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Separation of the n-Sphere by an (n - 1)-Sphere

James C. Cantrell
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by James C. Cantrell entitled "Separation of the n-Sphere by an (n - 1)-Sphere." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Orville G. Harrold, Major Professor

We have read this thesis and recommend its acceptance:

E. Cohen, Lee Simmons, & W.S. Makavier

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

December 5, 1961

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Accepted for the Council:

Hilton A. Smith
Dean of the Graduate School

SEPARATION OF THE n-SPHERE BY AN (n-1)-SPHERE

A Dissertation
Presented to
the Graduate Council of
University of Tennessee

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by
James C. Cantrell

December 1961

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CHAPTER I

INTRODUCTION

An $(n - 1)$-sphere is a topological image of $S^{n-1} = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}$, an open n-cell is a topological image of $\{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 < 1\}$, and a closed n-cell is a topological image of $\{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\}$.

In this thesis we consider certain $(n - 1)$-spheres embedded in S^n (we will frequently use the fact that S^n is topologically equivalent to the one point compactification of E^n). The problem is then to establish the existence or non-existence of certain topological properties of the two domains into which S^n is separated by the given $(n - 1)$-spheres.

For the cases $n = 1, 2$ it is known that each $(n - 1)$-sphere in S^n separates S^n into two domains, either of which is an open n-cell and has a closure which is a closed n-cell. That this is not the case for $n = 3$ is shown by numerous counter examples (see [2] and [5]*).

A 2-sphere K in S^3 that is locally polyhedral except at one, two or three points is considered in Chapter II and the following results are established. If K is locally polyhedral except at one point, then the closure of one component of $S^3 - K$ is a closed 3-cell and the other component is an open 3-cell. If K is locally polyhedral except at two points, then either the closure of one complementary domain is a closed

*Numbers in square brackets refer to numbers in the bibliography at the end of this paper.

3-cell or both complementary domains are open 3-cells. If K is locally polyhedral except at three points, then one of the complementary domains is an open 3-cell. This domain may or may not have a closure which is a closed 3-cell.

Let $A = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\}$,
$B = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq \frac{1}{4}\}$,
$C = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 4\}$, and
$D = \{(x_1, x_2, \ldots, x_n) \in E^n \mid x_1^2 + x_2^2 + \ldots + (x_n + 1)^2 \leq 4\}$. The Generalized Schoenflies Theorem states that if h is a homeomorphism of $\mathrm{Cl}\,(C \setminus B)$ into S^n, then the closure of either complementary domain of $h(\mathrm{Bd}\,A)$ is a closed n-cell. A proof of a special case of this theorem by Mazur [13] and a proof of the full theorem by Brown [8] point out that properties of the embedding homeomorphism of $S^{n-1} = \mathrm{Bd}\,A$ in S^n can be used to investigate the properties of the complementary domains. One is naturally led to the following question, if h is a homeomorphism of $\mathrm{Cl}\,(A \setminus B)$ into S^n, is the closure of the component of $S^n \setminus h(\mathrm{Bd}\,A)$ which contains $h(\mathrm{Bd}\,B)$ a closed n-cell? This question is answered affirmatively by Theorem 3.2. In fact the theorem follows from the Schoenflies Theorem and the two are therefore equivalent.

Two other embeddings of $\mathrm{Bd}\,A$ in S^n, $n > 3$, are considered in Chapter III: (1) a homeomorphism h of $\mathrm{Cl}\,(D \setminus B)$ into S^n, and (2) a homeomorphism h of $\mathrm{Cl}\,(D \setminus A)$ into S^n. In the first case it is shown that if h is semi-linear on each finite polyhedron of $(\mathrm{Int}\,A) \setminus B$, then the closure of either complementary domain of $h(\mathrm{Bd}\,A)$

is a closed n-cell. In the second case it is shown that if h is semi-linear on each finite polyhedron in a deleted neighborhood of $(0,0,\ldots,1)$ (see Definition 3.7), then the closure of the complementary domain of $h(\mathrm{Bd}\ A)$ which intersects $h(\mathrm{Bd}\ D)$ is a closed n-cell. The proofs of these theorems depend quite heavily on the fact that an arc in E^n, $n > 3$, which is locally polyhedral except at a single point is tame (see Lemma 3.3).

In Chapter IV three methods of constructing 3-spheres in S^4 from 2-spheres in S^3 are considered: (1) suspension of a 2-sphere in S^3, (2) rotation of a 2-cell in S^3 about the plane of its boundary, and (3) capping a cylinder over a 2-sphere in S^3. The construction methods in cases (1) and (2) were introduced by Artin [4] and have been used by him and by Andrews and Curtis [3] to construct 2-spheres in S^4 from 1-spheres and 1-cells in S^3. Their techniques are used to establish isomorphism theorems relating the fundamental groups of the complements of the constructed 3-spheres and the fundamental groups of the corresponding complements of the given 2-spheres. In Case (1) it is shown that the second homotopy groups of the complements of the constructed 3-spheres are trivial. Method (2) is also used to construct a 3-sphere in S^4, one complementary domain of which is simply connected but is not an open 4-cell. The third construction is considered because it seems to give the simplest scheme (in fact the only scheme of which I am aware) for showing the existence of a 3-sphere in S^4 such that one complementary domain has a closure which is a closed 4-cell, and the other complementary domain is an open 4-cell but its closure is not a closed 4-cell.

CHAPTER II

ALMOST LOCALLY POLYHEDRAL 2-SPHERES IN S^3

Let K be a set in a geometric complex C .

Definition 2.1. K is *locally polyhedral at a point* p *of* K if there is an open set U containing p such that $\mathrm{Cl}\,U \cap K$ is a polyhedron in C . K is said to be *locally polyhedral* if it is locally polyhedral at each point of its points.

Definition 2.2. K is *tamely embedded* in C if there is a homeomorphism of C onto itself that carries K onto a polyhedron.

Definition 2.3. K is *locally tamely embedded* in C if for each point p of K there is a neighborhood N of p and a homeomorphism h_p of $\mathrm{Cl}\,N$ onto a polyhedron in C , such that $h_p(\mathrm{Cl}\,N \cap K)$ is a polyhedron.

We will frequently have occasion to use the fact that a locally tamely embedded 2-manifold in a 3-manifold is tamely embedded [6, 15].

Lemma 2.1. *Let* T *be a torus in* E^3 *that is the union of two locally tame annuli,* A_1 *and* A_2 *, which meet along their common boundary curves* a_1 *and* a_2 *. Then* T *is tamely embedded in* E^3.

Proof. Let a_3 be a simple closed curve on A_2 which is homologous to both a_1 and a_2 on A_2 . Let A_{21} be the annulus on A_2 which is determined by a_1 and a_3 , and let A_{22} be the annulus on A_2 which is determined by a_2 and a_3 . Let f_1 be a space homeomorphism taking A_1 onto a polyhedral annulus. By Theorem 2 of [14], there exists a space homeomorphism f_2 which is the identity on $f_1(A_1)$ and

carries $f_1(A_2)$ onto a set which is locally polyhedral, except on $f_1(a_1) \cup f_1(a_2)$.

Let $\varepsilon = \frac{1}{2} \min \{\rho[f_2f_1(a_1), f_2f_1(A_{22})], \rho[f_2f_1(a_2), f_2f_1(A_{21})]\}$, and let U_1 be an ε-neighborhood of $f_2f_1(a_1)$ and U_2 be an ε-neighborhood of $f_2f_1(a_2)$. By Lemma 5.2 of [15], there is a space homeomorphism f_3 which is the identity on $f_2f_1(A_1) \cup (E^3 \setminus U_1)$ and carries $f_2f_1(A_1 \cup A_{21})$ onto a polyhedron. We again apply Lemma 5.2 of [15] to obtain a space homeomorphism f_4 which is the identity on $f_3f_2f_1(A_1) \cup (E^3 \setminus U_2)$ and carries $f_3f_2f_1(A_1 \cup A_{22})$ onto a polyhedron. The mapping $f = f_4f_3f_2f_1$ is then a space homeomorphism which carries T onto a polyhedron.

<u>Definition</u> 2.4. A k-manifold M in E^n is said to be <u>locally peripherally unknotted</u> at x if for each positive ε there is a closed n-cell of diameter less that ε whose interior contains x , such that the boundary of the n-cell and M meet in a locally peripherally unknotted cell or sphere, according as x lies on the boundary of M or not. A 0-cell or 0-sphere is considered to be locally peripherally unknotted. If M is locally peripherally unknotted at each of its points, then we say M is <u>locally peripherally unknotted</u> and use the corresponding abbreviation LPU .

An investigation of the proof of Theorem 1 of [12] shows that the conclusions of the theorem may be obtained under slightly weaker hypotheses. Since in the proof of the theorem the LPU property is used only at the points of U , the theorem may be restated as follows.

<u>Theorem</u> 2.1. <u>Let</u> M <u>be a topological</u> 2-<u>manifold without boundary in</u> E^3 <u>that is</u> LPU <u>on an open set</u> U <u>of</u> M . <u>Let</u> $\varepsilon > 0$ <u>and</u> A <u>a</u>

<u>component of</u> $E^3 \setminus M$. <u>Then there is a space homeomorphism</u> h <u>such that</u>

(1) $h(U) \subset A$,

(2) $\rho(x,h(x)) < \varepsilon$,

(3) $x \in M \setminus U$ <u>implies</u> $h(x) = x$.

<u>Definition</u> 2.5. Let Ψ be a semi-linear mapping of a right prism P onto the solid torus B such that, if corresponding points of the two bases of P are identified, the mapping then induced by Ψ is a homeomorphism. Let e be the boundary of the lower base of P . Those simple closed curves on $\mathrm{Bd}\, B$ which are homologous to $\Psi(e)$ are called <u>meridians</u> of B . A polyhedral disk D , such that $\mathrm{Int}\, D \subset \mathrm{Int}\, B$ and such that $\mathrm{Bd}\, D$ is a meridian of B , is called a <u>meridinal disk</u> of B .

<u>Definition</u> 2.6. Suppose that K is a polyhedral 3-cell in E^3 . By a <u>chord</u> of K is meant an oriented polygonal arc u whose end points lie on $\mathrm{Bd}\, K$, but which is otherwise contained in the interior of K . Let the end points of u be joined by an arc w on $\mathrm{Bd}\, K$. The chord u is said to be an <u>unknotted chord</u> of K if and only if $u \cup w$ is an unknotted simple closed curve (bounds a disk in E^3) . It is shown in [17, p. 155] that the knot type of $u \cup w$ is independent of the choice of $w \subset \mathrm{Bd}\, K$.

<u>Definition</u> 2.7. Let k_1 and k_2 be two knots in E^3 . Let S be a polyhedral 2-sphere in E^3 , and denote by C_1 and C_2 the closures of the two components of $E^3 \setminus S$. Choose a polygonal arc w on S with endpoints x and y . Then choose chords u_1 (from x to y) and u_2 (from y to x) of C_1 and C_2 respectively, each with endpoints x and y , such that $u_1 \cup w$ (oriented as u_1) is a representative of the

knot k_1 , and $u_2 \cup w$ (oriented as u_2) is a representative of k_2 . The knot represented by the oriented polygon $u_1 \cup u_2$ is defined to be the <u>product</u> of the knots k_1 and k_2 . It is shown in [17, p. 156] that the identity (the knot represented by a plane circle) cannot be expressed as a knot product containing non-identity factors.

Let $A' = \left\{(x,y,z) \in E^3 \mid x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 < \frac{1}{4}\right\}$, $C' = \left\{(x,y,z) \in E^3 \mid x^2 + y^2 + z^2 < 1\right\}$, and for $i = 1, 2, \ldots$ let π_i be the plane $y = \frac{i}{i+1}$. Let the following symbols denote the indicated subsets of E^3 .

D_i' : $\pi_i \cap \mathrm{Cl}\, A'$

d_i : $\mathrm{Bd}\, D_i'$

G_o' : Component of $\mathrm{Bd}\, A' \setminus d_1'$ which contains (0,0,0)

G_i' : open annulus on $\mathrm{Bd}\, A'$ determined by d_i' and d_{i+1}'

A_o' : component of $E^3 \setminus (G_o' \cup D_1')$ which does not contain (0, 1, 0).

A_i' : component of $E^3 \setminus (G_i' \cup D_i' \cup D_{i+1}')$ which does not contain (0, 1, 0)

E_i' : $\pi_i \cap \mathrm{Cl}\, C'$

e_i' : $\mathrm{Bd}\, E_i'$

H_o' : component of $\mathrm{Bd}\, C' \setminus e_1'$ which contains (0,-1,0)

H_i' : open annulus on $\mathrm{Bd}\, C'$ determined by e_i' and e_{i+1}'

J_i' : the frustum of a cone determined by e_i' and d_{i+1}'

T_o' : $H_o' \cup J_1' \cup \mathrm{Cl}\, G_o' \cup G_1'$

T_i' : $J_i' \cup H_i' \cup J_{i+1}' \cup G_{i+1}'$

R_i' : union of T_i' and its bounded complementary domain.

Let K be a 2-sphere in E^3 that is locally polyhedral except at a single point p. According to Lemma 3 of [11], there is a component E_p of $E^3 \setminus K$, and a sequence $D_1, D_2, \ldots$ of disjoint polyhedral disks in $\mathrm{Cl}\, E_p$, such that (1) for each i, $D_i \cap K$ is the boundary d_i of D_i, (2) the diameter of $p \cup D_i$ is less than $1/i$, and (3) for each i, d_{i+1} separates p from d_i. Let the following symbols denote the indicated subsets of $\mathrm{Cl}\, E_p$.

G_o : component of $K \setminus d_1$ which does not contain p

G_i : open annulus on K determined by d_i and d_{i+1}

A_o : component of $E^3 \setminus (G_o \cup D_1)$ which does not contain p

A_i : component of $E^3 \setminus (G_i \cup D_i \cup D_{i+1})$ which does not contain p.

