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Pseudoscalar Interaction in Nuclear Beta Decay

Chander Perkash Bhalla
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by Chander Perkash Bhalla entitled "Pseudoscalar Interaction in Nuclear Beta Decay." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

Dr. M.E. Rose, Major Professor

We have read this thesis and recommend its acceptance:

J.A. Cooley, Edward Harris, G.E. Albert

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

May 26, 1960

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Accepted for the Council:

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Dean of the Graduate School

PSEUDOSCALAR INTERACTION IN NUCLEAR BETA DECAY

A Dissertation

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Chander Perkash Bhalla

June 1960

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No Thin Front

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CHAPTER I

INTRODUCTION

The determination of the nature of beta interaction has been the subject of investigation for several years. The experimental confirmation of parity breakdown[1] in nuclear beta decay opened a new field of experimentation, and a very clear understanding of the main interactions has emerged from the "post" parity experiments. The experiments, to be briefly described below (Section I), lead uniquely to the vector and the axial vector interactions; but have no bearing on the pseudoscalar interaction. In Section II, it is discussed why these experiments do not have any bearing on the pseudoscalar interaction; then the experiments, which can best determine the existence, and hence, the contribution, of the pseudoscalar interaction are described (Section III). It is the purpose of this dissertation to discuss the pseudoscalar interaction by formulating[2] the theoretical expressions for these experiments and by comparing them with the existing experimental data.

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[1] C. S. Wu, E. Ambler, R. Hayward, D. D. Hoppes and R. P. Hudson, Phys. Rev. 105, 1413 (1957). The hypothesis of nonconservation of parity in β decay was originally suggested by T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956).

[2] We follow the formulation of the pseudoscalar interaction given by M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1315 (1954).

I. EXPERIMENTS INDICATING VECTOR AND AXIAL VECTOR INTERACTIONS

(a) One consequence of the parity breakdown is that the β particles are longitudinally polarized in the nuclear beta decay. The longitudinal polarization of β particles has been measured in many cases of "allowed" beta transitions,[3] and the results for β^- and β^+ are $-\frac{v}{c}$ and $\frac{v}{c}$ respectively within an experimental error of 10%. Here $\frac{v}{c}$ is the ratio of the particle velocity and the vacuum velocity of light.

(b) The particle of spin $\frac{1}{2}$ and mass zero accompanying β^- emission is called an antineutrino, and for β^+ emission, it is called a neutrino. To explain the experimental polarization data, the vector and the axial vector interactions require the neutrino to be "left-handed"; whereas the scalar and the tensor interactions demand the neutrino to be a "right-handed" particle. The left-handed and right-handed particles have negative and positive helicity respectively. The experimental observation of the neutrino helicity was made by Goldhaber, Grodzins and Sunyar.[4] This experiment involves Eu^{152} (0^-), which by K-capture goes

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[3] C. S. Wu, Proceedings Rehovoth Conference on Nuclear Structure (North-Holland Publishing Company, Amsterdam 1958), p. 359; and J. Heintze, Zeits. fur Physik 150, 134 (1958). For a recent summary of β polarization measurements, see A. I. Galonsky, A. R. Brosi, B. Ketelle and H. B. Willard (to be submitted for publication in Nuclear Physics).

[4] M. Goldhaber, L Grodzins, and A. W. Sunyar, Phys. Rev. 109, 1015 (1958). This has been confirmed by I. Marklund and L. A Page, Nuclear Physics 9, 88 (1958).

to the excited state of Sm^{152} (1^-); which in turn decays by a dipole gamma transition to the ground state of $Sm^{152}(0^+)$. By observing the resonance scattering of these gamma rays in Sm_2O_3, only those γ-rays, which go in opposite direction to that of the neutrino, were considered. The γ-ray helicity is the same as the neutrino helicity. The γ-ray helicity was found to be negative and therefore the neutrino helicity is negative.

(c) Thus the experimental data on β longitudinal polarization in allowed transitions and the helicity of the neutrino lead to the vector and the axial vector interactions. The relative sign and strength of the vector and axial vector interactions are fixed by the nuclear β transitions, where these interactions interfere. The most informative and carefully analyzed case is that of a polarized neutron transforming into a proton with the emission of an electron and an antineutrino. Burgy et al[5] measured the anisotropy of the electron and the antineutrino with respect to the spin direction of the neutron. The result of this experiment is that the relative sign of the vector and the axial vector coupling constant is negative. Comparison of "ft-values" (comparative half lives) of a neutron and O^{14} gives (1.21 ± 0.03) as the ratio of the absolute magnitudes of the axial vector and the vector coupling constants. The β interaction in the form of V - 1.2A law is consistent with the other

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[5]M. T. Burgy, V. E. Krohn, T. B. Novey, G. R. Ringo and V. L. Telegdi, Phys. Rev. 110, 1214 (1958) and Phys. Rev. Letters 1, 324 (1958).

experiments on "allowed" beta transitions.* To understand why the experiments on "allowed" transitions do not have any bearing on the pseudoscalar interaction, we give below the classification of the allowed and the forbidden beta transitions which is commonly used.

II. ALLOWED AND FORBIDDEN TRANSITIONS

The most general parity nonconserving interaction hamiltonian density for the nuclear β decay** is

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*A number of recent review articles appear in the literature. See references 7, 8, 9, 10, 11, 12.

[7] Invited papers at the Conference on Weak Interaction, Gatlinburg, Rev. Mod. Phys. 31, 782 (1959).

[8] M. E. Rose, Handbook of Physics (McGraw-Hill Book Co., New York, 1958) p. 9-90.

[9] D. L. Pursey, Proc. Royal Soc. of London, Series A, 246, 444 (1958).

[10] E. J. Konopinski, Annual Rev. Nuclear Science 9, 145 (1959).

[11] M. Deutsch and O. Kofoed-Hansen, Experimental Nuclear Physics Vol. III (John Wiley and Sons Inc., New York, 1959) p. 427.

[12] Y. Smorodinskii, Soviet Physics Uspekhi 67 (2), No. 1, 1 (1959).

**M. E. Rose, reference 8.

$$H = \sum_{x=1}^{5} (\Psi_P^* \Omega_x \Psi_N) \cdot (\Psi_e^* \Omega_x [C_x + C_x' \gamma_5] \Psi_\nu) + \text{hermitian conjugate} \quad (1.1a)$$

The summation over x implies the five possible interactions. The first term represents β^- emission, and the hermitian conjugate of this gives the β^+ emission. C_x and C_x' are generally called the parity conserving and parity nonconserving* coupling constants. Ω_x is a 4 x 4 matrix and in terms of the γ-matrices which obey the following commutation rules:

$$\gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\nu = 2 \delta_{\mu,\nu} \quad (\mu \text{ and } \nu = 1, 2, 3, 4)$$

Let $\Omega_x \equiv \gamma_4 O_x$. Then O_x has the following forms for the respective interactions

Scalar	1	(one component)
Vector	γ_μ	(four components)
Tensor	$\gamma_\mu \gamma_\nu ; \quad \mu \neq \nu$	(six components)
Axial Vector	$\gamma_\mu \gamma_5$	(four components)
Pseudoscalar	γ_5	(one component)

Making use** of $\gamma_k = -i\beta\vec{\alpha}_k$ (k = 1, 2, 3); $\gamma_4 = -\beta$, we write (1.1a) as

$$H = \sum_{x=1}^{5} H_x + \text{hermitian conjugate.} \quad (1.1b)$$

where

---

*It is the occurrence of the cross terms between C_x and C_x' that give the parity nonconserving effects.

**In this representation: $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4$; $\gamma_5\vec{\alpha} = \vec{\alpha}\gamma_5 = \vec{\sigma}$.

$$H_S = (\psi_P^* \beta \psi_N)(\psi_e^* \beta \,[C_S + C_S' \gamma_5]\, \psi_\nu \;)$$

$$H_V = (\psi_P^* \psi_N)(\psi_e^* \;[C_V + C_V' \gamma_5\;]\, \psi_\nu \;) \qquad - (\psi_P^* \vec{\alpha} \psi_N)\cdot(\psi_e^* \vec{\alpha} \,[C_V + C_V' \gamma_5]\, \psi_\nu \;)$$

$$H_T = (\psi_P^* \beta \vec{\sigma} \psi_N)\cdot(\psi_e^* \beta \vec{\sigma} \,[C_T + C_T' \gamma_5]\, \psi_\nu \;) + (\psi_P^* \beta \vec{\alpha} \psi_N)\cdot(\psi_e^* \beta \vec{\alpha} \,[C_T + C_T' \gamma_5]\, \psi_\nu \;)$$

$$H_A = (\psi_P^* \vec{\sigma} \psi_N)\cdot(\psi_e^* \vec{\sigma} \,[C_A + C_A' \gamma_5]\, \psi_\nu \;) - (\psi_P^* \gamma_5 \psi_N)(\psi_e^* \gamma_5 \,[C_A + C_A' \gamma_5]\, \psi_\nu \;)$$

$$H_P = \qquad (\psi_P^* \beta \gamma_5 \psi_N)(\psi_e^* \beta \gamma_5 \,[C_P + C_P' \gamma_5]\, \psi_\nu \;)$$

In equations (1.1), ψ_P^* and ψ_e^* represent the creation operators for a proton and an electron respectively, whereas ψ_N and ψ_ν are the destruction operators for a neutron and a neutrino in the negative energy state. One can consider a neutron and a proton as the two states of a nucleon, and we define an operator Q which transforms a neutron into a proton. Thus we can write the equation (1.1a) as

$$H_\beta = \sum_x \Omega_x \, Q \cdot (\psi_e^* \Omega_x \,[C_x + C_x' \gamma_5]\, \psi_\nu \;) + \text{h. c.} \tag{1.1c}$$

Strictly speaking, the setting up of the interaction hamiltonian density for the four fermions is a field theory problem and one requires second quantization of the field amplitudes to insure the Pauli exclusion principle and to describe properly the creation and the destruction of the particles. The usual field theoretic approach is to set up the first order perturbation theory formula for the transition probability between the initial and the final nuclear states. Then in this formula, ψ_e and ψ_ν are treated as the proper Dirac wave functions. Using the relativistic units ($\hbar = c = m_e = 1$) the transition probability for β^- emission between

the initial and the final nuclear states, represented as Ψ_i and Ψ_f, is given by the following:

$$w = 2\pi \, S \left| \int H_{\beta^-} \right|^2 \rho \tag{1.2a}$$

ρ is the density of the final states and S is the summation over all unobserved observables.

$$\int H_{\beta^-} = \sum_{k=1}^{A} \int d\tau_1 \ldots d\tau_A \Psi_f^* \sum_x \Omega_x(k)\; Q(k) \cdot (\Psi_e^* \Omega_x [C_x + C_x' \gamma_5] \Psi_\nu) \Psi_i \tag{1.2b}$$

In (1.2b) beta interaction operator is written in the space of the kth nucleon and the lepton covariant $(\Psi_e^* \Omega_x [C_x + C_x' \gamma_5] \Psi_\nu)$ is to be evaluated at the position x_k of the k-th nucleon. The integration is over all nuclear co-ordinates and it is to be summed over all nucleons.

The selection rules for the allowed and the first forbidden beta transitions depend on the rotational properties of the interaction operators in the nucleon space. The Coulomb effects on the β particle do not influence the rotational properties and as such, in the following discussion, the electron and neutrino are represented as Dirac plane waves.

$$\Psi_e = u_e \, e^{i\vec{p}\cdot\vec{r}} \tag{1.3a}$$

$$\Psi_\nu = u_\nu \, e^{-i\vec{q}\cdot\vec{r}} \tag{1.3b}$$

In (1.3), $\vec{p}$ and $\vec{q}$ are the (physical) momenta of the electron and the antineutrino respectively. Letting $\vec{P} = \vec{p} + \vec{q}$; substituting (1.3) in (1.2)

$$W = 2\pi \rho\, S \left| \sum_x (u_e^* \Omega_x [C_x + C_x' \gamma_5]\, u_\nu) \int \Psi_f^* \Omega_x\, e^{-i\vec{P}\cdot\vec{r}_k}\, Q \Psi_i \right|^2 \qquad (1.2c)$$

r_k is the position of the decaying nucleon and $r_k \leqslant$ nuclear radius (r). In the above we have introduced the conventional notation:

$$\sum_{k=1}^{A} \int d\,\tau_1 \ldots d\,\tau_A \Psi_f^* \Omega_x(k)\, e^{-i\vec{P}\cdot\vec{r}_k}\, Q(k) \Psi_i \equiv \int \Psi_f^* \Omega_x\, e^{-i\vec{P}\cdot\vec{r}}\, Q \Psi_i$$

P is at the most as large as 10 (in mc units), $r = 0.02 \frac{\hbar}{mc}$ for A = 210; therefore Pr $\sim$ 0.2. In general Pr $<<$ 1. Using the Raleigh expansion

$$e^{-i\vec{P}\cdot\vec{r}} = \sum_\ell (-i)^\ell\, (2\ell+1)\, j_\ell\,(Pr)\, P_\ell\,(\hat{P}\cdot\hat{r}) \qquad (1.4a)$$

$$e^{-i\vec{P}\cdot\vec{r}} = 4\pi \sum_\ell (-i)^\ell \frac{(Pr)^\ell}{(2\ell+1)!!} \sum_{M=\ell}^{-\ell} Y_\ell^{M\times}(\hat{P})\, Y_\ell^M(\hat{r})$$

$$e^{-i\vec{P}\cdot\vec{r}} = 4\pi \sum_M \frac{(-i)^\ell}{(2\ell+1)!!}\, \mathcal{Y}_\ell^{M\times}(\vec{P})\, \mathcal{Y}_\ell^M(\vec{r}) \qquad (1.4b)$$

In arriving at (1.4b), we have used

$$j_\ell\,(Pr) \simeq \frac{(Pr)^\ell}{(2\ell+1)!!}$$

$$P_\ell\,(\hat{P}\cdot\hat{r}) = \sum_M \frac{4\pi}{(2\ell+1)}\, Y_\ell^{M\times}(\hat{P})\, Y_\ell^M(\hat{r})$$

and

$$\mathcal{Y}_L^M(\vec{r}) \equiv r^\ell\, Y_L^M(\hat{r})$$

We write (1.4b) as

$$e^{-iP\cdot r} = \sum_{\ell,M} a_{\ell,M} \mathcal{Y}_\ell^M(\vec{r}) \tag{1.5a}$$

where

$$a_{\ell,M} \equiv \frac{4\pi(-i)^\ell}{(2\ell+1)!!} \mathcal{Y}_\ell^{M\times}(\vec{P}) \tag{1.5b}$$

Substituting (1.5a) in equation (1.2c)

$$W = 2\pi \, S \left| \sum_x (u_e^* \Omega_x [C_x + C_x' \gamma_5] u_\nu) \sum_{\ell,M} a_{\ell,M} \int \Psi_f^* \Omega_x \mathcal{Y}_\ell^M(\vec{r}) \, Q \Psi_i \right|^2 \rho \tag{1.2d}$$

Now Ω_x is a tensor of a certain rank, depending upon the interaction. For example, $\Omega_x = \beta, 1, \gamma_5, \beta\gamma_5$ are zero rank tensors, because these are scalar under three dimensional rotation. Let us denote them by $T_o(\Omega_x)$. Similarly when $\Omega_x = \beta\vec{\sigma}, \vec{\sigma}, \vec{\alpha}, \beta\vec{\alpha}$, Ω_x is a tensor of rank one and let us represent it by $T_1(\Omega_x)$. In the nuclear matrix element of equation (1.2d), we have

$$\Omega_x \mathcal{Y}_\ell^M(\vec{r}) = T_L^{\mu-M}(\Omega_x) \mathcal{Y}_\ell^M(\vec{r}) - \quad ; L = 0 \text{ or } L = 1.$$

$$= \sum_\lambda b_{L\ell\lambda} T_\lambda(\Omega_x, \vec{r}) \tag{1.6}$$

where $b_{L\ell\lambda}$ is a constant,* which for our discussion is irrelevant. Sub-

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*$b_{L\ell\lambda} = C(L\ell\lambda; \mu-M, M, \mu)$ is a Clebsh-Gordon Coefficient.

stituting (1.6) in (1.2d) we have

$$W = 2\pi\rho S\left|\sum_x (u_e^* \Omega_x [C_x + C_x' \gamma_5] u_\nu) \sum_{\ell, M, \lambda} a_{\ell, M}\, b_{L, \ell\lambda} \int \Psi_f^* T_\lambda^{\mu}(\Omega_x, \vec{r}) Q \Psi_i \right|^2 ,$$

(1.2e)

T_λ^{μ} are the components of tensor rank λ where, in steps of unity,

$$|L-\ell| \leq \lambda \leq L+\ell \qquad (1.7)$$

Thus L, ℓ and λ must form a triangle and we represent it as $\Delta(L\ell\lambda)$. This notation will be used wherever needed. To conserve angular momentum in the nuclear matrix element of Eq. (1.2e), we must have $\Delta(J_f \lambda J_i)$. J_f and J_i are the spins of the final and the initial nuclear states. If we represent the parity of $T_\lambda(\Omega_x, \vec{r})$ as $\pi(T_\lambda)$, then $\pi_f \pi_i = \pi(T_\lambda)$. In the Raleigh expansion, (1.5a), we get the leading term, when ℓ is zero. The higher the value of ℓ, the smaller is the term in the expansion. Also from Eq. (1.1') we observe that the even operators (in the Dirac sense, like β) appear on the left side and odd operators (in the Dirac sense, like α) appear on the right-hand side. As pointed out earlier, these operators in the nuclear space are 4 x 4 matrices. In the nuclear matrix elements the even operators connect the large component with the large component and the small component with the small components of the nuclear wave functions involved. On the other hand, the odd operators connect the large and small components. The small component is of the order $\frac{v}{c}$ of the large component. As such the even operators with the leading term

of the Raleigh expansion (for $\ell = 0$) give maximum contributions to the transition probability. Such transitions, for which the selections are given by $\Delta(J_f \lambda J_i)$ and $\pi_f \pi_i = \pi(T_\lambda)$, and which have the largest transition probability, are called allowed transitions.

Now we also note that for the pseudoscalar interaction, only an odd operator $\beta\gamma_5$ (a zero rank tensor) is involved. Therefore, the maximum contribution of this interaction arises from the selection rules $\Delta(J_f 0 J_i)$ and $\Delta\pi = -1$. Now, if there were no other interactions in nuclear β decay, then this transition ($\Delta J = 0$; $\Delta\pi = -1$) would be called an allowed transition. But we know that there are other interactions (the vector and the axial vector) which have much larger contributions; therefore the transitions with these selection rules are called first forbidden.

In the first forbidden beta transitions the contributions to the transition probability come from

(1) The even operators with the term in the Raleigh expansion for $\ell = 1$.

(2) The odd operators with the term in the Raleigh expansion for $\ell = 0$.

The selection rules for the first forbidden are $|\Delta J| = 0, 1, 2$ and $\Delta\pi = -1$.

In Table I we list the nuclear operators for the allowed (n = 0) and the first forbidden (n = 1) transitions. In Table II, we explicitly

TABLE I

NUCLEAR OPERATORS FOR ALLOWED (n = 0) AND FIRST FORBIDDEN (n = 1) TRANSITIONS

Ω_x	Interaction	$\pi(\Omega_x)$	T_λ	Rank λ	$\pi(T_\lambda)$	n
Even Operators						
β	Scalar	+	β	0	+	0
			$\beta\vec{r}$	1	-	1
1	Vector	+	1	0	+	0
			$\vec{r}$	1	-	1
$\beta\vec{\sigma}$	Tensor	+	$\beta\vec{\sigma}$	1	+	0
			$\beta\vec{\sigma}\cdot\vec{r}$	0	-	1
			$\beta\vec{\sigma}\times\vec{r}$	1	-	1
			$^*\beta(3\sigma_z r_z - \vec{\sigma}\cdot\vec{r})$	2	-	1
$\vec{\sigma}$	Axial Vector	+	$\vec{\sigma}$	1	+	0
			$\vec{\sigma}\cdot\vec{r}$	0	-	1
			$\vec{\sigma}\times\vec{r}$	1	-	1
			$^*(3\sigma_z r_z - \vec{\sigma}\cdot\vec{r})$	2	-	1
Odd Operators						
$\vec{\alpha}$	Vector	-	$\vec{\alpha}$	1	-	1
$\beta\vec{\alpha}$	Tensor	-	$\beta\vec{\alpha}$	1	-	1
γ_5	Axial Vector	-	γ_5	0	-	1
$\beta\gamma_5$	Pseudoscalar	-	$\beta\gamma_5$	0	-	1

*One component of the tensor has been shown.

TABLE II

ANGULAR MOMENTUM AND PARITY SELECTION RULES FOR ALLOWED AND FIRST FORBIDDEN TRANSITIONS

Interaction	T_λ	$\Delta\pi$	$\Delta(J_i J_f \lambda)$
Allowed:			
Scalar	β	1	$\Delta(J_i J_f 0)$
Vector	1	1	$\Delta(J_i J_f 0)$
Tensor	$\beta\vec{\sigma}$	1	$\Delta(J_i J_f 1)$
Axial Vector	$\vec{\sigma}$	1	$\Delta(J_i J_f 1)$
First Forbidden			
Scalar	$\beta\vec{r}$	-1	$\Delta(J_i J_f 1)$
Vector	$\vec{r}$	-1	$\Delta(J_i J_f 1)$
	$\vec{\alpha}$	-1	$\Delta(J_i J_f 1)$
Tensor	$\beta\,\vec{\sigma}\cdot\vec{r}$	-1	$\Delta(J_i J_f 0)$
	$\beta\vec{\sigma}\times\vec{r}$	-1	$\Delta(J_i J_f 1)$
	$^*\beta(3\,\sigma_z r_z - \vec{\sigma}\cdot\vec{r})$	-1	$\Delta(J_i J_f 2)$
Axial Vector	$\vec{\sigma}\cdot\vec{r}$	-1	$\Delta(J_i J_f 0)$
	$\vec{\sigma}\times\vec{r}$	-1	$\Delta(J_i J_f 1)$
	$^*(3\,\sigma_z r_z - \vec{\sigma}\cdot\vec{r})$	-1	$\Delta(J_i J_f 2)$
Pseudoscalar	$\beta\gamma_5$	-1	$\Delta(J_i J_f 0)$

*Only one component of the tensor is indicated.

give the selection rules for the allowed and the forbidden β transitions. In general, the nth forbidden transition has the following selection rules

$$\Delta J = n, n+1 \qquad \text{for } n > 1$$

$$\Delta \pi = (-)^n$$

III. EXPERIMENTS FOR INVESTIGATING THE PSEUDOSCALAR INTERACTION

Due to the parity selection rule, the experiments on the allowed beta transitions do not have any bearing on the existence of the pseudoscalar interaction. The operator $\beta\gamma_5$ (in the nucleon space) is a zero rank tensor and has odd parity. Therefore, the pseudoscalar interaction contributes when $\Delta J = 0$ and $\Delta \pi = -1$. The pseudoscalar and axial vector interactions contribute to nuclear transitions with $J_i = J_f = 0$ and $\pi_i\pi_f = -1$. Generally this type of transition is written as $0 \rightarrow 0$ (yes). Since the nuclear matrix elements are very hard to evaluate they are treated as parameters, which are adjusted to fit the experimental data. Though, in principle, the contribution of the pseudoscalar interaction can be determined also from transitions ($J_i = J_f \neq 0$; $\pi_i\pi_f = -1$) where the pseudoscalar, axial vector and vector interactions contribute, there are more unknown (nuclear matrix elements) parameters, thus making the analysis harder. The best cases for investigating the existence of the pseudoscalar interaction are $0 \rightarrow 0$ (yes) transitions because the vector interaction does not contribute at all. With the known negative helicity of the neutrino, the pseudoscalar and the axial vector interactions,

taken separately, give opposite longitudinal polarization of β particles. The relevant experimental data on $0 \rightarrow 0$ (yes) transitions are

(1) The β spectrum

(2) The longitudinal polarization of β particles.

At present, $0 \rightarrow 0$ (yes) transitions occur in the decay[13] of Pr^{144}, Ho^{166}, Ce^{144}, Eu^{152}, and possible Tl^{206}.

IV. STATEMENT OF THE PROBLEM

The problem, considered in this dissertation, is to investigate the existence of the pseudoscalar interaction in the interaction hamiltonian density for the processes of nuclear beta decay, by:

(1) Formulation of the theoretical expressions for β longitudinal polarization and the β spectrum in $0 \rightarrow 0$ (yes) transitions with the correct form of the pseudoscalar interaction[2] and the axial vector interaction.

(2) Making an extensive numerical analysis of the presently available experimental data, using the derived formulas, with the calculated electronic functions, which include accurately the nuclear finite size[*] and the effect due to finite deBroglie wavelength.[**]

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[13] D. Strominger, J. M. Hollander and G. T. Seaborg, Rev. Mod. Phys. 30, 585 (1958).

[*] M. E. Rose, Phys. Rev. 82, 389 (1951); M. E. Rose and D. K Holmes, Oak Ridge National Laboratory Report No. 1022 (Unpublished).

[**] M. E. Rose and C. L. Perry, Phys. Rev. 90, 479 (1953)

The remainder of this chapter summarizes the history and the present status of the pseudoscalar interaction in nuclear β decay. In Chapter II we find an expression for the interaction hamiltonian density, by removing the odd operators with the Foldy-Wouthuysen transformation. Also the representation and notation used is discussed. In Section I of Chapter III, time-dependent perturbation theory is outlined and the asymptotic wave function of the beta particle is given. After a brief discussion of the polarization operator in Section IIA, the main problem of longitudinal polarization and β spectrum is set up for $0 \rightarrow 0$ (yes) transitions. After assuming time reversal invariance in weak and strong interactions and the two-component theory of neutrino as valid, we give the resulting formulas for the β longitudinal polarization and spectrum in (3.35) and (3.37) on pages 89 and 90 respectively. The relevant experimental data on $0 \rightarrow 0$ (yes) transitions are summarized in Section II of Chapter IV. In Section III of Chapter IV, after a brief discussion of finite nuclear size and finite deBroglie wavelength corrections, the methods of analysis are described and results are given. Chapter V contains the conclusions of this investigation and a discussion of these conclusions. In Appendix A, for clarity, the symmetry relations of Clebsch-Gordon coefficients, Racah coefficients and X-coefficients are listed along with some "relations" which already exist in the literature. Appendix B contains the neutrino wave function in the negative energy state in (Section 1), and the general expression of β matrix elements for the

axial vector and the pseudoscalar interactions is worked out in detail in Section II In Section III of Appendix B the results of Section 2 are specialized to $0 \rightarrow 0$ (yes) transitions. Appendix C shows the details of a certain Racah recoupling in Section 1, and a short discussion of time reversal invariance in strong and weak interactions in Sections 2 and 3 respectively. Appendix D lists the expressions of certain functions (introduced in polarization expression of Chapter III) up to order R (the nuclear radius).

V. HISTORICAL BACKGROUND

In 1934, Fermi[14] formulated a field theory of beta decay in close analogy with the field theory of electromagnetic radiation. He considered only the vector interaction by taking the interaction hamiltonian density as a scalar product of two 4-vectors* $(\psi_P^* \gamma_4 \gamma_\mu \psi_N)$ and $(\psi_e^* \gamma_4 \gamma_\mu \psi_\nu)$. Soon it was realized that one could make other possible combinations[15] in the interaction hamiltonian density, and they are called the (1) scalar, (2) tensor, (3) axial vector, and (4) pseudoscalar interactions. In the

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[14] E. Fermi, Zeits. fur Physik 88, 161 (1934).

*A 4-vector transorms under Lorentz transformation as do x, y, z, ict.

[15] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 189 (1936).

setting up of the hamiltonian density, the following were assumed:

(1) The interaction hamiltonian density is hermitian so as to treat e^- and e^+ on the same footing.

(2) The beta interaction is direct, i.e., no derivatives of the field amplitudes exist.

(3) The beta interaction is local, i.e., the field amplitudes are taken at the same space-time point.

(4) The classical beta hamiltonian was, a priori, considered a scalar. The experiment[1] proved it otherwise.

In equation (1.1) setting C'_x equal to zero gives the classical beta interaction hamiltonian density.

In 1941 the theory of forbidden beta transitions was given by Konopinski and Uhlenbeck[16] and extended later by others.[17,18] The experimental data on β spectrum, half-life and electron-neutrino correlation were compared with the theory to determine the nature of β interaction. The energy dependence of the β spectrum in allowed transitions indicated that there is little or no interference between the vector and the scalar interactions nor between the axial vector and the tensor interactions. The above statement is generally expressed as

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[16] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. 60, 308 (1941).

[17] E. Greuling, Phys. Rev. 61, 568 (1942).

[18] D. L. Pursey, Phil. Mag. 42, 1193 (1951).

that the Fierz interference[19] terms are almost absent.[20] As late as September 1957, the scalar and the tensor interactions were considered as the main interactions[21] mostly due to erroneous results of Rustad and Ruby.[22]

From 1952 through 1957, several authors[2,23,24,25] expressed doubts about the correctness of the conventional treatment[16,18] of the pseudoscalar interaction. In the conventional treatment the parameters describing the lepton field were considered as independent of those describing the nucleon. In the "new" formulation of the pseudoscalar interaction a gradient operator appears, which operates on the lepton covariant.* As an illustration Rose and Osborn[2] applied the formula for the β spectrum in 0 → 0 (yes) transitions using the pseudoscalar and the tensor

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[19] M. Fierz, Zeits. fur Physik, 104, 553 (1937).

[20] J. B. Gerhart, Phys. Rev. 109, 897 (1958): gives $b_F = 0.00 \pm 0.12$. R. Sherr and R. H. Miller, Phys. Rev. 93, 1076 (1954): gives $b_{G.T.} = -0.01 \pm 0.02$.

[21] E. J. Konopinski, Proceedings Rehovoth Conference on Nuclear Structure (North-Holland Publishing Company, Amsterdam 1958) p. 318

[22] B. M. Rustad and S. L. Ruby, Phys. Rev. 89, 880 (1952).

[23] T. Ahrens, E. Feenberg, and H. Primakoff, Phys. Rev. 87, 663 (1952).

[24] T. Ahrens, Phys. Rev. 90, 974 (1953).

[25] G. Alaga and B. Jaksic, Glansik Mat-Fiz. i Astr. Tom. 12, No 1-2 (1957).

*Reference 2 contains an excellent discussion. See also Chapter II of this dissertation.

interactions to RaE, but it turned out later that in RaE (Bi^{210}) the beta transition was $1^- \rightarrow 0^+$. Laubitz[26] and Zyrianova[27] made a detailed analysis of $0 \rightarrow 0$ (yes) in Pr^{144} using the Rose and Osborn formula for the β spectrum. Alaga and Jaksic[25] applied essentially the Rose and Osborn formulation with some extra parameters describing the nuclear forces effect, to the analysis of β spectrum of $0 \rightarrow 0$ (yes) in Ho^{166}. Alaga, Sips and Tadic[28] also consider a tensor and the pseudoscalar interaction for the analysis of β spectrum of Pr^{144}. At present, our knowledge of nuclear hamiltonians is not adequate to calculate the nuclear matrix elements, and in the usual treatment of β decay, these nuclear matrix elements are considered as parameters. Alaga and Jaksic[25] introduce more parameters depending upon the nuclear forces. Now if one did know how to calculate the nuclear matrix elements, with some confidence, then it might be interesting to see how many other parameters (depending on the nuclear forces) are required to fit the experimental data. But with our present knowledge of nuclear forces, it is neither practical nor desirable to complicate the theoretical calculations with such parameters.

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[26] M. J. Laubitz, Proc. Phys. Soc. (London) A69, 789 (1956).

[27] L. N. Zyrianova, Bull. Acad. U.S.S.R.--Physical Series 20, 1280 (1956). (Translated by Columbia Technical Translation, New York).

[28] G. Alaga, L. Sips, D. Tadic, Glansik Mat-Fiz. i Astr. Ser. II, 12, 207 (1957).

After the experimental verification[1] of the breakdown of parity symmetry law in beta decay, a number of experiments on the longitudinal polarization of β particles, the anisotropy of β particles from oriented nuclei, and β-γ (circularly polarized) correlation were done. In the meantime Lee and Yang,[29] Salam[30] and also Landau[31] gave independently what is termed as the two-component theory of neutrino. Also following different approaches, Sudarshan and Marshak,[32] Feynman and Gell-Mann[33] and also Sakurai[34] proposed the vector and the axial vector theory of β decay. As pointed out earlier in this chapter, the experiments on the β longitudinal polarization in allowed transitions, the experimental determination of the neutrino-helicity and the anisotropy of e^- and $\bar{\nu}$ from the polarized neutron, uniquely indicate the vector and the axial vector interactions. These interactions are consistent with the electron-neutrino

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[29] T. D. Lee and C. N. Yang, Phys. Rev. 105, 1671 (1957).

[30] A. Salam, Nuovo Cimento 5, 299 (1957).

[31] L. Landau, Nuclear Physics 3, 127 (1957).

[32] R. E. Marshak and E. C. G. Sudarshan, Phys. Rev. 109, 1860 (1958).

[33] R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

[34] J. J. Sakurai, Nuovo Cimento 7, 649 (1958).

correlation experiments[35,36,37] on He^6, Ne^{19} and A^{35}.

As a consequence of the above development leading to the V-λA law of beta decay the previous estimates of the pseudoscalar interaction based on the analysis of a mixture of the tensor and the pseudoscalar interaction, are no longer correct. Recently the beta spectrum of Pr^{144} ($0^- \rightarrow 0^+$) has been studied experimentally.[38,39,40] Graham et al[38] set up an upper limit for the pseudoscalar interaction using the Rose and Osborn formula[2] with the axial vector and the pseudoscalar mixture. To investigate the existence of the pseudoscalar interaction one must consider all the experimental data in any particular β transition, and the best transition as pointed out earlier, is $0 \rightarrow 0$ (yes). Apart from the β spectrum, we have additional information about the β longitudinal polarization. A number of treatments[41,42,43] of

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[35] W. B. Hermannsfeldt, R. L. Burman, P. Stahelin, J. S. Allen and T. H. Bird, Phys. Rev. Letters 1, 61 (1958) and J. S. Allen, Rev. Mod. Phys. 31, 791 (1959). Also see F. Pleasonton, C. H. Johnson and A. H. Snell, Bull. Am. Phys. Soc. 4, 78 (1959).

[36] J. B. Gerhart, Phys. Rev. 109, 897 (1958).

[37] W. B. Hermannsfeldt, J. S. Allen and P. Stahlein, Phys. Rev. 107, 641 (1957).

[38] R. L. Graham, J. S. Geiger, and T. E. Eastwood, Can. J. Phys. 36, 1084 (1958).

[39] F. T. Porter and P. P. Day, Phys. Rev. 114, 1286 (1959).

[40] N. J. Freeman, Proc. Phys. Soc. 73, 600 (1959).

[41] G. E. Lee-Whiting, Can. J. Phys. 36, 1199 (1958).

[42] T. Kotani and M. Ross, Prog. Theor. Phys. 20, 643 (1958).

[43] V. B. Berestetsky, B. L. Ioffe, A. P. Rudik, and K. A. Ter-Martirosyan, Nuclear Phys. 5, 464 (1958).

the β longitudinal polarization in first forbidden transitions appear in the literature. In all of these treatments, the "conventional" form of the pseudoscalar interaction has been used instead of the correct formulation of the pseudoscalar interaction. Geshkenbein[44] gives the longitudinal polarization in $0 \rightarrow 0$ (yes) transition, still not using the correct form of the pseudoscalar interaction. Tadic[45] has analyzed the earlier less accurate (22%) measurement of the β longitudinal polarization[46] in $0 \rightarrow 0$ (yes) of Pr^{144}. His treatment, though it introduces parameters depending on the nuclear forces, is not rigorous and his analysis is inadequate because of the approximations used therein. Recently Bühring[47] measured the longitudinal polarization of β particles in the $0 \rightarrow 0$ (yes) transition of Ho^{166}. His analysis of the longitudinal polarization measurement is not correct because he uses the formulas of Lee-Whiting. Cuperman[48] has analyzed his measurement of the longitudinal polarization of β particles in the $(\frac{1}{2}^{+} \rightarrow \frac{1}{2}^{-})$ transition of Tl^{207}. Apart from

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[44] B. V. Geshkenbein, Zhur. Eksptl'i Teoret Fiz. 34, 1349 (1958).

[45] D. Tadic (Private communication to Dr. M. E. Rose).

[46] J. S. Geiger, G. T. Ewan, R. L. Graham and D. R. Mackenzie, Phys. Rev. 112, 1684 (1958).

[47] W. Bühring, Z. Phys. 155, 566 (1959).

[48] S. Cuperman (to be published in Phys. Rev.)

the fact that there are more parameters occurring, it appears that the correct formulation of the pseudoscalar interaction has not been used. Recently Mehlhop[49] reported the measurement on the longitudinal polarization in the $0^- \rightarrow 0^+$ transition of Pr^{144}. In his analysis, very crude approximations were used, along with the incorrect formulas of Lee-Whiting.[41] The effects of the finite deBroglie wavelength[50] and the finite size of the nucleus[51,52] are important in $0 \rightarrow 0$ (yes) beta transitions, and have not been properly considered.

Thus, until now, no consistent treatment for the search of the pseudoscalar interaction existed in which the correct formulation of the pseudoscalar interaction was used. This dissertation presents such a treatment in which all the relevant experimental data are analyzed with large scale computing programs using the accurate electronic functions.

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[49] W. A. W. Mehlhop, "A Measurement of the Longitudinal Polarization of Pr^{144} Beta Particles" (unpublished Ph.D. dissertation. Washington University, Saint Louis 1959).

[50] M. E. Rose and C. L. Perry, Phys. Rev. 90, 479 (1953).

[51] M. E. Rose, Phys. Rev. 82, 389 (1951).

[52] M. E. Rose and D. K. Holmes, Oak Ridge National Laboratory Report No. 1022 (unpublished).

CHAPTER II

BETA DECAY INTERACTION IN NONRELATIVISTIC FORM AND NOTATION USED

In this chapter, we start with a brief discussion of the difficulties which arise in obtaining the nonrelativistic limit of the nuclear matrix elements involving odd operators. The prescriptions of removing the odd operators in the hamiltonian by the Foldy-Wouthuysen (canonical) transformation for a free Dirac particle and for nuclear beta decay are given in Section I and II respectively. Section III contains the application of the results of Section II to the axial vector, vector and pseudoscalar interactions and it is explained, why the conventional treatment of the pseudoscalar interaction is not correct. For clarity, the notation and the representation, used in later chapters, is explained in Section IV.

The relativistic dynamics of a nucleon in a nucleus are not presently known and as such we are ignorant about the details of relativistic nucleon wavefunctions. In nuclear beta decay, the transition probability between the initial and final nuclear states depends upon the matrix elements of operators (4 x 4 matrices) in nucleon's space. Due to the above reason, these nuclear matrix elements are very hard to evaluate. However, there are some nuclear models like, the shell model, the Wigner model, the optical model and the unified model, which have some success in explaining some qualitative properties of nuclear structure. These

models are nonrelativistic in nature and it appears that the relativistic corrections are not very important. Thus, at least there is a possibility of evaluating the nuclear matrix elements, provided the nonrelativistic limits of these matrix elements are known. We represent the nuclear wave function

$$\Psi = \begin{pmatrix} v \\ u \end{pmatrix}$$

where v and u have two components. In the nonrelativistic limit:

$$\Psi \rightarrow \begin{pmatrix} 0 \\ u \end{pmatrix}$$

then v and u are called the small and the large components of Ψ. In the following the subscripts i and f refer to the initial and final nuclear states.

Consider a matrix element of an even operator in nucleon space, e.g., $\vec{\sigma} \cdot L(\vec{\sigma})Q$ (in the axial vector interaction)

$$L(\vec{\sigma}) \equiv (\Psi_e^* \vec{\sigma}\, [C_A + C_A' \gamma_5]\, \Psi_\nu)$$

$$\int \Psi_f^* \vec{\sigma} \cdot L(\vec{\sigma}) Q \Psi_i = \int u_f^* \vec{\sigma} \cdot L(\vec{\sigma})\, u_i + \int v_f^* \vec{\sigma} \cdot L(\vec{\sigma})\, v_i \qquad (2.1a)$$

In the above equation, u's and v's are two component functions and $\vec{\sigma}$ is a Pauli matrix (in the nucleon space) on the right-hand side of (2.1a). Also the first and second term in (2.1a) involve only the large and small components of the nuclear wavefunctions. To obtain the nonrelativistic limit, we can neglect the second term as compared to the first one

$$\int \Psi_f^* \vec{\sigma} \cdot L(\vec{\sigma}) \Psi_i \longrightarrow \int u_f^* \vec{\sigma} \cdot L(\vec{\sigma}) u_i \tag{2.1b}$$

In the pseudoscalar interaction, we are interested in the matrix elements of an odd operator (which anticommutes with β) $\beta\gamma_5 L(\beta\gamma_5)$.

$$\int \Psi_f^* \beta\gamma_5 L(\beta\gamma_5) \Psi_i = \int v_f^* L(\beta\gamma_5) u_i - \int u_f^* L(\beta\gamma_5) v_i \tag{2.2a}$$

where we have used the representation

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2b}$$

and the lepton covariant $L(\beta\gamma_5) \equiv (\Psi_e^* \beta\gamma_5 [C_P + C_P' \gamma_5] \Psi_\nu)$. In (2.2a), both the terms involve the large component and the small components of the nuclear wavefunctions, as such these terms are of the same order of magnitude. There is no simple prescription, as in the case of even operators, to obtain the nonrelativistic limit.

There are two methods, which, in principle, can be utilized for obtaining the nonrelativistic limit of the matrix elements of an odd operator in the nucleon space:

(1) By eliminating the small components by making use of the relation between the small and the large components of the nuclear wave function. As an illustration: if we take the wave equation for stationary states of a nucleon (in units $\hbar = m_e = c = 1$) as

$$-(\vec{\alpha}\cdot\vec{p} + \beta M - V)\Psi = W\Psi$$

$$\Psi \equiv \begin{pmatrix} v \\ u \end{pmatrix}; \text{ then } v = -(W-V+M)^{-1} \vec{\sigma}\cdot\vec{p}u \simeq -\frac{1}{2M}\vec{\sigma}\cdot\vec{p}u \tag{2.2c}$$

if we take $W - V - M \ll 2M$

Substituting (2.2c) in (2.2a) gives

$$\int \Psi_f^* \beta\gamma_5 L(\beta\gamma_5)\Psi_i = -\frac{1}{2M}\int (\vec{\sigma}\cdot\vec{p}\, u_f)^* L(\beta\gamma_5)u_i + \frac{1}{2M}\int u_f^* \vec{\sigma}\cdot\vec{p}\, L(\beta\gamma_5)u_i$$

$$\int \Psi_f^* \beta\gamma_5 L(\beta\gamma_5)\Psi_i = -\frac{1}{2M}\int u_f^* \left[\vec{\sigma}\cdot\vec{p}\, L(\beta\gamma_5)\right] u_i \qquad (2.2d)$$

In (2.2d) $\vec{p}$ operates only on $L(\beta\gamma_5)$ and if $L(\beta\gamma_5)$ is considered as a constant (as done in the conventional theory); then $\vec{\sigma}\cdot\vec{p}\, L(\beta\gamma_5) = 0$, and hence there is no contribution from the pseudoscalar interaction. The above procedure may not be quite correct when the fields are present and is very cumbersome in many body problems.

(2) By applying a Foldy-Wouthuysen (canonical) transformation to the total hamiltonian of the system comprised of the decaying nucleon, the lepton (e-ν) field and the lepton, so as to remove the odd operators. The odd operators can be eliminated up to any order in $\frac{1}{M}$. This procedure is theoretically more sound and we shall follow this prescription.

