the eigenvalue problem $Lu = \lambda Mu$, where $L$ and $M$ are ordinary self-adjoint differential operators and $M$ is semi-bounded below with positive lower bound. Since we have assumed only that $r > 0$, his results, as stated, actually do not include Theorem 2.64. Moreover, he does not state equations (2.69) or (2.70). In proving the analogue of the assertion connected with (2.49), Brauer appeals to a general theorem on weak solutions of partial differential equations; thus he is able to state that his main results are also valid for certain elliptic partial differential operators. Also, it might be mentioned that he does not give the Fubini argument on pp. 14, 15 or the interchange of sum and integral on pp. 16, 17. The method given on pp. 18-20 and 22-25, of extending the inversion formulas from restricted classes of functions in $\mathcal{H}$ to all of $\mathcal{H}$ is merely alluded to as standard in Brauer's paper.

Finally, Brauer discusses boundary conditions associated with $Lu = \lambda Mu$ and gives a theorem which is almost identical in statement and proof with Theorem 3 in [2]. Our Theorem 3.13, which concerns abstract operators in an arbitrary Hilbert space, includes both of these theorems.

From Brauer's paper we have also learned that the von Neumann theory

of generalized direct sums had been applied to elliptic partial differential operators by L. Gårding and published in mimeographed notes through the University of Maryland in 1954, through the University of Lund in 1956, and through the University of Colorado in 1957. The reader is referred to the bibliography of Brauer's paper for references.

# CHAPTER I

## PRELIMINARIES

### Linear Operators; Spectral Representation Of Self-Adjoint Operators

A (linear) operator or transformation $A$ in a Hilbert space is a map from a linear manifold $D_A \subset \mathcal{H}$ into $\mathcal{H}$ which is additive and homogeneous. For the basic facts about unbounded operators see [20], [19], or [16]. If $D_A$ is dense in $\mathcal{H}$, there is a well defined operator $A^*$, the adjoint of $A$ whose domain $D_{A^*}$ consists of those $v \in \mathcal{H}$ corresponding to which there exists a unique $v^* \in \mathcal{H}$ satisfying

$$(1.1) \qquad (Au, v) = (u, v^*) \quad \text{for all } u \in D_A .$$

$A^*$ is defined on $D_{A^*}$ into $\mathcal{H}$ by the equation $A^*v = v^*$. If $D_A$ is not dense in $\mathcal{H}$ there will not be a unique $v^*$ with the property 1.1; so that $A^*$ will not exist.

If $B$ is an operator such that $D_B \supset D_A$, and if $Bu = Au$ for all $u \in D_A$, then $B$ is called an extension of $A$, written $A \subseteq B$. If $D_A$ is dense in $\mathcal{H}$ and if $A \subseteq B$, then $B^* \subseteq A^*$. We write $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

If $\|Au\| = \|u\|$ for every $u \in D_A$, $A$ is called isometric. If $D_A$ is dense in $\mathcal{H}$ and $A \subseteq A^*$, $A$ is called symmetric. If $D_A$ is dense and $A = A^*$, $A$ is called self-adjoint. If $D_A$ is dense and $A \subseteq B$ where $B$ is symmetric, then $B \subseteq A^*$. A symmetric operator may or may not have self-adjoint extensions. In Chapter IV, where it is pertinent, an outline is presented of von Neumann's theory [20, Chapter IX] of the symmetric extensions of a given symmetric operator. This

theory leads to an elegant necessary and sufficient condition for the existence of self-adjoint extensions.

An operator $A$ is called closed if for each sequence $\{u_n\}$ of elements of $D_A$ which converges, say to $u$, and for which the sequence $\{Au_n\}$ converges, say to $v$, it is true that $u \in D_A$ and $Au = v$. An adjoint operator is always closed. If $A$ is closed and $D_A$ dense in $\mathcal{H}$, then $D_{A^*}$ is also dense in $\mathcal{H}$ and $A^{**} = A$. [19, p. 302]

If $A$ is self-adjoint it has a spectral representation which we proceed to describe.

A resolution of unity [19, p. 313] is a one parameter family of projections $E(\lambda)$ in $\mathcal{H}$, $-\infty < \lambda < +\infty$, satisfying

(a) $E(\lambda) \subseteq E(\mu)$ if $\lambda < \mu$

(b) $E(\lambda + 0) = E(\lambda)$

(c) $E(\lambda) \longrightarrow 0$ when $\lambda \longrightarrow -\infty$, and $E_\lambda \longrightarrow I$ (the identity operator) when $\lambda \longrightarrow +\infty$.

The spectral theorem [19, Section 120] states that if $A$ is a self-adjoint operator, there exists a resolution of unity $E(\lambda)$ such that $D_A$ consists of those $u \in \mathcal{H}$ satisfying

(1.2) $$\|Au\|^2 = \int_{-\infty}^{\infty} \lambda^2 d_\lambda \|E(\lambda)u\|^2 < +\infty ;$$

and if $u \in D_A$, $v \in \mathcal{H}$, then

(1.3) $$(Au, v) = \int_{-\infty}^{\infty} \lambda \, d_\lambda (E(\lambda)u, v) .$$

Equation 1.3 determines $E(\lambda)$ uniquely. The eigenvalues of $A$ are the points of discontinuity of $E(\lambda)$.

Generalized Direct Sums Of Hilbert Spaces

To von Neumann [22] is due the idea of a generalized direct sum of Hilbert spaces. (A lucid exposition is contained in [17, Section 2].) For each real $\lambda$ let $\widetilde{\mathcal{H}}^{(\lambda)}$ be a separable Hilbert space. Let $\sigma(\lambda)$ be a real valued, non-decreasing, right continuous, bounded function defined on $\Lambda$, the real $\lambda$ axis. We shall follow von Neumann [22] in calling a function with these properties an N-function. Let $\widetilde{X}$ denote the set-theoretic cartesian product $\widetilde{X} = \prod_{\lambda \in \Lambda} \widetilde{\mathcal{H}}^{(\lambda)}$. We use $\widetilde{u}, \widetilde{v}$ to denote elements of $\widetilde{X}$ whose respective $\lambda^{th}$ coordinates we denote by $\widetilde{u}(\lambda)$, $\widetilde{v}(\lambda)$. If $\widetilde{\mathcal{H}}$ is a subset of $\widetilde{X}$ satisfying the following axioms, $\widetilde{\mathcal{H}}$ is called a generalized direct sum of the $\widetilde{\mathcal{H}}^{(\lambda)}$, and we say that $\sigma$ belongs to $\widetilde{\mathcal{H}}$.

(a) $\widetilde{u} \in \widetilde{\mathcal{H}}$ and $\widetilde{v} \in \widetilde{\mathcal{H}}$ imply that the inner product (in $\widetilde{\mathcal{H}}^{(\lambda)}$) $(\widetilde{u}(\lambda), \widetilde{v}(\lambda))$ is a $\sigma$-measurable function of $\lambda$.

(b) $\widetilde{u} \in \widetilde{\mathcal{H}}$ implies $\int_\Lambda \|\widetilde{u}(\lambda)\|^2 d\sigma(\lambda) < +\infty$.

(c) $\widetilde{\mathcal{H}}$ is maximal with respect to properties (a) and (b) in the sense that if $\widetilde{u} \in \widetilde{X}$ is such that $(\widetilde{u}(\lambda), \widetilde{v}(\lambda))$ is $\sigma$-measurable for every $\widetilde{v} \in \widetilde{\mathcal{H}}$, and $\int_\Lambda \|\widetilde{u}(\lambda)\|^2 d\sigma(\lambda) < +\infty$, then $\widetilde{u} \in \widetilde{\mathcal{H}}$.

It follows from (a) and (b) that the integral

$$(1.4) \qquad (\widetilde{u}, \widetilde{v}) = \int_\Lambda (\widetilde{u}(\lambda), \widetilde{v}(\lambda)) d\sigma(\lambda) ,$$

exists and (using (c) also) that $\tilde{\mathcal{H}}$ with this inner product is a Hilbert space. As usual we identify elements the norm of whose difference vanishes. In particular completeness is proved [22, Lemma 2] much as is completeness of the ordinary $L^2$ spaces. It also follows from the completeness proof that if $\tilde{u}_n \longrightarrow \tilde{u}$ in $\tilde{\mathcal{H}}$ as $n \longrightarrow \infty$, then there exists a subsequence $\tilde{u}_{n_k}$ such that $\tilde{u}_{n_k}(\lambda) \longrightarrow \tilde{u}(\lambda)$ in $\tilde{\mathcal{H}}^{(\lambda)}$, except possibly for a $\lambda$ set of $\sigma$-measure zero. The linear operations in $\tilde{\mathcal{H}}$ are defined according to the rule that $a\tilde{u} + b\tilde{v} = \tilde{w}$ is equivalent to $a\tilde{u}(\lambda) + b\tilde{u}(\lambda) = \tilde{w}(\lambda)$.

Let $k(\lambda) = \dim \tilde{\mathcal{H}}^{(\lambda)}$ (the value $+\infty$ being allowed). Let

$$(1.5) \qquad \Delta_\ell = \{\lambda \in \Lambda : k(\lambda) = \ell\} \quad \ell = 1, 2, \ldots, +\infty .$$

For each $\lambda$ suppose $\tilde{\Psi}_1(\lambda), \ldots, \tilde{\Psi}_{k(\lambda)}(\lambda)$ is a basis in $\tilde{\mathcal{H}}^{(\lambda)}$ having the property that if $\tilde{u} \in \tilde{\mathcal{H}}$, then (a) and (b) are equivalent to both the conditions

(a') $(\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda))$ is $\sigma$-measurable for $m = 1, 2, \ldots, k(\lambda)$

(b') $\int_\Lambda \sum_{m=1}^{k(\lambda)} |(\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda))|^2 d\sigma(\lambda) < +\infty$ .

The systems $\{\tilde{\Psi}_m(\lambda)\}$ are then said to constitute a measurable family with respect to $\tilde{\mathcal{H}}$. Measurable families exist, and $k(\lambda)$ is $\sigma$-measurable [22, Theorem I]. Conversely [22, Theorem II] if $b_m(\lambda)$, $m = 1, \ldots, k(\lambda)$, are $\sigma$-measurable and satisfy

$\int_{\Lambda} \sum_{m=1}^{k(\lambda)} |b_1(\lambda)|^2 \, d\sigma(\lambda) < +\infty$ , and if $\{\tilde{\Psi}_m(\lambda)\}$ is a measurable family, then $\tilde{u}(\lambda) \equiv \sum_{m=1}^{k(\lambda)} b_m(\lambda) \tilde{\Psi}_m(\lambda)$ defines an element $\tilde{u} \varepsilon \tilde{\mathcal{H}}$ .

Let $E(\lambda)$ be a resolution of unity in a Hilbert space $\mathcal{H}$ . $E(\lambda)$ is said to be absolutely continuous with respect to an N-function $\tau(\lambda)$ if and only if $\|E(\lambda)u\|$ , considered as a function of $\lambda$ , is absolutely continuous with respect to $\tau(\lambda)$ for every $u \varepsilon \mathcal{H}$ . An N-function $\sigma(\lambda)$ is said to be equivalent to $E(\lambda)$ if and only if for every N-function $\tau(\lambda)$ , the absolute continuity of $E(\lambda)$ with respect to $\tau(\lambda)$ is equivalent to the absolute continuity of $\sigma(\lambda)$ with respect to $\tau(\lambda)$ .

Let $\tilde{E}(\lambda)$ be a resolution of unity in a generalized direct sum $\tilde{\mathcal{H}}$ to which $\sigma(\lambda)$ belongs. $\tilde{E}(\lambda)$ will be said to belong to $\tilde{\mathcal{H}}$ if $\tilde{u} \varepsilon \tilde{\mathcal{H}}$ and $\tilde{E}(\mu)\tilde{u} = \tilde{w}$ together imply $\tilde{w}(\lambda) = e_\mu(\lambda)\tilde{u}(\lambda)$ , where $e_\mu(\lambda)$ is the characteristic function of the $\lambda$ set $-\infty < \lambda \leq \mu$ . If $\mathcal{H}$ is a Hilbert space unitarily equivalent to $\tilde{\mathcal{H}}$ and $E(\lambda)$ the resolution of unity in $\mathcal{H}$ corresponding to $\tilde{E}(\lambda)$ in $\tilde{\mathcal{H}}$ , we will also say that $E(\lambda)$ belongs to $\tilde{\mathcal{H}}$ if $\tilde{E}(\lambda)$ belongs to $\tilde{\mathcal{H}}$ .

If $E(\lambda)$ is a resolution of unity in a separable Hilbert space $\mathcal{H}$ and $\sigma(\lambda)$ an N-function, then there exists a generalized direct sum $\tilde{\mathcal{H}}$ which is unitarily equivalent to $\mathcal{H}$ and to which $E(\lambda)$ and $\sigma(\lambda)$ belong if and only if $\sigma(\lambda)$ is equivalent to $E(\lambda)$ [22, Theorem III]. As a consequence of this theorem and the fact that there always exists a $\sigma(\lambda)$ equivalent to each $E(\lambda)$ [22, p. 407] we may state the following lemma, which will be fundamental to the proof of the

main theorem in Chapter II.