In the proof of Theorem 1 of [11] a homeomorphism σ, taking $\mathrm{Cl}\, A'$ onto $\mathrm{Cl}\, E_p$ (compactified at infinity if E_p is the unbounded component of $E^3 \setminus K$), was constructed which carries the "primed" subsets of $\mathrm{Cl}\, A'$ onto the corresponding "unprimed" subsets of $\mathrm{Cl}\, E_p$.

Lemma 2.2. There exists a 2-sphere L in E^3 such that E_p is contained in one complementary domain E of $E^3 \setminus L$ and $L \cap K = p$. Furthermore, there is a homeomorphism Ψ of $\mathrm{Cl}\, C'$ onto $\mathrm{Cl}\, E$ (compactified at infinity if necessary) such that Ψ agrees with σ on $\mathrm{Cl}\, A'$.

Proof. Let A denote the bounded component of $E^3 \setminus K$ and B the unbounded component. We will first assume $E_p = A$.

Let $\varepsilon_o, \varepsilon_1, \ldots$ be a sequence of positive numbers which converges to zero. By Theorem 2.1, there is a space homeomorphism h_o such that (1) $h_o(\mathrm{Cl}\, G_o \cup G_1) \subset B$, (2) $\rho(x, h_o(x)) < \varepsilon_o$, and

(3) $x \in K \setminus (\mathrm{Cl}\, G_o \cup G_1)$ implies $h_o(x) = x$. Since $\mathrm{Cl}\, G_o \cup \mathrm{Cl}\, G_1$ is locally polyhedral and h_o is a space homeomorphism, it follows that $h_o(\mathrm{Cl}\, G_o \cup \mathrm{Cl}\, G_1)$ is a locally tame disk. It follows, from Theorem 9.3 of [15], that $T_o = \mathrm{Cl}\, G_o \cup \mathrm{Cl}\, G_1 \cup h_o(\mathrm{Cl}\, G_o \cup \mathrm{Cl}\, G_1)$ is a tame 2-sphere. Hence the closure of the bounded complementary domain of T_o is a closed 3-cell [1].

Let h_o' be a homeomorphism of the disk $H_o' \cup J_1'$ onto $\mathrm{Cl}\,(G_o' \cup G_1')$ which is the identity on d_2' and carries e_1' onto d_1'. Now define a homeomorphism σ_o of T_o' onto T_o by the equations

$$\sigma_o(x) = h_o \sigma h_o'(x), \qquad x \in H_o' \cup J_1'$$
$$\sigma_o(x) = \sigma(x), \qquad x \in \mathrm{Cl}\,(G_o' \cup G_i').$$

Since the spheres T_o' and T_o are boundaries of closed 3-cells, σ_o can be extended to their respective interiors. This extension will also be denoted by σ_o.

For each positive integer i we will associate a mapping σ_i with $\sigma_{i-1}, \sigma_{i-2}, \ldots, \sigma_o$ by the following construction.

For $j = 0, 1, \ldots, i-1$ denote the following subsets of E^3 as indicated.

$$E_{j+1} : \sigma_j(E_{j+1}')$$
$$e_{j+1} : \sigma_j(e_{j+1}')$$
$$J_{j+1} : \sigma_j(J_{j+1}')$$
$$H_j : \sigma_j(H_j')$$
$$K_j : \left(\bigcup_{k=0}^{j} \mathrm{Cl}\, H_k\right) \cup J_{j+1} \cup \left(\bigcup_{k=j+2}^{\infty} \mathrm{Cl}\, G_k\right) \cup p$$
$$B_j : \text{unbounded component of } E^3 \setminus K_j$$

We again apply Theorem 2.1 to obtain a space homeomorphism h_i such that (1) $h_i(\text{Int } J_i \cup d_{i+1} \cup G_{i+1}) \subset B_{i-1}$, (2) $\rho(x,h_i(x)) < \varepsilon_i$, and (3) $x \in K_{i-1} \setminus (\text{Int } J_i \cup d_{i+1} \cup G_{i+1})$ implies $h_i(x) = x$. Since $J_i \cup \text{Cl } G_{i+1}$ is locally tame and h_i is a space homeomorphism, it follows that $h_i(J_i \cup \text{Cl } G_{i+1})$ is locally tame. These two locally tame annuli meet along their common boundary curves e_i and d_{i+2}, and hence their union is, by Lemma 2.1, a tame torus. Let us denote this torus by T_i. The bounded complementary domain of T_i is the common part of the interiors of the tame spheres

$$S_{i1} = E_i \cup J_i \cup G_{i+1} \cup D_{i+2}$$

and

$$S_{i2} = E_i \cup h_i(J_i \cup G_{i+1}) \cup D_{i+2} .$$

Furthermore, by the construction of the sphere S_{i2}, it is evident that the image under σ of the segment of the y-axis between d_i and d_{i+2} is an unknotted chord of each of the cells bounded by S_{i1} and S_{i2}. Hence, by Hilfsatz 1, p. 167 of [17], it follows that the union of T_i and its bounded complementary domain is an unknotted solid torus. Denote this solid torus by R_i.

Let h_i' be a homeomorphism of $\text{Cl } H_i' \cup J_{i+1}'$ onto $J_i' \cup \text{Cl } G_{i+1}'$ which leaves e_i' and d_{i+2}' fixed and carries e_{i+1}' onto d_{i+1}'. Now define a homeomorphism σ_i of T_i' onto T_i by the equations

$$\sigma_i(x) = h_i\sigma_{i-1}h_i'(x) \ , \quad x \in H_i'$$
$$\sigma_i(x) = h_i\sigma h_i'(x) \ , \quad x \in J_{i+1}'$$
$$\sigma_i(x) = \sigma_{i-1}(x) \ , \quad x \in J_i'$$
$$\sigma_i(x) = \sigma(x) \ , \quad x \in G_{i+1}' \ .$$

This gives a homeomorphism between the boundaries of the solid tori R_i' and R_i. To be able to extend this homeomorphism to their interiors it will suffice to exhibit a pair of meridian curves on Bd R_i' which are carried by σ_i onto meridian curves of Bd R_i [17].

Let k_{i1}' be the intersection of the half plane $x = 0$, $z > 0$ and T_i', and ℓ_{i1}' the intersection of the half plane $x = 0$, $z < 0$ and T_i'. The assertion is that k_{i1}' and ℓ_{i1}' are simple closed curves of the desired type. We will show that $\sigma_i(k_{i1}')$ is a meridian curve of Bd R_i. That $\sigma_i(\ell_{i1}')$ is also a meridian curve of Bd R_i would follow by a similar argument.

Let π be the half plane $x = 0$, $z > 0$ and let u_{i1}' be the oriented arc from $y' = \pi \cap d_{i+2}'$ to $x' = \pi \cap e_i'$ which lies in $\pi \cap (H_i' \cup J_{i+1}')$. Let w_i' be the arc from y' to x' which lies in $\pi \cap (J_i' \cup G_{i+1}')$. Let u_{i2}' be an oriented arc from x' to y' which leads from x' to the y-axis in E_i', then follows the y-axis to d_{i+2}', and then leads to y' in d_{i+2}'. Let $k_{i1}' = u_{i1}' \cup w_i'$, $k_{i2}' = u_{i2}' \cup w_i'$, and $k_{i3}' = u_{i1}' \cup u_{i2}'$, each with the orientation of u_{i1}' and u_{i2}'. Finally let u_{i1}, u_{i2}, w_i, k_{i1}, k_{i2}, and k_{i3} be the images under σ_i of the corresponding "primed" sets. Since k_{i3}' bounds a disk in the cell bounded by $E_i' \cup H_i' \cup J_{i+1}' \cup D_{i+2}'$, it follows that

k_{i3} bounds a disk in the cell bounded by S_{i2} . Hence k_{i3} represents the identity knot and we have it given as the product of the knots represented by k_{i1} and k_{i2} . Thus k_{i1} represents the identity knot. A disk F_i bounded by k_{i1} can then be found which, with the exception of the arc U_{i1} on its boundary, lies in the interior of S_{i2} . If F_i intersects $J_i \cup G_{i+1}$ only in w_i then F_i is a meridinal disk at R_i . Suppose, on the other hand, that there are components $a_1, a_2, \ldots, a_{ni}$ of $F_i \cap (J_i \cup G_{i+1})$ other than w_i . Then let a_j be a component which contains no other such component in its interior (relative to $J_i \cup G_{i+1}$) . Let X be the disk of $J_i \cup G_{i+1}$ bounded by a_j , and let Y be the subdisk of F_i bounded by a_j . Then define $F_i' = (F_i \setminus Y) \cup X$, and deform F_i' semilinearly away from $J_i \cup G_{i+1}$ in a sufficiently small neighborhood of X that no new intersections with S_{i1} or S_{i2} are introduced. The disk F_k'' thus produced is bounded by k_i, and has one less intersection with $J_i \cup G_{i+1}$ than F_i . In this way each of the a_j may be eliminated to obtain a disk F_i^* which, except for its boundary k_{i1} , is in the interior of R_i .

The extension of σ_i will also be denoted by σ_i .

The desired sphere L is taken to be $\bigcup_{i=0}^{\infty} \mathrm{Cl}\, H_i \cup p$ and Ψ is defined by the equations

$$\Psi(x) = \sigma(x) \;, \quad x \in \mathrm{Cl}\, A'$$
$$\Psi(x) = \sigma_0(x) \;, \quad x \in R_0'$$
$$\Psi(x) = \sigma_i(x) \;, \quad x \in R_i' \;, \; i = 1, 2, \ldots .$$

Lemma 2.3. *There is a continuous mapping* g *of* $\mathrm{Cl}\, C'$ *onto* $\mathrm{Cl}\, C'$ *such that*

(1) g *is fixed on* $\mathrm{Bd}\, C'$,

(2) g *is a homeomorphism of* $\mathrm{Cl}\, C' \setminus \mathrm{Cl}\, A'$ *onto* $\mathrm{Cl}\, C' \setminus (0, 1, 0)$, *and*

(3) $g(\mathrm{Cl}\, A') = (0, 1, 0)$.

Proof. For $x \in \mathrm{Cl}\, C' \setminus \mathrm{Cl}\, A'$ let X be the vector from $(0, 1, 0)$ to x and L the line determined by $(0, 1, 0)$ and x . Let x_1 be the point of intersection of L and $\mathrm{Bd}\, A'$ and x_2 the point of intersection of L and $\mathrm{Bd}\, C'$. Let $dx = \rho((0, 1, 0), x)$, $ex = \rho((0, 1, 0), x_1)$, and $fx = \rho((0, 1, 0), x_2)$. For $x \in \mathrm{Cl}(A')$ let $g(x) = (0, 1, 0)$ and for $x \in \mathrm{Cl}\, C' \setminus \mathrm{Cl}\, A'$ let $g(x)$ be the terminal point of the vector $\dfrac{(dx - ex)(fx + dx)}{2dx(fx - ex)} X$. It is evident that g has the desired properties.

Theorem 2.2. *Let* K *be a* 2-*sphere in* E^3 *that is locally polyhedral except at a single point* p . *Let the interior and exterior of* K *be* A *and* B *respectively*. *Then*,

(1) *either* $\mathrm{Cl}\, A$ *or* $\mathrm{Cl}\, B$ (*compactified at infinity*) *is a closed* 3-*cell, and*

(2) *the other complementary domain* (*compactified at infinity if necessary*) *is an open* 3-*cell*.

Proof. Statement (1) is Theorem 1 of [11].

Suppose A is the domain such that $\mathrm{Cl}\, A$ is a closed 3-cell. Let Y , L , and E be as in the conclusion of Lemma 2.2. Let g be the mapping of $\mathrm{Cl}\, C'$ onto $\mathrm{Cl}\, C'$ defined in Lemma 2.3. Define a continuous mapping f of E^3 onto E^3 by the equations

$$f(x) = x \qquad , \quad x \in E^3 \setminus E ,$$

$$f(x) = \Psi g \Psi^{-1}(x), \quad x \in E .$$

From the definitions of the mappings Ψ and g it is clear that f is a mapping of E^3 onto E^3 which takes B homeomorphically onto $E^3 \setminus p$. Thus B (compactified at infinity) is an open 3-cell.

A similar argument will apply in case $Cl\ B$ is a closed 3-cell, to show that A is then an open 3-cell.

Theorem 2.3. *If K is a 2-sphere in E^3 that is locally polyhedral except at two points p and q, then either*

(1) *$Cl\ A$ or $Cl\ B$ (compactified at infinity) is a closed 3-cell,*

or

(2) *both A and B (compactified at infinity) are open 3-cells.*

Proof. According to Lemma 2 of [11] we may associate with the point p a certain domain E_p of $E^3 \setminus K$ and a sequence $\{D_{pi}\}_{i=1}^{\infty}$ of disjoint polyhedral disks in $Cl\ E_p$ such that (1) for each i, $D_{pi} \cap K$ is the boundary d_{pi} of D_{pi}, (2) the diameter of $p \cup D_{pi}$ is less than $\frac{1}{i}$, and (3) for each i in $d_{p(i+1)}$ separates p from d_{pi} in K.

Similarly, let E_q be a domain of $E^3 \setminus K$ and $\{D_{qi}\}_{i=1}^{\infty}$ a sequence of disjoint polyhedral disks in $Cl\ E_q$ such that (1) for each i, $D_{qi} \cap K$ is the boundary d_{qi} of D_{qi}, (2) the diameter of $q \cup D_{qi}$ is less than $\frac{1}{i}$, and (3) for each i, $d_{q(i+1)}$ separates q from d_{qi} in K.

First suppose $E_p = E_q = A$. By taking subsequences, if necessary, we may assume that for each pair of integers i and j (1) $D_{pi} \cap D_{qj} = \square$, (2) the disk d_{qi} is in the closure of the bounded component of $E^3 \setminus (K \cup D_{pi})$ which has q as a limit point.