I. FOLDY-WOUTHUYSEN TRANSFORMATION (NO FIELDS)[1,2]

The equation of motion of a Dirac particle of mass M, with no

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[1] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

[2] M. E. Rose, Relativistic Electron Theory (to be published by John Wiley and Sons, New York) Sections 18 and 22.

fields present is

$$H \Psi = i \frac{\partial \Psi}{\partial t} \tag{2.3}$$

where in the standard notation[3] (using $\hbar = c = 1$)

$$H = - \beta M - \vec{\alpha} \cdot \vec{p} \tag{2.4}$$

In the equation (2.4) $\vec{\alpha}$ is an odd operator and we want to eliminate, for the time being, $\vec{\alpha} \cdot \vec{p}$ terms up to an order $\frac{1}{M}$. Consider a unitary transformation generated by S and $\frac{\partial S}{\partial t} = 0$

$$\Psi' = e^{S} \Psi \tag{2.5a}$$

then the equation of motion (2.3) can be written as

$$e^{S} H e^{-S} e^{S} \Psi = i \frac{\partial}{\partial t} e^{S} \Psi$$

$$H' \Psi' = i \frac{\partial \Psi'}{\partial t} \tag{2.5b}$$

where the new hamiltonian H' is given by

$$H' = e^{S} H e^{-S} \tag{2.5c}$$

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[3] L. I. Schiff, Quantum Mechanics (McGraw-Hill Book Company, New York 1955) Sec. 43.

$$H' = \left[1 + S + \frac{S^2}{2!} + \ldots\right] H \left[1 - S + \frac{S^2}{2!} - \ldots\right]$$

$$H' = H + (S, H) + \frac{1}{2}(S, (S, H)) + \ldots \qquad (2.6a)$$

where the commutator of S and H is written as

$$(S, H) \equiv SH - HS$$

We choose

$$S = - \frac{\beta}{2M} O_1 \equiv \frac{\beta}{2M} \vec{\alpha}\cdot\vec{p} \qquad (2.7)$$

where $O_1 \equiv - \vec{\alpha}\cdot\vec{p}$ is the odd part in the hamiltonian (2.4). We evaluate H' in equation (2.6a) up to order $\frac{1}{M}$

$$H' = H + \left(- \frac{\beta}{2M} O_1, H\right) + \frac{1}{2!}\left(- \frac{\beta}{2M} O_1, - \alpha\cdot p\right) + \ldots \qquad (2.6b)$$

Using

$$\beta O_1 = - O_1\beta$$

$$\beta\vec{\alpha}\cdot\vec{p} = - \vec{\alpha}\cdot\vec{p}\beta$$

$$\vec{\alpha}\cdot\vec{p}\ \vec{\alpha}\cdot\vec{p} = \vec{p}^2$$

$$\beta^2 = 1$$

$$\left(- \frac{\beta}{2M} O_1, H\right) = \left(- \frac{\beta}{2M} O_1, - \beta M - \vec{\alpha}\cdot\vec{p}\right) = - O_1 - \frac{1}{M} \beta \vec{p}^2 \qquad (2.7a)$$

$$\frac{1}{2!}\left(- \frac{\beta}{2M} O_1, - \vec{\alpha}\cdot\vec{p}\right) = \frac{\beta}{2M} \vec{p}^2 \qquad (2.7b)$$

Substituting (2.7) in (2.6b)

$$H' = H - O_1 - \frac{\beta}{M} \vec{p}^2 + \frac{\beta}{2M} \vec{p}^2 + \ldots$$

$$H' = -\beta M - \vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p} - \frac{1}{2M}\beta\vec{p}^{\,2} + \ldots$$

$$H' = -\beta M - \frac{\beta}{2M}\vec{p}^{\,2} + \ldots \qquad (2.6c)$$

We get the correct nonrelativistic hamiltonian by substituting $\beta \to -1$ in (2.6c) as $\frac{p^2}{2M} + M$. To remove odd operators in H' up to order $\left(\frac{1}{M}\right)^2$, this transformation is applied taking $S = -\frac{\beta}{2M}$ (odd part in H'). By successive application of the Foldy-Wouthuysen transformation, odd operators in the hamiltonian can be removed to any order in $\left(\frac{1}{M}\right)$. For large M (say for a nucleon), the terms containing $\left(\frac{1}{M}\right)^2$ or of higher orders are very small. A similar prescription can be applied when the fields are present by considering

$$S = -\frac{\beta}{2M}\left[O_1 + \text{odd operator involving fields}\right]$$

In the following, this procedure is applied to remove the odd operators in nucleon space for a system of nucleon "source" coupled to the lepton (e-ν) field.

II. FOLDY-WOUTHUYSEN TRANSFORMATION FOR NUCLEAR BETA DECAY[4,5]

The total hamiltonain is then composed of three parts (1) the nuclear hamiltonian (H_N), (2) β interaction hamiltonian (H_β), and (3) the lepton hamiltonian.

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[4] M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1315 (1954).

[5] G. Alaga and B. Jaksic, Glasnik Mat-Fiz. i Astr. Tom 12, No. 1-2, (1957).

$$H = H_N + H_\beta + H_\ell \tag{2.8a}$$

In the space of the decaying nucleon, considering the nucleon obeying the Dirac equation, we have

$$H_N = -\beta[M_n\tau_+ + M_p\tau_-] - \vec{\alpha}\cdot\vec{p} + V$$

$$H_N = -\beta M - \vec{\alpha}\cdot\vec{p} - \tfrac{1}{2}\beta(M_n - M_p)\tau_z + V \tag{2.8b}$$

In the above M_n and M_p are the neutron mass and proton mass. The rationalized relativistic units ($\hbar = c = m_{electron} = 1$) are used. $M \simeq \frac{M_n + M_p}{2}$, and V represents the nuclear potential. In the following, β^- emission is considered.

$$H_{\beta^-} = g\sum_x \Omega_x(N)Q\cdot L(\Omega_x) \tag{2.8c}$$

$$L(\Omega_x) \equiv (\psi_e^*\Omega_x[C_x + C_x'\gamma_5]\psi_\nu) \tag{2.8d}$$

Ω_x in the nucleon space, for clarity is written as $\Omega_x(N)$. β^+ emission is obtained by hermitian conjugating H_{β^-} in (2.8c). We write (2.8b) and (2.8c) as

$$H_N = H_N(\text{even}) + O_1 \tag{2.9a}$$

$$H_{\beta^-} = H_{\beta^-}(\text{even}) + O_2 \tag{2.9b}$$

where the odd part of H_N: $O_1 \equiv -\vec{\alpha}\cdot\vec{p}$ (2.9c)

and the odd part of H_{β^-} is O_2.

H_N(even) and H_{β^-}(even) in (2.9) represent the even parts of H_N and H_{β^-}.

Taking

$$S = -\frac{\beta}{2M}(O_1+O_2) \qquad (2.10a)$$

$$S = -\frac{\beta}{2M}(-\alpha\cdot p + O_2) \quad , \qquad (2.10b)$$

The calculation of H', the transformed hamiltonian, can be easily done by using:

$$H' = H + (S, H) + \tfrac{1}{2}(S, (S, H)) + \ldots. \qquad (2.11a)$$

$$H' \equiv H_N' + H_\beta' + H' \qquad (2.11b)$$

To remove odd operators in (2.11) up to order $\frac{1}{M}$ in a consistent manner, the following are used:

(1) The terms containing $\frac{1}{M^2}$ or higher orders are neglected.

(2) The terms of second order in the coupling constants are ignored.

(3) The term $-\frac{1}{2}(M_n-M_p)\beta\,\tau_z$ does not contribute to H_β'.

(4) There is no contribution to H_β' arising from H_ℓ .

(5) The odd part of the nuclear force operator does not contribute.[6,7]

Making use of the above:

$$(S, H_N) = -\frac{1}{2M}(-\beta\vec{\alpha}\cdot\vec{p} + \beta O_2, -\beta M - \vec{\alpha}\cdot\vec{p} + V_e)$$

$$(S, H_N) = \vec{\alpha}\cdot\vec{p} - O_2 - \frac{\beta}{2M}\vec{p}^{\,2} + \frac{\beta}{2M}(O_2\vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p}O_2) \qquad (2.12a)$$

In the commutator $(S, (S, H_N)$ we take up to order $\frac{1}{M}$

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[6]Z. V. Chraplyvy, Phys. Rev. 91, 388 (1953).

[7]Z. V. Chraplyvy, Phys. Rev. 92, 1310 (1953).

$$(S, \vec{\alpha}\cdot\vec{p} - O_2) = \frac{\beta\vec{p}^2}{M} - \frac{\beta}{M}(O_2\vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p}O_2) \qquad (2.12b)$$

$(S, H_{\beta^-}\,(\text{even}))$ is an odd operator because S is odd operator and it is of order $\frac{1}{M}$. The only contributing terms from this commutator are of the order $(\frac{1}{M})^2$ and so are neglected.

$$(S, O_2) = -\frac{1}{2M}(-\beta\vec{\alpha}\cdot\vec{p} + \beta O_2, O_2)$$

$$(S, O_2) = \frac{1}{2M}(\beta\vec{\alpha}\cdot\vec{p}, O_2) - \frac{1}{2M}(\beta O_2, O_2)$$

$$(S, O_2) = \frac{\beta}{2M}(\vec{\alpha}\cdot\vec{p}O_2 + O_2\vec{\alpha}\cdot\vec{p}). \qquad (2.12c)$$

Substituting (2.12) in (2.11),

$$H' = H + \vec{\alpha}\cdot\vec{p} - O_2 - \frac{\beta}{2M}\vec{p}^2 + \frac{\beta}{2M}(O_2, \vec{\alpha}\cdot\vec{p})_+ + \frac{\beta\vec{p}^2}{2M} - \frac{\beta}{2M}(O_2, \vec{\alpha}\cdot\vec{p})_+$$

$$+ \frac{\beta}{2M}(O_2, \vec{\alpha}\cdot\vec{p})_+ + \text{terms } \left(\sim\frac{1}{M^2}\right) \text{ or higher}$$

$$H' = \left[-\beta M + V_e - \frac{\beta}{2M}\vec{p}^2\right] + \left[H_\beta(\text{even}) + \frac{\beta}{2M}(O_2, \vec{\alpha}\cdot\vec{p})_+\right] + H_\ell \qquad (2.11c)$$

$$H'_N \equiv -\beta M + V_e - \frac{\beta}{2M}\vec{p}^2 \qquad (2.13a)$$

$$H'_\beta \equiv H_\beta(\text{even}) + \frac{\beta}{2M}(O_2, \vec{\alpha}\cdot\vec{p})_+ \qquad (2.13b)$$

$$H'_\ell \simeq H_\ell \qquad (2.13c)$$

The anticommutator $(O_2, \vec{\alpha}\cdot\vec{p})_+ \equiv O_2\vec{\alpha}\cdot\vec{p} + \vec{\alpha}\cdot\vec{p}O_2$ and $O_2 \equiv$ odd part of the H_{β^-}. Using (2.13b) the odd operators in the vector, axial vector, and the pseudoscalar interactions will now be transformed into even operators.

III. BETA-DECAY OPERATORS IN THE NONRELATIVISTIC FORM

For β^- emission, the interaction hamiltonian for the vector, axial vector and pseudoscalar interactions is $H_{\beta^-} = H_V + H_A + H_P$

$$H_V = 1Q\cdot L(1) - \vec{\alpha}Q\cdot L(\vec{\alpha}) \tag{2.14a}$$

$$H_A = \vec{\sigma}\, Q\cdot L(\vec{\sigma}) - \gamma_5 Q\cdot L(\gamma_5) \tag{2.14b}$$

$$H_P = \beta\gamma_5 Q\cdot L(\beta\gamma_5) \tag{2.14c}$$

where the lepton covariant $L(\Omega_x) \equiv (\psi_e^* \Omega_x\, [C_x + C_x'\gamma_5]\, \psi_\nu)$. In the following, (as usually is done), we suppress the operator Q which converts a neutron into a proton. (2.13b) gives

$$H_{\beta^-}' = H_{\beta^-}(\text{even}) + \frac{\beta}{2M}(O_2, \vec{\alpha}\cdot\vec{p})_+$$

For the vector interaction $O_2 = -\vec{\alpha}\cdot L(\vec{\alpha})$

$$(-\vec{\alpha}\cdot L(\vec{\alpha}), \vec{\alpha}\cdot\vec{p})_+ = -\left[2L(\vec{\alpha})\cdot\vec{p} + \vec{p}\cdot L(\vec{\alpha}) + i\,\vec{\sigma}\cdot\vec{p} \times L(\vec{\alpha})\right]$$

$$\frac{\beta}{2M}(-\vec{\alpha}\cdot L(\vec{\alpha}), \vec{\alpha}\cdot\vec{p})_+ = -\frac{\beta}{2M}\left[2L(\vec{\alpha})\cdot\vec{p} + \vec{p}\cdot L(\vec{\alpha}) + i\,\vec{\sigma}\cdot\vec{p} \times L(\vec{\alpha})\right] \tag{2.15a}$$

For the axial vector $O_2 = -\gamma_5 L(\gamma_5)$

$$(-\gamma_5 L(\gamma_5), \vec{\alpha}\cdot\vec{p})_+ = -\left[2L(\gamma_5)\vec{\sigma}\cdot\vec{p} + \vec{\sigma}\cdot\vec{p}L(\gamma_5)\right]$$

$$\frac{\beta}{2M}(-\gamma_5 L(\gamma_5), \vec{\alpha}\cdot\vec{p})_+ = -\frac{\beta}{2M}\left[2L(\gamma_5)\vec{\sigma}\cdot\vec{p} + \vec{\sigma}\cdot\vec{p}L(\gamma_5)\right] \tag{2.15b}$$

For the pseudoscalar interaction: $O_2 = \beta\gamma_5 L(\beta\gamma_5)$

$$(\beta\gamma_5 L(\beta\gamma_5),\ \vec{\alpha}\cdot\vec{p})_+ = -\beta\,\vec{\sigma}\cdot\vec{p}\ L(\beta\gamma_5)$$

$$\frac{\beta}{2M}(\beta\gamma_5 L(\beta\gamma_5),\vec{\alpha}\cdot\vec{p})_+ = -\frac{1}{2M}\vec{\sigma}\cdot\vec{p}\ L(\beta\gamma_5) \qquad (2.15c)$$

Up to order $\left(\frac{1}{M}\right)$, H_β' is from (2.14) and (2.15)

$$H_{\beta^-}' = H_V' + H_A' + H_P'$$

where

$$H_V' = 1\cdot L(1) - \frac{\beta}{2M}\left[2L(\vec{\alpha})\cdot\vec{p} + \left\{\vec{p}\cdot L(\vec{\alpha}) + i\vec{\sigma}\cdot\vec{p}\times L(\vec{\alpha})\right\}\right] \qquad (2.16a)$$

$$H_A' = \vec{\sigma}\cdot L(\vec{\sigma}) - \frac{\beta}{2M}\left[2L(\gamma_5)\vec{\sigma}\cdot\vec{p} + \left\{\vec{\sigma}\cdot\vec{p}\ L(\gamma_5)\right\}\right] \qquad (2.16b)$$

$$H_P' = \qquad -\frac{1}{2M}\left[\quad 0 \quad + \left\{\vec{\sigma}\cdot\vec{p}\ L(\beta\gamma_5)\right\}\right] \qquad (2.16c)$$

In (2.16), the even operators are the same as in H_{β^-}. There are two types of terms which arise in H_{β^-}' by removing the odd operators up to order $\left(\frac{1}{M}\right)$: (1) the operator $\vec{p} \equiv -i\nabla$ acting on the nuclear wavefunction, e.g., the first terms in the square brackets of (2.16). (2) The operator $\vec{p} = -i\nabla$ acting on the lepton covariant--the terms in the curly brackets of (2.16).

From the above considerations, the following major points come to light:

(a) For the allowed beta transition; the results of using (2.16) and of

the conventional theory[8] are the same.

(b) The second type of terms, (involving the gradient on the lepton covariant), are generally very small compared to the leading terms in transitions, where the even operators (in H_β) contribute. In the vector and the axial vector interactions, essentially H_β' gives the same results as the conventional treatment[8] (H_β), provided the second type of terms are neglected. Only the nuclear matrix elements have different forms. To illustrate this, consider the contribution of the axial vector interaction to $0 \rightarrow 0$ (yes) transitions. In the new formulation (H_β'), there will be the contribution of three matrix elements due to interaction operators: (1) $\vec{\sigma} \cdot \vec{r}$, (2) $\frac{1}{M} \vec{\sigma} \cdot \vec{p}$, (3) $\frac{1}{M} \vec{\sigma} \cdot \vec{r}$. (3) is the contribution to the matrix elements owing to a gradient operator on the lepton covariant. In such a case, one can neglect the contribution of $\frac{1}{M} \vec{\sigma} \cdot \vec{r}$ compared to the contribution of (1) $\vec{\sigma} \cdot \vec{r}$ and (2) $\frac{1}{M} \vec{\sigma} \cdot \vec{p}$. In the conventional theory, the contribution to $0 \rightarrow 0$ (yes) transitions is due to $\vec{\sigma} \cdot \vec{r}$ and γ_5 (e.g., see Tables I and II). By doing an explicit calculation, one sees that the contribution of γ_5 is the same as of $\frac{1}{M} \vec{\sigma} \cdot \vec{p}$. Thus in the presently considered case of $0 \rightarrow 0$ (yes) transitions the conventional treatment and the new formulation give the same results* for the axial

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[8] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. 60, 308 (1941). Also D. L. Pursey, Phil. Mag. 42, 1193 (1951).

*See M. E. Rose and R. K. Osborn, Reference 1, Section III.

vector interaction.

(c) In the pseudoscalar interaction, the only operator in H_β' is $-\frac{1}{2M}\vec{\sigma}\cdot\vec{p}\ L(\beta\gamma_5)$, in which a gradient appears on the lepton covariant. If the contribution of this interaction operator is neglected, then there is no contribution from the pseudoscalar interaction. In the conventional treatment of the pseudoscalar interaction, the lepton covariant $L(\beta\gamma_5)$ is considered as a constant, i.e., independent of the parameters describing the nucleon. In the nonrelativistic limit, the contribution of the conventional treatment of the pseudoscalar interaction vanishes. Thus, in the nonrelativistic limit, whereas very small correction terms appear to the conventional treatment of the vector and axial vector interactions, completely different contributions of the pseudoscalar interaction arise in the two treatments. Hence, these considerations point out why the conventional treatment of the pseudoscalar interaction is wrong.[9] Therefore, to study the existence and the contribution of the pseudoscalar interaction, the correct form of the interaction operator $-\frac{1}{2M}\vec{\sigma}\cdot\vec{p}\ L(\beta\gamma_5)$ must be employed. In this work, this has been done.

IV. NOTATION AND REPRESENTATION

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[9] M. Deutsch and O. Kofoed-Hansen, Experimental Nuclear Physics III (John Wiley and Sons, Inc. New York, 1959) p. 515.

A. Representation of Dirac Equation

The rationalized relativistic units are used $\hbar = c = m_e = 1$ (m_e - mass of an electron). The Dirac equation for a β^- particle in a central field V_c is

$$(-\alpha\cdot p - \beta + V_c)\Psi = i\,\frac{\partial\Psi}{\partial t} \tag{2.17}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.18a}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.18b}$$

In (2.18a); $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The commutation relations are

$$\alpha_i\beta + \beta\alpha_i = 0 \tag{2.19a}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\,\delta_{ij} \tag{2.19b}$$

$V_c = \frac{-\alpha Z}{r}$ for the Coulomb field: $\alpha \approx \frac{1}{137}$ the fine structure constant. $Z \cong$ the number of protons in the daughter nucleus. The solution of (2.17), for stationary states, can be obtained by separating the equation in polar co-ordinates[10,11]

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[10] M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957) p. 152.

[11] M. E. Rose, Phys. Rev. 51, 484 (1937).

$$\psi_{\varkappa}^{\mu} = \begin{pmatrix} -i\, f_{\varkappa}(r) & \chi_{-\varkappa}^{\mu} \\ g_{\varkappa}(r) & \chi_{\varkappa}^{\mu} \end{pmatrix} \tag{2.20a}$$

where

$$\begin{aligned} j &= |\varkappa| - \tfrac{1}{2} & & \\ \ell &= \varkappa & &\text{if } \varkappa > 0 \\ \ell &= -(\varkappa + 1) & &\text{if } \varkappa < 0 \end{aligned} \tag{2.20b}$$

$$\chi_j^{\mu} = \sum_{\tau} C(\ell \tfrac{1}{2} j; \mu-\tau, \tau)\, Y_{\ell}^{\mu-\tau}\, \chi_{\frac{1}{2}}^{\tau} \tag{2.20c}$$

$Y_{\ell}^{\mu-\tau}$ is a spherical harmonic. $\chi_{\frac{1}{2}}^{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$C(\ell \tfrac{1}{2} j; \mu-\tau, \tau)$ is a Clebsch-Gordon coefficient*

$$C(\ell \tfrac{1}{2} j; \mu-\tau, \tau) \equiv (\ell \tfrac{1}{2} j, \mu | \ell \tfrac{1}{2}, \mu-\tau, \tau).$$

Also in the representation (2.20a).

$$\begin{aligned} \vec{J}^2\, \chi_{\varkappa}^{\mu} &= j(j+1)\, \chi_{\varkappa}^{\mu} \\ \vec{L}^2\, \chi_{\varkappa}^{\mu} &= \ell_{\varkappa}\, (\ell_{\varkappa} + 1)\, \chi_{\varkappa}^{\mu} \\ \vec{S}^2\, \chi_{\varkappa}^{\mu} &= 3/4\, \chi_{\varkappa}^{\mu} \\ J_z\, \chi_{\varkappa}^{\mu} &= \mu\, \chi_{\varkappa}^{\mu} \\ K\, \psi_{\varkappa}^{\mu} \equiv \beta(\vec{\sigma} \cdot \vec{L} + 1)\, \psi_{\varkappa}^{\mu} &= \varkappa\, \psi_{\varkappa}^{\mu} \end{aligned} \tag{2.21}$$

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*We follow the conventions and notation of the Clebsch-Gordon coefficients of M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957).

$f_\varkappa(r)$ and $g_\varkappa(r)$ obey the following coupled equations

$$\frac{d}{dr} f_\varkappa = \frac{\varkappa - 1}{r} f_\varkappa - (W-1-V_c) g_\varkappa \tag{2.22a}$$

$$\frac{d}{dr} g_\varkappa = (W+1-V_c) f_\varkappa - \frac{\varkappa + 1}{r} g_\varkappa \tag{2.22b}$$

In this work, we take

$$V_c = -\frac{\alpha Z}{r} \quad ; \qquad r > R \text{ (nuclear radius)}$$

$$V_c = -\frac{\alpha Z}{2R}\left(3 - \frac{r^2}{R^2}\right) \quad ; \qquad r < R$$

The computation of $f_\varkappa$ and $g_\varkappa$, which are required for the analysis of β spectrum and longitudinal polarization, was done on the ORACLE. $f_\varkappa$ and $g_\varkappa$ are real.

For the Coulomb field $V_c = -\frac{\alpha Z}{r}$ and the normalization corresponding to one particle in a sphere of radius R, f and g are given by

$$r \begin{Bmatrix} f_\varkappa \\ g_\varkappa \end{Bmatrix} = \frac{(1 \mp W)^{\frac{1}{2}} (2pr)^{\gamma} e^{\pi y/2} \left|\Gamma(\gamma + iy)\right|}{2(WR)^{\frac{1}{2}} \Gamma(2\gamma + 1)}$$

$$\times \left\{ e^{-ipr+i\eta_\varkappa} (\gamma+iy) F(\gamma+1+iy, 2\gamma+1, 2ipr) \mp \text{complex conjugate} \right\} \tag{2.22c}$$

$$W = (p^2 + 1)^{\frac{1}{2}} \tag{2.23a}$$

$$y \equiv \frac{\alpha Z W}{p} \tag{2.23b}$$

$$\gamma_\varkappa = (\varkappa^2 - \alpha^2 Z^2)^{\frac{1}{2}} \tag{2.23c}$$

$$e^{2i\eta_\kappa} = -\frac{\kappa - i\alpha Z/p}{\gamma + i\alpha ZW/p} \tag{2.23d}$$

$$F(a, b, x) \quad 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \ldots \tag{2.23e}$$

For the neutrino $V_c = 0$, we represent the radial functions of the neutrino* as F_κ and G_κ

$$F_\kappa = S_\kappa \; q \; j_{\ell(-\kappa)}(qr) \tag{2.24a}$$

$$G_\kappa = q \; j_{\ell(\kappa)}(qr) \tag{2.24b}$$

q is the momentum of the neutrino and S_κ represents the sign of κ. The spherical Bessel function $j_\ell(x) = \sqrt{\frac{\pi}{2x}} \; J_{\ell+\frac{1}{2}}(x)$

$$j_\ell(x) \equiv \frac{x^\ell}{(2\ell+1)!!} \sum_{n=0}^{\infty} (-)^n \frac{(2\ell+1)!!}{(2n!!)(2\ell+2n+1)!!} x^{2n}$$

For $x \ll 1$;

$$j_\ell(x) \simeq \frac{x^\ell}{(2\ell+1)!!} \tag{2.25}$$

B. Irreducible Tensors and The Wigner-Eckart Theorem

First we describe the spherical representation of a vector $\vec{V}$. The three components V_m (for m = 1, 0, -1) are

$$V_1 = -\frac{1}{2}(V_x + iV_y) \tag{2.26a}$$

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*We take the Dirac wavefunction for the neutrino in our calculations and we discuss the relation to the two-component neutrino in Section C of Chapter III.

$$V_o = V_z \tag{2.26b}$$

$$V_{-1} = \frac{1}{2}(V_x - iV_y) \tag{2.26c}$$

The advantage of the spherical representation is that V_m transforms under a three-dimensional rotation as the three components of the spherical harmonic of order 1 or as an irreducible tensor of rank 1.

Also

$$\vec{A}\cdot\vec{B} = \sum_m (-)^m A_{-m} B_m \tag{2.26d}$$

An irreducible tensor operator of rank L is defined as a set of (2L+1) functions T_L^M, $(M = -L, -L+1, \ldots, L)$ which transform under the (2L + 1)-dimensional representation of the rotation group*

$$R\, T_L^M R^{-1} = \sum_{M'} D^L_{M',M}(\alpha\beta\gamma)\, T_L^{M'} \tag{2.27}$$

Thus an operator is an irreducible tensor of rank L, if it transforms like the spherical harmonic of order L.

The most important advantage that accrues from the introduction of these irreducible tensors is that one can make use of the Wigner-Eckart theorem, which is:

$$(j'm' \mid T_L^M \mid jm) = C(jLj';m,M,m')(j \| T_L \| j) \tag{2.28}$$

The quantity $(j' \| T_L \| j)$ is called the reduced matrix element of the tensor operator T_L^M and it is independent of M, m and m', as the notation implies.

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*M. E. Rose, Reference 10, p. 76-106.

The conservation of angular momentum is contained in the Clebsch-Gordon coefficient $C(jLj',mMm')$, i.e., $|j - L| \leqslant j' \leqslant (j + L)$ in steps of unity. Generally this fact is expressed as $\Delta(jLj')$ and this notation will be frequently used throughout this work.

Throughout, the rationalized relativistic units ($\hbar = m_e = c = 1$) are used. In the following chapters, M represents the nucleon mass (~ 1800) in units of the electron mass, $\xi = \frac{C_P}{MC_A}$ and $\lambda = \frac{i\int\gamma_5}{\int\vec{\sigma}\cdot\vec{r}}$.

We introduce also the following notation:

(1) $\ell = \ell_\kappa$ and $\bar{\ell} = \ell_{-\kappa}$ for the electron.

(2) $\ell_\nu = \ell_{\kappa_\nu}$ and $\bar{\ell}_\nu = \ell_{-\kappa_\nu}$ for the neutrino.

CHAPTER III

FORMULATION OF THE PROBLEM

In Section I, the first order time-dependent perturbation theory is outlined and the probability amplitude of the electron, due to β^- interaction, is given at large distances from the decaying nucleon. This asymptotic form of the probability amplitude (outgoing wave) is used in the calculations of the β longitudinal polarization and β spectrum in $0 \rightarrow 0$ (yes) transitions, (Section II).

I. FIRST ORDER TIME-DEPENDENT PERTURBATION THEORY

We follow Rose, Biedenharn and Arfken[1] and use $\hbar = c = m_e = 1$. For the time-dependent perturbation $H_1 e^{-ikt} + H_1^* e^{ikt}$, the wave equation is

$$(H_o + H_1 e^{-ikt} + H_1^* e^{ikt}) \Psi(\vec{r},t) = i \frac{\partial \Psi(\vec{r},t)}{\partial t} \tag{3.1a}$$

H_o - is the unperturbed hamiltonian.

Introduce the Fourier transform of $\Psi(\vec{r}, t)$ as

$$\Psi(\vec{r}, W) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\vec{r}, t)\, e^{iWt}\, dt \tag{3.2a}$$

then

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[1]M. E. Rose, L. C. Biedenharn and G. B. Arfken, Phys. Rev. 85, 5 (1952).

$$\Psi(\vec{r},t) = \int_{-\infty}^{\infty} \Psi(r,W)\, e^{-iWt}\, dW \qquad (3.2b)$$

Multiplying (3.1a) with $\frac{1}{2\pi} e^{iWt}$ and integrating over t,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H_o e^{iWt}\, \Psi(\vec{r},t)dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} H_1 e^{i(W-k)t}\, \Psi(\vec{r},t)dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} H_1^* e^{i(W+k)t}\, \Psi(\vec{r},t)dt$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t}\Psi(\vec{r},t)\right] e^{iWt}\, dt \qquad (3.1b)$$

Using (3.2) and

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\partial \Psi(\vec{r},t)}{\partial t}\, e^{iWt}\, dt = \frac{W}{2\pi}\int_{-\infty}^{\infty} \Psi(\vec{r},t)\, e^{iWt}\, dt$$

(3.1b) becomes

$$H_o \Psi(\vec{r},W) + H_1 \Psi(\vec{r},W-k) + H_1^* \Psi(\vec{r},W+k) = W\, \Psi(\vec{r},W) \qquad (3.1c)$$

W is the final state energy. Energy conservation dictates that either the second or the third term in (3.1c) contributes to any transition caused by the perturbation. Therefore considering the second term as contributing, (3.1c) becomes

$$(H_o - W)\, \Psi(\vec{r},W) = -\, H_1 \Psi(\vec{r}, W-k) \qquad (3.1d)$$

(3.1d) is an exact equation. In the first approximation,

$$\Psi(\vec{r}, W-k) \rightarrow \Psi_i(r, W_i)$$

where the initial energy of the system $W_i = W - k$. Ψ_i is the wave

function describing the initial system, before the perturbation is "switched" on.

$$(H_o - W)\Psi(\vec{r}, W) = - H_1 \Psi_i(\vec{r}, W_i) \tag{3.3a}$$

For an electron in a Coulomb field $V_c = -\frac{\alpha Z}{r}$

$$(\vec{\alpha}\cdot\vec{p} + \beta - V_c + W)\Psi(r, W) = H_1 \Psi_i(r, W_i) \tag{3.3b}$$

we have taken*

$$H_o = -\vec{\alpha}\cdot\vec{p} - \beta + V_c$$

The equation (3.3b) can be solved by finding the Green function $G(\vec{r}, \vec{r}\,')$

$$(\vec{\alpha}\cdot\vec{p} + \beta - V_c + W)\, G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')1 \tag{3.4a}$$

1 is a 4 x 4 unit matrix. Then

$$\Psi(r, W) = \int d\vec{r}\,'\, G(\vec{r},\vec{r}\,')\, H_1(\vec{r}\,')\, \Psi_i(\vec{r}\,',W) \tag{3.3c}$$

It has been shown, in detail,**,[2] that

$$G(r,r') = -\pi i \sum_{\varkappa,\mu} \overline{\Phi}^{\mu}_{\varkappa}(\vec{r})\; \Phi^{\mu *}_{\varkappa}(\vec{r}\,') \quad \text{for } r > r' \tag{3.4b}$$

where

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*For notation, see Chapter II, Section IV, of this dissertation.

**Rose et al., loc. cit.

[2] M. E. Rose, Relativistic Electron Theory (to be published by John Wiley and Sons, New York), Section 34.

$$\overline{\Phi}_{\varkappa}^{\mu} = \begin{pmatrix} -i\,\overline{f}_{\varkappa} & \chi_{-\varkappa}^{\mu} \\ \overline{g}_{\varkappa} & \chi_{\varkappa}^{\mu} \end{pmatrix}$$

$\overline{f}_{\varkappa}$ and $g_{\varkappa}$ are such that $\overline{\Phi}_{\varkappa}^{\mu}$ represents asymptotically an outgoing spherical wave[3,4]

$$r\,\overline{f}_{\varkappa} \xrightarrow[r\to\infty]{} -\left[\frac{W-1}{\pi p}\right]^{\frac{1}{2}} \frac{e^{ipr+\delta_{\varkappa}}}{i}$$

$$r\,\overline{g}_{\varkappa} \xrightarrow[r\to\infty]{} \left[\frac{W+1}{\pi p}\right]^{\frac{1}{2}} e^{ipr+\delta_{\varkappa}}$$

where

$$\delta_{\varkappa} = \frac{\alpha Z W}{p}\log 2pr - \arg\Gamma\left(\gamma + i\,\frac{\alpha Z W}{p}\right) + \eta_{\varkappa} - \pi\,\frac{\gamma}{2} \qquad (3.4c)$$

$$\overline{\Phi}_{\varkappa}^{\mu} = e^{i\delta_{\varkappa}}\,\frac{e^{ipr}}{r}\begin{pmatrix} \left[\frac{W-1}{\pi p}\right]^{\frac{1}{2}} & \chi_{-\varkappa}^{\mu} \\ \left[\frac{W+1}{\pi p}\right]^{\frac{1}{2}} & \chi_{\varkappa}^{\mu} \end{pmatrix} \qquad (3.5a)$$

and

$$\Phi_{\varkappa}^{\mu} = \begin{pmatrix} -i\,f_{\varkappa} & \chi_{-\varkappa}^{\mu} \\ g_{\varkappa} & \chi_{\varkappa}^{\mu} \end{pmatrix} \qquad (2.20a)$$

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[3]For the radial current only, the outgoing wave contributes. E. Greuling and M. L. Meeks, Phys. Rev. 82, 531 (1951).

[4]For the nonrelativistic case, H. A. Bethe, Ann. Physik 4, 443 (1930).

$$\gamma_{\varkappa} = \left[\varkappa^2 - (\alpha Z)^2\right]^{\frac{1}{2}} \quad ; \quad e^{2i\eta_{\varkappa}} = - \frac{\varkappa - i\alpha Z/p}{\gamma + i\alpha ZW/p}$$

Substituting (3.4b) in (3.3c)

$$\Psi_{\infty}(\vec{r},W) = - \pi\, i \int d\vec{r}\,' \sum_{\varkappa,\mu} \bar{\Phi}^{\mu}_{\varkappa}(\vec{r})\; \Phi^{*\mu}_{\varkappa}(r')\, H_1(\vec{r}\,')\, \Psi_i(\vec{r}\,')$$

$$\Psi_{\infty}(\vec{r},W) = - \pi\, i \sum_{\varkappa,\mu} \bar{\Phi}^{\mu}_{\varkappa}(\vec{r}) \left\langle \Phi^{\mu}_{\varkappa}(\vec{r}\,') \middle| H_1(\vec{r}\,') \middle| \Psi_i(\vec{r}\,') \right\rangle \qquad (3.3d)$$

Substituting (3.5a) in (3.3d)

$$\Psi_{\infty}(r,W) = - i\, \pi^{\frac{1}{2}} \frac{e^{ipr}}{r} \sum_{\varkappa,\mu} e^{i\delta_{\varkappa}} \left\langle \Phi^{\mu}_{\varkappa}(\vec{r}\,') \middle| H_1(\vec{r}\,') \middle| \Psi_i(\vec{r}\,') \right\rangle \begin{pmatrix} \left[\frac{W-1}{p}\right]^{\frac{1}{2}} \chi^{\mu}_{-\varkappa} \\ \left[\frac{W+1}{p}\right]^{\frac{1}{2}} \chi^{\mu}_{\varkappa} \end{pmatrix}$$

(3.3e)

The spinor $\begin{pmatrix} \left[\frac{W-1}{p}\right]^{\frac{1}{2}} \chi^{\mu}_{-\varkappa} \\ \left[\frac{W+1}{p}\right]^{\frac{1}{2}} \chi^{\mu}_{\varkappa} \end{pmatrix}$ is an eigenfunction of the free particle Dirac hamiltonian $-\vec{\alpha}\cdot\vec{p} - \vec{\beta}$, with eigenvalue W. This can be easily checked by taking $\vec{p}$ along $\vec{r}$ and using (A.2)

$$\sigma_r\, \chi^{\mu}_{-\varkappa} = - \chi^{\mu}_{\varkappa}$$

In nuclear β^- decay, a neutron transforms into a proton with the emission of β^- and $\bar{\nu}$ (antineutrino).

$$n \rightarrow p + e^- + \bar{\nu} \qquad (3.6)$$

The electron is in the Coulomb field of the daughter nucleus. According to the Dirac "hole" theory, the creation of $\bar{\nu}$ is equivalent to the destruction of a neutrino in negative energy state. The nuclear β^- decay problem (3.6), therefore, can be considered as one in which ν in a negative energy state (representing the initial state Ψ_i) is absorbed by a neutron, due to β interaction, making it a proton, and an electron appears in the final state (as $\Phi_\varkappa^\mu$). In (3.3e), therefore for nuclear β^- decay

$$\langle \Phi_\varkappa^\mu(\vec{r}\,') \,|\, H_1 \,|\, \Psi_i(r') \rangle \rightarrow \langle \Psi_f \,|\, H_{\beta^-} \,|\, \Psi_i \rangle \tag{3.7}$$

Ψ_f and Ψ_i represent the nuclear final and initial states respectively. Substituting (3.7) in (3.3e),

$$\Psi_\infty(\vec{r},W) = -\,i\,\pi^{\frac{1}{2}}\,\frac{e^{ipr}}{r}\sum_{\varkappa}\sum_{\mu} e^{i\delta_\varkappa}\langle \Psi_f \,|\, H_{\beta^-} \,|\, \Psi_i \rangle \begin{pmatrix} \left[\frac{W-1}{p}\right]^{\frac{1}{2}} \chi_{-\varkappa}^{\mu}(\hat{r}) \\ \left[\frac{W+1}{p}\right]^{\frac{1}{2}} \chi_{\varkappa}^{\mu}(\hat{r}) \end{pmatrix} \tag{3.8a}$$

After a brief discussion of the polarization operators, this asymptotic form of $\Psi(\vec{r},W)$ is used, in the next section, for calculating the β longitudinal polarization and spectrum in $0 \rightarrow 0$ (yes) transitions.

II. BETA LONGITUDINAL POLARIZATION IN NUCLEAR BETA DECAY

The breakdown of parity and charge conjugation symmetry laws are

now well established in nuclear beta decay. This implies[5] the existence of the pseudoscalar quantities[6,7,8] in the processes of nuclear β decay. One such pseudoscalar quantity is the longitudinal polarization ($\vec{S}\cdot\vec{p}$) of β particles[9,10] from unoriented nuclei.

A. Polarization Operator for Electrons

The covariant description of the spin of an electron has been given by Michel and Wightman[11] and also by Tolhoek.[12]

An operator $Q(n) = i\ \gamma_5 \gamma_\mu n_\mu$ is considered for the description of the spin. $(\overline{\Psi} Q(n) \Psi)$ transforms like a pseudoscalar quantity. n_μ is

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[5] C. S. Wu, E. Ambler, R. Hayward, D. D. Hoppes and R. P. Hudson, Phys. Rev. 105, 1413 (1957). In this experiment, the angular symmetry of β^- particles from oriented Co^{60} nucleus ($5^+ \rightarrow 4^+$) was observed. $\langle J \rangle \cdot \vec{p}$ is a pseudoscalar quantity because $\langle J \rangle$ - the average value of nuclear spin is an axial vector and $\vec{p}$ - the momentum is a polar vector. The experimental observation of $\langle J \rangle \cdot \vec{p}$ proved the nonconservation of parity in nuclear β decay.

[6] T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956).

[7] T. D. Lee, Conservation Laws in Weak Interactions (Unpublished). Lecture Notes at Harvard University, March 1957.

[8] T. D. Lee and C. N. Yang, Elementary Particles and Weak Interactions (Brookhaven National Laboratory 1957) B.N.L. 443 (T-91).

[9] J. D. Jackson, The Physics of Elementary Particles (Princeton University Press, New Jersey, 1958) p. 91

[10] J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr., Phys. Rev. 106, 517 (1957).

[11] L. Michel and A. S. Wightman, Phys. Rev. 98, 1190 (1955). Also see C. Bouchiat and L. Michel, Nuclear Physics 5, 416 (1958).

[12] H. A. Tolhoek, Revs. Mod. Phys. 28, 277 (1956).

a 4-vector and is $(\hat{n}, 0)$ in the rest system of the electron. One defines another 4-vector p_μ which is $(0, i)$ in the rest system of the electron.

Clearly in the rest frame of the electron $n_\mu p_\mu = 0$. Since it is a scalar product of two 4-vectors, therefore $n_\mu p_\mu = 0$ in any Lorentz reference frame, therefore only three parameters are required.

Using $\gamma_k = - i\beta\vec{\alpha}_k$; $\gamma_4 = - \beta$ and $\gamma_5\vec{\alpha} = \vec{\sigma}$

$$Q(n) = i\ \gamma_5\gamma_\mu n_\mu = - \beta\ \vec{\sigma}\cdot n - i\ \gamma_5\beta n_4 \tag{3.9}$$

In the rest frame of the electron, $\beta \rightarrow -1$ and $n_4 = 0$

$$Q(n) \rightarrow \vec{\sigma}\cdot\hat{n}$$

Thus $Q(n)$ gives the polarization operator in the rest system of the electron, if we take $\hat{n}$ along the spin direction. However, for the purpose of calculations* Rose introduced the polarization operator $\vec{\Theta}$ as

$$\vec{\Theta} = \vec{\sigma}\cdot\hat{e}_1\,\hat{e}_1 - \beta\,\vec{\sigma}\cdot\hat{e}_2\,\hat{e}_2 - \beta\,\vec{\sigma}\cdot\hat{e}_3\,\hat{e}_3 \tag{3.10a}$$

where $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ form an orthogonal right-handed set of unit vectors and $\hat{e}_1 \equiv \hat{p}$ (unit vector in the direction of momentum). First we show that $\vec{\Theta}$, indeed, reduces to the correct polarization operator in the rest frame of the electron and then we list its important properties.

$\vec{\sigma}$ can be written as

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*M. E. Rose, Reference 2, Sections 19 and 20. We use the representation in which Dirac hamiltonian for free particle is $-\vec{\alpha}\cdot\vec{p}-\beta$ and $\psi = \begin{pmatrix} v \\ u \end{pmatrix}$ which in the nonrelativistic limit goes to $\begin{pmatrix} 0 \\ u \end{pmatrix}$. The results, in the representation used in this book ($H_o = \vec{\alpha}\cdot\vec{p}+\beta$), can be converted in our representation by changing $\vec{\alpha} \rightarrow -\vec{\alpha}^o$ and $\beta \rightarrow -\beta$.

$$\vec{\sigma} = \vec{\sigma}\cdot\hat{e}_1\,\hat{e}_1 + \vec{\sigma}\cdot\hat{e}_2\,\hat{e}_2 + \vec{\sigma}\cdot\hat{e}_3\,\hat{e}_3$$

or

$$\beta\vec{\sigma} = \beta\vec{\sigma}\cdot\hat{e}_1\,\hat{e}_1 + \beta\vec{\sigma}\cdot\hat{e}_2\,\hat{e}_2 + \beta\vec{\sigma}\cdot\hat{e}_3\,\hat{e}_3$$

Substituting for $\beta\vec{\sigma}\cdot\hat{e}_2\,\hat{e}_2 + \beta\vec{\sigma}\cdot\hat{e}_3\,\hat{e}_3$ in (3.10a),

$$\vec{\Theta} = \vec{\sigma}\cdot\hat{e}_1\,\hat{e}_1 - \beta\vec{\sigma} + \beta\vec{\sigma}\cdot\hat{e}_1\,\hat{e}_1 \tag{3.10b}$$

In the rest system $\beta \rightarrow -1$,

$$\vec{\Theta} \rightarrow \vec{\sigma} \tag{3.10c}$$

The following are listed some useful relations[13] involving $\vec{\Theta}$:

(1) For positive energy states; $(i\gamma_5\gamma_\mu n_\mu - \vec{\Theta}\cdot\hat{n})$ acts as a null operator and as such they are equivalent.

(2) Each component of $\vec{\Theta}$ on the unit vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ commutes with the hamiltonian of the free particle.

$$(\vec{\Theta}\cdot\hat{e}_j,\ -\vec{\alpha}\cdot\vec{p} - \beta) = 0 \quad ; \quad j = 1, 2, 3$$

(3) $(\vec{\Theta}\cdot\hat{e}_k, \vec{\Theta}\cdot\hat{e}_\ell) = 2i\,\varepsilon_{k\ell m}\Theta_m$; (k, ℓ and m cyclic permutation)

(4) $(\vec{\Theta}\cdot\hat{e}_j)^2 = 1$

(5) The polarization "vector" is given by[14]

$$\vec{P} = \frac{\mathcal{S}(\Psi_\infty, \vec{\Theta}\,\Psi_\infty)}{\mathcal{S}(\Psi_\infty, \Psi_\infty)} \tag{3.12}$$

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[13] R. H. Good, Jr. and M. E. Rose, Nuovo Cimento 14, 872 (1959).

[14] See for the application to the polarization of conversion electrons following beta decay, R. L. Becker and M. E. Rose, Nuovo Cimento 13, 1182 (1959).

where $\mathcal{S}$ implies summation over all the observables not observed. For longitudinal polarization, we need only consider $\vec{\sigma}\cdot\hat{p}$. For Ψ_∞, $\hat{p}$ is in the same direction $\hat{r}$.