1.6 Lemma. Corresponding to any resolution of unity $E(\lambda)$ in a separable Hilbert space $\mathcal{H}$ there exists a generalized direct sum $\tilde{\mathcal{H}}$ which is unitarily equivalent to $\mathcal{H}$ and to which $E(\lambda)$ belongs.

Differential Operators

By a (linear, ordinary) differential operator $L$ is meant an expression of the form

$$(1.7) \qquad L \equiv p_0\left(\frac{d}{dx}\cdot\right)^n + p_1\left(\frac{d}{dx}\cdot\right)^{n-1} + \ldots + p_n \cdot \quad ,$$

where $p_i = p_i(x)$ is a complex valued function of class $C^{n-i}$ on an interval $a < x < b$ , $a = -\infty$ and $b = +\infty$ being allowed. The domain of $L$ consists of functions $u(x)$ for which $u^{(n-1)}(x)$ is absolutely continuous on $(a,b)$ . $L$ is called (formally) self-adjoint if it coincides with its Lagrange adjoint [4, pp. 84-86]

$$(1.8) \qquad L^* \equiv (-1)^n\left(\frac{d}{dx}\right)^n(\bar{p}_0\cdot) + (-1)^{n-1}\left(\frac{d}{dx}\right)^{n-1}(\bar{p}_1\cdot) + \ldots + \bar{p}_n\cdot \quad .$$

We suppose $p_0(x) \neq 0$ on $(a,b)$ .

Let $r(x) > 0$ be defined and measurable on $(a,b)$ and bounded on every compact subinterval of $(a,b)$ . We are interested in the eigenvalue problem $Ly = \lambda ry$ . [18; 12] . Let $\mathcal{H}$ be the Hilbert space of (equivalence classes of) complex valued measurable functions $u$ defined on $(a,b)$ and satisfying

$$(1.9) \qquad \int_a^b |u(x)|^2 \, r(x)dx < +\infty ,$$

The inner product in $\mathcal{H}$ is given by

$$(1.10) \qquad (u, v) = \int_a^b u(x)\overline{v(x)} \, r(x)dx$$

Following [18] (compare also [20] and [2]) let $D$ consist of those $u \in \mathcal{H}$ for which the following conditions are satisfied.

(i) $u$ and its derivatives to order $n - 1$ are continuous on $(a, b)$.

(ii) $u^{(n-1)}$ is absolutely continuous on every compact subinterval of $(a, b)$.

(iii) $\frac{1}{r} Lu \in \mathcal{H}$.

Let $T$ be the operator in $\mathcal{H}$, with domain $D = D_T$, defined by $Tu = \frac{1}{r} Lu$ for $u \in D$.

If $a < x_1 < x_2 < b$ and $u, v \in D$, then

$$(1.11) \qquad \int_{x_1}^{x_2} (\bar{v}Lu - uL\bar{v})dx = [u, v](x_2) - [u, v](x_1) ,$$

where $[u, v](x)$ is a certain bilinear form in $u, v,$ and their derivatives to order $n - 1$ [4, p. 86]. Since $v \in \mathcal{H}$ and $\frac{1}{r} Lu \in \mathcal{H}$, it follows from the Schwarz inequality that $\bar{v}Lu \in L(a, b)$. Similarly $uL\bar{v} \in L(a,b)$; so that from (1.11) the limits $[u,v](a) = \lim_{x \to a} [u,v](x)$

and $[u, v](b) = \lim_{x \to b} [u, v](x)$ exist. Define $\langle u,v \rangle \equiv [u,v](b) - [u,v](a)$.

Define

(1.12) $$D_0 \equiv \{u \in D : \langle u, v \rangle = 0 \text{ for all } v \in D\} .$$

Let $T_0$ be the restriction of $T$ to $D_0$. Then [2, Theorem 1] $T_0$ is a closed symmetric operator, $T_0^* = T$, and $T^* = T_0$.

In proving these facts Coddington introduces a function $K(x, y)$ defined for $a < x < b$, $a < y < b$, which he shows to have the property that if $\delta = [c,d]$ is a compact subinterval of $(a, b)$ and $v(x) \in L^2(\delta)$, then the function $w$ defined on $\delta$ by

(1.13) $$w(x) = \int_c^x K(x, y)v(y)dy$$

is such that $w^{(n-1)}$ is absolutely continuous on $\delta$, and on $\delta^0$, the interior of $\delta$, $w$ satisfies

(1.14) $$Lw = v ,$$

in the sense that $(Lw)(x)$ exists for almost all $x$ and is equal to $v(x)$ for almost all $x$. From an expression [2, formula 2.10] given by Coddington for the derivatives of $w$ it follows that

(1.15) $$w^{(j)}(c^+) = 0 , \quad j = 0, 1, \ldots, n-1 .$$

$K(x,y)$ has the explicit form

$$K(x,y) = \sum_{i,j=1}^{n} a_{ij} s_i(x) \overline{s_j(y)} , \tag{1.16}$$

where $s_1, \ldots, s_n$ is a set of linearly independent solutions of $Ly = 0$, and $a_{ij}$, $i, j = 1, \ldots, n$, are certain numbers.

## CHAPTER II

## THE EXPANSION THEOREM AND PARSEVAL EQUALITY

In this Chapter we show how von Neumann's theory of generalized direct sums of Hilbert spaces leads in a natural way to the expansion and Parseval theorems.

Let $L$ be a formally self-adjoint differential operator on $(a,b)$ with nonvanishing leading coefficient $p_0$. We follow the notation of the last section of Chapter I. Suppose that the operator $T_0$ in $\mathcal{H}$ has a self-adjoint extension $H$. This condition is always satisfied if $L$ has real coefficients [19, p. 325]. Conditions for the existence of self-adjoint extensions of a closed symmetric operator are described in Chapter IV, where references are also given.

Let $E(\lambda)$ be the resolution of unity corresponding to $H$ which occurs in the spectral theorem. By (1.6) there exists a generalized direct sum $\tilde{\mathcal{H}}$ which is unitarily equivalent to $\mathcal{H}$ and to which $E(\lambda)$ belongs. If $u \in \mathcal{H}$, let $\tilde{u}$ be its correspondent in $\tilde{\mathcal{H}}$. Denote by $\tilde{E}(\lambda)$ and $\tilde{H}$ the operators in $\tilde{\mathcal{H}}$ corresponding to $E(\lambda)$ and $H$ respectively. According to Chapter I, the assertion that $E(\lambda)$ belongs to $\tilde{\mathcal{H}}$ means that

$$(2.1) \qquad \tilde{E}(\mu)\tilde{u} = \tilde{w} \text{ implies } \tilde{w}(\lambda) = e_\mu(\lambda)\tilde{u}(\lambda) ,$$

where $e_\mu(\lambda)$ is the characteristic function of $(-\infty, \mu]$. By (1.4)

$$(2.2) \qquad (\tilde{w}, \tilde{v}) = \int_{\Lambda} (\tilde{w}(\lambda), \tilde{v}(\lambda)) d\sigma(\lambda) .$$

From 2.1 and 2.2

$$(2.3) \qquad \begin{aligned} (\tilde{E}(\mu)\tilde{u}, \tilde{v}) &= \int_{\Lambda} (e_{\mu}(\lambda)\tilde{u}(\lambda), \tilde{v}(\lambda)) d\sigma(\lambda) \\ &= \int_{-\infty}^{\mu} (\tilde{u}(\lambda), \tilde{v}(\lambda)) d\sigma(\lambda) ; \end{aligned}$$

or, in terms of the Radon-Nikodym derivative,

$$(2.4) \qquad \frac{d_{\mu}(\tilde{E}(\mu)\tilde{u}, \tilde{v})}{d\sigma(\mu)} = (\tilde{u}(\mu), \tilde{v}(\mu)) .$$

From the spectral representation (1.3) of $H$ and (2.4) we conclude that if $\tilde{u} \in D_{\tilde{H}}$ and $\tilde{v} \in \tilde{\mathcal{H}}$ then

$$(2.5) \qquad (\tilde{H}\tilde{u}, \tilde{v}) = \int_{-\infty}^{\infty} \mu \, d_{\mu}(\tilde{E}(\mu)\tilde{u}, \tilde{v}) = \int_{-\infty}^{\infty} \mu(\tilde{u}(\mu), \tilde{v}(\mu)) d\sigma(\mu) .$$

Thus

$$(2.6) \qquad \tilde{H}\tilde{u} = \tilde{w} \text{ is equivalent to } \tilde{w}(\lambda) = \lambda \tilde{u}(\lambda) .$$

Similarly we find, using (1.2) and (2.4), that

$$(2.7) \qquad \|\tilde{H}\tilde{u}\|^2 = \int_{-\infty}^{\infty} \mu^2 d_{\mu}(\tilde{E}(\mu)\tilde{u}, \tilde{u}) = \int_{-\infty}^{\infty} \mu^2 \|\tilde{u}(\mu)\|^2 d\sigma(\mu) .$$

Hence $D_{\tilde{H}}$ is the set of $\tilde{u} \in \tilde{\mathcal{H}}$ for which the last written integral is finite.

Let $c$ satisfy $a < c < b$, and define

$$(2.8) \qquad D_c \equiv \{u \in D_H : u^{(j)}(c) = 0, \quad j = 0, 1, \ldots, n-1\} .$$

Let $\delta$ be a non-degenerate compact subinterval of $(a, b)$ with one end-point at $c$. Let $K(x, y)$ be the function mentioned in the last section of Chapter I. Define the function $K^*(x, y)$ on $(a, b) \times (a, b)$ by

$$(2.9) \qquad K^*(x, y) = \begin{cases} K(x, y) & \text{if } c \leq y < x \\ -K(x,y) & \text{if } x < y \leq c \\ 0 & \text{for all other } x,y \text{ in } (a,b) \end{cases} .$$

For any $x \in \delta^{\circ}$ and $v \in L^2(\delta)$ it follows that

$$(2.10) \qquad \int_c^x K(x,y)v(y)dy = \int_\delta K^*(x,y)v(y)dy .$$

Thus the function $w$ defined by (1.13) may also be expressed in the form

$$(2.11) \qquad w(x) = \int_\delta K^*(x,y)v(y)dy .$$

By (1.14) it follows that

(2.12) if $w(x)$ satisfies (2.11), where $v \in L^2(\delta)$, then $Lw = v$.

Let $D$ be the manifold defined in the last section of Chapter I. Starting with $u \in D$ form

$$(2.13) \qquad Lu = v .$$

By property (iii) in the definition of $D$, $\frac{v}{r} \in \mathcal{H}$; that is

$$(2.14) \qquad \int_a^b \frac{|v(x)|^2}{r(x)} dx < +\infty .$$

Since $r$ is bounded on every compact subinterval of $(a, b)$, (2.14) implies $v \in L^2(\delta)$. Thus, if we define $w$ by (2.11) in terms of this $v$, by (2.12) $w$ satisfies $Lw = v$; and furthermore $w$ satisfies the initial conditions (1.14). Hence if, in addition, we assume $u \in D_c$ of (2.8), $u(x)$ and $w(x)$ satisfy the same linear differential equation (2.13) for almost all $x \in \delta$ and the same initial conditions (1.14). We conclude that $u(x) = w(x)$ for almost all $x \in \delta$. That is, using (2.11) and (2.13),

(2.15) if $u \in D_c$, then for almost all $x \in \delta$, $u(x) = \int_\delta K^*(x,y)(Lu)(y)dy$.