Let G_o be the annulus on K determined by D_{p1} and D_{q1} and for $i > 0$, let G_{pi} be the annulus on K determined by D_{pi} and $D_{p(i+1)}$ and G_{qi} the annulus determined by D_{qi} and $D_{q(i+1)}$. Denote the sphere $G_o \cup D_{p1} \cup D_{q1}$ by K_o, and for $i > 0$ denote the sphere $G_{pi} \cup D_{pi} \cup D_{p(i+1)}$ by K_{pi}, and the sphere $G_{qi} \cup D_{qi} \cup D_{q(i+1)}$ by K_{qi}. Let A_o be the bounded component of $E^3 \setminus K_o$, and for $i > 0$ let A_{pi} be the bounded component of $E^3 \setminus K_{pi}$ and A_{qi} the bounded component of $E^3 \setminus K_{qi}$. By [1] we know that $\mathrm{Cl}\, A_o$, $\mathrm{Cl}\, A_{pi}$, and $\mathrm{Cl}\, A_{qi}$, $i = 1, 2, \ldots$, are closed 3-cells.

Let $K' = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ and A' the bounded component of $E^3 \setminus K'$. For each $i > 0$ let π_{pi} be the plane perpendicular to the y-axis at $(0,\frac{i}{i+1},0)$ and π_{qi} the plane perpendicular to the y-axis at $(0,-\frac{i}{i+1},0)$. Define $D'_{pi} = \pi_{pi} \cap \mathrm{Cl}\, A'$, $D'_{qi} = \pi_{qi} \cap \mathrm{Cl}\, A'$, $i = 1, 2, \ldots$, and let the sets G'_o, G'_{pi}, K'_o, K'_{pi}, K'_{qi}, A'_o, A'_{pi} and A'_{qi} correspond to the "unprimed" sets above.

A homeomorphism of $\mathrm{Cl}\, A'$ onto $\mathrm{Cl}\, A$ is obtained by first using the lemma on page 40 of [10] to map the boundaries of A'_o, A'_{pi}, and A'_{qi}, $i = 1, 2, \ldots$, onto the boundaries of the corresponding A_o, A_{pi} and A_{qi} such that the disks D'_{pi} and D'_{qi}, $i = 1,2,\ldots$, are mapped onto the corresponding D_{pi} and D_{qi}. Then [1] is used to extend this homeomorphism to their respective interiors. This gives a homeomorphism h of

$$\mathrm{Cl}\, A'_o \cup \left[\bigcup_{i=1}^{\infty} \mathrm{Cl}\, A'_{pi}\right] \cup \left[\bigcup_{i=1}^{\infty} \mathrm{Cl}\, A'_{qi}\right] = \mathrm{Cl}\, A' \setminus [(0,-1,0) \cup (0,1,0)]$$

onto $\mathrm{Cl}\, A \setminus (p \cup q)$. By defining $h(0,-1,0) = q$ and $h(0,1,0) = p$ we have a homeomorphism of the closed 3-cell $\mathrm{Cl}\, A'$ onto $\mathrm{Cl}\, A$.

A similar argument may be used when $E_p = E_q = B$.

The alternative case $E_p \neq E_q$ will now be considered. Suppose $E_p = A$. We will show that A is an open 3-cell. A similar argument would show that B (compactified at infinity) is also an open 3-cell.

Let the sequence of polyhedral disks $\{D_{pi}\}_{i=1}^{\infty}$ be defined as above. We may assume that for each i , d_{pi} separates p and q in K . For each $i > 0$, let H_i be the component of $K \setminus d_{pi}$ which does not contain p . Denote $H_i \cup D_{pi}$ by K_i and the bounded component of $E^3 \setminus K_i$ by A_i . Since each K_i is locally polyhedral except at the point q , we have, by Theorem 2.2, that each A_i is an open 3-cell. Since A is the union of the increasing sequence of open 3-cells A_i , $i = 1, 2, \ldots$, it follows from [9] that A is an open 3-cell.

Let K be a 2-sphere in E^3 that is locally polyhedral except at the three points p, q and r . Associate with the points $p, q,$ and r , respectively, certain domains E_p , E_q , and E_r of $E^3 \setminus K$ and the sequences of polyhedral disks $\{D_{pi}\}_{i=1}^{\infty}$, $\{D_{qi}\}_{i=1}^{\infty}$ and $\{D_{ri}\}_{i=1}^{\infty}$ in accordance with Lemma 2 of [11].

Theorem 2.4.

(1) *If* $E_p = E_q = E_r$, *then* $\mathrm{Cl}\,(E_p)$ (*compactified at infinity if* $E_p = B$) *is a closed* 3-*cell*.

(2) *If* E_p, E_q, *and* E_r *do not coincide, say* $E_p = E_q \neq E_r$, *then* E_p (*compactified at infinity if* $E_p = B$) *is an open* 3-*cell*.

Proof of (1).

Suppose $E_p = E_q = E_r = A$. We may assume that for each triple i , j, k of positive integers that

(1) $(D_{pi} \cap D_{qj}) \cup (D_{pi} \cap D_{rk}) \cup (D_{qj} \cap D_{rk}) = \square$,

(2) $D_{pi} \cup D_{qj}$ is in the closure of the bounded component of $E^3 \setminus (K \cup D_{rk})$ which does not have r as a limit point,

(3) $D_{pi} \cup D_{rk}$ is in the closure of the bounded component of $E^3 \setminus (K \cup D_{qj})$ which does not have q as a limit point, and

(4) $D_{qj} \cup D_{rk}$ is in the closure of the bounded component of $E^3 \setminus (K \cup D_{pi})$ which does not have p as a limit point.

Let G_o be the component of $E^3 \setminus (K \cup d_{p1} \cup d_{q1} \cup d_{r1})$ which contains neither p, q, nor r. Let $K_o = G_o \cup D_{p1} \cup D_{q1} \cup D_{r1}$ and A_o the bounded component of $E^3 \setminus K_o$. For $i > 0$ define the sets G_{pi}, G_{qi}, G_{ri}, K_{pi}, K_{qi}, K_{ri}, A_{pi}, A_{qi}, and A_{ri} as indicated in the proof of statement (1) of Theorem 2.3.

Let $K' = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ and A' the interior of K'. For $i > 0$ let π_{pi} be the plane perpendicular to the y-axis at $(0, \frac{i+2}{i+3}, 0)$, π_{qi} the plane perpendicular to the y-axis at $(0, -\frac{i+2}{i+3}, 0)$, and π_{ri} the plane perpendicular to the z-axis at $(0, 0, \frac{i+2}{i+3})$. For $i > 0$ define $D'_{pi} = \pi_{pi} \cap \mathrm{Cl}\, A'$, $D'_{qi} = \pi_{qi} \cap \mathrm{Cl}\, A'$, and $D'_{ri} = \pi_{ri} \cap \mathrm{Cl}\, A'$. Let the sets G_o', G'_{pi}, G'_{qi}, G'_{ri} K_o', K'_{pi}, K'_{qi}, K'_{ri}, A_o', A'_{pi}, A'_{qi}, and A'_{ri} correspond to the "unprimed" sets above.

The spheres G_o', G'_{pi}, G'_{qi}, and G'_{ri} are mapped onto the corresponding spheres G_o, G_{pi}, G_{qi}, and G_{ri} by [10], and then [1] is used to extend this mapping to their respective interiors. This gives a

homeomorphism h of

$$\mathrm{Cl}\ A_o' \cup \left[\bigcup_{i=1}^{\infty} \mathrm{Cl}\ A_{pi}'\right] \cup \left[\bigcup_{i=1}^{\infty} \mathrm{Cl}\ A_{qi}'\right] \cup \left[\bigcup_{i=1}^{\infty} \mathrm{Cl}\ A_{pi}'\right]$$

$$= \mathrm{Cl}\ A' \setminus [(0,1,0) \cup (0,-1,0) \cup (0,0,1)]$$

onto $\mathrm{Cl}\ A \setminus (p \cup q \cup r)$. By defining $h(0, 1, 0) = p$, $h(0, -1, 0) = q$, and $h(0, 0, 1) = r$, we have a homeomorphism of $\mathrm{Cl}\ A'$ onto $\mathrm{Cl}\ A$.

<u>Proof of</u> (2).

Suppose $E_{p_\infty} = E_q = A$ and $E_r = B$. Let the sequences of polyhedral disks $\left\{D_{pi}\right\}_{i=1}^{\infty}$ and $\left\{D_{qi}\right\}_{i=1}^{\infty}$ be defined as above. For each pair i, j of positive integers we may assume that r is on the annulus D_{ij} of K determined by D_{pi} and D_{qj}. For each $i > 0$ let $K_i = D_{ii} \cup D_{pi} \cup D_{qi}$ and A_i the bounded component of $E^3 \setminus K_i$. Each K_i is a 2-sphere, locally polyhedral except at r. Hence by Theorem 2.2, each A_i is an open 3-cell. Since A is the union of the increasing sequence of open 3-cells A_i, it follows that A is an open 3-cell.

CHAPTER III

SOME EMBEDDINGS OF S^{n-1} IN S^n

Let us consider the following subsets of E^n :

$$A = \left\{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\right\} ,$$

$$B = \left\{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq \tfrac{1}{4}\right\} ,$$

$$C = \left\{(x_1, x_2, \ldots, x_n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq 4\right\} ,$$

$$D = \left\{(x_1, x_2, \ldots, x_n \mid x_1^2 + x_2^2 + \ldots + (x_n + 1)^2 \leq 4\right\} .$$

Let h be a homeomorphism taking Bd A into the n-sphere S^n and denote $h(\text{Bd } A)$ by S^{n-1} .

Definition 3.1. We say that h can be *tended in one direction along a cylinder* if there ists a homeomorphism f of the closed annulus determined by A and B into S^n , such that for each $x \in \text{Bd } A$, $f(x) = h(x)$.

Observe that the condition of Definition 3.1 is equivalent to the statement: there ists a homeomorphism f of $\text{Bd } A \times [0, 1]$ into S^n such that for each $x \in \text{Bd } A$, $f(x, 0) = h(x)$.

Definition 3.2. We say that h can be *extended in both directions along a cylinder* if there exists a homeomorphism f of the closed annulus determined by B and C into S^n such that for each $x \in \text{Bd } A$, $f(x) = h(x)$.

The usual formulation of the condition in Definition 3.2 is: there exists a homeomorphism f of $\text{Bd } A \times [-1, 1]$ into S^n , such that for

each $x \in \text{Bd } A$, $f(x, 0) = h(x)$.

Definition 3.3. If f is a continuous mapping of a topological space X into a topological space Y , then an inverse set (under f) is a set $M \subset X$, containing at least two points, and such that for some point $y \in f(X)$, $M = f^{-1}(y)$.

Definition 3.4. A set M is cellular in an n-dimensional metric space X if there exists n-cells $Q_1, Q_2, \ldots$ in X such that $Q_{i+1} \subset \text{Int } Q_i$, and $\bigcap_{i=1}^{\infty} Q_i = M$.

The concepts defined in Definitions 3.3 and 3.4 were used by M. Brown to prove the Generalized Schoenflies Theorem [8]. This theorem is stated as Theorem 3.1 below for the sake of completeness.

Theorem 3.1. If h can be extended in both directions along a cylinder, then the closure of either complementary domain of S^{n-1} is a closed n-cell.

Lemma 3.1. There exists a continuous mapping g of the annulus $(\text{Bd } A) \times [0, 1]$ onto a closed n-cell such that the only inverse set is $(\text{Bd } A) \times \{1\}$.

Proof. We may take the annulus to be the one determined by A and B , with $(\text{Bd } A) \times \{0\}$ identified with $\text{Bd } A$, and $(\text{Bd } A) \times 1$ identified with $\text{Bd } B$. For $x \in A \setminus B$ let X be the vector from the origin to x and let dx be the length of the vector X . For $x \in B$ let $g(x) = (0, 0, \ldots, 0)$ and for $x \in A \setminus B$ let $g(x)$ be the terminal point of the vector $(2\,dx - 1)X$.

Theorem 3.2. If h can be extended in one direction along a cylinder, then the closure of one complementary domain of S^{n-1} is a

closed n-cell.

More precisely, if E is the component of $S^n \setminus S^{n-1}$ which contains $f[(\mathrm{Bd}\ A) \times \{1\}]$, then $\mathrm{Cl}\ E$ is a closed n-cell.

Proof. Let E' be the complementary domain of $f[(\mathrm{Bd}\ A) \times \{1\}]$ which does not contain S^{n-1} . We first observe that $\mathrm{Cl}\ E'$ is a cellular subset of E . For, if E_i is the complementary domain of $f\left[(\mathrm{Bd}\ A) \times \left\{\frac{i}{i+1}\right\}\right]$ which contains E' , then, by Theorem 3.1, each $\mathrm{Cl}\ E_i$ is a closed n-cell. Furthermore $\mathrm{Cl}\ E_{i+1} \subset E_i$ and $\bigcap_{i=1}^{\infty} \mathrm{Cl}\ E_i = \mathrm{Cl}\ E'$.

Let g be a continuous mapping of $(\mathrm{Bd}\ A) \times [0, 1]$ onto an n-cell Q such that $(\mathrm{Bd}\ A) \times \{1\}$ is the only inverse set. Define a mapping k of $\mathrm{Cl}\ E$ onto Q by the equations

$$k(x) = gf^{-1}(x) , \quad x \in \mathrm{Cl}\ E \setminus \mathrm{Cl}\ E'$$

$$k(x) = g(\mathrm{Bd}\ A \times \{1\}), \quad x \in \mathrm{Cl}\ E .$$

The mapping k carries $\mathrm{Cl}\ E$ continuously onto the closed n-cell Q such that the only inverse set is the cellular subset $\mathrm{Cl}\ E'$ of E . Thus, by Theorem 2 of [8], $\mathrm{Cl}\ E$ is a closed n-cell.

The local connectedness property of an arc gives the following lemma.

Lemma 3.2. Suppose L is an arc in E^n and p is a point of L . Given $\varepsilon > 0$, there exists $\delta > 0$ such that, if L_1 is any subarc of L whose endpoints lie in $S_\delta(p)$, then $L_1 \subset S_\varepsilon(p)$.