B. Beta Longitudinal Polarization in 0 → 0 (yes) Transitions

The longitudinal polarization ($P_{\parallel}$) of β particles is given by

$$P_{\parallel} = \frac{\langle(\Psi_\infty,\ \vec{\sigma}\cdot\hat{r}\ \Psi_\infty)\rangle}{\langle(\Psi_\infty,\ \Psi_\infty)\rangle} \tag{3.12a}$$

where the unit vector $\hat{r}$ is in the direction of the momentum of the β particle. The round brackets indicate the scalar product with respect to the spinor indices only. The angular brackets in (3.12a) denote (1) the summation over all observables ($\varkappa_\nu$, μ_ν) of the neutrino (not observed) and (2) the average over the magnetic substates of the initial and final nuclear states. In 0 → 0 (yes) beta transitions, $M_i = M_f = 0$, so the averaging is trivial and it gives unity. The differential energy β spectrum is given by $\frac{4}{\pi} r^2 \langle(\Psi_\infty,\ \Psi_\infty)\rangle$

$$\Psi_\infty = -i\,\pi^{\frac{1}{2}}\,\frac{e^{ipr}}{r}\sum_{\varkappa,\mu} e^{i\delta_\varkappa}\langle\Psi_f|H_\beta{-}|\Psi_i\rangle\begin{pmatrix}\left[\frac{W-1}{p}\right]^{\frac{1}{2}}\chi^{\mu}_{-\varkappa}(\hat{r})\\ \left[\frac{W+1}{p}\right]^{\frac{1}{2}}\chi^{\mu}_{\varkappa}(\hat{r})\end{pmatrix} \tag{3.8a}$$

The vector interaction does not contribute* to 0 → 0 (yes) transitions. $\langle \Psi_f | H_{\beta^-} | \Psi_i \rangle$ is worked out in (B.2.7) of Appendix B for the axial vector and pseudoscalar interaction. We have used the conventional form of the axial vector and the correct formulation of the pseudoscalar interaction.[15] For 0 → 0 (yes) transitions, (Appendix B, equation (B.27))

$$\langle \Psi_f | H_{\beta^-} | \Psi_i \rangle$$

$$= \frac{1}{4\pi} (-)^{\mu+\ell+j} \delta_{\mu,-\mu_\nu} \Big\{ (iC_A \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C'_A \delta_{\varkappa,-\varkappa_\nu})$$

$$\times \Big[[6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1\, \ell; 00)\, W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \int \vec{\sigma} \cdot \hat{r} + (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) i \int \gamma_5 \Big]$$

$$+ \Big(i \frac{C_P}{2M} \delta_{\varkappa,\varkappa_\nu} - S_\varkappa \frac{C'_P}{2M} \delta_{\varkappa,-\varkappa_\nu}\Big) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) \int \vec{\sigma} \cdot \hat{r} \Big\} \tag{3.13a}$$

$S_\varkappa$ is the sign of $\varkappa$ and M is the nucleon mass in units of the electron mass.

$$\ell \equiv \ell_\varkappa \quad ; \quad \bar{\ell} \equiv \ell_{-\varkappa}$$

$\delta_{\varkappa,\varkappa_\nu}$ and $\delta_{\mu,-\mu_\nu}$ are Kronecker deltas. $\int \vec{\sigma} \cdot \hat{r}$ and $\int \gamma_5$ are reduced matrix elements and are independent of magnetic quantum numbers and in the theory of nuclear β decay, are considered as parameters. Since

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*This is an exact statement. In the nucleon space, the even operator of the vector interaction is 1 and for the parity change, $\vec{r}$ must occur at least once (or an odd number of times). But $\vec{r}$, being an irreducible tensor of rank 1, cannot make a Δ (0 1 0). Similarly $\vec{\alpha}$ - the odd operator of the vector interaction cannot make a Δ (0 1 0) and there is no combination of $\vec{\alpha}$ and $\vec{r}$ which can contribute to 0 → 0 (yes) transition. Also this can be seen from Tables I and II.

[15] M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1315 (1954). Also see Chapter II.

$\langle \Psi_f | H_{\beta^-} | \Psi_i \rangle$ depends on $\varkappa$, μ (of the electron) and $\varkappa_\nu$, μ_ν (of the neutrino) quantum numbers, it is convenient, for calculations, to define $\mathcal{J}(\varkappa, \varkappa_\nu)$ as

$$\langle \Psi_f | H_{\beta^-} | \Psi_i \rangle \equiv \frac{1}{4\pi} (-)^{\mu+\ell+j} \mathcal{J}(\varkappa, \varkappa_\nu) \delta_{\mu,-\mu_\nu} \qquad (3.13b)$$

where

$$\mathcal{J}(\varkappa, \varkappa_\nu) \equiv (iC_A \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C'_A \delta_{\varkappa,-\varkappa_\nu}) \Big[[6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1 \ell ; 00)$$

$$W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \int \vec{\sigma} \cdot \hat{r} + (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\, i \int \gamma_5 \Big]$$

$$+ \left(i \frac{C_P}{2M} \delta_{\varkappa,\varkappa_\nu} - S_\varkappa \frac{C'_P}{2M} \delta_{\varkappa,-\varkappa_\nu}\right) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) \int \vec{\sigma} \cdot \hat{r} \qquad (3.13c)$$

Substituting (3.13b) in (3.8a)

$$\Psi_\infty = -i \frac{\pi^{\frac{1}{2}}}{4\pi} \frac{e^{ipr}}{r} \sum_\varkappa \sum_\mu e^{i\delta_\varkappa} (-)^{\mu+\ell+j} \delta_{\mu,-\mu_\nu} \mathcal{J}(\varkappa, \varkappa_\nu) \begin{pmatrix} \left[\frac{W-1}{p}\right]^{\frac{1}{2}} \chi^{\mu}_{-\varkappa}(\hat{r}) \\ \left[\frac{W+1}{p}\right]^{\frac{1}{2}} \chi^{\mu}_{\varkappa}(\hat{r}) \end{pmatrix}$$

(3.8b)

Substituting (3.8b) in (3.12a) and cancelling the common factor $\frac{1}{16\pi} r^2$, we get for the β longitudinal polarization in $0 \to 0$ (yes) transitions

$$P_{\parallel} = \frac{N}{D} \qquad (3.12b)$$

where

$$N = -\frac{2W}{p} \sum_{\varkappa_\nu, \varkappa, \varkappa'} \sum_{\mu_\nu, \mu, \mu'} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{\mu - \mu' + j - j'} (-)^{\ell + \ell'}$$

$$\times \mathcal{J}^*(\varkappa', \varkappa_\nu)\, \mathcal{J}(\varkappa, \varkappa_\nu)\, \delta_{\mu, -\mu_\nu} \delta_{\mu', -\mu_\nu} (\chi^{\mu'}_{\varkappa'}, \chi^{\mu}_{-\varkappa}) \qquad (3.14b)$$

and

$$D = \sum_{\varkappa_\nu, \varkappa, \varkappa'} \sum_{\mu_\nu, \mu, \mu'} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{\mu - \mu' + j - j'} (-)^{\ell + \ell'}$$

$$\times \mathcal{J}^*(\varkappa', \varkappa_\nu)\, \mathcal{J}(\varkappa, \varkappa_\nu)\, \delta_{\mu', -\mu_\nu} \delta_{\mu, -\mu_\nu} (\chi^{\mu'}_{\varkappa'}, \chi^{\mu}_{\varkappa}) \qquad (3.15b)$$

In obtaining the above equations, we have used the following relations:

(1) $(\chi^{\mu'}_{-\varkappa'}, \sigma_r \chi^{\mu}_{-\varkappa}) = -(\chi^{\mu'}_{\varkappa'}, \chi^{\mu}_{-\varkappa})$

(2) $(\chi^{\mu'}_{\varkappa'}, \sigma_r \chi^{\mu}_{\varkappa}) = -(\chi^{\mu'}_{\varkappa'}, \chi^{\mu}_{-\varkappa})$

(3) $(\chi^{\mu'}_{-\varkappa'}, \chi^{\mu}_{-\varkappa}) = (\chi^{\mu'}_{\varkappa'}, \chi^{\mu}_{\varkappa})$

and the fact that $\mu' + j'$ is an integer.

The differential energy spectrum $N(W)$ is, then, given by

$$N(W) = \frac{4}{\pi} r^2 (\Psi_\infty , \Psi_\infty) \tag{3.11a}$$

$$N(W) = \frac{1}{4\pi^2} D \tag{3.11b}$$

The β longitudinal polarization ($P_{\parallel}$) in $0 \rightarrow 0$ (yes) transitions, is given by

$$P_{\parallel} = \frac{N}{D}.$$

Thus we wish to calculate N and D as given in (3.14b) and (3.15b) respectively. In the following pages, we show the details of these calculations and the resulting expressions for N and D are given in (3.14f) and (3.15f) on pages 66 and 67 respectively.

These results are simplified by using the assumptions of time-reversal invariance in the weak (β decay) interaction and in the strong (nuclear) interaction. N and D are given on pages 83 and 84 respectively.

The formulas of the β longitudinal polarization and the β spectrum in $0 \rightarrow 0$ (yes) beta transitions are, then, given on pages 89 and 90 respectively; we assume the validity of the two-component theory of the neutrino.

Making use of $\delta_{\mu,-\mu_\nu}$ and $\delta_{\mu',-\mu_\nu'}$

$$N = -\frac{2W}{p} \sum_{\varkappa_\nu, \varkappa, \varkappa'} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{j-j'} (-)^{\ell+\ell'} \mathcal{J}^*(\varkappa', \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\times \sum_{\mu_\nu} (\chi_{\varkappa'}^{-\mu_\nu}, \chi_{-\varkappa}^{-\mu_\nu})$$

Changing the summation letter μ_ν to μ

$$N = -\frac{2W}{p} \sum_{\varkappa_\nu, \varkappa', \varkappa} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{j-j'+\ell+\ell'} \mathcal{J}^*(\varkappa', \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\times \sum_{\mu} (\chi_{\varkappa'}^{\mu}, \chi_{-\varkappa}^{\mu}) \tag{3.14c}$$

Similarly

$$D = \frac{2W}{p} \sum_{\varkappa_\nu, \varkappa, \varkappa'} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{j-j'+\ell+\ell'} \mathcal{J}^*(\varkappa', \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\times \sum_{\mu} (\chi_{\varkappa'}^{\mu}, \chi_{\varkappa}^{\mu}) \tag{3.15c}$$

In (A.7) of Appendix A:

$$(\chi_{\varkappa'}^{\mu}, \chi_{\varkappa}^{\mu}) = \sum_{\lambda} (-)^{\mu+\frac{1}{2}} \left[\frac{(2\ell+1)(2\ell'+1)(2j+1)(2j'+1)}{4\pi(2\lambda+1)} \right]^{\frac{1}{2}} C(\ell'\ell\lambda;00)$$

$$\times C(j'j\lambda;-\mu,\mu)\, Y_\lambda^0(\hat{r})\, W(j'j\ell'\ell;\lambda\tfrac{1}{2})$$

Substituting for $(\chi^{\mu}_{\varkappa'}, \chi^{\mu}_{-\varkappa})$ in (3.14c), we get

$$N = -\frac{2W}{p} \sum_{\varkappa_\nu, \varkappa, \varkappa'} \sum_{\lambda} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{j-j'+\ell+\ell'} \mathcal{J}^*(\varkappa', \varkappa_\nu)\mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\times C(\ell'\bar{\ell}\lambda;00)\left[\frac{(2\bar{\ell}+1)(2\ell'+1)(2j+1)(2j'+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} Y^0_\lambda(\hat{r})$$

$$\times W(j'j\,\ell'\bar{\ell};\lambda\tfrac{1}{2}) \sum_{\mu} (-)^{\mu+\frac{1}{2}} C(j'j\lambda;-\mu,\mu) \qquad (3.14d)$$

Now noting that $C(j'j0;-\mu,\mu) = (-)^{j'+\mu}\left[2j+1\right]^{-\frac{1}{2}}$

$$\sum_{\mu}(-)^{\mu+\frac{1}{2}} C(j'j\lambda;-\mu,\mu) = \sum_{\mu}(-)^{\frac{1}{2}-j'}\left[2j+1\right]^{\frac{1}{2}} C(j'j0;-\mu,\mu)\, C(j'j\lambda;-\mu,\mu)$$

$$\sum_{\mu}(-)^{\mu+\frac{1}{2}} C(j'j\lambda;-\mu,\mu) = (-)^{\frac{1}{2}-j'}\left[2j+1\right]^{\frac{1}{2}} \delta_{\lambda,0} \qquad (3.16a)$$

Substituting the above in (3.14d), and summing over λ

$$N = -\frac{2W}{p} \sum_{\varkappa_\nu, \varkappa, \varkappa'} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{j-j'+\ell+\ell'} (-)^{\frac{1}{2}-j'} (2j+1)^{\frac{1}{2}} \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$$

$$\left[\frac{(2\bar{\ell}+1)(2\ell'+1)(2j+1)(2j'+1)}{4\pi}\right]^{\frac{1}{2}} C(\ell'\bar{\ell}\,0;00)\, \mathcal{J}^*(\varkappa', \varkappa_\nu)\, \mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\times \quad W(j'j\,\ell'\bar{\ell};0\tfrac{1}{2})$$

Using (A.1b) and (A.8f), we get

$$C(\ell'\bar{\ell}\,0;00) = \delta_{\varkappa,-\varkappa'}\, C(\bar{\ell}\bar{\ell}\,0;00)$$

$$C(\ell'\bar{\ell}\,0;00) = \delta_{\varkappa,-\varkappa'}\,(-)^{\bar{\ell}}\,[2\bar{\ell}+1]^{-\frac{1}{2}}$$

$$W(j'j\,\ell'\bar{\ell};0\tfrac{1}{2}) = \frac{(-)^{\frac{1}{2}-j-\bar{\ell}}\,\delta_{j,j'}\,\delta_{\ell',\bar{\ell}}}{[(2j+1)(2\bar{\ell}+1)]^{\frac{1}{2}}}$$

Substituting the above relations and simplifying, we obtain

$$N = \frac{W}{2\pi p}\sum_{\varkappa_\nu,\varkappa} e^{i(\delta_\varkappa - \delta_{-\varkappa})}\,(2j+1)\,\mathcal{J}^*(\varkappa' = -\varkappa,\,\varkappa_\nu)\,\mathcal{J}(\varkappa,\,\varkappa_\nu)$$

or

$$N = \frac{W}{2\pi p}\sum_{\varkappa_\nu,\varkappa} e^{i(\delta_\varkappa - \delta_{-\varkappa})}\,(2j+1)\,\mathcal{J}^*(-\varkappa,\,\varkappa_\nu)\,\mathcal{J}(\varkappa,\,\varkappa_\nu) \qquad (3.14e)$$

Now substituting $(\chi^{\mu}_{\varkappa'},\,\chi^{\mu}_{\varkappa})$ in (3.15c)

$$D = \frac{2W}{p}\sum_{\varkappa_\nu,\varkappa,\varkappa'} e^{i(\delta_\varkappa - \delta_{\varkappa'})}\,(-)^{j-j'+\ell+\ell'}\,\mathcal{J}^*(\varkappa',\,\varkappa_\nu)\,\mathcal{J}(\varkappa,\,\varkappa_\nu)$$

$$\times\sum_\lambda C(\ell'\ell\,\lambda;00)\left[\frac{(2\ell+1)(2\ell'+1)(2j+1)(2j'+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} Y^0_\lambda(\hat{r})\,W(j'j\,\ell'\ell;\lambda\tfrac{1}{2})$$

$$\times\sum_\mu (-)^{\mu+\frac{1}{2}}\,C(j'j\lambda;-\mu,\mu)$$

Using

$$\sum_\mu (-)^{\mu+\frac{1}{2}}\,C(j'j\lambda;-\mu,\mu) = (-)^{\frac{1}{2}-j'}\,\delta_{\lambda,0}\,[2j+1]^{\frac{1}{2}} \qquad (3.16a)$$

and summing over λ, we get

$$D = \frac{2W}{p} \sum_{\varkappa_\nu, \varkappa, \varkappa'} e^{i(\delta_\varkappa - \delta_{\varkappa'})} (-)^{j-j'+\ell+\ell'} \mathcal{J}^*(\varkappa', \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\times C(\ell'\ell 0;00) \left[\frac{(2\ell+1)(2\ell'+1)(2j+1)(2j'+1)}{4\pi}\right]^{\frac{1}{2}} \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} W(j'j\ell'\ell;0\tfrac{1}{2})$$

$$\times (-)^{\frac{1}{2}-j'} \left[2j+1\right]^{\frac{1}{2}} \tag{3.15d}$$

Using in (3.15d)

$$C(\ell'\ell 0;00) = \delta_{\varkappa, \varkappa'} C(\ell\ell 0;00) = (-)^{\ell} \delta_{\varkappa, \varkappa'} \left[2\ell+1\right]^{-\frac{1}{2}}$$

$$W(j'j\ell'\ell;0\tfrac{1}{2}) = \frac{\delta_{j,j'} \delta_{\ell',\ell} (-)^{\frac{1}{2}-j-\ell}}{\left[(2j+1)(2\ell+1)\right]^{\frac{1}{2}}}$$

For $\varkappa = \varkappa'$; $j = j'$; and summing over $\varkappa'$ gives

$$D = \frac{2W}{p} \sum_{\varkappa_\nu, \varkappa = \varkappa'} (-)^{\frac{1}{2}-j} (-)^{\frac{1}{2}-j-\ell} (-)^{\ell} \mathcal{J}^*(\varkappa'=\varkappa, \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu)$$

$$\frac{(2\ell+1)(2j+1)}{4\pi} \left[(2j+1)(2\ell+1)\right]^{-\frac{1}{2}} \left[2\ell+1\right]^{-\frac{1}{2}} \left[2j+1\right]^{\frac{1}{2}}$$

$$D = \frac{W}{2\pi p} \sum_{\varkappa_\nu, \varkappa} (2j+1) \mathcal{J}^*(\varkappa, \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu) \tag{3.15e}$$

$$P_{\parallel} = \frac{N}{D} \tag{3.12b}$$

where

N and D are given in (3.14e) and (3.15e) respectively. Now we define

$$\mathcal{J}_1(\varkappa) \equiv \sum_{\varkappa_\nu} \mathcal{J}^*(-\varkappa, \varkappa_\nu)\,\mathcal{J}(\varkappa, \varkappa_\nu) \tag{3.17a}$$

$$\begin{aligned}
\mathcal{J}_1(\varkappa) = \sum_{\varkappa_\nu} \Bigg\{ & \Big(-i\, C_A^* \delta_{-\varkappa,\varkappa_\nu} + S_\varkappa C_A'^* \delta_{\varkappa,\varkappa_\nu}\Big)\Big[\,[6(2\ell+1)]^{\frac{1}{2}}\, C(\ell 1 \bar{\ell};00) \\
& \times W(\ell 1 j \tfrac{1}{2}; \bar{\ell} \tfrac{1}{2})(f_{-\varkappa} G_{-\varkappa} + g_{-\varkappa} F_{-\varkappa})\Big(\int \vec{\sigma}\cdot\hat{r}\Big)^* + (f_{-\varkappa} F_{-\varkappa} - g_{-\varkappa} G_{-\varkappa})\Big(i\int \gamma_5\Big)^*\Big] \\
& + \Big(-i\, \frac{C_P^*}{2M}\delta_{-\varkappa,\varkappa_\nu} + S_\varkappa \frac{C_P'^*}{2M}\delta_{\varkappa,\varkappa_\nu}\Big)\frac{d}{dr}(f_{-\varkappa}F_{-\varkappa} + g_{-\varkappa}G_{-\varkappa})\Big(\int \vec{\sigma}\cdot\hat{r}\Big)^* \Bigg\} \\
\times \Bigg\{ & \Big(i\, C_A \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_A' \delta_{\varkappa,-\varkappa_\nu}\Big)\Big[\,[6(2\bar{\ell}+1)]^{\frac{1}{2}}\, C(\bar{\ell} 1 \ell;00) \\
& W(\bar{\ell} 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)\int \vec{\sigma}\cdot\hat{r} + (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\Big(i\int\gamma_5\Big)\Big] \\
& + \Big(i\, \frac{C_P}{2M}\delta_{\varkappa,\varkappa_\nu} - S_\varkappa \frac{C_P'}{2M}\delta_{\varkappa,-\varkappa_\nu}\Big)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\Big(\int\vec{\sigma}\cdot\hat{r}\Big) \Bigg\}
\end{aligned} \tag{3.17b}$$

We have for the radial functions of the neutrino,

$$F_{-\varkappa} = -S_\varkappa\, q\, j_\ell(qr) = -S_\varkappa G_\varkappa \tag{3.18a}$$

$$G_{-\varkappa} = q\, j_\ell(qr) = S_\varkappa F_\varkappa \tag{3.18b}$$

Using (3.18a) and (3.18b), we get

$$f_{-\varkappa} G_{-\varkappa} + g_{-\varkappa} F_{-\varkappa} = S_\varkappa (f_{-\varkappa} F_\varkappa - g_{-\varkappa} G_\varkappa) \tag{3.18c}$$

$$f_{-\varkappa} F_{-\varkappa} - g_{-\varkappa} G_{-\varkappa} = -S_\varkappa (f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa) \tag{3.18d}$$

$$f_{-\varkappa}F_{-\varkappa} + g_{-\varkappa}G_{-\varkappa} = S_{\varkappa}(-f_{\varkappa}G_{\varkappa} + g_{-\varkappa}F_{\varkappa}) \tag{3.18e}$$

Also from (A.1c) and (A.8d)

$$C(\ell 1 \bar{\ell};00) = -\left(\frac{2\bar{\ell}+1}{2\ell+1}\right)^{\frac{1}{2}} C(\bar{\ell} 1 \ell;00) \tag{3.16b}$$

$$W(\ell 1 j\tfrac{1}{2}; \bar{\ell}\tfrac{1}{2}) = -W(\bar{\ell} 1 j\tfrac{1}{2}; \ell\tfrac{1}{2}) \tag{3.16c}$$

Since $\ell + \bar{\ell}$ is odd integer.

Substituting (3.18) and (3.16b) and (3.16c) in (3.17b), we get

$$\mathcal{J}_1(\varkappa) = \sum_{\varkappa_\nu} \Big\{ S_\varkappa(-i\, C_A^* \delta_{-\varkappa,\varkappa_\nu} + S_\varkappa C_A'^* \delta_{\varkappa,\varkappa_\nu})\Big[-[6(2\bar{\ell}+1)]^{\frac{1}{2}}$$

$$\times C(\bar{\ell} 1 \ell;00)\, W(\bar{\ell} 1 j\tfrac{1}{2}; \ell\tfrac{1}{2})(-f_{-\varkappa}F_\varkappa + g_{-\varkappa}G_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right)^* - (f_{-\varkappa}G_\varkappa + g_{-\varkappa}F_\varkappa)\left(i\int \gamma_5\right)^*\Big]$$

$$+ S_\varkappa\left(-i\,\frac{C_P^*}{2M}\delta_{-\varkappa,\varkappa_\nu} + S_\varkappa \frac{C_P'^*}{2M}\delta_{\varkappa,\varkappa_\nu}\right)\frac{d}{dr}(-f_{-\varkappa}G_\varkappa + g_{-\varkappa}F_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right)^* \Big\}$$

$$\times \Big\{ (i\, C_A \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_A' \delta_{-\varkappa,\varkappa_\nu})\Big[[6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell} 1 \ell;00)\, W(\bar{\ell} 1 j\tfrac{1}{2}; \ell\tfrac{1}{2})$$

$$(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right) + (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\left(i\int \gamma_5\right)\Big]$$

$$+ \left(i\,\frac{C_P}{2M}\delta_{\varkappa,\varkappa_\nu} - S_\varkappa \frac{C_P'}{2M}\delta_{-\varkappa,\varkappa_\nu}\right)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right) \Big\}$$

where

$$\mathcal{J}_1(\varkappa) \equiv \sum_{\varkappa_\nu} \mathcal{J}^*(-\varkappa, \varkappa_\nu)\, \mathcal{J}(\varkappa, \varkappa_\nu) \tag{3.17a}$$

Carrying out the multiplication, we get

$$\mathcal{J}_1(\varkappa) = i(C_A^* C_A' + C_A C_A'^*) \Big\{ - 6(2\bar{\ell}+1) \left[C(\bar{\ell}\,1\,\ell\,;00)\, W(\bar{\ell}\,1 j \tfrac{1}{2};\ell \tfrac{1}{2})\right]^2$$

$$\times (-f_{-\varkappa} F_\varkappa + g_{-\varkappa} G_\varkappa)(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \left|\int \vec{\sigma}\cdot\hat{r}\right|^2$$

$$- (f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa)(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \left|\int \gamma_5\right|^2$$

$$- \left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\, W(\bar{\ell}\,1 j \tfrac{1}{2};\ell \tfrac{1}{2})(-f_{-\varkappa} F_\varkappa + g_{-\varkappa} G_\varkappa)$$

$$\times (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right)^*\left(i\int \gamma_5\right)$$

$$- 6(2\bar{\ell}+1)^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\, W(\bar{\ell}\,1 j \tfrac{1}{2};\ell \tfrac{1}{2})(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)$$

$$\times (f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa)\left(i\int \gamma_5\right)^*\left(\int \vec{\sigma}\cdot\hat{r}\right) \Big\}$$

$$+ \frac{i}{4M^2}(C_P^* C_P' + C_P' C_P^*)\Big\{\frac{d}{dr}(-f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\left|\int \vec{\sigma}\cdot\hat{r}\right|^2\Big\}$$

$$- \frac{i}{2M}(C_A^* C_P' + C_A'^* C_P)\Big\{ 6(2\bar{\ell}+1)^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\, W(\bar{\ell}\,1 j \tfrac{1}{2};\ell \tfrac{1}{2})(-f_{-\varkappa} F_\varkappa + g_{-\varkappa} G_\varkappa)$$

$$\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\Big\}\left|\int \vec{\sigma}\cdot\hat{r}\right|^2$$

$$- \frac{i}{2M}(C_A^* C_P' + C_A'^* C_P)\Big\{(f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right)\left(i\int \gamma_5\right)^*$$

$$+ \frac{i}{2M}(C_A C_P'^* + C_A' C_P^*)\Big\{\left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\, W(\bar{\ell}\,1 j \tfrac{1}{2};\ell \tfrac{1}{2})$$

$$\times (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)\frac{d}{dr}(-f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa)\left|\int \vec{\sigma}\cdot\hat{r}\right|^2 \Big\}$$

$$+ \frac{i}{2M}(C_A C_P'^* + C_A' C_P^*)\Big\{(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\frac{d}{dr}(-f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa)\left(\int \vec{\sigma}\cdot\hat{r}\right)^*\left(i\int \gamma_5\right)\Big\}$$

(3.17c)

We had

$$N = \frac{W}{2\pi p} \sum_{\varkappa} e^{i(\delta_\varkappa - \delta_{-\varkappa})} (2j+1)\, \mathcal{J}_1(\varkappa) \tag{3.14e}$$

where $\varkappa = -1, 1, -2, 2 \ldots$

and

$$j = |\varkappa| - \tfrac{1}{2}$$

We introduce

$$\mathcal{J}_1(\varkappa) \equiv i\Lambda_1(\varkappa) + \Lambda_2(\varkappa) \tag{3.19a}$$

Then we get

$$N = \frac{W}{2\pi p} \sum_{\varkappa} e^{i(\delta_\varkappa - \delta_{-\varkappa})} (2j+1) \left[i\Lambda_1(\varkappa) + \Lambda_2(\varkappa) \right] \tag{3.14f}$$

where from (3.17c), rearranging some terms and introducing the notations Re and Im as meaning the real and the imaginary parts, we have

$$\begin{aligned}
\Lambda_1(\varkappa) = & - (2\, \mathrm{Re}\, C_A C_A'^{*}) \Big\{ 6(2\bar{\ell}+1) \left[C(\bar{\ell}1\ell;00)\, W(\bar{\ell}1j\tfrac{1}{2};\ell\tfrac{1}{2}) \right]^2 (-f_{-\varkappa}F_\varkappa + g_{-\varkappa}G_\varkappa) \\
& \times (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \left| \int \vec{\sigma}\cdot\hat{r} \right|^2 + (f_{-\varkappa}G_\varkappa + g_{-\varkappa}F_\varkappa)(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \left| \int \gamma_5 \right|^2 \\
& + \left[6(2\bar{\ell}+1) \right]^{\frac{1}{2}} C(\bar{\ell}1\ell;00)\, W(\bar{\ell}1j\tfrac{1}{2};\ell\tfrac{1}{2}) \Big[(-f_{-\varkappa}F_\varkappa + g_{-\varkappa}G_\varkappa) \\
& \times (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \left(\int \vec{\sigma}\cdot\hat{r} \right)^{*} \left(i \int \gamma_5 \right) + (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)(f_{-\varkappa}G_\varkappa + g_{-\varkappa}F_\varkappa) \\
& \times \left(\int \vec{\sigma}\cdot\hat{r} \right) \left(i \int \gamma_5 \right)^{*} \Big] \Big\} +
\end{aligned}$$

$$+ \frac{2}{4M^2} (\mathrm{Re}\ C_P C_P'^{*}) \Big\{ \frac{d}{dr} (-f_\varkappa G_\varkappa + g_{-\varkappa} F_\varkappa) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) \ \Big| \int \vec{\sigma} \cdot \hat{r} \Big|^2 \Big\}$$

$$- \frac{1}{2M} \Big[\mathrm{Re}\ (C_A C_P'^{*} + C_A' C_P^{*}) \Big] \Big\{ [6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1 \ell ; 00)\ W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})$$

$$\times \Big[(-f_{-\varkappa} F_\varkappa + g_{-\varkappa} G_\varkappa) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) - (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \frac{d}{dr} (-f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa) \Big]$$

$$\times \Big| \int \vec{\sigma} \cdot \hat{r} \Big|^2 + (f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) \int \vec{\sigma} \cdot \hat{r}\ (i \int \gamma_5)^*$$

$$- (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \frac{d}{dr} (-f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa) \Big(\int \vec{\sigma} \cdot \hat{r} \Big)^* (i \int \gamma_5) \Big\} \qquad (3.19b)$$

$$\Lambda_2(\varkappa) \equiv - \frac{1}{2M} \Big[\mathrm{Im}\ (C_A C_P'^{*} + C_A' C_P^{*}) \Big] \Big\{ [6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1 \ell ; 00)\ W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})$$

$$\times (-f_{-\varkappa} F_\varkappa + g_{-\varkappa} G_\varkappa) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) + (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \frac{d}{dr} (-f_{-\varkappa} G \quad + g_{-\varkappa} F_\varkappa)$$

$$\times \Big| \int \vec{\sigma} \cdot \hat{r} \Big|^2 + (f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) \Big(\int \vec{\sigma} \cdot \hat{r} \Big) (i \int \gamma_5)^*$$

$$+ (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \frac{d}{dr} (-f_{-\varkappa} G_\varkappa + g_{-\varkappa} F_\varkappa) \Big(\int \vec{\sigma} \cdot \hat{r} \Big)^* (i \int \gamma_5) \Big\} \qquad (3.19c)$$

We had

$$D = \frac{W}{2\pi p} \sum_{\varkappa} (2j+1) \sum_{\varkappa_\nu} \mathcal{J}^*(\varkappa, \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu) \qquad (3.15e)$$

We define

$$\mathcal{J}_2(\varkappa) \equiv \sum_{\varkappa_\nu} \mathcal{J}^*(\varkappa, \varkappa_\nu) \mathcal{J}(\varkappa, \varkappa_\nu) \qquad (3.20a)$$

then, we get

$$D = \frac{W}{2\pi p} \sum_{\varkappa} (2j+1) \mathcal{J}_2(\varkappa) \qquad (3.15f)$$

$$\mathcal{J}_2(\varkappa) = \sum_{\varkappa_\nu} \Big\{ (-i\, C_A^* \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_A'^* \delta_{\varkappa,-\varkappa_\nu})$$

$$\times \Big[[6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1\, \ell\, ;00)\, W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})\, (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)(\textstyle\int \vec{\sigma}\cdot\hat{r})^*$$

$$+ (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)(i \textstyle\int \gamma_5)^* \Big]$$

$$+ \frac{1}{2M} (-i\, C_P^* \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_P'^* \delta_{\varkappa,-\varkappa_\nu}) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)(\textstyle\int \vec{\sigma}\cdot\hat{r})^* \Big\}$$

$$\times \Big\{ (i\, C_A \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_A' \delta_{\varkappa,-\varkappa_\nu}) \Big[[6(2\ell+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1\, \ell\, ;00)\, W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})$$

$$\times (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)(\textstyle\int \vec{\sigma}\cdot\hat{r}) + (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)(i \textstyle\int \gamma_5) \Big]$$

$$+ \frac{1}{2M} (i\, C_P \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_P' \delta_{\varkappa,-\varkappa_\nu}) \frac{d}{dr} (f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)(\textstyle\int \vec{\sigma}\cdot\hat{r}) \Big\} \qquad (3.20b)$$

There are two points to be noticed in the above equation.

(1) There will be no terms containing the primed and the unprimed coupling constants because of Kronecker deltas. This leads to a well known result that the measurement of the β spectrum does not show the effects of parity breakdown.

(2) $\mathcal{J}_2(\varkappa)$ is a positive definite quantity, as the intensity term should be

$$\mathcal{J}_2(\varkappa)$$

$$= (|C_A|^2 + |C_A'|^2) \Big\{ 6(2\bar{\ell}+1) [C(\bar{\ell}\, 1\, \ell\, ;00)\, W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})]^2 (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)^2 |\textstyle\int \vec{\sigma}\cdot\hat{r}|^2$$

$$+ (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)^2 |\textstyle\int \gamma_5|^2 + [6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell}\, 1\, \ell\, ;00)\, W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell \tfrac{1}{2})$$

$$\times (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \Big[(\textstyle\int \vec{\sigma}\cdot\hat{r})(i \textstyle\int \gamma_5)^* + (\textstyle\int \vec{\sigma}\cdot\hat{r})^* (i \textstyle\int \gamma_5) \Big] \Big\} +$$

$$+ \frac{1}{4M^2}\left(|C_P|^2 + |C_P'|^2\right)\left\{\left[\frac{d}{dr}\left(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa\right)\right]^2 \left|\int \vec{\sigma}\cdot\hat{r}\right|^2\right\}$$

$$+ \frac{2}{2M}\left[\mathrm{Re}\,\left(C_A C_P^* + C_A' C_P'^*\right)\right]\left\{\left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\, W(\bar{\ell}\,1 j\tfrac{1}{2};\ell\tfrac{1}{2})(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)\right.$$

$$\left.\times \frac{d}{dr}\left(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa\right)\left|\int \vec{\sigma}\cdot\hat{r}\right|^2\right\}$$

$$+ \frac{1}{2M}\left[\left(C_A C_P^* + C_A' C_P'^*\right)\left(\int \vec{\sigma}\cdot\hat{r}\right)\left(i\int \gamma_5\right)^* + \left(C_A^* C_P + C_A'^* C_P'\right)\left(\int \vec{\sigma}\cdot\hat{r}\right)^*\left(i\int \gamma_5\right)\right]$$

$$\times\left\{\left(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa\right)\frac{d}{dr}\left(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa\right)\right\} \qquad (3.20c)$$

Now we assume that time-reversal invariance is valid in nuclear β decay[15] and in strong interactions. If time reversal invariance holds in strong interactions then (1) all the combination of nuclear matrix elements (in the cross terms) are real.[16,17]

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[15] M. A. Clark, J. M. Robson, and R. Nathan, Phys. Rev. Letters 1, 100 (1958). The measurement involved the detection of a term like $\dfrac{D\,\vec{J}\cdot(\vec{p}_e \times \vec{p}_\nu)}{J\cdot E_e E_\nu}$ where for the vector and axial vector interactions in the decay of polarized neutron $D = -\dfrac{2\,\mathrm{Im}\,(C_V C_A^* + C_V' C_A'^*)}{|C_V|^2 + |C_V'|^2 + 3(|C_A|^2 + |C_A'|^2)}$; $\vec{p}_e$ and $\vec{p}_\nu$ - the momenta of the electron and antineutrino respectively and $\vec{J}$ being the spin direction of the neutrons. Under time reversal this quantity changes sign. $D = -0.02 \pm 0.28$, by experiment.

[16] C. L. Longmire and A. M. L. Messiah, Phys. Rev. 83, 464 (1951).

[17] For special case, see L. C. Biedenharn and M. E. Rose, Revs. Mod. Phys. 25, 729 (1953).

Therefore, $(i\int\gamma_5)(\int\vec{\sigma}\cdot\hat{r})^*$ is real* and for weak interactions(β decay) (2) the coupling constants are real.** Making use of the above, $\Lambda_2(\kappa)$ in (3.16c) vanishes because of the factor $\mathrm{Im}\,(C_A C'^*_P + C'_A C^*_P)$.

From (3.19b), we get

$$
\begin{aligned}
&\Lambda_1(\kappa)\\
&= -2\,C_A C'_A \Big\{ 6(2\bar{\ell}+1)\left[C(\bar{\ell}1\ell;00)\,W(\bar{\ell}1j\tfrac{1}{2};\ell\tfrac{1}{2})\right]^2 (-f_{-\kappa}F_\kappa+g_{-\kappa}G_\kappa)\\
&\quad\times(f_\kappa G_\kappa+g_\kappa F_\kappa)\left|\int\vec{\sigma}\cdot\hat{r}\right|^2 + (f_{-\kappa}G_\kappa+g_{-\kappa}F_\kappa)(f_\kappa F_\kappa-g_\kappa G_\kappa)\left|\int\gamma_5\right|^2\\
&\quad+\left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}1\ell;00)\,W(\bar{\ell}1j\tfrac{1}{2};\ell\tfrac{1}{2})\Big[(-f_{-\kappa}F_\kappa+g_{-\kappa}G_\kappa)\\
&\quad\times(f_\kappa F_\kappa-g_\kappa G_\kappa)+(f_\kappa G_\kappa+g_\kappa F_\kappa)(f_{-\kappa}G_\kappa+g_{-\kappa}F_\kappa)\Big](\int\vec{\sigma}\cdot\hat{r})^*(i\int\gamma_5)\Big\}\\
&\quad+\frac{1}{2M^2}C_P C'_P\Big\{\frac{d}{dr}(-f_{-\kappa}G_\kappa+g_{-\kappa}F_\kappa)\frac{d}{dr}(f_\kappa F_\kappa+g_\kappa G_\kappa)\left|\int\vec{\sigma}\cdot\hat{r}\right|^2\Big\}\\
&\quad-\frac{1}{2M}(C_A C'_P+C'_A C_P)\Big\{\left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}}C(\bar{\ell}1\ell;00)\,W(\bar{\ell}1j\tfrac{1}{2};\ell\tfrac{1}{2})\\
&\quad\times\Big[(-f_{-\kappa}F_\kappa+g_{-\kappa}G_\kappa)\frac{d}{dr}(f_\kappa F_\kappa+g_\kappa G_\kappa)-(f_\kappa G_\kappa+g_\kappa F_\kappa)\\
&\quad\times\frac{d}{dr}(-f_{-\kappa}G_\kappa+g_{-\kappa}F_\kappa)\Big]\left|\int\vec{\sigma}\cdot\hat{r}\right|^2+\Big[(f_{-\kappa}G_\kappa+g_{-\kappa}F_\kappa)\\
&\quad\times\frac{d}{dr}(f_\kappa F_\kappa+g_\kappa G_\kappa)-(f_\kappa F_\kappa-g_\kappa G_\kappa)\frac{d}{dr}(-f_{-\kappa}G_\kappa+g_{-\kappa}F_\kappa)\Big]\\
&\qquad(\int\vec{\sigma}\cdot\hat{r})^*(i\int\gamma_5)\Big\}
\end{aligned}
\tag{3.19d}
$$

---

*$(i\int\gamma_5)(\int\vec{\sigma}\cdot\hat{r})^*$ is real, as proved in Appendix C.

**T. D. Lee and C. N. Yang, Reference 8, p. 23. Also see Appendix C for proof.

And from (3.20c), we get, using these assumptions,

$$\mathcal{J}_2(\varkappa)$$

$$= (C_A^2 + C_A'^2)\Big\{6(2\bar{\ell}+1)\left[C(\bar{\ell}\,1\,\ell\,;00)\,W(\bar{\ell}\,1\,j\tfrac{1}{2};\ell\tfrac{1}{2})\right]^2 (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)^2 \left|\int\vec{\sigma}\cdot\hat{r}\right|^2$$

$$+ (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)^2 \left|\int\gamma_5\right|^2 + 2\left[6(2\ell+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\,W(\bar{\ell}\,1\,j\tfrac{1}{2};\ell\tfrac{1}{2})$$

$$\times (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\left(\int\vec{\sigma}\cdot\hat{r}\right)^* \left(i\int\gamma_5\right)\Big\}$$

$$+ \frac{1}{4M^2}(C_P^2 + C_P'^2)\Big\{\left[\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\right]^2 \left|\int\vec{\sigma}\cdot\hat{r}\right|^2 \Big\}$$

$$+ \frac{1}{M}(C_A C_P + C_A' C_P')\Big\{\left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\,W(\bar{\ell}\,1\,j\tfrac{1}{2};\ell\tfrac{1}{2})(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)$$

$$\times \frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\left|\int\vec{\sigma}\cdot\hat{r}\right|^2$$

$$+ (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\left(\int\vec{\sigma}\cdot\hat{r}\right)^* \left(i\int\gamma_5\right)\Big\} \qquad (3.20d)$$

We have, $j = |\varkappa| - \frac{1}{2}$ or $2j + 1 = 2|\varkappa|$

$$P_{\parallel} = \frac{N}{D}$$

where

$$N = \frac{W}{2\pi p}\sum_{\varkappa} e^{i(\delta_\varkappa - \delta_{-\varkappa})}\, 2|\varkappa|\, i\Lambda_1(\varkappa) \qquad (3.14f)$$

$$D = \frac{W}{2\pi p}\sum_{\varkappa} 2|\varkappa|\, \mathcal{J}_2(\varkappa) \qquad (3.15f)$$

$\Lambda_1(\varkappa)$ and $\mathcal{J}_2(\varkappa)$ are given in (3.19d) and (3.20d)

$\varkappa = -1, 1, -2, 2, \ldots$

We introduce

$$|\varkappa| \equiv k$$

Then

$$N = \frac{W}{\pi p} \sum_k k\, e^{i(\delta_k - \delta_{-k})}\, i\Lambda_1(\varkappa = k) + k\, e^{-i(\delta_k - \delta_{-k})}\, i\Lambda_1(\varkappa = -k)$$

$$N = \frac{W}{\pi p} \sum_k i\, k \left[e^{i(\delta_k - \delta_{-k})} \Lambda_1(\varkappa = k) + e^{-i(\delta_k - \delta_{-k})} \Lambda_1(\varkappa = -k) \right] \qquad (3.14g)$$

Similarly, we obtain

$$D = \frac{W}{\pi p} \sum_k k \left[\mathcal{J}_2(\varkappa = k) + \mathcal{J}_2(\varkappa = -k) \right] \qquad (3.15g)$$

For $\Lambda_1(\varkappa = -k)$, we replace, in (3.19d), $\varkappa$ by $-k$

we see that

$$(2\ell+1)^{\frac{1}{2}}\, C(\ell 1 \bar{\ell}\,;00)\, W(\ell 1 j \tfrac{1}{2}; \bar{\ell}\, \tfrac{1}{2}) = (2\bar{\ell}+1)^{\frac{1}{2}}\, C(\bar{\ell} 1 \ell\,;00)\, W(\bar{\ell} 1 j \tfrac{1}{2}; \ell\, \tfrac{1}{2}) \qquad (3.18a)$$

For the neutrino radial functions, we have

$$F_{-\varkappa} = -\, S_\varkappa\, G_\varkappa \qquad (3.18b)$$

$$G_{-\varkappa} = S_\varkappa\, F_\varkappa \qquad (3.18c)$$

where $S_\varkappa$ is the sign of $\varkappa$.

It is very easy to see, by substituting the above that

$$\Lambda_1(\varkappa = -k) = -\Lambda_1(\varkappa = k) \qquad (3.19e)$$

Substituting (3.19e) in (3.14g), we get

$$N = \frac{W}{\pi p} \sum_k ik \left[e^{i(\delta_k - \delta_{-k})} - e^{-i(\delta_k - \delta_{-k})} \right] \Lambda_1(\varkappa = k)$$

$$N = -\frac{2W}{\pi p}\sum_k k \sin(\delta_k - \delta_{-k})\Lambda_1(\varkappa = k) \tag{3.19f}$$

$\Lambda_1(\varkappa)$ is given in (3.19d).