Define $f_x(y) = f_x^\delta(y)$ by

$$(2.16) \qquad f_x(y) = \begin{cases} \overline{K^*(x, y)} & \text{if } x \in \delta \text{ and } y \in \delta \\ 0 & \text{if } x \notin \delta \text{ or } y \notin \delta . \end{cases}$$

Then if $u \in D_H$, using $(\ ,\ )$ to denote the inner product in both

$\mathcal{H}$ and $\widetilde{\mathcal{H}}$ ,

$$\int_\delta (Lu)(y)K^*(x, y)dy = \int_a^b (Lu)(y)\overline{f_x(y)}dy = \int_a^b (Hu)(y)\overline{f_x(y)}\, r(y)dy$$

$$= (Hu,f_x) = (\widetilde{H}\tilde{u},\tilde{f}_x) = \int_\Lambda \lambda(\tilde{u}(\lambda),\tilde{f}_x(\lambda))d\sigma(\lambda) \ ,$$

the last equality following from (2.5). Hence we see from (2.15) that

(2.17)
$$\text{if } u \in D_c \text{ , then for almost all } x \in \delta \ ,$$
$$u(x) = \int_\Lambda \lambda\, (\tilde{u}(\lambda), \tilde{f}_x(\lambda))d\sigma(\lambda) \ .$$

We now make some estimates so that we may apply the Fubini theorem later. Let $u \in D_H$ , $v \in \mathcal{H}$ . Applying first the Schwarz inequality in $\widetilde{\mathcal{H}}(\lambda)$ , then the Schwarz inequality for integrals, then (2.7), (1.4), and the unitary equivalence of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ we see that

(2.18)
$$\int_\delta \left\{\int_\Lambda |\lambda\cdot(\tilde{u}(\lambda), \tilde{f}_x(\lambda))| \cdot |v(x)| \cdot r(x)d\sigma(\lambda)\right\} dx$$

$$\leq \int_\delta \left\{\int_\Lambda \|\lambda\tilde{u}(\lambda)\| \cdot \|\tilde{f}_x(\lambda)\|\, d\sigma(\lambda)\right\} |v(x)| \cdot r(x)dx$$

$$\leq \int_\delta \left\{\int_\Lambda \lambda^2 \|\tilde{u}(\lambda)\|^2 d\sigma(\lambda)\right\}^{\frac{1}{2}} \left\{\int_\Lambda \|\tilde{f}_x(\lambda)\|^2 d\sigma(\lambda)\right\}^{\frac{1}{2}} |v(x)| \cdot r(x)dx$$

$$= \|\widetilde{H}\tilde{u}\| \cdot \int_\delta \|f_x\| \cdot |v(x)| \cdot r(x)dx \ .$$

Application of Schwarz's integral inequality to the last integral yields

$$(2.19)\qquad \begin{aligned} &\int_\delta \left\{ \int_\Lambda |\lambda(\tilde{u}(\lambda), \tilde{f}_x(\lambda))| \cdot |v(x)| \cdot r(x)\, d\sigma(\lambda) \right\} dx \\ &\le \| \tilde{H}\tilde{u} \| \left\{ \int_\delta \| f_x \|^2 r(x) dx \right\}^{\frac{1}{2}} \left\{ \int_\delta |v(x)| r(x) dx \right\}^{\frac{1}{2}} . \end{aligned}$$

Since $v \in \mathcal{H}$, the last written factor is finite. As for the other factor, notice first that $\| f_x \|^2 = \int_a^b | f_x(y)|^2 dy = \int_\delta |K(x,y)|^2 dy$ ; so that the square of the first factor is $\int_\delta r(x) \left\{ \int_\delta |K(x,y)|^2 dy \right\} dx = \int_{\delta \times \delta} |K(x,y)|^2 r(x) dx dy$ , which is finite since $K(x, y)$ is piecewise continuous by (1.16). Thus the integral on the left side of (2.19) is finite.

Let $u \in D_c$ , $v \in D_\delta$ , where $D_\delta$ is defined by

$$(2.20)\qquad D_\delta = \left\{ v \in D : v(x) = 0 \text{ for almost all } x \notin \delta \right\}.$$

Then by the unitary equivalence of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ and by (2.17)

$$\begin{aligned} (\tilde{u}, \tilde{v}) = (u, v) &= \int_a^b u(x)\overline{v(x)} r(x) dx = \int_\delta u(x)\overline{v(x)}\, r(x) dx \\ &= \int_\delta \left\{ \int_\Lambda \lambda(\tilde{u}(\lambda), \tilde{f}_x(\lambda))\, d\sigma(\lambda) \right\} \overline{v(x)} r(x) dx . \end{aligned}$$

Because of the finiteness of the left side of (2.19), we may apply the Fubini theorem to conclude that

$$(2.21)\qquad (\tilde{u}, \tilde{v}) = \int_\Lambda \left\{ \int_\delta \lambda\, (\tilde{u}(\lambda), \tilde{f}_x(\lambda))\, \overline{v(x)}\, r(x) dx \right\} d\sigma(\lambda) .$$

According to Chapter I we may introduce a measurable family $\{\tilde{\Psi}_m(\lambda)\}$ in $\tilde{\mathcal{H}}$, in terms of which it is evident that if $\tilde{u}$, $\tilde{v} \in \tilde{\mathcal{H}}$, then

$$(2.22)\qquad (\tilde{u}, \tilde{v}) = \int_\Lambda \left\{ \sum_{m=1}^{k(\lambda)} (\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda))\overline{(\tilde{v}(\lambda), \tilde{\Psi}_m(\lambda))} \right\} d\sigma(\lambda) .$$

Thus (2.17) states that

if $u \in D_c$ then for almost all $x \in \delta$,

$$(2.23)\qquad u(x) = \int_\Lambda \left\{ \sum_{m=1}^{k(\lambda)} (\lambda \tilde{u}(\lambda), \tilde{\Psi}_m(\lambda))\overline{(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))} \right\} d\sigma(\lambda) ;$$

and (2.21) states that

$$(2.24)\qquad (\tilde{u},\tilde{v}) = \int_\Lambda \left[ \int_\delta \left\{ \sum_{m=1}^{k(\lambda)} (\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda))\overline{(\lambda v(x) r(x) \tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))} \right\} dx \right] d\sigma(\lambda) .$$

If we may interchange summation and integration in (2.24) we may write

$$(2.25)\qquad (\tilde{u},\tilde{v}) = \int_\Lambda \left\{ \sum_{m=1}^{k(\lambda)} (\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda)) \int_\delta \overline{(\lambda v(x) r(x) \tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))dx} \right\} d\sigma(\lambda) .$$

To justify the interchange we use the dominated convergence theorem. Since the interchange takes place within a $\sigma(\lambda)$ integral, it is sufficient to show dominance by an x-integrable function for almost all

$\lambda[\sigma]$. (This notation shall mean almost all $\lambda$ with respect to $\sigma$.) Thus we estimate by the Cauchy inequality

$$\sum_{m=1}^{k(\lambda)} \left| (\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda)) \cdot \overline{(\lambda \tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))} \right| \cdot |v(x)| \cdot r(x)$$

$$(2.26) \quad \leq |v(x)| \cdot r(x) \left\{ \sum_{m=1}^{k(\lambda)} \left| (\tilde{u}(\lambda), \tilde{\Psi}_m(\lambda)) \right|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{m=1}^{k(\lambda)} \left| (\lambda \tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda)) \right|^2 \right\}^{\frac{1}{2}}$$

$$= |v(x)| \cdot r(x) \cdot \| (\tilde{u}(\lambda) \| \cdot |\lambda| \cdot \| \tilde{f}_x(\lambda) \| \ .$$

It suffices, therefore, to show that this last expression is an integrable function of $x$ for almost all $\lambda[\sigma]$. But this expression is the integrand in the second term of (2.18), and we have already shown that the integral appearing there is finite. As a consequence of the Fubini theorem the integrand is integrable in $x$ for almost all $\lambda[\sigma]$. Thus (2.25) is validated if $u \in D_c$ and $v \in D_\delta$.

Now $D_c$ is dense in $\mathcal{H}$; so if $v \in D_\delta$, the $\tilde{u}$ for which (2.25) is true are dense in $\tilde{\mathcal{H}}$. Also it is clear that if $\tilde{w} \in \tilde{\mathcal{H}}$, $\tilde{w}$ determines $\tilde{w}(\lambda)$ uniquely for almost all $\lambda[\sigma]$, and $\tilde{w}(\lambda)$ has a unique Fourier expansion in $\tilde{\mathcal{H}}^{(\lambda)}$. Thus, comparison of (2.25) and (2.22) shows that if $v \in D_\delta$, then for almost all $\lambda[\sigma]$

$$(2.27) \qquad (\tilde{v}(\lambda), \tilde{\Psi}_m(\lambda)) = \int_\delta (\lambda\, v(x) r(x) \tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda)) dx \ .$$

Using (2.27) in the equality $\tilde{v}(\lambda) = \sum_{m=1}^{k(\lambda)} (\tilde{v}(\lambda), \tilde{\Psi}_m(\lambda)) \tilde{\Psi}_m(\lambda)$, and then

writing u instead of v , yields the important result that if $u \in D_\delta$, then

for almost all $\lambda\ [\sigma]$

(2.28)
$$\tilde{u}(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{\int_\delta u(x)r(x)(\lambda\tilde{f}_x(\lambda), \tilde{\psi}_m(\lambda))dx\right\} \tilde{\psi}_m(\lambda) .$$

This equation is an inverse formula to (2.23). Our next task is to extend these formulas to all $u \in \mathcal{H}$ . When the extended formulas have been derived, a simple change of measures will yield the main theorem.

Notice that (2.23) may be written in the form

$$u(x) = \int_\Lambda (\lambda\tilde{u}(\lambda), \tilde{f}_x(\lambda))d\sigma(\lambda) , \tag{2.29}$$

valid for almost all $x \in \delta$ when $u \in D_c$ . Also (2.28) may be written symbolically in the form

$$\tilde{u}(\lambda) = \lambda\int_\delta u(x)r(x)\tilde{f}_x(\lambda)dx . \tag{2.30}$$

The symmetry of (2.29) and (2.30) is quite evident if we introduce the expression $\tilde{\Gamma}(x,\lambda) = \lambda\tilde{f}_x(\lambda)$ and write (2.29) and (2.30) symbolically as inner products in the forms $u(x) = (\tilde{u}, \tilde{\Gamma}(x,\ ))$ and $\tilde{u}(\lambda) = (u, \overline{\tilde{\Gamma}(\ ,\lambda)})$ . Of course, in general $\tilde{\Gamma}(x,\ ) \notin \tilde{\mathcal{H}}$ ; so these formulas have only heuristic value.

If $u \in \mathcal{H}$ which vanishes almost everywhere outside $\delta$ , there

exists a sequence $\{u_n\}$, with $u_n \in D_c \cap D_\delta$, for which $u_n \longrightarrow u$ in $\mathcal{H}$. Since $\| u_n - u_m \| = \| \tilde{u}_n - \tilde{u}_m \|$, $\{\tilde{u}_n\}$ is a Cauchy sequence in $\tilde{\mathcal{H}}$, converging say to $\tilde{v}$. But since $\| \tilde{v} - \tilde{u} \| \leq \| \tilde{v} - \tilde{u}_n \| + \| \tilde{u}_n - \tilde{u} \| = \| \tilde{v} - \tilde{u}_n \| + \| u_n - u \|$, which approaches $0$ as $n \longrightarrow \infty$, we see that $\tilde{v} = \tilde{u}$; that is $\tilde{u}_n \longrightarrow \tilde{u}$.

Since $u_n \in D_\delta$, (2.28) holds for each $u_n$; that is, for almost all $\lambda[\sigma]$ and each $n = 1, 2, \ldots,$

(2.31) $$\tilde{u}_n(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{ \int_\delta \lambda \, u_n(x) r(x) (\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda)) dx \right\} \tilde{\Psi}_m(\lambda) .$$

Since $\tilde{u}_n \longrightarrow \tilde{u}$, as remarked in Chapter 1 there exists a subsequence $\tilde{u}_{n_k}$ such that $\tilde{u}_{n_k}(\lambda) \longrightarrow \tilde{u}(\lambda)$ for almost all $\lambda[\sigma]$. We may, and shall, suppose this subsequence chosen from the beginning so that

(2.32) $\tilde{u}_n(\lambda) \longrightarrow \tilde{u}(\lambda)$ if $\lambda$ belongs to a certain set $\Sigma$ whose complement is of $\sigma$-measure zero.

Thus if $\lambda \in \Sigma$ every Fourier coefficient of $\tilde{u}_n(\lambda)$ approaches the corresponding Fourier coefficient of $\tilde{u}(\lambda)$; that is, by (2.31),

(2.33) $$\int_\delta \lambda \, u_n(x) r(x) (\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda)) dx , \quad m = 1, 2, \ldots, k(\lambda) ,$$

approaches the $m^{th}$ Fourier coefficient of $u(\lambda)$ with respect to the basis $\{\tilde{\Psi}_j(\lambda)\}$. Thus when we show that the integral in (2.33) approaches

(2.34) $\qquad \int_\delta \lambda u(x)r(x)(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))dx\,, \qquad m = 1, 2, \ldots, k(\lambda),$

we will know that (2.34) is the $m^{th}$ Fourier coefficient of $\tilde{u}(\lambda)$ ; that is that

(2.35) (2.28) holds for every $u \in \mathfrak{H}$ which vanishes almost everywhere outside $\delta$ ,

not merely for $u \in D_\delta$ .

It remains to show that the integral (2.33) approaches the integral (2.34) as $n \longrightarrow \infty$. The square of the modulus of the difference of these integrals is

$$(2.36) \qquad \left| \int_\delta \{u_n(x) - u(x)\} \lambda\, r(x)(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))dx \right|^2 .$$

Applying the Schwarz integral inequality and then the Schwarz inequality in $\tilde{\mathfrak{H}}(\lambda)$ we see that (2.36) does not exceed

$$(2.37) \qquad \begin{aligned} &|\lambda|^2 \left\{ \int_\delta |u_n(x) - u(x)|^2 dx \right\} \left\{ \int_\delta |(r(x)\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))|^2 dx \right\} \\ &\leq |\lambda|^2 \| u_n - u \|^2 \int_\delta \{r(x)\}^2 \| \tilde{f}_x(\lambda) \|^2 \, dx \,. \end{aligned}$$

The last written integral is finite by the argument following (2.19), wherein $r$ is now replaced by $r^2$ ; and $\| u_n - u \| \longrightarrow 0$ . (2.35) is therefore proved.

