5-2014

Positive Periodic Solutions for a Higher Order Functional Difference Equation

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POSITIVE PERIODIC SOLUTIONS FOR A HIGHER ORDER FUNCTIONAL DIFFERENCE EQUATION

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Abstract. We consider a higher order functional difference equations on $\mathbb{Z}$ with an eigenvalue parameter λ in the equation. Sufficient conditions are obtained for the existence of at least one or two positive periodic solutions of the equation for different values of λ. The nonlinear function in the equation is allowed to be sign-changing in some of our results. Our proofs utilize Krasnosel'skii's fixed point theorem.

1. Introduction

This paper was written at the University of Tennessee, Chattanooga as part of the REU program in Number Theory and Difference Equations. Michael Ruddy and Alexander Ruys de Perez also contributed to the main part of this paper under the mentorship of Dr. Lingju Kong. The subject of this paper (solutions to higher a order functional difference equation) is motivated by the work of Dr. Kong as well as that of the authors cited throughout this paper.

It is well known that nonlinear difference equations of order greater than one are of paramount importance in applications where the $(n+1)^{\text{st}}$ generation (or state) of the system depends on the previous k generations (or states). Such equations also appear naturally as discrete analogue and as numerical solutions of differential equations and delay differential equations which model various diverse phenomena in statistics, computing, electrical circuit analysis, dynamical systems, economics, biology (see, for example, [10,12]). Because of their applications, the existence of solutions of such equations has been investigated by many researchers in recent years. For instance, the reader may refer to [3, 6–9, 11, 16–18] and the cited references for some recent work. In this paper, we are concerned with a higher order functional difference equation. To introduce our equation, we let $a \neq 1$, $b \neq 1$ be any fixed positive numbers, and m, k, ω be any fixed positive integers, and for any $u : \mathbb{Z} \to \mathbb{R}$, define

$$Lu(n) = u(n+m+k) - au(n+m) - bu(n+k) + abu(n).$$

1991 *Mathematics Subject Classification.* 39A23, 39A10.

Key words and phrases. Positive periodic solutions, higher order, functional difference equations, cone, fixed point theorem.

Here, we study the existence of positive periodic solutions of the higher order functional difference equation

$$Lu(n) = \lambda f(n, u(n - \tau(n))) + r(n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f : \mathbb{Z} \times [0, \infty) \to \mathbb{R}$, $\tau : \mathbb{Z} \to \mathbb{Z}$, and $r : \mathbb{Z} \to \mathbb{R}$ are ω-periodic on $\mathbb{Z}$, and $f(n, x)$ is continuous in x.

Equation (1.1) with $\lambda = 1$ and $r(n) \equiv 0$, i.e., the equation

$$Lu(n) = f(n, u(n - \tau(n))), \quad n \in \mathbb{Z}, \tag{1.2}$$

has been recently studied by Wang and Chen in [18]. When the nonlinear function $f(n, x)$ has a constant sign, they obtained sufficient conditions for the existence of positive periodic solutions by using Krasnosel'skii's fixed point theorem. Applying the same fixed point theorem, in this paper, we consider equation (1.1) and establish several new criteria for the existence of at least one or two positive periodic solutions for different values of λ. We allow that the nonlinearity $f(n, x)$ can be sign-changing on $\mathbb{Z} \times [0, \infty)$ in our Theorems 2.1–2.3; see Section 2 below for details.

We comment that problems with sign-changing nonlinearities occur in models for steady–state diffusion with reactions [2] and interest in obtaining conditions for the existence of positive solutions of such problems has been ongoing for many years. For a small sample of such work, we refer the reader to the papers [1, 4, 13–15, 19] and the references therein. Our proofs are partly motivated by these works.

As simple illustrations of our theorems, we derive sufficient conditions for the existence of positive periodic solutions of the functional difference equation

$$Lu(n) = \lambda \left(c(n)(u(n - \tau(n)))^{\mu(u(n-\tau(n)))} - d(n) \right) + r(n), \quad n \in \mathbb{Z}, \tag{1.3}$$

where $c : \mathbb{Z} \to (0, \infty)$ and $d : \mathbb{Z} \to [0, \infty)$ are ω-periodic on $\mathbb{Z}$, and $\mu : [0, \infty) \to [0, \infty)$ is continuous,.

The rest of this paper is organized as follows. In Section 2, we present our main results, and the proofs of the main results together with several technical lemmas are given in Section 3.