Lemma 3.3. *Let* L *be an arc in* E^n, $n > 3$, *such that* L *is locally polyhedral except at a single point* p. *Then, given* $\varepsilon > 0$, *there exists a homeomorphism* h *of* E^n *onto* E^n *such that* h *is fixed outside* $S_\varepsilon(p)$ *and* $h(L)$ *is polyhedral.*

Proof. We will prove the lemma for p an interior point. Essentially the same proof may be applied in case p is an endpoint. Let a and b be the endpoints of L and V_1 the closed cubical neighborhood centered at p of diameter $\varepsilon_1 = \varepsilon$. For $i = 2, 3, 4, \ldots$ let δ_i be given by Lemma 3.2 for $\varepsilon = \varepsilon_{i-1}$, and let $\varepsilon_i = \min\left(\delta_i, \frac{\varepsilon_{i-1}}{2}\right)$. Let V_i be the closed cubical neighborhood of p of diameter ε_i.

By making use of semi-linear deformations in small neighborhoods of the Bd V_{2i}, if necessary, we may assume that $L \cap \mathrm{Vd}\, V_{2i}$ is a finite set of points, and that no pair of components of $L \setminus V_{2i}$ share a common endpoint. For each integer i let $u_{i1}, \ldots, u_{ie}$ be the closures of the components of $L \setminus V_{2i}$ which have both endpoints on Bd V_{2i}. Observe that each of these components is contained in the half open annulus Int $V_{2i-1} \setminus$ Int V_{2i}. Let w_{i1} be a polyhedral arc in Bd V_{2i} which connects the endpoints of u_{i1} and, except for these two points, is disjoint from L. The resulting simple closed curve $d_{i1} = u_{i1} \cup w_{i1}$ bounds a polyhedral 2-cell D_{i1} in Int V_{2i-1}, since $n > 3$ [10].

If $(D_{i1} \cap \mathrm{Bd}\, V_{2i}) \setminus w_{i1} \neq \square$, the components that are either points, arcs, or 2-dimensional subsets of D_{i1} may be eliminated by semi-linear deformations in small neighborhoods of these components. The components that are simple closed curves may be eliminated as follows. Let c be a component which contains no other such component in its interior

(relative to D_{i1}). Let Y be the subdisk of D_{i1} bounded by c, and let r be a point in the complementary domain of $\text{Bd } V_{2i}$ opposite to the one containing Y. Select r sufficiently close to $\text{Bd } V_{2i}$ for X (the join of r and c) to meet D_{i1} only in c. Define $D'_{i1} = (D_{i1} \setminus Y) \cup X$, and deform D'_{i1} semi-linearly away from $\text{Bd } V_{2i}$ in a sufficiently small neighborhood of c, so that no new intersections are introduced. The disk D''_{i1} thus obtained is bounded by d_{i1} and intersects $\text{Bd } V_{2i}$ in exactly those components, other than c, in which D_{i1} intersected $\text{Bd } V_{2i}$. After a finite number of steps we obtain a disk D^*_{i1} which, except for w_{i1}, is contained in the open annulus $\text{Int}(V_{2i-1} \setminus V_{2i})$. Since $\dim D^*_{i1} = 2$, $\dim L = 1$, and $n > 3$, we may assume that D^*_{i1} intersects L only in u_{i1}. Let $\eta > 0$ be such that the η-neighborhood S_{i1} of D^*_{i1} intersects $L \cap (V_{2i-1} \setminus V_{2i})$ only in u_{i1}, and such that S_{i1} is contained in $\text{Int}(V_{2i-1} \setminus V_{2i+1})$. By a sequence of simplicial moves across the 2-simplexes of D^*_{i1} the arc u_{i1} may be moved onto the arc w_{i1}. By making use of a corresponding semi-linear space homeomorphism, we may deform u_{i1} onto w_{i1} and then into $\text{Int } V_{2i}$ by a semi-linear homeomorphism which is the identity outside S_{i1} [Lemma 3, 19]. The components $u_{i2}, \ldots, u_{ie}$ are successively moved into $\text{Int}(V_{2i} - V_{2i+1})$ by a technique similar to that used on u_{i1}. We are careful in each move to leave the remaining components fixed. This is to keep from introducing new intersections with $\text{Bd } V_{2i}$. We denote the composition of these moves by f_i, and observe that f_i is a semi-linear space homeomorphism and is fixed outside $\text{Int}(V_{2i-1} \setminus V_{2i+1})$. Also, if a_i is the first point of $L \cap \text{Bd } V_{2i}$ relative to the order

of L from a to p, and b_i is the last point (equivalently b_i is the first point of $L \cap \operatorname{Bd} V_{2i}$, relative to the order of L from b to p), then $f_i(L) \cap \operatorname{Bd} V_{2i} = a_i \cup b_i$.

We define a mapping f of E^n onto E^n by the equations

$$f(x) = x, \qquad x \in E^n \setminus V_1,$$

$$f(x) = f_i(x), \; X \in V_{2i-1} \setminus V_{2i+1}, \quad i = 1, 2, \ldots,$$

$$f(p) = p.$$

It is clear, since, for each i, f_i is fixed on $\operatorname{Bd} V_{2i-1} \cup \operatorname{Bd} V_{2i+1}$ and f_i eliminates all but two points of intersection of L and $\operatorname{Bd} V_{2i}$, that f is a space homeomorphism, semilinear except at p, and that $f(L) \cap \operatorname{Bd} V_{2i} = a_i \cup b_i$.

We now consider the arc $f(L)$. Let L_{i1} be the subarc of $f(L)$ from a_i to a_{i+1} and let L_{i2} be the subarc of L from b_i to b_{i+1}. Let x_i be the point of intersection of the segment $\overline{a_1 p}$ with $\operatorname{Bd} V_{2i}$ and let y_i be the point of intersection of the $\overline{b_1 p}$ with $\operatorname{Bd} V_{2i}$. Let ϕ_i be a semi-linear space homeomorphism which is fixed outside $V_{2i-1} \setminus V_{2i+1}$ and which carries $\operatorname{Bd} V_{2i}$ onto $\operatorname{Bd} V_{2i}$, with $\phi_i(a_i) = x_i$ and $\phi_i(b_i) = y_i$. Since $a_1 = x_1$ and $b_1 = y_1$, we will assume that ϕ_1 is the identity homeomorphism. We may assume that the arcs $\overline{x_i x_{i+1}}$ and $\phi_{i+1}\phi_i(L_{i1})$ meet only in their endpoints, that $\overline{y_i y_{i+1}}$ and $\phi_{i+1}\phi_i(L_{i2})$ meet only in their endpoints, that $\overline{x_i x_{i+1}}$ does not meet $\phi_{i+1}\phi_i(L_{i1})$, and that $\overline{y_i y_{i+1}}$ does not meet $\phi_{i+1}\phi_i(L_{i1})$. The simple closed curve $\phi_{i+1}\phi_i(L_{i1}) \cup \overline{X_i X_{i+1}}$ bounds a polyhedral disk D_{i1}, which, because of the restrictions on dimensions, may be taken to be

disjoint from $\phi_{i+1}\phi_i(L_{i2}) \cup \overline{y_i y_{i+1}}$. Furthermore, in the light of the elimination of component scheme used above, D_{i1} may be selected so that $D_{i1} \cap (\mathrm{Bd}\ V_{2i} \cup \mathrm{Bd}\ V_{2i+2}) = x_i \cup x_{i+1}$. The arc $\phi_{i+1}\phi_i(L_{i1})$ is then moved across the disk D_{i1} onto the arc $\overline{x_i x_{i+1}}$ by a space homeomorphism Ψ_{i1} , which is the identity outside $V_{2i} \setminus V_{2i+2}$ and on $\phi_{i+1}\phi_i(L_{i2})$. Similarly $\phi_{i+1}\phi_i(L_{i2})$ is moved onto $\overline{y_i y_{i+1}}$ by a space homeomorphism Ψ_{i2}, which is fixed outside $V_{2i} \setminus V_{2i+2}$ and on $\overline{x_i x_{i+1}}$. The composition $\Psi_{i2}\Psi_{i1}$ is denoted by Ψ_i .

A mapping g is defined by the equations

$$g(x) = x , \qquad x \in E^n \setminus V_2$$
$$g(x) = \Psi_i \phi_{i+1}\phi_i(x) , \quad x \in V_{2i} - V_{2i+2} , \quad i = 1, 2, \ldots,$$
$$g(p) = p .$$

Since $\Psi_i \phi_{i+1}\phi_i$ and $\Psi_{i+1}\phi_{i+2}\phi_{i+1}$ agree on the common part of their domains of definition, Bd V_{2i+2} (each reduces to ϕ_{i+1} on this set), it is clear that g is a space homeomorphism. Also g carries $f(L)$ onto the sum of four polyhedral arcs: (1) the subarc of $f(L)$ from a to $a_1 = x_1$, (2) $\overline{x_1 p}$, (3) $\overline{p y_1}$, and (4) the subarc of $f(L)$ from $b_1 = y_1$ to b . The desired space homeomorphism h is taken to be the composition gf . Since each of f and g is fixed outside V_1 , all the requirements of the lemma are met.

A technique similar to that used in the proofs of Lemma 2.3 and Lemma 3.1 may be used to prove the following lemmas.

<u>Lemma</u> 3.4. <u>There is a continuous mapping</u> g <u>of</u> D <u>onto</u> D <u>such that</u>

(1) g <u>is fixed on</u> Bd D ,

(2) g <u>is a homeomorphism of</u> $D \setminus A$ <u>onto</u> $D \setminus (0,0, \ldots,0,1)$ <u>and</u>

(3) $g(A) = (0, 0, \ldots, 0, 1)$.

Lemma 3.5. *Let* L' *be the segment of the* x_n*-axis from* $(0, 0, \ldots, 0, \frac{1}{2})$ *to* $(0, 0, \ldots, 0, 1)$. *Then, there is a continuous mapping of* $\mathrm{Cl}\,(D \setminus B)$ *onto* $\mathrm{Cl}(D \setminus A)$, *such that* (1) g *is fixed on* $\mathrm{Bd}\, D$, (2) $g(\mathrm{Bd}\, B) = \mathrm{Bd}\, A$, *and* (3) L' *is the only inverse set under* g .

Definition 3.5. We say that h can be *extended in one direction along a cylinder and in the opposite direction along a cylinder truncated at* $(0, 0, \ldots, 0, 1)$ if there exists a homeomorphism f carrying the closed annulus determined by B and D into S^n , such that f agrees with h on $\mathrm{Bd}\, A$.

Theorem 3.3. *Suppose* h *can be extended in one direction along a cylinder and in the opposite direction along a cylinder truncated at* $(0, 0, \ldots, 0, 1)$. *Let* G *be the component of* $S^n \setminus S^{n-1}$ *which intersects* $f(\mathrm{Bd}\, D)$. *Then* G *is an open* n-*cell*.

Proof. Let J be the closure of the component of $S^n \setminus S^{n-1}$ which contains $f(\mathrm{Bd}\, B)$. By Theorem 3.2, J is a closed n-cell and hence there is an extension Ψ of h , which carries A homeomorphically onto J . Define a homeomorphism ϕ of D into S^n by the equations

$$\phi(x) = f(x) , \quad x \in D \setminus A$$
$$\phi(x) = \Psi(x) , \quad x \in A .$$

Let $\phi(0, 0, \ldots, 0, 1) = p$, and use the mapping ϕ and the mapping g of Lemma 3.4 to define a mapping k of S^n onto S^n as follows ,

$$k(x) = x \qquad , \quad x \in S^n \setminus \phi(D) \, ,$$
$$k(x) = \phi g \phi^{-1}(x) \, , \quad x \in \phi(D) \, .$$

The mapping k carries S^n onto S^n, leaves p fixed, and has J as the only inverse set. Hence G is carried homeomorphically onto $S^n \setminus p$ and is an open n-cell.

Let B_1 be the closed n-cell in E^n, which is centered at the origin and has radius three-fourths. Let L_1' be the segment of the x_n-axis from $(0, 0, \ldots, 0, 3/4)$ to $(0, 0, \ldots, 0, 1)$, and $L_1 = f(L_1')$. Let h, G, and p be as in Theorem 3.3, and let g be given by Lemma 3.4, with B and L' replaced by B_1 and L_1' respectively.

Theorem 3.4. *If H is the closure of the component of $S^n \setminus f(\mathrm{Bd}\, B_1)$ which contains G, then H is a closed n-cell, and $(\mathrm{Cl}\, G) \setminus p$ is topologically equivalent to $H \setminus L_1$.*

Proof. That H is a closed n-cell follows immediately from Theorem 3.2.

Let I be the component of $S^n \setminus f(\mathrm{Bd}\, D)$ which does not intersect S^{n-1}. The mapping k of H onto $\mathrm{Cl}\, G$ defined by

$$k(x) = x \qquad , \quad x \in I \, ,$$
$$k(x) = fgf^{-1}(x), \quad x \in H \setminus I \, ,$$

is a continuous mapping of H onto $\mathrm{Cl}\, G$ such that the only inverse set is L_1 and $k(L_1) = p$. Hence, k is a homeomorphism of $H \setminus L_1$ onto $(\mathrm{Cl}\, G) \setminus p$.

In case there exists a continuous mapping ℓ of H onto H such that L_1 is the only inverse set under ℓ, then we can state that $\mathrm{Cl}\ G$ is a closed n-cell. In fact, the product mapping ℓk^{-1} is a homeomorphism of $\mathrm{Cl}\ G$ onto H.