Recall that, according to (2.16), the $\tilde{f}_x(\lambda) = \tilde{f}^\delta_x(\lambda)$ which oc-

curs in (2.28) depends on $\delta$. We wish now to find $\tilde{f}_x(\lambda)$ which is independent of any particular compact subinterval, and for which (2.28) holds, at least under the restrictions which we have thus far placed on $u$. Recall that we know (2.28) to be valid for every $u \in \mathcal{H}$ vanishing outside $\delta$ and almost all $\lambda\,[\sigma]$. Let $\delta_1$, $\delta_2$, and $\delta_3 = \delta_1 \cap \delta_2$ be non-degenerate compact subintervals of $(a, b)$. Let $u \in \mathcal{H}$ and vanish almost everywhere outside $\delta_3$. Since $u$ also vanishes almost everywhere outside $\delta_1$ and $\delta_2$, (2.28) holds when $\delta$ is replaced by $\delta_1$ or by $\delta_2$ provided $\tilde{f}_x(\lambda)$ is replaced by $\tilde{f}_x^{\delta_1}(\lambda)$ or $\tilde{f}_x^{\delta_2}(\lambda)$ respectively. Since equality of two elements of $\tilde{\mathcal{H}}^{(\lambda)}$ implies equality of their Fourier coefficients, we find for almost all $\lambda\,[\sigma]$

$$(2.38)\qquad \int_{\delta_1} \lambda\, u(x) r(x) (\tilde{f}_x^{\delta_1}(\lambda), \tilde{\Psi}_m(\lambda))dx = \int_{\delta_2} \lambda u(x) r(x) (\tilde{f}_x^{\delta_2}(\lambda), \tilde{\Psi}_m(\lambda))dx .$$

Since $u$ vanishes almost everywhere outside $\delta_3$, (2.38) is equivalent to

$$(2.39)\qquad \int_{\delta_3} \lambda u(x) r(x) (\tilde{f}_x^{\delta_1}(\lambda), \tilde{\Psi}_m(\lambda))dx = \int_{\delta_3} \lambda u(x) r(x) (\tilde{f}_x^{\delta_2}(\lambda), \tilde{\Psi}_m(\lambda))dx .$$

The validity of (2.39) for almost all $\lambda\,[\sigma]$ for each member of a countable set of $u$ which is dense in the space of all $u$ which vanish almost everywhere outside $\delta_3$ implies that for almost all $\lambda\,[\sigma]$ and almost all $x \in \delta_3$

$$(\tilde{f}_x^{\delta_1}(\lambda), \tilde{\Psi}_m(\lambda)) = (\tilde{f}_x^{\delta_2}(\lambda), \tilde{\Psi}_m(\lambda)) \,, \quad m = 1, 2, \ldots, k(\lambda) \,. \tag{2.40}$$

Since $\{\tilde{\Psi}_j(\lambda)\}$ is a basis in $\tilde{\mathcal{H}}^{(\lambda)}$, we conclude that

(2.41) $\lambda \tilde{f}_x^{\delta_1}(\lambda) = \lambda \tilde{f}_x^{\delta_2}(\lambda)$ for almost all $\lambda[\sigma]$ and almost all $x \varepsilon \delta_3$.

Thus there exists $\tilde{f}_x(\lambda)$ defined for $a < x < b$ such that if $\delta$ is a compact subinterval of $[a, b]$ and $x \varepsilon \delta$, then $\lambda \tilde{f}_x^{\delta}(\lambda) = \lambda \tilde{f}_x(\lambda)$ for almost all $\lambda[\sigma]$. We refrain from cancelling $\lambda$ in this equality since $\sigma$ may have a jump at $\lambda = 0$. However, since the product $\lambda \tilde{f}_x(\lambda)$ occurs in (2.28), we conclude that (2.35) is true if the $\tilde{f}_x(\lambda)$ occurring in (2.28) is now interpreted as the extended function.

In general if $u$ is an arbitrary element of $\mathcal{H}$, the integral in (2.23) does not converge. However, we may still prove that if $u \varepsilon \mathcal{H}$, and if, for a compact interval $\Delta \subset \Lambda$,

$$\begin{aligned} u_\Delta(x) &\equiv \int_\Delta \left\{ \sum_{m=1}^{k(\lambda)} (\lambda \tilde{u}(\lambda), \tilde{\Psi}_m(\lambda)) \overline{(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))} \right\} d\sigma(\lambda) \,, \\ &= \int_\Delta (\lambda \tilde{u}(\lambda), \tilde{f}_x(\lambda)) d\sigma(\lambda) \,, \end{aligned} \tag{2.42}$$

then

$$u_\Delta \longrightarrow u \text{ in } \mathcal{H} \text{ as } \Delta \text{ expands to } (-\infty, \infty) \,. \tag{2.43}$$

To begin the proof let w be an element of $\mathcal{H}$ which vanishes almost everywhere outside a compact subinterval $\delta$ of (a, b) . By the unitary equivalence of $\mathcal{H}$ and $\tilde{\mathcal{H}}$

(2.44) $$\int_a^b u(x)\overline{w(x)}\, r(x)dx = \int_\Lambda (\tilde{u}(\lambda), \tilde{w}(\lambda))d\sigma(\lambda) .$$

On the other hand, using (2.42),

$$\int_a^b u_\Delta(x)\overline{w(x)}r(x)dx = \int_\delta u_\Delta(x)\overline{w(x)}r(x)dx$$

$$= \int_\delta \overline{w(x)}r(x) \int_\Delta \left\{\sum_{m=1}^{k(\lambda)} (\lambda\,\tilde{u}(\lambda), \tilde{\psi}_m(\lambda))\overline{(\tilde{f}_x(\lambda), \tilde{\psi}_m(\lambda))}\right\} d\sigma(\lambda)dx$$

$$= \int_\Delta \left\{\sum_{m=1}^{k(\lambda)} [(\tilde{u}(\lambda), \tilde{\psi}_m(\lambda)) \int_\delta \overline{(\lambda\, w(x)r(x)\tilde{f}_x(\lambda), \tilde{\psi}_m(\lambda))}dx]\right\} d\sigma(\lambda) ;$$

where the interchange of order of integration is justified by (2.19) and the comments following that equation, and the interchange of order of of summation and integration by (2.26) and the comments which follow it. Applying (2.28) now to the integrand in the last written integral we conclude that

(2.45) $$\int_a^b u_\Delta(x)\overline{w(x)}r(x)dx = \int_\Delta (\tilde{u}(\lambda),\tilde{w}(\lambda))d\sigma(\lambda)$$

Subtracting (2.45) from (2.44) results in

$$(2.46) \qquad \int_a^b \{u(x) - u_\Delta(x)\} \overline{w(x)} r(x)dx = \int_{\Delta'} (\tilde{u}(\lambda), \tilde{w}(\lambda)) d\sigma(\lambda) ,$$

where $\Delta' = \Lambda - \Delta$ .

Hence, by the Schwarz inequality and unitary equivalence of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ ,

$$\left| \int_a^b \{u(x) - u_\Delta(x)\} \overline{w(x)} r(x)dx \right|^2$$

$$\leq \int_{\Delta'} \| \tilde{w}(\lambda) \|^2 d\sigma(\lambda) \int_{\Delta'} \| \tilde{u}(\lambda) \|^2 d\sigma(\lambda)$$

$$\leq \| \tilde{w} \|^2 \int_{\Delta'} \| \tilde{u}(\lambda) \|^2 d\sigma(\lambda) = \| w \|^2 \int_{\Delta'} \| \tilde{u}(\lambda) \|^2 d\sigma(\lambda) .$$

If we now let $w(x) \equiv \chi_\delta(x) \{u(x) - u_\Delta(x)\}$ , it follows that $\| w \|^4 \leq \| w \|^2 \int_{\Delta'} \| \tilde{u}(\lambda) \|^2 d\sigma(\lambda)$ , or

$$\int_\delta | u(x) - u_\Delta(x) |^2 dx = \| w \|^2 \leq \int_{\Delta'} \| \tilde{u}(\lambda) \|^2 d\sigma(\lambda) .$$

The last written integral does not depend on $\delta$ ; so we may let $\delta$ expand to $(a, b)$ and obtain

$$\| u - u_\Delta \|^2 = \int_a^b | u(x) - u_\Delta(x) |^2 dx \leq \int_{\Delta'} \| \tilde{u}(\lambda) \|^2 d\sigma(\lambda) .$$

Since $\tilde{u} \in \tilde{\mathcal{H}}$ we conclude that $\| u - u_\Delta \| \longrightarrow 0$ as $\Delta$ expands to

$(-\infty, \infty)$ . Hence (2.43) is proved.

We now extend (2.28) to all $u \in \mathcal{H}$ in the following way. Define $u_\delta(x) \equiv \chi_\delta(x)u(x)$ . By (2.35), for almost all $\lambda[\sigma]$

$$(2.47) \qquad \tilde{u}_\delta(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{ \int_\delta \lambda u(x)r(x)(\tilde{f}_x(\lambda), \tilde{\psi}_m(\lambda))dx \right\} \tilde{\psi}_m(\lambda) ,$$

Let $\delta'$ be the complement of $\delta$ in (a, b). Then

$$\|\tilde{u} - \tilde{u}_\delta\|^2 = \|u - u_\delta\|^2 = \int_{\delta'} |u(x)|^2 dx ;$$

so that $\tilde{u}_\delta \longrightarrow \tilde{u}$ in $\tilde{\mathcal{H}}$ as $\delta$ expands to (a, b) .

We summarize what we have proved in the following lemma.

2.48 Lemma. If $u \in \mathcal{H}$ , (2.47) defines an element $\tilde{u}_\delta$ in $\tilde{\mathcal{H}}$ which converges in $\tilde{\mathcal{H}}$ to $\tilde{u}$ as $\delta$ expands to (a, b) , and $\|u\| = \|\tilde{u}\|$ . If $u_\Delta(x)$ is defined by (2.42) in terms of this $\tilde{u}$ , then $u_\Delta$ converges in $\mathcal{H}$ to $u$ as $\Delta$ expands to $(-\infty, \infty)$ . For each $\tilde{u} \in \tilde{\mathcal{H}}$ there exists a unique $u \in \mathcal{H}$ , the limit of $u_\Delta$ defined by (2.42), having the properties just described.

We show next that for almost all $\lambda[\sigma]$ , $k(\lambda) \leq n$ , the order of $L$ . In fact we shall prove that for almost all $\lambda[\sigma]$ , each of the functions

$$(2.49) \qquad \varphi_m(x) = \varphi_m(x,\lambda) \equiv \overline{(\tilde{f}_x^\delta(\lambda), \tilde{\psi}_m(\lambda))} , \quad m = 1,2, \ldots, k(\lambda) ,$$

satisfies $L\varphi_m = \lambda r \varphi_m$ for almost all $x \in \delta$ , and that for almost

all $\lambda[\sigma]$, $\varphi_1(x), \ldots, \varphi_{k(\lambda)}(x)$ are linearly independent.

According to (2.35), with $u(\ )$ replaced by $f_x^\delta(\ )$, for almost all $\lambda[\sigma]$

$$\tilde{f}_x^\delta(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{ \int_\delta f_x^\delta(y) r(y) (\lambda \tilde{f}_y^\delta(\lambda), \tilde{\Psi}_m(\lambda)) dy \right\} \tilde{\Psi}_m(\lambda) ,$$

or

$$(\tilde{f}_x^\delta(\lambda), \tilde{\Psi}_m(\lambda)) = \int_\delta f_x^\delta(y) (\tilde{f}_y^\delta(\lambda), \tilde{\Psi}_m(\lambda)) \lambda r(y) dy ;$$

that is, by (2.49) and (2.16)

$$\overline{\varphi_m(x)} = \int_\delta f_x^\delta(y) (\tilde{f}_y^\delta(\lambda), \tilde{\Psi}_m(\lambda)) \lambda r(y) dy = \int_\delta f_x^\delta(y) \overline{\varphi_m(y)} \lambda r(y) dy$$

$$= \overline{\int_\delta K^*(x,y) \varphi_m(y) \lambda r(y) dy} .$$

Taking conjugates and applying (2.12) with $v = \lambda r \varphi_m$ and $w = \varphi_m$, we see that

$$L \varphi_m = \lambda r \varphi_m \qquad m = 1, 2, \ldots, k(\lambda) , \tag{2.50}$$

since by (2.49) $|v(x)|^2 = |r(x) \varphi_m(x)|^2 \leq \{r(x)\}^2 \|f_x^\delta\|^2$, and the latter function is integrable by the statements following (2.19).

To prove the linear independence for almost all $\lambda[\sigma]$ of $\varphi_1(x,\lambda), \ldots, \varphi_{k(\lambda)}(x,\lambda)$ it suffices, in view of the $\sigma$-measurabi-

lity of $\Delta_\ell$ of (1.5), to show that $\varphi_1(x,\lambda), \ldots, \varphi_\ell(x,\lambda)$ are linearly independent on each compact non-degenerate interval $\delta$ for almost all $\lambda[\sigma]$ in $\Delta_\ell$, $\ell = 1, 2, \ldots, \infty$. We wish to show that for almost all $\lambda[\sigma]$ in $\Delta_\ell$

$$\sum_{m=1}^{\ell} c_m(\lambda)\varphi_m(x,\lambda) = 0 \tag{2.51}$$

implies $c_m(\lambda) = 0$, $m = 1, \ldots, \ell$. Suppose first that $\ell$ is finite, We construct a $\sigma$-measurable solution vector $C(\lambda)$ of (2.51) which is not null for each $\lambda$ for which (2.51) has a non-trivial solution. Once this construction is completed, the remainder of the proof is easy.