2. Main results

In this paper, for any $c, d \in \mathbb{Z}$ with $c \leq d$, let $[c, d]_{\mathbb{Z}}$ denote the discrete interval $\{c, \ldots, d\}$. For the function $r(n)$ given in equation (1.1), define a function $\gamma : \mathbb{Z} \to \mathbb{R}$ by

$$\gamma(n) = \sum_{i=1}^{\omega} \sum_{j=1}^{\omega} G(i,j) r(n_{ij}), \tag{2.1}$$

where

$$G(i,j) = \frac{a^\omega b^\omega a^{-i} b^{-j}}{(1-a^\omega)(1-b^\omega)}, \quad i,j \in [1,\omega]_{\mathbb{Z}}, \tag{2.2}$$

and

$$n_{ij} = n + (i-1)k + (j-1)m. \tag{2.3}$$

We make the following assumptions.

(H1) $(k, \omega) = (m, \omega) = 1$, where (x, y) denotes the greatest common divisor of x and y;

(H2) either

(a) $(a-1)(b-1) > 0$, $\gamma(n) \geq 0$ on $\mathbb{Z}$, and there exists $M : \mathbb{Z} \to (0, \infty)$ such that $M(n+\omega) = M(n)$ and

$$f(n, x + \gamma(n - \tau(n))) \geq -M(n) \quad \text{for } (n, x) \in \mathbb{Z} \times [0, \infty),$$

or

(b) $(a-1)(b-1) < 0$, $\gamma(n) \geq 0$ on $\mathbb{Z}$, and there exists $M : \mathbb{Z} \to (0, \infty)$ such that $M(n+\omega) = M(n)$ and

$$f(n, x + \gamma(n - \tau(n))) \leq M(n) \quad \text{for } (n, x) \in \mathbb{Z} \times [0, \infty);$$

(H3) there exists $n_1 \in [1, \omega]_{\mathbb{Z}}$ such that $\lim_{x\to\infty} f(n_1, x + \gamma(n_1 - \tau(n_1)))/x = \infty$;

(H4) $f(n, \gamma(n - \tau(n))) > 0$ for all $n \in [1, \omega]_{\mathbb{Z}}$;

(H5) there exists $n_2 \in [1, \omega]_{\mathbb{Z}}$ such that $\lim_{x\to\infty} f(n_2, x + \gamma(n_2 - \tau(n_2))) = \infty$;

(H6) $\lim_{x\to\infty} f(n, x + \gamma(n - \tau(n)))/x = 0$ for all $n \in [1, \omega]_{\mathbb{Z}}$;

(H7) $\lim_{x\to 0^+} f(n, x + \gamma(n - \tau(n)))/x = 0$ for all $n \in [1, \omega]_{\mathbb{Z}}$.

Remark 2.1. Although $\gamma(n)$ is assumed to be nonnegative in (H2), $r(n)$ can be a sign-changing function on $\mathbb{Z}$.

Now, we state our main results. Our first two theorems concern the case where λ is small and the last two are for the case of λ large.

Theorem 2.1. *Assume that (H1)–(H3) hold. Then there exists* $\lambda_1 > 0$ *such that for* $\lambda \in (0, \lambda_1)$*, equation* (1.1) *has at least one positive periodic solution* $y(n)$ *satisfying* $y(n) > \gamma(n)$ *on* $\mathbb{Z}$*.*

Theorem 2.2. *Assume that (H1)–(H4) hold. Then there exists* $\lambda_2 \in (0, \lambda_1]$ *such that for* $\lambda \in (0, \lambda_2)$*, equation* (1.1) *has at least two positive periodic solution* $y_i(n)$ *satisfying* $y_i(n) > \gamma(n)$ *for* $i = 1, 2$ *and* $n \in \mathbb{Z}$*.*

Theorem 2.3. *Assume that (H1), (H2), (H5), and (H6) hold. Then there exists* $\lambda_3 > 0$ *such that for* $\lambda \in (\lambda_3, \infty)$*, equation* (1.1) *has at least one positive periodic solution* $y(n)$ *satisfying* $y(n) > \gamma(n)$ *on* $\mathbb{Z}$*.*

Theorem 2.4. *Assume that (H1), (H2) with* $M(n) \equiv 0$ *on* $\mathbb{Z}$*, (H5), (H6), and (H7) hold. Then there exists* $\lambda_4 \in [\lambda_3, \infty)$ *such that for* $\lambda \in (\lambda_4, \infty)$*, equation* (1.1) *has at least two positive periodic solution* $y_i(n)$ *satisfying* $y_i(n) > \gamma(n)$ *for* $i = 1, 2$ *and* $n \in \mathbb{Z}$*.*

Let

$$\mu_0 = \lim_{x \to 0^+} \mu(x) \quad \text{and} \quad \mu_\infty = \lim_{x \to \infty} \mu(x).$$

The following corollary is a direct consequence of Theorems 2.1–2.4.