Let us now suppose that the extension f of h is semi-linear on each finite polyhedron of $\mathrm{Int}(A \setminus B)$. Then $f(\mathrm{Bd}\ B_1)$ is a polyhedron and L_1 is locally polyhedral except at p. Let $\varepsilon > 0$ be such that $S\,\varepsilon\,(p) \subset \mathrm{Int}\ H$ and let ϕ be a homeomorphism of S^n onto S^n such that ϕ is fixed outside $S\,\varepsilon\,(p)$ and $\phi(L_1)$ is polyhedral. Let q be the endpoint of L_1 which lies on $\mathrm{Bd}\ H$ and let Q be a polyhedral n-cell in H, such that $q \in \mathrm{Bd}\ Q$, $\phi(L_1)\setminus q \subset \mathrm{Int}\ Q$, and Q has a subdivision isomorphic to a subdivision of a simplex (see Lemma 5.32 of [10]). Let Ψ be a simi-linear homeomorphism of Q onto a simplex R. The arc $\Psi\phi(L_1)$ is then polyhedral in R and, together with the linear segment $\overline{\Psi\phi(q)\Psi\phi(p)}$ bounds a polyhedral disk D in R which, except for $\Psi\phi(q)$, lies in the interior of R. There is then a homeomorphism η of R onto R such that η is fixed on $\mathrm{Bd}\ R$ and carries $\Psi\phi(L_1)$ onto the segment $\overline{\Psi\phi(q)\Psi\phi(p)}$. It is then easy to find a continuous mapping Θ of R onto R such that Θ is fixed on $\mathrm{Bd}\ R$, $\Theta(\overline{\Psi\phi(q)\Psi\phi(p)})$ $= \Psi\phi(q)$, and $\overline{\Psi\phi(q)\Psi\phi(p)}$ is the only inverse set. The mapping ℓ, defined by $\ell(x) = \Psi^{-1}\Theta\eta\Psi\phi(x)$, $x \in Q$, and $\ell(x) = x$, $x \in H\setminus Q$, is a continuous mapping of H onto H such that L_1 is the only inverse set. Thus we have the following theorem.

<u>Theorem 3.5.</u> <u>Let</u> h <u>be</u> <u>a</u> <u>homeomorphism</u> <u>embedding</u> $\mathrm{Bd}\ A$ <u>in</u> S^n, $n > 3$. <u>If</u> h <u>can be extended in one direction along a cylinder and in</u>

the opposite direction along a cylinder truncated at $(0, 0, \ldots, 0, 1)$, such that the extension is locally semi-linear on $\operatorname{Int} A \setminus B$, then the closure of either complementary domain of $h(\operatorname{Bd} A)$ is a closed n-cell.

Definition 3.6. We say that h can be extended in one direction along a cylinder truncated at $(0, 0, \ldots, 0, 1)$, if there exists a homeomorphism f carrying the closed pinched annulus determined by D and A into S^n, such that f agrees with h on $\operatorname{Bd} A$.

Definition 3.7. Let f be the extension homeomorphism of Definition 3.6. If there exists a neighborhood N of $(0, 0, \ldots, 0, 1)$ in E^n such that f is semi-linear on each finite polyhedron of $\operatorname{Int}(D \setminus A) \cap N$, then we say that f is semi-linear on a deleted neighborhood of $(0, \ldots, 1)$.

Theorem 3.6. Let h be a homeomorphism embedding $\operatorname{Bd} A$ in S^n, $n > 3$, such that h can be extended in one direction along a cylinder truncated at $(0, 0, \ldots, 0, 1)$, and let G be the component of $S^n \setminus f(\operatorname{Bd} A)$ which intersects $f(\operatorname{Bd} D)$. If f is semi-linear on a deleted neighborhood of $(0, 0, \ldots, 0, 1)$, then $\operatorname{Cl} G$ is a closed n-cell.

Proof. Let D_1 be a cell, obtained from D by a slight contraction on E^n toward $(0, 0, \ldots, 0, 1)$, such that $[\operatorname{Bd} D_1 \setminus (0,0,\ldots,0,1)]$ is contained in $D \setminus A$. Let G_1 and G_2 respectively be the components of $S^n \setminus f(\operatorname{Bd} D_1)$ and $S^n \setminus f(\operatorname{Bd} D)$, which are contained in G. We now observe that $\operatorname{Cl} G_1$ is homeomorphic to $\operatorname{Cl} G$. For, if g is a space homeomorphism which is fixed on $\operatorname{Bd} D$ and carries $\operatorname{Bd} D_1$ onto $\operatorname{Bd} A$, then the mapping ϕ defined by

$$\phi(x) = x, \qquad x \in G_2$$
$$\phi(x) = fgf^{-1}(x), \quad x \in \operatorname{Cl}(G_1 \setminus G_2),$$

carries $\mathrm{Cl}\ G_1$ homeomorphically onto $\mathrm{Cl}\ G$. This suggests the following observation: if one attaches a copy of $\mathrm{Cl}\ G_1$ to $\mathrm{Cl}\ (D_1 \setminus A)$ along $\mathrm{Bd}\ D_1$ with f^{-1}, the set thus obtained is equivalent to $\mathrm{Cl}\ G_1$ (it is simply $\mathrm{Cl}\ G$). This will be used to show that $\mathrm{Cl}\ G_1$ is a closed n-cell, and hence that $\mathrm{Cl}\ G$ is a closed n-cell.

Let N be a neighborhood of $(0, 0, \ldots, 0, 1)$ such that f is semi-linear on $\mathrm{Int}(D \setminus A) \cap N$. Let S_1, S_2, and S_3 be three n-simplexes in $\mathrm{Cl}\ (D_1 \setminus A) \cap N$, such that S_1 has $(0, 0, \ldots, 0, 1)$ as one vertex, $S_1 \setminus (0, 0, \ldots, 0, 1) \subset \mathrm{Int}(D_1 \setminus A)$, $\mathrm{Bd}\ S_1 \cap \mathrm{Bd}\ S_2 = (0, 0, \ldots, 0, 1)$, $S_2 \setminus (0, 0, \ldots, 0, 1) \subset \mathrm{Int}\ S_1$, and $S_3 \subset \mathrm{Int}\ S_2$. Let k be the component of $S^n \setminus f(\mathrm{Bd}\ S_2)$ which contains G_1. Then by Theorem 3.5, $\mathrm{Cl}\ k$ is a closed n-cell. Let $H = S^n \setminus \mathrm{Cl}\ G$, then $\mathrm{Cl}\ k$ can be realized by taking $P = \mathrm{Cl}\ (D_1 \setminus A) \setminus \mathrm{Int}\ S_2$ and attaching $\mathrm{Cl}\ H$ to P along $\mathrm{Bd}\ A$ with f^{-1}, and attaching $\mathrm{Cl}\ G_1$ to P along $\mathrm{Bd}\ D_1$ with f^{-1}. The set P is a closed n-cell (the closure of the exterior of S_2) with the interiors of two cells sharing a common boundary point with $\mathrm{Bd}\ S_2$, removed. The cell obtained from P by attaching $\mathrm{Cl}\ G_1$ and $\mathrm{Cl}\ H$ to the interior boundary spheres of P with f^{-1} will be denoted by $\overline{P}$.

Let E be the part of the solid unit ball in E^n centered at $(0, 0, \ldots, 1, 0)$, determined by $x_n \geq 0$. Let $\{q_i\}_0^\infty$ be a sequence of points $(x_1 = x_2 = \ldots = x_{n-2} = 0) \cap \mathrm{Bd}\ E$ such that, if q_i is represented by $(0, 0, \ldots, a_{(n-1)i}, a_{ni})$, then $a_{(n-1)0} = 2$ and the $a_{(n-1)i}$ converge monotonically to zero through positive values, and $a_{ni} > 0$, $i = 0$.

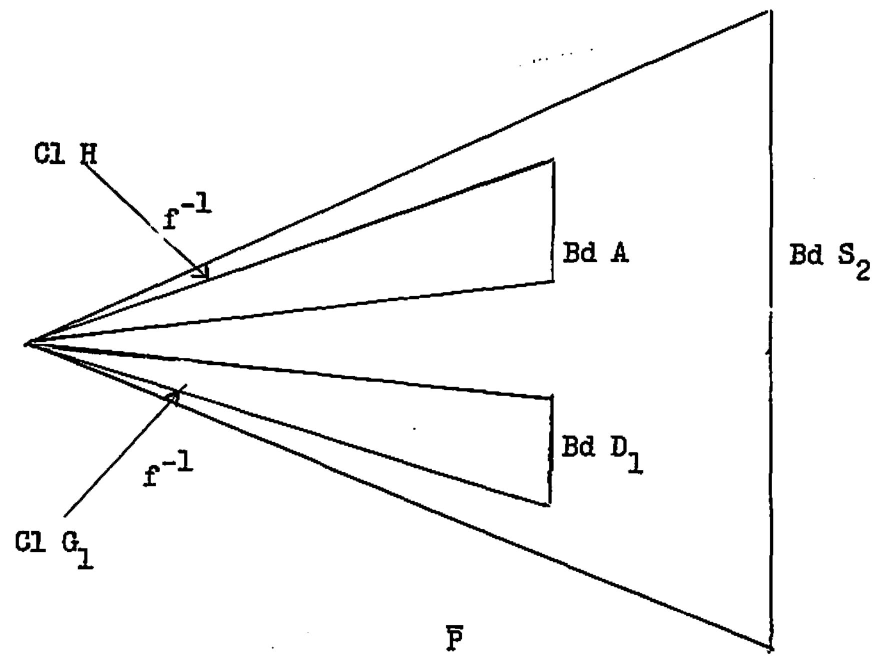

Figure 1

A Chambered n-cell.

We then section E into a countable number of n-cells by projecting the $(n - 1)$-plane $x_n = x_{n-1} = 0$ onto each of the p_i. The section determined by p_{i-1} and p_i is denoted by C_i. We then delete from C_i the interior of a cell C_i', similar in shape to C_i and, except for the boundary point $(0, 0, \ldots, 0, 0)$, contained in the interior of C_i. Any two adjacent sections then form a copy of P, and are labeled by P_i, P_i', as in Figure 2. Notice that P_i and P_i' have $w_{2i} = \mathrm{Bd}\, C'_{2i}$ in common, and P_i' and P_{i+1} have $w_{2i+1} = \mathrm{Bd}\, C'_{2i+1}$ in common. Let ϕ_i be a homeomorphism of P_i onto P_i' which leaves w_{2i} fixed and carries w_{2i-1} onto w_{2i+1}. Let Ψ_i be a homeomorphism of P_i' onto P_{i+1} which leaves w_{2i+1} fixed and carries w_{2i} onto w_{2i+2}.

We identify P_1 with P, with w_1 identified with $\mathrm{Bd}\, D_1$ and w_2 identified with $\mathrm{Bd}\, A$. The sets $\mathrm{Cl}\, G_1$ and $\mathrm{Cl}\, H$ are then sewn to P along w_1 and w_2, respectively, with f^{-1}. The resulting n-cell is denoted by $\overline{P_1}$. The sets $\mathrm{Cl}\, G_1$ and $\mathrm{Cl}\, H$ are then sewn into alternate holes bounded by w_{2i+1} and w_{2i+2} by the attaching homeomorphisms

$$\phi_i \cdots \phi_2\phi_1 f^{-1} : \mathrm{Bd}\, G_1 \longrightarrow w_{2i+1},$$

$$\Psi_i \cdots \Psi_2\Psi_1 f^{-1} : \mathrm{Bd}\, H \longrightarrow w_{2i+2}.$$

The sets thus obtained from the P_i and P_i' are denoted by $\overline{P_i}$ and $\overline{P_i}'$, and the union of the $\overline{P_i}$ is denoted by E_1.

Since ϕ_1 is the identity on w_2, we can extend ϕ_1 to a homeomorphism of $\overline{P_1}$ onto $\overline{P_1}'$, and conclude that $\overline{P_1}'$ is also a closed n-cell. In a similar manner we extend the homeomorphism Ψ_i to a homeo-