Since

$$\left\|\sum_{m=1}^{\ell} c_m(\lambda)\varphi_m(\ ,\lambda)\right\|^2 = \sum_{i,j=1}^{\ell} a_{ij}c_i(\lambda)\overline{c_j(\lambda)}, \tag{2.52}$$

where $(a_{ij}) = (a_{ij}(\lambda))$ is the Gramian matrix whose elements are

$$a_{ij} = (\varphi_i, \varphi_j), \tag{2.53}$$

the inner product being formed in $L^2(\delta)$, $(a_{ij})$ is positive semi-definite; so that, by (2.52), (2.51) is equivalent to the system of equations

$$\sum_{j=1}^{\ell} a_{ij}(\lambda)c_j(\lambda) = 0, \quad i = 1, \ldots, \ell. \tag{2.54}$$

By (a') in the definition in Chapter I of a measurable family, (2.49), (2.53), and the Fubini theorem, each $a_{ij}(\lambda)$ is $\sigma$-measurable; so that the rank $\rho(\lambda)$ of $(a_{ij}(\lambda))$ is also $\sigma$-measurable. In fact, if we define

(2.55) $$E_r = \{\lambda \in \Delta_\ell : \rho(\lambda) \geq r\},$$

then $E_r = \emptyset$ when $r > \ell$ ; $E_r = \Delta_\ell$ when $r \leq 0$ ; and $E_r = \{\lambda \in \Delta_\ell$ : some $r \times r$ minor of $(a_{ij}(\lambda))$ is not zero$\}$ when $1 \leq r \leq \ell$ . Thus $\rho(\lambda)$ is $\sigma$-measurable, and hence, for each $r = 0, 1, \ldots, \ell$, the set

(2.56) $$F_r = \{\lambda \in \Delta_\ell : \rho(\lambda) = r\}$$

is $\sigma$-measurable.

(2.57) $F_0, \ldots, F_\ell$ are disjoint and $\bigcup_{r=0}^{\ell} F_r = \Delta_\ell$.

Suppose $\lambda \in \Delta_\ell$, and is such that (2.54) has a non-trivial solution. Then $\lambda \in F_r$ , where $r \leq \ell - 1$ . Choose an $r + 1$ by $r + 1$ sub-matrix $\alpha$ of $(a_{ij}(\lambda))$ . Consider a fixed $i$ satisfying $1 \leq i \leq \ell$. Define an $\ell$-vector $D(\lambda)$ so that the $j^{th}$ component of $D$ is $0$ if $a_{ij}$ is not an element of $\alpha$ , but the $j^{th}$ component is the cofactor of $a_{ij}$ in $\alpha$ when $a_{ij}$ is an element of $\alpha$ . Consider all such $\ell$-vectors

$D(\lambda)$ which can be obtained in this way by letting $i$ vary from 1 to $\ell$ and $\alpha$ vary over all $r+1$ by $r+1$ submatrices of $a_{ij}(\lambda)$. List these $\ell$-vectors in some order, say $D^1(\lambda), \ldots, D^{p_r}(\lambda)$. At least one $D^i(\lambda)$ is not null since $\rho(\lambda) = r$. Thus if we define

$$(2.58) \qquad G_i^r = \{\lambda \in F_r : D^i(\lambda) \neq 0\}, \quad i = 1, \ldots, p_r,$$

then each $G_i^r$ is $\sigma$-measurable and

$$(2.59) \qquad \bigcup_{i=1}^{p_r} G_i^r = F_r .$$

Define a solution vector $C(\lambda)$ of (2.54) in the following way:

(a) If $\lambda \in F_r, \; r = 0, \ldots, \ell - 1,$

$$\text{let } C(\lambda) = \begin{cases} D^1(\lambda) & \text{if } \lambda \in G_1^r \\ D^i(\lambda) & \text{if } \lambda \in G_i^r - \left(\bigcup_{j=1}^{i-1} G_j^r\right), \; i = 2, \ldots, p_r . \end{cases}$$

According to (2.59), (a) defines $C(\lambda)$ on all of $F_r$.

(b) If $\lambda \in F_\ell$, let $C(\lambda) = 0$.

According to (2.57) (a) and (b) define $C(\lambda)$ on all of $\Delta_\ell$.

(c) If $\lambda \notin \Delta_\ell$, let $C(\lambda) = 0$.

Clearly, since all sets mentioned in (a), (b), and (c) are $\sigma$-measurable, $C(\lambda)$ is $\sigma$-measurable. $C(\lambda)$ is a solution vector of

(2.54), that is, of (2.51); and $C(\lambda) \neq 0$ for each $\lambda$ for which (2.51) has a non-trivial solution. We shall now prove that $C(\lambda) = 0$ for almost all $\lambda[\sigma]$, and that $\Delta_\infty$ is of $\sigma$-measure zero. When these two facts have been proved, the desideratum mentioned in connection with (2.51) will be attained.

To prove that $C(\lambda) = 0$ for almost all $\lambda[\sigma]$, define

$$B(\lambda) = \begin{cases} \frac{1}{|C(\lambda)|} C(\lambda) & \text{if } C(\lambda) \neq 0 \\ 0 & \text{if } C(\lambda) = 0 \end{cases}, \tag{2.60}$$

where $|C(\lambda)|$ means the Euclidean length of $C(\lambda)$. $B(\lambda)$ is $\sigma$-measurable. We wish to show that $B(\lambda) = 0$ for almost all $\lambda[\sigma]$. By the definition in Chapter I of an N-function

$$\int_\Lambda |B(\lambda)|^2 d\sigma(\lambda) \leq \int_\Lambda d\sigma(\lambda) < +\infty;$$

that is, if we let $b_i(\lambda)$, $i = 1, \ldots, \ell$, be the components of $B(\lambda)$,

$$\int_\Lambda \sum_{i=1}^{k(\lambda)} |b_i(\lambda)|^2 d\sigma(\lambda) < \infty, \tag{2.61}$$

the justification for summing to $k(\lambda)$ being the fact that $B(\lambda) = 0$ if $\lambda \notin \Delta_\ell$. Thus, according to Chapter I, $\tilde{b}(\lambda) \equiv \sum_{i=1}^{k(\lambda)} b_i(\lambda) \Psi_i(\lambda)$ determines an element $\tilde{b} \in \tilde{\mathcal{H}}$. By (2.51), (2.60), and (2.49),

$0 = \sum_{i=1}^{k(\lambda)} b_i(\lambda)\varphi_i(x,\lambda) = \sum_{i=1}^{k(\lambda)} b_i(\lambda)\overline{(\tilde{f}_x(\lambda), \tilde{\varphi}_m(\lambda))}$. Thus

$(\tilde{b}(\lambda), \tilde{f}_x(\lambda)) = 0$, and so by (2.42) $b_\Delta(x) = 0$. Hence by (2.43), $b = 0$ in $\mathcal{H}$, which implies $\tilde{b} = 0$ in $\tilde{\mathcal{H}}$, which implies $\tilde{b}(\lambda) = 0$ for almost all $\lambda[\sigma]$. Thus $b_i(\lambda) = 0$ for almost all $\lambda[\sigma]$, $i = 1, \ldots, k(\lambda)$. That is, $B(\lambda) = 0$ for almost all $\lambda$, so that, by (2.60), $C(\lambda) = 0$ for almost all $\lambda$.

It remains to show that $\Delta_\infty$ is of $\sigma$-measure zero. The argument is quite similar to the one just completed. Let $\lambda \varepsilon \Delta_\infty$. Since by (2.50), the $\varphi_m(x,\lambda)$ are, for almost all $\lambda[\sigma]$, solutions of an ordinary linear differential equation of degree n, there exist for almost all $\lambda \varepsilon \Delta_\infty[\sigma]$, $c_1(\lambda), \ldots, c_{n+1}(\lambda)$, not all zero such that

(2.62) $$\sum_{i=1}^{n+1} c_i(\lambda)\varphi_i(x,\lambda) = 0 .$$

The previous argument shows that (2.62) implies

(2.63) $0 = c_1(\lambda) = c_2(\lambda) = \ldots = c_{n+1}(\lambda)$ for almost all $\lambda \varepsilon \Delta_\infty[\sigma]$.

We must conclude that $\Delta_\infty$ has $\sigma$-measure zero. Thus the assertion made in connection with (2.49) is proved.

We now derive the fundamental theorem. Let $\rho(\lambda) = (\rho_{jk}(\lambda))$, $j, k = 1, \ldots, n$, be an Hermitian matrix function of $\lambda$ defined on $\Lambda$, and non-decreasing in the sense that $\nu > \mu$ implies that $\rho(\nu) - \rho(\mu)$

is positive semi-definite. Let $\Gamma$ denote the family of n-vector functions of $\lambda$, $\omega(\lambda) = (\omega_1(\lambda), \ldots, \omega_n(\lambda))$ which are $\rho$-measurable in the sense that $\omega_j(\lambda)$ is measurable with respect to $\rho_{ji}$ and $\rho_{ij}$, $i = 1, \ldots, n$. Introduce an inner product in $\Gamma$ according to the formula

$$(\omega, \chi) = \int_\Lambda \sum_{j,k=1}^{n} \omega_j(\lambda)\overline{\chi_k(\lambda)}\, d\rho_{jk}(\lambda).$$

Let $\Omega$ be the set of elements $\omega \in \Gamma$ for which $\|\omega\| = (\omega, \omega)^{\frac{1}{2}}$ is finite, where as usual we identify elements of $\Gamma$ the norm of whose difference vanishes. The main theorem may be stated as follows [11; 2].

2.64 <u>Theorem</u>. <u>Let</u> $s_j(x,\lambda)$, $j = 1, \ldots, n$, <u>be a system of linearly independent solutions of</u> $Ly = \lambda ry$. <u>Then there exists an Hermitian, non-decreasing spectral matrix</u> $\rho(\lambda)$ <u>and the associated Hilbert space</u> $\Omega$ <u>of n-vectors</u> $\omega$ <u>such that if</u> $u \in \mathfrak{H}$ <u>and we define</u>

(2.65) $$\omega_j^\delta(\lambda) = \int_\delta \overline{s_j(x,\lambda)}\, u(x)r(x)dx,$$

<u>then the vector</u> $\omega^\delta = (\omega_1^\delta, \ldots, \omega_n^\delta) \in \Omega$, <u>and</u> $\omega^\delta$ <u>converges in the norm of</u> $\Omega$ <u>as</u> $\delta$ <u>expands to</u> (a, b), <u>say</u>

(2.66) $$\omega^\delta \longrightarrow \omega \quad \text{in } \Omega.$$

If $\omega = (\omega_1, \ldots, \omega_n)$, $u(x)$ may be recovered according to the formula

$$(2.67) \qquad u(x) = \int_\Lambda \sum_{j,k=1}^{n} s_k(x,\lambda)\,\omega_j(\lambda)\,d\rho_{jk}(\lambda) ,$$

where the integral converges in mean in $\mathfrak{H}$. The Parseval relation is also valid,

$$(2.68) \qquad \|\omega\| = \|u\| .$$

Reciprocally, if $\omega \in \Omega$ the integral (2.67) exists in mean, the $u$ which is defined thereby belongs to $\mathfrak{H}$, and in terms of $u$, $\omega$ may be recovered by (2.65), (2.66). Moreover, if $E(\lambda)$ is the resolution of unity corresponding to $H$, the self adjoint extension of $T_0$ of (1.12), then for almost all $x$

$$(2.69) \qquad (E(\Delta)u)(x) = \int_\Delta \sum_{j,k=1}^{n} s_k(x,\lambda)\,\omega_j(\lambda)\,d\rho_{jk}(\lambda)$$

where we have used $E(\Delta)$ to mean $E(\nu) - E(\mu)$ if $\Delta = (\mu, \nu]$; and if $u \in D_H$, then for almost all $x$,

$$(2.70) \qquad (Hu)(x) = \int_\Lambda \sum_{j,k=1}^{n} s_k(x,\lambda)\,\omega_j(\lambda)\,\lambda\,d\rho_{jk}(\lambda) ,$$

where the integral converges in mean in $\mathfrak{H}$. Thus $H$ has the spec-

tral (diagonal) form in $\Omega$.