Corollary 2.1. *Assume that (H1) holds,* $(a-1)(b-1) > 0$*, and* $\gamma(n) \geq 0$ *on* $\mathbb{Z}$*. Then, we have*

(a) *if* $\mu_\infty > 1$*, then there exists* $\lambda_1 > 0$ *such that equation* (1.3) *has at least one positive periodic solution for* $\lambda \in (0, \lambda_1)$*;*

(b) *if* $\mu_\infty > 1$ *and*

$$c(n)(\gamma(n - \tau(n)))^{\mu(\gamma(n-\tau(n)))} > d(n) \quad \text{for } n \in [1, \omega]_{\mathbb{Z}},$$

then there exists $\lambda_2 > 0$ *such that equation* (1.3) *has at least two positive periodic solutions for* $\lambda \in (0, \lambda_2)$*;*

(c) *if* $0 < \mu_\infty < 1$*, then there exists* $\lambda_3 > 0$ *such that equation* (1.3) *has at least one positive periodic solution for* $\lambda \in (0, \lambda_3)$*;*

(d) *if* $\mu_0 > 1$*,* $0 < \mu_\infty < 1$*, and* $d(n) = \gamma(n) = 0$ *on* $[1, \omega]_{\mathbb{Z}}$*, there exists* $\lambda_4 > 0$ *such that equation* (1.3) *has at least two positive periodic solutions for* $\lambda \in (\lambda_4, \infty)$*.*

3. Proofs of the main results

Throughout this section, let X be the set of all real ω-periodic sequences. Then, equipped with the maximum norm $||u|| = \max_{n\in[1,\omega]_{\mathbb{Z}}} |u(n)|$, X is a Banach space.

For convenience, we also introduce the following notations

$$\delta_1 = \frac{a^\omega b^\omega}{|(1-a^\omega)(1-b^\omega)|} \min\{a^{-1},\ a^{-\omega}\} \min\{b^{-1},\ b^{-\omega}\}$$

and

$$\delta_2 = \frac{a^\omega b^\omega}{|(1-a^\omega)(1-b^\omega)|} \max\{a^{-1},\ a^{-\omega}\} \max\{b^{-1},\ b^{-\omega}\}.$$

From (2.2), we have

$$\begin{cases} \delta_1 \le G(i,j) \le \delta_2 \quad \text{for } i,j \in [1,\omega]_{\mathbb{Z}} & \text{if (H2)(a) holds,} \\ \delta_1 \le -G(i,j) \le \delta_2 \quad \text{for } i,j \in [1,\omega]_{\mathbb{Z}} & \text{if (H2)(b) holds.} \end{cases} \tag{3.1}$$

For any $h \in X$, it is easy to see that

$$\sum_{i=1}^{\omega}\sum_{j=1}^{\omega} h(n_{ij}) = \omega \sum_{i=1}^{\omega} h(i).$$

This identity will be frequently used in the proofs of our results.

Lemma 3.1 below can be proved using [18, Lemma 2.1].

Lemma 3.1. *Assume that (H1) holds. Then, for any $h \in X$, $u(n)$ is a periodic solution of the equation*

$$Lu(n) = h(n), \quad n \in \mathbb{Z},$$

if and only if $u(n)$ is a solution of the summation equation

$$u(n) = \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j)h(n_{ij}),$$

where $G(i,j)$ and n_{ij} are defined by (2.2) and (2.3), respectively.

Lemma 3.2. *Assume that (H1) and (H2) hold. We have*

(a) *if (H2)(a) holds, then the equation*

$$Lu(n) = M(n), \quad n \in \mathbb{Z}, \tag{3.2}$$

has a unique positive periodic solution $\phi(n)$ such that $\phi(n) \le \rho$, where

$$\rho = \delta_2\omega^2||M||. \tag{3.3}$$

(b) *if (H2)(b) holds, then the equation*

$$Lu(n) = -M(n), \quad n \in \mathbb{Z},$$

has a unique positive periodic solution $\phi(n)$ *such that* $\phi(n) \le \rho$*, where* ρ *is defined by* (3.3).

