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The Complementing Condition in Elasticity

Lavanya Ramanan
University of Tennessee - Knoxville, lramanan@utk.edu

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I am submitting herewith a thesis written by Lavanya Ramanan entitled "The Complementing Condition in Elasticity." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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The Complementing Condition in Elasticity

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I dedicate this thesis to my parents, brother and Paramesh

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Abstract

We consider a boundary value problem of nonlinear elasticity on a domain Ω [omega] in $\mathbb{R}^3$ [3-dimensional space] and compute the Complementing Condition for the linearized equations at a point x_0 [x zero] on $\partial\Omega$ [boundary of omega]. We assume a stored energy function $W(F)$ depending on the first, second and third invariants of the deformation F and that the strong-ellipticity condition holds in Ω. A surface traction boundary condition is imposed at x_0.

The Complementing Condition is calculated from a system of 3 second-order ordinary differential equations ($0 \leq t < \infty$[infinity]) with boundary conditions at $t = 0$ and constant coefficients; the condition is satisfied if and only if the only bounded exponential solution is trivial. We compute the Complementing Condition in terms of parameters in W.

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Chapter 1

Introduction

Consider a body in a region $\Omega \subset \mathbb{R}^3$, its reference configuration. Let $f : \Omega \to \mathbb{R}^3$ be a deformation of the body. The 3×3 matrix $F(x) = \nabla f(x)$ with components $\nabla f(x)_{ij} = \frac{\partial f_i(x)}{\partial x_j}$ is called the deformation gradient and belongs to Lin^+ as det $\nabla f > 0$ on Ω. Here

$$\text{Lin} = \{3 \times 3 \text{ real matrices}\}$$

$$\text{Lin}^+ = \{F \in \text{ Lin } : \text{ det } F > 0\}$$

$$\text{and } K \cdot H = \sum_{i,j=1}^{3} K_{ij} H_{ij} \text{ for } K, H \in \text{ Lin}$$

In this thesis, we assume a stored energy function for material of the body :

$$W(F) = \Phi(\frac{1}{2}\ F \cdot F) + \Omega(\frac{1}{4}(FF^T) \cdot (FF^T)) + \Psi(\text{det } F), \quad F \in \text{ Lin}^+$$

The Piola-Kirchhoff stress is given by

$$S(F) = \frac{d}{dF} W(F),$$

$$\text{i.e. } S(F) \cdot H = \frac{d}{dt} W(F + tH)|_{t=0}$$

where $S : \text{ Lin}^+ \to \text{ Lin }, F \in \text{ Lin}^+, H \in \text{ Lin }$.

The elasticity tensor is given by

$$C(F) = \frac{dS(F)}{dF};$$

It is the linearization of S about F :

$$C[H] = \frac{d}{dt} S(F + tH)|_{t=0} \quad \in \text{ Lin }, \text{ for } H \in \text{ Lin}$$

We assume C is strongly elliptic, that is, $(a \otimes b) \cdot C[a \otimes b] > 0, \ \ \forall a, b \in \mathbb{R}^3 \setminus \{0\}$ where $(a \otimes b) \in \text{ Lin}, (a \otimes b)_{ij} = a_i b_j$.

We study the boundary value problem

$$\begin{aligned} \text{div } S(\nabla f) &= -b \text{ on } \Omega, \\ S(\nabla f)n &= t \text{ on } \partial\Omega \end{aligned} \tag{1}$$

for a given body force, $b : \overline{\Omega} \to \mathbb{R}^3$ and applied surface traction, $t : \partial\Omega \to \mathbb{R}^3$ Here n is the outward unit normal to $\partial\Omega$.

Fix $\mu, \omega > 0$ and let $F_0 =$ diag $[\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}, \omega]$, $f_0(x) = F_0 x$, $x \in \Omega, \nabla f_0 = F_0$.

The linearized equations about f_0 are obtained by letting $f = f_0 + tu$ and applying $\frac{d}{dt}...|_{t=0}$ to the left side of equation (1).

This yields the linearized boundary value problem :

$$\begin{aligned} \text{div } C[\nabla u] &= 0 \text{ on } \Omega \\ C[\nabla u]n &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

for $u : \Omega \to \mathbb{R}^3; C$ is the linearization of S about F_0.

We calculate the Complementing Condition on $\partial\Omega$ for equation (2) for different cases when Ω is a cylinder $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 < R^2,\ \ 0 < x_3 < L\}$ and $n = e_1$ and $n = e_3$.

Then we assume a stored energy function

$$W(F) = \frac{1}{2}\ F.F + \Psi(detF)$$

and calculate the Complementing Condition when $n = e_1$.

The Complementing Condition was first formulated by Lopatinskii(1953), Shapiro(1953) and Agmon, Douglis, Nirenberg(1964). It was used to establish estimates for elliptic boundary value problems such as equation (2). It was shown that ellipticity and the Complementing Condition are necessary and sufficient for the estimates to hold (Friedman(1969), "Partial Differential Equations", Thompson(1969), "Some existence theorems for the traction boundary value problem of linearized elastostatics").
Agmon's condition was first defined in Agmon(1958), "The coerciveness problem for integro-differential forms", and Agmon(1962), "On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems", De Figueiredo (1963), "The coerciveness problem for forms over vector valued functions". For system (2) this is :

$$\int_{\Omega} \nabla u \cdot C[\nabla u]\, dx \geq c_0 \int_{\Omega} |\nabla u|^2\, dx + c_1 \int_{\Omega} |u|^2\, dx$$

for all $u \in W^{1,2}(\Omega)$ with $c_0 > 0, c_1 \in \mathbb{R}$.
In Simpson and Spector(1987), "On the positivity of the second variation in finite elasticity" it is shown this holds iff C is strongly elliptic and satisfies the Complementing and Agmon's conditions on $\partial\Omega$ (assuming $\partial\Omega$ is C^1-smooth). (Simpson and Spector(2008), Friedman(1969))

In the paper, "On Bifurcation in Finite Elasticity: Buckling of a Rectangular Rod", Henry C. Simpson and Scott J. Spector consider a two dimensional material with stored-energy function $W(F) = W(\frac{1}{2}F \cdot F,\ \det\ F)$ where F is a two-by-two matrix and describe the strong ellipticity and Complementing Conditions.

In the paper of Pablo V. Negron-Marrero and Error Montes-Pizarro, they study the deformation of cylinders for a stored energy function of a homogeneous and isotropic material of the body, with $W(F) = av_1 + b(v_1^2 - v_2) + \Psi(v_3),\ \ v_1 = \frac{1}{2}\ F.F, v_2 = \frac{1}{4}(FF^T) \cdot (FF^T), v_3 = \det\ F,\ \ a > 0, b \geq 0.$

In this thesis, we study the deformation in cylinders by calculating the Complementing Condition on $\partial\Omega$ with $F = \operatorname{diag}\ [\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}, \omega]$ for $\mu, \omega > 0$.

We start with Chapter 2 and 3 which describe the definitions of Complementing Condition and Agmon's Condition. In Chapter 4, we describe a boundary value problem. In Chapter 5 and 6, we calculate the Complementing Condition for our boundary value problem when $n = e_1$ and $n = e_3$ respectively. Lastly, in Chapter 7, we consider a slightly altered stored energy function and calculate the Complementing Condition for $n = e_1$.

Chapter 2

Preliminaries

Let

$$\text{Lin} = \{3 \times 3 \text{ real matrices}\},$$

$$\text{Lin}^+ = \{F \in \text{ Lin } : \det F > 0\},$$

with inner product, $H \cdot K = \text{ tr } (H^T K)$

$$= \text{ tr } (HK^T) = \textstyle\sum_{i,j=1}^3 H_{ij} K_{ij} \text{ for } H, K \in \text{ Lin}$$

and norm $|H|^2 = H \cdot H = \sum_{i,j=1}^3 H_{ij}^2$.

For column vectors $a, b \in \mathbb{R}^3$ we define $a \otimes b = ab^T \in$ Lin by

$(a \otimes b)_{ij} = a_i b_j$ (Note $a \otimes b$ has rank ≤ 1).

Also we define Cof $F = (\det F) F^{-T}$ for $F \in$ Lin (Cof F = matrix of cofactors of F).

Assume a material has a stored energy function $W :$ Lin$^+ \to \mathbb{R}$, and assume W is C^2−smooth on Lin$^+$.

The Piola-Kirchhoff stress, $S :$ Lin$^+ \to$ Lin is given by

$$S(F) = \frac{d}{dF} W(F), \quad F \in \text{ Lin}^+$$

In particular, for $F \in$ Lin$^+$, $H \in$ Lin,

$$S(F) \cdot H = \frac{d}{dt} W(F + tH)|_{t=0}$$

and $S(F)$ has components

$$S(F)_{ij} = \frac{\partial W(F)}{\partial F_{ij}}, i, j = 1, 2, 3$$

Fix $F_0 \in$ Lin$^+$.

We can expand each component of S in a Taylor series about $F = F_0$:

$S(F)_{ij} = S(F_0)_{ij} + \sum_{k,l=1}^{3} \frac{\partial S_{ij}}{\partial F_{kl}}(F_0)(F - F_0)_{kl} + 0(\|F - F_0\|^2)$,

for F near F_0 (i.e. $\|F - F_0\|$ small).

The first order terms give the linearization of S about F_0:

$$C[H] = \frac{d}{dt} S(F_0 + tH)|_{t=0}$$

$$C[H]_{ij} = \sum_{k,l=1}^{3} \frac{\partial S_{ij}}{\partial F_{kl}}(F_0) H_{kl} = \sum_{k,l=1}^{3} C_{ijkl} H_{kl}$$

$$\text{where } C_{ijkl} = \frac{\partial S_{ij}}{\partial F_{kl}}(F_0), \quad i,j,k,l = 1,2,3, \quad H \in Lin$$

Here $H \longmapsto C[H]$: Lin $\rightarrow$ Lin is linear in H.

Thus C is a 4-tensor (with components C_{ijkl}) that maps Lin linearly into itself. Also C is symmetric:

$C_{ijkl} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} = C_{klij}$

$K \cdot C[H] = H \cdot C[K] = \sum_{ijkl=1}^{3} C_{ijkl} K_{ij} H_{kl} \quad \forall H, K \in$ Lin.

Let $\Omega \subset \mathbb{R}^3$, be an open, bounded set and suppose $\partial\Omega$ is C^2- smooth. Consider the boundary value problem

$$\text{div } S(\nabla f) = -b \text{ on } \Omega$$
$$S(\nabla f)n = t \text{ on } \partial\Omega$$

for a given body force, $b : \overline{\Omega} \rightarrow \mathbb{R}^3$ and given applied surface traction, $t : \partial\Omega \rightarrow \mathbb{R}^3$;

div $S(\nabla f) = -b$ is the balance of momentum equation, $S(\nabla f)n = t$ is the surface traction boundary condition. Here n is the outward unit normal to $\partial\Omega$. We wish to solve for $f : \overline{\Omega} \rightarrow \mathbb{R}^3$ which is the deformation of the body.

Its derivative, the deformation gradient is the matrix $\nabla f(x)$ with components $\nabla f(x)_{ij} = \frac{\partial f_i(x)}{\partial x_j}, 1 \le i,j \le 3$

$\nabla f(x) \in$ Lin $^+ \quad \forall x \in \overline{\Omega}$.

Fix a deformation $f_0 : \overline{\Omega} \to \mathbb{R}^3$ and assume f_0 is C^2−smooth.

The linearized equations about f_0 are obtained by applying $\frac{d}{dt}...|_{t=0}$ to the left side of the balance of momentum equation with $f = f_0 + tu,$ where $u : \overline{\Omega} \to \mathbb{R}^3$ is an infinitesimal displacement from f_0.

$\frac{d}{dt}(\text{div } S(\nabla(f_0 + tu))|_{t=0} = \text{div } \frac{d}{dt} S(\nabla f_0 + t\nabla u)|_{t=0} = \text{ div } C[\nabla u],$

$\frac{d}{dt} S(\nabla(f_0 + tu))n|_{t=0} = C[\nabla u]n.$

This yields the linearized boundary value problem about f_0 :

$$\text{div } C[\nabla u] = 0 \text{ on } \Omega$$
$$C[\nabla u]n = 0 \text{ on } \Omega$$

$\nabla u(x)_{ij} = \frac{\partial u_i(x)}{\partial x_j}, \quad \nabla u(x) \in \text{ Lin} \qquad \forall x \in \overline{\Omega}$

We denote

$$\mathscr{L}[u] = \text{div } C[\nabla u]$$
$$\mathscr{B}[u] = C[\nabla u]n$$

Definition 2.1 :

The 4 - tensor C is strongly-elliptic (SE) iff

$$(a \otimes b) \cdot C[a \otimes b] > 0 \quad \forall a, b \in \mathbb{R}^3 \setminus \{0\}.$$

Definition 2.2 :

For $b \in \mathbb{R}^3$ define the 3×3 matrix $M(b)$ by

$$M(b)a = C[a \otimes b]b \quad \forall a \in \mathbb{R}^3$$

($M(b)$ is also defined by this formula in case $a, b \in \mathbb{C}^3$). $M(b)$ is the acoustic tensor; it is a function of $b \in \mathbb{R}^3$ (or $\mathbb{C}^3$).

Theorem 2.3.

For $a, b \in \mathbb{C}^3$

a) $M(b)$ is symmetric

b) $(a \otimes b) \cdot C[a \otimes b] = a^T M(b) a$

c) $\mathscr{L}[e^{b \cdot x} a] = M(b) a e^{b \cdot x}, \quad \mathscr{B}[e^{b \cdot x} a] = C[a \otimes b] n e^{b \cdot x}$.

Proof. c) Let $u(x) = e^{b \cdot x} a, \quad a, b \in \mathbb{C}^3$.

Then $\nabla u_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} e^{b \cdot x} a_i = b_j a_i e^{b \cdot x}$

$\implies \nabla u = a b^T e^{b \cdot x} = (a \otimes b) e^{b \cdot x}$

$\mathscr{L}[u]_i = (\text{div } C[\nabla u])_i = \sum_j \frac{\partial}{\partial x_j} C[\nabla u]_{ij}$

$= \sum_j C[a \otimes b]_{ij} \ \frac{\partial}{\partial x_j} e^{b \cdot x}$

$= \sum_j C[a \otimes b]_{ij} \ b_j e^{b \cdot x}$

$= (C[a \otimes b] b)_i \ e^{b \cdot x}$

$\implies \mathscr{L}[u] = C[a \otimes b] \ b e^{b \cdot x} = M(b) \ a e^{b \cdot x}$.

$\mathscr{B}[u]_i = \sum_j C[\nabla u]_{ij} n_j e^{b \cdot x} = (C[a \otimes b] n)_i e^{b \cdot x}$.

b) Let $c \in \mathbb{C}^3$. Then $c^T M(b) a = c^T C[a \otimes b] b = (c \otimes b) \cdot C[a \otimes b]$.

a) From b) and symmetry of C,

$c^T M(b) a = (a \otimes b) \cdot C[c \otimes b] = a^T M(b) c$.

Thus $M(b) = M(b)^T$. □

Chapter 3

Complementing and Agmon's Conditions

Suppose $n \in \mathbb{R}^3$ is a unit vector, $|n| = 1$, and C is a constant 4-tensor which is strongly elliptic. Denote the halfspace

$$H = \{x \in \mathbb{R}^3 : x \cdot n > 0\}$$

with boundary hyperplane

$$\partial H = \{x \in \mathbb{R}^3 : x \cdot n = 0\}$$

Definition 3.1 :

A function $u : \overline{H} \to \mathbb{C}^3$ is a bounded exponential iff u is of the form

$$u(x) = z(n \cdot x) e^{i\xi \cdot x}, x \in \overline{H} \tag{3}$$

for some bounded function $z : [0, \infty) \to \mathbb{C}^3$ and some $\xi \in \mathbb{R}^3 \setminus \{0\}$ such that $\xi \perp n$ ($\xi^T n = \xi \cdot n = 0$, ξ is parallel to ∂H); z is C^2–smooth on $[0, \infty)$.

We denote the normal variable $t = n \cdot x \in [0, \infty)$ for $x \in \overline{H}$. Then $t = 0$ for $x \in \partial H$ and $z = z(t)$ is a function of $t \in [0, \infty)$.

Consider the boundary value problem on $\overline{H}$:

$$\begin{aligned} \text{div } C[\nabla u] &= \alpha u \quad \text{on } H \ , \\ C[\nabla u] n &= 0 \quad \text{on } \partial H \end{aligned} \tag{4}$$

where $u : \overline{H} \to \mathbb{C}^3$ is a bounded exponential and $\alpha \in \mathbb{R}, \alpha \geq 0$.

Definition 3.2 : Assume C is strongly elliptic.

The pair (C, n) satisfies the **Complementing Condition** iff for $\alpha = 0$ and each $\xi \in \mathbb{R}^3 \setminus \{0\}, \xi \perp n$, the only bounded exponential solution of equation (4) is $u \equiv 0$ on $\overline{H}$
(i.e. $z \equiv 0$ on $[0, \infty)$).

(C,n) satisfies **Agmon's Condition** iff for every $\alpha > 0$ and each $\xi \in \mathbb{R}^3 \setminus \{0\}, \xi \perp n$, the only bounded exponential solution of equation (4) is $u \equiv 0$ on $\overline{H}$.

In order to determine if the Complementing or Agmon's Condition is satisfied, substitute (3) into equation (4) which yields an ODE system for $z(t), t \in [0,\infty)$, with boundary condition at $t = 0$. This system consists of 3 second-order ODE's (on $[0,\infty)$) and 3 first-order boundary conditions (at $t = 0$) each with constant (complex) coefficients depending on n, ξ.

By Theorem 3.3 below, the ODE has 6 linearly independent solutions, and by strong ellipticity of C, three are exponentially decaying to 0 as $t \to \infty$ and three are exponentially unbounded as $t \to \infty$. We denote the exponentially decreasing solutions by $z^{(1)}, z^{(2)}, z^{(3)}$.

Thus for bounded z, we have

$z(t) = c_1 z^{(1)}(t) + c_2 z^{(2)}(t) + c_3 z^{(3)}(t)$

$c_i \in \mathbb{C}, i = 1,2,3.$

Applying the boundary conditions to z yields 3 algebraic linear equations for c_1, c_2, c_3. A nontrivial solution z of equation (3) exists iff $c = (c_1, c_2, c_3) \neq 0$.

Thus the Complementing and Agmon's Conditions are algebraic conditions depending on C, n only.