Proof. According to (2.50), for almost all $\lambda[\sigma]$ $\overline{(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))}$ is a solution of $Ly = \lambda ry$; so that for almost all $\lambda[\sigma]$ we may write

$$(2.71) \qquad \overline{(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))} = \sum_{j=1}^{n} c_{mj}(\lambda)s_j(x,\lambda) \ , \quad m = 1, \ldots, k(\lambda).$$

Differentiation of the $r^{th}$ one of these equations $n - 1$ times with respect to $x$ leads, for fixed $x$, to a system of $n$ linear equations for the $c_{rj}(\lambda)$, $j = 1, \ldots, n$, whose coefficient matrix has for determinant the Wronskian of $s_1, \ldots, s_n$ which is independent of $\lambda$ and not zero. Thus the $c_{rj}(\lambda)$ are polynomials in the x-derivatives of $\overline{(\tilde{f}_x(\lambda), \tilde{\Psi}_m(\lambda))}$ and $s_1, \ldots, s_n$ of order not exceeding $n - 1$. Thus by standard theorems [4, Chapter 1, Theorem 8.4 and following remarks] on the analyticity of solutions with respect to a parameter which occurs analytically,

(2.72) $c_{mj}(\lambda)$ is an entire function of $\lambda$ for each $m = 1, \ldots, k(\lambda)$, $j = 1, \ldots, n$.

Substitute (2.71) in (2.47) to obtain

$$\tilde{u}_\delta(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{ \sum_{j=1}^{n} \lambda \overline{c_{mj}(\lambda)} \int_\delta u(x)r(x)\overline{s_j(x, \lambda)}dx \right\} \tilde{\Psi}_m(\lambda) \ ,$$

or by (2.65)

$$(2.73) \qquad \tilde{u}_\delta(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{ \sum_{j=1}^{n} \lambda \overline{c_{mj}(\lambda)} \omega_j^\delta(\lambda) \right\} \tilde{\psi}_m(\lambda)$$

for almost all $\lambda[\sigma]$.

Define

$$(2.74) \qquad \rho_{jk}(\lambda) = \int_0^{\lambda+} \mu^2 \sum_{m=1}^{k(\mu)} \overline{c_{mj}(\mu)} c_{mk}(\mu) d\sigma(\mu) .$$

The existence of this integral is assured by (2.72). Clearly $(\rho_{jk}(\lambda))$ is Hermitian. That it is non-decreasing follows from the equalities

$$\sum_{j,k=1}^{n} \left\{ \rho_{jk}(\nu) - \rho_{jk}(\mu) \right\} \bar{t}_j t_k = \sum_{j,k=1}^{n} \bar{t}_j t_k \int_{\mu+}^{\nu+} \lambda^2 \sum_{m=1}^{k(\lambda)} \overline{c_{mj}(\lambda)} c_{mk}(\lambda) d\sigma(\lambda)$$

$$= \int_{\mu+}^{\nu+} \lambda^2 \sum_{m=1}^{k(\lambda)} \left\{ \left( \sum_{k=1}^{n} c_{mk}(\lambda) t_k \right) \overline{\left( \sum_{j=1}^{n} c_{mj}(\lambda) t_j \right)} \right\} d\sigma(\lambda) \geq 0$$

Using (2.73) and (2.74) we now may write

$$(2.75) \quad \|\tilde{u}_\delta\|^2 = \int_\Lambda \|\tilde{u}_\delta(\lambda)\|^2 d\sigma(\lambda)$$

$$= \int_\Lambda \left\{ \sum_{m=1}^{k(\lambda)} \lambda^2 \sum_{j=1}^{n} \overline{c_{mj}(\lambda)} \omega_j^\delta(\lambda) \sum_{k=1}^{n} c_{mk}(\lambda) \overline{\omega_k^\delta(\lambda)} \right\} d\sigma(\lambda)$$

$$= \int_\Lambda \left\{ \sum_{j,k=1}^{n} \omega_j^\delta(\lambda) \overline{\omega_k^\delta(\lambda)} \sum_{m=1}^{k(\lambda)} \lambda^2 \overline{c_{mj}(\lambda)} c_{mk}(\lambda) \right\} d\sigma(\lambda) =$$

$$= \int_{\Lambda} \sum_{j,k=1}^{n} \omega_j^{\delta}(\lambda)\, \overline{\omega_k^{\delta}(\lambda)}\, d\rho_{jk}(\lambda) \, .$$

The finiteness of the last written integral shows $\omega^{\delta} \varepsilon \Omega$. Moreover it follows from (2.64) that

$$(2.76) \qquad \|\tilde{u}_{\delta}\| = \|\omega^{\delta}\| \quad .$$

By (2.48) $\tilde{u}_{\delta} \longrightarrow \tilde{u}$ in $\tilde{\mathcal{H}}$ as $\delta$ expands to (a, b) ; and now that we have the Parseval relation (2.76) for elements of the form $\tilde{u}_{\delta}$, the argument just above (2.31) shows that $\omega^{\delta}$ converges in $\Omega$ , say to $\omega$ , and hence, applying (2.76) again,

$$(2.77) \qquad \|\omega\| = \|\tilde{u}\| = \|u\| \, .$$

By the third sentence following (1.4) and what we have just proved we may allow $\delta$ to expand to (a, b) through a suitable sequence of compact intervals, thus seeing from (2.73) that for almost all $\lambda[\sigma]$ , and hence, by (2.74), for almost all $\lambda[\rho_{jk}]$ , $j, k = 1, \ldots, n$ , that

$$(2.78) \qquad \tilde{u}(\lambda) = \sum_{m=1}^{k(\lambda)} \left\{ \sum_{j=1}^{n} \lambda\, \overline{c_{mj}(\lambda)}\, \omega_j(\lambda) \right\} \tilde{\psi}_m(\lambda) \, .$$

We now find from (2.71) and (2.78) that

$$(\lambda\tilde{u}(\lambda),\tilde{f}_x(\lambda)) = \sum_{m=1}^{k(\lambda)}\left\{\sum_{j=1}^{n}\lambda^2\overline{c_{mj}(\lambda)}\,\omega_j(\lambda)\right\}\left\{\sum_{i=1}^{n}c_{mi}(\lambda)s_i(x,\lambda)\right\}$$

$$(2.79)\qquad = \sum_{i,j=1}^{n} s_i(x,\lambda)\omega_j(\lambda)\sum_{m=1}^{k(\lambda)}\lambda^2\,\overline{c_{mj}(\lambda)}c_{mi}(\lambda)\,.$$

Thus, from (2.42) and (2.74) we find that

$$u_\Delta(x) = \int_\Delta (\lambda\tilde{u}(\lambda),\tilde{f}_x(\lambda))d\sigma(\lambda)$$

$$= \int_\Delta\left\{\sum_{i,j=1}^{n} s_i(x,\lambda)\omega_j(\lambda)\sum_{m=1}^{k(\lambda)}\lambda^2\overline{c_{mj}(\lambda)}c_{mi}(\lambda)\right\}d\sigma(\lambda)$$

$$= \int_\Delta \sum_{i,j=1}^{n} s_i(x,\lambda)\omega_j(\lambda)d\rho_{ji}(\lambda)\,.$$

According to (2.48) $u_\Delta \longrightarrow u$ in $\mathcal{H}$ as $\Delta$ expands to $(a, b)$, and the assertion connected with (2.67) is proved.

According to (2.77) $\tilde{\mathcal{H}}$ is mapped isometrically onto a subspace $\Omega'$ of $\Omega$, and (2.78) gives the explicit mapping if $\omega \in \Omega'$. Consider now an arbitrary $\omega \in \Omega$. Define $u$ in terms of this $\omega$ by (2.78). Then the chain of equalities (2.75), read in reverse and omitting the $\delta$'s, shows that $\tilde{u} \in \tilde{\mathcal{H}}$, and moreover that the map from $\Omega$ onto $\tilde{\mathcal{H}}$ so defined is an isometry. Thus there exists an isometry of $\Omega$ onto $\tilde{\mathcal{H}}$ which is an extension of an isometry of $\Omega'$ onto $\tilde{\mathcal{H}}$. We conclude that $\Omega = \Omega'$, and (2.78) provides the expli-

cit unitary map of $\Omega$ onto $\widetilde{\mathcal{H}}$. Since (2.78) was originally proved to hold for an arbitrary $\tilde{u} \varepsilon \widetilde{\mathcal{H}}$, and since $\tilde{u}$ cannot correspond to two different elements of $\Omega$, we see that $u$ bears the relation to $\omega$ which is asserted in the theorem.

To prove (2.69) apply (2.42) to $E(\Delta)u$, remembering (2.1). The result is

$$(2.80) \qquad (E(\Delta)u)_{\Delta'}(x) = \int_{\Delta'} (\lambda \chi_{\Delta}(\lambda)\tilde{u}(\lambda), \tilde{f}_x(\lambda))d\sigma(\lambda) .$$

By (2.48) the left side of (2.80) tends in $\mathcal{H}$ to $(E(\Delta)u)(x)$ as $\Delta' \longrightarrow (-\infty, \infty)$, but it is clear that when $\Delta' \supset \Delta$ the right side of (2.80) is $\int_{\Delta} (\lambda\tilde{u}(\lambda), \tilde{f}_x(\lambda))d\sigma(\lambda)$. Thus by (2.79) and (2.74)

$$(E(\Delta)u)(x) = \int_{\Delta} (\lambda\tilde{u}(\lambda), \tilde{f}_x(\lambda))d\sigma(\lambda)$$

$$(2.81) \qquad = \int_{\Delta} \left\{ \sum_{i,j=1}^{n} s_i(x,\lambda)\omega_j(\lambda) \sum_{m=1}^{k(\lambda)} \lambda^2 \overline{c_{mj}(\lambda)} c_{mi}(\lambda) \right\} d\sigma(\lambda)$$

$$= \int_{\Delta} \sum_{i,j=1}^{n} s_i(x,\lambda)\,\omega_j(\lambda)\,d\rho_{ji}(\lambda) .$$

This equality proves (2.69). To prove (2.70) we notice that by (2.42) and (2.6)

$$(Hu)_{\Delta}(x) = \int_{\Delta} (\lambda \cdot \lambda\tilde{u}(\lambda), \tilde{f}_x(\lambda))d\sigma(\lambda) ,$$

and, just as in (2.81), we conclude that

$$(Hu)_\Delta(x) = \int_\Delta \sum_{i,j=1}^{n} s_i(x,\lambda)\omega_j(\lambda)\lambda\, d\rho_{ji}(\lambda) .$$

An application of (2.48) proves (2.70) and completes the proof of the theorem.

The spectrum $s(H)$ of a self-adjoint operator $H$ in an arbitrary Hilbert space may be defined as the set of points of non-constancy of the resolution of the identity $E(\lambda)$ of $H$. The points of $s(H)$ at which $E(\lambda)$ is continuous constitute the continuous spectrum of $H$; the remaining points of $s(H)$ constitute the point spectrum. The point spectrum consists of the eigenvalues of $H$ [19, pp. 356, 357].

The spectrum of a self-adjoint extension $H$ of a differential operator is usually defined [4, p. 252; 21, p. 58; 7; 23] as the set of points of non-constancy of the spectral matrix $\rho(\lambda)$, when $\rho$ is uniquely determined by $H$. (Sufficient conditions for uniqueness of $\rho$ are given in [2, Theorem 7; 4, Chapter 10, section 3].) The continuous spectrum is then defined to be those points of the spectrum at which $\rho$ is continuous and the point spectrum to be the remaining points.

It might be expected that the separate definitions would agree, at least under fairly general conditions on $L$, but we have seen no proof of this fact in the literature. Stone indicates [20, p. 530] that the problem had not been investigated at that time. While we have not been able to prove the uniqueness of $\rho$, we can prove that the spectrum, continuous spectrum, and point spectrum defined in terms of the spectral

matrix $\rho$ constructed in the proof of Theorem 2.64, agree with the corresponding concepts defined in terms of H as an abstract operator, at least for all points except $\lambda = 0$.

(2.82) Corollary to Theorem 2.64. The real number $\lambda_0 \neq 0$ is

(i) a point of constancy,

(ii) a point of continuous increase,

(iii) a point of discontinuity

of $E(\lambda)$ if and only if it satisfies the same condition with respect to $\rho(\lambda)$.

Proof: By the construction of $\widetilde{\mathfrak{H}}$ given at the beginning of this chapter and the paragraph preceding Lemma 1.6, $\sigma(\lambda)$ is equivalent to $E(\lambda)$. Define

$$\hat{\sigma}(\lambda) = \begin{cases} \sigma(\lambda), & \text{if } \lambda < 0 \\ \sigma(\lambda) - \{\sigma(0^+) - \sigma(0^-)\} & \text{if } \lambda \geq 0 . \end{cases}$$

Then it suffices to prove that $\hat{\sigma}$ and $\rho$ are equivalent; that is that each is absolutely continuous with respect to the other. The absolute continuity of each $\rho_{jk}$, $j, k = 1, \ldots, n$, with respect to $\hat{\sigma}$ is immediate from (2.74). On the other hand the functions appearing on the left of the $k(\lambda)$ equations (2.71) are linearly independent for almost all $\lambda[\sigma]$; so that in particular for almost all $\lambda[\sigma]$ not every coefficient $c_{11}(\lambda), \ldots, c_{1n}(\lambda)$ is zero. Thus every $\lambda$-set $\Gamma$ of positive $\hat{\sigma}$-measure, and hence of positive $\sigma$-measure, contains a sub-

set of positive $\sigma$-measure on which one, at least, of these functions, say $c_{ii}$ is not zero. Application of (2.74) with $j = k = i$ yields

$$\rho_{ii}(\Gamma) = \int_{\Gamma} \lambda^2 \sum_{m=1}^{k(\lambda)} | c_{mi}(\lambda)|^2 \, d\sigma(\lambda) .$$

Since the non-negative integrand is positive on a set of positive $\sigma$-measure, $\rho_{ii}(\Gamma) > 0$ . Thus $\sigma$ is absolutely continuous with respect to $\rho$ . The corollary is thereby proved.