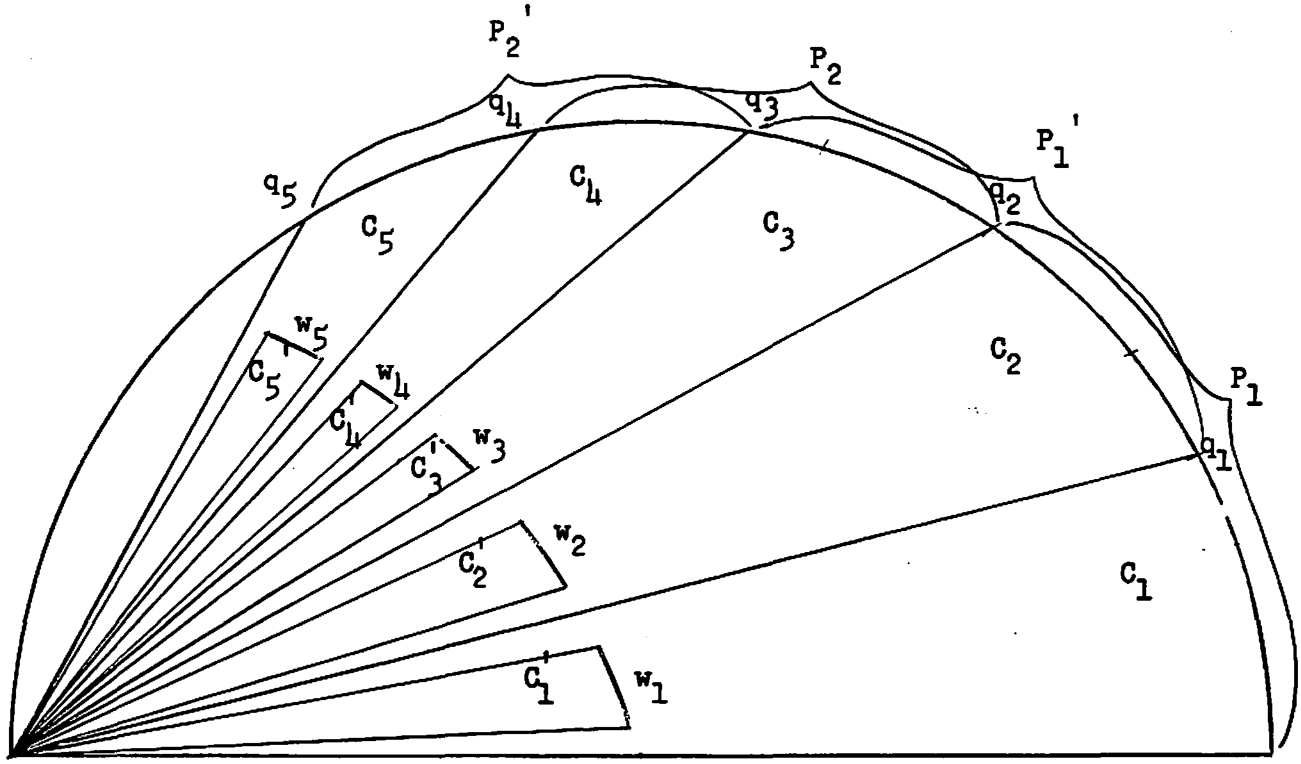

Figure 2

A Countable Partition of an n-cell

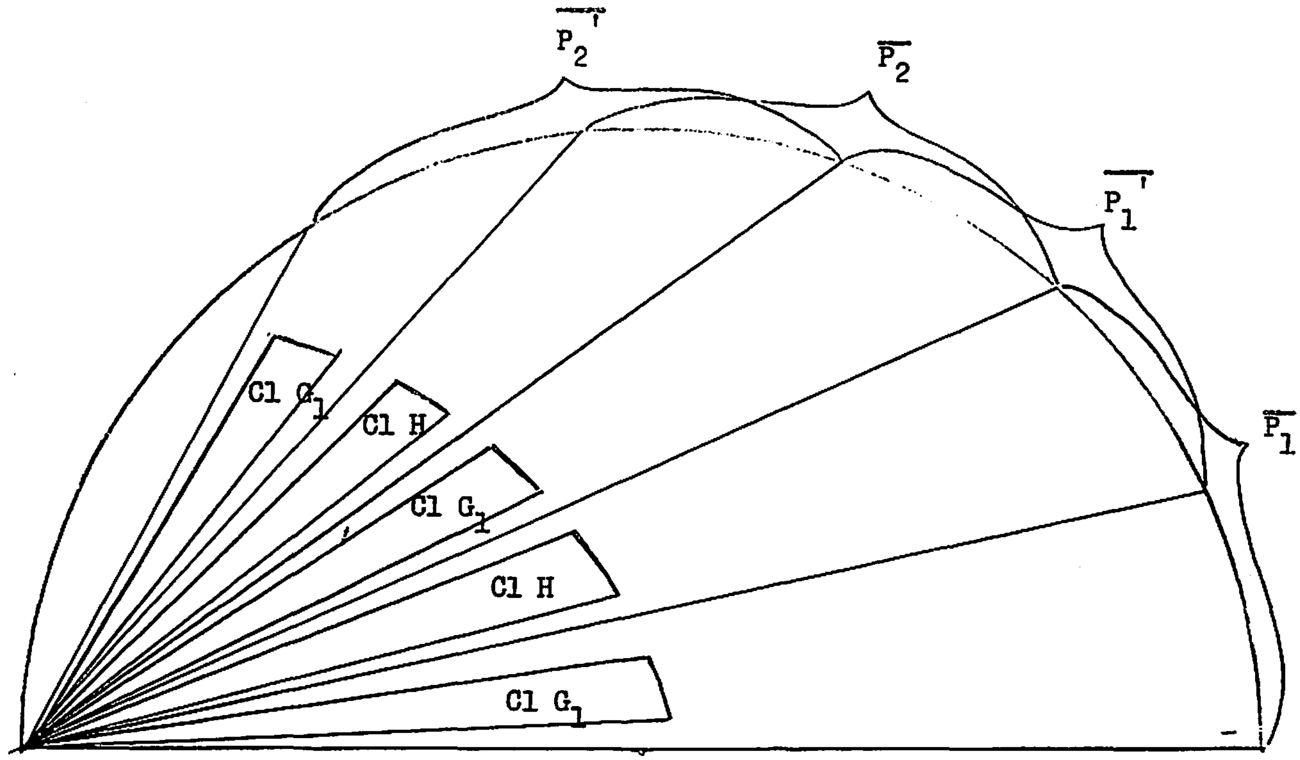

Figure 3

A Modified n-cell.

morphism of $\overline{P_i}'$ onto $\overline{P_{i+1}}$ and extend the homeomorphism ϕ_i to a homeomorphism of $\overline{P_i}$ onto $\overline{P}'_{i+1}$. It then follows that each of the $\overline{P_i}$ and $\overline{P_i}'$ is a closed n-cell.

We now observe that E_1 is a closed n-cell. This is established by constructing a homeomorphism of E onto E_1. We map the boundary of $C_{2i-1} \cup C_{2i}$ onto the boundary of $\overline{P_i}$ with the identity homeomorphism. Since $C_{2i-1} \cup C_{2i}$ and $\overline{P_i}$ are n-cells, this homeomorphism between their boundaries can be extended to a homeomorphism between the cells. These extensions for $i = 1, 2, \ldots$ yield a homeomorphism from E onto E_1.

We next observe that E_1 is a copy of $\mathrm{Cl}(D_1' \setminus A')$ with $\mathrm{Cl}\, G_1$ sewn along one of the boundary spheres. This can be established by showing that E_1, with G_1 removed from $\overline{P_1}$, is homeomorphic to E with $\mathrm{Int}\, C_1'$ removed. Let λ be the identity mapping on $C_1 \setminus \mathrm{Int}\, C_1'$ and on $\mathrm{Bd}(C_{2i} \cup C_{2i+1})$, $i = 1, 2, \ldots$. Since $C_{2i} \cup C_{2i+1}$ and $\overline{P_i}'$ are closed n-cells and λ restricts to a homeomorphism between their boundaries, λ can be extended over their interiors. The extensions over each of the $C_{2i} \cup C_{2i+1}$ yield the desired homeomorphism.

We have seen that E_1 may first be viewed as a closed n-cell and secondly as $\mathrm{Cl}\, G_1$ sewn into a boundary sphere of a copy of $\mathrm{Cl}\,(D_1 \setminus A)$. We previously observed that a set of the second type is homeomorphic to $\mathrm{Cl}\, G_1$. Hence $\mathrm{Cl}\, G_1$, or equivalently $\mathrm{Cl}\, G$, is a closed n-cell.

CHAPTER IV

SOME 3-SPHERES IN S^4

4.1. Three-Spheres in S^4 Obtained by Suspension

Definition 4.1. In E^4 we take coordinates x_1, x_2, x_3, x_4 and let E^3 be described by $x_4 = 0$. Let $a = (0, 0, 0, 1)$ and $b = (0, 0, 0, -1)$. For a set A in E^3 the suspension of A in E^4 is the join of A and $a \cup b$ (the collection of line segments $\overline{ax}$ and $\overline{bx}$, $x \varepsilon A$). The abbreviation Susp A will be used for the suspension of A in E^4.

If $A = \{(x_1, x_2, x_3, 0) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, then it is clear that Susp A is a 3-sphere in E^4 and that Susp(Int A) = Int(Susp A) is an open 4-cell. Furthermore, the suspension of the union of A and its interior is a closed 4-cell.

Lemma 4.1. *If A_1 and A_2 are homeomorphic subsets of E^3, then Susp A_1 and Susp A_2 are homeomorphic subsets of E^4.*

Proof. Let g be a homeomorphism of A_1 onto A_2. For $O_{i\alpha}$ an open set in A_i and $-1 \leq t_1 < t_2 \leq 1$ let $O_{i\alpha}(t_1, t_2)$ be the part of Susp $O_{i\alpha}$ which lies between the 3-planes $x_4 = t_1$ and $x_4 = t_2$. If either $t_1 = -1$ or $t_2 = 1$, then we will add to $O_{i\alpha}(t_1,t_2)$ the point b or a as $t_1 = -1$ or $t_2 = 1$. The collection of sets

$$\left\{O_{i\alpha}(t_1,t_2) \mid O_{i\alpha} \text{ open in } A_i,\ -1 \leq t_1 < t_2 \leq 1\right\}$$

forms a basis for the topology of A_i.

Let z_1 be a point of Susp A_1. Then there exists an $x_1 \varepsilon A_1$ and a $-1 \leq t \leq 1$ such that z_1 is the intersection of $x_4 = t$ and a segment $\overline{x_1 a}$ or $\overline{x_1 b}$, according as t is positive or non-positive. In the first case we associate with z_1 the intersection of $x_4 = t$ and $\overline{g(x_1)a}$. In the latter case we associate with z_1 the intersection of $x_4 = t$ and $\overline{g(x_1)b}$. The mapping thus defined carries Susp A_1 onto Susp A_2 in a one-to-one manner and carries the basis elements of A_1 onto the basis elements of A_2 in a one-to-one manner.

Let L be the x_4 axis and let M denote the part of L with $|x_4| \geq 1$.

Lemma 4.2. *Let S be a 2-sphere in E^3 and K = Susp S. For each $\varepsilon > 0$ there exists a set T_ε in the ε-neighborhood of $K \cup M$ such that T_ε is homeomorphic with $S \times E^1$ and there exists a homotopic deformation of $E^4 \setminus T_\varepsilon$ onto $E^3 \setminus S$.*

Proof. Let $0 < t_1 < 1$ and sufficiently close to 1 for the set $P(a) = \{(x_1, x_2, x_3, x_4) \varepsilon K \mid x_4 \geq t_1\}$ to be in the ε-neighborhood of a. Let $-1 < t_2 < 0$ and sufficiently close to -1 for the set $P(b) = \{(x_1, x_2, x_3, x_4) \varepsilon K \mid x_4 \leq t_2\}$ to be in the ε-neighborhood of b. Let $Q(a)$ be those points of $P(a)$ with x_4 coordinate t_1, and $Q(b)$ those points of $P(b)$ with x_4 coordinate t_2. Let $R(a)$ be the union of all half-lines which are directed in the positive x_4 direction and have their endpoint in $Q(a)$, and let $R(b)$ be the union of all half-lines which are directed in the negative x_4 direction and have their endpoint in $Q(b)$. The set T_ε is then defined to be

$$\{K \setminus [P(a) \cup P(b)]\} \cup [R(a) \cup R(b)] .$$

From the definition of T_ε it is easy to see that there is a homeomorphism f of E^4 onto E^4 which is the identity on E^3 and carries T onto $S \times E^1$. For $0 \leq t \leq 1$ let $\bar{t}$ be the transformation which carries (x_1, x_2, x_3, x_4) onto (x_1, x_2, x_3, tx_4). The desired deformation G is then defined by $G(x, t) = f^{-1}\bar{t}f(x)$.

Definition 4.2. Let A and B be two arcwise connected spaces with $A \subset B$. Let $p \varepsilon A$ be used as the base point for computing the fundamental groups $\pi_1(A)$ and $\pi_1(B)$. The injection homomorphism of $\pi_1(A)$ into $\pi_1(B)$ is the homomorphism induced by the identity mapping of A into B.

Theorem 4.1. *Let S be a 2-sphere in E^3 and $K = \text{Susp } S$. Let A_1 and A_2 be the bounded and unbounded components of $E^3 \setminus S$ respectively, and B_1, B_2 the corresponding components of $E^4 \setminus K$. Then the injection homomorphism $i_j : \pi_1(A_j) \longrightarrow \pi_1(B_j)$, $j = 1, 2$, is an onto isomorphism.*

Proof. First consider the sets A_1 and B_1. Let W be an element of $\pi_1(B_1)$ and let w be a representative of W. Let w' be the path in A_1 which is the image of w under the deformation G of Lemma 4.2. Then w' is also a representative of W. If W' is the element of $\pi_1(A_1)$ represented by w', then $i_1(W') = W$, by the definition of i_1, and i_1 is an onto homomorphism.

Let W' be an element of $\pi_1(A_1)$ such that $i_1(W')$ is the identity element E of $\pi_1(B_1)$, and let w' be a representative of W'. Then w' bounds a singular 2-cell D in B_1. Let D' be the image of D under the deformation G. Since w' is fixed under G,

w' bounds the singular disk D' in A_1. Hence, W' is the identity element E' of $\pi_1(A_1)$ and the kernel of i_1 is E'.

Now consider A_2 and B_2. Let W be an element of $\pi_1(B_2)$, and let W be represented by a polygonal path w in B_2. Since w and M are 1-dimensional subsets of the 4-dimensional set B_2 we may, by deforming w away from M if necessary assume that $\rho(w, M) > 0$. By selecting $\varepsilon < \rho(w, M)$ and selecting T_ε and G by Lemma 4.2, we can deform w by G into A_2 and thus obtain a path w' representing an element W' such that $i_2(W') = W$.

Let W' be an element of $\pi_1(A_2)$ such that $i_2(W') = E$, and let W' be represented by a polygonal path w' in A_2. Then w' bounds a singular 2-cell D in B_2. By the Deformative Theorem [18, p. 115], we may assume that D is a simplicial 2-complex. Again, since the dimensions of D and M add up to three, we may assume that $\rho(M, D) = \varepsilon > 0$. Then, by Lemma 4.2, we can find a G which deforms D into A_2 and leaves w' fixed. Thus w' represents the identity element of $\pi_1(A_2)$, and i_2 is an isomorphism.

If E^4 is compactified with a point at infinity, then E^4 becomes S^4 and E^3 becomes S^3, and the corresponding proofs for Lemma 4.2 and Theorem 4.1 can be carried out with S^4 and S^3 replacing E^4 and E^3.

<u>Theorem</u> 4.2. <u>Let</u> A_1, A_2, B_1, B_2 <u>denote the components of</u> $S^3 \setminus S$ <u>and</u> $S^4 \setminus K$ <u>as indicated in Theorem</u> 4.1. <u>Then the second homotopy groups</u> $\pi_2(B_1)$ <u>and</u> $\pi_2(B_2)$ <u>are trivial</u>.

Proof. It is proved in [16, p. 19] that each of $\pi_2(A_1)$ and $\pi_2(A_2)$ is trivial. The proof then will be to show that each singular 2-sphere in B_1 or B_2 can be deformed into A_1 or A_2 respectively without crossing K.

Let D be a singular 2-sphere in B_1. Then by Lemma 4.2, there exists a deformation G which deforms D into A_1. The situation is quite similar for B_2. Let D be a singular 2-sphere in B_2. Again by the Deformation Theorem, we may assume that D is a simplicial 2-complex in B_2 and, since the dimensions of D and $\bar{K}$ ($K \cup$ the point at infinity) add up to three, we may assume that D and $\bar{K}$ do not intersect. Let $\varepsilon = \rho(D, \bar{K})$ and let G be given by Lemma 4.2. The deformation G deforms D continuously into A_2 and the theorem is proved.