Now we simplify the expression for $\Lambda_1(\varkappa)$ for $\varkappa = k$

$$(-f_{-k}F_k+g_{-k}G_k)(f_kG_k+g_kF_k)$$

$$= - f_{-k}g_kF_k^2 + g_{-k}f_kG_k^2 - (f_kf_{-k}-g_kg_{-k})F_kG_k \tag{3.21a}$$

$$(f_{-k}G_k+g_{-k}F_k)(f_kF_k-g_kG_k)$$

$$= f_kg_{-k}F_k^2 - g_kf_{-k}G_k^2 + (f_kf_{-k}-g_kg_{-k})F_kG_k \tag{3.21b}$$

$$(-f_{-k}F_k+g_{-k}G_k)(f_kF_k-g_kG_k) + (f_kG_k+g_kF_k)(f_{-k}G_k+g_{-k}F_k)$$

$$= - (f_kf_{-k}-g_kg_{-k})F_k^2 + (f_kf_{-k}-g_kg_{-k})G_k^2 + 2(f_kg_{-k}+g_kf_{-k})F_kG_k \tag{3.21c}$$

Now

$$\frac{d}{dr}(f_kF_k+g_kG_k) = f_kF_k'+f_k'F_k+g_kG_k'+g_k'G_k$$

The prime in the above equations means differentiation with respect to r. To evaluate this, we use the coupled equations[18] where V is the potential energy.

$$f_k' = \frac{k-1}{r} f_k - (W-V-1)g_k \tag{3.22a}$$

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[18] M. E. Rose, <u>Elementary Theory of Angular Momentum,</u> (John Wiley and Sons, New York 1957) p. 153. Also see M. E. Rose, reference 2, Section 26.

$$g_k' = (W-V+1)f_k - \frac{k+1}{r} g_k \tag{3.22b}$$

For the neutrino: q = the momentum of the neutrino (in $m_e c$ units)

$$F_k' = \frac{k-1}{r} F_k - q\, G_k \tag{3.22c}$$

$$G_k' = q\, F_k - \frac{k+1}{r} G_k \tag{3.22d}$$

We define

$$u \equiv W - V - q \tag{3.22e}$$

Using (3.22), and rearranging terms,

$$\frac{d}{dr}(f_kF_k+g_kG_k)\,\frac{d}{dr}(-f_{-k}G_k+g_{-k}F_k)$$

$$= 2\left[\frac{k-1}{r}\left\{u(f_kf_{-k}-g_kg_{-k}) + (f_kf_{-k}+g_kg_{-k})\right\} + 4\,\frac{(k-1)^2}{r^2}\, f_kg_{-k}-(u^2-1)f_{-k}g_k\right] F_k^2$$

$$+ 2\left[\frac{k+1}{r}\left\{u(f_kf_{-k}-g_kg_{-k}) + (f_kf_{-k}+g_kg_{-k})\right\} - 4\,\frac{(k+1)^2}{r^2}\, f_{-k}g_k-(u^2-1)f_kg_{-k}\right] G_k^2$$

$$+ \left[4\,\frac{(k^2-1)}{r^2}(f_kf_{-k}-g_kg_{-k}) + (u^2-1)(f_kf_{-k}-g_kg_{-k}) + 2u(f_kf_{-k}+g_kg_{-k})\right.$$

$$\left. + 4\,\frac{(k-1)}{r}\,u\, f_kg_{-k} - 4\,\frac{(k+1)}{r}\,u\, f_{-k}g_k\right] F_kG_k \tag{3.21d}$$

Similarly

$$(-f_{-k}F_k+g_{-k}G_k)\,\frac{d}{dr}(f_kF_k+g_kG_k) - (f_kG_k+g_kF_k)\,\frac{d}{dr}(-f_{-k}G_k+g_{-k}F_k)$$

$$= -\,2\left[f_{-k}g_k + 2\,\frac{(k-1)}{r}(f_kf_{-k}+g_kg_{-k})\right]F_k^2 + 2\left[f_kg_{-k} - \frac{(k+1)}{r}(f_kf_{-k}+g_kg_{-k})\right]G_k^2$$

$$-\,2\left[u(f_kf_{-k}+g_kg_{-k}) + (f_kf_{-k}-g_kg_{-k})\right] F_kG_k \tag{3.21e}$$

$$(f_{-k}G_k+g_{-k}F_k)\frac{d}{dr}(f_kF_k+g_kG_k) - (f_kF_k-g_kG_k)\frac{d}{dr}(-f_{-k}G_k+g_{-k}F_k)$$
$$= \left[-u(f_kf_{-k}+g_kg_{-k}) - (f_kf_{-k}-g_kg_{-k})\right]F_k^2 + \left[u(f_kf_{-k}+g_kg_{-k})\right.$$
$$\left. + (f_kf_{-k}-g_kg_{-k})\right]G_k^2 + \left[-\frac{4}{r}(f_kf_{-k}+g_kg_{-k}) + 2f_kg_{-k}+2f_{-k}g_k\right]F_kG_k \qquad (3.21f)$$

In nuclear β decay, the lepton functions are evaluated at the nuclear radius. In this work, we shall denote the radius of the nucleus as R. In the relativistic units, the nuclear radius is given by

$$R = 0.43\alpha A^{1/3}\ \left(\frac{\hbar}{mc}\right).$$

where $$\alpha \backsim \frac{1}{137}.$$

We define the following:*

$$A_{k-1} \equiv (p^2F_o)^{-1}\, R^{2-2k}\, f_kg_{-k}\,\mathrm{Sin}(\delta_k-\delta_{-k}) \qquad (3.23a)$$

$$B_{k-1} \equiv (p^2F_o)^{-1}\, R^{-2k}\, f_{-k}g_k\,\mathrm{Sin}(\delta_k-\delta_{-k}) \qquad (3.23b)$$

$$C_{k-1} \equiv (p^2F_o)^{-1}\, R^{1-2k}\, (f_kf_{-k}+g_kg_{-k})\,\mathrm{Sin}(\delta_k-\delta_{-k}) \qquad (3.23c)$$

$$D_{k-1} \equiv (p^2F_o)^{-1}\, R^{1-2k}\, (f_kf_{-k}-g_kg_{-k})\,\mathrm{Sin}(\delta_k-\delta_{-k}) \qquad (3.23d)$$

F_o is the Fermi function

$$F_o \equiv 4\,(2pR)^{2(\gamma_1-1)}\,\exp\frac{\pi\alpha ZW}{p}\,\frac{|\Gamma(\gamma_1+i\alpha ZW/p)|^2}{\Gamma^2(2\gamma_1+1)} \qquad (3.23e)$$

$$\gamma_1 = \left[1-(\alpha Z)^2\right]^{\frac{1}{2}}$$

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*See Appendix D for analytical expression up to order R.

Now we define

$$\Lambda(k) \equiv \Lambda_1(k)\ (p^2F_o)^{-1}\ R^{2-2k}\ \mathrm{Sin}(\delta_k - \delta_{-k}) \tag{3.24a}$$

So that

$$N = -\frac{2W}{\pi p}(p^2F_o)\sum_k k\ R^{2k-2}\Lambda(k) \tag{3.19g}$$

Substituting (3.21) in (3.19d) and using (3.23), we get

$$\begin{aligned}
&\Lambda(k) \\
&\quad = -2\,C_A C_A' \Big\{ -6(2\bar{\ell}+1)\left[C(\bar{\ell}\,1\,\ell\,;00)\ W(\bar{\ell}\,1\,j\tfrac{1}{2};\ell\,\tfrac{1}{2})\right]^2 \\
&\qquad \times (B_{k-1}\,R^2F_k^2 - A_{k-1}\,G_k^2 + D_{k-1}\,RF_kG_k)\ \left|\int\vec{\sigma}\cdot\hat{r}\right|^2 \\
&\qquad + (A_{k-1}\,F_k^2 - B_{k-1}\,R^2G_k^2 + D_{k-1}\,RF_kG_k)\ \left|\int\gamma_5\right|^2 \\
&\qquad + \left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell\,;00)\ W(\bar{\ell}\,1\,j\tfrac{1}{2};\ell\,\tfrac{1}{2})\Big[-D_{k-1}\,RF_k^2 \\
&\qquad + D_{k-1}\,RG_k^2 + (2A_{k-1} + R^2B_{k-1})\,F_kG_k\Big]\left(\int\vec{\sigma}\cdot\hat{r}\right)^*\left(i\int\gamma_5\right)\Big\} \\
&+ \frac{1}{2M^2}\,C_PC_P'\ \Big\{\Big[2(k-1)(uD_{k-1} + C_{k-1}) + 4(k-1)^2\,R^{-2}A_{k-1} \\
&\quad - (u^2-1)B_{k-1}\,R^2\Big]F_k^2 + \Big[2(k+1)(uD_{k-1} + C_{k-1}) \\
&\quad - 4(k+1)^2B_{k-1} + (u^2-1)A_{k-1}\Big]G_k^2 + \Big[4(k^2-1)R^{-1}D_{k-1} \\
&\quad + (u^2-1)RD_{k-1} + 2uRC_{k-1} + 4(k-1)R^{-1}A_{k-1}u \\
&\quad - 4(k+1)RuB_{k-1}\Big]F_kG_k\Big\}\left|\int\vec{\sigma}\cdot\hat{r}\right|^2\ -
\end{aligned}$$

$$- \frac{1}{2M} (C_A C_P' + C_A' C_P) \Big\{ 2 \left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\ 1 \ell\ ;00)\ W(\bar{\ell}\, 1 j \tfrac{1}{2}; \ell\ \tfrac{1}{2})$$

$$\times \Big[- (R^2 B_{k-1} + (k-1) C_{k-1}) F_k^2 + (A_{k-1} - (k+1) C_{k-1}) G_k^2$$

$$- (uRC_{k-1} + RD_{k-1}) F_k G_k \Big] \left| \int \vec{\sigma} \cdot \hat{r} \right|^2$$

$$+ \Big[- (uRC_{k-1} + RD_{k-1}) F_k^2 + (uRC_{k-1} + RD_{k-1})\ G_k^2$$

$$+ (- 4C_{k-1} + 2R^2 B_{k-1} + 2A_{k-1}) F_k G_k \Big] \left(\int \vec{\sigma} \cdot \hat{r} \right)^* i \int \gamma_5 \Big\} \qquad (3.24b)$$

where

$$u \equiv W - V - q \qquad (3.22e)$$

In (3.15g)

$$D = \frac{W}{\pi p} \sum_k k \left[\mathcal{J}_2(\varkappa = k) + \mathcal{J}_2(\varkappa = - k) \right]$$

$\mathcal{J}_2(\varkappa)$ is given in (3.20d)

Now, as before, the following are to be evaluated at the nuclear radius.

$$(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)^2 = f_\varkappa^2 G_\varkappa^2 + g_\varkappa^2 F_\varkappa^2 + 2 f_\varkappa g_\varkappa F_\varkappa G_\varkappa \qquad (3.25a)$$

$$(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)^2 = f_\varkappa^2 F_\varkappa^2 + g_\varkappa^2 G_\varkappa^2 - 2 f_\varkappa g_\varkappa F_\varkappa G_\varkappa \qquad (3.25b)$$

$$(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)$$

$$= (f_\varkappa^2 - g_\varkappa^2) F_\varkappa G_\varkappa - f_\varkappa g_\varkappa G_\varkappa^2 + f_\varkappa g_\varkappa F_\varkappa^2 \qquad (3.25c)$$

$$\left[\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)\right]^2$$

$$= \left\{ \left[\frac{2(\varkappa-1)}{r} f_\varkappa - (u-1)g_\varkappa\right] F_\varkappa + \left[(u+1)f_\varkappa - \frac{2(\varkappa+1)}{r} g_\varkappa\right] G_\varkappa \right\}^2$$

$$= \left[4\frac{(\varkappa-1)^2}{r^2} f_\varkappa^2 + (u-1)^2 g_\varkappa^2 - 4\frac{(\varkappa-1)}{r}(u-1)f_\varkappa g_\varkappa\right] F_\varkappa^2$$

$$+ \left[(u+1)^2 f_\varkappa^2 + 4\frac{(\varkappa+1)^2}{r^2} g_\varkappa^2 - 4\frac{(\varkappa+1)}{r}(u+1)f_\varkappa g_\varkappa\right] G_\varkappa^2$$

$$+ 2\left[2\frac{(\varkappa-1)}{r}(u+1)f_\varkappa^2 + 2\frac{(\varkappa+1)}{r}(u-1)g_\varkappa^2 - \left(4\frac{(k^2-1)}{r^2} + u^2-1\right)f_\varkappa g_\varkappa\right] F_\varkappa G_\varkappa \tag{3.25d}$$

$$(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)$$

$$= \left[2\frac{(\varkappa-1)}{r} f_\varkappa g_\varkappa - (u-1)g_\varkappa^2\right] F_\varkappa^2$$

$$+ \left[(u+1)f_\varkappa^2 - 2\frac{(\varkappa+1)}{r} f_\varkappa g_\varkappa\right] G_\varkappa^2$$

$$+ \left[2\frac{(\varkappa-1)}{r} f_\varkappa^2 + 2f_\varkappa g_\varkappa - 2\frac{(\varkappa+1)}{r} g_\varkappa^2\right] F_\varkappa G_\varkappa \tag{3.25e}$$

$$(f_\varkappa F_\varkappa - g_\varkappa G_\varkappa)\frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa)$$

$$= \left[2\frac{(\varkappa-1)}{r} f_\varkappa^2 - (u-1)f_\varkappa g_\varkappa\right] F_\varkappa^2 - \left[(u+1)f_\varkappa g_\varkappa - 2\frac{(\varkappa+1)}{r} g_\varkappa^2\right] G_\varkappa^2$$

$$+ \left[(u+1)f_\varkappa^2 - 4\frac{\varkappa}{r} f_\varkappa g_\varkappa + (u-1)g_\varkappa^2\right] F_\varkappa G_\varkappa \tag{3.25f}$$

Using (3.18a) and noting from (3.18b) and (3.18c)

$$F_{-k}^2 = G_k^2 \quad ; \quad G_{-k}^2 = F_k^2$$

$$F_{-k}G_{-k} = -F_k G_k$$

and with the following definitions[19]

$$L_{k-1} = (2p^2F_o)^{-1}\, R^{2-2k}\, (f_k^2 + g_{-k}^2) \tag{3.26a}$$

$$M_{k-1} = (2p^2F_o)^{-1}\, R^{-2k}\, (f_{-k}^2 + g_k^2) \tag{3.26b}$$

$$N_{k-1} = (2p^2F_o)^{-1}\, R^{1-2k}\, (f_{-k}g_{-k} - f_kg_k) \tag{3.26c}$$

$$P_{k-1} = (2p^2F_o)^{-1}\, R^{2-2k}\, (g_{-k}^2 - f_k^2) \tag{3.26d}$$

$$Q_{k-1} = (2p^2F_o)^{-1}\, R^{-2k}\, (g_k^2 - f_{-k}^2) \tag{3.26e}$$

$$R_{k-1} = (2p^2F_o)^{-1}\, R^{1-2k}\, (f_{-k}g_{-k} + f_kg_k) \tag{3.26f}$$

$$\begin{aligned}
&\left[\mathcal{J}_2(\varkappa = k) + \mathcal{J}_2(\varkappa = -k)\right] (2p^2F_o)^{-1} R^{2-2k} \\
&= (C_A^2 + C_A'^2)\Big\{ 6(2\bar{\ell}+1)\left[C(\bar{\ell}\,1\,\ell;00)\, W(\bar{\ell}\,1j\tfrac{1}{2};\ell\,\tfrac{1}{2})\right]^2 \Big[M_{k-1}\, R^2F_k^2 \\
&\quad + L_{k-1}\, G_k^2 - 2N_{k-1}\, RF_kG_k\Big] \left|\int \vec{\sigma}\cdot\hat{r}\right|^2 \\
&\quad + \left[L_{k-1}\, F_k^2 + M_{k-1}\, R^2G_k^2 + N_{k-1}\, RF_kG_k\right] \left|\int \gamma_5\right|^2 \\
&\quad + 2\left[6(2\bar{\ell}+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,1\,\ell;00)\, W(\bar{\ell}\,1j\tfrac{1}{2};\ell\,\tfrac{1}{2})\Big[-N_{k-1}\, RF_k^2 \\
&\quad + N_{k-1}RG_k^2 + (L_{k-1} - M_{k-1}R^2)F_kG_k \quad \left(\int \vec{\sigma}\cdot\hat{r}\right)^* \left(i\int \gamma_5\right)\Big\} +
\end{aligned}$$

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[19] E. Greuling, Phys. Rev. 61, 568 (1942); D. L. Pursey, Phil. Mag. 42, 1193 (1951). Also see M. E. Rose and R. K. Osborn, reference 15, equation 59.

$$+ \frac{1}{4M^2} (C_P^2 + C_P'^2) \Big\{ \Big[4(k-1)^2 R^{-2} L_{k-1} + (u^2+1) R^2 M_{k-1} - 2uR^2 Q_{k-1}$$

$$+ 4(k-1)uN_{k-1} + 4(k-1)R_{k-1} \Big] F_k^2 + \Big[(u^2+1) L_{k-1}$$

$$- 2uP_{k-1} + 4(k+1)^2 M_{k-1} + 4(k+1)uN_{k-1} - 4(k+1)R_{k-1} \Big] G_k^2$$

$$+ 2 \Big[2(k-1)uR^{-1} L_{k-1} - 2(k-1)R^{-1} P_{k-1} + 2(k+1)uRM_{k-1}$$

$$- 2(k+1)RQ_{k-1} + 4(k^2-1)R^{-1} N_{k-1} + (u^2-1)RN_{k-1} \Big] F_k G_k \Big\} \Big| \int \vec{\sigma} \cdot \hat{r} \Big|^2$$

$$+ \frac{1}{M} (C_A C_P + C_A' C_P') \Big\{ \big[6(2\bar{\ell}+1) \big]^{\frac{1}{2}} C(\bar{\ell} 1 \ell; 00)\, W(\bar{\ell} 1 j \tfrac{1}{2}; \ell \tfrac{1}{2}) \Big\{ \Big[2(k-1)R_{k-1}$$

$$- uR^2 Q_{k-1} + R^2 M_{k-1} \Big] F_k^2 + \Big[- uP_{k-1} + L_{k-1} - 2(k+1)R_{k-1} \Big] G_k^2$$

$$+ \Big[- 2(k-1)R^{-1} P_{k-1} - 2RN_{k-1} - 2(k+1)RQ_{k-1} \Big] F_k G_k \Big\} \Big| \int \vec{\sigma} \cdot \hat{r} \Big|^2$$

$$\Big\{ \Big[- 2(k-1)R^{-1} P_{k-1} - uRR_{k-1} - RN_{k-1} \Big] F_k^2$$

$$- \Big[uRR_{k-1} - RN_{k-1} - 2(k+1)RQ_{k-1} \Big] G_k^2$$

$$\Big[- uP_{k-1} + L_{k-1} - 4kR_{k-1} + uR^2 Q_{k-1} - R^2 M_{k-1} \Big] F_k G_k \Big\}$$

$$\Big(\int \vec{\sigma} \cdot \hat{r} \Big)^* \Big(i \int \gamma_5 \Big) \Big\} \qquad (3.27a)$$

From (3.15g)

$$D = \frac{W}{\pi p} (2p^2 F_o) \sum_k R^{2k-2} \Big\{ \Big[\mathcal{J}_2(\varkappa = k) + \mathcal{J}_2(\varkappa = -k) \quad (2p^2 F_o)^{-1} R^{2-2k} \Big\} \qquad (3.15b)$$

where the expression in the curly brackets in (3.15b) is given by (3.27a)

$$N = -\frac{2W}{\pi p}(p^2 F_o)\sum_k k\, R^{2k-2} \Lambda(k) \qquad (3.19g)$$

where $\Lambda(k)$ is given by (3.24b)

and

$$P_{\parallel} = \frac{N}{D}$$

Now

$$\ell = \varkappa \qquad ; \qquad \varkappa > 0$$

$$\ell = -(\varkappa + 1) \qquad ; \qquad \varkappa < 0$$

$$\ell_{-\varkappa} \equiv \bar{\ell}$$

Also

$$F_k = q\, j_{k-1}(qR) \simeq q(qR)^{k-1}\left[(2k-1)!!\right]^{-1}$$

$$G_k = q\, j_k(qR) \simeq q(qR)^k \left[(2k+1)!!\right]^{-1}$$

In (3.19g) we evaluate the first term of the series, corresponding to $k = 1$.

Then, for $k = 1$ $\quad \bar{\ell} = 0, \quad \ell = 1, \qquad j = \frac{1}{2}$

$$6(2\bar{\ell}+1)^{\frac{1}{2}}\, C(\bar{\ell}1\ell;00)\, W(\bar{\ell}1j\tfrac{1}{2};\ell\tfrac{1}{2})$$

$$= (6)^{\frac{1}{2}}\, C(011;00)\, W(01j\tfrac{1}{2};1\tfrac{1}{2})$$

$$= (6)^{\frac{1}{2}}\, (-)^1\, C(101;00)\, (-)^{\frac{1}{2}-1-\frac{1}{2}}\, W(11\tfrac{1}{2}\tfrac{1}{2};0\tfrac{1}{2})$$

$$= (6)^{\frac{1}{2}}\, (6)^{-\frac{1}{2}} = 1 \qquad (3.28a)$$

Also

$$F_1 = q^2 \tag{3.28b}$$

$$G_1^2 = q^4R^2/9 \tag{3.28c}$$

$$F_1G_1 = q^3F/3 \tag{3.28d}$$

Substituting (3.28) in (3.24b) for k = 1

$$N(k=1) = -\frac{2W}{\pi p}(p^2F_o)\Lambda\ (k=1)$$

using (3.24b)

$\Lambda\ (k = 1)$

$$\begin{aligned}
&= 2q^2\, C_AC_A' \Big\{ (B_o - \frac{q^2}{9} A_o + \frac{q}{3} D_o)R^2 \,\Big|\int\vec{\sigma}\cdot\hat{r}\Big|^2 \\
&\quad - (A_o - \frac{q^2}{9} R^4B_o + \frac{q}{3} R^2D_o)\,\Big|\int\gamma_5\Big|^2 \\
&\quad - (-D_o + \frac{q^2}{9} R^2D_o + \frac{2}{3} qA_o + \frac{2}{3} qR^2B_o)\, R\, (\int\vec{\sigma}\cdot\hat{r})^* (i\int\gamma_5) \Big\} \\
&- \frac{q^2}{2M^2} C_PC_P' \Big\{ (u^2-1)B_o + \frac{q^2}{9}\Big[-4(uD_o+C_o) + 16B_o - (u^2-1)A_o\Big] \\
&\quad + \frac{q}{3}\Big[-(u^2-1)D_o - 2uC_o + 8uB_o\Big] \Big\} R^2 \,\Big|\int\vec{\sigma}\cdot\hat{r}\Big|^2 \\
&+ \frac{q^2}{2M}(C_AC_P' + C_A'C_P) \Big\{ 2\Big[B_o + \frac{q^2}{9}(-A_o + 2C_o) + \frac{q}{3}(uC_o+D_o)\Big] R^2 \,\Big|\int\vec{\sigma}\cdot\hat{r}\Big|^2 \\
&\quad + \Big[uC_o + D_o + \frac{q^2}{9}(-uC_o - D_o)R^2 + \frac{q}{3}(4C_o - 2R^3B_o - 2A_o) \\
&\quad \times R\, (\int\vec{\sigma}\cdot\hat{r})^*\; i\int\gamma_5 \Big\}
\end{aligned} \tag{3.24c}$$

Define

$$R\int\vec{\sigma}\cdot\hat{r} \equiv \int\vec{\sigma}\cdot\vec{r}$$

Since $R \ll 1$ for $A = 144$, $R = 0.016\,(\frac{\hbar}{mc})$ units, we neglect all terms of order R^2 or higher orders. In this approximation, the second term of the series in (3.19g), namely, $2\,R^2\Lambda\,(k = 2)$ gives only

$$2\cdot R^2 \frac{1}{2M^2} C_P C_P' \cdot 4R^{-2} A_1 \cdot \frac{1}{9} q^4 R^2 \quad \left|\int \vec{\sigma}\cdot\hat{r}\right|^2$$

$$= \frac{q^2}{2M^2} C_P C_P' \left[\frac{8}{9} q^2 A_1\right] \quad \left|\int \vec{\sigma}\cdot\vec{r}\right|^2$$

However, this term compared with the leading term in (3.19g) is small and as such is neglected. Therefore substituting (3.24c) in (3.19g), we get

$$\begin{aligned}
N = -\frac{4W}{\pi p}(p^2q^2F_o)\Big[& C_A C_A' \Big\{(B_o - \frac{q^2}{9}A_o + \frac{q}{3}D_o)\left|\int \vec{\sigma}\cdot\vec{r}\right|^2 \\
& - A_o\left|\int \gamma_5\right|^2 + (D_o - \frac{2}{3}q\,A_o)(\int \vec{\sigma}\cdot\vec{r})^*\, i\int \gamma_5 \Big\} \\
& - \frac{1}{4M^2} C_P C_P' \Big\{(u^2-1)B_o + \frac{q^2}{9}\left[16B_o - 4(uD_o+C_o) - (u^2-1)A_o\right] \\
& + \frac{q}{3}\left[8uB_o - (u^2+1)D_o - 2uC_o\right]\Big\}\left|\int \vec{\sigma}\cdot\vec{r}\right|^2 \\
& + \frac{1}{2M}(C_A C_P' + C_A' C_P)\Big\{\left[B_o + \frac{q^2}{9}(2C_o - A_o) + \frac{q}{3}(uC_o + D_o)\right]\left|\int \vec{\sigma}\cdot\vec{r}\right|^2 \\
& + \frac{1}{2}\left[uC_o + D_o + \frac{2q}{3}(2C_o - A_o)\right.\ (\int \vec{\sigma}\cdot\vec{r})^*\,(i\int \gamma_5)\Big\}\Big]
\end{aligned} \qquad (3.29a)$$

Similarly we take the term for $k = 1$ in (3.15b) using (B.27a), (3.28) and neglecting terms of order R^2 or higher orders.

$$D = \frac{2W}{\pi P}(p^2q^2F_o)\Bigg[(C_A^2+C_A'^2)\Big\{(M_o + \frac{1}{9}q^2L_o - \frac{2}{3}qN_o)\ \Big|\int\vec{\sigma}.\vec{r}\Big|^2$$

$$+ \ L_o \ \Big|\int\gamma_5\Big|^2 + 2(-N_o + \frac{1}{3}qL_o)\Big(\int\vec{\sigma}.\vec{r}\Big)^*\Big(i\int\gamma_5\Big)\Big\}$$

$$+ \ \frac{1}{4M^2}(C_P^2+C_P'^2)\Big\{(u^2+1)M_o - 2uQ_o + \frac{q^2}{9}\Big[(u^2+1)L_o$$

$$- 2uP_o + 16M_o + 8uN_o - 8R_o\Big] + \frac{2q}{3}\Big[4(uM_o-Q_o) + (u^2-1)N_o\Big]\Big\}\Big|\int\vec{\sigma}.\vec{r}\Big|^2$$

$$+ \ \frac{1}{M}(C_AC_P+C_A'C_P')\Big\{\Big[-uQ_o+M_o+\frac{q^2}{9}\Big[-uP_o+L_o - 4R_o\Big] - \frac{2q}{3}(N_o+2Q_o)\Big]\Big|\int\vec{\sigma}.\vec{r}\Big|^2$$

$$- \Big[uR_o+N_o + \frac{q^2}{9} R^2(uR_o-N_o-4Q_o)$$

$$+ \ \frac{q}{3}(uP_o-L_o+4R_o-R^2(uQ_o-M_o))\Big]\Big\}\Big(\int\vec{\sigma}.\vec{r}\Big)^*\Big(i\int\gamma_5\Big) \qquad (3.30a)$$

where L_o, M_o, N_o, P_o, Q_o, and R_o, A_o, B_o, C_o, and D_o are defined on pages 79 and 75 respectively.

The longitudinal polarization of β particles in $0 \rightarrow 0$ (yes) transitions assuming time reversal invariance in strong and weak interactions is

$$P_{\parallel} = \frac{N}{D}$$

where N and D are given in (3.29a) and (3.30a) respectively and from (3.11) the differential energy β spectrum N(W) is given by

$$N(W) = \frac{1}{4\pi^2} D$$

Now we specialize these results assuming the validity of the two-component theory of neutrino.

C. The Two-Component Theory of Neutrino

After the experimental confirmation of parity breakdown* in nuclear β decay, it was proposed[20,21,22] that a neutrino can exist either in a positive helicity or a negative helicity state but not in both. Whether a neutrino is left-handed or right-handed, is to be decided by experiment. Goldhaber, Grodzins and Sunyar[23] measured the neutrino helicity to be negative.

The wave equation of neutrino, with negative helicity, is

$$H \Phi_\nu = i \frac{\partial \Phi_\nu}{\partial t}$$

$$H = - \vec{\sigma} \cdot \vec{p}$$

We have taken the mass of neutrino to zero and use $\hbar = c = 1$. Then for the stationary state of energy $W = q$ and $\Phi_\nu = \phi\, e^{-iqt}$; $- \vec{\sigma} \cdot \hat{q}\, \phi = \phi$. Clearly the helicity operator $\vec{\sigma} \cdot \hat{q}$ commutes with the hamiltonian $(- \vec{\sigma} \cdot \hat{q})$ and as such helicity is a good quantum number. (In fact, in our case it is -1).

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*C. S. Wu et al, reference 5.

[20] T. D. Lee and C. N. Yang, Phys. Rev. 105, 1671 (1957).

[21] A. Salam, Nuovo Cimento 5, 299 (1957).

[22] L. Landau, Nuclear Physics 3, 127 (1957).

[23] M. Goldhaber, L. Grodzins, and A. W. Sunyar, Phys. Rev. 109, 1015 (1958). Also these results have been confirmed by I. Marklund and L. A. Page, Nuclear Physics 9, 88 (1958).

Since, in our calculations, the Dirac wave function for the neutrino was taken, we show the connection of the two-component theory and the Dirac theory of the neutrino.

For stationary states* of W = q, the Dirac equation gives

$$- \vec{\alpha}\cdot\hat{q}\, \psi_\nu = \psi_\nu \tag{3.32a}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

then

$$\psi_\nu = \begin{pmatrix} \phi_\nu \\ - \vec{\sigma}\cdot\hat{q}\, \phi_\nu \end{pmatrix}$$

Since γ_5 commutes with $\vec{\alpha}$, therefore from (3.32a), $\gamma_5 \psi_\nu$ is also a solution. If we impose the condition

$$\gamma_5 \psi_\nu = \epsilon \psi_\nu \quad ; \qquad \epsilon = \pm 1 \tag{3.32b}$$

Then we show, below, that ψ_ν represents a two-component neutrino of negative helicity for $\epsilon = 1$

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From (3.32b)

$$\frac{1}{2}(1 + \gamma_5)\psi_\nu = \frac{1}{2}(1 + \epsilon)\psi_\nu$$

---

*We consider here positive energy states.

Consider $\epsilon = 1$

$$\Psi_\nu = \frac{1}{2}(1+\gamma_5)\Psi_\nu = \begin{pmatrix} \frac{1}{2}(1-\vec{\sigma}\cdot\hat{q})\,\phi_\nu \\ \\ \frac{1}{2}(1-\vec{\sigma}\cdot\hat{q})\,\phi_\nu \end{pmatrix} \tag{3.33}$$

In (3.33), the upper and the lower components of Ψ_ν are the same. The helicity of ν , for $\epsilon = 1$, is negative.*

In nuclear β decay, the interaction hamiltonian for an interaction can be written as

$$H_x = \Omega_x(N)Q\cdot(\Psi_e^*\Omega_x[C_x+C_x'\gamma_5]\Psi_\nu\ )$$

$$= \Omega_x(N)Q\cdot(\Psi_e^*\Omega_x\left[\frac{1}{2}(C_x+C_x')(1+\gamma_5) + \frac{1}{2}(C_x-C_x')(1-\gamma_5)\right]\Psi_\nu\ ) \tag{3.34}$$

Thus to find the condition so that neutrino has negative helicity (eigenvalue of $\vec{\sigma}\cdot\hat{q}$ is -1) for positive energy states,** from (3.34), we observe

---

*In fact $\frac{1}{2}(1+\gamma_5)$ and $\frac{1}{2}(1-\gamma_5)$ are the projection operators for spin of the neutrino and these projections operators acting on Ψ_ν select states of negative and positive helicity respectively. Thus taking $\epsilon = 1$ (or -1) in (3.32b) gives for the neutrino a negative (or positive) helicity.

**In fact $W = \pm q$ in (3.32a) corresponding to positive and negative energy states of neutrino. Then $\vec{\sigma}\cdot\hat{q}\ \frac{1}{2}(1 \pm \gamma_5)\Psi_\nu = \mp\ Sw\ \frac{1}{2}(1 \pm \gamma_5)\Psi_\nu$ where Sw is the sign of W. From this it is clear that $\frac{1}{2}(1 \pm \gamma_5)\Psi_\nu$ are the eigenfunctions of the helicity operator $\vec{\sigma}\cdot\hat{q}$. For Sw positive: $\frac{1}{2}(1 \pm \gamma_5)\Psi_\nu$ represents states of $(\mp)$ helicity of ν . For Sw negative: $\frac{1}{2}(1 \pm \gamma_5)\Psi_\nu$ represent states of $(\pm)$ helicity of ν. For an antinuetrino in positive energy state: $\frac{1}{2}(1 \pm \gamma_5)\Psi_{\bar{\nu}}$ represent $(\pm)$ helicity of $\bar{\nu}$.

that $(C_x+C'_x)$ and $(C_x-C'_x)$ are the amplitudes for positive (negative) and negative (positive) helicity of antineutrino (neutrino). Therefore the longitudinal polarization of $\bar{\nu}$ is

$$P_{\bar{\nu}} = \frac{(\text{Intensity spins} \parallel \hat{q}) - (\text{Intensity spins antiparallel } \hat{q})}{(\text{Intensity spins} \parallel \hat{q}) + (\text{Intensity spins antiparallel } \hat{q})}$$

$$P = \frac{|C_x+C'_x|^2 - |C_x-C'_x|^2}{|C_x+C_x|^2 + |C_x-C'_x|^2}$$

Assuming time reversal invariance in nuclear beta decay, $(C_x = C_x^*;\ C'_x = C_x^*)$ we get

$$P_{\bar{\nu}} = \frac{2C_xC'_x}{C_x^2+C_x'^2}$$

From the above equation, we see that

(1) $P_{\bar{\nu}} = 1$ for $C_x = C'_x$

and (2) $P_{\bar{\nu}} = -1$ for $C_x = -C'_x$

The choice $C_x = C'_x$ is the correct choice, consistent with the experimental determination of the negative helicity of the neutrino. Thus the results derived in using Dirac wave functions for the neutrino in nuclear β decay, can be converted into those using the two-component theory by substituting $C_x = C'_x$ for all the interactions. In our case of the $0 \rightarrow 0$ (yes) transition, we get

$$C_A = C'_A$$

and

$$C_P = C'_P$$

D. Formulas* for Longitudinal Polarization and β Spectrum in 0 → 0 (yes) Transitions, Using the Validity of the Two-Component Theory of Neutrino

Cancelling common factors ,

$$\frac{4W}{\pi p}(p^2q^2F_o)\,C_A^2\;\left|\int \vec{\sigma}\cdot\vec{r}\right|^2;\quad \text{in (3.29a) and (3.30a)}$$

after setting

$$C_A = C_A' \quad ; \quad C_P = C_P' \quad ; \text{ we get,}$$

$$P_{\parallel} = \frac{N'}{D'}$$

where

$$N' \equiv -\Bigg[\left\{B_o - \frac{q^2}{9}A_o + \frac{q}{3}D_o - A_o\frac{\left|\int\gamma_5\right|^2}{\left|\int\vec{\sigma}\cdot\vec{r}\right|^2} + \left(D_o - \frac{2}{3}q\,A_o\right)\frac{i\int\gamma_5}{\int\vec{\sigma}\cdot\vec{r}}\right\}$$

$$-\frac{1}{4}\left(\frac{C_P}{MC_A}\right)^2\Big\{(u^2-1)B_o + \frac{q^2}{9}\left[16\,B_o - 4(uD_o+C_o) - (u^2-1)A_o\right]$$

$$+\frac{q}{3}\left[8uB_o - (u^2+1)D_o - 2uC_o\right]\Big\}$$

$$+\left(\frac{C_P}{MC_A}\right)\Big\{B_o + \frac{q^2}{9}(2C_o-A_o) + \frac{q}{3}(uC_o+D_o)$$

$$+\frac{1}{2}\left[uC_o+D_o+\frac{2q}{3}(2C_o-A_o)\right]\frac{i\int\gamma_5}{\int\vec{\sigma}\cdot\vec{r}}\Big\}\Bigg] \tag{3.36a}$$

$$D' \equiv M_o + \frac{1}{9}q^2L_o - \frac{2}{3}qN_o + L_o\frac{\left|\int\gamma_5\right|^2}{\left|\vec{\sigma}\cdot\vec{r}\right|^2} + 2\left(-N_o + \frac{1}{3}qL_o\right)\frac{i\int\gamma_5}{\int\vec{\sigma}\cdot\vec{r}} +$$

---

*To check these formulas in the limit $Z \to 0$, it is necessary to take $\varkappa = \pm 2$; in those terms which vanish for $|\varkappa| = 1$, for example to N' we should add $-\frac{1}{4}\left(\frac{C_P}{MC_A}\right)^2 \cdot \frac{8}{9}q^2A_1$ and for D'; we should consider $\frac{1}{4}\left(\frac{C_P}{MC_A}\right)^2 \frac{8}{9}q^2L_1$.

$$+ \frac{1}{4}\left(\frac{C_P}{MC_A}\right)^2 \Big\{ (u^2+1)M_o - 2uQ_o + \frac{q^2}{9}\Big[(u^2+1)L_o - 2uP_o + 16M_o$$

$$+ 8uN_o - 8R_o\Big] + \frac{2q}{3}\Big[4(uM_o - Q_o) + (u^2-1)N_o\Big] \Big\}$$

$$+ \frac{C_P}{MC_A}\Big\{ M_o - uQ_o + \frac{q^2}{9}(-uP_o + L_o - 4R_o) - \frac{2}{3}q(N_o + 2Q_o)$$

$$- \Big[uR_o + N_o + \frac{q^2R^2}{9}(uR_o - N_o - 4Q_o) - \frac{q}{3}\Big\{-uP_o + L_o - 4R_o$$

$$+ R^2(uQ_o - M_o)\Big\}\Big] \frac{i\int\gamma_5}{\int\vec{\sigma}\cdot\vec{r}} \Big\} \qquad (3.36b)$$

The differential energy β spectrum

$$N(W) = \frac{1}{4\pi^2} D$$

$$N(W) \equiv \frac{pWq^2F_o}{2\pi^3} C_\beta$$

C_β is called the Shape factor.* For allowed transitions it is $L_o \simeq \frac{1+\gamma_1}{2}$.

$$C_\beta = C_A^2 D' \left|\int\vec{\sigma}\cdot\vec{r}\right|^2 \qquad (3.37)$$

D' is given, above, in (3.36b).

In (3.36)

$$u \equiv W - V - q$$

$$R = 0.43(137)^{-1} A^{1/3}$$

For β^- emission, $V = -\alpha Z/R$ and for β^+ emission, $V = \alpha Z/R$

*This shape factor was derived by M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1315 (1954).

The functions A_o, B_o, C_o, D_o and L_o, M_o, N_o, P_o, Q_o, R_o are defined in equations (3.23) and (3.26). Units are such that ($\hbar = m_e = c = 1$). $W = (p^2 + 1)^{\frac{1}{2}}$ and in these formulas $\dfrac{i \int \gamma_5}{\int \vec{\sigma} \cdot \vec{r}}$ is a real number, because of the assumption of the validity of time reversal invariance in strong interactions. The neutrino energy represented by q is equal to $W_o - W$, where W_o is the end-point beta energy.

CHAPTER IV

EXPERIMENTAL DATA ON $0 \rightarrow 0$ (YES) BETA TRANSITIONS ITS ANALYSIS WITH THE DEVELOPED THEORY AND THE RESULTS

In the previous chapter, we developed the theoretical formulas for the β longitudinal polarization and the shape factor for $0 \rightarrow 0$ (yes) transitions using the axial vector interaction and the correct form of the pseudoscalar interaction. In this chapter, Section I gives the presently available experimental data on the β longitudinal polarization and the β spectrum of $0 \rightarrow 0$ (yes) transitions. Section II starts with a short discussion of the finite nuclear size corrections and the finite deBroglie wavelength effects; then the numerical coefficients for the formulas of the β longitudinal polarization and the β shape factor are given in Table IV for Pr^{144} $(0^- \rightarrow 0^+)$ and in Table V for Ho^{166} $(0^- \rightarrow 0^+)$. In Section III, the methods employed for the analysis of the experimental data are described and the results of the extensive computations are graphically presented. The main results are given in Section IV.

In the final chapter, we give the conclusions and a discussion of these conclusions.

I. EXPERIMENTAL DATA ON $0 \rightarrow 0$ (YES) TRANSITIONS

The $0 \rightarrow 0$ (yes) beta transitions have been established[1] in the decays of Pr^{144}, Ho^{166}, Eu^{152} and Ce^{144}. A $0 \rightarrow 0$ (yes) transition has been re-

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[1] D. Strominger, J. M. Hollander and G. T. Seaborg, Revs. Modern Phys. 30, 585 (1958). See pages 724, 743, 732 and 723.

ported[2] in Tl^{206}.

The relevant experimental data for our purpose of investigating the existence of the pseudoscalar interaction are (1) the β longitudinal polarization[*] and (2) the β shape factor in $0 \rightarrow 0$ (yes) transitions.

(1) $Pr^{144} \rightarrow Nd^{144}$ (β^-)

The decay scheme of Porter and Day[3] is given in Figure 1. The maximum β^- kinetic energy is 3 Mev in the $0^- \rightarrow 0^+$ branch. Pr^{144} has a half-life time of 17 minutes. The advantage of the study of Pr^{144} ($0^- \rightarrow 0^+$) is that this transition is very intense ($\sim 98\%$). The log ft value is 6.6.

A number of groups[3, 4, 5] have studied the β^- spectrum and the general conclusion is that the β^- shape factor of the $0^- \rightarrow 0^+$ branch is independent of the β^- energy within 6%. For our analysis we take the shape factor as given by Porter and Day.[3]

Table III summarizes the β^- longitudinal polarization measurements of the $0^- \rightarrow 0^+$ branch.

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[2] L. Zyrianova, Bull. Acad. Sciences U.S.S.R. 20, 1280 (1956). (Translated by Columbia Technical Translation, New York).

[*] For a recent review article, see L. A. Page, Revs. Mod. Phys. 31, 759 (1959).

[3] F. T. Porter and P. P. Day, Phys. Rev. 114, 1286 (1959).

[4] R. L. Graham, J. S. Geiger and T. A. Eastwood, Can. J Phys. 36, 1084 (1958). This paper contains the references to the previous work.

[5] N. J. Freeman, Proc. Phys. Soc. 73, 600 (1959).

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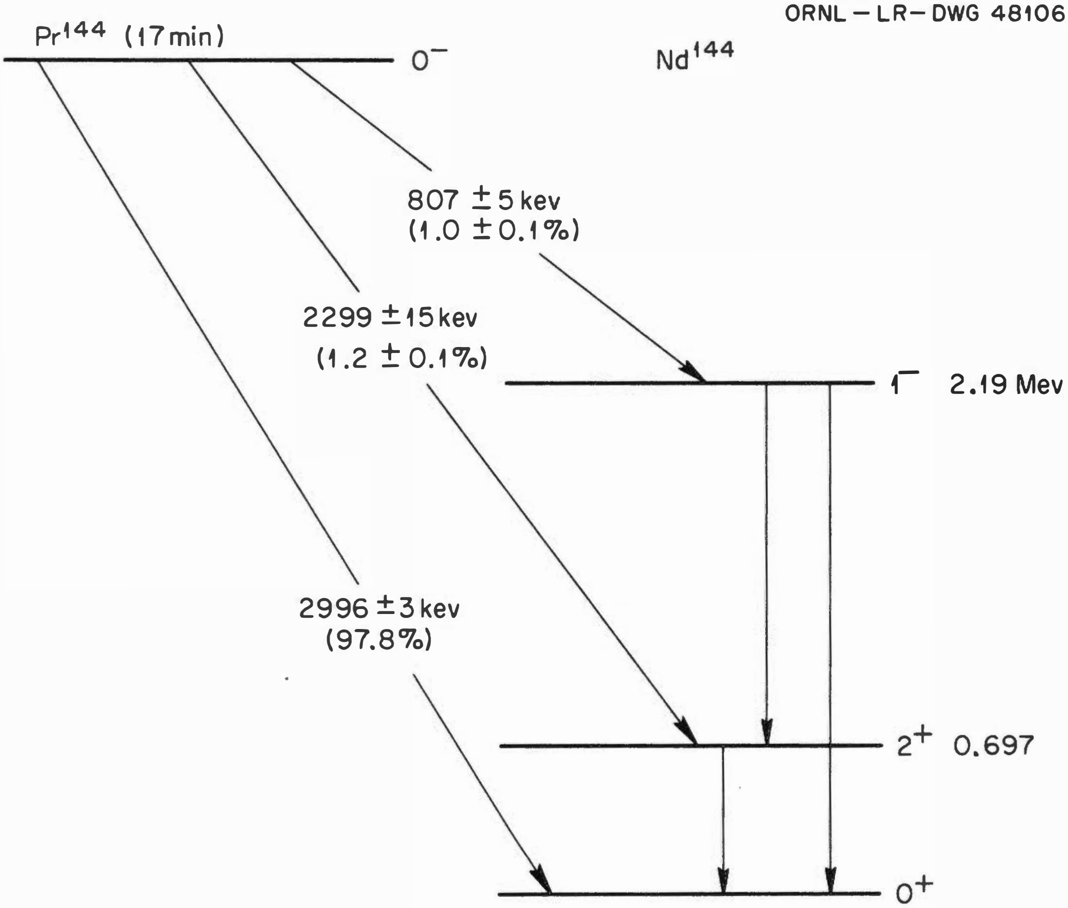

Figure 1. Decay Scheme of Pr^{144} of Porter and Day, *Phys. Rev.* 114, 1286 (1959). The numbers in the parenthesis refer to intensity.