Proof. We only prove part (a). Part (b) can be proved similarly. In view of Lemma 3.1, it is obvious that

$$\phi(n) = \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j)M(n_{ij})$$

is the unique positive periodic solution of (3.2). By (3.1), we have

$$\phi(n) \le \delta_2 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} M(n_{ij}) \le \delta_2\omega \sum_{i=1}^{\omega} M(i) \le \delta_2\omega^2||M||.$$

This completes the proof of the lemma. □

Define a cone $K \subset X$ by

$$K = \{u \in X \ : \ u(n) \ge \delta||u|| \text{ on } \mathbb{Z}\}, \tag{3.4}$$

where $\delta = \delta_1/\delta_2$. For $\lambda > 0$, let $v(n) = \lambda\phi(n)$. Consider the equations

$$Lu(n) = \lambda\big[f\big(n, [u(n-\tau(n)) - v(n-\tau(n))]^* + \gamma(n-\tau(n))\big) + M(n)\big], \ n \in \mathbb{Z}, \tag{3.5}$$

and

$$Lu(n) = \lambda\big[f\big(n, [u(n-\tau(n)) - v(n-\tau(n))]^* + \gamma(n-\tau(n))\big) - M(n)\big], \ n \in \mathbb{Z}, \tag{3.6}$$

where $\gamma(n)$ is defined by (2.1) and

$$[w(n)]^* = \begin{cases} w(n), & w(n) \ge 0, \\ 0, & w(n) < 0. \end{cases}$$

Lemma 3.3. *Assume that (H1) and (H2) hold. Suppose further that*

either (3.5) *has a solution* $u(n)$ *satisfying* $u(n) > v(n)$ *on* $\mathbb{Z}$ *when (H2)(a) holds, or* (3.6) *has a solution* $u(n)$ *satisfying* $u(n) > v(n)$ *on* $\mathbb{Z}$ *when (H2)(b) holds.*

Then $y(n) = u(n) - v(n) + \gamma(n)$ *is a positive periodic solution of* (1.1) *satisfying* $y(n) > \gamma(n)$ *for* $n \in \mathbb{Z}$.

Proof. Clearly $y(n) > \gamma(n)$ for $n \in \mathbb{Z}$. Note that

$$\begin{aligned} Ly(n) &= Lu(n) - Lv(n) + L\gamma(n) \\ &= \lambda[f(n, u(n-\tau(n)) - v(n-\tau(n)) + \gamma(n-\tau(n))) + M(n)] - \lambda M(n) + r(n) \\ &= \lambda f(n, y(n-\tau(n))) + r(n) \end{aligned}$$

if (H2)(a) holds, and

$$\begin{aligned} Ly(n) &= Lu(n) - Lv(n) + L\gamma(n) \\ &= \lambda[f(n, u(n-\tau(n)) - v(n-\tau(n)) + \gamma(n-\tau(n))) - M(n)] + \lambda M(n) + r(n) \\ &= \lambda f(n, y(n-\tau(n))) + r(n) \end{aligned}$$

if (H2)(b) holds. The conclusion is proved. □

The following well known Krasnosel'skii fixed point theorem will be used in the proofs of our theorems. The reader may refer to [5, Theorem 2.3.4.] for its proof.

Lemma 3.4. *Let X be a Banach space, and let $K \subset X$ be a cone. Assume that Ω_1, Ω_2 are two bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $||Tu|| \leq ||u||$ *for* $u \in K \cap \partial\Omega_1$ *and* $||Tu|| \geq ||u||$ *for* $u \in K \cap \partial\Omega_2$, *or*

(ii) $||Tu|| \geq ||u||$ *for* $u \in K \cap \partial\Omega_1$ *and* $||Tu|| \leq ||u||$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Below, in the proofs of Theorems 2.1–2.4 , we only prove the case when (H2)(a) holds. The proofs are similar when (H2)(b) holds.