In [5] there are examples of 2-spheres in S^3 such that one complementary domain has a non-trivial fundamental group. An elementary modification of these examples will give 2-spheres in S^3 such that the fundamental group of either complementary domain is non-trivial. These examples plus Theorem 4.1 give the existence of 3-spheres in S^4 such that either one or both complementary domains have non-trivial fundamental groups. However, Theorem 4.2 tells us that both complementary domains of these examples will have trivial second homotopy groups.

4.2. Three-Spheres in S^4 Obtained by Rotation

Definition 4.3. Let $E^3_+ = \left\{(x_1, x_2, x_3, 0) \varepsilon E^4 \mid x_3 \geq 0\right\}$ and let P be the plane $x_3 = x_4 = 0$. Let M be a subset of E^3_+ and

define $R(M)$ as follows: $R(M) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \in E^4 \mid \bar{x}_1 = x_1,$ $\bar{x}_2 = x_2, \bar{x}_3 = x_3 \cos t, \bar{x}_4 = x_3 \sin t$ for some $(x_1, x_2, x_3, 0) \in M$ and $0 \leq t < 2\pi\}$.

The following theorem is an immediate consequence of Definition 4.3.

Theorem 4.3. *Let M be the hemisphere in E^3_+ defined by the equation $x_1{}^2 + x_2{}^2 + x_3{}^2 = 1$. Then $R(M)$ is the 3-sphere $x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 = 1$ in E^4.*

Furthermore, if D is the bounded complementary domain of $d = \{(x_1, x_2) \in P \mid x_1{}^2 + x_2{}^2 = 1\}$ in P, A_1 the bounded complementary domain of $M \cup D$ in E^3_+, and A_2 the unbounded complementary domain of $M \cup D$ in E^3_+, then $R(A_1 \cup D)$ and $R(A_2)$ are respectively the bounded and unbounded complementary domains of $R(M)$ in E^4.

A proof similar to that of Lemma 4.1 can be used to establish the following lemma.

Lemma 4.3. *Suppose A_1 and A_2 are homeomorphic subsets of E^3_+ with f a homeomorphism of A_1 onto A_2 and h the restriction of f to $A_1 \cap P$. If h is a homeomorphism of $A_1 \cap P$ onto $A_2 \cap P$, then $R(A_1)$ and $R(A_2)$ are homeomorphic subsets of E^4.*

Let M, D, d, A_1, and A_2 be as in Theorem 4.3, and let M' be a 2-cell in E^3_+ such that $M' \cap P = \mathrm{Bd}\, M' = d'$. Let D' be the bounded complementary domain of $P \setminus d'$ and $A_1{}'$, $A_2{}'$ the bounded and unbounded complementary domains of $M' \cup D'$ in E^3_+ respectively. A combination of Lemma 4.3 and Theorem 4.3 yields that $R(M')$ is a

3-sphere in E^4. Denote the bounded component of $E^4 \setminus R(M')$ by B_1 and the unbounded component by B_2.

<u>Theorem</u> 4.4. $\pi_1(A_i') \approx \pi_1(B_i)$, $i = 1, 2$.

<u>Proof</u>. First consider A_1' and B_1 and select a point p in A_1' as the base point for computing $\pi_1(A_1')$ and $\pi_1(B_1)$. Let L be an element of $\pi_1(B_1)$, and let ℓ be a polygonal representative of L. Let E^4_+ be the collection of points in E^4 with positive fourth coordinates, and let E^4_- be those points with negative fourth coordinates. We will say that a is an exceptional point of ℓ if $a \in \ell \cap A_1$ and each interval on ℓ about a contains points of E^4_-. Let a be an exceptional point of ℓ, and let q traverse ℓ in the direction determined by the requirement that q approach a through points in E^4_-. The exceptional point a of ℓ will then be classified according as

(1) q passes from a immediately back into E^4_-,

(2) q moves from a along a polygonal curve in A_1' to another exceptional point and then into E^4_-,

(3) q moves from a along a polygonal curve u_a in A_1' to a vertex b and then into E^4_+, or

(4) q passes from a immediately into E^4_+.

In cases (1) and (2) a may be eliminated as an exceptional point by decreasing fourth coordinates slightly in a neighborhood of a. An exceptional point of type (3) may be reclassified as type (4) by rotating u_a about a so that $u_a \setminus a \subset E^4_+$. We then may assume that the exceptional points, $a_1, a_2, \ldots, a_n$, of ℓ are all of type (4).

For each exceptional point a_i of ℓ let b_i be a vertex in D and u_i a directed polygonal arc in A_1' from a_i to b_i. We then take as our representative of L the curve m obtained from ℓ by inserting at each a_i the arc $u_i u_i^{-1}$.

For each $x \varepsilon E^4$ let $y_x = (x_1, x_2, x_3, 0)$ and t_x be the unique point in E_+^3 and real number $0 \leq t_x < 2\pi$ respectively, such that $x = (x_1, x_2, x_3 \cos t_x, x_3 \sin t_x)$. We will say that x is obtained by rotating y_x about P through an angle t_x and write $x = R_{t_x}(y_x)$. The continuous mapping $x \longrightarrow y_x$ of E^4 onto E_+^3 will be denoted by R^{-1}.

We now return to the curve m and define a homotopic deformation carrying m into E_+^3. For $x \varepsilon m \setminus (\bigcup_{i=1}^{n} u_i^{-1})$ and $0 \leq t \leq 2\pi$ let $R_{m_t}^{-1}(x) = R(t_x - t)(y_x)$ if $0 \leq t < t_x$, and $R_{m_t}^{-1}(x) = y_x$ if $t_x \leq t \leq 2\pi$. For $x \varepsilon \bigcup_{n=1}^{n} u_i^{-1}$ let $R_{m_t}^{-1}(x) = R_{(2\pi - t)}(x)$. Observe that for each m, $R_{m_{2\pi}}^{-1}$ is the restriction of R^{-1} to m, and hence $m \sim R^{-1}(m)$ in B_1.

Let h be the homomorphism of $\pi_1(B_1)$ onto $\pi_1(A_1')$ defined by associating the element L of $\pi_1(B_1)$ with the homotopy class of $\pi_1(A_1)$ determined by $R^{-1}(m)$. We need to establish that h is well defined (if $\ell \sim \ell'$ in B_1, then $R^{-1}(m) \sim R^{-1}(m')$ in A_1'), and that h is a homomorphism $(R^{-1}(mm') \sim R^{-1}(m)R^{-1}(m'))$. The second condition, in fact the equality between $R^{-1}(mm')$ and $R^{-1}(m)R^{-1}(m')$, follows immediately from the definition of R^{-1}. To establish the first condition, suppose that $m' \sim m$ or equivalently $m'm^{-1} \sim 0$ in B_1 and let f be a continuous mapping of the boundary of the unit circle C into B_1 such that $f(\mathrm{Bd}\, C) = m'm^{-1}$. Then there exists a continuous extension g of f carrying C into B_1. The mapping $R^{-1}g$ then carries C into A_1' with $\mathrm{Bd}\, C$ being carried onto

$R^{-1}(m'm^{-1})$. Hence $R^{-1}(m'm^{-1}) \sim 0$ in A_1' , or equivalently $R^{-1}(m')R^{-1}(m^{-1}) \sim 0$ in A_1' .

We now observe that if i denotes the injection homomorphism of $\pi_1(A_1')$ into $\pi_1(B_1)$, then each of hi and ih is the identity homomorphism and hence each of i and h is an onto isomorphism. To see that hi is the identity mapping, let $K \in \pi_1(A_1')$ and let k be a polygonal representative of K . Then $i(K)$ is the element of $\pi_1(B_1)$ determined by k , and $hi(K)$ is the element of $\pi_1(A')$ determined by $R^{-1}(k) = k$. Now consider an element $L \in \pi_1(B)$, and let us determine $ih(L)$. Let ℓ represent L and replace ℓ by a simple closed curve m by the above rule. Then $h(L)$ is the element of $\pi_1(A_1)$ determined by $R^{-1}(m)$, and $ih(L)$ is the element of $\pi_1(B_1)$ determined by $R^{-1}(m)$. This is the element L , since $R^{-1}(m) \sim m$ in B_1 .

The fact that $\pi_1(A_2') \approx \pi_1(B_2)$ follows by a similar argument.

The proof of Theorem 4.4 may be used to prove the following argument.

Theorem 4.5. Let M be a closed subset of E^3_+ and A a component of $E^3_+ \setminus M$. If P is arcwise accessible from each point of A , then $\pi_1(A) \approx \pi_1[R(A)]$.

Let S be a 2-sphere in E^3_+ which is locally polyhedral except at a finite number of points, and which is embedded in E^3_+ such that $S \cap P = D$ is a 2-cell. Let $M = Cl(S \setminus D)$ and let A_1 and A_2 , respectively, denote the bounded and unbounded components of $E^3_+ \setminus S$. Then $R(M)$ is a 3-sphere in E^4 and, if B_1 is the component of

$E^4 \setminus R(M)$ corresponding to A_i, then, by Theorem 4.4, $\pi_1(B_i) \approx \pi_1(A_i)$.

One may again select well known 2-spheres in E^3 to construct examples of 3-spheres in E^4 such that either one or both complementary domains will have non-trivial fundamental groups.

In passing, we observe one difference between the spheres Susp S and $R(M)$. Associated with each exceptional point $p \in M$ there will be an arc, Susp p, of exceptional points on Susp S and a simple closed curve, $R(p)$, of exceptional points on $R(M)$.

We now use the rotation of a disk about P to construct a 3-sphere in S^4, one complementary domain of which is simply connected but is not an open 4-cell. Let us first embed the 2-sphere S, discussed as Example 3.3 in [5], in E^3_+ as indicated in Figure 4. The sphere S is to intersect P in a 2-cell D and $S \setminus D$ is denoted by M. The proof in [5] that the exterior of S in E^3 is simply connected may be used directly to show that A_2 (the exterior of S in E^3_+) is simply connected. Hence, by Theorem 4.4, B_2 (the exterior of $R(M)$ in $E^4(S^4)$) is simply connected.

The cross section $[M \cup R_\pi(M)]$ of $R(M)$ in $E^3(S^3)$ is shown in Figure 5.

Let A_2' denote the exterior of $M \cup R\pi(M)$ in E^3. It is shown in [5] (Example 1.3) that C_o cannot be contracted to a point in $A_2' \setminus [W \cup R_\pi(W)]$. This fact is now used to show that $R(W)$ is contained in no closed 4-cell subset of B_2 whose complement in B_2 is simply connected. Hence, B_2 is not an open 4-cell.

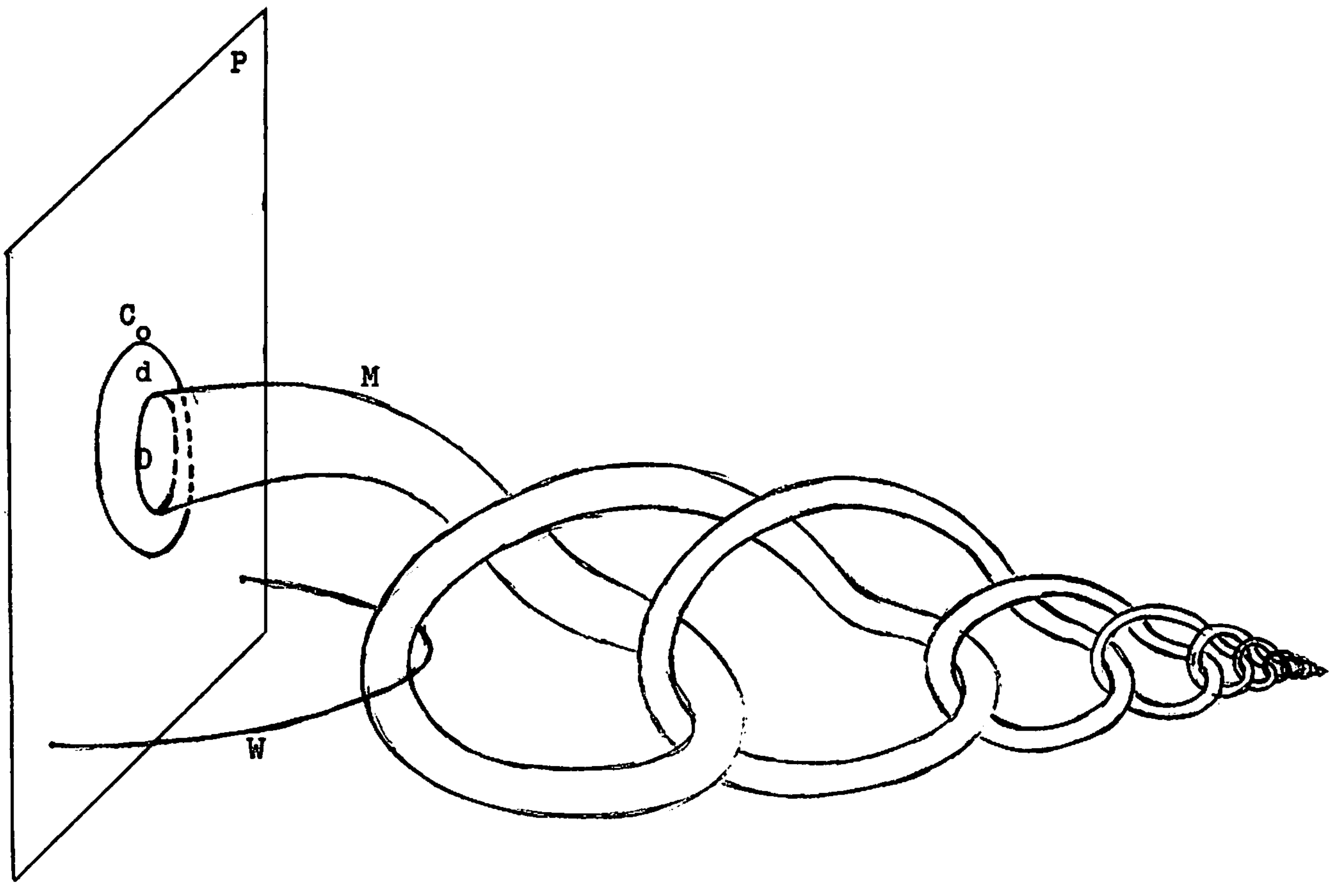

Figure 4

A Wild 2-Sphere in S^3.