To calculate these conditions, we find, from equations (3) and (4) :

$$\begin{aligned} \text{div } C[\nabla u] &= \mathscr{L}_\xi[z] e^{i\xi\cdot x}, \quad t \in [0,\infty) \\ C[\nabla u] n &= \mathscr{B}_\xi[z] e^{i\xi\cdot x}, \quad t = 0 \end{aligned} \tag{5}$$

where $\mathscr{L}_\xi$ is a second order ODE operator, and $\mathscr{B}_\xi$ is a first order boundary operator on z .

Now equation (4) becomes equivalent to :

$$\begin{aligned} \mathscr{L}_\xi[z] &= \alpha z, \quad t \in [0,\infty) \\ \mathscr{B}_\xi[z] &= 0 \quad \text{at } t = 0 \end{aligned} \tag{6}$$

Thus the Complementing Condition is satisfied iff for $\alpha = 0$ and each $\xi \in \mathbb{R}^3 \setminus \{0\}, \xi \perp n$, the only exponentially decreasing solution z of equations (6) is

$z \equiv 0$ on $[0, \infty)$. Agmon's Condition is satisfied iff for every $\alpha > 0$ and each $\xi \in \mathbb{R}^3 \smallsetminus \{0\}, \xi \perp n$, the only exponentially decreasing solution of equations (6) is $z \equiv 0$ on $[0, \infty)$.

To determine $\mathscr{L}_\xi, \mathscr{B}_\xi$ explicitly first note by equation (3)
$\nabla u(x) = [z'(t) \otimes n + iz(t) \otimes \xi]e^{i\xi \cdot x}$

where $' = \frac{d}{dt}$ in all that follows.

Proof. With $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}$,

$(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{\partial z_i}{\partial x_j} e^{i\xi \cdot x} + z_i e^{i\xi \cdot x} \frac{\partial (i\xi \cdot x)}{\partial x_j}$

$= [z_i' \frac{\partial}{\partial x_j}(n \cdot x) + z_i \frac{\partial}{\partial x_j}(i\xi \cdot x)]e^{i\xi \cdot x}$

$= [z_i' n_j + iz_i \xi_j]e^{i\xi \cdot x}$

$= [(z' \otimes n)_{ij} + i(z \otimes \xi)_{ij}]e^{i\xi \cdot x}$

□

Then,

$$\begin{aligned} \mathscr{L}_\xi[z] = C[z'' \otimes n + iz' \otimes \xi]n + iC[z' \otimes n + iz \otimes \xi]\xi, \\ \mathscr{B}_\xi[z] = C[z' \otimes n + iz \otimes \xi]n \quad \text{at } t = 0. \end{aligned} \tag{7}$$

Proof. Put $H(t) = z'(t) \otimes n + iz(t) \otimes \xi$,

$(\text{div } C[\nabla u])_i = \sum_j \frac{\partial}{\partial x_j} \{C[H(t)]_{ij} e^{i\xi \cdot x}\}$

$= \sum_j \{\frac{d}{dt} C[H(t)]_{ij} \frac{\partial}{\partial x_j}(n \cdot x)e^{i\xi \cdot x} + C[H(t)]_{ij} \frac{\partial}{\partial x_j}(i\xi \cdot x)e^{i\xi \cdot x}\}$

$= \sum_j \{C[H'(t)]_{ij} n_j + iC[H(t)]_{ij} \xi_j\} \ e^{i\xi \cdot x}$

$= [C[H'(t)]n + iC[H(t)\xi]_i \ e^{i\xi \cdot x}$

$\implies \ \text{div } C[\nabla u] = \{C[H'(t)]n + iC[H(t)]\xi\} \ e^{i\xi \cdot x}.$

$(C[\nabla u]n)_i = (C[z' \otimes n + iz \otimes \xi]n)_i \ e^{i\xi \cdot x}$ from formula for ∇u above.

$\implies (C[\nabla u])n = \{C[H(t)]n\}e^{i\xi\cdot x}$.

Then see equation (5).

$\square$

To find nontrivial solutions z of equation (6), we try

$$z(t) = e^{rt}a \tag{8}$$

for some $r \in \mathbb{C}, a \in \mathbb{C}^3 \setminus \{0\}$.

Then $t = n \cdot x$ and equation (3) $\implies u(x) = e^{(rn+i\xi)\cdot x}a = e^{b.x}a$
where $b = rn + i\xi \in \mathbb{C}^3$

Then following Theorem 2.3 and its proof we have $\nabla u(x) = a \otimes b\ e^{b\cdot x}$,

div $C[\nabla u] = \mathscr{L}[e^{b\cdot x}a] = M(b)\ a\ e^{b\cdot x} = M(b)\ a\ e^{rt}e^{i\xi\cdot x}$, for $t \geq 0$ and

$C[\nabla u]n = \mathscr{B}(e^{b\cdot x}a) = C[a \otimes b]n\ e^{i\xi\cdot x}$ at $t = 0$.

Thus by equation (5),

$$\begin{aligned} \mathscr{L}_\xi[e^{rt}a] &= M(b)ae^{rt}, \quad t \in [0, \infty), \\ \mathscr{B}_\xi[e^{rt}a] &= C[a \otimes b]n, \quad t = 0, \end{aligned} \tag{9}$$

$b = rn + i\xi$.

Thus we have nontrivial solutions of equation $(6)_1$ of the form $z(t) = e^{rt}a$ iff

$$[M(rn + i\xi) - \alpha I]a = 0 \tag{10}$$

for some $a \in \mathbb{C}^3, a \neq 0$ iff $p(r) =$ det $[M(rn + i\xi) - \alpha I] = 0$;

p is the characteristic polynomial of $\mathscr{L}_\xi$.

Theorem 3.3.

Assume C is strongly elliptic and $\alpha \geq 0$. Then

a) p is a 6^{th} degree polynomial in r.

b) p has no roots r with Re $r = 0$.

c) p has 3 roots r with Re $r < 0$ and 3 roots with Re $r > 0$.

Proof. First we note that by strong ellipticity and Theorem 2.3(b), $M(b)$ is positive definite for each real $b \in \mathbb{R}^3 \setminus \{0\}$. Also $M(cb) = c^2 M(b)$ for $c \in \mathbb{C}$.

a) The coefficient of r^6 in $p(r)$ is det $M(n)$ which is nonzero since $M(n)$ is positive definite.

b) If $r \in \mathbb{C}$ is a root of $p(r)$ then $[M(rn + i\xi) - \alpha I]a = 0$ holds for some $a \in \mathbb{C}^3 \setminus \{0\}$. If Re $r = 0$, say $r = i\rho$ for $\rho \in \mathbb{R}$, then $M(b) = -M(e)$ for $e = \rho n + \xi \in \mathbb{R}^3$. Thus $[M(e) + \alpha I]a = 0$ and $\overline{a}^T[M(e) + \alpha I]a = 0$. Also $e \neq 0$ since $\xi \neq 0, \xi \perp n$. Thus $M(e)$ is positive definite so $\overline{a}^T M(e)a > 0$. But $\alpha \geq 0$ contradicts the above equation.

c) Since $\overline{M(b)a} = \overline{C[a \otimes b]b} = C[\overline{a} \otimes \bar{b}]\bar{b} = M(\bar{b})\overline{a}$ then $\overline{M(b)} = M(\bar{b})$. Also $M(-b) = M(b)$ so $\overline{p(r)} = p(-\overline{r})$. Thus if r is a root of p, so is $-\overline{r}$ and Re $r < 0 \implies$ Re $(-\overline{r}) > 0$.

$\square$

The Lame' 4-tensor of linear elasticity is given by

$$C[H] = \mu(H + H^T) + \lambda(H \cdot I)I, H \in \text{ Lin } .$$

for some $\mu, \lambda \in \mathbb{R}$, see Gurtin(1981). Then we have (see also Mikhlin(1993), Thompson(1969), Simpson and Spector(1987))

Theorem 3.4.

a) C is strongly elliptic iff $\mu > 0$ and $2\mu + \lambda > 0$.

b) For any unit vector $n \in \mathbb{R}^3, (C, n)$ satisfies the Complementing Condition iff $\mu + \lambda \neq 0$; (C, n) satisfies Agmon's condition iff $\mu + \lambda \geq 0$.

Chapter 4

The Boundary Value Problem

In this thesis we assume a stored energy function of the form

$$W(F) = \Phi(\frac{1}{2}\ F \cdot F) + \Omega(\frac{1}{4}(FF^T)\cdot(FF^T)) + \Psi(detF),$$

i.e. $W(F) = \sigma(v_1, v_2, v_3) = \Phi(v_1) + \Omega(v_2) + \Psi(v_3)$

where $v_1 = \frac{1}{2}F \cdot F, v_2 = \frac{1}{4}(FF^T)\cdot(FF^T), v_3 = detF, \quad F \in$ Lin$^+$ for some functions $\Phi, \Omega, \Psi : (0, \infty) \to \mathbb{R}$ (Gurtin(1981) Chapter IX(25)), v_i are invariants of FF^T.

We assume Φ, Ω, Ψ are C^2smooth on $(0, \infty)$.

Denote $\sigma_1 = \frac{\partial \sigma}{\partial v_1} = \Phi'(v_1), \sigma_2 = \frac{\partial \sigma}{\partial v_2} = \Omega'(v_2), \sigma_3 = \frac{\partial \sigma}{\partial v_3} = \Psi'(v_3), \sigma_{ii} = \frac{\partial^2 \sigma}{\partial v_i^2}, \quad i = 1, 2, 3.$

Then the Piola-Kirchhoff stress is given by
$S(F) = \sigma_1 F + \sigma_2 FF^T F + \sigma_3 CofF$. (See Simpson(2008))

We linearize the equations as in Chapter 2 about

$$F_0 = \text{ diag } [\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}, \omega]$$

for $\mu, \omega > 0$. (See Negron-Marrero, P.V and E.Montes-Pizarro(2011)).

Then the elasticity tensor,

$$\begin{aligned} C[H] = \frac{d}{dt} S(F_0 + tH)|_{t=0} &= \sigma_1 H + \sigma_2(HF_0^T F_0 + F_0 H^T F_0 + F_0 F_0^T H) \\ &+\sigma_3(\text{det } F_0)[(F_0^{-T} \cdot H)F_0^{-T} - F_0^{-T} H^T F_0^{-T}] \\ &+\sigma_{11}(G^1 \cdot H)G^1 + \sigma_{22}(G^2 \cdot H)G^2 + \sigma_{33}(G^3 \cdot H)G^3, \\ G^1 = F_0, G^2 &= F_0 F_0^T F_0, G^3 = \text{ Cof } F_0 = (\text{det } F_0)F_0^{-T}. \end{aligned} \tag{11}$$

We define the constants

$Y = \sigma_{11}\mu + \sigma_{22}\mu^3 + \sigma_{33}\mu\omega^2$

$V = \sigma_{11}\omega^2 + \sigma_{22}\omega^6 + \sigma_{33}\mu^2$

$Z = \sigma_{11}\mu^{\frac{1}{2}}\omega + \sigma_{22}\mu^{\frac{3}{2}}\omega^3 + \sigma_{33}\mu^{\frac{3}{2}}\omega$

$X = \mu^{\frac{1}{2}}\sigma_3 + Z$

$\beta_1 = \sigma_1 + (\mu + \omega^2)\sigma_2$

$\beta_3 = \sigma_1 + 2\mu\sigma_2$

$\tau_1 = \sigma_1 + 3\mu\sigma_2 + Y$

$\tau_3 = \sigma_1 + 3\omega^2\sigma_2 + V$

$\nu_1 = \mu^{\frac{1}{2}}\omega\sigma_2 - \mu^{\frac{1}{2}}\sigma_3$

$\nu_3 = \mu\sigma_2 - \omega\sigma_3$

$\gamma = \omega\sigma_3 + Y$

$N = X + \nu_1$

$a = \tau_1 - \gamma.$

Note $\nu_3 + \gamma = \tau_1 - \beta_3$.

Theorem 4.1. The elasticity tensor in equation (11) is given by

$$C[H] = H_{11}\begin{bmatrix} \tau_1 & & \\ & \gamma & \\ & & X \end{bmatrix} + H_{22}\begin{bmatrix} \gamma & & \\ & \tau_1 & \\ & & X \end{bmatrix} + H_{33}\begin{bmatrix} X & & \\ & X & \\ & & \tau_3 \end{bmatrix} +$$
$$H_{12}\begin{bmatrix} 0 & \beta_3 & 0 \\ \nu_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + H_{21}\begin{bmatrix} 0 & \nu_3 & 0 \\ \beta_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + H_{13}\begin{bmatrix} 0 & 0 & \beta_1 \\ 0 & 0 & 0 \\ \nu_1 & 0 & 0 \end{bmatrix} + H_{31}\begin{bmatrix} 0 & 0 & \nu_1 \\ 0 & 0 & 0 \\ \beta_1 & 0 & 0 \end{bmatrix} +$$
$$H_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_1 \\ 0 & \nu_1 & 0 \end{bmatrix} + H_{32}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu_1 \\ 0 & \beta_1 & 0 \end{bmatrix}, \qquad H \in \text{Lin}.$$

Proof. Let $H \in \text{Lin}$, $H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$.

Recalling $F_0 = \text{ diag } [\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}, \omega]$ we have

$G^1 = F_0 = \text{ diag } [\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}, \omega]$

$G^2 = F_0 F_0^T F_0 = \text{ diag } [\mu^{\frac{3}{2}}, \mu^{\frac{3}{2}}, \omega^3]$

$G^3 = \text{ Cof } F_0 = \text{ diag } [\mu^{\frac{1}{2}}\omega, \mu^{\frac{1}{2}}\omega, \mu]$

Then $(\det F_0)[(F_0^{-T}\cdot H)F_0^{-T} - F_0^{-T}H^TF_0^{-T}] = (\det F_0)^{-1}[(G^3\cdot H)G^3 - G^3H^TG^3]$

$$= [\mu^{-\frac{1}{2}}(H_{11}+H_{22}) + \omega^{-1}H_{33}]\begin{bmatrix} \mu^{\frac{1}{2}}\omega & 0 & 0 \\ 0 & \mu^{\frac{1}{2}}\omega & 0 \\ 0 & 0 & \mu \end{bmatrix} - \frac{1}{\mu\omega}G^3\begin{bmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & H_{22} & H_{32} \\ H_{13} & H_{23} & H_{33} \end{bmatrix}G^3$$

$$= \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \mu^{\frac{1}{2}} \end{bmatrix}(H_{11}+H_{22}) + \begin{bmatrix} \mu^{\frac{1}{2}} & 0 & 0 \\ 0 & \mu^{\frac{1}{2}} & 0 \\ 0 & 0 & \mu\omega^{-1} \end{bmatrix}H_{33}$$

$$-\frac{1}{\mu\omega}\begin{bmatrix} \mu\omega^2H_{11} & \mu\omega^2H_{21} & \mu^{\frac{3}{2}}\omega H_{31} \\ \mu\omega^2H_{12} & \mu\omega^2H_{22} & \mu^{\frac{3}{2}}\omega H_{32} \\ \mu^{\frac{3}{2}}\omega H_{13} & \mu^{\frac{3}{2}}\omega H_{23} & \mu^2H_{33} \end{bmatrix}.$$

Next,

$HF_0^TF_0 + F_0H^TF_0 + F_0F_0^TH =$

$$\begin{bmatrix} 3\mu H_{11} & 2\mu H_{12}+\mu H_{21} & (\omega^2+\mu)H_{13}+\mu^{\frac{1}{2}}\omega H_{31} \\ 2\mu H_{21}+\mu H_{12} & 3\mu H_{22} & (\omega^2+\mu)H_{23}+\mu^{\frac{1}{2}}\omega H_{32} \\ (\omega^2+\mu)H_{31}+\mu^{\frac{1}{2}}\omega H_{13} & (\omega^2+\mu)H_{32}+\mu^{\frac{1}{2}}\omega H_{23} & 3\omega^2H_{33} \end{bmatrix}$$

Next,
$\sigma_{11}(G^1\cdot H)G^1 + \sigma_{22}(G^2\cdot H)G^2 + \sigma_{33}(G^3\cdot H)G^3 =$

$$\begin{aligned} \operatorname{diag}\,[&\sigma_{11}\mu^{\frac{1}{2}}(G^1\cdot H) + \sigma_{22}\mu^{\frac{3}{2}}(G^2\cdot H) + \sigma_{33}\mu^{\frac{1}{2}}\omega(G^3\cdot H), \\ &\sigma_{11}\mu^{\frac{1}{2}}(G^1\cdot H) + \sigma_{22}\mu^{\frac{3}{2}}(G^2\cdot H) + \sigma_{33}\mu^{\frac{1}{2}}\omega(G^3\cdot H), \\ &\sigma_{11}\omega(G^1\cdot H) + \sigma_{22}\omega^3(G^2\cdot H) + \sigma_{33}\mu(G^3\cdot H)] \end{aligned}$$

$$= \begin{bmatrix} Y(H_{11}+H_{22})+ZH_{33} & 0 & 0 \\ 0 & Y(H_{11}+H_{22})+ZH_{33} & 0 \\ 0 & 0 & Z(H_{11}+H_{22})+VH_{33} \end{bmatrix}$$

because :

i) $\sigma_{11}\mu^{\frac{1}{2}}(G^1 \cdot H) + \sigma_{22}\mu^{\frac{3}{2}}(G^2 \cdot H) + \sigma_{33}\mu^{\frac{1}{2}}\omega(G^3 \cdot H)$

$= (\sigma_{11}\mu + \sigma_{22}\mu^3 + \sigma_{33}\mu\omega^2)(H_{11} + H_{22}) + (\sigma_{11}\mu^{\frac{1}{2}}\omega + \sigma_{22}\mu^{\frac{3}{2}}\omega^3 + \sigma_{33}\mu^{\frac{3}{2}}\omega)H_{33}$

ii) $\sigma_{11}\omega(G^1 \cdot H) + \sigma_{22}\omega^3(G^2 \cdot H) + \sigma_{33}\mu(G^3 \cdot H) =$
$(\sigma_{11}\mu^{\frac{1}{2}}\omega + \sigma_{22}\mu^{\frac{3}{2}}\omega^3 + \sigma_{33}\mu^{\frac{3}{2}}\omega)(H_{11} + H_{22}) + (\sigma_{11}\omega^2 + \sigma_{22}\omega^6 + \sigma_{33}\mu^2)H_{33}$

Let $B = C[H]$. Then by equation (11)

$B_{11} = (\sigma_1 + Y + 3\mu\sigma_2)H_{11} + (\omega\sigma_3 + Y)H_{22} + (\mu^{\frac{1}{2}}\sigma_3 + Z)H_{33}$

$B_{22} = (\omega\sigma_3 + Y)H_{11} + (\sigma_1 + Y + 3\mu\sigma_2)H_{22} + (\mu^{\frac{1}{2}}\sigma_3 + Z)H_{33}$

$B_{33} = (\mu^{\frac{1}{2}}\sigma_3 + Z)(H_{11} + H_{22}) + (\sigma_1 + V + 3\omega^2\sigma_2)H_{33}$

$B_{12} = (\sigma_1 + 2\mu\sigma_2)H_{12} - (\omega\sigma_3 - \mu\sigma_2)H_{21}$

$B_{21} = (\sigma_1 + 2\mu\sigma_2)H_{21} - (\omega\sigma_3 - \mu\sigma_2)H_{12}$

$B_{13} = (\sigma_1 + \omega^2\sigma_2 + \mu\sigma_2)H_{13} - (\mu^{\frac{1}{2}}\sigma_3 - \mu^{\frac{1}{2}}\omega\sigma_2)H_{31}$

$B_{31} = (\sigma_1 + \omega^2\sigma_2 + \mu\sigma_2)H_{31} - (\mu^{\frac{1}{2}}\sigma_3 - \mu^{\frac{1}{2}}\omega\sigma_2)H_{13}$

$B_{23} = (\sigma_1 + \omega^2\sigma_2 + \mu\sigma_2)H_{23} - (\mu^{\frac{1}{2}}\sigma_3 - \mu^{\frac{1}{2}}\omega\sigma_2)H_{32}$

$B_{32} = (\sigma_1 + \omega^2\sigma_2 + \mu\sigma_2)H_{32} - (\mu^{\frac{1}{2}}\sigma_3 - \mu^{\frac{1}{2}}\omega\sigma_2)H_{23}$

□

Theorem 4.2.