Proof of Theorem 2.1. By Lemma 3.3, it suffices to show that (3.5) has a solution $u(n)$ satisfying $u(n) > v(n)$ on $\mathbb{Z}$. To this end, define an operator $T : K \to X$ by

$$Tu(n) = \lambda \sum_{i=1}^{\omega} \sum_{j=1}^{\omega} G(i,j) \big[f(n_{ij}, [u(n_{ij}^{\tau}) - v(n_{ij}^{\tau})]^{*} + \gamma(n_{ij}^{\tau})) + M(n_{ij}) \big], \tag{3.7}$$

where

$$n_{ij}^{\tau} = n_{ij} - \tau(n_{ij}) \quad \text{with } n_{ij} \text{ being defined by (2.3).}$$

Then, T is obviously completely continuous, and by Lemma 3.1, finding a fixed point of T in X is equivalent to finding a periodic solution of (3.5). Moreover, from (3.1), we see that

$$Tu(n) \leq \lambda \delta_2 \sum_{i=1}^{\omega} \sum_{j=1}^{\omega} \big[f(n_{ij}, [u(n_{ij}^{\tau}) - v(n_{ij}^{\tau})]^{*} + \gamma(n_{ij}^{\tau})) + M(n_{ij}) \big]$$

and

$$Tu(n) \geq \lambda\delta_1 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} \big[f(n_{ij}, [u(n_{ij}^{\tau}) - v(n_{ij}^{\tau}))]^* + \gamma(n_{ij}^{\tau})) + M(n_{ij})\big]$$

for $n \in \mathbb{Z}$. Thus, $Tu(n) \geq \delta||Tu||$ on $\mathbb{Z}$. This shows that $T(K) \subset K$.

In the remainder of the proof, let

$$0 < \lambda < \lambda_1 := \min\left\{\frac{1}{2\rho},\ \frac{1+\|\gamma\|}{\delta_2\omega\sum_{i=1}^{\omega} g(i)}\right\}, \tag{3.8}$$

where ρ is defined by (3.3) and

$$g(i) = \sup_{x\in[0,1+\|\gamma\|]} \big[f(i, x+\gamma(i)) + M(i)\big].$$

Clearly, $g(n+\omega) = g(n)$ for $n \in \mathbb{Z}$. Let

$$\Omega_1 = \{u \in X \ :\ \|u\| < 1 + \|\gamma\|\}.$$

Then, for $u \in K \cap \partial\Omega_1$, $0 \leq [u(i-\tau(i)) - v(i-\tau(i))]^* \leq u(i-\tau(i)) \leq \|u\| = 1 + \|\gamma\|$. Thus, for $n \in \mathbb{Z}$, from (3.1), (3.7), and (3.8), we obtain that

$$\begin{aligned} Tu(n) &\leq \lambda\delta_2 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} \big[f(n_{ij}, [u(n_{ij}^{\tau}) - v(n_{ij}^{\tau}))]^* + \gamma(n_{ij}^{\tau})) + M(n_{ij})\big] \\ &= \lambda\delta_2\omega \sum_{i=1}^{\omega} \big[f(i, [u(i-\tau(i)) - v(i-\tau(i))]^* + \gamma(i)) + M(i)\big] \\ &\leq \lambda\delta_2\omega \sum_{i=1}^{\omega} g(i) \\ &\leq 1 + \|\gamma\| = \|u\|. \end{aligned}$$

Hence,

$$\|Tu\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.9}$$

Now choose $A > 0$ large enough so that

$$\frac{1}{2}A\lambda\delta_1\omega\delta \geq 1. \tag{3.10}$$

By (H3), there exists $N > 0$ such that

$$f(n_1, x + \gamma(n_1 - \tau(n_1)) \geq Au \quad \text{for } x \in (N, \infty). \tag{3.11}$$

Let $R > 0$ satisfy

$$R > \max\{1 + \|\gamma\|,\ 2\lambda\rho/\delta,\ 2N/\delta\}$$

and define

$$\Omega_2 = \{u \in X \ :\ \|u\| < R\}.$$

Then, for $u \in K \cap \partial\Omega_2$ and $n \in \mathbb{Z}$, by Lemma 3.2 (a) and (3.4), we see that

$$u(n) - v(n) = u(n) - \lambda\phi(n) \geq u(n) - \lambda\rho \geq u(n) - \lambda\rho\frac{u(n)}{\delta R} \geq \frac{1}{2}u(n), \tag{3.12}$$