## CHAPTER III

## BOUNDARY CONDITIONS

Coddington [2, Theorem 3] has characterized the self-adjoint extensions of $T_o$ (Chapter I). His method uses the facts that $\mathcal{H}$ is a function space and $T_o$ a differential operator. The theorem in question may be proved quite abstractly for closed symmetric operators in an arbitrary Hilbert space, using nothing deeper than the simplest facts about finite dimensional spaces.

Let $T_o$ be a symmetric operator in a Hilbert space $\mathcal{H}$ with domain $D_o$ and with adjoint $T = T_o^*$. Let $D \equiv D_T$. If $u, v \in D$, define $\{u, v\} \equiv (Tu, v) - (u, Tv)$. Since $\{v, u\} = -\overline{\{u, v\}}$, $\{v, u\} = 0$ if and only if $\{u, v\} = 0$. We are interested in conditions which characterize the self-adjointness of an extension $H$ of $T_o$. Let $\Delta \equiv D_H$, and $\Delta^* \equiv D_{H^*}$. Suppose that $T_o$ is closed so that $T^* = T_o$ (Chapter I). Since all self-adjoint extensions of $T_o$ are contained in $T$ we need (and shall) consider only those $H$ satisfying $T_o \subseteq H \subseteq T$. Taking adjoints shows that $T_o \subseteq H^* \subseteq T$, since $T^* = T_o$. Thus $H$ is self-adjoint if and only if $\Delta = \Delta^*$. Now by definition of adjoint it follows that

$$\Delta^* = \{u \in D : (Hv, u) = (v, Tu) \text{ for all } v \in \Delta\}$$

$$= \{u \in D : (Tv, u) = (v, Tu) \text{ for all } v \in \Delta\} .$$

Thus

$$(3.1) \qquad \Delta^* = \{u \in D : \{v, u\} = 0 \text{ for all } v \in \Delta\} .$$

Let us suppose dim $D/D_0$ to be finite (a condition which is satisfied in the case of differential operators). (3.1) shows that $\Delta^*$ is obtained from D by imposing conditions on $u \in D$ of the form $\{v, u\} = 0$ for $v \in \Delta$. Such conditions, because of the way in which they arise in connection with differential operators [2, p. 194 and p. 198], will be called boundary conditions. A finite set of such boundary conditions on elements $u \in D$, say $\{v_i, u\} = 0$, $i = 1, \ldots, k$, will be called dependent with respect to D if there exist complex numbers $c_1, \ldots, c_k$, not all zero, such that $\sum_{i=1}^{k} c_i \{v_i, u\} = 0$ for all $u \in D$. Otherwise the boundary conditions will be called independent. In general not all the boundary conditions occurring in the characterization (3.1) of $\Delta^*$ will be independent. In fact, we prove the following lemma.

3.2 Lemma. <u>The maximum number of boundary conditions of the form $\{v, u\} = 0$, with $v \in \Delta$, which are independent with respect to D and which suffice to characterize $\Delta^*$ as in (3.1) is equal to dim $D/\Delta^*$ (which is finite since dim $D/D_0$ is finite).</u>

<u>Proof.</u> Let dim $D/\Delta^* = k$. Suppose that there exist $v_1, \ldots, v_m \in \Delta$ such that the boundary conditions

$$(3.3) \qquad \{v_j, u\} = 0 , \quad j = 1, \ldots, m ,$$

are independent with respect to D, and that $m < k$. Let $w_1, \ldots, w_k$ be in D and linearly independent modulo $\Delta^*$; so that $\Delta^*$ and $w_1, \ldots, w_k$ together span D. There exists a non-trivial solution $c_1, \ldots, c_m$ of the equations

$$(3.4) \qquad \sum_{j=1}^{m} c_j \{v_j, w_i\} = 0, \quad i = 1, \ldots, k;$$

Moreover, by (3.1), every element $w \in \Delta^*$ satisfies

$$(3.5) \qquad \sum_{j=1}^{m} c_j \{v_j, w\} = 0.$$

From (3.4) and (3.5) it follows that every $u \in D$ satisfies

$$\sum_{j=1}^{m} c_j \{v_j, u\} = 0;$$

that is, the boundary conditions are dependent with respect to D. This is a contradiction; so that we must conclude $m \leq k$.

Now let m denote the maximum number of independent boundary conditions of the form (3.3) such that $\Delta^*$ may be represented in the form

$$(3.6) \qquad \Delta^* = \{u \in D : \{u, v_j\} = 0, \ v_j \in \Delta, \ j = 1, \ldots, m\}.$$

Suppose $m < k$. Let $w_1, \ldots, w_k \in D$ be linearly independent modulo

$\Delta^*$. There exists a non-trivial solution $c_1, \ldots, c_k$ of the equations

$$(3.7) \qquad \sum_{i=1}^{k} c_i \left\{ w_i, v_j \right\} = 0 , \qquad j = 1, \ldots, m ;$$

so that

$$(3.8) \qquad \left\{ \sum_{i=1}^{k} c_i w_i , v_j \right\} = 0 , \qquad j = 1, \ldots, m .$$

By (3.6) we conclude that $\sum_{i=1}^{k} c_i w_i \in \Delta^*$, contradicting the independence of $w_1, \ldots, w_k$ modulo $\Delta^*$. We therefore have $k = m$, and the lemma is proved.

3.9 <u>Lemma</u>. <u>Let</u> $v_1, \ldots, v_n \in \Delta$. <u>Then</u> $v_1, \ldots, v_n$ <u>are linearly independent modulo</u> $D_0$ <u>if and only if the boundary conditions</u> $\left\{v_j, u\right\} = 0$, $j = 1, \ldots, n$, <u>are independent with respect to</u> $D$.

<u>Proof</u>. If $w \in D$, application of (3.1) to $D$ in place of $\Delta$ shows that

$$(3.10) \qquad w \in D^* = D_0 \text{ if and only if } \left\{w, u\right\} = 0 \text{ for all } u \in D .$$

This fact, applied to $w = \sum_{j=1}^{n} c_j v_j$, proves the lemma.

Lemmas 3.2 and 3.9 imply that

$$(3.11) \qquad \dim \Delta/D_0 = \dim D/\Delta^* .$$

3.12 Definition. Let $\alpha_1, \ldots, \alpha_k \in D$. If the boundary conditions $\{u, \alpha_i\} = 0$, $i = 1, \ldots, k$, on elements $u \in D$ are independent with respect to $D$ and if $\{\alpha_i, \alpha_j\} = 0$, $i, j = 1, \ldots, k$, the boundary conditions will be called self-adjoint [2, p. 198].

3.13 Theorem. Let $T_0$ be a closed symmetric operator in $\mathcal{H}$ with adjoint $T \equiv T_0^*$. (See Chapter I.) Let $D \equiv D_T$ and $D_0 \equiv D_{T_0}$. Suppose that $\dim {}^{D}/D_0$ is finite. Let $H$ be an operator satisfying $T_0 \subseteq H \subseteq T$; let $\Delta \equiv D_H$ and $\Delta^* \equiv D_{H^*}$. Then $H$ is self-adjoint if and only if $\dim {}^{D}/D_0 = 2k$ and there exist $\alpha_1, \ldots, \alpha_k \in D$ such that the boundary conditions $\{u, \alpha_j\} = 0$ are self-adjoint, $j = 1, \ldots, k$, and

$$\Delta = \left\{u \in D : \{u, \alpha_j\} = 0 , \quad j = 1, \ldots, k\right\} . \tag{3.14}$$

Proof. Necessity: Assume $H = H^*$; so that

$$\Delta = \Delta^* . \tag{3.15}$$

Let $\dim {}^{\Delta}/D_0 = k$. By (3.11) and (3.15) $\dim {}^{D}/D_0 = \dim {}^{D}/\Delta^* + \dim {}^{\Delta^*}/D_0$ $= \dim {}^{\Delta}/D_0 + \dim {}^{\Delta}/D_0 = 2k$. Let $\alpha_1, \ldots, \alpha_k \in \Delta$ and be linearly independent modulo $D_0$. If $u, v \in \Delta$, then by (3.1) and (3.15), $\{u,v\} = 0$. In particular $\{\alpha_i, \alpha_j\} = 0$, $i, j, = 1, \ldots, k$. By Lemma 3.9 the boundary conditions $\{u,\alpha_j\} = 0$, $j = 1, \ldots, k$, are independent with respect to $D$; hence they are self-adjoint. Finally, it follows from (3.10) that $u \in D$ implies $\{u, v\} = 0$ for all $v \in D_0$. We see therefore, using

(3.1) and the fact that $D_o$ and $\alpha_1, \ldots, \alpha_k$ span $\Delta$ , that

$$\begin{aligned}\Delta = \Delta^* &= \{u \in D : \{v, u\} = 0 , \text{ all } v \in \Delta\} = \\ &= \{u \in D : \{u, v\} = 0 , \text{ all } v \in \Delta\} \\ &= \{u \in D : \{u, a_j\} = 0 , \ j = 1, \ldots, k\}.\end{aligned}$$

Sufficiency: Suppose that $\dim {}^D/D_o = 2k$ and that $\Delta$ is defined by k self-adjoint boundary conditions as in (3.14). By Lemma 3.9, $\alpha_1, \ldots, \alpha_k$ are linearly independent modulo $D_o$ . Let $\tilde{\Delta}$ be the linear manifold spanned by $D_o$ and $\alpha_1, \ldots, \alpha_k$ . Then

$$\dim {}^{\tilde{\Delta}}/D_o = k . \tag{3.16}$$

By (3.14) and the self-adjointness condition, $\alpha_j \in \Delta$ , $j = 1, \ldots, k$ . Thus $\tilde{\Delta} \subset \Delta$ . We shall show that

$$\tilde{\Delta} = \Delta . \tag{3.17}$$

It suffices, in view of (3.16), to prove $\dim {}^{\Delta}/D_o = k$ . By the definition (3.14) of $\Delta$ and the independence of the k boundary conditions occurring in that definition, it follows, exactly as in the proof of Lemma 3.2, that $\dim {}^D/\Delta = k$ . Using this fact we are able to write $2k = \dim {}^D/D_o = \dim {}^D/\Delta + \dim {}^{\Delta}/D_o = k + \dim {}^{\Delta}/D_o$ ; from which we conclude that

(3.18) $$\dim \Delta/D_o = k \ .$$

Thus (3.17) is proved. From (3.16) and the definition of $\hat{\Delta}$ it follows that each $u \in D$ can be written in the form $u = u_o + \sum_{i=1}^{k} b_i\alpha_i$, where $u_o \in D_o$ and $b_1, \ldots, b_k$ are scalars. If $v \in \Delta$ has the corresponding representation $v = v_o + \sum_{j=1}^{k} c_j\alpha_j$, we find that

$$\{v, u\} = \{v_o, u_o\} + \sum_{j=1}^{k} c_j \{\alpha_j, u_o\} + \sum_{i=1}^{k} \bar{b}_i \{v_o, \alpha_i\}$$

$$+ \sum_{i,j=1}^{k} c_j\bar{b}_i \{\alpha_j, \alpha_i\} \ .$$

By (3.10) any brace with $u_o$ or $v_o$ inside is zero; and by the self-adjointness property, the last sum vanishes. Thus for each pair of elements $u, v \in \Delta$, $\{u, v\} = 0$. Hence if $u \in \Delta$, $u$ also satisfies the condition expressed in (3.1) that it be in $\Delta^*$; that is, $\Delta \subset \Delta^*$. On the other hand, by (3.11) and (3.18) we see that $\dim D/\Delta^* = \dim \Delta/D_o = k$; so that $2k = \dim D/D_o = \dim D/\Delta^* + \dim \Delta^*/\Delta + \dim \Delta/D_o = k + \dim \Delta^*/\Delta + k$. Thus $\Delta^* = \Delta$. Since $H$ and $H^*$ have a common extension $T$, this equality implies $H^* = H$. The theorem is thereby proved.

CHAPTER IV

# SPECTRA OF SELF-ADJOINT EXTENSIONS OF A SYMMETRIC OPERATOR

## Von Neumann's Theory of Symmetric Extensions

Proofs of statements made in this section may be found in [20, Chapter IX], [16, Chapter VI], [1, Sections 67 and 78-80], and [19, Section 123].