If C in Theorem 4.1 is strongly elliptic then $\beta_1 > 0, \beta_3 > 0, \tau_1 > 0, \tau_3 > 0$.

Proof. We recall C is strongly elliptic iff $H \cdot C[H] > 0$ for all
$H = a \otimes b \neq 0, \quad a, b \in \mathbb{R}^3$.

$H = e_1 \otimes e_1 \implies \tau_1 > 0, H = e_3 \otimes e_3 \implies \tau_3 > 0,$

$H = e_1 \otimes e_3 \implies \beta_1 > 0, H = e_1 \otimes e_2 \implies \beta_3 > 0.$

e_1, e_2, e_3 are the unit vectors in $\mathbb{R}^3$.

□

Chapter 5

Calculating the Complementing Condition when $\mathbf{n} = \mathbf{e}_1$

In this Chapter we calculate the Complementing Condition for (C, n) with $n = e_1$ and C in Theorem 4.1. As in Chapter 3, this is equivalent to showing that for each $\xi \in \mathbb{R}^3 \smallsetminus \{0\}, \xi \perp n$

$$\begin{aligned} \mathscr{L}_\xi[z] = 0, \quad t \in [0, \infty) \\ \mathscr{B}_\xi[z] = 0 \quad \text{at } t = 0, \end{aligned} \tag{12}$$

the only exponentially decreasing solution of equation (12) is the trivial solution $z \equiv 0$ on $[0, \infty)$.

Let $n = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\xi = \begin{bmatrix} 0 \\ \xi_2 \\ \xi_3 \end{bmatrix}$

$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}, \quad t \in [0, \infty).$

Let

$$H = z' \otimes n + iz \otimes \xi = \begin{bmatrix} z_1' & i\xi_2 z_1 & i\xi_3 z_1 \\ z_2' & i\xi_2 z_2 & i\xi_3 z_2 \\ z_3' & i\xi_2 z_3 & i\xi_3 z_3 \end{bmatrix}$$

Then by equation (7)

$\mathscr{L}_\xi[z] = C[H']n + iC[H]\xi$

$= C[z'' \otimes n]n + iC[z' \otimes \xi]n + iC[z' \otimes n]\xi - C[z \otimes \xi]\xi$

$\mathscr{B}_\xi[z] = C[H]n = C[z' \otimes n]n + iC[z \otimes \xi]n$

From the formula for $C[H]$ in Theorem 4.1, we have

$$\begin{aligned}
\mathscr{L}_\xi[z]_1 &= C[H']_{11} + iC[H]_{12}\xi_2 + iC[H]_{13}\xi_3 \\
= \tau_1 z_1'' - (\beta_3\xi_2^2 + \beta_1\xi_3^2)z_1 &+ i\xi_2(\nu_3 + \gamma)z_2' + i\xi_3(X + \nu_1)z_3' \\
\mathscr{L}_\xi[z]_2 &= C[H']_{21} + iC[H]_{22}\xi_2 + iC[H]_{23}\xi_3 \\
= \beta_3 z_2'' - (\tau_1\xi_2^2 + \beta_1\xi_3^2)z_2 &+ i\xi_2(\tau_1 - \beta_3)z_1' - \xi_2\xi_3(X + \nu_1)z_3 \\
\mathscr{L}_\xi[z]_3 &= C[H']_{31} + iC[H]_{32}\xi_2 + iC[H]_{33}\xi_3 \\
= \beta_1 z_3'' - (\beta_1\xi_2^2 + \tau_3\xi_3^2)z_3 &+ i\xi_3(X + \nu_1)z_1' - \xi_2\xi_3(X + \nu_1)z_2 \\
\mathscr{B}_\xi[z]_1 &= C[H]_{11} = \tau_1 z_1' + i\gamma\xi_2 z_2 + iX\xi_3 z_3 \\
\mathscr{B}_\xi[z]_2 &= C[H]_{21} = \beta_3 z_2' + i\nu_3\xi_2 z_1 \\
\mathscr{B}_\xi[z]_3 &= C[H]_{31} = \beta_1 z_3' + i\nu_1\xi_3 z_1
\end{aligned} \tag{13}$$

Then by equation (12), $\mathscr{L}_\xi[z] = 0$ iff

$$\begin{aligned}
\tau_1 z_1'' - (\beta_3\xi_2^2 + \beta_1\xi_3^2)z_1 + i\xi_2(\nu_3 + \gamma)z_2' + i\xi_3(X + \nu_1)z_3' &= 0 \\
\beta_3 z_2'' - (\tau_1\xi_2^2 + \beta_1\xi_3^2)z_2 + i\xi_2(\tau_1 - \beta_3)z_1' - \xi_2\xi_3(X + \nu_1)z_3 &= 0 \\
\beta_1 z_3'' - (\beta_1\xi_2^2 + \tau_3\xi_3^2)z_3 + i\xi_3(X + \nu_1)z_1' - \xi_2\xi_3(X + \nu_1)z_2 &= 0
\end{aligned} \tag{14}$$

for $t \geq 0$

and $\mathscr{B}_\xi[z] = 0$ iff

$$\begin{aligned}
\tau_1 z_1' + i\gamma\xi_2 z_2 + iX\xi_3 z_3 &= 0 \\
\beta_3 z_2' + i\nu_3\xi_2 z_1 &= 0 \\
\beta_1 z_3' + i\nu_1\xi_3 z_1 &= 0
\end{aligned} \tag{15}$$

at $t = 0$

We seek decaying exponential solutions of equation (14) of the form

$$z(t) = e^{rt}p, \qquad t \geq 0 \tag{16}$$

for $r \in \mathbb{C}$, $\operatorname{Re} r < 0, p \in \mathbb{C}^3$. (See equation (8) with a replaced by p).

By equation (9) we have

$$\begin{aligned} \mathscr{L}_\xi[e^{rt}p] &= M(rn + i\xi)pe^{rt}, \qquad t \geq 0, \\ \mathscr{B}_\xi[e^{rt}p] &= C[p \otimes (rn + i\xi)]n, \qquad t = 0 \end{aligned} \tag{17}$$

and M is the acoustic tensor in Definition 2.2. We define, for $r \in \mathbb{C}$, the 3×3 complex matrices $A(r), B(r)$ by

$A(r) = M(rn + i\xi)$

$B(r)p = C[p \otimes (rn + i\xi)]n, \quad p \in \mathbb{C}^3.$

So equation (17) becomes

$$\begin{aligned} \mathscr{L}_\xi[e^{rt}p] &= A(r)pe^{rt}, \quad t \geq 0 \\ \mathscr{B}_\xi[e^{rt}p] &= B(r)p, \quad t = 0 \end{aligned} \tag{18}$$

It follows that equation (16) is a solution of equation (14) iff

$$A(r)p = 0. \tag{19}$$

To find $A(r), B(r)$ we substitute equation (16) with $p = e_1, e_2, e_3$ into equation (13) and use equation (18) and $\nu_3 + \gamma = \tau_1 - \beta_3$ to get

$$A(r) = \begin{bmatrix} \tau_1 r^2 - (\beta_3\xi_2^2 + \beta_1\xi_3^2) & i\xi_2(\tau_1 - \beta_3)r & i\xi_3 N r \\ i\xi_2(\tau_1 - \beta_3)r & \beta_3 r^2 - (\tau_1\xi_2^2 + \beta_1\xi_3^2) & -\xi_2\xi_3 N \\ i\xi_3 N r & -\xi_2\xi_3 N & \beta_1 r^2 - (\beta_1\xi_2^2 + \tau_3\xi_3^2) \end{bmatrix}$$

$$B(r) = \begin{bmatrix} \tau_1 r & i\xi_2\gamma & i\xi_3 X \\ i\xi_2\nu_3 & \beta_3 r & 0 \\ i\xi_3\nu_1 & 0 & \beta_1 r \end{bmatrix}$$

We are looking for nontrivial solutions (16) of equation (14) which requires $p \neq 0$ in equation (19) $\Longrightarrow$

$$\det A(r) = 0 \tag{20}$$

As in Theorem 3.3, this is the 6^{th} degree characteristic polynomial in r. It has exactly 3 roots $r_1, r_2, r_3 \in \mathbb{C}$ with $\operatorname{Re} r_i < 0$. By strong ellipticity no root has real part equal to zero. Let

S = set of solutions of equation (14) on $[0, \infty)$ which are exponentially decreasing as $t \to \infty$.

S is a vector space over $\mathbb{C}$ and by Theorem 3.3, $\dim S = 3$.

Theorem 5.1. Letting $\rho = r^2 - \xi_2^2$ we have

$$\det A(r) = (\beta_3 \rho - \beta_1 \xi_3^2)(\beta_1 \tau_1 \rho^2 - \lambda \xi_3^2 \rho + \beta_1 \tau_3 \xi_3^4) \tag{21}$$

$$\text{with } \lambda = \beta_1^2 + \tau_1 \tau_3 - N^2$$

$\det A(r) = 0$ has roots $\pm r_1, \pm r_2, \pm r_3 \in \mathbb{C}$ with $\operatorname{Re} r_i < 0, i = 1, 2, 3$.

There are four cases.

Case i) $\xi_3 \neq 0, N \neq 0, r_1 \neq r_3$.

$r_1^2 = \xi_2^2 + k_1 \xi_3^2, r_2^2 = \xi_2^2 + \frac{\beta_1}{\beta_3} \xi_3^2, r_3^2 = \xi_2^2 + k_3 \xi_3^2$

where $k_1, k_3 \in \mathbb{C}$ are the roots of

$$\beta_1 \tau_1 k^2 - \lambda k + \beta_1 \tau_3 = 0 \tag{22}$$

With

$$p_1 = \begin{bmatrix} r_1 \\ i\xi_2 \\ ic_1\xi_3 \end{bmatrix}, \quad p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} r_3 \\ i\xi_2 \\ ic_3\xi_3 \end{bmatrix} \tag{23}$$

$$c_i = \frac{\tau_1 k_i - \beta_1}{N}, \quad i = 1, 3 \tag{24}$$

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3\}$

Case ii) $\xi_3 \neq 0, N = 0$

$r_1^2 = \xi_2^2 + \frac{\beta_1}{\tau_1}\xi_3^2, \quad r_2^2 = \xi_2^2 + \frac{\beta_1}{\beta_3}\xi_3^2, \quad r_3^2 = \xi_2^2 + \frac{\tau_3}{\beta_1}\xi_3^2$

With

$$p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} r_1 \\ i\xi_2 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}$$

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3\}$

Case iii) $\xi_3 \neq 0, N \neq 0, r_1 = r_3$

$r_1^2 = \xi_2^2 + k_1\xi_3^2, \quad r_2^2 = \xi_2^2 + \frac{\beta_1}{\beta_3}\xi_3^2, \quad r_3^2 = \xi_2^2 + k_3\xi_3^2$ with k_1, k_3 as in Case i).

With p_1, p_2 in equation (23) and

$$p(r) = \begin{bmatrix} r \\ i\xi_2 \\ ic(r)\xi_3 \end{bmatrix}$$

$c(r) = \frac{\tau_1\rho(r) - \beta_1\xi_3^2}{N\xi_3^2}$,

$\rho(r) = r^2 - \xi_2^2$, we have $S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_1}\}$

Case iv) $\xi_3 = 0$

$r_1^2 = r_2^2 = r_3^2 = |\xi|^2$

With

$$p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p(r) = \begin{bmatrix} \beta_3 r^2 - \tau_1 \xi_2^2 \\ -i(\tau_1 - \beta_3) r \xi_2 \\ 0 \end{bmatrix}$$

$S = \text{ span } \{e^{r_2 t} p_2, e^{r_3 t} p_3, \frac{\partial}{\partial r}[e^{rt} p(r)]|_{r=r_1}\}$

Proof. To find det $A(r)$ we reduce $A(r)$ by row and column operations via $E(r)$ below to a simpler matrix $D(r)$.
Let

$$E(r) = \begin{bmatrix} 1 & -i\xi_2 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D(r) = E(r)^T A(r) E(r) = \begin{bmatrix} \eta_1(\rho) & -i\xi_2\eta_2(\rho) & i\xi_3 N r \\ -i\xi_2\eta_2(\rho) & \rho\eta_2(\rho) & 0 \\ i\xi_3 N r & 0 & \eta_3(\rho) \end{bmatrix}$$

where

$$\rho = r^2 - \xi_2^2$$

$$\eta_1(\rho) = \tau_1 \rho - \beta_1 \xi_3^2 + (\tau_1 - \beta_3)\xi_2^2$$

$$\eta_2(\rho) = \beta_3 \rho - \beta_1 \xi_3^2$$

$$\eta_3(\rho) = \beta_1 \rho - \tau_3 \xi_3^2$$

Expanding the determinant of $D(r)$ about the third row $\Longrightarrow$

(and for brevity writing $\eta_i = \eta_i(\rho)$)

det $D(r) = [$ det $E(r)]^2$ det $A(r) = \eta_2[\xi_3^2 N^2 r^2 \rho + \eta_3(\rho\eta_1 + \xi_2^2 \eta_2)]$

$= \eta_2 r^2[\xi_3^2 N^2\rho + \eta_3(\tau_1\rho - \beta_1\xi_3^2)]$

(since $\rho\eta_1 + \xi_2^2\eta_2 = r^2\eta_1 + \xi_2^2(\eta_2 - \eta_1) = r^2\eta_1 - \xi_2^2(\tau_1 - \beta_3)r^2$

$= r^2(\tau_1\rho - \beta_1\xi_3^2)$)

$= \eta_2 r^2[\xi_3^2 N^2\rho + (\beta_1\rho - \tau_3\xi_3^2)(\tau_1\rho - \beta_1\xi_3^2)]$

$= r^2(\beta_3\rho - \beta_1\xi_3^2)[\beta_1\tau_1\rho^2 - \lambda\xi_3^2\rho + \beta_1\tau_3\xi_3^4]$

where $\lambda = \beta_1^2 + \tau_1\tau_3 - N^2$.

Since det $E(r) = r$, we get equation (21) .

Let the roots of equation (21) be $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$. Then the roots of det $A(r)$ are $\pm r_1, \pm r_2, \pm r_3 \in \mathbb{C}$ where $r_i^2 = \rho_i + \xi_2^2$ and Re $r_i < 0, \quad i = 1, 2, 3$.

i) ρ_2 satisfies

$$\beta_3\rho - \beta_1\xi_3^2 = 0 \implies \rho_2 = \frac{\beta_1}{\beta_3}\xi_3^2 \tag{25}$$

ii) ρ_1, ρ_3 satisfy

$$\beta_1\tau_1\rho^2 - \lambda\xi_3^2\rho + \beta_1\tau_3\xi_3^4 = 0 \tag{26}$$

and putting $k_1 = \frac{\rho_1}{\xi_3^2}, k_3 = \frac{\rho_3}{\xi_3^2}$

is equivalent to $k_1, k_3 \in \mathbb{C}$ satisfying equation (22).

k_1, k_3 can be complex. Now putting $r_i^2 = \rho_i + \xi_2^2$ we get :

$$r_1^2 = \xi_2^2 + k_1\xi_3^2, r_2^2 = \xi_2^2 + \frac{\beta_1}{\beta_3}\xi_3^2, r_3^2 = \xi_2^2 + k_3\xi_3^2. \tag{27}$$

This gives the 6 roots of equation (20) det $A(r) = 0$ which occur in 3 pairs $\pm r_1, \pm r_2, \pm r_3$ with Re $r_i < 0, \quad i = 1, 2, 3$.

Case i) $\xi_3 \neq 0, N \neq 0, r_1 \neq r_3$:
The roots r_1, r_2, r_3 are given in equation (27).
Next we find the null vectors p_i of $A(r_i)$ satisfying equation (19). This is equivalent to finding q_i such that $D(r_i)q_i = 0$,
then putting $p_i =$ const. $E(r_i)q_i$

Finding the null vector p_2 :

$r = r_2 \implies \eta_2(\rho_2) = 0 \implies$

$$D(r_2) = \begin{bmatrix} \eta_1 & 0 & i\xi_3 N r_2 \\ 0 & 0 & 0 \\ i\xi_3 N r_2 & 0 & \eta_3 \end{bmatrix}$$

Here $\eta_i = \eta_i(\rho_2)$.

Then $q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow$

$$p_2 = E(r_2)q_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}$$

Finding the null vectors p_1, p_3 :

Letting $r = r_i, \rho_i = k_i \xi_3^2, \quad i = 1,3$ in $D(r_i)$, we can take

$$q_i = \begin{bmatrix} \rho_i \\ i\xi_2 \\ i\xi_3 r_i c_i \end{bmatrix}$$

where $c_i = \frac{\tau_1 \rho_i - \beta_1 \xi_3^2}{N\xi_3^2}$ (This uses that ρ_i satisfies equation (26), $i = 1,3$.)