TABLE III

EXPERIMENTA DATA ON BETA LONGITUDINAL POLARIZATION

OF Pr^{144} $0^- \rightarrow 0^+$

Longitudinal Polarization in units of (v/c)	Energy Range of β particles	Method of Measurement
$-(0.986 \pm 0.030)$	1 Mev to 3 Mev	Møller Scattering[1]
$-(1. \pm {}^{0.00}_{0.13})$	0.3 Mev to 3 Mev	Circulation Polarization[2] of Bremsstrahlung
$-(0.96 \pm 0.04)$	0.3 Mev to 3 Mev	Multiple - Mott Scattering[3]
$-(0.90 \pm 0.22)$	0.3 Mev to 3 Mev	Møller Scattering[4]
$-(0.77 \pm 0.21)$	0.4 Mev to 1.1 Mev	Møller Scattering[5]
$-(1.08 \pm -.26)$	1.2 Mev to 3 Mev	Møller Scattering[5]

[1] W. A. W. Mehlhop, Ph.D. dissertation (Unpublished) Washington University, Saint Louis (1959).

[2] S. G. Cohen and R. Wiener, Nuclear Physics 15, 79 (1960).

[3] J. Heintze, Z. Physik 150, 134 (1958).

[4] Geiger, et. al., Phys. Rev. 112, 1684 (1958).

[5] Frauenfelder, et. al., Phys. Rev. 107, 643 (1957).

The most accurate measurement of the β^- longitudinal polarization of the ($0^- \rightarrow 0^+$) branch is due to Mehlhop[6] et al. In this experiment a relative measurement of the β^- longitudinal polarization was made with respect to the β^- particles of the $2^- \rightarrow 0^+$ (unique) transition of Y^{90}. The β^- longitudinal polarization in Y^{90} ($2^- \rightarrow 0^+$) was assumed to be $-\frac{v}{c}$. Mehlhop found the average of $\frac{P_{\parallel}}{v/c}$ over the β^- spectrum of Pr^{144} ($0^- \rightarrow 0^+$), for the kinetic energy of the β^- particles from 1 Mev to the end point (~ 3 Mev), to be

$$\left\langle \frac{P_{\parallel}}{v/c} \right\rangle = -(0.986 \pm 0.030)$$

v/c is the ratio of the β^- particle velocity to the vacuum velocity of light.

(2) <u>$Ho^{166} \rightarrow Er^{166}$</u> ($\beta^-$)

The decay scheme[7] of Ho^{166} is shown in Figure 7. Its half-life time is 27 hours and its log ft value is 8.2. The β^- spectrum of the $0^- \rightarrow 0^+$ branch ($\sim 47\%$ intense) has not been very carefully studied experimentally.* We do not, therefore, make a detailed analysis of the shape factor.

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[6] W. A. W. Mehlhop, E. D. Lambe and T. Pond, Bull. Am. Phys. Society 5, 9 (1960). Also W. A. W. Mehlhop, Ph.D. dissertation, (Unpublished), Washington University, Saint Louis, 1959.

[7] J. M. Cork, M. K. Brice, R. G. Helmer and R. W. Woods, Jr., Phys. Rev. 110, 526 (1958). Also see Strominger et al., op. cit. p. 743.

*Private Communication from Dr. R. L Graham.

Bühring[8] measured the β^- longitudinal polarization in the $0^- \rightarrow 0^+$ branch and the results are:

$$\left\langle \left(-\frac{P_{\parallel}}{v/c} \right)_{Ho^{166}} \right\rangle = (0.99 \pm 0.02) \left\langle \left(-\frac{P_{\parallel}}{v/c} \right)_{P^{32}} \right\rangle$$

The range of the β^- energy was from 0.18 Mev to 1.8 Mev and the angular brackets mean the averaging over this range of the β^- energy. P^{32} $(1^+ \rightarrow 0^+)$ is an allowed β^- transition and we assume, for our analysis, $P_{\parallel} = -v/c$ in this case.

(3) $\underline{Eu^{152} \rightarrow Gd^{152}}$ (β^-)

The decay scheme[9] is given in the tables of isotopes by Strominger et al. The intensity of the $0^- \rightarrow 0^+$ transition is $\sim 70\%$ and the half-life time is 9.2 hours. No accurate measurement of the β^- spectrum is available. Also the β^- longitudinal polarization has not been measured. We do not carry out any analysis of such cases where the accurate measurements of these are not available.

(4) $\underline{Ce^{144} \rightarrow Pr^{144}}$ (β^-)

Strominger[10] et al. give the decay scheme. The half-life time is about 285 days. No detailed study of the beta spectrum has been made of the (76%) intense $0^+ \rightarrow 0^-$ transition and also no measurement of the beta longitudinal polarization has been reported.

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[8]W. Bühring, Z. Physik <u>155</u>, 566 (1959).

[9]Strominger et al., *op. cit.*, p. 732.

[10]Strominger et al., *op. cit.*, p. 723.

(5) $Tl^{206} \rightarrow Pb^{206}$ (β^-)

The decay scheme is given by Zyrianova.* The half life time is 4.2 minutes and the log ft value is 5.2. The beta end point energy is 1.5 Mev and the β^- spectrum is a simple one. The beta energy differential spectrum is of the allowed shape within 5%. But a careful analysis is desirable to establish the decay scheme. Accurate measurements of the β^- longitudinal polarization and β^- spectrum are required.

Now we describe the method of computation and the nuclear finite size and the finite deBroglie wavelength effects.

II. METHOD OF COMPUTATION

The formulas of the β longitudinal polarization and the β shape factor, as developed in this work, are given for $0 \rightarrow 0$ (yes) beta transitions on pages 89 and 90. In these expressions, we have A_0, B_0, C_0, D_0, L_0, M_0, N_0, P_0, Q_0 and R_0, which depend on the electronic radial functions evaluated at the nuclear radius. For the β decays of Pr^{144} and Ho^{166}, these electronic radial function f_1, g_1, f_{-1}, and g_{-1}, evaluated at the nuclear radius $0.43\ \alpha\ A^{\frac{1}{3}} \left(\frac{\hbar}{mc}\right)$, were computed taking into consideration (1) the finite nuclear size effect, by considering the nucleus as a sphere with a uniform charge distribution and (2) the finite deBroglie wavelength effects. Sin (δ_1 - δ_{-1}) was computed, only considering the Coulomb effects. In the following, we briefly describe these two effects and then give the numerical coefficients of the formulas.

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*Zyrianova, loc. cit.

A. Nuclear Finite Size Effects

In our calculations, we have taken for the electron

$$\psi_{\varkappa}^{\mu} = \begin{pmatrix} -i\, f_{\varkappa}(r)\chi^{\mu}_{-\varkappa} \\ g_{\varkappa}(r)\, \chi^{\mu}_{\varkappa} \end{pmatrix}$$

where $f_{\varkappa}(r)$ and $g_{\varkappa}(r)$ are real and are the radial functions. It has been shown by Rose[11] that the indicial behavior of the radial wavefunctions for any central field $V(r)$ is given by the following:

(a) For $\varkappa = k$; k is a positive integer,

using the notation $x \equiv \frac{r}{R}$

and $w(x) \equiv (2j + 2) \quad x^{2j + 1}$,

$$f_k \sim C_1 \quad R^{j + \frac{1}{2}}$$

$$g_k \sim \frac{C_1}{(2j + 2)} \quad (R)^{j + \frac{3}{2}} \int_0^1 V(x) \quad w(x)\, dx \tag{4.1a}$$

(b) and for $\varkappa = -k$

$$f_{-k} \sim \frac{C_2}{2j + 2} \quad R^{j + \frac{3}{2}} \int_0^1 V(x) \quad w(x)\, dx$$

$$g_{-k} \sim C_2 \quad R^{j + \frac{1}{2}} \tag{4.1b}$$

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[11] M. E. Rose, Phys. Rev. 82, 389 (1951).

From (4.1) it is clear that g_k and f_{-k} are "field sensitive" and for large angular momentum, the weighting function approaches a delta function. However, for small values of angular momentum, this effect becomes important. In $0 \rightarrow 0$ (yes) transitions, the finite nuclear size corrections are therefore important.

We take R the nuclear radius as $0.43\,\alpha A^{\frac{1}{3}}$ ($\frac{\hbar}{mc}$ units) and a uniform charge distribution in the nucleus, which gives:

$$V = -\frac{\alpha Z}{r}; \quad r > R \tag{4.2}$$

$$V = -\frac{\alpha Z}{2r}\left(3 - \frac{r^2}{R^2}\right); \quad r < R$$

$\alpha \approx \frac{1}{137}$ and Z is the number of protons of the daughter nucleus. The details of the computation of the radial functions, using the potential in (4.2) are given elsewhere.[12]

B. <u>Finite deBroglie Wavelength Effects</u>.

Usually, the analytic expressions are given[13,14] for Lo, Mo, No, Po, Qo, Ro and also* Ao, Bo, Co, and Do by considering the first term in the power series expansion of these combinations of the radial functions.

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[12] C. P. Bhalla and M. E. Rose, Oak Ridge National Laboratory Report (to be issued).

[13] E. Greuling, Phys. Rev. <u>61</u>, 568 (1942).

[14] D. L. Pursey, Phil. Mag. <u>42</u>, 1193 (1951).

*Appendix D. of this dissertation.

The validity of this procedure rests on the fact that the beta particle deBroglie wavelength is very large compared to the nuclear radius or to be more exact $PR \ll 1$. However, where we have either large β momentum, or for small momentum but αZ not small, these approximate expressions are not quite valid. The desirable procedure is, therefore, not to terminate the power series expansion by considering the first term only, but to take into account a large number of the terms in this expansion. The correction arising from this procedure has been called the finite deBroglie wavelength effect.[15]

In our analysis, these corrections have been properly considered by computing all the required functions on the ORACLE of the Oak Ridge National Laboratory.

C. β Longitudinal Polarization and Spectrum Formulas of $Pr^{144}(0^- \rightarrow 0^+)$ and $Ho^{166}(0^- \rightarrow 0^+)$ with Numerical Coefficients

We write the formulas for the β longitudinal polarization and spectrum, as developed in this work as:

$$P_{\parallel} = -\frac{a_0 + a_1 \lambda^2 + a_2 \lambda - a_3 \xi^2 + (a_4 + a_5 \lambda)\xi}{b_0 + b_1 \lambda^2 + b_2 \lambda + b_3 \xi^2 + (b_4 + b_5 \lambda)\xi} \tag{4.3}$$

and the shape factor C_{β^-} as

$$C_{\beta^-} = b_0 + b_1 \lambda^2 + b_2 \lambda + b_3 \xi^2 + (b_4 + b_5 \lambda)\xi \tag{4.4}$$

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[15] M. E. Rose and C. L. Perry, Phys. Rev. 90, 479 (1953).

where we have introduced

$$\lambda \equiv \frac{i \int \gamma_5}{\int \vec{\sigma} . \vec{r}} \tag{4.5a}$$

$$\xi \equiv \frac{C_P}{MC_A} \tag{4.5b}$$

and

$$a_o \equiv B_o + \frac{q}{3} D_o - \frac{q^2}{9} A_o \tag{4.6a}$$

$$a_1 \equiv -A_o \tag{4.6b}$$

$$a_2 \equiv D_o - \frac{2}{3} q A_o \tag{4.6c}$$

$$a_3 \equiv \frac{1}{4} \Big\{ (u^2 - 1) B_o + \frac{1}{3} q \left[8u B_o - (u^2 + 1) D_o - 2u C_o \right] + \frac{q^2}{9} \left[16 B_o - 4(u D_o + C_o) - (u^2 - 1) A_o \right] \Big\} \tag{4.6d}$$

$$a_4 \equiv B_o + \frac{q}{3} (u C_o + D_o) + \frac{q^2}{9} (2 C_o - A_o) \tag{4.6e}$$

$$a_5 \equiv \frac{1}{2} \left[u C_o + D_o + \frac{2}{3} q (2 C_o - A_o) \right] \tag{4.6f}$$

$$b_o \equiv M_o - \frac{2}{3} q N_o + \frac{1}{9} q^2 L_o \tag{4.7a}$$

$$b_1 \equiv L_o \tag{4.7b}$$

$$b_2 \equiv -2 (N_o - \frac{1}{3} q L_o) \tag{4.7c}$$

$$b_3 \equiv \frac{1}{4} \Big\{ (u^2 + 1) M_o - 2u Q_o + \frac{2}{3} q \left[4u M_o - 4Q_o + (u^2 - 1) N_o \right] + \frac{q^2}{9} \left[(u^2 + 1) L_o - 2u P_o + 16 M_o + 8u N_o - 8R_o \right] \tag{4.7d}$$

$$b_4 \equiv M_o - uQ_o - \frac{2}{3} q (N_o + 2 Q_o) + \frac{q^2}{9} (-uP_o + L_o - 4R_o) \tag{4.7e}$$

$$b_5 \equiv -\left[uR_o + N_o - \frac{1}{3} q \left\{ - uP_o + L_o - 4R_o + R^2(uQ_o - M_o)\right\} + \frac{1}{9} q^2 R^2 (uR_o - N_o - 4Q_o)\right] \tag{4.7f}$$*

L_o, M_o, N_o, P_o, Q_o, R_o, and A_o, B_o, C_o, and D_o, are defined in (3.26) and (3.23) on pages 79 and 75 respectively.

These coefficients, as defined in (4.6) and (4.7) are given in Table IV for Pr^{144} ($0^- \rightarrow 0^+$), taking $W_o = 6.854$ (mc^2 units), and in Table V for H_o^{166} ($0^- \rightarrow 0^+$), taking $W_o = 4.514$ (mc^2 units). The β momentum p is in mc units. These coefficients are given for various values of p, assuming a uniform charge distribution in the nucleus, with radius $0.428\,\alpha\, A^{\frac{1}{3}} (\frac{\hbar}{mc})$ and taking into account the finite deBroglie wavelength effects.

III. METHOD OF ANALYSIS OF DATA AND RESULTS

In (4.3) and (4.4) we have two unknown quantities (1) $\xi = \frac{C_P}{MC_A}$, the ratio of the coupling constants of the pseudoscalar interaction and the axial vector interaction divided by the nucleon mass in ($\sim$ 1800) units of m, and (2) λ, the ratio of the nuclear matrix elements. Our motivation is to investigate the existence of the pseudoscalar interaction. Therefore, first we investigate whether or not the axial vector interaction alone can explain the experimental data ($\xi \simeq 0$); then we want to find an upper limit of the value of ξ which is consistent with the experimental

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*In (4.7f), the terms containing R^2 can be neglected.

TABLE IV

Pr^{144} $0^- \rightarrow 0^+$. NUMERICAL COEFFICIENTS FOR BETA LONGITUDINAL POLARIZATION AND SHAPE FACTOR FORMULAS*

p	a_0	a_1	a_2	a_3	a_4	a_5	b_0	b_1	b_2	b_3	b_4	b_5
1.0	112.3	0.6400	16.97	14290	91.54	1.922	153.4	0.9026	23.53	20780	2260	175.7
1.5	131.5	0.7487	19.85	17740	108.5	2.182	153.9	0.8976	23.50	21830	1768	141.1
2.0	140.7	0.7992	21.21	20250	117.7	2.261	153.8	0.8917	23.43	23050	1445	116.7
2.5	145.1	0.8234	21.86	22390	123.3	2.248	153.6	0.8854	23.32	24480	1217	99.48
2.783	146.6	0.8310	22.08	23520	125.7	2.229	153.3	0.8816	23.25	25320	1116	91.95
3.0	147.3	0.8348	22.19	24370	127.3	2.203	153.1	0.8787	23.18	26010	1050	87.00
3.5	148.5	0.8395	22.33	26380	130.5	2.128	152.5	0.8720	23.05	27680	923.5	77.60
4.0	148.9	0.8402	22.37	28410	133.2	2.047	151.8	0.8651	22.92	29490	825.4	70.33
4.5	148.9	0.8385	22.35	30510	135.8	1.964	151.1	0.8582	22.76	31400	746.6	64.52
5.0	148.7	0.8354	22.29	32730	138.2	1.870	150.3	0.8512	22.62	33480	681.8	59.83
5.5	148.2	0.8312	22.20	35020	140.6	1.770	149.5	0.8442	22.45	35670	627.9	55.94
6.0	147.6	0.8264	22.09	37430	142.9	1.690	148.7	0.8372	22.31	37990	582.0	52.67
6.5	147.0	0.8211	21.97	39960	145.3	1.595	147.8	0.8302	22.15	40460	542.6	49.89

*Equations (4.3) and (4.4). These coefficients have been calculated considering (1) the nuclear radius to be $0.428\alpha A^{1/3}(\hbar/mc)$, (2) the corrections due to the finite nuclear size and (3) the finite deBroglie wavelength effects.

TABLE V

Ho^{166} $0^- \rightarrow 0^+$. NUMERICAL COEFFICIENTS FOR BETA LONGITUDINAL POLARIZATION AND SHAPE FACTOR FORMULAS*

p	a_0	a_1	a_2	a_3	a_4	a_5	b_0	b_1	b_2	b_3	b_4	b_5
0.76	95.95	0.5323	14.30	16800	84.46	1.441	152.6	0.8758	23.12	28130	2809	217.9
1.0	111.8	0.6200	16.66	20050	99.02	1.661	152.8	0.8734	23.10	28670	2505	195.1
1.5	130.8	0.7239	19.47	24820	117.5	1.882	153.1	0.8673	23.05	30080	1983	156.2
2.0	139.6	0.7713	20.76	28220	127.4	1.936	152.9	0.8602	22.93	31750	1615	128.7
2.5	143.9	0.7932	21.37	31080	133.7	1.919	152.4	0.8525	22.80	33620	1354	109.3
3.0	145.9	0.8026	21.65	33700	138.1	1.869	151.6	0.8446	22.62	35620	1163	95.15
3.5	146.8	0.8055	21.76	36290	141.6	1.801	150.9	0.8364	22.47	37820	1018	84.52
4.0	147.0	0.8046	21.76	38910	144.7	1.724	149.9	0.8282	22.28	40120	905.6	76.26

*Equations (4.3) and (4.4). These coefficients have been calculated considering (1) the nuclear radius to be $0.428\alpha A^{1/3}(\hbar/mc)$, (2) the corrections due to the finite nuclear size and (3) the finite deBroglie wavelength effects.

data. However, there is one difficulty: λ has also to be treated as a parameter.

Nuclear Matrix Elements

No reliable calculations of λ, the ratio of the nuclear matrix elements have been done because not enough is known about the nuclear forces. However, several attempts have been made to calculate the value of λ, using certain assumptions. Equations (4.9) and (4.10) give these results. We write

$$\lambda \equiv \frac{i\int\gamma_5}{\int\vec{\sigma}\cdot\vec{r}} = -\tfrac{1}{2}\Lambda\left(\frac{\alpha Z}{R}\right) \qquad (4.8)$$

Then, on the basis of the single particle (j - j coupling scheme) model, Rose and Osborn[16] give: ($\hbar = m_c = c = 1$)

$$\Lambda = 2\left[\left(R\frac{\mathfrak{J}_0}{\mathfrak{J}_1}\right) + \frac{R}{\alpha Z}(W_o - \Delta)\right) \qquad (4.9a)$$

$\left(R\frac{\mathfrak{J}_0}{\mathfrak{J}_1}\right) \simeq 1$ to $4/3$. W_o is the maximum β total energy.

Δ is the mass difference of neutron and proton and R is the nuclear radius in units of $\left(\frac{\hbar}{mc}\right)$.

Using a semi-empirical energy formula, Ahrens and Feenberg[17] obtain:

$$\Lambda \doteq 2\left[1.2 + \frac{R}{\alpha Z}(W_o - \Delta)\right] \qquad (4.9b)$$

Pursey,[18] using an explicit form of the nuclear hamiltonian and single

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[16] M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1326 (1954).

[17] T. Ahrens and E. Feenberg, Phys. Rev. 86, 64 (1952).

[18] D. L. Pursey, Phil. Mag. 42, 1193 (1951).

particle wavefunctions, gives

$$\Lambda = 2\left[1 + \frac{R}{\alpha Z}(W_o - \Delta)\right] \qquad (4.9c)$$

For Pr^{144} ($0^- \rightarrow 0^+$), Pearson[19] has calculated λ, assuming the single particle potential to be an infinitely deep square well. With this assumption:

$$\frac{i\int\gamma_5}{\int\vec{\sigma}.\vec{r}} = 2.5 \qquad (4.10a)$$

And using the harmonic oscillator potential of equal strengths for the parent and the daughter nucleus, he obtains:

$$\lambda = 8 \qquad (4.10b)$$

But from the equations (4.9) one gets for Pr^{144} ($0^- \rightarrow 0^+$)

$$\lambda \simeq -30 \text{ to } -37 \qquad (4.10c)$$

and for Ho^{166} ($0^- \rightarrow 0^+$)

$$\lambda \simeq -31 \text{ to } -37 \qquad (4.11)$$

The Coulomb contribution provides the dominant term in the expressions of λ as given in equations (4.9). This fact favors a value of λ from -30 to -40 for Pr^{144} and Ho^{166}.

But it must be remembered that these calculated values of λ cannot be accepted with much confidence, because of the lack of the knowledge concerning the nuclear forces; therefore, we shall consider λ as a para-

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[19] J. M. Pearson, Can. J. Physics <u>38</u>, 148 (1960).

meter in the following analysis, rather than relying on certain calculated values.

A. Analysis of Data on Pr^{144} $(0^- \rightarrow 0^+)$

1. β^- longitudinal polarization

(a) Pure Axial Vector Interaction. (i.e. $\xi = 0$)

The calculated β^- longitudinal polarization ($P_{\parallel}$) in units of $-v/c$, is plotted (Figure 2) versus β^- momentum in mc units for $\lambda = 10$, 30, 110, -30, -50 and -150. The region of β^- kinetic energy for the measurement of Mehlhop* et al. is from 1 Mev to ~ 3 Mev, and this is shown in the Figure. From Figure 2, we observe that in the β^- energy range of Mehlhop's data, the calculated $-\frac{P_{\parallel}}{v/c}$ deviates less than 0.25% from 1.000 for large values of $|\lambda|$ (>100). The upper and the lower limits of $\left\langle -\frac{P_{\parallel}}{v/c} \right\rangle$ of this data are 1.016 and 0.956 respectively. A large number of the values of λ, the ratio of the nuclear matrix elements, can be found, for which the calculated values of $\left\langle -\frac{P_{\parallel}}{v/c} \right\rangle$ lie well within these limits (of Mehlhop's data). Therefore, the pure axial vector interaction can easily explain his data.*

(b) Axial Vector and Pseudoscalar Interactions. ($\xi \neq 0$)

Using both the axial vector and the pseudoscalar interactions, we now investigate the regions of the values of ξ and λ, for which the calculated $\left\langle -\frac{P_{\parallel}}{v/c} \right\rangle$ lies within the experimental limits. Figure 3 shows

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*Mehlhop et al., op. cit. And also Mehlhop, op. cit.

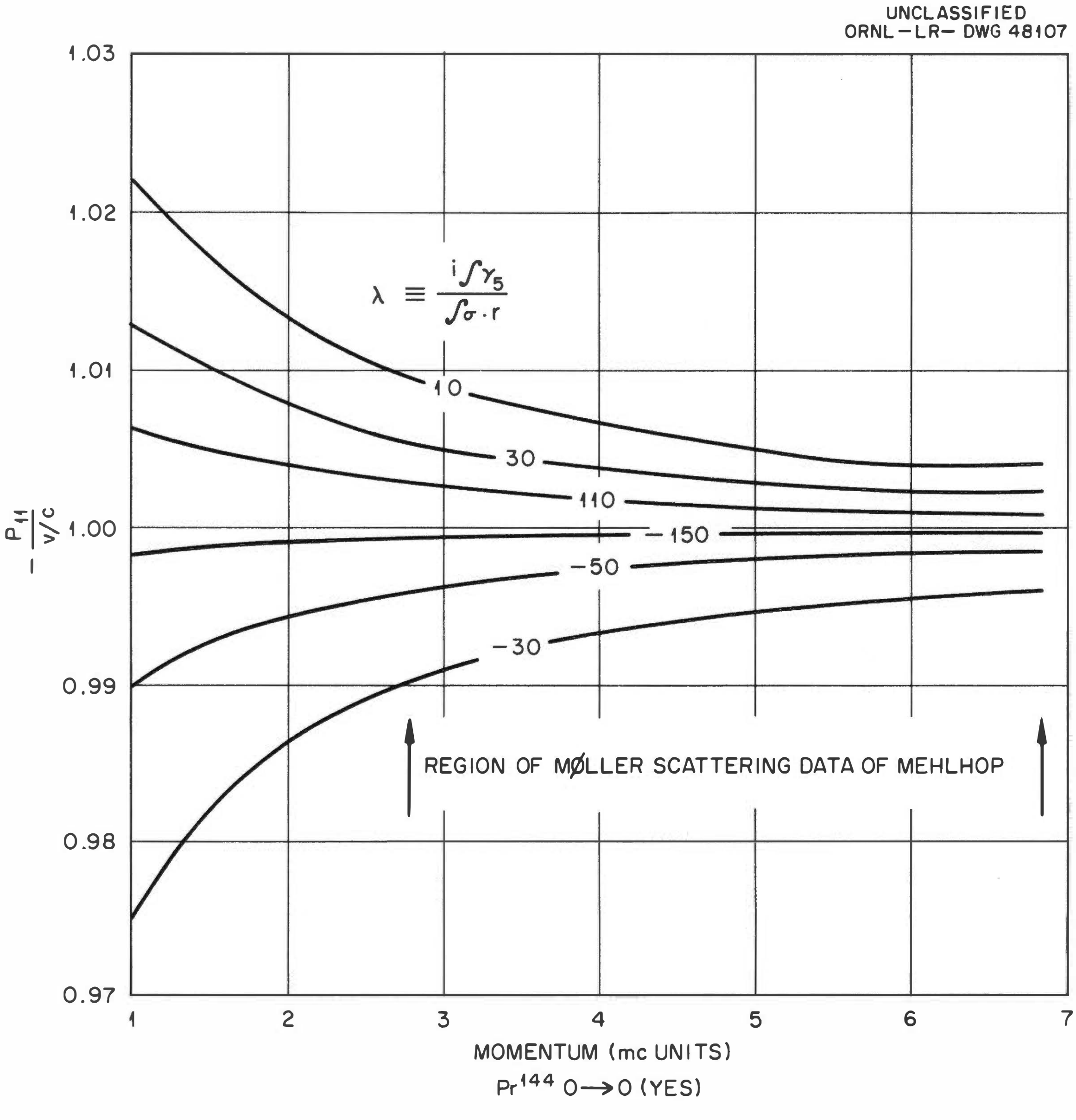

Figure 2. Calculated Longitudinal Polarization in Units of $-v/c$ versus β Momentum for A Interaction Only. The numbers attached to the curves give the ratio of the nuclear matrix elements (λ).

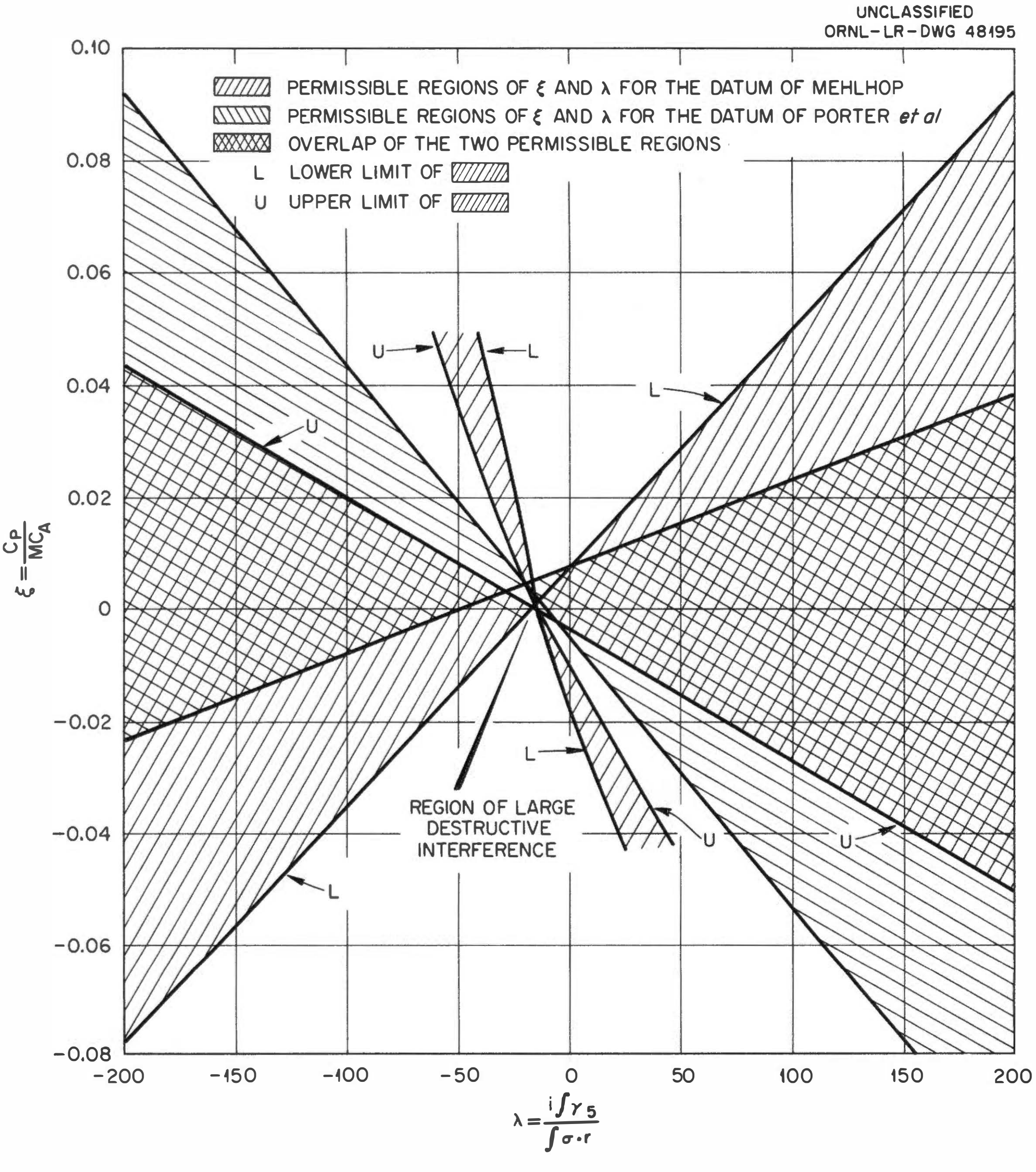

Figure 3. Pr^{144} $0^- \longrightarrow 0^+$. The permissible values of the parameters ξ, the ratio of the coupling constants, and λ, the ratio of the nuclear matrix elements for the polarization and shape factor data of Mehlhop and Porter *et al.*, respectively.

these permissible regions. In this Figure, the curves designated by L and U represent the loci of the points in the (ξ, λ) plane, for the lower and upper limits (0.956 and 1.016 respectively) of $\langle -\frac{P_{\parallel}}{v/c}\rangle$. The lightly shaded regions, in between the curves designated by L and U, give the permissible values of ξ and λ for the longitudinal polarization measurement of Mehlhop.

Figures 4, 5, and 6 respectively, give the typical curves of the calculated $-\frac{P_{\parallel}}{v/c}$ versus β^- momentum for the following:

(1) $\xi = 0.05$; $\lambda = 90, 100, -65, -150$ and -190

(2) $\xi = -0.05$; $\lambda = 150, 175, 190, -110, -150$ and -190

and (3) $\xi = 0.002, 0.004, 0.006, 0.008, -0.002, -0.004, -0.006$, and -0.008 for $\lambda = -35$

It is interesting to notice that for the pure pseudoscalar interaction, our calculated value of $P_{\parallel}$ is $(0.97)\,\frac{v}{c}$ at $p = 1.0$; however, using plane waves* for the electron we get $P_{\parallel} = 0.8\ v/c$ at $p = 1.0$.

2. β^- shape factor

We compare the calculated shape factor curve indirectly with the experimental data. The comparison is made between the calculated shape factor and a cubic (in p) fit through the experimental data. The cubic fit to the data on the shape factor data of Porter and Day is as follows:

$$C_{\beta^-} = 9459.32 - 375.752\,p + 89.840\,p^2 - 8.4994\,p^3 \qquad (4.12)$$

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*This implies neglecting the Coulomb effect and other corrections, used in our analysis.

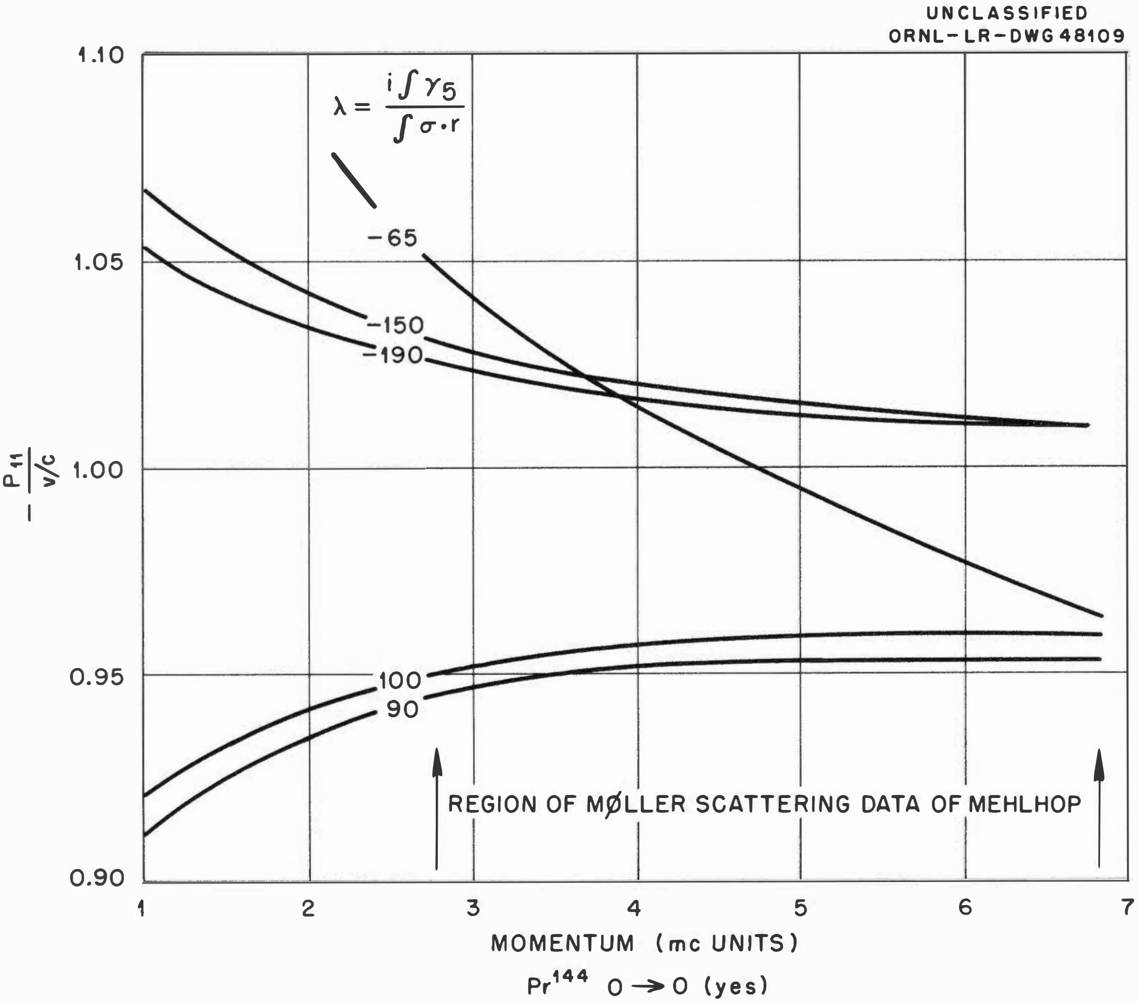

Figure 4. Calculated Longitudinal Polarization in Units of $-v/c$ versus β Momentum for $C_P/MC_A = 0.05$. The numbers attached to the curves give λ.

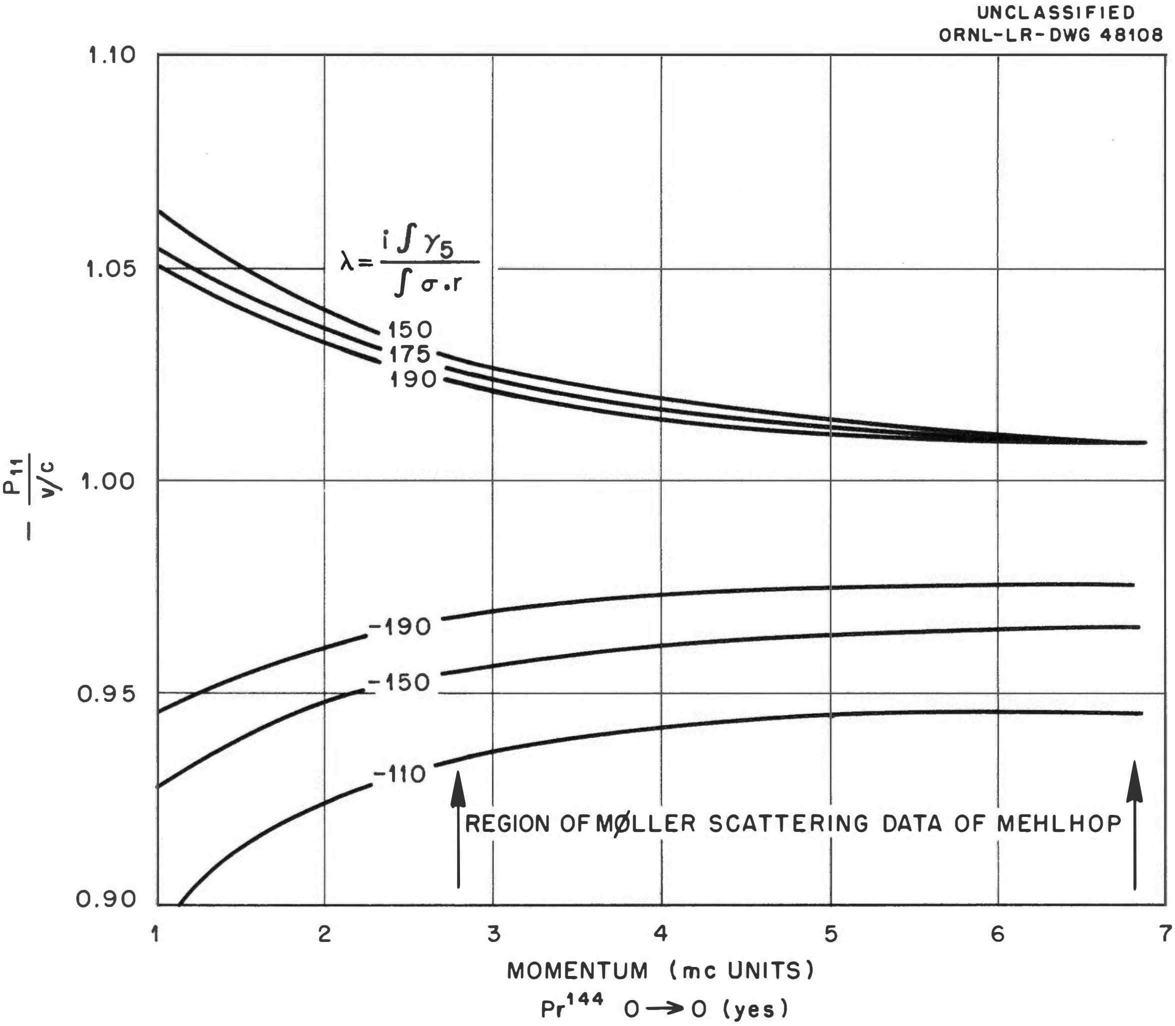

Figure 5. Calculated Longitudinal Polarization in Units of $-v/c$ versus β Momentum for $C_P/MC_A \simeq -0.05$. The numbers attached to the curves give λ.

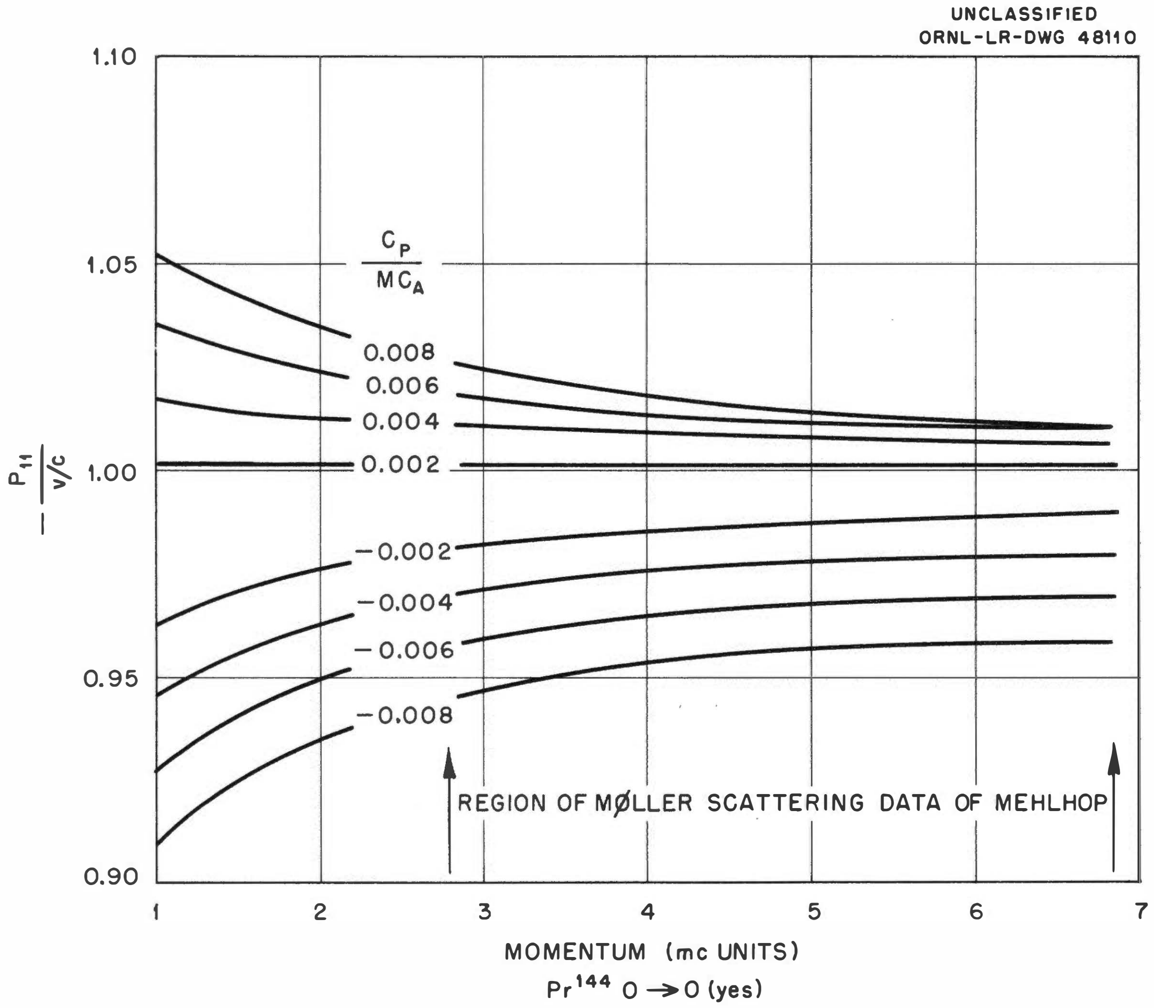

Figure 6. Calculated Longitudinal Polarization in Units of $-v/c$ versus β Momentum for $\lambda = -35$. The numbers attached to the curves give C_P/MC_A.