As in Chapter I let $A$ be a closed, symmetric operator in a Hilbert space $\mathcal{H}$. If $R_-$ and $R_+$ denote the respective ranges of $A - i$ and $A + i$, then $R_-$ and $R_+$ are closed subspaces of $\mathcal{H}$. Since the eigenvalues of a symmetric operator are real, $A + i$ maps $D_A$ in a one to one manner onto $R_+$. Thus $(A + i)^{-1}$ exists. The operator $V = (A - i)(A + i)^{-1}$, the Cayley transform of $A$, is an isometry with domain $R_+$ and range $R_-$. $V$ determines $A$ uniquely according to the formula $Au = i(I + V)(I - V)^{-1}u$ for $u \in D_A$. Moreover, there is a one to one correspondence between the isometric extensions $\tilde{V}$ of $V$ and the symmetric extensions $\tilde{A}$ of $A$, which is such that $\tilde{V}$ corresponds to $\tilde{A}$ if and only if $\tilde{V}$ is the Cayley transform of $\tilde{A}$. $\tilde{A}$ is self-adjoint, moreover, if and only if $\tilde{V}$ is unitary, that is, if and only if $D_{\tilde{V}} = \mathcal{H}$ and the range $R_{\tilde{V}} = \mathcal{H}$. From these facts it follows that in order to determine the self-adjoint extensions of $A$, it is sufficient merely to determine the unitary extensions of $V$. But every unitary extension of $V$ can clearly be obtained by mapping the orthogonal complement $R_+^{\perp}$ of the domain $R_+$ of $V$ isometrically onto the orthogonal complement $R_-^{\perp}$ of the range $R_-$ of $V$. If $V'$ is an operator effecting this map, then a unitary extension $\tilde{V}$ of $V$ is obtained by defining $\tilde{V}$ to agree with $V$

on $R_+$ and to agree with $V'$ on $R_+^{\perp}$. Of course, a necessary and sufficient condition for the existence of such a $V'$ is that $m = \dim R_+^{\perp}$ [6, Section 16] and $n = \dim R_-^{\perp}$ be equal, not necessarily finite, cardinal numbers. $m$ and $n$ are known as the defect numbers of $A$. The ordered pair $(m, n)$ is known as the defect index of $A$. Notice that $m = n = 0$ is a necessary and sufficient condition for $A$ to be self-adjoint.

We assume henceforth that $m = n$. Von Neumann has given an explicit expression for the domain $D_{\tilde{A}}$ of the self-adjoint extension $\tilde{A}$ of $A$ whose Cayley transform is the $\tilde{V}$ described above. First, $D_{A^*}$ can be written as a direct sum

$$(4.1) \qquad D_{A^*} = D_A \oplus R_+^{\perp} \oplus R_-^{\perp} .$$

Moreover, $R_+^{\perp}$ is the eigenspace of $A^*$ corresponding to the eigenvalue $i$ and $R_-^{\perp}$ the eigenspace of $A^*$ corresponding to the eigenvalue $-i$. Furthermore, if $u \in D_{A^*}$ is represented according to the decomposition (4.1) in the form

$$(4.2) \qquad u = u_0 + u_+ + u_- ,$$

then [20, p. 344]

$$(4.3) \qquad \operatorname{Im}(A^*u,u) = \| u_+ \|^2 - \| u_- \|^2 .$$

If $\widetilde{A}$ is a self-adjoint extension of $A$, then [1, p. 242]

$$(4.4) \qquad D_{\widetilde{A}} = D_A \oplus (I - V')R_+^{\perp} ,$$

where $V'$ is the isometry occurring in the description of $\widetilde{V}$.

<u>The Spectrum Of A Self-Adjoint Operator</u>

Let $H$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. The resolvent set of $H$ [19, Section 132] consists of those complex numbers $\ell$ such that $(H - \ell)^{-1}$ is defined and bounded on $\mathcal{H}$. The complement in the complex plane of the resolvent set is the spectrum $s(H)$ of $H$. This definition agrees with the one given in Chapter II [20, Theorem 5.11]. $s(H)$ is a non-empty subset of the real axis and contains all the eigenvalues of $H$ [19, Section 132]. The limit spectrum, or essential spectrum, of $H$ consists of all points in the derived set $s(H)'$ ($\pm \infty$ being included when appropriate), as well as the eigenvalues of $H$ of infinite multiplicity. Such eigenvalues cannot occur with differential operators.

<u>Spectra Of Self-Adjoint Extensions</u>

Weyl showed [23] for the second order real differential operator mentioned in the introduction, defined on a half-axis, and in the limit point case, that the essential spectrum is independent of the particular self-adjoint extension of the operator. Recently, Heinz [8] showed that the independence of the essential spectrum of a self-adjoint extension of a closed symmetric operator $A$ with respect to the particular self-adjoint

extension is actually a property shared by all $A$ having equal and finite defect numbers. Heinz derives his result by use of the spectral representation of self-adjoint operators.

Hartman and Wintner [7] have complemented Weyl's result by proving that, in the case considered by Weyl, every real number not in the essential spectrum of $L$ is an eigenvalue of some self-adjoint extension of $L$. Their argument uses differential equations. See also [3], where a different proof is given.

Actually the theorem of Hartman and Wintner is true in the situation considered by Heinz. We prove the following theorem.

4.5 Theorem: _Let $A$ be a closed symmetric operator with equal defect numbers $m = n > 0$. Let $\lambda$ be a real number not in the essential spectrum of some self-adjoint extension $B$ of $A$. Then there exists a self-adjoint extension $\tilde{A}$ of $A$ such that $\lambda$ is an eigenvalue of finite multiplicity of $\tilde{A}$._

_Proof._ If $\lambda \varepsilon s(B)$, we may take $\tilde{A} = B$, since the only points of $s(B)$ which are not in the essential spectrum are eigenvalues of finite multiplicity of $B$ [19, Section 133]. Suppose then that $\lambda \not\varepsilon s(B)$, so that $(B - \lambda)^{-1}$ is defined and bounded on all of $\mathcal{H}$. Since $m \neq 0$ there exists a non-zero element $v \varepsilon R_+^{\perp}$, which therefore satisfies

$$A^*v = iv . \tag{4.6}$$

Clearly $(B - \lambda)^{-1}(i - \lambda)v \neq 0$, and $i$ is not an eigenvalue of $B$; so

that $u$, defined by

$$(4.7) \qquad u = (B - i)(B - \lambda)^{-1}(i - \lambda)v ,$$

is not zero. Since $(B - \lambda)^{-1}$ is a right [19, p. 295] (though not in general a left) inverse of $B - \lambda$, $(B - \lambda)(B - \lambda)^{-1}v = v$ ; from which

$$(4.8) \qquad B(B - \lambda)^{-1}v = \lambda(B - \lambda)^{-1}v + v .$$

Subtracting $i(B - \lambda)^{-1}v$ from both sides of (4.8) leads to

$$(B - i)(B - \lambda)^{-1}v = (\lambda - i)(B - \lambda)^{-1}v + v ,$$

or

$$(4.9) \qquad v - (B - i)(B - \lambda)^{-1}v = (i - \lambda)(B - \lambda)^{-1}v .$$

Since the right side of (4.9) belongs to $D_B$, the left side does also, and

$$(4.10) \qquad (B - i)\left\{v - (B - i)(B - \lambda)^{-1}v\right\} = (B - i)(B - \lambda)^{-1}(i - \lambda)v = u ,$$

where we have used (4.7). Again using (4.7) in the term in braces in (4.10), one obtains

$$(4.11) \qquad (B - i)\left\{v + (\lambda - i)^{-1}u\right\} = u .$$

Since in general $v \notin D_B$, we may not distribute $B - i$ in the braces. However, since $B \subseteq A^*$, (4.11) implies

$$(4.12) \qquad (A^* - i)\left\{v + (\lambda - i)^{-1}u\right\} = u .$$

Thus $v + (\lambda - i)^{-1}u \in D_{A^*}$. Since $v \in R_+^{\perp} \subset D_{A^*}$, we find that $u \in D_{A^*}$; (4.12) and (4.6) then imply

$$(A^* - i)(\lambda - i)^{-1}u = u .$$

Thus

$$(A^* - i)u = (\lambda - i)u ,$$

or finally

$$(4.13) \qquad A^*u = \lambda u .$$

This equation implies first that $(A^*u, u) = \lambda(u, u)$ is real; so that by (4.3)

$$(4.14) \qquad \| u_+ \| = \| u_- \| .$$

Since $m = n$ there exist isometries of $R_+^{\perp}$ onto $R_-^{\perp}$; and

according to (4.14), we may choose one, say $V'$ , such that $V'u_+ = -u_-$ . Thus $u$ , as expressed in (4.2), has the obvious representation

$$u = u_0 + (I - V')u_+ ,$$

with $u_0 \in D_{\bar{A}}$ . By virtue of (4.4), this equation implies $u \in D_{\tilde{A}}$ . Since $\tilde{A} \subseteq A^*$ , (4.13) then shows that $\tilde{A}u = \lambda u$ . The theorem is thereby proved.

# BIBLIOGRAPHY

## BIBLIOGRAPHY


[1] Achieser, N. I. and Glasmann, I. M., Theorie der linearen Operatoren im Hilbert-Raum, (translated from the Russian by L. Kaloujnine, H. L. Burmeister, W. Richter, G. Schwarze, H. Becker, and G. Köthe) Berlin, Akademie-Verlag, 1954.

[2] Coddington, E. A., The spectral representation of ordinary self-adjoint differential operators, Annals of Mathematics, Vol. 60, No. 1, 2nd series (1954) pp. 192-211.

[3] Coddington, E. A. and Levinson, N., On the spectrum of singular differential equations, Canadian Journal of Mathematics, Vol. III, No.3 (1951) pp. 335-338.

[4] Coddington, E. A. and Levinson, N., Theory of ordinary differential equations, New York, McGraw Hill, 1955.

[5] Glazman, I. M., (This spelling is the English transliteration of the name of the second author in [1].) On the theory of singular differential operators, Uspehi Matematičeskih Nauk (N. S.), Vol. 5, No. 6 (40), (1950) pp. 102-135, American Mathematical Society Translation No. 96.

[6] Halmos, P. R., Introduction to Hilbert space and the theory of spectral multiplicity, New York, Chelsea, 1951.

[7] Hartman, P. and Wintner, A., An oscillation theorem for continuous spectra, Proceedings of the National Academy of Sciences, Vol. 33 (1947) pp. 376-379.

[8] Heinz, E., Zur Theorie der Hermiteschen Operatoren des Hilbertschen Raumes, Nachrichten der Akademie der Wissenschaften in Göttingen. Mathematisch-Physikalische Klasse. IIa. No. 2 (1951).

[9] Kamke, E., Differentialgleichungen reeler Funktionen, Leipzig (1930), reprinted New York, Chelsea, 1947.

[10] Kodaira, K., The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices American Journal of Mathematics, Vol. 71 (1949) pp. 921-945.

[11] Kodaira, K., On ordinary differential equations of any even order and the corresponding eigenfunction expansions, American Journal of Mathematics, Vol. 72 (1950) pp. 502-544.

[12] Krumhaar, H., Zur Theorie der gewöhnlichen selbstadjungierten Differentialoperatoren gerader ordnung, Mathematische Annalen, Vol. 130 (1955) pp. 109-136.

[13] Levinson, N., A simplified proof of the expansion theorem for singular second order linear differential equations, Duke Mathematical Journal, Vol. 18 (1951) pp. 57-71.

[14] Levinson, N. Addendum to above, Vol. 18 (1951) pp. 719-722.

[15] Levinson, N. The expansion theorem for singular self-adjoint linear differential operators, Annals of Mathematics, Vol. 60, No. 2, 2nd series (1954) pp. 300-315.

[16] Nagy, B., Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 5 (1942).

[17] Naĭmark, M. A. and Fomin, S. V., Continuous direct sums of Hilbert spaces and some of thei applications, Uspehi Matematičeskih Nauk, Vol. 10, No. 2 (64)(1955) pp. 111-143; American Mathematical Society Translations, Series 2, Vol. 5, pp. 35-65.

[18] Rellich, F., Spectral theory of a second-order ordinary differential operator, New York University lecture notes, New York, 1953.

[19] Riesz, F. a d Nagy, B., Leçons d'analyse fonctionelle, 3rd edition, Budapest, Akadémiai Kiadó, 1955.

[20] Stone, M. H., Linear transformations in Hilbert space and their applications to analysis, New York, American Mathematical Society, 1932, Vol. XV.

[21] Titchmarsh, E. C., Eigenfunction expansions associated with second order differential equations, Oxford, Clarendon Press, 1946.

[22] Von Neumann, J., On Rings of Operators. Reduction Theory, Annals of Mathematics, Vol. 50, No. 2 (1949) pp. 401-485.

[23] Weyl, H., Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, Mathematische Annalen, Vol. 68 (1910) pp. 220-269.

[24] Weyl, H., Ueber gewöhnliche Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen, Nachrichten der Akademie der Wissenschaften in Göttingen. Mathematisch-Physikalische Klasse. IIa. (1910) pp. 442-467.

[25] Yosida, K., On Titchmarsh-Kodaira's formula concerning Weyl-Stone's eigenfunction expansion, Nagoya Mathematical Journal, Vol. 1 (1950) pp. 49-58.