Then $p_i =$ const. $E(r_i)q_i =$ const. $\begin{bmatrix} r_i^2 \\ i\xi_2 r_i \\ i\xi_3 r_i c_i \end{bmatrix}$

Put the constant $= r_i^{-1} \Longrightarrow$

$$p_i = \begin{bmatrix} r_i \\ i\xi_2 \\ ic_i\xi_3 \end{bmatrix}, \quad i = 1, 3.$$

with $\rho_i = k_i\xi_3^2$ this gives p_1, p_3 in equation (23). Thus since $A(r_i)p_i = 0$ for $i = 1, 2, 3$, then equation (19) $\implies e^{r_i t}p_i, i = 1, 2, 3$ are exponentially decreasing solutions of equation (14) in S. They are linearly independent on $[0, \infty)$ since $r_1 \neq r_3$ and $\{p_1, p_2\}, \{p_2, p_3\}$ are linearly independent in $\mathbb{C}^3$ due to $\det \begin{vmatrix} r_i & -i\xi_2 \\ i\xi_2 & r_2 \end{vmatrix} = r_i r_2 - \xi_2^2 \neq 0$ for $i = 1, 3$. Since dim $S = 3$ they form a basis for S and S is spanned by $e^{r_i t}p_i, i = 1, 2, 3$.

Case ii) $\xi_3 \neq 0, N = 0$:

When $N = 0$, equation (22) becomes $(\tau_1 k - \beta_1)(\beta_1 k - \tau_3) = 0$. So

$k_1 = \frac{\beta_1}{\tau_1}$; $\quad k_3 = \frac{\tau_3}{\beta_1}$ and the roots are in equation (27).

$r_1^2 = \xi_2^2 + \frac{\beta_1}{\tau_1}\xi_3^2, \quad r_2^2 = \xi_2^2 + \frac{\beta_1}{\beta_3}\xi_3^2, \quad r_3^2 = \xi_2^2 + \frac{\tau_3}{\beta_1}\xi_3^2$, Re $r_i < 0$.

To find null vectors $p_i \in \mathbb{C}^3$ of $A(r_i)$ we have

$$A(r) = \begin{bmatrix} \tau_1 r^2 - (\beta_3\xi_2^2 + \beta_1\xi_3^2) & i\xi_2(\tau_1 - \beta_3)r & 0 \\ i\xi_2(\tau_1 - \beta_3)r & \beta_3 r^2 - (\tau_1\xi_2^2 + \beta_1\xi_3^2) & 0 \\ 0 & 0 & \beta_1 r^2 - (\beta_1\xi_2^2 + \tau_3\xi_3^2) \end{bmatrix}$$

We get

$$p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} r_1 \\ i\xi_2 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}$$

As in Case i) $\{e^{r_i t}p_i\}_{i=1}^3$ are linearly independent on $[0,\infty)$, using the fact that p_1, p_2, p_3 are linearly independent in $\mathbb{C}^3$

$$\left(\begin{vmatrix} r_1 & -i\xi_2 \\ i\xi_2 & r_2 \end{vmatrix} = r_1 r_2 - \xi_2^2 = |r_1||r_2| - \xi_2^2 > |\xi_2||\xi_2| - \xi_2^2 = 0 \text{ since } |r_i| > |\xi_2|\right).$$

Thus they form a basis for S.

Case iii) $\xi_3 \neq 0, N \neq 0, r_1 = r_3$:

The roots r_i are as in Case i). Also, with $p_1, p_2 \in \mathbb{C}^3$ in equation (23), $e^{r_1 t}p_1, e^{r_2 t}p_2 \in S$ are linearly independent on $[0,\infty)$. To find a third linearly independent solution in S

$$\text{let } p(r) = \begin{bmatrix} r \\ i\xi_2 \\ ic(r)\xi_3 \end{bmatrix}, c(r) = \frac{\tau_1\rho(r) - \beta_1\xi_3^2}{N\xi_3^2}, \quad \rho(r) = r^2 - \xi_2^2$$

Then

$A(r)p(r) =$

$$\begin{bmatrix} \tau_1 r^2 - (\beta_3\xi_2^2 + \beta_1\xi_3^2) & i\xi_2(\tau_1 - \beta_3)r & i\xi_3 N r \\ i\xi_2(\tau_1 - \beta_3)r & \beta_3 r^2 - (\tau_1\xi_2^2 + \beta_1\xi_3^2) & -\xi_2\xi_3 N \\ i\xi_3 N r & -\xi_2\xi_3 N & \beta_1 r^2 - (\beta_1\xi_2^2 + \tau_3\xi_3^2) \end{bmatrix} \begin{bmatrix} r \\ i\xi_2 \\ ic(r)\xi_3 \end{bmatrix}$$

$$= \frac{i}{N\xi_3}(\beta_1\tau_1\rho^2 - \lambda\xi_3^2\rho + \beta_1\tau_3\xi_3^4) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \rho = r^2 - \xi_2^2.$$

Since $r_1 = r_3$ then $k_1 = k_3$ is a double root of equation (22) so $\beta_1\tau_1 k^2 - \lambda k + \beta_1\tau_3 = \beta_1\tau_1(k - k_1)^2$.

Similarly $\beta_1\tau_1\rho^2 - \lambda\xi_3^2\rho + \beta_1\tau_3\xi_3^4 = \beta_1\tau_1(\rho - \rho_1)^2$. Thus

$$A(r)p(r) = \frac{i\beta_1\tau_1}{N\xi_3}(\rho-\rho_1)^2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So $A(r)p(r)|_{r=r_1} = \frac{\partial}{\partial r}[A(r)p(r)]|_{r=r_1} = 0$ since $r = r_1 \implies \rho = \rho_1$.

Now by equation (18) with $p = p(r)$, differentiating $\frac{\partial}{\partial r}$ gives

$\mathscr{L}_\xi[\frac{\partial}{\partial r}(e^{rt}p(r))] = \frac{\partial}{\partial r}[A(r)p(r)e^{rt}], t \geq 0$

and therefore

$\mathscr{L}_\xi[\frac{\partial}{\partial r}(e^{rt}p(r))|_{r=r_1}] = 0$ on $[0,\infty)$.

Thus we let a third linearly independent solution of S be

$$\frac{\partial}{\partial r}(e^{rt}p(r))|_{r=r_1} = te^{r_1t}p_1 + e^{r_1t}\begin{bmatrix} 1 \\ 0 \\ \frac{2i\tau_1 r_1}{N\xi_3} \end{bmatrix}$$

since $\rho(r_1) = \rho_1 = k_1\xi_3^2, \quad c(r_1) = \frac{\tau_1\rho_1-\beta_1\xi_3^2}{N\xi_3^2} = \frac{\tau_1k_1-\beta_1}{N} = c_1 \implies p(r_1) = p_1$ in equation (23).

Since $p_1, p_2 \in \mathbb{C}^3$ are linearly independent then $e^{r_1t}p_1, e^{r_2t}p_2, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_1}$ are linearly independent on $[0,\infty)$ and form a basis for S.

Case iv) $\xi_3 = 0$:

When $\xi_3 = 0$, then $r_i^2 = \xi_2^2, \quad i = 1,2,3$, Re $r_i < 0 \implies$

$r_1 = r_2 = r_3 = -|\xi_2|$. Also,

$$A(r) = \begin{bmatrix} \tau_1r^2-\beta_3\xi_2^2 & i\xi_2(\tau_1-\beta_3)r & 0 \\ i\xi_2(\tau_1-\beta_3)r & \beta_3r^2-\tau_1\xi_2^2 & 0 \\ 0 & 0 & \beta_1(r^2-\xi_2^2) \end{bmatrix}$$

Then $A(r_i)$ has null vectors $p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

and $e^{r_2 t}p_2, e^{r_3 t}p_3 \in S$ and are linearly independent on $[0, \infty)$. As in Case iii) another solution is $\frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_1}$

with $p(r) = \begin{bmatrix} \beta_3 r^2 - \tau_1\xi_2^2 \\ -i(\tau_1 - \beta_3)r\xi_2 \\ 0 \end{bmatrix},$

This follows from

$$A(r)p(r) = \tau_1\beta_3(r^2 - \xi_2^2)^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So $A(r)p(r)|_{r=r_1} = \frac{\partial}{\partial r}[A(r)p(r)]|_{r=r_1} = 0.$

Now $\frac{\partial}{\partial r}(e^{rt}p(r))|_{r=r_1} = te^{r_1 t}p(r_1) + e^{r_1 t}p'(r_1)$

$$= te^{-|\xi_2|t}(\tau_1 - \beta_3) \begin{bmatrix} -\xi_2^2 \\ i\xi_2|\xi_2| \\ 0 \end{bmatrix} + e^{-|\xi_2|t} \begin{bmatrix} -2\beta_3|\xi_2| \\ -i(\tau_1 - \beta_3)\xi_2 \\ 0 \end{bmatrix},$$

Then $e^{r_2 t}p_2, e^{r_3 t}p_3, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_1}$ are linearly independent on $[0, \infty)$ (considering the cases $\tau_1 - \beta_3 = 0, \neq 0$ separately).

□

From Theorem 5.1, equation (14) has three linearly independent solutions

$$z^{(i)}(t), \quad i = 1, 2, 3 \tag{28}$$

on $[0, \infty)$ such that $z^{(i)}$ are exponentially decreasing.

We substitute these into the boundary operator $\mathscr{B}_\xi$ in equation (13) at $t = 0$. We write

$$B_\xi[z] = \begin{bmatrix} \tau_1 z_1' + i\xi_2 \gamma z_2 + i\xi_3 X z_3 \\ \beta_3 z_2' + i\xi_2 \nu_3 z_1 \\ \beta_1 z_3' + i\xi_3 \nu_1 z_1 \end{bmatrix} \tag{29}$$

and

$C_\xi = [\mathscr{B}_\xi[z^{(1)}]|_{t=0}, \mathscr{B}_\xi[z^{(2)}]|_{t=0}, \mathscr{B}_\xi[z^{(3)}]|_{t=0}]$;

C_ξ is a 3×3 matrix.

In the following Theorem we find C_ξ using the four cases in Theorem 5.1.

Theorem 5.2.

Case i) $\xi_3 \neq 0, N \neq 0, r_1 \neq r_3$

$$C_\xi = \begin{bmatrix} a\xi_2^2 + (\tau_1 k_1 - X c_1)\xi_3^2 & -ia\xi_2 r_2 & a\xi_2^2 + (\tau_1 k_3 - X c_3)\xi_3^2 \\ ia\xi_2 r_1 & a\xi_2^2 + \beta_1 \xi_3^2 & ia\xi_2 r_3 \\ i\xi_3 r_1(\nu_1 + \beta_1 c_1) & \nu_1 \xi_2 \xi_3 & i\xi_3 r_3(\nu_1 + \beta_1 c_3) \end{bmatrix}$$

$a = \tau_1 - \gamma, c_i = \frac{\tau_1 k_i - \beta_1}{N}, \quad i = 1, 3.$

Case ii) $\xi_3 \neq 0, N = 0$

$$C_\xi = \begin{bmatrix} a\xi_2^2 + \beta_1\xi_3^2 & -ia\xi_2 r_2 & i\xi_3 X \\ ia\xi_2 r_1 & a\xi_2^2 + \beta_1\xi_3^2 & 0 \\ i\xi_3 r_1\nu_1 & \nu_1\xi_2\xi_3 & \beta_1 r_3 \end{bmatrix}$$

$a = \tau_1 - \gamma$

Case iii) $\xi_3 \neq 0, N \neq 0, r_1 = r_3$

$$C_\xi = \begin{bmatrix} a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2 & -ia\xi_2 r_2 & \frac{\partial}{\partial r}[\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]|_{r=r_1} \\ ia\xi_2 r_1 & a\xi_2^2 + \beta_1\xi_3^2 & \frac{\partial}{\partial r}[ia\xi_2 r]|_{r=r_1} \\ i\xi_3 r_1(\nu_1 + \beta_1 c_1) & \nu_1\xi_2\xi_3 & \frac{\partial}{\partial r}[i\xi_3 r(\nu_1 + \beta_1 c(r))]|_{r=r_1} \end{bmatrix}$$

$a = \tau_1 - \gamma, c(r) = \frac{\tau_1\rho(r) - \beta_1\xi_3^2}{N\xi_3^2}, \quad \rho(r) = r^2 - \xi_2^2.$

Case iv) $\xi_3 = 0$

$$C_\xi = \begin{bmatrix} ia\xi_2|\xi_2| & 0 & \frac{\partial}{\partial r}[\tau_1\beta_3 r^3 + ((\tau_1 - \beta_3)\gamma - \tau_1^2)\xi_2^2 r]|_{r=r_1} \\ a\xi_2^2 & 0 & \frac{\partial}{\partial r}[-i\xi_2(\beta_3\gamma r^2 + \nu_3\tau_1\xi_2^2)]|_{r=r_1} \\ 0 & -\beta_1|\xi_2| & 0 \end{bmatrix}$$

$a = \tau_1 - \gamma, r_1 = -|\xi_2|.$

Proof. In the following calculations we use from equation (18)

$\mathscr{B}_\xi[e^{rt}p] = B(r)p \quad$ at $t = 0$ and in cases where $p = p(r)$,

$$\mathscr{B}_\xi[\frac{\partial}{\partial r}[e^{rt}p(r)]] = \frac{\partial}{\partial r}[B(r)p(r) \text{ at } t = 0. \tag{30}$$

Also $a = \tau_1 - \gamma$.

Case i) : $\xi_3 \neq 0, N \neq 0, r_1 \neq r_3$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2 \\ ia\xi_2 r_1 \\ i\xi_3 r_1(\nu_1 + \beta_1 c_1) \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} -ia\xi_2 r_2 \\ a\xi_2^2 + \beta_1\xi_3^2 \\ \nu_1\xi_2\xi_3 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(3)}] = B(r_3)p_3 = \begin{bmatrix} a\xi_2^2 + (\tau_1 k_3 - Xc_3)\xi_3^2 \\ ia\xi_2 r_3 \\ i\xi_3 r_3(\nu_1 + \beta_1 c_3) \end{bmatrix}$$

Case ii) : $\xi_3 \neq 0, N = 0$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} a\xi_2^2 + \beta_1\xi_3^2 \\ ia\xi_2 r_1 \\ i\xi_3 r_1\nu_1 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} -ia\xi_2 r_2 \\ a\xi_2^2 + \beta_1\xi_3^2 \\ \nu_1\xi_2\xi_3 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(3)}] = B(r_3)p_3 = \begin{bmatrix} i\xi_3 X \\ 0 \\ \beta_1 r_3 \end{bmatrix}$$

Case iii) : $\xi_3 \neq 0, N \neq 0, r_1 = r_3$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} a\xi_2^2 + (\tau_1 k_1 - Xc_1) \\ ia\xi_2 r_1 \\ i\xi_3 r_1(\nu_1 + \beta_1 c_1) \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} -ia\xi_2 r_2 \\ a\xi_2^2 + \beta_1\xi_3^2 \\ \nu_1\xi_2\xi_3 \end{bmatrix}$$

From equation (30), $\mathscr{B}_\xi[z^{(3)}] = \frac{\partial}{\partial r}[B(r)p(r)]|_{r=r_1}$

$$\text{where } B(r)p(r) = \begin{bmatrix} \tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2 \\ ia\xi_2 r \\ i\xi_3 r(\nu_1 + \beta_1 c(r)) \end{bmatrix}$$

Case iv) : $\xi_3 = 0$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_2)p_2 = \begin{bmatrix} ia\xi_2|\xi_2| \\ a\xi_2^2 \\ 0 \end{bmatrix}$$

using $\beta_3 + \nu_3 = \tau_1 - \gamma$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_3)p_3 = \begin{bmatrix} 0 \\ 0 \\ -\beta_1|\xi_2| \end{bmatrix}$$

From equation (30), $\mathscr{B}_\xi[z^{(3)}] = \frac{\partial}{\partial r}[B(r)p(r)]|_{r=r_1}$

$$B(r)p(r) = \begin{bmatrix} \tau_1\beta_3 r^3 + [(\tau_1 - \beta_3)\gamma - \tau_1^2]\xi_2^2 r \\ -i\xi_2(\beta_3\gamma r^2 + \nu_3\tau_1\xi_2^2) \\ 0 \end{bmatrix}$$

□

Next we note equations (14), (15) have only the trivial exponentially decreasing solution z iff det $C_\xi \neq 0$. Since $\{z^{(1)}, z^{(2)}, z^{(3)}\}$ is a basis for S then any exponentially decreasing solution z of equation (14) can be written

$$z(t) = c_1 z^{(1)}(t) + c_2 z^{(2)}(t) + c_3 z^{(3)}(t) \text{ with } c_i \in \mathbb{C}.$$

Then $\mathscr{B}_\xi[z]|_{t=0} = \sum_{i=1}^3 c_i \mathscr{B}_\xi[z^{(i)}]|_{t=0} = C_\xi c$ where $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{C}^3$. Thus z satisfies equation (15) iff $C_\xi c = 0$. Finally, z is trivial iff $c = 0$, and $C_\xi c = 0$ has only the trivial solution $c = 0$ iff det $C_\xi \neq 0$.

Theorem 5.3.

(C, n) satisfies the Complementing Condition iff det $C_\xi \neq 0$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}, \xi \perp n$.

Proof. By Definition 3.2 and the remarks after equation (6), (C, n) satisfies the Complementing Condition iff for each $\xi \in \mathbb{R}^3 \setminus \{0\}, \xi \perp n$, the only exponentially decreasing solution z of equations (14), (15) is trivial $z \equiv 0$ on $[0, \infty)$ iff for each such ξ, det $C_\xi \neq 0$. □

In view of Theorem 5.3 we calculate det C_ξ.

Theorem 5.4.

With the 3×3 matrix, C_ξ in all four cases of Theorem 5.2, we find det C_ξ.
$a = \tau_1 - \gamma$.