The mean sum squared of the residuals* of this fit from the experimental data is 1.217.

For our analysis, we arbitrarily normalize the shape factor (4.12) and the calculated shape factor (from Table IV) to unity at p = 5.0.

Thus we get from (4.12): C_{β^-} normalized to unity at p = 5.0.

$$C_{\beta^-} = \frac{1}{8764.14}\left[\,9459.32 - 375.752p + 89.840p^2 - 8.4994p^3\,\right] \qquad (4.13)$$

For p = 1.0 to p = 6.5 in the steps of 0.5, we calculate $\left(\frac{\Delta X_i}{X_i}\right)^2$, where ΔX_i is the deviation of the calculated shape factor (from our formulas) from the value (X_i) given by the cubic fit. Then we compute

$$\overline{\Delta} \equiv \frac{1}{11} \sum_{p=1}^{p=6.5} \left(\frac{\Delta X_i}{X_i}\right)^2$$

for the cases of interest. We consider the calculated shape factor a "reasonable" fit if $\overline{\Delta} < 0.005$; this generally corresponds to the maximum deviation $\frac{\Delta X_i}{X_i}$ being less 4%.

(a) Pure A. Interaction ($\xi \equiv \frac{C_P}{MC_A} = 0$)

We find, for the pure axial vector interaction, that for $\lambda > 0$, we always get a reasonable fit and for $\lambda = -50$, we also get such a fit to the shape factor.

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*The mean sum squared of the residuals is defined as $\dfrac{\sum_1^{46} (n_c - n)^2}{(46 - 4)}$; where n_c and n are the computed values and the experimental values of the β shape factor. There were 46 experimental points in this experimental shape factor.

This implies that the pure axial vector interaction can explain the experimental shape factor of Porter and Day.

It is interesting to observe that it appears in the literature that there is no fit to the shape factor for $\lambda < 0$, contrary to our finding ($\lambda < -50$). In the previous works, $-\lambda$ was only considered in the region of a large destructive interference ($\sim$10 to 30), where there is no fit. We have investigated taking $-\lambda$ to be as large as 200 and find that for $\lambda < -50$, there is a reasonable fit.

(b) A and P Interactions ($\xi \neq 0$)

We now consider the axial vector and the pseudoscalar interactions. The results of the computations are summarized in Figure 3, page 110. In this Figure, we show by the shaded regions, the permissible values of and λ for a reasonable fit to the data of Porter and Day.

The overlapping regions of the β shape factor fit and a fit to the polarization data of Mehlhop, is shown as a crosshatched region. The values of ξ in this crosshatched region depend on λ. It is interesting to notice that we can find values of ξ for $\lambda = -35$ which are consistent* with the experimental data. In the previous works, no such value of ξ $\left(= \frac{C_P}{MC_A}\right)$ was reported.

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*See Figure 6 for a plot of $-\frac{P_{\parallel}}{v/c}$ versus β^- momentum for $\lambda = -35$ and a number of values of ξ .

B. Analysis of Data on Ho^{166} $(0^- \rightarrow 0^+)$

The decay scheme of Ho^{166} of Cork et al.* is given in Figure 7.

β^- Longitudinal Polarization

(a) Pure A. Interaction ($\xi = 0$)

The calculated β^- longitudinal polarization in units of $- v/c$ is plotted (Fig. 8) versus β^- momentum in mc units for $\lambda = 10$, 30, 130, -30, -50 and 130. The range of the β^- kinetic energy for the β^-longitudinal polarization data of Buhring is 0.18 Mev to 1.8 Mev. A large number of the values of λ , the ratio of the nuclear matrix elements, can be found for which the calculated values of $\left\langle - \frac{P_{\parallel}}{v/c} \right\rangle$ lies within the experimental limits. Therefore, as before for Pr^{144} $(0^- \rightarrow 0^+)$, the pure axial vector interaction can easily explain Bühring's data.**

(b) A and P Interactions ($\xi \neq 0$)

Using both the axial vector and the pseudoscalar interaction, the regions of ξ and λ , for which the calculated value of $\left\langle - \frac{P_{\parallel}}{v/c} \right\rangle$ lies within the experimental limits, are shown (Fig. 9) by the shaded region. The curves denoted by L and U give the loci of the points in the (ξ, λ) plane for the lower and the upper limit (0.97 and 1.01 respectively) of the β^- longitudinal polarization data of Bühring.

In Figure 9, the region of large destructive interference, $(-25 < \lambda < -5)$

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*Cork et al., loc. cit. Modified by Strominger, op. cit. 743.

**Bühring, loc. cit.

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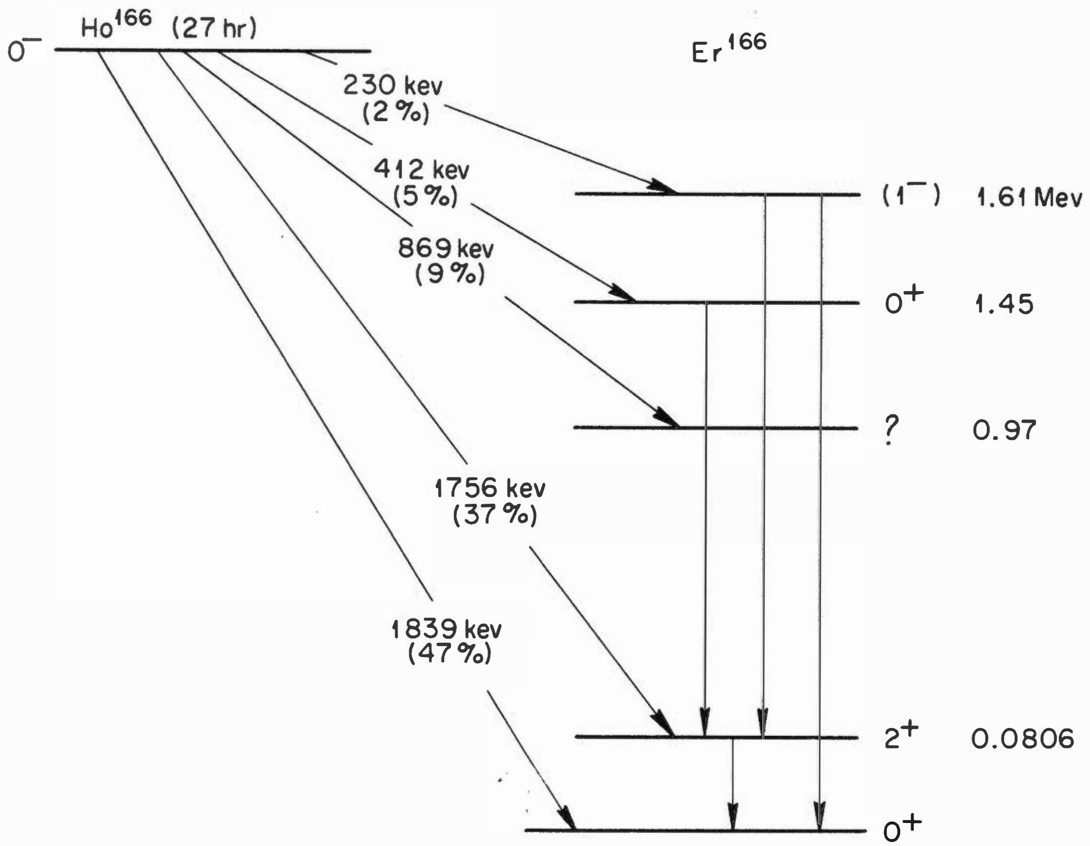

Figure 7. Decay Scheme of Ho^{166} of Cork ***et al., Phys. Rev.*** **110, 526 (1958). The numbers in the parenthesis refer to intensity.**

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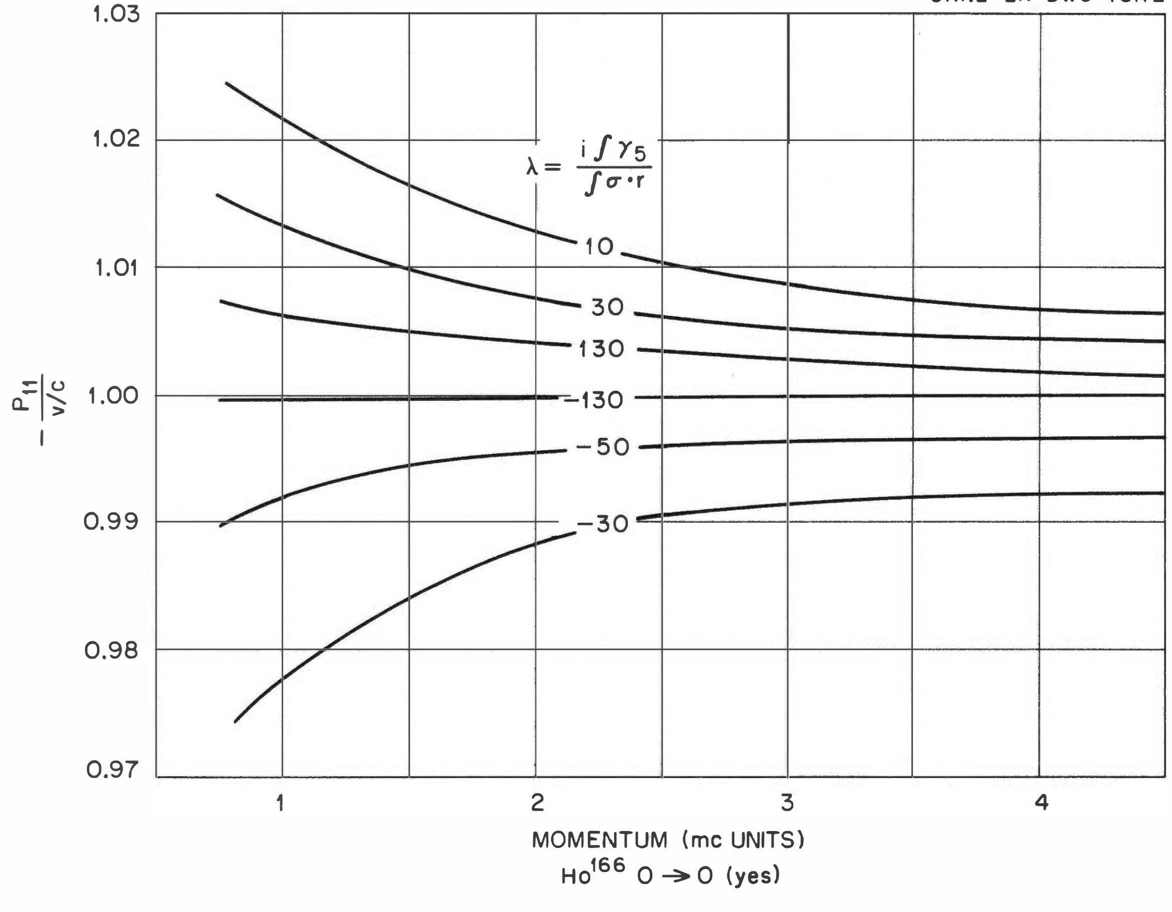

Figure 8. Calculated Longitudinal Polarization in Units of –v/c versus β Momentum for A Interaction Only. The numbers attached to the curves give the ratio of the nuclear matrix elements (λ).

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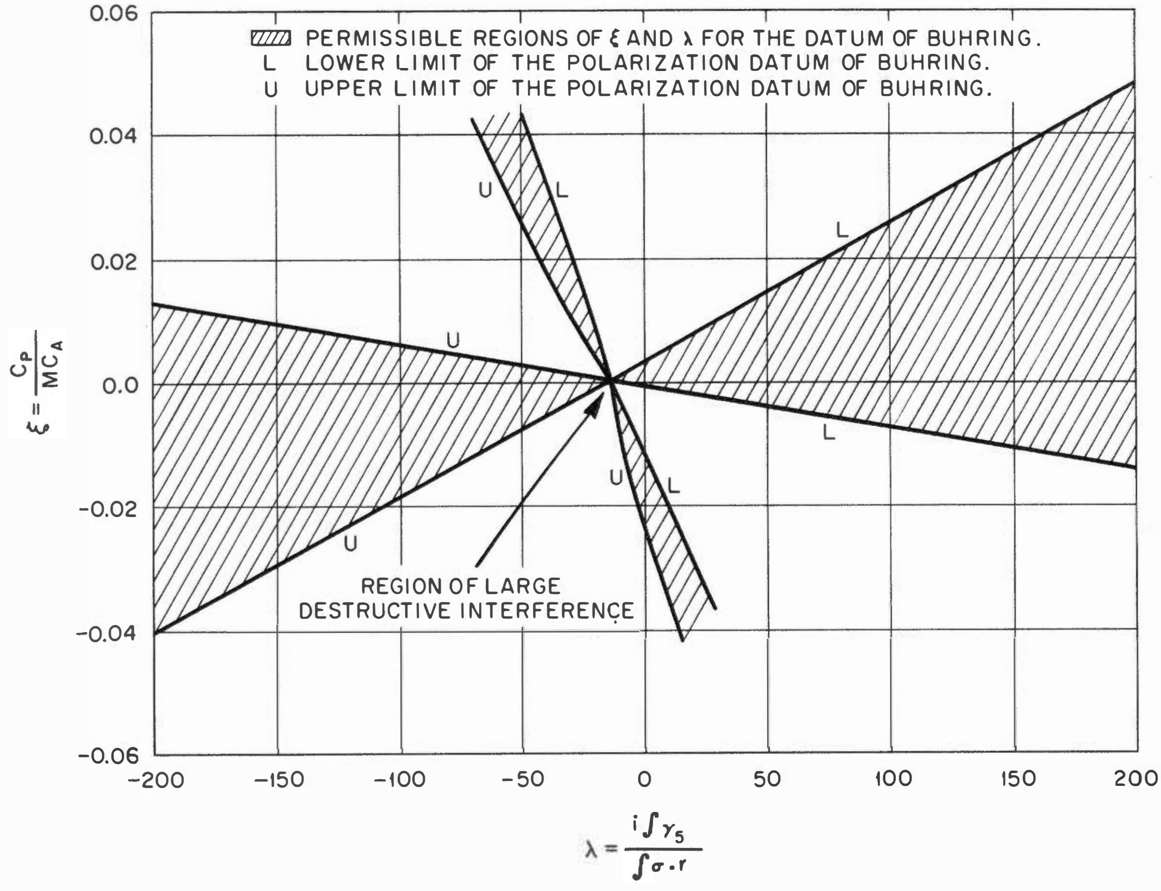

Figure 9. Ho^{166} $0^- \longrightarrow 0^+$. The permissible values of the parameters ξ, the ratio of the coupling constants, and λ, the ratio of the nuclear matrix elements for the plarization datum of Buhring.

is so indicated.

We do not analyze any shape factor for Ho^{166} $(0^- \rightarrow 0^+)$, because so far no accurate measurement exists.

IV. RESULTS

(1) We find that the pure axial vector interaction can explain the existing data on Pr^{144} $(0^- \rightarrow 0^+)$ and Ho^{166} $(0^- \rightarrow 0^+)$.

(2) We find the upper limit of $\frac{C_P}{MC_A}$, which is consistent with the experimental data.

<u>For Pr^{144} $(0^- \rightarrow 0^+)$</u>

(a) $\frac{C_P}{MC_A} = -0.05$ for λ as large as 200

(b) $\frac{C_P}{MC_A} = 0.045$ for $-\lambda$ as large as 200

<u>For Ho^{166} $(0^- \rightarrow 0^+)$</u>

(a') $\frac{C_P}{MC_A} = 0.048$ for λ as large as 200

(b') $\frac{C_P}{MC_A} = -0.04$ for $-\lambda \leq 200$

CHAPTER V

SUMMARY AND CONCLUSIONS

In Section I, we summarize what has been covered in the previous chapters. Then in Section II, we list the main points of this work including the assumptions used in this investigation of the existence of the pseudoscalar interaction. The conclusions of this work are given in Section III, followed by a brief discussion of these conclusions.

I. SUMMARY

In Chapter I, the statement of the problem considered in this dissertation is given: the investigation of the existence of the pseudoscalar interaction by the formulation of the theoretical expressions for the β longitudinal polarization and the β spectrum in $0 \rightarrow 0$ (yes) beta transitions and by a comparison of the existing experimental data with the developed theory.

In Chapter II, we explain the reasons for the incorrectness of the conventional treatment of the pseudoscalar interaction, and then give the correct form of the operator for this interaction.

In Chapter III, using the correct form of the pseudoscalar interaction and the conventional form of the axial vector interaction, we develop the β longitudinal polarization and the β shape factor formulas for $0 \rightarrow 0$ (yes) beta transitions.

In Chapter IV, we analyze the experimental data on Pr^{144} $(0^- \rightarrow 0^+)$ and Ho^{166} $(0^- \rightarrow 0^+)$ with the formulas of Chapter III, which in these cases are tabulated on pages 104 and 105 respectively.

II. MAIN POINTS OF THE PROCEDURE

1. We have used the conventional form of the axial vector interaction and the correct[1] operator for the pseudoscalar interaction. The derived formulas of the β longitudinal polarization and the β spectrum are given in (3.35) and (3.37), on pages 89 and 90 respectively, for the $0 \rightarrow 0$ (yes) beta transition. These developed formulas are expressed in terms of (1) ξ , the ratio of the coupling constants of the pseudoscalar interaction and the axial vector interaction, divided by the nucleon mass (~ 1800) in units of electron mass, (2) λ , the ratio of the nuclear matrix elements and (3) certain functions, which are defined in (3.23) and (3.26), depending on the electronic radial functions, evaluated at the nuclear radius.

2. All the electronic radial functions,[2] used in the numerical analysis, were computed on the ORACLE of the Oak Ridge National Laboratory, taking into account the following:

(a) the nucleus was considered to be a sphere with a uniform charge distribution - the nuclear finite size effect[3]

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[1]M. E. Rose and R. K. Osborn, Phys. Rev. 93, 1315 (1954).

[2]The details of the actual computations are given by C. P. Bhalla and M. E. Rose, Oak Ridge National Laboratory Report (to be issued).

[3]M. E. Rose and D. K. Holmes, Oak Ridge National Laboratory Report 1022 (unpublished).

(b) the nuclear radius was taken to be $0.428\,\alpha\, A^{\frac{1}{3}}\,(\frac{\hbar}{mc})$

and (c) the finite deBroglie wavelength effects[4] were properly considered.

3. A comparison of the developed theory was made with the following experimental data:

(a) Pr^{144} $(0^- \rightarrow 0^+)$

(1) The β^- longitudinal polarization datum of Mehlhop et al.[5]

(2) The β^- shape factor of Porter and Day.[6]

(b) Ho^{166} $(0^- \rightarrow 0\ )$

Only the β^- longitudinal polarization measurement of Bühring.[7]

4. Time reversal invariance is valid in strong interactions. This implies that the ratio of the nuclear matrix elements is real.

5. Time reversal invariance holds in the nuclear beta decay. This assumption gives the reality condition on the coupling constants.

6. The two component theory of the neutrino is used. This assumption gave $C_A = C_A'$ and $C_P = C_P'$ in the theoretical formulas,

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[4]M. E. Rose and C. L. Perry, Phys. Rev. 90, 479 (1953).

[5]W. A. W. Mehlhop, E. D. Lambe, and T. Pond, Bull. Am. Phys. Soc. 5, 9 (1960). And also W. A. W. Mehlhop, dissertation, Washington University, Saint Louis, 1959.

[6]F. T. Porter and P. P. Day, Phys. Rev. 114, 1286 (1959).

[7]W. Bühring, Z. Physik 155, 566 (1959).

developed using the 4-component Dirac neutrino.

Now we present the conclusions of this investigation.

III. CONCLUSIONS

1. We have developed the theoretical formulas for the β longitudinal polarization and the β shape factor in $0 \rightarrow 0$ (yes) transitions, without any known significant approximations, using the correct form of the operator for the pseudoscalar interaction and the conventional form of the axial vector interaction.

2. By the application of these formulas to the most accurate existing experimental data on $0 \rightarrow 0$ (yes) beta transitions, we have been able to conclude that:

(A) The absence of the pseudoscalar interaction in nuclear beta decay is consistent with the existing experimental data. This data does not contradict the V-1.2 A law, which is well established by the experiments on the allowed beta transitions.

(B) A new upper limit on the ratio of the coupling constants of the pseudoscalar interaction and the axial vector interaction can be set and this is

$$\left|\frac{C_P}{C_A}\right| < 90$$

which is about half the previous extimates, as reported in the literature; then the contribution* of the pseudoscalar interaction is $<$ .002.

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*The ratio of the shape factor for the pure pseudoscalar interaction to the shape factor for the pure axial vector interaction at the β kinetic energy of 1 Mev for Pr^{144} $(0^- \rightarrow 0^+)$.

3. Within the framework of the developed formulas, it is possible to decide the question of the existence of the pseudoscalar interaction in nuclear beta decay, provided that

(A) The β longitudinal polarization in $0 \rightarrow 0$ (yes) beta transitions is measured with an accuracy* of $\sim 1\%$ at four or five different values of the β momentum, throughout the beta spectrum.

(B) The accurate β spectrum measurements are performed, as in $Pr^{144}(0^- \rightarrow 0^+)$, for the other $0 \rightarrow 0$ (yes) beta transitions.

Discussion of the Conclusions

This work represents a consistent detailed analysis of the pseudoscalar interaction in nuclear beta decay. The essential limitations which influence the results of this analysis are the following:

1. The ratio of the nuclear matrix element has to be treated as a parameter.

2. The two accurate (2 to 3%) measurements of the β^- longitudinal polarization in Pr^{144} $(0^- \rightarrow 0^+)$ and Ho^{166} $(0^- \rightarrow 0^+)$ give the average of $\frac{P_{\parallel}}{v/c}$ over the beta kinetic energies from 1 Mev to 3 Mev and from 0.18 to 1.8 Mev respectively. These measurements do not provide a sensitive test of the existence of the pseudoscalar interaction, because a wide range of the values of $\frac{C_P}{C_A}$ and the ratio of the nuclear matrix elements

*The Oak Ridge group has achieved an accuracy of $\sim 1.2\%$ in the β longitudinal polarization measurement for $P^{32}(1^+ \rightarrow 0^+)$ at $v/c = 0.8912$, by Mott Scattering.

can be found which give the calculated $\left\langle \frac{P_{||}}{v/c} \right\rangle$ consistent with the measurements. However, if the β longitudinal polarization is measured at about four different beta momenta, (say by Mott Scattering), then these parameters cannot take on a wide range of values and still give a fit to the experimental data. This can be readily understood, because the pure pseudoscalar interaction and the pure axial vector interaction, taken separately, give the opposite signs for the beta longitudinal polarization.

Also the accurate β shape factor measurements of the $0 \rightarrow 0$ (yes) beta transitions are extremely useful to investigate the possible contribution of the pseudoscalar interaction, provided that the beta longitudinal polarization measurements are available for these cases, (as explained above).

Finally, we wish to point out that the necessary accuracy of the measurements in $0 \rightarrow 0$ (yes) transitions required to settle the question of the existence of the pseudoscalar interaction in nuclear β decay, is within reach of experimental measurement.

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APPENDICES

APPENDIX A

The following relations, which have been useful in this work, are listed. The proofs and discussions appear in the literature.

Clebsch-Gordon Coefficients or C-Coefficients

Throughout this work, notation of C-Coefficients as given by Rose[1] is used; and for reference, it is compared with that of Condon and Shortley.[2]

$$C(j_1 j_2 j; m_1, m_2, m) \equiv (j_1 j_2 m_1 m_2 \mid j_1 j_2 j m)$$

Since $m_1 + m_2 = m$; for brevity $C(j_1 j_2 j; m_1 m_2 m)$ is also written as $C(j_1 j_2 j; m_1 m - m_1)$.

The following* are the symmetry coefficients of C-Coefficients.

$$C(j_1 j_2 j_3; m_1 m_2 m_3) = (-)^{j_1+j_2-j_3} C(j_1 j_2 j_3; -m_1, -m_2, -m_3) \qquad \text{(A.1a)}$$

$$= (-)^{j_1+j_2-j_3} C(j_2 j_1 j_3; m_2, m_1, m_3) \qquad \text{(A.1b)}$$

$$= (-)^{j_1-m_1} \left(\frac{2j_3+1}{2j_2+1}\right)^{\frac{1}{2}} C(j_1 j_3 j_2; m_1, -m_3, -m_2) \qquad \text{(A.1c)}$$

Making use of (A.1a) through (A.1c), additional relations can be derived

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[1]M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York 1957) p. 37.

[2]E. U. Condon and G. H. Shortley, Theory of Atomic Spectra (Cambridge University Press, 1935)

*M. E. Rose, op. cit. p. 38-39.

$$C(j_1j_2j_3;m_1,m_2,m_3) = (-)^{j_2+m_2}\left(\frac{2j_3+1}{2j_1+1}\right)^{\frac{1}{2}} C(j_3j_2j_1;-m_3,m_2,-m_1) \quad \text{(A.1d)}$$

$$= (-)^{j_1-m_1}\left(\frac{2j_3+1}{2j_2+1}\right)^{\frac{1}{2}} C(j_3j_1j_2;m_3,-m_1,m_2) \quad \text{(A.1e)}$$

$$= (-)^{j_2+m_2}\left(\frac{2j_3+1}{2j_1+1}\right)^{\frac{1}{2}} C(j_2j_3j_1;-m_1,m_3,m_1) \quad \text{(A.1f)}$$

In (A.1) the phases are real and the parity C-Coefficient

$C(\ell_1\ell_2\ell_3;000) = 0$ unless $\ell_1 + \ell_2 + \ell_3$ is even.

$$\sigma_r \chi_\varkappa^\mu = - \chi_{-\varkappa}^\mu \quad \text{(A.2)}$$

(Reference 1. p. 154)

$$(j'm' \mid J_\mu \mid jm) = \delta_{j',j}\, \delta_{m',m+\mu}\ (-)^\mu \left[j(j+1)\right]^{\frac{1}{2}} C(j1j;m+\mu,-\mu) \quad \text{(A.5)}$$

(Reference 1. p. 85)

$$Y_{\ell_1 m_1}(\theta,\varphi) Y_{\ell_2 m_2}(\theta,\varphi) = \sum_\ell \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi\,(2\ell+1)}\right]^{\frac{1}{2}} C(\ell_1\ell_2\ell;m_1m_2)C(\ell_1\ell_2\ell;oo)Y_{\ell,m_1+m_2}(\theta,\varphi) \quad \text{(A.6)}$$

(Reference 1. p. 61)

$$(\chi_\varkappa^\mu, \chi_{\varkappa'}^{\mu'}) = \sum_\lambda (-)^{\mu+\frac{1}{2}} \left[\frac{(2\ell+1)\,(2\ell'+1)\,(2j+1)\,(2j'+1)}{4\pi\,(2\lambda+1)}\right]^{\frac{1}{2}} C(\ell\ell'\lambda;00) \; C(jj'\lambda;-\mu,\mu')\ Y_{\lambda,\mu'-\mu}\ W(jj'\ell\ell';\lambda\tfrac{1}{2}) \quad \text{(A.7)}$$

(Proof given on page 158 of Reference 1)

$W\ (jj'\ell\ell';\lambda\tfrac{1}{2})$ is a Racah Coefficient and is defined* as

$$C(abe;\alpha\beta)\ C(edc;\alpha+\beta,\delta) = \sum_{f}\left[(2e+1)(2f+1)\right]^{\frac{1}{2}} C(bdf;\beta\delta)$$

$$\times\ C(afc;\alpha,\beta+\delta)W\ (abcd;ef) \tag{A.8a}$$

$$W\ (abcd;ef) = 0 \quad \text{unless} \quad \Delta(abe), \Delta(edc), \Delta(bdf), \Delta(afc) \tag{A.8b}$$

Symmetry Relations of Racah-Coefficients**

In W (abcd;ef) the following permutations of (a,b,c,d;ef) are permissable without any phase change.

(badc;ef), (cdab;ef), (dcba;ef), (acbd;fe)

(cadb;fe), (bdac;fe), (dbca;fe) (A.8c)

The Racah coefficients of the following argument permutations give

$(-)^{b+c-e-f}\ W(abcd;ef)$

(aefd;bd), (eadf;bc), (fdae;bc), (dfea;bc)

(afed;cb), (fade;cb), (edaf;cb), (defa;cb) (A.8d)

The Racah coefficients of the following argument permutations give

$(-)^{a+d-e-f}\ W(abcd;ef)$

(ebcf;ad), (befc;ad), (cfeb;ad), (fcbe;ad)

(ecbf;da), (cefb;da), (bfec;da), (fbce;da) (A.8e)

Also,

$$W\ (abcd;of) = \frac{(-)^{f-b-d}\,\delta_{ab}\,\delta_{ed}}{\left[(2b+1)(2d+1)\right]^{\frac{1}{2}}} \tag{A.8f}$$

(Reference 1. p. 113)

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*See M. E. Rose, op. cit., p. 110.

**See M. E. Rose, op. cit., p. 226.

We give, below, the gradient formula[3]

$$\nabla_m \Phi(r)\, Y_\ell^m(\vec{r}) = \left(\frac{\ell+1}{2\ell+3}\right)^{\frac{1}{2}} \left(\frac{d\Phi}{dr} - \ell\frac{\Phi}{r}\right) C(\ell\, 1\, \ell+1; Mm) Y_{\ell+1}^{M+m}(\vec{r})$$
$$- \left(\frac{\ell}{2\ell-1}\right)^{\frac{1}{2}} \left(\frac{d\Phi}{dr} + \frac{\ell+1}{r}\Phi\right) C(\ell\, 1\, \ell-1; Mm) Y_{\ell-1}^{M+m}(\vec{r}) \qquad \text{(A.9a)}$$

<u>Definition of X-Coefficient</u>*

$$X \equiv X(\ell_1 s_1 j_1; \ell_2 s_2 j_2; LSJ)$$
$$= (-)^{\sigma} \sum_t (2t+1)\, W(s_1 \ell_2 j_1 L; t \ell_1)\, W(\ell_2 s_1 j_2 S; t s_2) W(L j_1 S j_2; tJ) \qquad \text{(A.10a)}$$
$$\sigma = \ell_1 + s_1 + j_1 + \ell_2 + s_2 + j_2 + L + S + J \qquad \text{(A.10b)}$$

If the nine arguments are arranged in a square array:

$$\begin{bmatrix} \ell_1 & s_1 & j_1 \\ \ell_2 & s_2 & j_2 \\ L & S & J \end{bmatrix}$$

then the entries in any column or any row form a triangle. Any pair of rows or columns can be interchanged, introducing a phase $(-)^{\sigma}$.

As a special case

$$X(abc; dec; gg0) = \frac{(-)^{c+g-a-e}\, W(abde; cg)}{\left[(2c+1)(2g+1)\right]^{\frac{1}{2}}} \qquad \text{(A.10c)}$$

(Reference 1. p. 192)

[3] M. E. Rose and R. K. Osborn, Phys. Rev. <u>93</u>, 1315 (1954) equation 47. Also M. E. Rose Reference 1. p. 120

* M. E. Rose, Reference 1. p. 191-192.

APPENDIX B

In the process $n \rightarrow p + e^- + \bar{\nu}$, a neutron transforms into a proton, with the emission of an electron and an antineutrino. By the Dirac "hole" theory, the creation of the antiparticle (antineutrino) is equivalent to the destruction of the particle (neutrino) in a negative energy state. By the charge conjugation operator, the wave function of a neutrino in negative energy state is obtained in Section 1 (B.1c). Using the Dirac wavefunction of the electron in a Coulomb field, and the neutrino wavefunction (B.1c), the general matrix elements for the axial vector and pseudoscalar interactions are set up (B.23). In Section 3, the H_{β^-} matrix elements of $0 \rightarrow 0$ (yes) transitions are obtained from (B.23).

1. NEUTRINO WAVEFUNCTION IN NEGATIVE ENERGY STATE

The charge conjugation operator is $i\,\alpha_2 K_o$ where K_o is a complex conjugating operator.

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\psi_\nu^c = i\beta\,\alpha_2 K_o \psi_\nu = i\,\beta\alpha_2\,\psi_\nu^* \qquad \text{(B.1a)}$$

$$\psi_\nu = \begin{pmatrix} -i\,F_\kappa(r)\,\chi_{-\kappa}^{\mu} \\ G_\kappa(r)\,\chi_{\kappa}^{\mu} \end{pmatrix}$$

$$\chi^{\mu}_{j_\varkappa} = \sum_{\tau} C(\ell_\varkappa \tfrac{1}{2}\ j_\varkappa\ ;\ \mu - \tau, \tau)\ Y_{\ell_\varkappa}^{\mu-\tau}\ \chi^{\tau}_{\frac{1}{2}}$$

$F_\varkappa(r)$ and $G_\varkappa(r)$ are real functions:

$$F_\varkappa = S_\varkappa\ q\ j_{\ell(-\varkappa)}\ (qr)$$

$$G_\varkappa = q\ j_{\ell(\varkappa)}(qr)$$

$$\psi_\nu^{c} = i\beta\alpha_2\ \psi_\nu^{*}$$

$$\psi_\nu^{c} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} i\ F_\varkappa \chi^{\mu *}_{-\varkappa} \\ G_\varkappa \chi^{\mu *}_{\varkappa} \end{pmatrix}$$

$$\psi_\nu^{c} = i\begin{pmatrix} G_\varkappa\ \sigma_2\ \chi^{\mu}_{\varkappa} \\ -i\ F_\varkappa\ \sigma_2\ \chi^{\mu}_{-\varkappa} \end{pmatrix} \tag{B.1b}$$

Using $\sigma_2\ \chi^{\tau}_{\frac{1}{2}} = i\ (-)^{-\tau + \frac{1}{2}}\ \chi^{-\tau}_{\frac{1}{2}}$ (B.2a)

and $Y^{M\,*}_{L} = (-)^{M}\ Y^{-M}_{L}$ (B.2b)

$$\sigma_2\ \chi^{\mu *}_{\varkappa} = i\ (-)^{\mu-\frac{1}{2}} \sum_{\tau} C(\ell_\varkappa \tfrac{1}{2} j; \mu - \tau, \tau)\ \chi^{-\tau}_{\frac{1}{2}}\ \ Y_{\ell_\varkappa}^{-\mu-\tau} \tag{B.3a}$$

Changing the summation letter $\tau \rightarrow -\tau$, we get

$$\sigma_2\ \chi^{\mu *}_{\varkappa} = i\ (-)^{\mu-\tau} \sum_{\tau} C(\ell_\varkappa \tfrac{1}{2}\ j; \mu + \tau, -\tau)\ \chi^{\tau}_{\frac{1}{2}}\ \ Y_{\ell_\varkappa}^{-\mu-\tau}$$

Using the symmetry property of C-coefficient (A.1a), we obtain

$$\sigma_2 \chi_{\varkappa}^{\mu\,\times} = i\,(-)^{\mu-\frac{1}{2}} \sum_{\tau} (-)^{\ell_{\varkappa}+\frac{1}{2}-j}\, C(\ell_{\varkappa}\tfrac{1}{2}j;\, -\mu-\tau, \tau)\, \chi_{\frac{1}{2}}^{\tau}\; Y_{\ell_{\varkappa}}^{-\mu-\tau}$$

$$\sigma_2 \chi_{\varkappa}^{\mu\,\times} = i\,(-)^{\mu+\ell_{\varkappa}-j} \sum_{\tau} C(\ell_{\varkappa}\tfrac{1}{2}j;-\mu-\tau, \tau)\, \chi_{\frac{1}{2}}^{\tau}\; Y_{\ell_{\varkappa}}^{-\mu-\tau}$$

$$\sigma_2 \chi_{\varkappa}^{\mu\,\times} = i\,(-)^{\mu+\ell_{\varkappa}-j}\; \chi_{\varkappa}^{-\mu} \tag{B.3b}$$

Also

$$\sigma_2 \chi_{-\varkappa}^{\mu\,\times} = i\,(-)^{\mu+\ell_{-\varkappa}-j}\; \chi_{-\varkappa}^{-\mu} \tag{B.4a}$$

$$\sigma_2 \chi_{-\varkappa}^{\mu\,\times} = i\,(-)^{\mu-\ell_{\varkappa}-j-1}\; \chi_{-\varkappa}^{-\mu} \tag{B.4b}$$

We have used $\ell_{\varkappa} + \ell_{-\varkappa} + 1 = 2\,|\varkappa| =$ even integer.

$$\sigma_2 \chi_{-\varkappa}^{\mu\,\times} = i\,(-)^{j+\ell_{\varkappa}+1-\mu}\; \chi_{-\varkappa}^{-\mu} \tag{B.4c}$$

(B.3b) can be written considering that $\ell_{\varkappa}$ is an integer, therefore

$$\sigma_2 \chi_{\varkappa}^{\mu} = i\,(-)^{j+\ell_{\varkappa}-\mu}\; \chi_{\varkappa}^{-\mu} \tag{B.3c}$$

Substituting (B.3c) and (B.4c) in (B.1b)

$$\Psi_{\nu}^{c} = (-)^{j_{\nu}+\ell_{\varkappa_{\nu}}-\mu_{\nu}+1} \begin{pmatrix} G_{\varkappa_{\nu}}\, \chi_{\varkappa_{\nu}}^{-\mu_{\nu}} \\ i\,F_{\varkappa_{\nu}}\, \chi_{-\varkappa_{\nu}}^{-\mu_{\nu}} \end{pmatrix} \tag{B.1c}$$

We have introduced the subscript ν for clarity.

2. GENERAL NUCLEAR MATRIX ELEMENTS FOR A AND P INTERACTIONS

For the axial vector interaction

$$H_A = \vec{\sigma} \cdot (\psi_e^* \vec{\sigma} [C_A + C_A' \gamma_5] \psi_\nu^c) Q - \gamma_5 (\psi_e^* \gamma_5 [C_A + C_A' \gamma_5] \psi_\nu^c) Q \tag{B.5a}$$

For the pseudoscalar interaction, the hamiltonian is

$$H_P = \frac{i}{2M} \vec{\sigma} \cdot \nabla (\psi_e^* \beta \gamma_5 [C_P + C_P' \gamma_5] \psi_\nu^c) Q \tag{B.5b}$$

In the above Q is an operator which transforms a neutron state into a proton state.

For the axial vector and pseudoscalar interactions,

$$H_{\beta-} = H_A + H_P \tag{B.6a}$$

We introduce the following notation:

$$\mathcal{H}_A (1) \equiv C_A \vec{\sigma} \cdot (\psi_e^* \vec{\sigma} \psi_\nu^c) Q \tag{B.7a}$$

$$\mathcal{H}_A (2) \equiv C_A' \vec{\sigma} \cdot (\psi_e^* \vec{\sigma} \gamma_5 \psi_\nu^c) Q \tag{B.8a}$$

$$\mathcal{H}_A (3) \equiv -C_A \gamma_5 \cdot (\psi_e^* \gamma_5 \psi_\nu^c) Q \tag{B.9a}$$

$$\mathcal{H}_A (4) \equiv -C_A' \gamma_5 \cdot (\psi_e^* \gamma_5 \gamma_5 \psi_\nu^c) Q \tag{B.10a}$$

$$\mathcal{H}_P (1) \equiv \frac{i}{2M} C_P \vec{\sigma} \cdot \nabla (\psi_e^* \beta \gamma_5 \psi_\nu^c) Q \tag{B.11a}$$

$$\mathcal{H}_P (2) \equiv \frac{i}{2M} C_P' \vec{\sigma} \cdot \nabla (\psi_e^* \beta \gamma_5 \gamma_5 \psi_\nu^c) Q \tag{B.12a}$$

In the following, $\mathcal{H}_A (1)$, $\mathcal{H}_A (2)$, $\mathcal{H}_A (3)$, $\mathcal{H}_A (4)$, $\mathcal{H}_P (1)$ and $\mathcal{H}_P (2)$ are evaluated and the results are given in (B.7d), (B.8d),

(B.9b), (B.10c), (B.11c) and (B.12d) respectively.