which in turn implies that

$$u(n) - v(n) \geq \frac{1}{2}\delta\|u\| = \frac{1}{2}\delta R > N.$$

Hence, from (H2)(a), (3.1), (3.7), and (3.10)–(3.12), it follows that

$$\begin{aligned}
Tu(n) &\geq \lambda\delta_1 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega}[f(n_{ij}, [u(n_{ij}^{\tau}) - v(n_{ij}^{\tau})]^* + \gamma(n_{ij}^{\tau})) + M(n_{ij})] \\
&= \lambda\delta_1\omega \sum_{i=1}^{\omega} \left[f(i, [u(i-\tau(i)) - v(i-\tau(i))]^* + \gamma(i-\tau(i))) + M(i)\right] \\
&\geq \lambda\delta_1\omega[f(n_1, u(n_1-\tau(n_1)) - v(n_1-\tau(n_1)) + \gamma(n_1-\tau(n_1))) + M(n_1)] \\
&\geq \lambda\delta_1\omega f(n_1, u(n_1-\tau(n_1)) - v(n_1-\tau(n_1)) + \gamma(n_1-\tau(n_1))) \\
&\geq A\lambda\delta_1\omega(u(n_1-\tau(n_1)) - v(n_1-\tau(n_1))) \\
&\geq \frac{1}{2}A\lambda\delta_1\omega u(n_1-\tau(n_1)) \\
&\geq \frac{1}{2}A\lambda\delta_1\omega\delta\|u\| \geq \|u\|.
\end{aligned}$$

Thus,

$$\|Tu\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_2. \tag{3.13}$$

In view of (3.9) and (3.13), T has a fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ by Lemma 3.4. From Lemma 3.2 (a) and (3.8), we have

$$u(n) \geq 1 + \|\gamma\| > 1/2 > \lambda\rho \geq \lambda\phi(n) = v(n) \quad \text{for } n \in \mathbb{Z}.$$

Then, by Lemma 3.3, (1.1) has a solution $y(n) = u(n) - v(n) + \gamma(n) > \gamma(n)$ on $\mathbb{Z}$. This completes the proof of the theorem. □

Remark 3.1. For the solution $y(n)$ of (1.1) given in the proof of Theorem 2.1, we have

$$\|y\| \geq \|u - v\| \geq \|u\| - \|v\| \geq 1 + \|\gamma\| - 1/2 = 1/2 + \|\gamma\|.$$

Note that (H4) implies that there exists a constant $q > 0$ such that

$$f(n, x + \gamma(n - \tau(n))) > 0 \quad \text{for } n \in \mathbb{Z} \text{ and } x \in [0, q]. \tag{3.14}$$

Now define

$$f^*(n, x + \gamma(n)) = \begin{cases} f(n, x + \gamma(n)) & \text{for } n \in \mathbb{Z} \text{ and } x \in [0, q], \\ f(n, q + \gamma(n)) & \text{for } n \in \mathbb{Z} \text{ and } x \in (q, \infty), \end{cases}$$

and consider the equation

$$Lu(n) = \lambda f^*(n, u(n-\tau(n)) + \gamma(n-\tau(n))), \quad n \in \mathbb{Z}. \tag{3.15}$$

It is clear that $f^*(n, x+\gamma(n)) \geq 0$ for $n \in \mathbb{Z}$ and $x \in [0, \infty)$.

Lemma 3.5. *If* (3.15) *has a solution* $0 \leq u(n) \leq q$ *on* $\mathbb{Z}$*, then* $y(n) = u(n) + \gamma(n)$ *is a solution of* (1.1)*.*

Proof. Note that

$$\begin{aligned} Ly(n) &= Lu(n) + L\gamma(n) \\ &= \lambda f^*(n, u(n-\tau(n)) + \gamma(n-\tau(n))) + r(n) \\ &= \lambda f(n, u(n-\tau(n)) + \gamma(n-\tau(n))) + r(n) \\ &= \lambda f(n, y(n-\tau(n))) + r(n). \end{aligned}$$

The conclusion is proved. □

Proof of Theorem 2.2. For λ_1 defined in (3.8), by Theorem 2.1 and Remark 3.1, we know that, for $\lambda \in (0, \lambda_1)$, (1.1) has a positive periodic solution $y_1(n)$ satisfying $y_1(n) > \gamma(n)$ on $\mathbb{Z}$ and

$$\|y_1\| \geq 1/2 + \|\gamma\|. \tag{3.16}$$

In what follows, we show the existence of a second positive periodic solution of (1.1). Define a completely continuous operator $S : X \to K$ by

$$Su(n) = \lambda \sum_{i=1}^{\omega} \sum_{j=1}^{\omega} G(i,j) f^*(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})).$$

By Lemmas 3.1 and 3.5, we see that finding a periodic solution of (1.1) is equivalent to finding a fixed point $u(n)$ of S in X with $0 \leq u(n) \leq q$ on $\mathbb{Z}$. Moreover, as showing $T(K) \subset K$ in the proof of Theorem 2.1, we have that $S(K) \subset K$.