Case i) det C_ξ =

$(i\beta_1\xi_3)\{-r_1[a\xi_2^2 + (\tau_1 k_3 - Xc_3)\xi_3^2]\,[ac_1\xi_2^2 + (\nu_1 + \beta_1 c_1)\xi_3^2] + r_3[a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2]\,[ac_3\xi_2^2 + (\nu_1 + \beta_1 c_3)\xi_3^2] + a^2(c_1 - c_3)\xi_2^2 r_1 r_2 r_3\}$

Case ii) det C_ξ =

$-a^2\beta_1 r_1 r_2 r_3 \xi_2^2 + \beta_1 r_3(a\xi_2^2 + \beta_1\xi_3^2)^2 + \beta_1 X\nu_1 r_1 \xi_3^4$

Case iii) det C_ξ =

$(i\beta_1\xi_3)\ \frac{\partial}{\partial r}\{-r_1[\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]\,[ac_1\xi_2^2 + (\nu_1 + \beta_1 c_1)\xi_3^2] + r[a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2]\,[ac(r)\xi_2^2 + (\nu_1 + \beta_1 c(r))\xi_3^2] + a^2(c_1 - c(r))\xi_2^2 r_1 r_2 r\}|_{r=r_1}$

$c(r) = \frac{\tau_1\rho(r) - \beta_1\xi_3^2}{N\xi_3^2}, \quad \rho(r) = r^2 - \xi_2^2$

Case iv) det C_ξ =

$a\beta_1|\xi_2|\xi_2^4\,[-\tau_1^2 + (3\beta_3 + \gamma)\tau_1 + \beta_3\gamma]$

Proof. **Case i)** Expanding det C_ξ about its 3$^{\text{rd}}$ column gives

det $C_\xi = [a\xi_2^2 + (\tau_1 k_3 - Xc_3)\xi_3^2]\,(i\xi_3 r_1)\,[a\nu_1\xi_2^2 - (\nu_1 + \beta_1 c_1)\,(a\xi_2^2 + \beta_1\xi_3^2)] - ia\xi_2 r_3(\xi_2\xi_3)\,[\nu_1(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2) - ar_1r_2(\nu_1 + \beta_1 c_1)] + i\xi_3 r_3(\nu_1 + \beta_1 c_3)\,[(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2)\,(a\xi_2^2 + \beta_1\xi_3^2) - a^2\xi_2^2 r_1 r_2]$

$= (i\xi_3)\{-\beta_1 r_1[a\xi_2^2 + (\tau_1 k_3 - Xc_3)\xi_3^2]\,[ac_1\xi_2^2 + (\nu_1 + \beta_1 c_1)\xi_3^2] - ar_3\xi_2^2\,[\nu_1\,(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2) - ar_1r_2(\nu_1 + \beta_1 c_1)] + r_3(\nu_1 + \beta_1 c_3)\,[(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2)\,(a\xi_2^2 + \beta_1\xi_3^2) - a^2\xi_2^2 r_1 r_2]$

$= (i\xi_3)\{-\beta_1 r_1[a\xi_2^2 + (\tau_1 k_3 - Xc_3)\xi_3^2]\,[ac_1\xi_2^2 + (\nu_1 + \beta_1 c_1)\xi_3^2] + \beta_1 r_3(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2)\,[ac_3\xi_2^2 + (\nu_1 + \beta_1 c_3)\xi_3^2] + a^2 r_1 r_2 r_3 \xi_2^2\beta_1(c_1 - c_3)\}$

Case ii) Expanding det C_ξ about the second row $\Longrightarrow$

det C_ξ =
$-(ia\xi_2 r_1)[-ia\xi_2 r_2\beta_1 r_3 - i\xi_3 X\nu_1\xi_2\xi_3] + (a\xi_2^2 + \beta_1\xi_3^2)\,[\beta_1 r_3(a\xi_2^2 + \beta_1\xi_3^2) - i\xi_3 X i\xi_3 r_1\nu_1]$

$= -a^2\beta_1 r_1 r_2 r_3\xi_2^2 + \beta_1 r_3(a\xi_2^2 + \beta_1\xi_3^2)^2 + \beta_1 X\nu_1 r_1\xi_3^4$

Case iii) Let

$$C_\xi(r) = \begin{bmatrix} a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2 & -ia\xi_2 r_2 & \tau_1 r^2 - \gamma\xi_3^2 - Xc(r)\xi_3^2 \\ ia\xi_2 r_1 & a\xi_2^2 + \beta_1\xi_3^2 & ia\xi_2 r \\ i\xi_3 r_1(\nu_1 + \beta_1 c_1) & \nu_1\xi_2\xi_3 & i\xi_3 r(\nu_1 + \beta_1 c(r)) \end{bmatrix}$$

Then expanding det C_ξ in Case iii) about its 3$^{\text{rd}}$ column we see det $C_\xi = \frac{\partial}{\partial r}$ det $C_\xi(r)|_{r=r_1}$

Expanding det $C_\xi(r)$ about the 3$^{\text{rd}}$ column gives

det $C_\xi(r) = [\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]\,(i\xi_3 r_1)\,[a\nu_1\xi_2^2 - (\nu_1 + \beta_1 c_1)(a\xi_2^2 + \beta_1\xi_3^2)] - ia\xi_2 r(\xi_2\xi_3)\,[\nu_1(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2) - ar_1 r_2(\nu_1 + \beta_1 c_1)] + i\xi_3 r(\nu_1 + \beta_1 c(r))\,[(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2)\,(a\xi_2^2 + \beta_1\xi_3^2) - a^2\xi_2^2 r_1 r_2]$

$= (i\xi_3)\{-\beta_1 r_1[\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]\,[ac_1\xi_2^2 + (\nu_1 + \beta_1 c_1)\xi_3^2] - ar\xi_2^2[\nu_1(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2) - ar_1 r_2(\nu_1 + \beta_1 c_1)] + r(\nu_1 + \beta_1 c(r))\,[(a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2)\,(a\xi_2^2 + \beta_1\xi_3^2) - a^2\xi_2^2 r_1 r_2]\}$

$= (i\xi_3)\{-\beta_1 r_1[\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]\,[ac_1\xi_2^2 + (\nu_1 + \beta_1 c_1)\xi_3^2] + \beta_1 r[a\xi_2^2 + (\tau_1 k_1 - Xc_1)\xi_3^2]\,[ac(r)\xi_2^2 + (\nu_1 + \beta_1 c(r))\xi_3^2] + a^2 r_1 r_2 r\xi_2^2\beta_1(c_1 - c(r))\}$

Case iv) Let

$$\tilde{\mathrm{C}}_\xi(r) = \begin{bmatrix} ia\xi_2|\xi_2| & 0 & \tau_1\beta_3 r^3 + ((\tau_1 - \beta_3)\gamma - \tau_1^2)\xi_2^2 r \\ a\xi_2^2 & 0 & -i\xi_2(\beta_3\gamma r^2 + \nu_3\tau_1\xi_2^2) \\ 0 & -\beta_1|\xi_2| & 0 \end{bmatrix}$$

Then by expanding det C_ξ in Case iv) about its 3$^{\text{rd}}$ column shows det $C_\xi = \frac{\partial}{\partial r}$ det $\tilde{\mathrm{C}}_\xi(r)|_{r=r_1}$.

det $\tilde{\mathrm{C}}_\xi(r) = a\beta_1|\xi_2|\xi_2^2[|\xi_2|(\beta_3\gamma r^2 + \nu_3\tau_1\xi_2^2) - (\tau_1\beta_3 r^3 + ((\tau_1 - \beta_3)\gamma - \tau_1^2)\xi_2^2 r)]$.

Then differentiating $\frac{\partial}{\partial r}$ at $r = r_1 = -|\xi_2| \implies$

det $C_\xi = a\beta_1|\xi_2|\xi_2^4\,[-\tau_1^2 + (3\beta_3 + \gamma)\tau_1 + \beta_3\gamma]$.

$\square$

In the next Theorem we summarize the results of this Chapter. By combining Theorems 5.3, 5.4 we have

Theorem 5.5.

Assume C in Theorem 4.1, C is strongly elliptic and $n = e_1$. Let $k_1, k_3 \in \mathbb{C}$ be the roots of $\beta_1\tau_1 k^2 - \lambda k + \beta_1\tau_3 = 0, \quad \lambda = \beta_1^2 + \tau_1\tau_3 - N^2$. Denote $a = \tau_1 - \gamma$. All Re $(...)^{\frac{1}{2}} > 0$.

a) Assume $N \neq 0, k_1 \neq k_3$. Then (C, n) satisfies the Complementing Condition iff

$a[-\tau_1^2 + (3\beta_3 + \gamma)\tau_1 + \beta_3\gamma] \neq 0$ and

$(x + k_1)^{\frac{1}{2}}[ax + (\tau_1 k_3 - Xc_3)][ac_1 x + (\nu_1 + \beta_1 c_1)] - (x + k_3)^{\frac{1}{2}}[ax + (\tau_1 k_1 - Xc_1)][ac_3 x + (\nu_1 + \beta_1 c_3)] - a^2(c_1 - c_3)x(x + k_1)^{\frac{1}{2}}(x + k_3)^{\frac{1}{2}}(x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}} \neq 0$

for all $x \in [0, \infty)$. Here $c_i = \frac{\tau_1 k_i - \beta_1}{N}, \quad i = 1, 3$.

b) Assume $N = 0$. Then (C, n) satisfies the Complementing Condition iff

$a[-\tau_1^2 + (3\beta_3 + \gamma)\tau_1 + \beta_3\gamma] \neq 0$ and

$X\nu_1(x + k_1)^{\frac{1}{2}} + (x + k_3)^{\frac{1}{2}}(ax + \beta_1)^2 - a^2 x(x + k_1)^{\frac{1}{2}}(x + k_3)^{\frac{1}{2}}(x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}} \neq 0$

for all $x \in [0, \infty)$. Here $k_1 = \frac{\beta_1}{\tau_1}, k_3 = \frac{\tau_3}{\beta_1}$.

c) Assume $N \neq 0, k_1 = k_3$. Then (C, n) satisfies the Complementing Condition iff

$a[-\tau_1^2 + (3\beta_3 + \gamma)\tau_1 + \beta_3\gamma] \neq 0$ and

$\frac{\partial}{\partial k}\{(x + k_1)^{\frac{1}{2}}[ax + (\tau_1 k - X\hat{c}(k))][ac_1 x + (\nu_1 + \beta_1 c_1)] - (x + k)^{\frac{1}{2}}[ax + (\tau_1 k_1 - Xc_1)][a\hat{c}(k)x + (\nu_1 + \beta_1\hat{c}(k))] + \frac{a^2\tau_1}{N}(k_1 - k)x(x + k_1)^{\frac{1}{2}}(x + k_3)^{\frac{1}{2}}(x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}}\}|_{k=k_1} \neq 0$

for all $x \in [0, \infty)$. Here $\hat{c}(k) = \frac{\tau_1 k - \beta_1}{N}$.

Proof. We use the results of Theorem 5.3, 5.4 and require det $C_\xi \neq 0$ for all possible $(\xi_2, \xi_3) \in \mathbb{R}^2 \setminus \{0\}$ in order for the Complementing Condition to be satisfied. In particular, when $\xi_3 = 0$, Theorem 5.4 (iv) implies

$$a[-\tau_1^2 + (3\beta_3 + \gamma)\tau_1 + \beta_3\gamma] \neq 0 \tag{31}$$

in all cases a), b), c).

a) $N \neq 0, k_1 \neq k_3$.

In Theorem 5.4 (i) we put $x = \frac{\xi_2^2}{\xi_3^2} \in [0, \infty)$.

Then $r_i = -(x + k_i)^{\frac{1}{2}}|\xi_3|$, $i = 1, 3$, $r_2 = -(x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}}|\xi_3|$ and $\det C_\xi \neq 0$ iff

$(x + k_1)^{\frac{1}{2}}[ax + (\tau_1 k_3 - Xc_3)][ac_1 x + (\nu_1 + \beta_1 c_1)] - (x + k_3)^{\frac{1}{2}}[ax + (\tau_1 k_1 - Xc_1)][ac_3 x + (\nu_1 + \beta_1 c_3)] - a^2(c_1 - c_3)x(x + k_1)^{\frac{1}{2}}(x + k_3)^{\frac{1}{2}}(x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}} \neq 0.$

This must hold for all $x \in [0, \infty)$. The case $\xi_3 = 0$ is in equation (31). Also $\operatorname{Re} r_i < 0 \Leftrightarrow \operatorname{Re} (x + k_i)^{\frac{1}{2}} > 0$, $\operatorname{Re} (x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}} > 0$.

b) $N = 0$.

In Theorem 5.4 (ii) we put $x = \frac{\xi_2^2}{\xi_3^2} \in [0, \infty)$ $\quad r_i = -(x + k_i)^{\frac{1}{2}}|\xi_3|$,

$k_1 = \frac{\beta_1}{\tau_1}, k_2 = \frac{\beta_1}{\beta_3}, k_3 = \frac{\tau_3}{\beta_1}$ in $\det C_\xi$.

The case $\xi_3 = 0$ is in equation (31).

c) $N \neq 0, k_1 = k_3$.

In Theorem 5.4 (iii) we put $x = \frac{\xi_2^2}{\xi_3^2} \in [0, \infty)$ and
$r = -(x + k)^{\frac{1}{2}}|\xi_3|$, $k > 0$, $\operatorname{Re} (x + k)^{\frac{1}{2}} > 0$. Then $\frac{\partial}{\partial r} = (2r\xi_3^{-2})\frac{\partial}{\partial k}$ and
$r = r_1 \Leftrightarrow k = k_1$. Also we put $c(r) = \frac{\tau_1 \rho(r) - \beta_1 \xi_3^2}{N\xi_3^2} = \frac{\tau_1(r^2 - \xi_2^2) - \beta_1 \xi_3^2}{N\xi_3^2} = \frac{\tau_1 k - \beta_1}{N} = \hat{c}(k)$

so $\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2 = (\tau_1 - \gamma)\xi_2^2 + (\tau_1 k - X\hat{c}(k))\xi_3^2 = [ax + (\tau_1 k - X\hat{c}(k))]\xi_3^2$,

$c_1 - c(r) = \frac{\tau_1 k_1 - \beta_1}{N} - \frac{\tau_1 k - \beta_1}{N} = \frac{\tau_1}{N}(k_1 - k)$.

Then $\det C_\xi \neq 0$ in Theorem 5.4 (iii) iff

$\frac{\partial}{\partial k}\{(x + k_1)^{\frac{1}{2}}[ax + (\tau_1 k - X\hat{c}(k))][ac_1 x + (\nu_1 + \beta_1 c_1)] - (x + k)^{\frac{1}{2}}[ax + (\tau_1 k_1 - Xc_1)][a\hat{c}(k)x + (\nu_1 + \beta_1 \hat{c}(k))] + \frac{a^2\tau_1}{N}(k_1 - k)x(x + k_1)^{\frac{1}{2}}(x + k_3)^{\frac{1}{2}}(x + \frac{\beta_1}{\beta_3})^{\frac{1}{2}}\}|_{k=k_1} \neq 0.$

This must hold for all $x \in [0, \infty)$. The case $\xi_3 = 0$ is in equation (31). We note k is a real variable and $k_1 \in \mathbb{R}$ (since k_1 is a double root of equation (22) which has real coefficients).

□

Chapter 6

Calculating the Complementing Condition when $\mathbf{n} = \mathbf{e}_3$

In this Chapter we calculate the Complementing Condition for (C, n) with $n = e_3$ and C in Theorem 4.1. The calculations are similar to those in Chapter 5.

$$n = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix}$$

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}, t \geq 0.$$

Let

$$H = z' \otimes n + iz \otimes \xi = \begin{bmatrix} i\xi_1 z_1 & i\xi_2 z_1 & z_1' \\ i\xi_1 z_2 & i\xi_2 z_2 & z_2' \\ i\xi_1 z_3 & i\xi_2 z_3 & z_3' \end{bmatrix}$$

Then by equation (7) and the formula for $C[H]$ in Theorem 4.1, we have

$$
\begin{aligned}
\mathscr{L}_\xi[z]_1 &= C[H']_{13} + iC[H]_{11}\xi_1 + iC[H]_{12}\xi_2 \\
&= \beta_1 z_1'' + i\xi_1(X+\nu_1)z_3' - (\tau_1\xi_1^2 + \beta_3\xi_2^2)z_1 - \xi_1\xi_2(\nu_3+\gamma)z_2 \\
\mathscr{L}_\xi[z]_2 &= C[H']_{23} + iC[H]_{21}\xi_1 + iC[H]_{22}\xi_2 \\
&= \beta_1 z_2'' + i\xi_2(X+\nu_1)z_3' - \xi_1\xi_2(\nu_3+\gamma)z_1 - (\beta_3\xi_1^2 + \tau_1\xi_2^2)z_2 \\
\mathscr{L}_\xi[z]_3 &= C[H']_{33} + iC[H]_{31}\xi_1 + iC[H]_{32}\xi_2 \\
&= \tau_3 z_3'' + i\xi_1(X+\nu_1)z_1' + i\xi_2(X+\nu_1)z_2' - \beta_1(\xi_1^2+\xi_2^2)z_3 \\
\mathscr{B}_\xi[z]_1 &= C[H]_{13} = \beta_1 z_1' + i\xi_1\nu_1 z_3 \\
\mathscr{B}_\xi[z]_2 &= C[H]_{23} = \beta_1 z_2' + i\xi_2\nu_1 z_3 \\
\mathscr{B}_\xi[z]_3 &= C[H]_{33} = i\xi_1 X z_1 + i\xi_2 X z_2 + \tau_3 z_3'
\end{aligned}
\tag{32}
$$

$\mathscr{L}_\xi[z] = 0 \Leftrightarrow$

$$
\begin{aligned}
\beta_1 z_1'' + i\xi_1(X+\nu_1)z_3' - (\tau_1\xi_1^2 + \beta_3\xi_2^2)z_1 - \xi_1\xi_2(\nu_3+\gamma)z_2 &= 0 \\
\beta_1 z_2'' + i\xi_2(X+\nu_1)z_3' - \xi_1\xi_2(\nu_3+\gamma)z_1 - (\beta_3\xi_1^2 + \tau_1\xi_2^2)z_2 &= 0 \\
\tau_3 z_3'' + i\xi_1(X+\nu_1)z_1' + i\xi_2(X+\nu_1)z_2' - \beta_1(\xi_1^2+\xi_2^2)z_3 &= 0
\end{aligned}
\tag{33}
$$

for $t \geq 0$

$\mathscr{B}_\xi[z] = 0 \Leftrightarrow$

$$
\begin{aligned}
\beta_1 z_1' + i\xi_1\nu_1 z_3 &= 0 \\
\beta_1 z_2' + i\xi_2\nu_1 z_3 &= 0 \\
\tau_3 z_3' + i\xi_1 X z_1 + i\xi_2 X z_2 &= 0
\end{aligned}
\tag{34}
$$

at $t = 0$

We seek decaying exponential solutions of equation (18) of the form

$$
z(t) = e^{rt}p, \qquad t \geq 0 \tag{35}
$$

for $r \in \mathbb{C}$, $\text{Re } r < 0, p \in \mathbb{C}^3$.