First we evaluate $\mathcal{H}_A$ (1) and show the details of the calculations.

$$\mathcal{H}_A (1) \equiv c_A \; \vec{\sigma} \cdot (\psi^* \; \vec{\sigma} \; \psi_\nu^c) \, Q \tag{B.7a}$$

We shall suppress Q in the following:

Using spherical basis, we write (B.7a) as

$$\mathcal{H}_A (1) = c_A \sum_m (-)^m \sigma_{-m} (\psi^* \; \sigma_m \; \psi_\nu^c) \tag{B.7b}$$

$$(\psi^* \; \sigma_m \; \psi_\nu^c) =$$

$$\begin{pmatrix} -i f_\varkappa \chi^\mu_{-\varkappa} \\ g_\varkappa \chi^\mu_\varkappa \end{pmatrix}^* \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix} (-)^{j_\nu + \ell_{\varkappa_\nu} - \mu_\nu + 1} \begin{pmatrix} G_{\varkappa_\nu} \chi^{-\mu_\nu}_{\varkappa_\nu} \\ i F_{\varkappa_\nu} \chi^{-\mu_\nu}_{-\varkappa_\nu} \end{pmatrix}$$

$$(\psi^* \; \sigma_m \; \psi_\nu^c) =$$

$$(-)^{j_\nu + \ell_{\varkappa_\nu} - \mu_\nu + 1} \left[i f_\varkappa G_\varkappa \; (\chi^\mu_-, \sigma_m \chi^{-\mu}_{\varkappa_\nu}) + i g_\varkappa F_{\varkappa_\nu} (\chi^\mu_{\varkappa}, \sigma_m \chi^{-\mu}_{-\kappa_\nu}) \right] \tag{B.13a}$$

Now we evaluate

$$(\chi^\mu_\varkappa, \sigma_m \chi^{\mu'}_{\varkappa'}) =$$

$$\sum_\gamma \sum_{\gamma'} C(\ell \tfrac{1}{2} j; \mu-\gamma, \gamma) \, C(\ell' \tfrac{1}{2} j'; \mu'-\gamma', \gamma') \, Y_\ell^{\mu-\gamma\,\times} Y_{\ell'}^{\mu'-\gamma'}$$

$$\times \, (\chi^\gamma_{\frac{1}{2}}, \sigma_m \chi^{\gamma'}_{\frac{1}{2}}) \tag{B.14a}$$

Using (A.5), which gives

$$(\chi_{\frac{1}{2}}^{\gamma}, \sigma_m \chi_{\frac{1}{2}}^{\gamma'}) =$$

$$\delta_{\gamma, \gamma' + m} (-)^m \sqrt{3} \quad C(\tfrac{1}{2}\ 1\ \tfrac{1}{2} ; \gamma' + m, -m)$$

Substituting the above in (B.14a)

$$(\chi_{\varkappa}^{\mu}, \sigma_m \chi_{\varkappa'}^{\mu'}) =$$

$$\sum_{\gamma} \sum_{\gamma'} \delta_{\gamma, \gamma' + m} (-)^m (3)^{\frac{1}{2}} \quad C(\ell\ \tfrac{1}{2}\ j;\mu-\gamma, \gamma) C(\ell'\ \tfrac{1}{2}\ j';\mu'-\gamma', \gamma')$$

$$\times \ C(\tfrac{1}{2}\ 1\ \tfrac{1}{2};\ \gamma' + m, -m) \quad Y_{\ell}^{\mu-\gamma\,*} \quad Y_{\ell'}^{\mu'-\gamma'} \qquad \text{(B.14b)}$$

Summing over γ', (making use of $\delta_{\gamma, \gamma' + m}$) and using (A.6)

$$(\chi_{\varkappa}^{\mu}, \sigma_m \chi_{\varkappa'}^{\mu'}) =$$

$$(-)^m \sum_{\lambda} \left[\frac{3(2\ell + 1)(2\ell' + 1)}{4\pi(2\lambda + 1)}\right]^{\frac{1}{2}} C(\ell'\ell\lambda\ ;00)\ Y_{\lambda}^{\mu'-\mu+m}$$

$$\times \sum_{\gamma} (-)^{\mu-\gamma} \quad C(\ell\ell'\lambda; \gamma-\mu, \mu'-\gamma+m) C(\ell\ \tfrac{1}{2}\ j;\mu-\gamma, \gamma) C(\ell'\ \tfrac{1}{2}\ j';\mu'-\gamma+m, \gamma-m)$$

$$\times \ C(\tfrac{1}{2}\ 1\ \tfrac{1}{2}\ ;\gamma, -m) \qquad \text{(B.14c)}$$

Using (C.1c) for the summation over γ

$$(\chi_{\varkappa}^{\mu}, \sigma_m \chi_{\varkappa'}^{\mu'}) = (-)^{\mu-\mu'+1+\ell+m} \sum_{\lambda} \left[\frac{6(2\ell + 1)(2\ell' + 1)(2j' + 1)}{4\pi}\right]^{\frac{1}{2}} \times$$

$$\times\ C(\ell\,\ell'\lambda\,;00)\ Y_{\lambda}^{\mu'-\mu+m} \sum_{s} (2s+1)^{\frac{1}{2}}\ C(\lambda\, s j;\mu-\mu'-m,\mu'+m)$$

$$\times\ C(j'\,1\ s;-\mu',-m)W(\lambda\,\ell' j\tfrac{1}{2};\ell s)W(\ell' j'\tfrac{1}{2}\ 1;\tfrac{1}{2}s) \qquad \text{(B.14d)}$$

From (B.7b) and (B.13a)

$$\mathcal{H}_A(1) = (-)^{j_{\varkappa_\nu}+\ell_{\varkappa_\nu}-\mu_\nu+1}\ C_A \sum_m\ i f_\varkappa\ G_{\varkappa_\nu}(-)^m \sigma_{-m}(\chi^{\mu}_{-\varkappa}, \sigma_m\, \chi^{-\mu}_{\varkappa_\nu})$$

$$+\ i\, g_\varkappa F_{\varkappa_\nu}\ (-)^m\ \sigma_{-m}\,(\chi^{\mu}_{\varkappa}, \sigma_m\, \chi^{-\mu}_{-\varkappa_\nu}) \qquad \text{(B.7c)}$$

Using (B.14d) changing $\varkappa \to -\varkappa$; $\varkappa' \to \varkappa_\nu$, $\mu' \to -\mu_\nu$

$$\sum_m (-)^m\ \sigma_{-m}\,(\chi^{\mu}_{-\varkappa}, \sigma_m\ \chi^{-\mu}_{\varkappa_\nu})$$

$$= \sum_m (-)^m\ \sigma_{-m}\ (-)^{\mu+\mu_\nu+1+\ell_\nu+m} \sum_\lambda \frac{6(2\bar{\ell}+1)(2\ell_\nu+1)(2j_\nu+1)}{4\pi}\ C(\bar{\ell}\,\ell_\nu\lambda\,;00)$$

$$Y_{\lambda}^{-\mu_\nu-\mu+m} \sum_s (2s+1)^{\frac{1}{2}}\ C(\lambda\, s j;\mu+\mu_\nu-m,-\mu_\nu+m)C(j_\nu\, 1\ s;\mu_\nu,-m)$$

$$\times\ W(\lambda\,\ell_\nu\ j\tfrac{1}{2};\bar{\ell}\,s)W(\ell_\nu\ j\ \tfrac{1}{2}\ 1;\tfrac{1}{2}s) \qquad \text{(B.15a)}$$

We have introduced the following notation

$$\bar{\ell} \equiv \ell_{-\varkappa}\ ; \qquad \ell \equiv \ell_\varkappa$$

$$\bar{\ell}_\nu \equiv \ell_{-\varkappa_\nu}; \qquad \ell_\nu \equiv \ell_{\varkappa_\nu} \qquad \text{(B.16a)}$$

Also $\bar{\ell}+\ell+1$ = even integer (B.16b)

Now, in general, the definition of an irreducible tensor is

$$T^{M}_{L\lambda}(\hat{r},\vec{\sigma}) = \sum_{m'} C(\lambda 1L;M-m',m')\, Y^{M-m'}_{\lambda}(\hat{r})\, \sigma_{m'}$$

and

$$Y^{M-m}_{\lambda}(\hat{r})\, \sigma_{m} = \sum_{L} C(\lambda 1L;M-m,m)\, T^{M}_{L\lambda}(\hat{r},\vec{\sigma})$$

L is the rank of the irreducible tensor and the parity of

$T^{M}_{L,\lambda}(\hat{r},\vec{\sigma})$ is $(-)^{\lambda}$, since $\vec{\sigma}$ is an even operator.*

In (B.15a) we combine

$$\sigma_{-m}\, Y^{m-\mu-\mu_\nu}_{\lambda} = \sum_{L} C(\lambda 1L;m-\mu-\mu_\nu,-m)\, T^{-\mu-\mu_\nu}_{L\lambda} \qquad \text{(B.16c)}$$

$$T^{-\mu-\mu_\nu}_{L\lambda} = \sum_{m} C(\lambda 1L;m-\mu-\mu_\nu,-m)\, Y^{m-\mu-\mu_\nu}_{\lambda}\, \sigma_{-m} \qquad \text{(B.16d)}$$

Therefore, (B.15a) becomes

$$\sum_{m} (-)^{m} \sigma_{-m} (\chi^{\mu}_{-\varkappa}\, \sigma_{m}\, \chi^{-\mu}_{\varkappa_\nu}) = \sum_{m}\sum_{\lambda}\sum_{L} (-)^{\mu+\mu_\nu+1+\ell_\nu}$$

$$\times \left[\frac{6(2\bar{\ell}+1)(2\ell_\nu+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}} C(\bar{\ell}\ell_\nu\lambda;00)\, T^{-\mu-\mu_\nu}_{L\lambda}$$

$$\times \sum_{s} (2s+1)^{\frac{1}{2}} C(\lambda 1L;m-\mu-\mu_\nu,-m)\, C(\lambda sj;\mu+\mu_\nu-m,-\mu_\nu+m)$$

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* σ_i commutes with β. $(i = 1, 2, 3)$

$$\times\, C(j_\nu 1\, s;\mu_\nu, - m)\;\; W(\lambda\, \ell_\nu\, j\tfrac{1}{2};\, \bar{\ell}\, s)\;\; W(\ell_\nu\, j_\nu \tfrac{1}{2} 1\, ;\tfrac{1}{2} s) \qquad \text{(B.15b)}$$

Again we use (A.8a) on the three C-coefficient in the summation over s.

$$C(\lambda 1\, L;m - \mu - \mu_\nu\, , - m)C(\lambda\, s j;\mu + \mu_\nu - m)C(j_\nu 1\, s;\mu_\nu - m)$$

$$= (-)^{\lambda+s-j}\;\; (-)^{j_\nu + 1-s}\;\; C(\lambda 1\, L;m - \mu - \mu_\nu, - m)C(j_\nu 1\, s; - \mu_\nu ,m)$$

$$\times\, C(s\, \lambda\, j; - \mu_\nu + m,\mu + \mu_\nu -m)$$

$$= (-)^{-j+\lambda+j_\nu +1}\;\; C(\lambda 1\, L;m-\mu-\mu_\nu ,-m)\; \sum_{s'} \left[(2s'+1)(2s+1)\right]^{\frac{1}{2}} C(1\lambda\, s';-\mu_\nu,\mu+\mu_\nu +m)$$

$$\times\, C(j_\nu\, s' j;-\mu_\nu ,\mu+\mu_\nu)\quad W(j_\nu 1\, j\, \lambda;s's)$$

Substituting the above in (B.15b) and making use of

$$\sum_m C(\lambda 1\, L;m-\mu-\mu_\nu ,-m)\;\; C(\lambda 1\, s';m-\mu-\mu_\nu ,-m) \;=\; \delta_{Ls'}$$

$$\sum_m (-)^m\;\; \sigma_{-m}\, (\chi^{\mu}_{-\varkappa},\, \sigma_m\, \chi^{-\mu}_{\varkappa_\nu})$$

$$= \sum_L \sum_\lambda\;\; (-)^{\mu+\mu_\nu + \ell_\nu + \lambda + j_\nu - j} \left[\frac{6(2\bar{\ell} +1)(2\ell_\nu +1)(2j_\nu +1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} T^{-\mu-\mu_\nu}_{L\,\lambda}$$

$$\times\;\; C(\bar{\ell}\, \ell_\nu\, \lambda\, ;00)\; C\, (j_\nu\, L j;-\mu_\nu,\mu + \mu_\nu\,)$$

$$\times \sum_s\; (2s+1)W(\lambda\, \ell_\nu\, j\tfrac{1}{2};\, \bar{\ell}s)W(\ell_\nu\;\, j\tfrac{1}{2}1;\tfrac{1}{2}s)W(j_\nu\, 1 j\, \lambda\, ;LS) \qquad \text{(B.15c)}$$

By definition of X-coefficient (A.10); and using (A.8c)

$$\sum_s\; (2s+1)W(\lambda\, \ell_\nu\, j\tfrac{1}{2};\, \bar{\ell}s)W(\ell_\nu\, j\tfrac{1}{2}1;\tfrac{1}{2}s)W(j_\nu\, 1 j\, \lambda\, ;LS)$$

$$= (-)^{L+\lambda+j_\nu+j+\ell_\nu+\bar{\ell}} \quad X(L1\lambda;j_\nu\tfrac{1}{2}\ell_\nu;j\tfrac{1}{2}\bar{\ell})$$

Substituting the above in (B.15c) and using (A1.a)

$$\sum_m (-)^m \sigma_{-m} (\chi^{\mu}_{-\varkappa}, \sigma_m \chi^{-\mu}_{\varkappa_\nu}) = \sum_L \sum_\lambda (-)^{\mu+\mu_\nu+j_\nu+\bar{\ell}+j}$$

$$\times \left[\frac{6(2\bar{\ell}+1)(2\ell_\nu+1)(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\,\lambda;00)$$

$$\times C(j_\nu Lj;\mu_\nu,-\mu-\mu_\nu)\, T^{-\mu-\mu}_{L\lambda} \quad X(L1\lambda;j_\nu\tfrac{1}{2}\ell_\nu;j\tfrac{1}{2}\bar{\ell}) \qquad \text{(B.15d)}$$

From (B.15d); $\varkappa \to -\varkappa$, $\varkappa_\nu \to -\varkappa_\nu$

$$\sum_m (-)^m \sigma_{-m} (\chi^{\mu}_{\varkappa}, \sigma_m \chi^{-\mu_\nu}_{-\varkappa_\nu}) = \sum_L \sum_\lambda (-)^{\mu+\mu_\nu+j_\nu+\ell+j}$$

$$\times \left[\frac{6(2\ell+1)(2\bar{\ell}_\nu+1)(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu\,\lambda;00)$$

$$\times C(j_\nu Lj;\mu_\nu,-\mu-\mu_\nu)\, T^{-\mu-\mu_\nu}_{L\lambda} \quad X(L1\lambda;j_\nu\tfrac{1}{2}\bar{\ell}_\nu;j\tfrac{1}{2}\ell) \qquad \text{(B.15e)}$$

Substituting (B.15d) and (B.15e) in (B.7c) and using $2j_\nu+1$ = even

$$\mathcal{H}_A(1) = C_A\, i\, (-)^{j+\ell_\nu+\mu} \sum_\lambda \sum_L \left[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(j_\nu Lj;\mu_\nu,-\mu-\mu_\nu)$$

$$T^{-\mu-\mu_\nu}_{L\lambda} \Big\{ (-)^{\bar{\ell}} \left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\,\lambda;00) f_\varkappa G_{\varkappa_\nu}\, X(L1\lambda;j_\nu\tfrac{1}{2}\ell_\nu;j\tfrac{1}{2}\bar{\ell})$$

$$+ (-)^{\ell} \left[(2\ell+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu\,\lambda\,;00) g_\varkappa F_{\varkappa_\nu} X(L1\lambda\,; j_\nu \tfrac{1}{2} \bar{\ell}_\nu\,; j \tfrac{1}{2} \ell)\Big\} \quad \text{(B.7d)}$$

Next we calculate

$$\mathcal{H}_A\,(2) \equiv C_A'\,\vec{\sigma}\cdot(\psi_e^* \vec{\sigma}\gamma_5 \psi_\nu^c) \quad \text{(B.8a)}$$

$$\mathcal{H}_A\,(2) = C_A' \sum_m (-)^m \sigma_{-m} (\psi_e^* \sigma_m \gamma_5 \psi_\nu^c) \quad \text{(B.8b)}$$

Using ψ_e and ψ_ν^c; (B.8b) becomes

$$\mathcal{H}_A\,(2) = C_A' \sum_m (-)^m \sigma_{-m} (-)^{j_\nu+\ell_\nu-\mu_\nu+1} \Big[-f_\varkappa F_{\varkappa_\nu} (\chi^\mu_{-\varkappa}, \sigma_m \chi^{-\mu}_{-\varkappa_\nu})$$

$$+ g_\varkappa G_{\varkappa_\nu} (\chi^\mu_\varkappa, \sigma_m \chi^{-\mu}_{\varkappa_\nu}) \Big] \quad \text{(B.8c)}$$

Using (B.14d) and proceeding exactly as before, we get

$$\mathcal{H}_A\,(2) \equiv C_A'\,\vec{\sigma}\cdot(\psi_e^* \vec{\sigma}\gamma_5 \psi_\nu^c)$$

$$= C_A' (-)^{\ell_\nu+\mu+j} \sum_\lambda \sum_L \left[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(j_\nu L j; \mu_\nu, -\mu-\mu_\nu)\; T_{L\lambda}^{-\mu-\mu_\nu}$$

$$\times \Big\{ (-)^{\bar{\ell}+1} \left[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu\lambda\,;00) f_\varkappa F_{\varkappa_\nu} X(L1\lambda\,; j_\nu \tfrac{1}{2} \bar{\ell}_\nu\,; j \tfrac{1}{2} \bar{\ell})$$

$$+ (-)^{\ell} \left[(2\ell+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\ell_\nu\lambda;00) g_\varkappa G_{\varkappa_\nu} X(L1\lambda\,; j \tfrac{1}{2} \ell_\nu\,; j \tfrac{1}{2} \ell)\Big\} \quad \text{(B.8d)}$$

Now we consider $\mathcal{H}_A\,(3) \equiv -C_A\,\gamma_5(\psi_e^* \gamma_5 \psi_\nu^c)$

$$(\psi_e^* \gamma_5 \psi_\nu^c) = (-)^{j_\nu+\ell_\nu-\mu_\nu+1} \begin{pmatrix} -if_\varkappa \chi^\mu_{-\varkappa} \\ g_\varkappa \chi^\mu_\varkappa \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_{\varkappa_\nu} \chi^{-\mu_\nu}_{\varkappa_\nu} \\ iF_{\varkappa_\nu} \chi^{-\mu_\nu}_{-\varkappa_\nu} \end{pmatrix}$$

$$(\psi_e^* \gamma_5 \psi_\nu^c) = (-)^{j_\nu+\ell_\nu-\mu_\nu+1} \Big\{ -f_\varkappa F_{\varkappa_\nu} (\chi^{\mu}_{-\varkappa}, \chi^{-\mu_\nu}_{-\varkappa_\nu}) + g_\varkappa G_{\varkappa_\nu} (\chi^{\mu}_{\varkappa}, \chi^{-\mu_\nu}_{\varkappa_\nu}) \qquad \text{(B.17a)}$$

Using $(\chi^{\mu}_{\varkappa}, \chi^{\mu'}_{\varkappa'})$ i.e. (A.6):

$$\mathcal{H}_A(3) = -C_A (-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}} \sum_{\lambda} \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)} \right]^{\frac{1}{2}} C(jj_\nu\lambda\,;-\mu,-\mu_\nu)\gamma_5 Y_\lambda^{-\mu-\mu_\nu}(\hat{r})$$

$$\times \Big[-f_\varkappa F_{\varkappa_\nu} \left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} \quad C(\bar{\ell}\,\bar{\ell}_\nu\lambda\,;00)W(jj_\nu\,\bar{\ell}\,\bar{\ell}_\nu;\lambda\tfrac{1}{2})$$

$$+ g_\varkappa G_{\varkappa_\nu} \left[(2\ell+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\ell_\nu\lambda\,;00)W(jj_\nu\,\ell\,\ell_\nu\,;\lambda\tfrac{1}{2}) \Big]$$

Again, we introduce the irreducible tensor

$$\gamma_5 Y_\lambda^{-\mu-\mu_\nu}(\hat{r}) = T_{\lambda\lambda}^{-\mu-\mu_\nu}(\hat{r},\gamma_5) \qquad \text{(B.17b)}$$

The rank of the irreducible tensor is and its parity is $(-)^{\lambda+1}$ because γ_5 is a tensor of rank zero and is an odd operator (in the Dirac sense).*

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*Writing the Dirac equation in Co-variant form $(\gamma_\mu \frac{\partial}{\partial x_\mu} + 1)\Psi = 0$; it is easily seen that under the Parity operator P: $P\Psi(\vec{r}) = \gamma_4\Psi(-\vec{r})$. An operator Ω in the Dirac space, then, transforms as $\gamma_4\Omega\gamma_4$ under the parity transformation. Since γ_5 anticommutes with γ_4, therefore, $\gamma_4\gamma_5\gamma_4 = -\gamma_5$: and hence the parity of γ_5 is odd and is odd operator (in the Dirac sense).

Substituting (B.17b) we get

$$\mathcal{H}_A\ (3) =$$

$$-C_A\ (-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j\ \ +1)}{4\pi(2\ \ \ +1)}\right]^{\frac{1}{2}} C(jj_\nu\lambda\,;-\mu,-\mu_\nu)T^{-\mu-\mu_\nu}_{\lambda\lambda}\ (\hat{r},\gamma_5)$$

$$\times \Big[-f_\varkappa F_{\varkappa_\nu}\ \left[(2\bar{\ell}\ +1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\ \bar{l}_\nu\lambda\,;00)W(jj\ \bar{\ell}\ \bar{\ell}_\nu\,;\ \lambda\tfrac{1}{2})$$

$$+ g_\varkappa G_{\varkappa_\nu}\left[(2\ell+1)(2\ell_\nu\ +1)\right]^{\frac{1}{2}} C(\ell\ell\lambda\ ;00)W(jj\ \ell\ \ell_\nu\ \ ;\ \lambda\tfrac{1}{2})\ \ \Big] \tag{B.9b}$$

$$\mathcal{H}_A\ (4) \equiv\ -C'_A\ \gamma_5(\psi^*_e\gamma_5\gamma_5\,\psi^c_\nu)$$

$$= -C'_A\gamma_5(\psi^*_e\,\psi^c_\nu) \tag{B.10a}$$

$$\mathcal{H}_A\ (4) = -C'_A\ (-)^{j_\nu+\ell_\nu-\mu_\nu+1}\left\{ if_\varkappa G_{\varkappa_\nu}(\chi^{\mu}_{-\varkappa}\,\chi^{-\mu}_{\varkappa_\nu}) + ig_\varkappa F_{\varkappa_\nu}(\chi^{\mu}_{\varkappa},\chi^{-\mu_\nu}_{-\varkappa_\nu})\right\} \tag{B.10b}$$

Using (A.7) in (B.10b)

$$\mathcal{H}_A\ (4) = -iC'_A\ (-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(jj_\nu\,\lambda,-\mu,-\mu_\nu)$$

$$T^{-\mu-\mu_\nu}_{\lambda,\lambda}(\hat{r},\gamma_5) \Big[f_\varkappa G_{\varkappa_\nu}\left[(2\,\bar{\ell}+1)(2\,\ell_\nu+1)\right]^{\frac{1}{2}}\ C(\ \bar{l}\ \ell_\nu\lambda\ ;00)W(jj\ell\ \bar{\ell}_\nu\ \ ;\ \lambda\tfrac{1}{2})$$

$$+ g_\varkappa F_{\varkappa_\nu}\left[(2\ell+1)(2\,\bar{\ell}_\nu\ +1)\right]^{\frac{1}{2}} C(\ell\ \bar{\ell}_\nu\lambda\ ;00)W(jj_\nu\ell\,\bar{\ell}_\nu\,;\ \lambda\tfrac{1}{2})\Big] \tag{B.10c}$$

For the axial vector interaction we have

$$H_A = \mathcal{H}_A\ (1) + \mathcal{H}_A\ (2) + \mathcal{H}_A\ (3) + \mathcal{H}_A\ (4) \tag{B.5a}$$

where, explicit expressions for $\mathcal{H}_A$ (1), $\mathcal{H}_A$ (2), $\mathcal{H}_A$ (3) and $\mathcal{H}'_A$ (4) are given in (B.7d), (B.8d), (B.9b) and (B.10c) respectively.

In the following, the pseudoscalar interaction is treated.

$$H_P = \frac{i}{2M} \vec{\sigma} \cdot \nabla (\psi_e^* \beta \gamma_5 (C_P + C'_P \gamma_5) \psi_\nu^c) \tag{B.5b}$$

$$H_P = \mathcal{H}_P\ (1) + \mathcal{H}_P\ (2)$$

$$\mathcal{H}_P\ (1) \equiv \frac{i}{2M} C_P\ \vec{\sigma} \cdot \nabla (\psi_e^* \beta \gamma_5 \psi_\nu^c) \tag{B.11a}$$

$$\mathcal{H}_P\ (2) \equiv \frac{i}{2M} C'_P\ \vec{\sigma} \cdot \nabla (\psi_e^* \beta\ \psi_\nu^c) \tag{B.12a}$$

$$(\psi_e^* \beta \gamma_5\ \psi_\nu^c) = \begin{pmatrix} -if_\varkappa \chi^\mu_{-\varkappa} \\ g_\varkappa \chi^\mu_\varkappa \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-)^{j_\nu + \ell_\nu - \mu_\nu + 1} \begin{pmatrix} G_{\varkappa_\nu} \chi^{-\mu_\nu}_{\varkappa_\nu} \\ iF_{\varkappa_\nu} \chi^{-\mu_\nu}_{-\varkappa_\nu} \end{pmatrix}$$

$$(\psi_e^*\ \beta \gamma_5\ \psi_\nu^c) = (-)^{j_\nu + \ell_\nu - \mu_\nu} \Big\{ f_\varkappa F_{\varkappa_\nu} (\chi^\mu_{-\varkappa}, \chi^{-\mu_\nu}_{-\varkappa_\nu}) + g_\varkappa G_{\varkappa_\nu} (\chi^\mu_\varkappa, \chi^{-\mu_\nu}_{\varkappa_\nu}) \Big\}$$

Substituting the above in (B.11a), we get

$$\mathcal{H}_P\ (1) = i\ \frac{C_P}{2M} (-)^{j_\nu + \ell_\nu - \mu_\nu} \Big[\vec{\sigma} \cdot \nabla \Big\{ f_\varkappa F_{\varkappa_\nu} (\chi^\mu_{-\varkappa}, \chi^{-\mu_\nu}_{-\varkappa_\nu}) + g_\varkappa G_{\varkappa_\nu} (\chi^\mu_\varkappa, \chi^{-\mu_\nu}_{\varkappa_\nu}) \Big\} \Big] \tag{B.11b}$$

Using (A.7) and $\vec{\sigma} \cdot \nabla = \sum_m (-)^m \sigma_{-m} \nabla_m$

$$\mathcal{H}_P\ (1) = i\ (-)^{j_\nu + \ell_\nu - \mu_\nu - \mu - \frac{1}{2}} \frac{C_P}{2M} \left[\frac{(2j+1)(2j_\nu+1)}{4\pi} \right]^{\frac{1}{2}} \sum_m (-)^m \sigma_{-m} \nabla_m \Big[\sum_\lambda C(j j_\nu \lambda, -\mu, -\mu_\nu)$$

$$Y_\lambda^{-\mu-\mu_\nu} \Big\{ \left[\frac{(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)}{2\lambda+1} \right]^{\frac{1}{2}} C(\bar{\ell}\ \bar{\ell}_\nu\ \lambda; 00) f_\varkappa F_{\varkappa_\nu}\ W(j j_\nu \bar{\ell}\ \bar{\ell}_\nu; \lambda \tfrac{1}{2}) \quad +$$

$$+\left[\frac{(2\ell+1)(2\ell_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\ell\ell_\nu\lambda;00)g_\varkappa G_{\varkappa_\nu}\, W(jj_\nu\,\ell\ell_\nu\,;\lambda\tfrac{1}{2})\Big\}\Big] \qquad \text{(B.11c)}$$

We introduce

$$R_1 \equiv \left[\frac{(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu\lambda;00)f_\varkappa F_{\varkappa_\nu}\, W(jj_\nu\bar{\ell}\,\bar{\ell}_\nu;\lambda\tfrac{1}{2}) \qquad \text{(B.18a)}$$

$$R_2 \equiv \left[\frac{(2\ell+1)(2\ell_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\ell\ell_\nu\lambda;00)g_\varkappa F_{\varkappa_\nu}\, W(jj_\nu\ell\ell_\nu;\lambda\tfrac{1}{2}) \qquad \text{(B.18b)}$$

then

$$\mathcal{H}_P(1) = i\,(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu-\frac{1}{2}}\frac{C_P}{2M}\left[\frac{(2j+1)(2j\;+1)}{4\pi}\right]^{\frac{1}{2}}\sum_m(-)^m C(jj_\nu\lambda\,,-\mu,-\mu_\nu)$$

$$\times\quad \sigma_{-m}\nabla_m Y_\lambda^{-\mu-\mu_\nu}(\hat{r})\,(R_1+R_2) \qquad \text{(B.11d)}$$

Using the gradient formula (A.9a) and

$$\sigma_{-m}\,Y_{\lambda+1}^{-\mu-\mu_\nu+m}(\hat{r}) = \sum_L C(\lambda+1,1,L;-\mu-\mu_\nu+m,-m)T_{L,\lambda+1}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma})$$

and

$$\sigma_{-m}\,Y_{\lambda-1}^{-\mu-\mu_\nu+m}(\hat{r}) = \sum_L C(\lambda-1,1,L;-\mu-\mu_\nu+m,-m)T_{L,\lambda-1}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma}) \qquad \text{(B.18c)}$$

$$\mathcal{H}_P(1) = i\,(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu-\frac{1}{2}}\frac{C_P}{2M}\left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}}\sum_\lambda\sum_m(-)^m C(jj_\nu\lambda;-\mu,-\mu_\nu)$$

$$\times\Bigg\{\left[\frac{\lambda+1}{2\lambda+3}\right]^{\frac{1}{2}}\sum_L C(\lambda\,1,\lambda+1;-\mu-\mu_\nu,m)C(\lambda+1,1L;-\mu-\mu_\nu+m,-m)T_{L\,\lambda+1}^{-\mu-\mu_\nu}D_+(\lambda)(R_1+R_2)$$

$$-\left[\frac{\lambda}{2\lambda-1}\right]^{\frac{1}{2}}\sum_L C(\lambda\,1\,\lambda-1;-\mu-\mu_\nu,m)C(\lambda-1,1,L;-\mu-\mu_\nu+m,-m)T_{L\,\lambda-1}^{-\mu-\mu}D_+(\lambda)(R_1+R_2)\Bigg\}$$

where

$$D_+ (\lambda) \equiv \frac{d}{dr} - \frac{\lambda}{r} \qquad \text{(B.18d)}$$

$$D_+ (\lambda) \equiv \frac{d}{dr} + \frac{\lambda+1}{r} \qquad \text{(B.18e)}$$

$$T^{-\mu-\mu_\nu}_{L\,\lambda\pm 1} \equiv T^{-\mu-\mu_\nu}_{L\,\lambda\pm 1}(\hat{r},\vec{\sigma}) \qquad \text{(B.18f)}$$

Using (A.1d) and (A.1a)

$$\mathcal{H}_P (1) = i\,(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu-\frac{1}{2}} \frac{C_P}{2M} \left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}} \sum_{\lambda, L} \sum_{M} C(jj_\nu\lambda\,;-\mu,-\mu_\nu)$$

$$\times\ (-)^m \left\{ \left[\frac{\lambda+1}{2\lambda+3} \cdot \frac{2\lambda+3}{2\lambda+1}\right]^{\frac{1}{2}} (-)^{1-m} C(\lambda+1\ 1\ \lambda\,;-\mu-\mu_\nu+m,-m)\, C(\lambda+1\ 1L;-\mu-\mu_\nu+m,-m) \right.$$

$$\times\ T^{-\mu-\mu}_{L\ \lambda+1}\ D-(\lambda)\ (R_1+R_2) - \left[\frac{\lambda}{2\lambda-1} \cdot \frac{2\lambda-1}{2\lambda+1}\right]^{\frac{1}{2}} (-)^{1-m} C(\lambda-1\ 1\lambda\,;-\mu-\mu_\nu+m,-m)$$

$$\left. \times\ C(\lambda-1\ 1L;-\mu-\mu_\nu+m,-m)\ T^{-\mu-\mu_\nu}_{L\ \lambda-1}\ D_+\ (\lambda)\ (R_1+R_2) \right\} \qquad \text{(B.11e)}$$

Using

$$\sum_m C(\lambda+1\ 1\lambda\,;-\mu-\mu_\nu+m,-m)\,C(\lambda+1\ 1L;-\mu-\mu_\nu+m,-m) = \delta_{\lambda,L}$$

and

$$\sum_m C(\lambda-1\ 1\lambda\,;-\mu-\mu_\nu+m,-m)\,C(\lambda-1\ 1L;-\mu-\mu_\nu+m,-m) = \delta_{\lambda,L} \qquad \text{(B.18g)}$$

We get

$$\mathcal{H}_P (1) = (-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \frac{iC_P}{2M} \left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}} \sum_{\lambda} C(jj_\nu\lambda\,;-\mu,-\mu_\nu) \quad \times$$

$$\times\left[\left[\frac{\lambda+1}{2\lambda+1}\right]^{\frac{1}{2}} T^{-\mu-\mu_\nu}_{L,\lambda-1}(\hat{r},\vec{\sigma})D_-(\lambda)(R_1+R_2) - \left[\frac{\lambda}{2\lambda+1}\right]^{\frac{1}{2}} T^{-\mu-\mu_\nu}_{L,\lambda-1}(\hat{r},\vec{\sigma})D_+(\lambda)(R_1+R_2)\right]$$

where R_1, R_2, $D_-(\lambda)$, $D_+(\lambda)$ and $T^{-\mu-\mu_\nu}_{L,\lambda\pm1}$ are defined in (B.18a), (B.18b), (B.18d), (B.18e) and (B.18c) respectively.

Now we calculate

$$\mathcal{H}_P(2) \equiv \frac{i}{2M}C_P'\,\vec{\sigma}\cdot\nabla(\psi_e^*\beta\,\psi_\nu^c) \tag{B.12a}$$

$$(\psi_e^*\beta\,\psi_\nu^c) = (-)^{j_\nu+\ell_\nu-\mu_\nu+1}\left[if_\varkappa G_{\varkappa_\nu}(\chi^{\mu}_{-\varkappa},\chi^{-\mu_\nu}_{\varkappa_\nu}) - ig_\varkappa F_{\varkappa_\nu}(\chi^{\mu}_{\varkappa},\chi^{-\mu_\nu}_{-\varkappa_\nu})\right]$$

Using (A.7)

$$(\psi_e^*\beta\,\psi_\nu^c) = i\,(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}}\left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}}\sum_\lambda C(jj_\nu\lambda;-\mu,-\mu_\nu)Y_\lambda^{-\mu-\mu_\nu}$$

$$\times\left\{\left[\frac{(2\bar{\ell}+1)(2\ell_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}}C(\bar{\ell}\ell_\nu\lambda;00)\,f_\varkappa G_{\varkappa_\nu}W(jj_\nu\bar{\ell}\ell_\nu;\lambda\tfrac{1}{2})\right.$$

$$\left.-\left[\frac{(2\ell+1)(2\bar{\ell}_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}}C(\ell\bar{\ell}_\nu\lambda;00)\,g_\varkappa F_{\varkappa_\nu}W(jj_\nu\ell\bar{\ell}_\nu;\lambda\tfrac{1}{2})\right\}$$

Substituting the above in (B.12a) and using

$$\vec{\sigma}\cdot\nabla = \sum_m(-)^m\sigma_{-m}\nabla_m$$

$$R_3 \equiv \left[\frac{(2\bar{\ell}+1)(2\ell_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}}C(\bar{\ell}\ell_\nu\lambda;00)\,f_\varkappa G_{\varkappa_\nu}W(jj_\nu\bar{\ell}\ell_\nu;\lambda\tfrac{1}{2}) \tag{B.19a}$$

$$R_4 \equiv -\left[\frac{(2\ell+1)(2\bar{\ell}_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}}C(\ell\bar{\ell}_\nu\lambda;00)\,g_\varkappa F_{\varkappa_\nu}W(jj_\nu\ell\bar{\ell}_\nu;\lambda\tfrac{1}{2}) \tag{B.19b}$$

$$\mathcal{H}_P(2) = -(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}}\left(\frac{C_P'}{2M}\right)\left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}}\sum_{\lambda,m}(-)^m C(jj_\nu\lambda;-\mu,-\mu_\nu)$$

$$\times\ \sigma_{-m}\nabla_m Y_\lambda^{-\mu-\mu_\nu}(R_3+R_4) \qquad \text{(B.12b)}$$

Applying the gradient formula (A.9a)

$$\nabla_m Y_\lambda^{-\mu-\mu_\nu}(R_3+R_4) = \left[\frac{\lambda+1}{2\lambda+3}\right]^{\frac{1}{2}} C(\lambda\ 1\ \lambda+1;-\mu-\mu_\nu,m)Y_{\lambda+1}^{-\mu-\mu_\nu+m}\ D_-(\lambda)(R_3+R_4)$$

$$-\left[\frac{\lambda}{2\lambda+1}\right]^{\frac{1}{2}} C(\lambda\ 1\ \lambda-1;-\mu-\mu_\nu,m)Y_{\lambda-1}^{-\mu-\mu_\nu+m}\ D_+(\lambda)(R_3+R_4)$$

$D_-(\lambda)$ and $D_+(\lambda)$ are given in (B.18d) and (B.18e)

Substituting the above in (B.12b) and using (B.18c)

$$\mathcal{H}_P(2) = (-)^{j+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}}\left(-\frac{C_P'}{2M}\right)\left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}}\sum_L\sum_\lambda\sum_m(-)^m C(jj_\nu\lambda;-\mu,-\mu_\nu)$$

$$\times\left\{\left[\frac{\lambda+1}{2\lambda+3}\right]^{\frac{1}{2}} C(\lambda\ 1\ \lambda+1;-\mu-\mu_\nu,m)C(\lambda+1\ 1L;-\mu-\mu_\nu+m,-m)T_{L\ \lambda+1}^{-\mu-\mu_\nu}\ D_-(\lambda)(R_3+R_4)\right.$$

$$\left.-\left[\frac{\lambda}{2\lambda-1}\right]^{\frac{1}{2}} C(\lambda\ 1\ \lambda-1;\mu-\mu_\nu,m)C(\lambda-1\ 1L;-\mu-\mu_\nu+m,-m)T_{L\ \lambda-1}^{-\mu-\mu_\nu}\ D_+(\lambda)(R_3+R_4)\right\} \qquad \text{(B.12c)}$$

As before in $\mathcal{H}_P(1)$, using (B.18g), (A.1a) and (A.1d)

$$\mathcal{H}_P(2) = (-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}}\left(\frac{C_P'}{2M}\right)\left[\frac{(2j+1)(2j_\nu+1)}{4\pi}\right]^{\frac{1}{2}}\sum_\lambda C(jj_\nu\lambda,-\mu,-\mu_\nu)$$

$$\left\{\left[\frac{\lambda+1}{2\lambda+1}\right]^{\frac{1}{2}} T_{\lambda,\lambda+1}^{-\mu-\mu_\nu}\ D_-(\lambda)(R_3+R_4) - \left[\frac{\lambda}{2\lambda+1}\right]^{\frac{1}{2}} T_{\lambda,\lambda-1}^{-\mu-\mu_\nu}\ D_+(\lambda)(R_3+R_4)\right\} \qquad \text{(B.12d)}$$

R_3 and R_4 are defined in (B.19a) and (B.19b).

For the axial vector and pseudoscalar interactions, the β^- hamiltonian

$$H_{\beta^-} = \vec{\sigma}\cdot(\psi_e^* \vec{\sigma}[C_A + C_A'\gamma_5]\,\psi_\nu^c) - \gamma_5(\psi_e^*\gamma_5[C_A + C_A'\gamma_5]\,\psi_\nu^c)$$

$$+ \frac{i}{2M}\,\vec{\sigma}\cdot\nabla(\psi_e^*\beta\gamma_5[C_P + C_P'\gamma_5]\,\psi_\nu^c)$$

becomes from (B.7d), (B.8d), (B.9b), (B.10c), (B.11c) and (B.12d)

$$H_{\beta^-} = \Big\{ iC_A\,(-)^{j+\ell_\nu+\bar{\ell}+\mu} \sum_L \sum_\lambda \Big[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\Big]^{\frac{1}{2}} C(j_\nu L j;\mu_\nu,-\mu-\mu_\nu)\,T_{L\lambda}^{-\mu-\mu_\nu}$$

$$\times\Big[\big[(2\bar{\ell}+1)(2\ell_\nu+1)\big]^{\frac{1}{2}}\, C(\bar{\ell}\ell_\nu\lambda;00) f_\varkappa G_{\varkappa_\nu}\, X(L1\lambda;j_\nu\tfrac{1}{2}\ell_\nu;j\tfrac{1}{2}\bar{\ell})$$

$$-\big[(2\ell+1)(2\bar{\ell}_\nu+1)\big]^{\frac{1}{2}}\, C(\ell\bar{\ell}_\nu\lambda;00) g_\varkappa F_{\varkappa_\nu}\, X(L1\lambda;j_\nu\tfrac{1}{2}\bar{\ell}_\nu;j\tfrac{1}{2}\ell)\Big]\Big\}$$

$$+\Big\{ C_A'\,(-)^{j+\ell_\nu+\bar{\ell}+\mu-1} \sum_L \sum_\lambda \Big[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\Big]^{\frac{1}{2}} C(j_\nu L j;\mu_\nu,-\mu-\mu_\nu)\,T_{L\lambda}^{\mu-\mu_\nu}$$

$$\times\Big[\big[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\big]^{\frac{1}{2}}\, C(\bar{\ell}\bar{\ell}_\nu\lambda;00) f_\varkappa F_{\varkappa_\nu}\, X(L1\lambda;j_\nu\tfrac{1}{2}\bar{\ell}_\nu;j\tfrac{1}{2}\bar{\ell})$$

$$+\big[(2\ell+1)(2\ell_\nu+1)\big]^{\frac{1}{2}}\, C(\ell\ell_\nu\lambda;00) g_\varkappa G_{\varkappa_\nu}\, X(L1\lambda;j_\nu\tfrac{1}{2}\ell_\nu;j\tfrac{1}{2}\ell)\Big]\Big\}$$

$$-\Big\{ C_A\,(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}} \sum_\lambda \Big[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\Big]^{\frac{1}{2}} C(jj_\nu\lambda;-\mu,-\mu_\nu)\,T_{\lambda,\lambda}^{-\mu_\nu-\mu}(\hat{r},\gamma_5)$$

$$\times\Big[-\big[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\big]^{\frac{1}{2}}\, C(\bar{\ell}\bar{\ell}_\nu\lambda;00) f_\varkappa F_{\varkappa_\nu} W(jj_\nu\bar{\ell}\bar{\ell}_\nu;\lambda\tfrac{1}{2})$$

$$+\big[(2\ell+1)(2\ell_\nu+1)\big]^{\frac{1}{2}}\, C(\ell\ell_\nu\lambda;00) g_\varkappa G_{\varkappa_\nu} W(jj_\nu\ell\ell_\nu;\lambda\tfrac{1}{2})\Big]\Big\}$$

$$-\Big\{ iC_A'\,(-)^{j_\nu+\ell_\nu-\mu_\nu-\mu+\frac{1}{2}} \sum_\lambda \Big[\frac{(2j+1)(2j\ \ +1)}{4\pi(2\ \ +1)}\Big]^{\frac{1}{2}} C(jj_\nu\lambda;-\mu,-\mu_\nu)\,T_{\lambda,\lambda}^{-\mu_\nu-\mu}(\hat{r},\gamma_5)\times$$

$$\times \left[\left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\,\lambda;00) f_\varkappa G_{\varkappa_\nu} W(jj_\nu\,\bar{\ell}\,\ell_\nu;\lambda\,\tfrac{1}{2}) + \left[(2\ell+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu\,\lambda;00) g_\varkappa F_{\varkappa_\nu} W(jj_\nu\,\ell\,\bar{\ell}_\nu;\lambda\,\tfrac{1}{2}) \right] \Bigg\}$$

$$+ \Bigg\{ i\,\frac{C_P}{2M}\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(jj_\nu\lambda;-\mu,-\mu_\nu)$$

$$\times \left[(\lambda+1)^{\frac{1}{2}} T^{-\mu-\mu_\nu}_{\lambda,\lambda+1} D_-(\lambda) - \lambda^{\frac{1}{2}}\, T^{-\mu-\mu_\nu}_{\lambda,\lambda-1} D_+(\lambda) \right] (R_1+R_2) \Bigg\}$$

$$+ \Bigg\{ \frac{C'_P}{2M}\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(jj_\nu\lambda;-\mu,-\mu_\nu)$$

$$\left[(\lambda+1)^{\frac{1}{2}}\, T^{-\mu-\mu_\nu}_{\lambda,\lambda+1} D_-(\lambda) - (\lambda)^{\frac{1}{2}}\, T^{-\mu-\mu_\nu}_{\lambda,\lambda-1} D_+(\lambda) \right] (R_3+R_4) \Bigg\} \qquad \text{(B.6b)}$$

where

$$R_1 \equiv \left[\frac{(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu\,\lambda;00) f_\varkappa F_{\varkappa_\nu} W(jj_\nu\,\bar{\ell}\,\bar{\ell}_\nu;\lambda\,\tfrac{1}{2}) \qquad \text{(B.18a)}$$

$$R_2 \equiv \left[\frac{(2\ell+1)(2\ell_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\ell\,\ell_\nu\,\lambda;00) g_\varkappa G_{\varkappa_\nu} W(jj_\nu\,\ell\,\ell_\nu;\lambda\,\tfrac{1}{2}) \qquad \text{(B.18b)}$$

$$R_3 \equiv \left[\frac{(2\bar{\ell}+1)(2\ell_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\,\lambda;00) f_\varkappa G_{\varkappa_\nu} W(jj_\nu\,\bar{\ell}\,\ell_\nu;\lambda\,\tfrac{1}{2}) \qquad \text{(B.19a)}$$

$$R_4 \equiv -\left[\frac{(2\ell+1)(2\bar{\ell}_\nu+1)}{2\lambda+1}\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu\,\lambda;00) g_\varkappa F_{\varkappa_\nu} W(jj_\nu\,\ell\,\bar{\ell}_\nu;\lambda\,\tfrac{1}{2}) \qquad \text{(B.19b)}$$

$$D_+(\lambda) \equiv \frac{d}{dr} + \frac{\lambda+1}{r} \qquad \text{(B.18e)}$$

$$D_-(\lambda) \equiv \frac{d}{dr} - \frac{\lambda}{r} \qquad \text{(B.18d)}$$

$$T_{L\lambda}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma}) \equiv \sum_{m} C(\lambda 1 L; m-\mu-\mu_\nu, -m) Y_{\lambda}^{m-\mu-\mu_\nu}(\hat{r})\ \sigma_{-m} \qquad (B.16d)$$

$$T_{\lambda\lambda}^{-\mu-\mu_\nu}(\hat{r},\gamma_5) \equiv Y_{\lambda}^{-\mu-\mu_\nu}(\hat{r})\ \gamma_5 \qquad (B.17b)$$

Thus far, no retardation expansion has been made and the results given above are the most general for the interaction hamiltonian for the axial vector and pseudoscalar interactions.

Now to get a matrix element between nuclear states specified by J_i, M_i and J_f, M_f; we shall use the following notation:

$$\sum_{k}^{A} \int \Psi_{J_f}^{M_f *} \Omega_L^M(k)\ Q(k)\ \Psi_{J_i}^{M_i}\ d\tau_1 \cdots d\tau_A \equiv \int \Psi_{J_f}^{M_f *}\ \Omega_L^M\ \Psi_{J_i}^{M_i}$$

Where in the above, $\Omega_L^M(k)$ is an operator acting in the space of the kth nucleon. $Q(k)$ is the operator which converts the kth nucleon from one charge state to the other. There is a summation over all the nucleons.

In (B.6b), we have the following:

L - rank of the irreducible tensor

λ - order of spherical harmonic for the axial vector interaction.

ℓ, ℓ_ν - refers to the orbital angular monentum of the electron and neutrino for $\varkappa$ and $\varkappa_\nu$

$\bar{\ell}, \bar{\ell}_\nu$ are $\ell_{-\varkappa}$ and $\ell_{-\varkappa_\nu}$ respectively.

By Wigner-Eckart theorem;

$$(J_f\ M_f |\ T_L^M\ | J_i\ M_i) = C(J_i\ L\ J_f; M_i\ M\ M_f)\ (J_f \|\ T_L \| J_i)$$

and also since nuclear states have definite parity:

$$\pi_i \; \pi(T) = \pi_f$$

π_i, π_f, $\pi(T)$ are the parities of the initial, and final nuclear states and the irreducible tensor respectively.