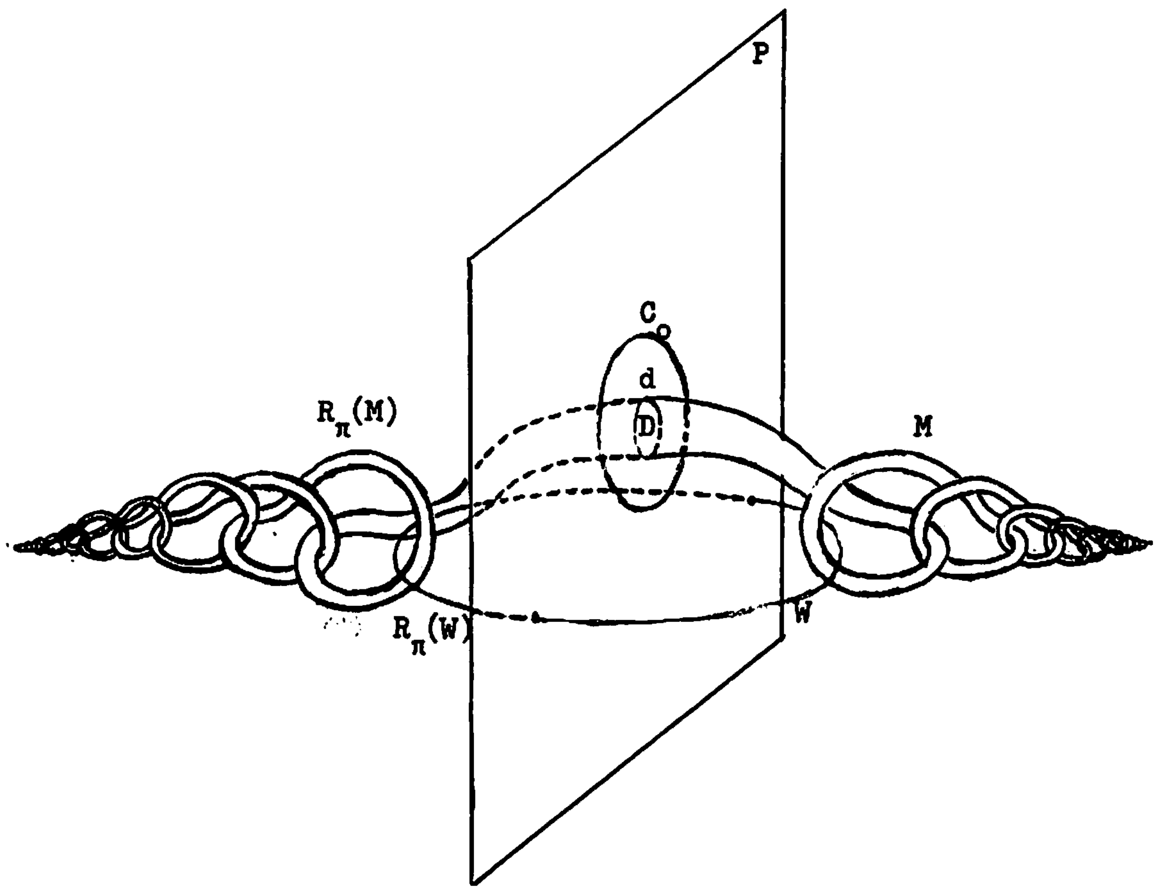

Figure 5

A Cross Section of a Wild 3-Sphere in S^4.

Suppose that such a 4-cell J did exist. Choose the base point for computing $\pi_1(B_2 \setminus J)$ in P and so close to d that there is a path c in $(B_2 \setminus J) \cap P$ which represents C_0 in $\pi_1\left\{A_2' \setminus [W \cup R\pi(W)]\right\}$. Let E be a unit disk in E^2 with boundary e, and let h be a continuous mapping of e onto c. Since $\pi_1(B_2 \setminus J)$ is trivial, there exists an extension H of h which carries E into $B_2 \setminus J$. We then follow H by R^{-1} and obtain a singular 2-cell, $R^{-1}H(E)$, in $A_2 \setminus R^{-1}(J)$ which is bounded by c. Since $A_2 \setminus R^{-1}(J) \subset A_2 \setminus W$, we see that c can be contracted to a point in $A_2 \setminus W$ and hence in the larger set $A_2' \setminus [W \cup R\pi(W)]$. This contradiction establishes the desired conclusion.

4.3. Three-Spheres Obtained by Capping a Cylinder

In E^n we again take coordinates $x_1, x_2, \ldots, x_n$ and let E^{n-1} be described by $x_n = 0$.

Lemma 4.4. *Let S be an $(n-2)$-sphere in E^{n-1} with the bounded and unbounded components of $E^{n-1} \setminus S$ denoted by A_1 and A_2 respectively. If $\mathrm{Cl}\, A_2$ (compactified at infinity) is a closed $(n-1)$-cell, then $\left\{S \times [0, 1]\right\} \cup \left\{A_1 \times [1]\right\}$ is a closed $(n-1)$-cell.*

Proof. Let h be a homeomorphism of $\mathrm{Cl}\, A_2$ onto a standard unit ball B in E^{n-1}. Let $S_1 = \mathrm{Bd}\, B$ and let S_2 be the sphere concentric with S_1 and with radius one-half. Then $h^{-1}(S_2)$ is a sphere in A_2, and h^{-1} restricted to S_2 can be extended in both directions along a cylinder (h^{-1} is such an extension). If C is the closure at the component of $E^{n-1} \setminus h^{-1}(S_2)$ which contains A_1, then, by Theorem 3.2,

C is a closed $(n-1)$-cell. We now observe that C consists of a closed annulus $(h^{-1}(B \setminus \text{Int } S_2))$ with $\text{Cl } A_1$ sewn along one boundary component (along $h^{-1}(S_1) = S$), and is therefore homeomorphic with $\{S \times [0, 1]\} \cup \{\text{Cl } A_1 \times [1]\}$.

<u>Theorem</u> 4.6. <u>Let</u> S <u>be an</u> $(n-2)$-<u>sphere in</u> E^{n-1} <u>with the bounded and unbounded components of</u> $E^{n-1} \setminus S$ <u>denoted by</u> A_1 <u>and</u> A_2 <u>respectively</u>. <u>If</u> $\text{Cl } A_2$ (<u>compactified at infinity</u>) <u>is a closed</u> $(n-1)$-<u>cell</u>, <u>then</u> $\{S \times [-1, 1]\} \cup \{\text{Cl } A_1 \times [-1]\} \cup \{\text{Cl } A_1 \times [1]\}$ <u>is an</u> $(n-1)$-<u>sphere in</u> E^n.

<u>Proof</u>. By Lemma 4.4, each of $\{S \times [-1,0]\} \cup \{\text{Cl } A_1 \times [-1]\}$ and $\{S \times [0, 1]\} \cup \{\text{Cl } A_1 \times [1]\}$ is a closed $(n-1)$-cell. These two cells intersect along their common boundary sphere S, and hence their union is an $(n-1)$-sphere.

We now consider a 2-sphere S, locally polyhedral except at a single point, in E^3 (S^3) such that the bounded complementary domain A_1 is an open 3-cell, $\text{Cl } A_1$ is not a closed 3-cell, the unbounded complementary domain A_2 (compactified at infinity) is an open 3-cell, and $\text{Cl } A_2$ is a closed 3-cell. The assertion is that the 3-sphere

$$T = \{S \times [-1,1]\} \cup \{A_1 \times [1]\} \cup \{A_1 \times [-1]\}$$

is embedded in S^4 such that, if B_1 and B_2 respectively are the components of $S^4 \setminus T$ which contain A_1 and A_2, then B_1 is an open 4-cell, $\text{Cl } B_1$ is not a closed 4-cell, and $\text{Cl } B_2$ is a closed 4-cell.

Since B_1 is the product of the open 3-cell A_1 and the open interval $(-1, 1)$, it follows immediately that B_1 is an open 4-cell.

If $\mathrm{Cl}\, B_1 = \mathrm{Cl}\, A_1 \times [-1,1]$ were a closed 4-cell, a theorem due to Bing [7] would imply that $\mathrm{Cl}\, A_1$ is a closed 3-cell. Thus we have a contradiction of our assumption on the embedding of S in E^3.

We now show that $\mathrm{Cl}\, B_2$ is a closed 4-cell by constructing a homeomorphism $f : T \times [0, \frac{1}{2}] \longrightarrow \mathrm{Cl}\, B_2$ such that the mapping f_o defined by $f_o(y) = f(y, 0)$ is the identity mapping on T, and then applying Theorem 3.2. Since $\mathrm{Cl}\, A_2$ is a closed 3-cell, there exists a homeomorphism $h : S \times [0, \frac{1}{2}] \longrightarrow \mathrm{Cl}\, A_2$ such that $h_o(x) = h(x, 0) = x$ for all $x \in S$. For $y \in T$, let x be the point of $\mathrm{Cl}\, A_1$ which lies under y ($y = (x, t)$ for some $t \in [-1, 1]$). We define f by the following equations:

(1) $f_r(y) = (x, 1 + r)$, $\quad y = (x, 1)$, $\quad x \in A_1$,

(2) $f_r(y) = (x, -1 - r)$, $\quad y = (x, -1)$, $\quad x \in A_1$,

(3) $f_r(y) = (h_r(x), t)$, $\quad x \in S$, $\quad -1 + r < t < 1 - r$,

(4) $f_r(y) = (h_{(1-t)}(x), 2t - (1 - r))$, $\quad x \in S$, $\quad 1 - r \leq t \leq 1$,

(5) $f_r(y) = (h_{(1-t)}(x), 2t - (r - 1))$, $\quad x \in S$, $\quad -1 \leq t \leq -1 + r$.

To show that f is a one-to-one mapping of $T \times [0, \frac{1}{2}]$ into $\mathrm{Cl}\, B_2$ we must show that if $y_1 = (x_1, t_1)$, $y_2 = (x_2, t_2)$, and $f_{r_1}(y_1) = f_{r_2}(y_2)$, then $x_1 = x_2$, $t_1 = t_2$, and $r_1 = r_2$. Since f_r cannot decrease second coordinates of points of $\{S \times [0, 1]\} \cup \{A_1 \times [1]\}$ and cannot increase second coordinates of $\{S \times [-1, 0]\} \cup \{A \times [-1]\}$, we may assume that both t_1 and t_2 are non-negative, or that both are negative. We will only consider the first case; the latter would follow

by a similar argument. If $f_{r_1}(y_1) = f_{r_2}(y_2)$ is a point of $A_1 \times [1 + r_o]$ for some $r_o \in [0, \frac{1}{2}]$, then by (1), $r_1 = r_o = r_2$, $t_1 = t_2 = 1$, and $x_1 = x_2$.

For x_1, x_2 in S and $f_{r_1}(y_1) = f_{r_2}(y_2)$, we must have $x_1 = x_2$, since $f_{r_1}(y_1)$ and $f_{r_2}(y_2)$ lie over points of the arcs $h_t(x_1)$, $0 \leq t \leq \frac{1}{2}$, and $h_t(x_2)$, $0 \leq t \leq \frac{1}{2}$, respectively. Since h is a homeomorphism, these arcs intersect if and only if $x_1 = x_2$.

We now consider two special cases $t_1 = t_2$ and $r_1 = r_2$. If $t_1 = t_2$, we may assume $r_1 \leq r_2$. There are then three possibilities: (a) $0 \leq t_1 < 1 - r$, $0 \leq t_2 = t_1 < 1 - r_2$, (b) $0 \leq t_1 < 1 - r_1$, $1 - r_2 \leq t_2 = t_1 \leq 1$, (c) $1 - r_1 \leq t_1 \leq 1$, $1 - r_2 \leq t_2 = t_1 \leq 1$. For (a) we have $h_{r_1}(x_1) = h_{r_2}(x_1)$ and $r_1 = r_2$, since h is one-to-one on $S \times [0, \frac{1}{2}]$. For (b) we have $r_1 = 1 - t_1$, $t_1 = 2t_1 - (1 - r_2)$ and for (c) we have $2t_1 - (1 - r_1) = 2t_1 - (1 - r_2)$, each of which leads to $r_1 = r_2$. If $r_1 = r_2$, then $t_1 = t_2$, since each f_r is one-to-one.

We now return to the general case $y_1 = (x_1, t_1)$, $y_2 = (x_1, t_2)$, $x \in S$ and $h_{r_1}(y_1) = h_{r_2}(y_2)$. We may assume $t_1 \leq t_2$. Equations (3) and (4) them imply the following possibilities: (a) $t_1 < 1 - r_1$, $t_2 < 1 - r_2$, (b) $t_1 < 1 - r_1$, $t_2 \geq 1 - r_2$, (c) $t_1 \geq 1 - r_1$, $t_2 \geq 1 - r_2$. In (a), $t_1 = t_2$ (the second coordinates of $f_{r_1}(y_1)$ and $f_{r_2}(y_2)$ must be equal), and hence $r_1 = r_2$. In (b), $r_1 = 1 - t_2$, and $t_1 = 2t_2 - (1 - r_2)$ imply that $t_1 = 2(1-r_1) - (1-r_2) = (1-r_1) + (r_2-r_1)$. Since $t_1 < 1 - r_1$, we must have $r_2 - r_1 < 0$, or $1 - r_1 < 1 - r_2$. This leads to $t_2 = 1 - r_1 < 1 - r_2$, which contradicts our assumption that $t_2 \geq 1 - r_2$. Hence (b) cannot occur. In (c) we have

$$1 - t_1 = 1 - t_2 \text{ , or } t_1 = t_2 \text{ ,}$$

since the first coordinates of $f_{r_1}(y_1)$ and $f_{r_2}(y_1)$ must be equal. Since $t_1 = t_2$, we must also have $r_1 = r_2$.

The continuity of f follows rather quickly from the definition of f in terms of the continuous mapping h and a set of linear equations.

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