As in equation (18)

$A(r) = M(rn + i\xi)$,

$B(r)p = C[p \otimes (rn + i\xi)]n$ for $r \in \mathbb{C}, p \in \mathbb{C}^3$ and

$$\begin{aligned} \mathscr{L}_\xi[e^{rt}p] &= A(r)pe^{rt}, \qquad t \geq 0, \\ \mathscr{B}_\xi[e^{rt}p] &= B(r)p, \qquad t = 0 \end{aligned} \tag{36}$$

Then equation (33) implies

$$A(r)p = 0. \tag{37}$$

Substituting equation (35) into equation (32) with $\nu_3 + \gamma = \tau_1 - \beta_3, X + \nu_1 = N$ and using equation (36) yields the formula for $A(r), B(r)$:

$$A(r) = \begin{bmatrix} \beta_1 r^2 - (\tau_1\xi_1^2 + \beta_3\xi_2^2) & -\xi_1\xi_2(\tau_1 - \beta_3) & i\xi_1 N r \\ -\xi_1\xi_2(\tau_1 - \beta_3) & \beta_1 r^2 - (\beta_3\xi_1^2 + \tau_1\xi_2^2) & i\xi_2 N r \\ i\xi_1 N r & i\xi_2 N r & \tau_3 r^2 - \beta_1(\xi_1^2 + \xi_2^2) \end{bmatrix}$$

$$B(r) = \begin{bmatrix} \beta_1 r & 0 & i\xi_1\nu_1 \\ 0 & \beta_1 r & i\xi_2\nu_1 \\ i\xi_1 X & i\xi_2 X & \tau_3 r \end{bmatrix}$$

We are looking for nontrivial solutions (35) of equation (33) which requires $p \neq 0$ in equation (37) $\implies$

$$\det A(r) = 0 \tag{38}$$

Let

S = set of solutions of equation (33) on $[0, \infty)$ which are exponentially decreasing as $t \to \infty$.

By Theorem 3.3, dim $S = 3$.

Theorem 6.1.

$$\det A(r) = (\beta_1 r^2 - \beta_3|\xi|^2)(\beta_1\tau_3 r^4 - \lambda|\xi|^2 r^2 + \beta_1\tau_1|\xi|^4), \tag{39}$$

with $\lambda = \beta_1^2 + \tau_1\tau_3 - N^2, |\xi|^2 = \xi_1^2 + \xi_2^2$.
$\det A(r)$ has roots $\pm r_1, \pm r_2, \pm r_3 \in \mathbb{C}$ with $\mathrm{Re}\ r_i < 0, i = 1, 2, 3$.

Case i) $N \neq 0, r_2 \neq r_3$

$r_1^2 = \frac{\beta_3}{\beta_1}|\xi|^2, r_2^2 = k_2|\xi|^2, r_3^2 = k_3|\xi|^2$.

$k_2, k_3 \in \mathbb{C}$ are the roots of

$$\beta_1\tau_3 k^2 - \lambda k + \beta_1\tau_1 = 0 \tag{40}$$

$\lambda = \beta_1^2 + \tau_1\tau_3 - N^2$

With

$$p_1 = \begin{bmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} \xi_1 \\ \xi_2 \\ ic_2 r_2 \end{bmatrix}, \quad p_3 = \begin{bmatrix} \xi_1 \\ \xi_2 \\ ic_3 r_3 \end{bmatrix} \tag{41}$$

$c_i = \frac{\beta_1 k_i - \tau_1}{N k_i}, \quad i = 2, 3$

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3\}$

Case ii) $N = 0$

$r_1^2 = \frac{\beta_3}{\beta_1}|\xi|^2, \quad r_2^2 = \frac{\tau_1}{\beta_1}|\xi|^2, \quad r_3^2 = \frac{\beta_1}{\tau_3}|\xi|^2$

$$p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix}$$

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3\}$.

Case iii) $N \neq 0, r_2 = r_3$

$r_1^2 = \frac{\beta_3}{\beta_1}|\xi|^2, r_2^2 = k_2|\xi|^2, r_3^2 = k_3|\xi|^2$ with k_2, k_3 as in Case i). With p_1, p_2 as in equation (41) and

$$p(r) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ ic(r) \end{bmatrix}$$

$c(r) = \frac{\beta_1 r^2 - \tau_1|\xi|^2}{Nr}$

We have $S = span\{e^{r_1 t}p_1, e^{r_2 t}p_2, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_2}\}$

Proof. To find det $A(r)$ we reduce $A(r)$ by row and column operations via $E(r)$ to a simpler matrix $D(r)$.

Let

$$E(r) = \begin{bmatrix} 1 & -\xi_2 & 0 \\ 0 & \xi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D(r) = E(r)^T A(r) E(r) = \begin{bmatrix} \eta_1(r) & -\xi_2\eta_2(r) & i\xi_1 Nr \\ -\xi_2\eta_2(r) & |\xi|^2\eta_2(r) & 0 \\ i\xi_1 Nr & 0 & \eta_3(r) \end{bmatrix}$$

where

$$|\xi|^2 = \xi_1^2 + \xi_2^2$$

$$\eta_1(r) = \beta_1 r^2 - (\tau_1\xi_1^2 + \beta_3\xi_2^2)$$

$$\eta_2(r) = \beta_1 r^2 - \beta_3|\xi|^2$$

$$\eta_3(r) = \tau_3 r^2 - \beta_1|\xi|^2$$

Expanding the determinant of $D(r)$ about the third row $\Longrightarrow$

$\det D(r) = [\det E(r)]^2 \det A(r) = \eta_2[\xi_1^2 N^2 r^2|\xi|^2 + \eta_1\eta_3|\xi|^2 + \xi_2^2\eta_2\eta_3)]$

$\det E(r) = \xi_1$.

Substituting for η_1, η_2, η_3, we get :

$\det A(r) = (\beta_1 r^2 - \beta_3|\xi|^2)(\beta_1\tau_3 r^4 - \lambda|\xi|^2 r^2 + \beta_1\tau_1|\xi|^4) = 0, \quad \lambda = \beta_1^2 + \tau_1\tau_3 - N^2$

Let the roots of equation (39) be $\pm r_1, \pm r_2, \pm r_3 \in \mathbb{C}$, Re $r_i < 0$.

i) r_1 satisfies

$$\beta_1 r^2 - \beta_3|\xi|^2 = 0 \Longrightarrow r_1^2 = \frac{\beta_3}{\beta_1}|\xi|^2 \tag{42}$$

ii) r_2, r_3 satisfy $\beta_1\tau_3 r^4 - \lambda|\xi|^2 r^2 + \beta_1\tau_1|\xi|^4 = 0$ and putting $k_2 = \frac{r_2^2}{|\xi|^2}, k_3 = \frac{r_3^2}{|\xi|^2}$,

this is equivalent to $k_2, k_3 \in \mathbb{C}$ satisfying equation (40) :

We have

$$r_1^2 = \frac{\beta_3}{\beta_1}|\xi|^2, r_2^2 = k_2|\xi|^2, r_3^2 = k_3|\xi|^2 \tag{43}$$

Case i): $N \neq 0, r_2 \neq r_3$
The roots r_1, r_2, r_3 are in equation (43).
Next we find the null vectors p_i of $A(r_i)$ satisfying equation (37). This is equivalent to finding q_i such that $D(r_i)q_i = 0$,
then putting $p_i =$ const. $E(r_i)q_i$

Finding the null vector p_1 :

$r = r_1 \Longrightarrow \eta_2(r_1) = 0 \Longrightarrow$

$$D(r_1) = \begin{bmatrix} \eta_1 & 0 & i\xi_1 N r_1 \\ 0 & 0 & 0 \\ i\xi_1 N r_1 & 0 & \eta_3 \end{bmatrix}$$

Here $\eta_i = \eta_i(r_i)$.

Then $q_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow$

$$p_1 = E(r_1)q_1 = \begin{bmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{bmatrix}$$

Finding the null vectors p_2, p_3 :

Letting $r = r_i = k_i|\xi|^2, \quad i = 2,3$ in $D(r_i)$, we can take

$$q_i = \begin{bmatrix} |\xi|^2 \\ \xi_2 \\ ic_i r_i \xi_1 \end{bmatrix} \quad \text{where } c_i = \tfrac{\beta_1 k_i - \tau_1}{N k_i}.$$

$$p_i = \text{ const. } E(r_i)q_i = \text{ const. } \begin{bmatrix} \xi_1^2 \\ \xi_1\xi_2 \\ ic_i r_i \xi_1 \end{bmatrix}$$

Putting the constant $= \xi_1^{-1} \Longrightarrow$

$$p_i = \begin{bmatrix} \xi_1 \\ \xi_2 \\ ic_i r_i \end{bmatrix}, \quad i = 2,3$$

Thus $e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3$ are exponentially decreasing solutions of equation (33) in

S. They are linearly independent on $[0,\infty)$ since $r_2 \neq r_3$ and $\{p_1, p_2\}, \{p_1, p_3\}$are linearly independent in $\mathbb{C}^3$. Then S is spanned by $e^{r_i t} p_i, i = 1, 2, 3$.

Case ii) : $N = 0$

Equation (40) becomes $(\beta_1 k - \tau_1)(\tau_3 k - \beta_1) = 0$ so $k_2 = \frac{\tau_1}{\beta_1}, k_3 = \frac{\beta_1}{\tau_3}$. The roots are in equation (43). To find null vectors $p_i \in \mathbb{C}^3$ of $A(r_i)$ we have

$$A(r) = \begin{bmatrix} \beta_1 r^2 - (\tau_1 \xi_1^2 + \beta_3 \xi_2^2) & -\xi_1 \xi_2 (\tau_1 - \beta_3) & 0 \\ -\xi_1 \xi_2 (\tau_1 - \beta_3) & \beta_1 r^2 - (\beta_3 \xi_1^2 + \tau_1 \xi_2^2) & 0 \\ 0 & 0 & \tau_3 r^2 - \beta_1 (\xi_1^2 + \xi_2^2) \end{bmatrix}$$

We get

$$p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad p_1 = \begin{bmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix}$$

Since p_1, p_2, p_3 are linearly independent in $\mathbb{C}^3$, then $e^{r_1 t} p_1, e^{r_2 t} p_2, e^{r_3 t} p_3$ are linearly independent on $[0,\infty)$ and they form a basis for S.

Case iii) : $N \neq 0, r_2 = r_3$

The roots r_i are as in Case i). With p_1, p_2 in equation (41), $e^{r_1 t} p_1, e^{r_2 t} p_2 \in S$ are linearly independent on $[0,\infty)$. A third linearly independent solution in S is $\frac{\partial}{\partial r}[e^{rt} p(r)]|_{r=r_2}$ where

$$p(r) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ ic(r) \end{bmatrix}, \quad c(r) = \frac{\beta_1 r^2 - \tau_1 |\xi|^2}{Nr}$$

This follows from

$$A(r)p(r) = \frac{i}{Nr}(\beta_1\tau_3 r^4 - \lambda|\xi|^2 r^2 + \tau_1\beta_1|\xi|^4)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But since $r_2 = r_3$ then $k_2 = k_3$ is a double root of equation (40) so $\beta_1\tau_3 k^2 - \lambda k + \beta_1\tau_1 = \beta_1\tau_3(k - k_2)^2$ and similarly

$\beta_1\tau_3 r^4 - \lambda|\xi|^2 r^2 + \tau_1\beta_1|\xi|^4 = \beta_1\tau_3(r^2 - r_2^2)^2$.

Thus $A(r)p(r) = \frac{i\beta_1\tau_3}{Nr}(r^2 - r_2^2)^2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So $A(r)p(r)|_{r=r_2} = \frac{\partial}{\partial r}[A(r)p(r)]|_{r=r_2} = 0$.

As in the proof of Case iii) in Theorem 5.1 we see $\frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_2}$ is a solution in S. Also it equals $te^{r_2 t}p_2 + e^{r_2 t}p'(r_2)$ since

$c(r_2) = \frac{\tau_1 r_2^2 - \tau_1|\xi|^2}{Nr_2} = \frac{\tau_1 k_2 - \tau_1}{Nk_2} r_2 = c_2 r_2 \implies p(r_2) = p_2$. Since $p_1, p_2 \in \mathbb{C}^3$ are linearly independent then $e^{r_1 t}p_1, e^{r_2 t}p_2, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_2}$ are linearly independent on $[0, \infty)$ and form a basis for S.

□

Theorem 6.1 implies equation (33) has three linearly independent solutions $z^{(1)}(t), z^{(2)}(t), z^{(3)}(t)$ on $[0, \infty)$ which are exponentially decreasing.

We substitute these into $\mathscr{B}_\xi$ in equation (32) at $t = 0$,

$$\mathscr{B}_\xi[z] = \begin{bmatrix} \beta_1 r & 0 & i\xi_1\nu_1 \\ 0 & \beta_1 r & i\xi_2\nu_1 \\ i\xi_1 X & i\xi_2 X & \tau_3 r \end{bmatrix}$$

and

$C_\xi = [\mathscr{B}_\xi[z^{(1)}]|_{t=0}, \mathscr{B}_\xi[z^{(2)}]|_{t=0}, \mathscr{B}_\xi[z^{(3)}]|_{t=0}]$;

C_ξ is a 3×3 matrix.

In the following Theorem we find C_ξ using Theorem 6.1.

Theorem 6.2

Case i) $N \neq 0, r_2 \neq r_3$

$$C_\xi = \begin{bmatrix} -\beta_1 r_1 \xi_2 & r_2(\beta_1 - \nu_1 c_2)\xi_1 & r_3(\beta_1 - \nu_1 c_3)\xi_1 \\ \beta_1 r_1 \xi_1 & r_2(\beta_1 - \nu_1 c_2)\xi_2 & r_3(\beta_1 - \nu_1 c_3)\xi_2 \\ 0 & i(\tau_3 c_2 r_2^2 + X|\xi|^2) & i(\tau_3 c_3 r_3^2 + X|\xi|^2) \end{bmatrix}$$

$c_i = \frac{\beta_1 k_i - \tau_1}{N k_i}$, $i = 2, 3$

Case ii) $N = 0$

$$C_\xi = \begin{bmatrix} -\beta_1 r_1 \xi_2 & \beta_1 r_2 \xi_1 & i\nu_1 \xi_1 \\ \beta_1 r_1 \xi_1 & \beta_1 r_2 \xi_2 & i\nu_1 \xi_2 \\ 0 & iX|\xi|^2 & \tau_3 r_3 \end{bmatrix}$$

Case iii) $N \neq 0, r_2 = r_3$

$$C_\xi = \begin{bmatrix} -\beta_1 r_1 \xi_2 & r_2(\beta_1 - \nu_1 c_2)\xi_1 & \frac{\partial}{\partial r}[(\beta_1 r - \nu_1 c(r))\xi_1]|_{r=r_2} \\ \beta_1 r_1 \xi_1 & r_2(\beta_1 - \nu_1 c_2)\xi_2 & \frac{\partial}{\partial r}[(\beta_1 r - \nu_1 c(r))\xi_2]|_{r=r_2} \\ 0 & i(\tau_3 c_2 r_2^2 + X|\xi|^2) & \frac{\partial}{\partial r}[i(\tau_3 c(r) r + X|\xi|^2)]|_{r=r_2} \end{bmatrix}$$

$c(r) = \frac{\beta_1 r^2 - \tau_1 |\xi|^2}{N r}$.

Proof. In the following we use equation (18) $\mathscr{B}_\xi[e^{rt}p] = B(r)p$ at $t = 0$ and equation (30)

$$\mathscr{B}_\xi[\frac{\partial}{\partial r}[e^{rt}p(r)]] = \frac{\partial}{\partial r}[B(r)p(r)] \text{ at } t = 0.$$

Case i) : $N \neq 0, r_2 \neq r_3$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} -\beta_1 r_1 \xi_2 \\ \beta_1 r_1 \xi_1 \\ 0 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} r_2(\beta_1 - \nu_1 c_2)\xi_1 \\ r_2(\beta_1 - \nu_1 c_2)\xi_2 \\ i(\tau_3 c_2 r_2^2 + X|\xi|^2) \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(3)}] = B(r_3)p_3 = \begin{bmatrix} r_3(\beta_1 - \nu_1 c_3)\xi_1 \\ r_3(\beta_1 - \nu_1 c_3)\xi_2 \\ i(\tau_3 c_3 r_3^2 + X|\xi|^2) \end{bmatrix}$$

Case ii): $N = 0$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} -\beta_1 r_1 \xi_2 \\ \beta_1 r_1 \xi_1 \\ 0 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} \beta_1 r_2 \xi_1 \\ \beta_1 r_2 \xi_2 \\ iX|\xi|^2 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(3)}] = B(r_3)p_3 = \begin{bmatrix} i\nu_1\xi_1 \\ i\nu_1\xi_2 \\ \tau_3 r_3 \end{bmatrix}$$

Case iii) : $N \neq 0, r_2 = r_3$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} -\beta_1 r_1 \xi_2 \\ \beta_1 r_1 \xi_1 \\ 0 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} r_2(\beta_1 - \nu_1 c_2)\xi_1 \\ r_2(\beta_1 - \nu_1 c_2)\xi_2 \\ i(\tau_3 c_2 r_2^2 + X|\xi|^2) \end{bmatrix}$$

From equation (30) $\mathscr{B}_\xi[z^{(3)}] = \frac{\partial}{\partial r}[B(r)p(r)]|_{r=r_2}$ where

$$B(r)p(r) = \begin{bmatrix} (\beta_1 r - \nu_1 c(r))\xi_1 \\ (\beta_1 r - \nu_1 c(r))\xi_2 \\ i(X|\xi|^2 + c(r)\tau_3 r) \end{bmatrix}$$

$\square$

Theorem 6.3. We calculate det C_ξ

Case i) det C_ξ =

$$\frac{i\beta_1 r_1|\xi|^4}{N} \ \{[\tau_3\beta_1 k_2 + (XN - \tau_1\tau_3)]\ (\beta_1 - \nu_1 c_3)r_3 - [\tau_3\beta_1 k_3 + (XN - \tau_1\tau_3)]\ (\beta_1 - \nu_1 c_2)r_2\}$$

Case ii)
det $C_\xi = -\beta_1 r_1 |\xi|^4 \ [X\nu_1 + \beta_1\sqrt{\tau_1\tau_3}]$

Case iii) det C_ξ

$= i\beta_1 r_1 |\xi|^2 \frac{\partial}{\partial r} \left\{ \left[\tau_3 c_2 r_2^2 + X|\xi|^2\right] \left(\beta_1 r - \nu_1 c(r)\right) - \left[\tau_3 c(r) r + X|\xi|^2\right] \left(\beta_1 - \nu_1 c_2\right) r_2 \right\}\Big|_{r=r_2}$