We shall use also the standard notation

$$(J_f \| T_L \| J_i) \equiv \int T_L$$

To find the matrix elements of $H_{\beta-}$ as given in (B.6b), we are interested in the following:

(1) $$(J_f \; M_f \,|\, T_{L\lambda}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma}) \,|\, J_i \; M_i) = C(J_i \; L J_f; M_i, -\mu-\mu_\nu, M_f) \int T_{L\lambda}(\hat{r},\vec{\sigma}) \qquad \text{(B.20a)}$$

and

$$\pi_i \; \pi_f = (-)^\lambda \quad ; \Delta(J_i \; L J_f) \qquad \text{(B.20b)}$$

(2) $$(J_f \; M_f \,|\, T_{\lambda,\lambda}^{-\mu-\mu_\nu}(\hat{r},\gamma_5) \,|\, J_i \; M_i) = C(J_i \; \lambda \; J_f; M_i, -\mu-\mu_\nu, M_f) \int T_{\lambda,\lambda}(\hat{r},\gamma_5) \qquad \text{(B.21a)}$$

and

$$\pi_i \; \pi_f = (-)^{\lambda+1} \quad ; \Delta(J_i \; \lambda \; J_f) \qquad \text{(B.21b)}$$

(3) $$(J_f \; M_f \,|\, T_{\lambda,\,\lambda+1}^{-\mu-\mu}(\hat{r},\vec{\sigma}) \,|\, J_i \; M_i) = C(J_i \; \lambda \; J_f; M_i, -\mu-\mu_\nu, M_f) \int T_{\lambda,\,\lambda+1}(\hat{r},\vec{\sigma}) \qquad \text{(B.22a)}$$

and

$$\pi_i \; \pi_f = (-)^{\lambda+1} \quad ; \Delta(J_i \; \lambda \; J_f) \qquad \text{(B.22b)}$$

(4) $$(J_f \; M_f \,|\, T_{\lambda,\,\lambda-1}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma}) \,|\, J_i \; M_i) = C(J_i \; \lambda \; J_f; M_i, -\mu-\mu_\nu, M_f) \int T_{\lambda,\,\lambda-1}(\hat{r},\vec{\sigma}) \qquad \text{(B.23a)}$$

and

$$\pi_i \; \pi_f = (-)^{\lambda-1} \quad ; \Delta(J_i \; \lambda \; J_f) \qquad \text{(B.23b)}$$

In (B.20), (B.21), (B.22) and (B.23), $M_f = M_i + (-\mu-\mu_\nu)$. Using (B.20a), (B.21a), (B.22a), (B.23a) and (B.6b), the matrix elements of H_{β^-} for the axial vector and the pseudoscalar interaction involving a nuclear transition from a state (J_i, π_i) to (J_f, π_f) is

$$\left\langle J_f\, M_f \right| H_{\beta^-} \left| J_i\, M_i \right\rangle$$

$$= \Big\{ i\, C_A\, (-)^{j+\ell_\nu+\bar{\ell}+\mu} \sum_L \sum_\lambda \left[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(j_\nu L j; \mu_\nu, -\mu-\mu_\nu) C(J_i L J_f; M_i, -\mu-\mu_\nu, M_f)$$

$$\times \int T_{L,\lambda}(\hat{r}, \vec{\sigma}) \Big[\left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\lambda; 00) f_\varkappa G_{\varkappa_\nu} X(L\,1\,\lambda; j_\nu \tfrac{1}{2}\, \ell_\nu; j\, \tfrac{1}{2}\, \bar{\ell})$$

$$- \left[(2\ell+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu\,\lambda; 00) g_\varkappa F_{\varkappa_\nu} X(L\,1\,\lambda; j_\nu \tfrac{1}{2}\, \bar{\ell}_\nu; j \tfrac{1}{2}\, \ell) \Big] \Big\}$$

$$+ \Big\{ C_A'\, (-)^{j+\ell_\nu+\bar{\ell}+\mu-1} \sum_L \sum_\lambda \left[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(j_\nu L j; \mu_\nu, -\mu-\mu_\nu) C(J_i L J_f; M_i, -\mu-\mu_\nu, M_f)$$

$$\times \int T_{L,\lambda}(\hat{r}, \vec{\sigma}) \Big[\left[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu\,\lambda; 00) f_\varkappa F_{\varkappa_\nu} X(L\,1\,\lambda; j_\nu \tfrac{1}{2}\, \bar{\ell}_\nu; j \tfrac{1}{2}\, \ell)$$

$$+ \left[(2\ell+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\ell_\nu\,\lambda; 00) g_\varkappa G_{\varkappa_\nu} X(L\,1\,\lambda; j_\nu \tfrac{1}{2}\, \ell_\nu; j \tfrac{1}{2}\, \ell) \Big] \Big\}$$

$$- \Big\{ C_A\, (-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(j j_\nu \lambda; -\mu, -\mu_\nu) C(J_i \lambda J_f; M_i, -\mu-\mu_\nu, M_f)$$

$$\times \int T_{\lambda,\lambda}(\hat{r}, \gamma_5) \Big[- \left[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu\lambda; 00) f_\varkappa F_{\varkappa_\nu} W(j j_\nu \bar{\ell}\, \bar{\ell}_\nu; \lambda \tfrac{1}{2})$$

$$+ \left[(2\ell+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\ell_\nu\,\lambda; 00) g_\varkappa G_{\varkappa_\nu} W(j j_\nu \ell\, \ell_\nu; \lambda \tfrac{1}{2}) \Big] \Big\}$$

$$- \Big\{ i C_A'\, (-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(j j_\nu \lambda; -\mu, -\mu_\nu) C(J_i \lambda J_f; M_i, -\mu-\mu_\nu, M_f)$$

$$\int T_{\lambda,\lambda}(\hat{r}, \gamma_5) \Big[\left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\,\lambda; 00) f_\varkappa G_{\varkappa_\nu} W(j j_\nu \bar{\ell}\, \ell_\nu; \lambda \tfrac{1}{2})$$

$$+ \left[(2\ell+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu\,\lambda; 00) g_\varkappa F_{\varkappa_\nu} W(j j_\nu \ell\, \bar{\ell}_\nu; \lambda \tfrac{1}{2}) \Big] \Big\} +$$

$$+\left\{ i\,\frac{C_P}{2M}\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(jj_\nu\lambda\,;-\mu,-\mu_\nu)\right.$$

$$\times\left[(\lambda+1)^{\frac{1}{2}}\, C(J_i\lambda J_f;M_i,-\mu-\mu_\nu,M_f)\int T_{\lambda,\lambda+1}(\hat{r},\vec{\sigma})D_-(\lambda)\right.$$

$$\left.\left. -(\lambda)^{\frac{1}{2}}\, C(J_i\lambda J_f;M_i,-\mu-\mu_\nu,M_f)\int T_{\lambda,\lambda-1}(\hat{r},\vec{\sigma})D_+(\lambda)\right]\times(R_1+R_2)\right\}$$

$$+\left\{ \frac{C_P'}{2M}\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(jj_\nu\lambda\,;-\mu,-\mu_\nu)\right.$$

$$\times\left[(\lambda+1)^{\frac{1}{2}}\, C(J_i\lambda J_f;M_i,-\mu-\mu_\nu,M_f)\int T_{\lambda,\lambda+1}(\hat{r},\vec{\sigma})D_-(\lambda)\right.$$

$$\left.\left. -(\lambda)^{\frac{1}{2}}\, C(J_i\lambda J_f;M_i,-\mu-\mu\ ,M_f)\int T_{\lambda,\lambda-1}(\hat{r},\vec{\sigma})D_+(\lambda)\right](R_3\ R_4)\right\}$$

(B.6c)

3. MATRIX ELEMENTS OF H_{β^-} FOR $0\rightarrow 0$ (YES) TRANSITIONS

Now we specialize the equation (B.6c) for $J_i = J_f = 0$ and $\pi_i\,\pi_f = -1$.

Two types of irreducible tensors occur for the axial vector interaction. Namely, $T_{L\,\lambda}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma})$ and $T_{\lambda,\lambda}^{-\mu-\mu_\nu}(\hat{r},\gamma_5)$.

$$(J_f\,M_f\,|T_{L\,\lambda}^{-\mu-\mu_\nu}(\hat{r},\vec{\sigma})|\,J_i\,M_i) = C(J_iLJ_f;M_i,-\mu-\mu_\nu,M_f)\int T_{L\,\lambda}(\hat{r},\vec{\sigma}) \qquad \text{(B.20a)}$$

In this case $\pi_i\,\pi_f = (-)^\lambda$ and $\Delta(J_iLJ_f)$ (B.20b)

$\lambda\rightarrow 1$ so that $\pi_i\,\pi_f = -1$

$L\rightarrow 0$ (since $J_i = J_f = 0$) and the projection of a zero rank tensor is also zero; therefore, $\mu+\mu_\nu = 0$.

By definition:

$$T_{01}^{0}(\hat{r},\vec{\sigma}) = \sum_m C(1\,1\,0;m,-m)\,Y_1^m(\hat{r})\,\sigma_{-m}$$

$$= \sum_m (-)^{1-m} \sqrt{\tfrac{1}{3}}\; C(1\,0\,1,m,0)\sqrt{\tfrac{3}{4\pi}}\; \hat{r}_m\; \sigma_{-m}$$

$$= -\sqrt{\tfrac{1}{4\pi}}\; \sum_m (-)^m \hat{r}_m\; \sigma_{-m}$$

$$T^0_{01}\;(\hat{r},\vec{\sigma}) = -\sqrt{\tfrac{1}{4\pi}}\quad \vec{\sigma}.\hat{r}$$

Matrix elements of $T^0_{01}\;(\hat{r},\vec{\sigma})$ for $J_i = J_f = 0$ are

$$(00\,|T^0_{01}\;(\hat{r},\vec{\sigma})|00) = -\delta_{\mu,-\mu_\nu}\sqrt{\tfrac{1}{4\pi}}\quad \int \vec{\sigma}.\hat{r} \tag{B.20c}$$

Similarly for the case

$$(J_f\; M_f\,|\,T^{-\mu-\mu_\nu}_{\lambda\,\lambda}(\hat{r}.\gamma_5)|\;J_i\;M_i) = C(J_i\lambda J_f;M_i,-\mu-\mu_\nu,M_f)\int T_{\lambda\lambda}(\hat{r},\gamma_5) \tag{B.21a}$$

and as pointed out earlier:

$$\pi_i\pi_f = (-)^{\lambda+1} \quad ;\text{ and } \Delta(J_i\lambda J_f) \tag{B.21b}$$

In $0 \rightarrow 0$ (yes), $\pi_i\;\pi_f = -1$; therefore, $\lambda = 0$

and also $\mu = -\mu_\nu$.

$$T^0_{00}\;(\hat{r},\gamma_5) = Y^0_0\;(\hat{r})\;\gamma_5 = \sqrt{\tfrac{1}{4\pi}}\quad \gamma_5$$

$$(00\,|\,T^0_{00}\;(\hat{r},\gamma_5)|00) = \sqrt{\tfrac{1}{4\pi}}\;\;\delta_{\mu,-\mu_\nu}\int\gamma_5 \tag{B.21c}$$

In the pseudoscalar interaction, there are two irreducible tensors $T^{-\mu-\mu_\nu}_{\lambda,\;\lambda+1}\;(\hat{r},\vec{\sigma})$ and $T^{-\mu-\mu_\nu}_{\lambda,\;\lambda-1}\;(\hat{r},\vec{\sigma})$; where $\pi_i\;\pi_f = (-)^{\lambda+1}$. Thus we consider $\lambda = 0$, and as before $\mu = -\mu_\nu$

$$T^0_{01} = -\sqrt{\tfrac{1}{4\pi}}\quad \vec{\sigma}.\hat{r}$$

$$(00\,|\,T^0_{01}(\hat{r},\vec{\sigma})\,|\,00) = -\sqrt{\tfrac{1}{4\pi}}\;\delta_{\mu,-\mu_\nu}\int\vec{\sigma}.\hat{r} \tag{B.22c}$$

But $T^{-\mu-\mu_\nu}_{\lambda,\lambda-1}(\hat{r},\vec{\sigma}) \equiv \sum_m C(\lambda 1 \lambda-1, -\mu-\mu_\nu+m, -m)\, Y^{-\mu-\mu_\nu+m}_{\lambda-1}(\hat{r})\, \sigma_{-m}$

For $\lambda = 0$; $Y^{-\mu-\mu_\nu+m}_{\lambda-1}(r) = 0$ and as such there is no contribution of

$T^{-\mu-\mu}_{\lambda,\lambda-1}(\hat{r},\vec{\sigma})$ to $0 \rightarrow 0$ (yes) transitions.

The coefficient $C(\ell_1\ell_2\ell_3;00)$ is called the parity C-coefficient, if ℓ_1, ℓ_2, and ℓ_3 are integers.

Using (A.1a)

$$C(\ell_1\ell_2\ell_3,00) = (-)^{\ell_1+\ell_2-\ell_3} C(\ell_1\ell_2\ell_3,00)$$

For the equation to be true, $\ell_1 + \ell_2 - \ell_3$ = even.

and so $\ell_1 + \ell_2 + \ell_3$ = even.

Also $C(J_1 0 J_3; m_1 0 m_3) = \delta_{J_1,J_3}\,\delta_{m_1,m_3}$ (B.23c)

Consider in (B.6c) for $0 \rightarrow 0$ (yes) beta transition

$$\int \mathcal{H}_A(1)$$

$$\equiv \Big\{ i\, C_A\, (-)^{j+\ell_\nu+\bar{\ell}+\mu} \sum_L \sum_\lambda \left[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(j_\nu L j;\mu_\nu,-\mu-\mu_\nu) C(J_i L J_f; M_i,-\mu-\mu_\nu, M_f)$$

$$\int T_{L\lambda}(\hat{r},\vec{\sigma}) \times \Big[\left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu\lambda;00) f_\varkappa G_{\varkappa_\nu} X(L1\lambda; j_\nu \tfrac{1}{2} \ell_\nu; j \tfrac{1}{2} \bar{\ell})$$

$$-\left[(2\ell+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\ell\bar{\ell}_\nu\lambda;00) g_\varkappa F_{\varkappa_\nu} X(L1\lambda; j_\nu \tfrac{1}{2} \bar{\ell}_\nu; j \tfrac{1}{2} \ell) \qquad \text{(B.24a)}$$

Since in the above $\lambda = 1$, the parity C-coefficients gives

$$C(\bar{\ell}\ell_\nu 1;00) = \delta_{\varkappa,\varkappa_\nu}\, C(\bar{\ell}\ell 1;00) \qquad \text{(B.25a)}$$

$$C(\ell\bar{\ell}_\nu 1;00) = \delta_{\varkappa,\varkappa_\nu}\, C(\ell\bar{\ell} 1;00) \qquad \text{(B.25b)}$$

and $C(j\,0j;\mu\,,-\mu-\mu_\nu) = \delta_{jj_\nu}\,\delta_{\mu,-\mu}$ (B.25c)

then X-coefficients become

$$X(011;j\tfrac{1}{2}\ell;j\tfrac{1}{2}\bar{\ell}) = X(\bar{\ell}\tfrac{1}{2}j;\ell\tfrac{1}{2}j;110)$$

$$X(011;j\tfrac{1}{2}\ell;j\tfrac{1}{2}\bar{\ell}) = (-)^{j-\bar{\ell}+\frac{1}{2}}\,\frac{W(\bar{\ell}\tfrac{1}{2}\ell\tfrac{1}{2};j1)}{[3(2j+1)]^{1/2}}$$

Using (A.8c)

$$X(011;j\tfrac{1}{2}\ell;j\tfrac{1}{2}\bar{\ell}) = (-)^{j-\bar{\ell}+\frac{1}{2}}\,\frac{W(\bar{\ell}\ell\tfrac{1}{2}\tfrac{1}{2};1j)}{[3(2j+1)]^{1/2}} \qquad \text{(b.25d)}$$

Similarly,

$$X(011;j\tfrac{1}{2}\bar{\ell};j\tfrac{1}{2}\ell) = (-)^{j-\ell+\frac{1}{2}}\,\frac{W(\bar{\ell}\ell\tfrac{1}{2}\tfrac{1}{2};1j)}{[3(2j+1)]^{1/2}} \qquad \text{(B.25e)}$$

Also $C(j0j;\mu_\nu,0) = 1$ (B.25f)

Substituting (B.25) and L = 0 in (B.24a), we get

$$\int \mathcal{H}_A(1) = i\,C_A\,(-)^{j+\ell+\bar{\ell}+\mu}\sqrt{\frac{6(2j+1)}{4\pi}}\;\delta_{\mu,-\mu_\nu}\,(-)\sqrt{\frac{1}{4\pi}}\;\int \vec{\sigma}\cdot\hat{r}\;\delta_{\varkappa,\varkappa_\nu}$$

$$\times\Bigg[\,[(2\bar{\ell}+1)(2\ell+1)]^{\frac{1}{2}}\,C(\bar{\ell}\ell 1;00)f_\varkappa G_\varkappa(-)^{j-\bar{\ell}+\frac{1}{2}}\,\frac{W(\bar{\ell}\ell\tfrac{1}{2}\tfrac{1}{2};1j)}{[3(2j+1)]^{1/2}}$$

$$-\,[(2\ell+1)(2\bar{\ell}+1)]^{\frac{1}{2}}\,C(\ell\bar{\ell}1;00)g_\varkappa F_\varkappa\,(-)^{j-\ell+\frac{1}{2}}\,\frac{W(\bar{\ell}\ell\tfrac{1}{2}\tfrac{1}{2};1j)}{[3(2j+1)]^{\frac{1}{2}}}\Bigg]$$

Using (A.1b)

$$C(\ell\bar{\ell}\,1,00) = (-)^{\ell+\bar{\ell}-1}\,C(\bar{\ell}\ell 1,00) = C(\bar{\ell}\ell\,1,00)$$

$$\int \mathcal{H}_A(1) = -i\,C_A\,(-)^{j+\ell+\bar{\ell}+\mu+j-\bar{\ell}+\frac{1}{2}}\,\delta_{\mu,-\mu_\nu}\,\delta_{\varkappa,\varkappa_\nu}\left[\frac{(2\ell+1)(2\bar{\ell}+1)}{8\pi^2}\right]^{\frac{1}{2}}$$

$$\times\;C(\bar{\ell}\ell\,1,00)W(\bar{\ell}\ell\tfrac{1}{2}\tfrac{1}{2};1j)\left[f_\varkappa G_\varkappa - (-)^{\bar{\ell}-\ell}\;g_\varkappa F_\varkappa\right]\int \vec{\sigma}\cdot\hat{r}$$

Using $\bar{\ell} + \ell + 1$ = even and $2j + 1$ = even integer

$$\int \mathcal{H}_A(1) = i\, C_A\, (-)^{\ell+\mu+\frac{1}{2}}\, \delta_{\mu,-\mu}\, \delta_{\varkappa,\varkappa_\nu} \left[\frac{(2\ell+1)(2\bar{\ell}+1)}{8\pi^2}\right]^{\frac{1}{2}}$$

$$\times\, C(\bar{\ell}\ell 1,00) W(\bar{\ell}\ell \tfrac{1}{2}\tfrac{1}{2};1j)(f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \int \vec{\sigma}\cdot\hat{r} \qquad \text{(B.24b)}$$

Similarly we consider

$$\int \mathcal{H}_A(2) \equiv C_A'\, (-)^{j_\nu+\ell_\nu+\bar{\ell}+\mu-1} \sum_L \sum_\lambda \left[\frac{6(2j_\nu+1)(2L+1)}{4\pi}\right]^{\frac{1}{2}} C(j_\nu L j;\mu_\nu,-\mu-\mu_\nu)$$

$$\times\, C(J_i L J_f;M_i,-\mu-\mu_\nu,M_f) \int T_{L,\lambda}(\hat{r},\vec{\sigma}) \Big[\left[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu\lambda,00) f_\varkappa F_{\varkappa_\nu}$$

$$\times\, X(L1\lambda;j_\nu \tfrac{1}{2}\bar{\ell}_\nu;j\tfrac{1}{2}\bar{\ell}) + \left[(2\ell+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\ell_\nu\lambda;00) g_\varkappa G_{\varkappa_\nu} X(L1\lambda;j\tfrac{1}{2}\ell_\nu;j\tfrac{1}{2}\ell) \Big] \qquad \text{(B.24c)}$$

Again for $0 \rightarrow 0$ (yes) transitions,

$\lambda = 1$; $L = 0$

The parity C-coefficients give

$$C(\bar{\ell}\,\bar{\ell}_\nu' 1;00) = \delta_{\varkappa,-\varkappa_\nu} C(\bar{\ell}\,\ell\, 1,00)$$

$$C(\ell\,\ell_\nu\, 1,00) = \delta_{\varkappa,-\varkappa_\nu} C(\ell\,\bar{\ell}\, 1,00) = \delta_{\varkappa,-\varkappa_\nu} C(\bar{\ell}\,\ell\, 1,00)$$

$$C(j_\nu 0 j,\mu_\nu,-\mu-\mu_\nu) = \delta_{j_\nu,j}\cdot\delta_{\mu,-\mu_\nu}$$

$X(011,j\frac{1}{2}\ell;j\frac{1}{2}\bar{\ell})$ and $X(011,j\frac{1}{2}\bar{\ell};j\frac{1}{2}\ell)$ are given in (B.25d) and (B.25e).

$$C(j0j;\mu_\nu 0) = 1.$$

Substituting these relations and (B.20c) in (B.24c)

$$\int \mathcal{H}_A(2) = C_A'\, (-)^{\ell+\mu+\frac{1}{2}}\, \delta_{\mu,-\mu_\nu}\, \delta_{\varkappa,-\varkappa_\nu} \left[\frac{(2\ell+1)(2\bar{\ell}+1)}{8\pi^2}\right]^{\frac{1}{2}}$$

$$C(\bar{\ell}\ell 1,00) W(\bar{\ell}\ell \tfrac{1}{2}\tfrac{1}{2};1j)(f_\varkappa F_{-\varkappa} - g_\varkappa G_{-\varkappa}) \int \vec{\sigma}\cdot\hat{r} \qquad \text{(B.24d)}$$

$$\int \mathcal{H}_A\,(3) \equiv -C_A\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi\,(2\lambda+1)}\right]^{\frac{1}{2}} C(j j_\nu \lambda\,; -\mu, -\mu_\nu)$$

$$\times\ C(J_i \lambda J_f; M_i, -\mu-\mu_\nu, M_f) \int T_{\lambda,\lambda}(\hat{r}, \gamma_5) \Big[-\left[(2\bar{\ell}+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\bar{\ell}_\nu \lambda\,;00)$$

$$\times\ f_\varkappa F_{\varkappa_\nu} W(j j\, \bar{\ell}\,\bar{\ell}_\nu\,; \lambda\, \tfrac{1}{2}) + \left[(2\ell+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\ell_\nu \lambda\,;00)\, g_\varkappa G_{\varkappa_\nu} W(j j_\nu \ell\,\ell_\nu\,; \lambda\,\tfrac{1}{2})\Big] \qquad \text{(B.24e)}$$

From (B.21): $\lambda = 0$; after simplification, as shown above,

we get

$$\int \mathcal{H}_A\,(3) = \frac{-C_A}{4\pi}\,(-)^{j+\ell+\mu}\, \delta_{\mu,-\mu_\nu} \delta_{\varkappa,\varkappa_\nu}\, (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) \int \gamma_5 \qquad \text{(B.24f)}$$

Similarly, we evaluate

$$\int \mathcal{H}_A\,(4) \equiv -iC_A'\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi(2\lambda+1)}\right]^{\frac{1}{2}} C(j j_\nu \lambda\,; -\mu, -\mu_\nu)$$

$$\times\ C(J_i \lambda J_f; M_i, -\mu-\mu_\nu, M_f) \int T_{\lambda,\lambda}(\hat{r}, \gamma_5)$$

$$\times \Big[\left[(2\bar{\ell}+1)(2\ell_\nu+1)\right]^{\frac{1}{2}} C(\bar{\ell}\,\ell_\nu \lambda\,;00) f_\varkappa G_{\varkappa_\nu} W(j j_\nu \bar{\ell}\,\ell_\nu\,; \lambda\,\tfrac{1}{2})$$

$$+ \left[(2\ell+1)(2\bar{\ell}_\nu+1)\right]^{\frac{1}{2}} C(\ell\,\bar{\ell}_\nu \lambda\,;00)\, g_\varkappa F_{\varkappa_\nu} W(j j_\nu \ell\,\bar{\ell}_\nu\,; \lambda\,\tfrac{1}{2})\Big]$$

Substituting the values of Racah Coefficients and simplifying,

we get:

$$\int \mathcal{H}_A\,(4) = -i\,\frac{C_A'}{4\pi}\,(-)^{j+\ell+\mu}\, \delta_{\mu,-\mu_\nu} \delta_{\varkappa,-\varkappa_\nu}\, (f_\varkappa G_{-\varkappa} + g_\varkappa F_{-\varkappa}) \int \gamma_5 \qquad \text{(B.24g)}$$

Now we consider the terms due to the pseudoscalar interaction.

For the general case we have:

$$\int \mathcal{H}_P\,(1) \equiv \frac{iC_P}{2M}\,(-)^{j_\nu+\ell_\nu-\mu-\mu_\nu+\frac{1}{2}} \sum_\lambda \left[\frac{(2j+1)(2j_\nu+1)}{4\pi\,(2\lambda+1)}\right]^{\frac{1}{2}} C(j j_\nu \lambda\,; -\mu, -\mu_\nu)\ \times$$

$$\times\left[(\lambda+1)^{\frac{1}{2}}C(J_i\lambda J_f;M_i,-\mu-\mu_\nu,M_f)\int T_{\lambda,\lambda+1}(\hat{r},\vec{\sigma})D_-(\lambda) - (\lambda)^{\frac{1}{2}}C(J_i\lambda J_f;M_i,-\mu-\mu_\nu,M_f)\right.$$

$$\left.\int T_{\lambda,\lambda-1}(\hat{r},\vec{\sigma})\,D_+(\lambda)\right](R_1+R_2) \tag{B.26a}$$

R_1, R_2, $D_+(\lambda)$ and $D_-(\lambda)$ are defined in (B.18a), (B.18b), (B.18e), and (B.18d).

Using (B.22c) and simplifying, we get for $0\rightarrow 0$ (yes) beta transitions

$$\int\mathcal{H}_P(1) = i\,\frac{C_P'}{8\pi M}\,(-)^{j+\ell+\mu}\,\delta_{\varkappa,\varkappa_\nu}\delta_{\mu,-\mu_\nu}\,\frac{d}{dr}\left[f_\varkappa F_\varkappa + g_\varkappa G_\varkappa\right]\int\vec{\sigma}\cdot\hat{r} \tag{B.26b}$$

Similarly we consider $\int\mathcal{H}_P(2)$ and we get in $0\rightarrow 0$ (yes) transitions,

$$\int\mathcal{H}_P(2) = \frac{C_P'}{8\pi M}\,(-)^{j+\ell+\mu+1}\,\delta_{\varkappa,-\varkappa_\nu}\delta_{\mu,-\mu_\nu}\,\frac{d}{dr}\left[f_\varkappa G_{-\varkappa} - g_\varkappa F_{-\varkappa}\right]\int\vec{\sigma}\cdot\hat{r} \tag{B.26d}$$

For $0\rightarrow 0$ (yes) beta transitions, we get

$$\langle\Psi_f|H_{\beta^-}|\Psi_i\rangle = \int\mathcal{H}_A(1) + \int\mathcal{H}_A(2) + \int\mathcal{H}_A(3) + \int\mathcal{H}_A(4) + \int\mathcal{H}_P(1) + \int\mathcal{H}_P(2)$$

where the terms on the righthand side are given by (B.24b), (B.24d), (B.24f), (B.24g), (B.26b) and (B.26d) respectively.

Now

$$F_\varkappa = S_\varkappa\, q\, j_{\bar{\ell}}(qr)$$

$$G_\varkappa = q\, j_\ell(qr)$$

therefore, $\quad F_{-\varkappa} = -S_\varkappa\, q\, j_\ell(qr) = -S_\varkappa G_\varkappa$

and $\quad G_{-\varkappa} = q\, J_{\bar{\ell}}(qr) = S_\varkappa F_\varkappa$

Making use of the above and using (A.1c) and (A.8d), we obtain the nuclear β matrix elements for $0\rightarrow 0$ (yes) transitions,

$$\langle \Psi_f | H_{\beta^-} | \Psi_i \rangle$$

$$= \frac{1}{4} (-)^{\mu+\ell+j} \delta_{\mu,-\mu_\nu} \Big\{ (iC_A \delta_{\varkappa,\varkappa_\nu} - S_\varkappa C_A' \delta_{\varkappa,\varkappa_\nu}) \Big[[6(2\bar{\ell}+1)]^{\frac{1}{2}} C(\bar{\ell} 1 \ell ;00)$$

$$W(\bar{\ell} 1 j\tfrac{1}{2}; \ell \tfrac{1}{2}) (f_\varkappa G_\varkappa + g_\varkappa F_\varkappa) \int \vec{\sigma}.\hat{r} + (f_\varkappa F_\varkappa - g_\varkappa G_\varkappa) i \int \gamma_5$$

$$+ (i \frac{C_P}{2M} \delta_{\varkappa,\varkappa_\nu} - S_\varkappa \frac{C_P'}{2M} \delta_{\varkappa,-\varkappa_\nu}) \frac{d}{dr}(f_\varkappa F_\varkappa + g_\varkappa G_\varkappa) \int \vec{\sigma}.\hat{r} \Big\} \qquad (B.27)$$

APPENDIX C

1. <u>Evaluation of $\mathcal{S}$.</u>

We show the details of some Racah recoupling, which have been used.

$$\mathcal{S} \equiv \sum_{\tau} (-)^{\mu-\tau} C(\ell\, \ell'\lambda;\tau-\mu,\mu'-\tau+m)C(\ell\,\tfrac{1}{2}j;\mu-\tau,\tau)C(\ell'\tfrac{1}{2}j';\mu'-\tau+m,\tau-m)$$
$$\times\ C(\tfrac{1}{2}\,1\,\tfrac{1}{2};\tau,-m) \qquad \text{(C.1a)}$$

$C(\ell\,\ell'\lambda;\tau-\mu,\mu'-\tau+m)C(\ell\tfrac{1}{2}\,j;\mu-\tau,\tau)$

$$= (-)^{\ell'+\mu'-\tau+m}\left[\frac{2\lambda+1}{2\ell+1}\right]^{\frac{1}{2}} C(\lambda\,\ell'\ell;\mu-\mu'-m,\mu'-\tau+m)C(\ell\tfrac{1}{2}\,j;\mu-\tau,\tau)$$

$$= (-)^{\ell'+\mu'-\tau+m}\left[\frac{2\lambda+1}{2\ell+1}\right]^{\frac{1}{2}}\sum_{s_1}\left[(2s_1+1)(2\ell+1)\right]^{\frac{1}{2}} C(\lambda s_1 j;\mu-\mu'-m,\mu'+m)$$

$$\times\ C(\ell'\,\tfrac{1}{2}\,s_1;\mu'-\tau+m,\tau)\ W(\lambda\,\ell'\,j\tfrac{1}{2};\ell\,s_1) \qquad \text{(C.2a)}$$

In arriving at (C.2a) we have used (A.1d) and(A.8a).

Similarly,

$$C(\ell'\,\tfrac{1}{2}\,j';\mu'-\tau+m,\tau-m)C(\tfrac{1}{2}\,1\,\tfrac{1}{2};\tau,-m) = (-)^{\ell'-m'+\tau+1}\sqrt{\frac{2j'+1}{2}}$$

$$\times\ C(\ell'\,j'\,\tfrac{1}{2};-\tau+\mu'+m,-\mu')C(\tfrac{1}{2}\,1\,\tfrac{1}{2};-\tau+m,-m)$$

$$= (-)^{\ell'-\mu'+\tau+1}\sum_{s_2}\left[(2j'+1)(2s_2+1)\right]^{\frac{1}{2}} C(j'1\,s_2;-\mu',-m)$$
$$\times\ C(\ell' s_2\,\tfrac{1}{2};-\tau+\mu'+m,-\mu'-m)\ W(\ell'\,j'\,\tfrac{1}{2}\,1;\tfrac{1}{2}\,s_2) \qquad \text{(C.2b)}$$

Substituting (C.2) in (C.1a)

$$\mathcal{S} = \sum_{s_1 s_2} (-)^{\mu+1+m}\left[(2\lambda+1)(2j'+1)(2s_1+1)(2s_2+1)\right]^{\frac{1}{2}} C(\lambda s_1 j;\mu-\mu'-m,\mu'+m)$$

$$\times\, C(j'1\, s_2;-\mu', -m)\; W(\lambda \ell' j\tfrac{1}{2};\ell s_1)\; W(\ell' j'\tfrac{1}{2} 1; \tfrac{1}{2} s_2)$$

$$\sum_{\gamma} (-)^{-\gamma} C(\ell'\tfrac{1}{2} s_1;\mu - \gamma + m, \gamma)\; C(\ell' s_2 \tfrac{1}{2};-\gamma + \mu' + m, -\mu' - m) \qquad \text{(C.1b)}$$

Using (A.1c)

$$\sum_{\gamma} (-)^{\gamma} C(\ell' \tfrac{1}{2} s_1;\ \mu - \gamma + m, \gamma)\; C(\ell' s_2 \tfrac{1}{2};\ -\gamma + \mu' + m,\ -\mu' - m)$$

$$= \sum_{\gamma} (-)^{\gamma} \quad (-)^{\ell' + \gamma - \mu' - m} \sqrt{\frac{2}{2s_2 + 1}}\; C(\ell' \tfrac{1}{2} s_2;\mu' - \gamma + m, \gamma)$$

$$C(\ell' \tfrac{1}{2} s_1,\ \mu' - \gamma + m, \gamma)$$

$$= (-)^{\ell' - \mu' - m} \left[\frac{2}{2s_2 + 1}\right]^{\frac{1}{2}} \delta_{s_1 s_2}$$

Substituting the above in (C.1b) and summing over s_2

$$\mathcal{S} = \sum_{s} (-)^{\mu + 1 + \ell' - \mu'} \left[(2\lambda + 1)(2j' + 1)(2s + 1)2\right]^{\frac{1}{2}} C(\lambda s j;\mu - \mu' - m, \mu' + m)$$

$$\times\, C(j'\, 1\, s;\ -\mu', -m)\; W(\ell' j\tfrac{1}{2};\ \ell s)\; W(\ell' j'\tfrac{1}{2} 1;\ \tfrac{1}{2} s) \qquad \text{(C.1c)}$$

2. Time Reversal Invariance in Strong Interactions

The present evidence is that time reversal invariance holds in strong (nuclear) interactions to a high degree of accuracy.[1]

The time reversal operator, for a Dirac wave function, is a non-linear operator and is $i\,\sigma_2 K$, where K means complex conjugation.

$$\psi'(t' = -t) = T\psi(t) = i\,\sigma_2\, \psi^{*}(t)$$

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[1] T. D. Lee and C. N. Yang, Elementary Particles and Weak Interactions, Brookhaven National Laboratory, B.N.L. 443 (T-91) (1957) p. 16.

Therefore, if we consider a nuclear matrix element of an operator Ω between initial (Ψ_i) and final (Ψ_f) nuclear states, then under time reversal it goes to

$$\int \Psi_f^{*} \Omega \Psi_i \longrightarrow \int (T\Psi_f)^{*} \Omega (T\Psi_i)$$
$$= \int (i\sigma_2 \Psi_f^{\times})^{*} \Omega \, i \sigma_2 \Psi_i^{\times}$$
$$= \int \tilde{\Psi}_f \sigma_2 \Omega \sigma_2 \Psi_i^{\times}$$
$$= \left(\int \Psi_f^{*} \sigma_2 \Omega^{\times} \sigma_2 \Psi_i \right)^{\times}$$
$$= \left(\int \Psi_f^{*} \Omega_T \Psi_i \right)^{\times}$$

where $\Omega_T = \sigma_2 \Omega^{\times} \sigma_2$. In the above * means hermitian conjugate and $\times$ implies complex conjugation. In our problem, we have:

$$i \int \gamma_5 \left(\int \vec{\sigma}.\vec{r}\right)^{\times} \longrightarrow i\left(\int (\gamma_5)_T\right)^{\times} \left(\int \vec{\sigma}.\vec{r})_T\right)$$

$$= -i \left(\int \gamma_5\right)^{\times} \int \vec{\sigma}.\vec{r} = \left(i \int \gamma_5\right)^{\times} \left(\int \vec{\sigma}.\vec{r}\right) \qquad (C.2)$$

We have used

$$(\gamma_5)_T = \sigma_2 \gamma_5 \sigma_2 = \gamma_5$$

and

$$(\sigma.r)_T = \sigma_2 \vec{\sigma}.\vec{r} \sigma_2 = -\vec{\sigma}.\vec{r}$$

From (C.2) we get $\left(i\int \gamma_5\right)\left(\int \vec{\sigma}.\vec{r}\right)^{\times}$ = its complex conjugate and, therefore $i\int \gamma_5 \left(\int \vec{\sigma}.\vec{r}\right)^{\times}$ is real.

3. <u>Time Reversal Invariance in Weak Interactions</u>

We prove, that under the assumption of time-reversal invariance

being valid in nuclear β decay, all the coupling constants are real.

The β interaction hamiltonian density is

$$H_\beta = \sum_x (\psi_P^* \Omega_x \psi_N) \cdot (\psi_e^* \Omega_x [C_x + C_x' \gamma_5] \psi_\nu)$$
$$+ (\psi_N^* \Omega_x \psi_P) \cdot (\psi_\nu^* [C_x^* + C_x'^* \gamma_5] \Omega_x \psi_e) \qquad (C.3)$$

In (C.3) the first term represents β^- emission and the second term (which in hermitian conjugate of the first) represents β^+ emission. In the above, we have taken Ω_x to be hermitian.

Under time reversal, the Dirac wave function transforms as follows:

$$\psi'(t' = -t) = T\psi(t) = i \sigma_2 \psi^\times(t)$$

Let us now consider a matrix element

$$(\psi_1, \Omega\psi_2) \equiv (\psi_1^* \Omega \psi_2)$$

$$\rightarrow (T\psi_1, \Omega T \psi_2)$$

$$= (\psi_1^\times, \sigma_2 \Omega \sigma_2 \psi_2^\times)$$

$$= (\psi_1, \sigma_2 \Omega \sigma_2 \psi_2)^\times \qquad (C.4a)$$

Now $(\psi_1, \sigma_2 \Omega^\times \sigma_2 \psi_2)^\times$ is a 1×1 matrix: therefore, complex conjugation is equivalent to hermitian conjugation.

Introducing $\Omega_T = \sigma_2 \Omega^\times \sigma_2$

$$(\psi_1, \Omega \psi_2) \rightarrow (\psi_1, \Omega_T \psi_2)^*$$
$$= (\psi_2, \Omega_T^* \psi_1)$$

In (C.4b) we note that ψ_1 and ψ_2 have reversed their positions.

$$\Omega_T^* = (\sigma_2 \Omega^{\times} \sigma_2)^*$$

$$\Omega_T^* = \sigma_2 \Omega^{\times} \sigma_2 = \Omega_T$$

In general σ_2 either commutes with $\Omega^{\times}$; in which case $\Omega_T^* = \Omega_T = \Omega^{\times}$; or σ_2 anticommutes with $\Omega^{\times}$, in which case $\Omega_T^* = \Omega_T = -\Omega$.

Also, Ω is either real or pure imaginary. In the former case $\Omega^{\times} = \Omega$ and in the latter, $\Omega^{\times} = -\Omega$. But in any case, at the most

$$\Omega_T^* = \Omega_T = \pm \Omega \tag{C.5a}$$

Similarly,

$$(\Omega \gamma_5)_T = \pm \gamma_5 \Omega \tag{C.5b}$$

Substituting (C.4b), (C.5a), and (C.5b) in (C.3), the $\pm$ signs cancel out, as they occur twice (once in the lepton covariant and also in the nuclear space).

$$H_\beta \rightarrow \sum_x (\psi_N^* \Omega_x \psi_P) \cdot (\psi_\nu^* [C_x^* + C_x'^* \gamma_5] \Omega_x \psi_e) + (\psi_P \Omega_x \psi_N) \cdot (\psi_e^* \Omega_x [C_x + C_x' \gamma_5] \psi_\nu) \tag{C.6}$$

Comparing (C.3) and (C.6), they are exactly the same, provided:

$$C_x^* = C_x$$

and $C_x'^* = C_x'$

Hence the coupling constants C_x and C_x' (for all interactions) are real, provided H_β is invariant under time reversal operation.

APPENDIX D

We introduced A_o, B_o, C_o and D_o in the expression of longitudinal polarization. We give, below, their analytical expressions up to order R (the nuclear radius).

$$A_{k-1} \equiv (p^2F_o)^{-1} R^{2-2k} f_k g_{-k} \operatorname{Sin}(\delta_k - \delta_{-k})$$

$$\rightarrow -\frac{p}{W} \frac{k(k+\gamma)}{2} \left[2^{2k} (p)^{2k-2} \left(\frac{(k-1)!}{2k!}\right)^2 \frac{F_{k-1}}{F_o} \right]$$

$$B_{k-1} \equiv (p^2Fo)^{-1} R^{-2k} f_{-k} g_k \operatorname{Sin}(\delta_k - \delta_{-k})$$

$$\rightarrow \frac{p}{W}\left[\frac{k-\gamma}{2R^2} + \frac{\alpha Z}{R} W \frac{(2\gamma+1-2k)}{2\gamma+1} + \frac{\gamma(\gamma+1-k)p^2}{(1+2\gamma)^2} \right.$$

$$\left. - (\alpha Z)^2 W^2 \frac{(4\gamma+3)(\gamma+1-k)}{(1+2\gamma)^2(\gamma+1)} \right] \cdot 2^{2k} p^{2k-2} \left(\frac{(k-1)!}{2k!}\right)^2 \frac{F_{k-1}}{F_o}$$

$$C_{k-1} \equiv (p^2Fo)^{-1} R^{1-2k} (f_k f_{-k} + g_k g_{-k}) \operatorname{Sin}(\delta_k - \delta_{-k})$$

$$\rightarrow -\frac{p}{W} \cdot \frac{2\gamma k}{2\gamma+1} \left[2^{2k} p^{2k-2} \left(\frac{(k-1)!}{2k!}\right)^2 \frac{F_{k-1}}{Fo} \right]$$

$$D_{k-1} \equiv (p^2Fo)^{-1} R^{1-2k} (f_k f_{-k} - g_k g_{-k}) \operatorname{Sin}(\delta_k - \delta_{-k})$$

$$\rightarrow \frac{p}{W} k\left\{\frac{\alpha Z}{R} + \frac{2W}{2\gamma+1}(\gamma - 2\alpha^2 Z^2)\right\} \times \left[2^{2k} p^{2k-2} \left(\frac{k-1!}{2k!}\right)^2 \frac{F_{k-1}}{Fo} \right]$$

In the above,

$$\gamma_{|k|} = \left[k^2 - \alpha^2 Z^2\right]^{\frac{1}{2}}$$

$$\alpha \approx \frac{1}{137}$$

Z corresponds to the daughter nucleus.

$$R = \frac{0.43}{137} A^{1/3}$$

$$W = [p^2 + 1]^{1/2}$$

$$\delta_\kappa - \delta_{-k} \equiv \eta_k - \eta_{-k}; \text{ where } e^{2i\eta_\varkappa} = -\frac{\varkappa - i\alpha Z/p}{\gamma + i\alpha ZW/p}$$

For

$$\alpha Z \ll 1 \quad ,$$

$$\gamma_1 \approx 1 - \tfrac{1}{2}(\alpha Z)^2$$

and $\quad F_1 \approx F_o$

then,

$$A_o \to -\frac{p}{W}$$

$$B_o \to \frac{p}{W}\left[\left(\frac{\alpha Z}{2R}\right)^2 + \frac{1}{3}\left(\frac{\alpha Z}{R}\right)W + \frac{1}{9}p^2\right]$$

$$C_o \to -\frac{p}{W}\left[2/3\right]$$

$$D_o \to \frac{p}{W}\left[\left(\frac{\alpha Z}{R}\right) + \frac{2}{3}W\right]$$

$$A_1 \to -\frac{p}{W}\left[\frac{1}{9}p^2\right]$$

L_{k-1}, M_{k-1} and N_{k-1} are given by Greuling.[1] P_{k-1}, Q_{k-1}, and R_{k-1} are given by Pursey[2] and these have been tabulated for Coulomb functions by Rose, Perry and Dismuke.[3] The nuclear finite size

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[1]E. Greuling, Phys. Rev. 61, 568 (1942).

[2]D. L. Pursey, Phil. Mag. 42, 1193 (1951).

[3]M. E. Rose, C. L. Perry and N. Dismuke, Oak Ridge National Laboratory Report No. 1459 (Unpublished).

corrections have also been given by Rose and Holmes.[4, 5]

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[4] M. E. Rose and D. K. Holmes, Phys. Rev. **83**, 190 (1951) and Oak Ridge National Laboratory Report No. 1022 (Unpublished).

[5] C. P. Bhalla and M. E. Rose (Oak Ridge National Laboratory Report to be issued); containing tables of functions, with finite nuclear size corrections, for polarization and β spectrum.