$c(r) = \frac{\beta_1 r^2 - \tau_1 |\xi|^2}{N r}$

Proof. **Case i)** Expanding det C_ξ about the 3rd row $\implies$

det $C_\xi = -i(\tau_3 c_2 r_2^2 + X|\xi|^2)\left[\beta_1 r_1 r_3 (\beta_1 - \nu_1 c_3)(-\xi_2^2 - \xi_1^2)\right] + i(\tau_3 c_3 r_3^2 + X|\xi|^2)\left[\beta_1 r_1 r_2 (\beta_1 - \nu_1 c_2)(-\xi_2^2 - \xi_1^2)\right]$

$= i\beta_1 r_1 |\xi|^2 \left[(\tau_3 c_2 r_2^2 + X|\xi|^2)(\beta_1 - \nu_1 c_3) r_3 - (\tau_3 c_3 r_3^2 + X|\xi|^2)(\beta_1 - \nu_1 c_2) r_2\right]$

$= i\beta_1 r_1 |\xi|^4 \left[\left(\frac{\tau_3 \beta_1}{N} k_2 + \left(X - \frac{\tau_1 \tau_3}{N}\right)\right)(\beta_1 - \nu_1 c_3) r_3 - \left(\frac{\tau_3 \beta_1}{N} k_3 + \left(X - \frac{\tau_1 \tau_3}{N}\right)\right)(\beta_1 - \nu_1 c_2) r_2\right]$

Case ii) Expanding det C_ξ about the 3rd row $\implies$

det $C_\xi = -iX|\xi|^2 \left[(-\beta_1 r_1 \xi_2)(i\nu_1 \xi_2) - (\beta_1 r_1 \xi_1)(i\nu_1 \xi_1)\right] + \tau_3 r_3 \left[(-\beta_1 r_1 \xi_2)(\beta_1 r_2 \xi_2) - (\beta_1 r_1 \xi_1)(\beta_1 r_2 \xi_1)\right]$

$= -X|\xi|^2 \beta_1 \nu_1 r_1 (\xi_2^2 + \xi_1^2) - \tau_3 r_3 \beta_1^2 r_1 r_2 (\xi_2^2 + \xi_1^2)$

$= -\beta_1 r_1 |\xi|^2 \left[X\nu_1 |\xi|^2 + \tau_3 \beta_1 r_2 r_3\right]$

$= -\beta_1 r_1 |\xi|^4 \left[X\nu_1 + \beta_1 \sqrt{\tau_1 \tau_3}\right]$

using $r_2 = -\sqrt{\frac{\tau_1}{\beta_1}}|\xi|,\ r_3 = -\sqrt{\frac{\beta_1}{\tau_3}}|\xi|$

Case iii) Let

$$C_\xi(r) = \begin{bmatrix} -\beta_1 r_1 \xi_2 & r_2(\beta_1 - \nu_1 c_2)\xi_1 & (\beta_1 r - \nu_1 c(r))\xi_1 \\ \beta_1 r_1 \xi_1 & r_2(\beta_1 - \nu_1 c_2)\xi_2 & (\beta_1 r - \nu_1 c(r))\xi_2 \\ 0 & i(\tau_3 c_2 r_2^2 + X|\xi|^2) & i(\tau_3 c(r) r + X|\xi|^2) \end{bmatrix}$$

Expanding det C_ξ in Case iii) about its 3rd column, we see det $C_\xi = \frac{\partial}{\partial r}$ det $C_\xi(r)|_{r=r_2}$.

Expanding det $C_\xi(r)$ about the 3rd row

det $C_\xi(r) = -i(\tau_3 c_2 r_2^2 + X|\xi|^2)\left[\beta_1 r_1 (\beta_1 r - \nu_1 c(r))(-\xi_2^2 - \xi_1^2)\right] + i(\tau_3 c(r) r + X|\xi|^2)\left[\beta_1 r_1 r_2 (\beta_1 - \nu_1 c_2)(-\xi_2^2 - \xi_1^2)\right]$

$$= i\beta_1 r_1|\xi|^2 \left[\left(\tau_3 c_2 r_2^2 + X|\xi|^2\right)\left(\beta_1 r - \nu_1 c(r)\right) - \left(\tau_3 c(r) r + X|\xi|^2\right)\left(\beta_1 - \nu_1 c_2\right) r_2 \right]$$

□

Theorem 6.4.

Assume C in Theorem 4.1, C is strongly elliptic and $n = e_3$. Let $k_2, k_3 \in \mathbb{C}$ be the roots of $\beta_1\tau_3 k^2 - \lambda k + \beta_1\tau_1 = 0, \quad \lambda = \beta_1^2 + \tau_1\tau_3 - N^2$.

a) Assume $N \neq 0, k_2 \neq k_3$.

Then (C, n) satisfies the Complementing Condition iff

$$[\tau_3\beta_1 k_2 + (XN - \tau_1\tau_3)](\beta_1 - \nu_1 c_3)k_3^{\frac{1}{2}} - [\tau_3\beta_1 k_3 + (XN - \tau_1\tau_3)](\beta_1 - \nu_1 c_2)k_2^{\frac{1}{2}} \neq 0$$

Here Re $k_i^{\frac{1}{2}} > 0, \;\; c_i = \frac{\beta_1 k_i - \tau_1}{Nki}, \;\; i = 2, 3.$

b) Assume $N = 0$.

Then (C, n) satisfies the Complementing Condition iff $X\nu_1 + \beta_1\sqrt{\tau_1\tau_3} \neq 0$.

c) Assume $N \neq 0, k_2 = k_3$.

Then (C, n) satisfies the Complementing Condition iff

$$\frac{\partial}{\partial r}\{[\tau_3 c_2 k_2 + X](\beta_1 r - \nu_1 c(r)) + [\tau_3 c(r) r + X](\beta_1 - \nu_1 c_2)k_2^{\frac{1}{2}}\}|_{r=k_2} \neq 0$$

.

Here $k_2^{\frac{1}{2}} > 0, \quad c_2 = \frac{\beta_1 k_2 - \tau_1}{Nk_2}, \;\; c(r) = \frac{\beta_1 r^2 - \tau_1}{Nr}$.

Proof. We use Theorems 6.3, 5.3 and require det $C_\xi \neq 0$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ in order for the Complementing Condition to be satisfied.

a) $N \neq 0, k_2 \neq k_3$.

In Theorem 6.3 (i) we have $r_i = -k_i^{\frac{1}{2}}|\xi| \;\; (|\xi|^2 = \xi_1^2 + \xi_2^2), \;\; i = 2, 3$. Then Re $r_i < 0 \implies$ Re $k_i^{\frac{1}{2}} > 0$, and det $C_\xi \neq 0$ for all $|\xi| \neq 0$ gives a).

b) $N = 0$.

This follows from Theorem 6.3(ii).

c) $N \neq 0, k_2 = k_3$.

Here k_2 is real since it is a double root of equation (40) which has real coefficients. In Theorem 6.3(iii) we have $r_2 = -k_2^{\frac{1}{2}}|\xi|$ and Re $r_2 < 0 \implies k_2^{\frac{1}{2}} > 0$. Also by dividing r and r_2 by $|\xi|$ we may assume $|\xi| = 1$.

$\square$

Chapter 7

Case with Stored Energy Function where $\sigma_1 = 1$

Assume a stored energy function

$$W(F) = \frac{1}{2}\ F.F + \Psi(\det\ F),$$

i.e. $W(F) = \sigma(v_1, v_2, v_3) = v_1 + \Psi(v_3)$

where $v_1 = \frac{1}{2}F.F, v_3 =\ \det\ F, \quad F \in\ \mathrm{Lin}^+$ for some C^2–smooth function $\Psi : (0, \infty) \to \mathbb{R}$

Denote $\sigma_1 = \frac{\partial\sigma}{\partial v_1} = 1, \sigma_2 = \frac{\partial\sigma}{\partial v_2} \equiv 0, \sigma_3 = \frac{\partial\sigma}{\partial v_3} = \Psi'(v_3)$.

and $\sigma_{11} = 0, \sigma_{33} = \Psi''(v_3)$

The Piola-Kirchhoff stress, $S(F) = \sigma_1 F + \sigma_3 Cof F = \sigma_1 F + \sigma_3 \cdot (det F)F^{-T}$.

We linearize about $F_0 =\ \mathrm{diag}\ [\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}, \omega]$ for $\mu, \omega > 0$.

Then the elasticity tensor, $C[H] = \frac{d}{dt}S(F_0 + tH)|_{t=0}$

$= H + \sigma_3(\ \det\ F_0)[(F_0^{-T} \cdot H)F_0^{-T} - F_0^{-T}H^T F_0^{-T}] + \sigma_{33}(G^3 \cdot H)G^3,$

$G^3 = Cof F_0 = (\det\ F_0)F_0^{-T}$.

We have the constants

$Y = \sigma_{33}\mu\omega^2$

$V = \sigma_{33}\mu^2$

$Z = \sigma_{33}\mu^{\frac{3}{2}}\omega$

$X = \mu^{\frac{1}{2}}\sigma_3 + Z$

$\beta_1 = \beta_3 = 1$

$\tau_1 = 1 + Y$

$\tau_3 = 1 + V$

$\nu_1 = -\mu^{\frac{1}{2}}\sigma_3$

$\nu_3 = -\omega\sigma_3$

$\gamma = \omega\sigma_3 + Y$

$a = \tau_1 - \gamma = 1 - \omega\sigma_3,\ b = 1 - \frac{\mu}{\omega}\sigma_3$

Then $\nu_3 + \gamma = \tau_1 - \beta_3 = Y$, and

$X + \nu_1 = Z$

Theorems 4.1, 4.2 still hold with the the above constants.

We calculate the Complementing Condition for (C, n) with C in Theorem 4.1 and $n = e_1$. The calculations are similar to those in Chapters 5, 6.

Let $n = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\xi = \begin{bmatrix} 0 \\ \xi_2 \\ \xi_3 \end{bmatrix}$

$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}, t \in [0, \infty)$

We get exactly the equations (14), (15) :

$\mathscr{L}_\xi[z] = 0 \Leftrightarrow$

$$\begin{aligned} \tau_1 z_1'' - (\beta_3\xi_2^2 + \beta_1\xi_3^2)z_1 + i\xi_2(\nu_3 + \gamma)z_2' + i\xi_3(X + \nu_1)z_3' = 0 \\ \beta_3 z_2'' - (\tau_1\xi_2^2 + \beta_1\xi_3^2)z_2 + i\xi_2(\tau_1 - \beta_3)z_1' - \xi_2\xi_3(X + \nu_1)z_3 = 0 \\ \beta_1 z_3'' - (\beta_1\xi_2^2 + \tau_3\xi_3^2)z_3 + i\xi_3(X + \nu_1)z_1' - \xi_2\xi_3(X + \nu_1)z_2 = 0 \end{aligned} \tag{44}$$

for $t \geq 0$

$\mathscr{B}_\xi[z] = 0 \Leftrightarrow$

$$\begin{aligned} \tau_1 z_1' + i\gamma\xi_2 z_2 + iX\xi_3 z_3 = 0 \\ \beta_3 z_2' + i\nu_3\xi_2 z_1 = 0 \\ \beta_1 z_3' + i\nu_1\xi_3 z_1 = 0 \end{aligned} \tag{45}$$

at $t = 0$

We seek decaying exponential solutions of equation (44)

$$z(t) = e^{rt}p, \qquad t \geq 0 \tag{46}$$

for $r \in \mathbb{C}$, Re $r < 0, p \in \mathbb{C}^3$.

Exactly as in equation (18), (36) we have
$A(r) = M(rn + i\xi)$, $\;\; B(r)p = C[p(rn + i\xi)]p$ for $r \in \mathbb{C}, p \in \mathbb{C}^3$ and

$$\begin{aligned} \mathscr{L}_\xi[e^{rt}p] = A(r)pe^{rt}, \qquad t \geq 0, \\ \mathscr{B}_\xi[e^{rt}p] = B(r)p, \qquad t = 0 \end{aligned} \tag{47}$$

Then equation (44) implies

$$A(r)p = 0 \tag{48}$$

Substituting equation (46) into equation (13) with $\nu_3 + \gamma = \tau_1 - \beta_3 = Y, X + \nu_1 = Z$ and using equation (47) we get

$$A(r) = \begin{bmatrix} \tau_1 r^2 - (\xi_2^2 + \xi_3^2) & i\xi_2 Y r & i\xi_3 Z r \\ i\xi_2 Y r & r^2 - (\tau_1\xi_2^2 + \xi_3^2) & -\xi_2\xi_3 Z \\ i\xi_3 Z r & -\xi_2\xi_3 Z & r^2 - (\xi_2^2 + \tau_3\xi_3^2) \end{bmatrix},$$

$$B(r) = \begin{bmatrix} \tau_1 r & i\xi_2\gamma & i\xi_3 X \\ i\xi_2\nu_3 & r & 0 \\ i\xi_3\nu_1 & 0 & r \end{bmatrix},$$

As in Theorem 3.3 det $A(r)$ is a 6^{th} degree polynomial in r and it has exactly three roots $r_1, r_2, r_3 \in \mathbb{C}$ with Re $r_i < 0$. We let

S = set of solutions of equation (44) on $[0, \infty)$ which are exponentially decreasing as $t \to \infty$.

dim $S = 3$.

Theorem 7.1.

Letting $\rho = r^2 - \xi_2^2$,

$$\det\, A(r) = (\rho - \xi_3^2)^2(\tau_1\rho - \tau_3\xi_3^2). \tag{49}$$

det $A(r)$ has roots $\pm r_1, \pm r_2, \pm r_3 \in \mathbb{C}$ with Re $r_i < 0, i = 1, 2, 3$. There are 4 cases.

Case i) $\xi_3 \neq 0, Z \neq 0, r_1 \neq r_3$

$r_1^2 = r_2^2 = |\xi|^2 = \xi_2^2 + \xi_3^2, \quad r_3^2 = \xi_2^2 + \frac{\tau_3}{\tau_1}\xi_3^2$. With

$$p_1 = \begin{bmatrix} r_1 \\ i\xi_2 \\ i\frac{\omega}{\mu^{\frac{1}{2}}}\xi_3 \end{bmatrix}, \quad p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} r_3 \\ i\xi_2 \\ i\frac{\mu^{\frac{1}{2}}}{\omega}\xi_3 \end{bmatrix} \tag{50}$$

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3\}$.

Case ii) $\xi_3 \neq 0, Z = 0$

$r_1^2 = r_2^2 = r_3^2 = |\xi|^2 = \xi_2^2 + \xi_3^2$. With

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, e^{r_3 t}p_3\}$.

Case iii) $\xi_3 \neq 0, Z \neq 0, r_1 = r_3$

$r_1^2 = r_2^2 = |\xi|^2 = \xi_2^2 + \xi_3^2, \quad r_3^2 = \xi_2^2 + \frac{\tau_3}{\tau_1}\xi_3^2.$

With p_1, p_2 in equation (50) and

$$p(r) = \begin{bmatrix} r \\ i\xi_2 \\ ic(r)\xi_3 \end{bmatrix}, \quad c(r) = \frac{\tau_1\rho(r) - \xi_3^2}{Z\xi_3^2},$$

$\rho(r) = r^2 - \xi_2^2$ we have

$S =$ span $\{e^{r_1 t}p_1, e^{r_2 t}p_2, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_1}\}$.

Case iv) $\xi_3 = 0$

$r_1^2 = r_2^2 = r_3^2 = \xi_2^2$. With

$$p_2 = \begin{bmatrix} -i\xi_2 \\ r_2 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p(r) = \begin{bmatrix} r^2 - \tau_1\xi_2^2 \\ -i\xi_2 Yr \\ 0 \end{bmatrix},$$

$S =$ span $\{e^{r_2 t}p_2, e^{r_3 t}p_3, \frac{\partial}{\partial r}[e^{rt}p(r)]|_{r=r_1}\}$.

Proof. The proof follows that of Theorem 5.1. Note $N = X + \nu_1 = Z$. Then det $A(r)$ is in equation (21). Also

$\beta_1 = \beta_3 = 1, \ \lambda = \beta_1^2 + \tau_1\tau_3 - Z^2 = 1 + (1+Y)(1+V) - \sigma_{33}^2\mu^3\omega^2 = 1 + (1+Y)(1+V) - YV$

$= 2 + Y + V = \tau_1 + \tau_3, \beta_1\tau_1\rho^2 - \lambda\xi_3^2\rho + \beta_1\tau_3\xi_3^4 = (\rho^2 - \xi_3^2)(\tau_1\rho^2 - \tau_3\xi_3^2)$ and equation (21) gives (49).

Case i) $\xi_3 \neq 0, Z \neq 0, r_1 \neq r_3$

The roots of equation (49) are $\rho_1 = \rho_2 = \xi_3^2, \rho_3 = \frac{\tau_3}{\tau_1}\xi_3^2$ and

$r_i^2 = \rho_i + \xi_2^2 \implies r_1^2 = r_2^2 = \xi_2^2 + \xi_3^2 = |\xi|^2, \quad r_3^2 = \xi_2^2 + \frac{\tau_3}{\tau_1}\xi_3^2.$

In the notation in Theorem 5.1, $k_1 = 1, k_3 = \frac{\tau_3}{\tau_1}$. The null vectors with p_i of $A(r_i)$ are in equation (23) with $c_1 = \frac{\tau_1 k_1 - \beta_1}{N} = \frac{\tau_1 - 1}{Z} = \frac{Y}{Z} = \frac{\omega}{\mu^{\frac{1}{2}}}, \quad c_3 = \frac{\tau_1 k_3 - \beta_1}{N} = \frac{\tau_3 - 1}{Z} = \frac{V}{Z} = \frac{\mu^{\frac{1}{2}}}{\omega}$.

This gives equation (50).

Case ii) $\xi_3 \neq 0, Z = 0$

$Z = 0 \implies \sigma_{33} = 0$ (since $\mu, \omega > 0$)
$\implies Y = V = 0 \implies \tau_1 = \tau_3 = 1 \implies k_1 = k_3 = 1$
$\implies r_1^2 = r_2^2 = r_3^2 = \xi_2^2 + \xi_3^2 = |\xi|^2.$

Also $A(r) = (r^2 - |\xi|^2)I$, I = identity matrix. Linearly independent null vectors of $A(r_i)$ are $p_1 = e_1, p_2 = e_2, p_3 = e_3$ since $A(r_i) = 0$.

Case iii) $\xi_3 \neq 0, Z \neq 0, r_1 = r_3$

The roots are in Case i). This case is exactly as in Theorem 5.1 with

$$p(r) = \begin{bmatrix} r \\ i\xi_2 \\ ic(r)\xi_3 \end{bmatrix},$$

$c(r) = \frac{\tau_1\rho(r) - \beta_1\xi_3^2}{N\xi_3^2} = \frac{\tau_1\rho(r) - \xi_3^2}{Z\xi_3^2}, \quad \rho(r) = r^2 - \xi_2^2$

Case iv) $\xi_3 = 0$

Exactly as in Theorem 5.1, $r_i^2 = \xi_2^2, \quad i = 1, 2, 3$

with $p(r) = \begin{bmatrix} r^2 - \tau_1\xi_2^2 \\ -iYr\xi_2 \\ 0 \end{bmatrix}$, using $Y = \tau_1 - \beta_3$.

□

Theorem 7.1 implies equation (44) has 3 linearly independent solutions $z^{(1)}(t), z^{(2)}(t), z^{(3)}(t)$ on $[0, \infty)$ which are exponentially decreasing. We substitute these into $\mathscr{B}_\xi$ in equation (45) at $t = 0$

$$\mathscr{B}_\xi[z] = \begin{bmatrix} \tau_1 z_1' + i\gamma\xi_2 z_2 + iX\xi_3 z_3 \\ z_2' + i\nu_3\xi_2 z_1 \\ z_3' + i\nu_1\xi_3 z_1 \end{bmatrix}$$

and put

$$C_\xi = [\mathscr{B}_\xi[z^{(1)}]|_{t=0}, \mathscr{B}_\xi[z^{(2)}]|_{t=0}, \mathscr{B}_\xi[z^{(3)}]|_{t=0}]$$

We find C_ξ using Theorem 7.1.

Theorem 7.2

Case i) $\xi_3 \neq 0, Z \neq 0, r_1 \neq r_3$

$$C_\xi = \begin{bmatrix} a|\xi|^2 & i\xi_2|\xi|a & a\xi_2^2 + b\xi_3^2 \\ -ia\xi_2|\xi| & a\xi_2^2 + \xi_3^2 & ia\xi_2 r_3 \\ -i\xi_3|\xi|\frac{\omega}{\mu^{12}}b & \nu_1\xi_2\xi_3 & i\xi_3 r_3 \frac{\mu^{12}}{\omega}a \end{bmatrix}$$

$a = \tau_1 - \gamma, b = 1 - \frac{\mu}{\omega}\sigma_3$

Case ii) $\xi_3 \neq 0, Z = 0$

$$C_\xi = \begin{bmatrix} -|\xi| & i\xi_2\gamma & i\xi_3 X \\ i\xi_2\nu_3 & -|\xi| & 0 \\ i\xi_3\nu_1 & 0 & -|\xi| \end{bmatrix}$$

Case iii) $\xi_3 \neq 0, Z \neq 0, r_1 = r_3$

$$C_\xi = \begin{bmatrix} a|\xi|^2 & i\xi_2|\xi|a & \frac{\partial}{\partial r}[\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]|_{r=r_1} \\ -ia\xi_2|\xi| & a\xi_2^2 + \xi_3^2 & \frac{\partial}{\partial r}[ia\xi_2 r]|_{r=r_1} \\ -i\xi_3|\xi|\frac{\omega}{\mu^{12}}b & \nu_1\xi_2\xi_3 & \frac{\partial}{\partial r}[i\xi_3 r(\nu_1 + c(r))]|_{r=r_1} \end{bmatrix}$$

$a = \tau_1 - \gamma, b = 1 - \frac{\mu}{\omega}\sigma_3$

$c(r) = \frac{\tau_1\rho(r) - \xi_3^2}{Z\xi_3^2}, \quad \rho(r) = r^2 - \xi_2^2.$

Case iv) $\xi_3 = 0$

$$C_\xi = \begin{bmatrix} ia\xi_2|\xi_2| & 0 & \frac{\partial}{\partial r}[\tau_1 r^3 + ((\tau_1-1)\gamma - \tau_1^2)|\xi|^2 r]|_{r=r_1} \\ a|\xi_2|^2 & 0 & \frac{\partial}{\partial r}[-i\xi_2(\gamma r^2 + \nu_3\tau_1\xi_2^2)]|_{r=r_1} \\ 0 & -|\xi_2| & 0 \end{bmatrix}$$

$a = \tau_1 - \gamma, r_1 = -|\xi_2|$

Proof. We use equation (18) $\mathscr{B}_\xi[e^{rt}p] = B(r)p$ at $t = 0$ and equation (30) $\mathscr{B}_\xi[\frac{\partial}{\partial r}[e^{rt}p(r)]] = \frac{\partial}{\partial r}[B(r)p(r)]$ at $t = 0$.

Case i) : $\xi_3 \neq 0, Z \neq 0, r_1 \neq r_3$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} a|\xi|^2 \\ -ia\xi_2|\xi| \\ -i\xi_3|\xi|\frac{\omega}{\mu^{12}}b \end{bmatrix}$$

using $\nu_3 + 1 = \tau_1 - \gamma = a,\ \ b = 1 - \frac{\mu}{\omega}\sigma_3,\ \ X\frac{\omega}{\mu^{\frac{1}{2}}} = \gamma$.

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} i\xi_2|\xi|a \\ a\xi_2^2 + \xi_3^2 \\ \nu_1\xi_2\xi_3 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(3)}] = B(r_3)p_3 = \begin{bmatrix} a\xi_2^2 + b\xi_3^2 \\ ia\xi_2 r_3 \\ i\xi_3 r_3\frac{\mu^{12}}{\omega}a \end{bmatrix}$$

using $\tau_3 - X\frac{\mu^{\frac{1}{2}}}{\omega} = b$

Case ii) : $\xi_3 \neq 0, Z = 0$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_1)p_1 = \begin{bmatrix} -|\xi| \\ i\xi_2\nu_3 \\ i\xi_3\nu_1 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_2)p_2 = \begin{bmatrix} i\xi_2\gamma \\ -|\xi| \\ 0 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(3)}] = B(r_3)p_3 = \begin{bmatrix} i\xi_3 X \\ 0 \\ -|\xi| \end{bmatrix}$$

Case iii) : $\xi_3 \neq 0, Z \neq 0, r_1 = r_3$

$\mathscr{B}_\xi[z^{(1)}], \mathscr{B}_\xi[z^{(2)}]$ are the same as in Case i).

From equation (30), $\mathscr{B}_\xi[z^{(3)}] = \frac{\partial}{\partial r}[B(r)p(r)]|_{r=r_1}$

where $B(r)p(r) = \begin{bmatrix} \tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2 \\ ia\xi_2 r \\ i\xi_3 r(\nu_1 + c(r)) \end{bmatrix}$

Case iv) : $\xi_3 = 0$

$$\mathscr{B}_\xi[z^{(1)}] = B(r_2)p_2 = \begin{bmatrix} ia\xi_2|\xi_2| \\ a|\xi_2|^2 \\ 0 \end{bmatrix}$$

$$\mathscr{B}_\xi[z^{(2)}] = B(r_3)p_3 = \begin{bmatrix} 0 \\ 0 \\ -|\xi_2| \end{bmatrix}$$

From equation (30), $\mathscr{B}_\xi[z^{(3)}] = \frac{\partial}{\partial r}[B(r)p(r)]|_{r=r_1}$

$$\text{where } B(r)p(r) = \begin{bmatrix} \tau_1 r^3 + ((\tau_1 - 1)\gamma - \tau_1^2)|\xi|^2 r \\ -i\xi_2(\gamma r^2 + \nu_3\tau_1\xi_2^2) \\ 0 \end{bmatrix}$$

using $\gamma = Y - \nu_3,\ Y = \tau_1 - 1.$

□

We calculate $\det C_\xi$.

Theorem 7.3.

Denote $a = 1 - \omega\sigma_3$.

Case i) det C_ξ

$= i\xi_3|\xi|\frac{\omega}{\mu^{\frac{1}{2}}}\ [a^2 r_3(\xi_2^2 + \frac{\mu}{\omega^2}\xi_3^2)|\xi| + (a\xi_2^2 + b\xi_3^2)^2]$

Case ii) det $C_\xi = -|\xi|[(1-\omega^2\sigma_3^2)\xi_2^2 + (1-\mu\sigma_3^2)\xi_3^2]$

Case iii) det C_ξ

$= i\xi_3|\xi|\frac{\omega}{\mu^{\frac{1}{2}}}\ \frac{\partial}{\partial r}\{a|\xi|r\ [a\xi_2^2 + \frac{\mu^{\frac{1}{2}}}{\omega}(\nu_1 + c(r))\xi_3^2] + [\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]\ [a\xi_2^2 + b\xi_3^2]\}|_{r=r_1}$

Case iv) det $C_\xi = a|\xi_2|\xi_2^4[-\tau_1^2 + (3+\gamma)\tau_1 + \gamma]$

Proof. **Case i)** To find det C_ξ, multiply the first column of C_ξ by $(\frac{-i\xi_2}{|\xi|})$ and add to the second column $\implies$

$$\det C_\xi = \det \begin{bmatrix} a|\xi|^2 & 0 & a\xi_2^2 + b\xi_3^2 \\ -ia\xi_2|\xi| & \xi_3^2 & ia\xi_2 r_3 \\ -i\xi_3|\xi|\frac{\omega}{\mu^{\frac{1}{2}}}b & -\frac{\omega}{\mu^{\frac{1}{2}}}\xi_2\xi_3 & i\xi_3 r_3 \frac{\mu^{\frac{1}{2}}}{\omega}a \end{bmatrix}$$

Expanding this about the first row $\Longrightarrow$

$\det C_\xi = i\xi_3|\xi|\frac{\omega}{\mu^{\frac{1}{2}}}\,[a^2 r_3(\xi_2^2 + \frac{\mu}{\omega^2}\xi_3^2)|\xi| + (a\xi_2^2 + b\xi_3^2)^2]$

Case ii) Expanding about the third row

$\det C_\xi = (i\xi_3\nu_1)(|\xi| i\xi_3 X) - |\xi|(|\xi|^2 + \xi_2^2\gamma\nu_3) = -|\xi|[(1+\gamma\nu_3)\xi_2^2 + (1+\nu_1 X)\xi_3^2].$

Then use $Z = Y = V = 0,\ \ \gamma = \omega\sigma_3,\ \ X = \mu^{\frac{1}{2}}\sigma_3$ to get
$-|\xi|[(1-\omega^2\sigma_3^2)\xi_2^2 + (1-\mu\sigma_3^2)\xi_3^2].$

Case iii) Let

$$C_\xi(r) = \begin{bmatrix} a|\xi|^2 & i\xi_2|\xi|a & \tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2 \\ -ia\xi_2|\xi| & a\xi_2^2 + \xi_3^2 & ia\xi_2 r \\ -i\xi_3|\xi|\frac{\omega}{\mu^{\frac{1}{2}}}b & \nu_1\xi_2\xi_3 & i\xi_3 r(\nu_1 + c(r)) \end{bmatrix}$$

Expanding $\det C_\xi$ in case iii) about its 3$^{\text{rd}}$ column we see
$\det C_\xi = \frac{\partial}{\partial r}\det C_\xi(r)|_{r=r_1}$. To find $\det C_\xi(r)$, multiply the first column of $C_\xi(r)$ by $(\frac{-i\xi_2}{|\xi|})$ and add to its second column $\Longrightarrow$

$\det C_\xi(r)$

$= i\xi_3|\xi|\frac{\omega}{\mu^{\frac{1}{2}}}\{a|\xi|r[a\xi_2^2 + \frac{\mu^{\frac{1}{2}}}{\omega}\,(\nu_1 + c(r))\xi_3^2] + [\tau_1 r^2 - \gamma\xi_2^2 - Xc(r)\xi_3^2]\,[a\xi_2^2 + b\xi_3^2]\}.$

Case iv) This follows from the proof of Theorem 5.4 (iv) with $\beta_1 = \beta_3 = 1$.

□

Theorem 7.4.

Assume C in Theorem 4.1, C is strongly elliptic and $n = e_1$. Denote
$a = 1 - \omega\sigma_3,\ \ b = 1 - \frac{\mu}{\omega}\sigma_3$. All $(...)^{\frac{1}{2}} > 0$.

a) Assume $Z \neq 0, \tau_1 \neq \tau_3$.

Then(C, n) satisfies the Complementing Condition iff

$a[-\tau_1^2 + (3+\gamma)\tau_1 + \gamma] \neq 0$ and

$-a^2(x + \frac{\tau_3}{\tau_1})^{\frac{1}{2}}(x + \frac{\mu}{\omega^2})(x+1)^{\frac{1}{2}} + (ax+b)^2 \neq 0$

for all $x \in [0, \infty)$.

b) Assume $Z = 0$.

Then (C, n) satisfies the Complementing Condition iff

$(1 - \omega^2\sigma_3^2)(1 - \mu\sigma_3^2) > 0$

c) Assume $Z = 0, \tau_1 = \tau_3$.

Then (C, n) satisfies the Complementing Condition iff

$a[-\tau_1^2 + (3+\gamma)\tau_1 + \gamma] \neq 0$ and

$\frac{\partial}{\partial k}\{-a(x+k)^{\frac{1}{2}}[ax + \frac{\mu^{\frac{1}{2}}}{\omega}(\nu_1 + \hat{c}(k))](x+1)^{\frac{1}{2}} + [ax + (\tau_1 k - X\hat{c}(k))](ax+b)\}|_{k=1} \neq 0$

for all $x \in [0, \infty)$. Here $\hat{c}(k) = \frac{\tau_1 k - 1}{Z}$.

Proof. We use Theorems 7.3, 5.3 and require det $C_\xi \neq 0$ for all $(\xi_2, \xi_3) \in \mathbb{R}^2 \setminus \{0\}$ in order for the Complementing Condition to be satisfied. In particular, when $\xi_3 = 0$, Theorem 7.3 (iv) implies

$$a[-\tau_1^2 + (3+\gamma)\tau_1 + \gamma] \neq 0 \tag{51}$$

in all cases a), b), c). Also $r_1 = r_3 \Leftrightarrow \tau_1 = \tau_3$.

a) $Z \neq 0, \tau_1 \neq \tau_3$.

In Theorem 7.3 (i) we put $x = \frac{\xi_2^2}{\xi_3^2} \in [0, \infty)$. Then
$r_3 = -(x + \frac{\tau_3}{\tau_1})^{\frac{1}{2}}|\xi_3|, \quad |\xi| = (x+1)^{\frac{1}{2}}|\xi_3|$ with each $(..)^{\frac{1}{2}} > 0$. Then det $C_\xi \neq 0$ iff

$$-a^2(x+\tfrac{\tau_3}{\tau_1})^{\frac{1}{2}}(x+\tfrac{\mu}{\omega^2})(x+1)^{\frac{1}{2}}+(ax+b)^2\neq 0.$$

This must hold for all $x\in[0,\infty)$. The case $\xi_3=0$ is in equation (51).

b) $Z=0$.

In Theorem 7.3 (ii), $\det C_\xi\neq 0$ for all $(\xi_2,\xi_3)\in\mathbb{R}^2\setminus\{0\}$ iff $1-\omega^2\sigma_3^2$ and $1-\mu\sigma_3^2$ are nonzero and have the same sign. Also $\tau_1=1,\gamma=\omega\sigma_3,a=1-\omega\sigma_3\implies$ the left side of equation (51) $=2(1-\omega^2\sigma_3^2)\implies$ equation (51) is satisfied.

c) $Z\neq 0,\tau_1=\tau_3$.

In Theorem 7.3 (iii) we put
$x=\frac{\xi_2^2}{\xi_3^2}\in[0,\infty),\quad r=-(x+k)^{\frac{1}{2}}|\xi_3|,\quad k>0,\quad (x+k)^{\frac{1}{2}}>0.$

Then $\frac{\partial}{\partial r}=(2r\xi_3^{-2})\frac{\partial}{\partial k}$ and $r=r_1\Leftrightarrow k=1$ (since $r_1=-|\xi|,\ |\xi|^2=\xi_2^2+\xi_3^2$).
Also we put $c(r)=\frac{\tau_1\rho(r)-\xi_3^2}{Z\xi_3^2}=\frac{\tau_1(r^2-\xi_2^2)-\xi_3^2}{Z\xi_3^2}=\frac{\tau_1k-1}{Z}=\hat{c}(k)$ so

$$\begin{aligned}\tau_1r^2-\gamma\xi_2^2-Xc(r)\xi_3^2&=(\tau_1-\gamma)\xi_2^2+(\tau_1k-X\hat{c}(k))\xi_3^2\\&=[ax+(\tau_1k-X\hat{c}(k))]\xi_3^2,\quad |\xi|=(x+1)^{\frac{1}{2}}|\xi_3|.\end{aligned}$$

Then $\det C_\xi\neq 0$ in Theorem 7.3 (iii) iff

$$\tfrac{\partial}{\partial k}\{-a(x+1)^{\frac{1}{2}}(x+k)^{\frac{1}{2}}\quad[ax+\tfrac{\mu^{\frac{1}{2}}}{\omega}(\nu_1+\hat{c}(k))]+[ax+(\tau_1k-X\hat{c}(k))](ax+b)\}|_{k=1}\neq 0.$$

This must hold for all $x\in[0,\infty)$. The case $\xi_3=0$ is in equation (51).

$\square$

Chapter 8

Conclusion

We calculated the Complementing Condition for the boundary value problem given by equations (1), (2). We have considered different cases when Ω is a cylinder and the unit normal, $n = e_1$ and $n = e_3$. The assumed stored energy function for material of the body depends on first, second and third invariants of the deformation F.

We then calculated the Complementing Condition for a special case of stored energy function when the unit normal, $n = e_1$.

We have found solution sets of ODE system, substituted into the boundary conditions and found the Complementing Condition equivalent to det $C_\xi \neq 0$ for all ξ in all cases. The Complementing Condition results are summarized in Theorems 5.5, 6.4, 7.4.

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Vita

Lavanya Ramanan was born in Chennai, India on July 27, 1987. She completed her High School from Delhi Tamil Education Association (D.T.E.A) School in New Delhi. She earned a Bachelor's degree in Mechanical and Automation Engineering from Indira Gandhi Institute of Technology (affiliated to Guru Gobind Singh Indraprastha University), Delhi in 2008. She then worked as a Design Engineer at TATA Consultancy Services for 3 years in Bangalore, India.

To pursue her passion in Mathematics, she enrolled at the University of Tennessee, Knoxville in Fall 2011 as a graduate student to study MS in Mathematics. She also worked as a Graduate Research Assistant at the Office of Information Technology, UTK. Upon graduation, she plans to apply her mathematical skills to solve engineering problems.