Let

$$H = \min\{0.4,\ q\}$$

and

$$0 < \lambda < \lambda_2 := \min\left\{\lambda_1,\ \frac{H}{\delta_2 \omega^2 M_1}\right\},$$

where

$$M_1 = \max\{f^*(n, u(n-\tau(n)) + \gamma(n-\tau(n))) : n \in \mathbb{Z} \text{ and } 0 \leq u \leq H\}.$$

Choose

$$\Omega_3 = \{u \in X \ : \ \|u\| < H\}.$$

Then, for $u \in K \cap \partial\Omega_3$, we have

$$\begin{aligned}
Su(n) &\leq \lambda\delta_2 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} f^*(n_{ij}, u(n_{ij}^{\tau})) + \gamma(n_{ij}^{\tau})) \\
&= \lambda\delta_2\omega \sum_{i=1}^{\omega} f^*(i, u(i-\tau(i)) + \gamma(i-\tau(i))) \\
&\leq \lambda\delta_2\omega^2 M_1 \leq H = \|u\|.
\end{aligned}$$

Hence,

$$\|Su\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_3. \tag{3.17}$$

From (3.14), it follows that

$$\lim_{x\to 0^+} \frac{f^*(n, x + \gamma(n-\tau(n)))}{x} = \infty, \quad n \in \mathbb{Z}.$$

Thus, there exists a constant $0 < r_0 < H$ such that

$$f^*(n, x + \gamma(n-\tau(n))) \geq \eta x, \quad (n, x) \in \mathbb{Z} \times [0, r_0]$$

where $\eta > 0$ is large enough so that

$$\lambda\delta_1\omega^2\eta\delta \geq 1.$$

Let

$$\Omega_4 = \{u \in X \ : \ \|u\| < r_0\}.$$

Then, for $u \in K \cap \partial\Omega_4$, we have

$$\begin{aligned}
Su(n) &\geq \lambda\delta_1 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} f^*(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \\
&= \lambda\delta_1\omega \sum_{i=1}^{\omega} f^*(i, u(i-\tau(i)) + \gamma(i-\tau(i))) \\
&\geq \lambda\delta_1\omega\eta \sum_{i=1}^{\omega} u(i-\tau(i)) \\
&\geq \lambda\delta_1\omega^2\eta\delta\|u\| \geq \|u\|.
\end{aligned}$$

Hence,

$$\|Su\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_4. \tag{3.18}$$

In view of (3.17) and (3.18), S has a fixed point $u \in K \cap (\overline{\Omega}_3 \setminus \Omega_4)$ by Lemma 3.4. Clearly, $r_0 \le \|u\| \le H = \min\{0.4,\ q\}$, and Lemma 3.1 implies that $u(n)$ is a periodic solution of (3.15). From Lemma 3.5, $y_2(n) = u(n) + \gamma(n) > \gamma(n)$ is a solution of (1.1). Note that

$$\|y_2\| \le \|u\| + \|\gamma\| \le 0.4 + \|\gamma\|.$$

Then, from (3.16), we see that $y_2(n) \not\equiv y_1(n)$ on $\mathbb{Z}$, i.e., $y_2(n)$ is a second positive periodic solution of (1.1). This completes the proof of the theorem. $\square$

Proof of Theorem 2.3. Let the cone K and the completely continuous operator T be defined by (3.4) and (3.7), respectively. Then, as before, $T(K) \subset K$ for $\lambda > 0$. Let

$$B_1 = \frac{4\rho}{\delta\delta_1\omega} + \frac{\|\gamma\|}{\delta_1\omega}.$$

Then, (H5) implies that there exists $C_1 > 0$ such that

$$f(n_2, x + \gamma(n_2 - \tau(n_2))) \ge B_1 \quad \text{for } x \in (C_1, \infty).$$

Let

$$\lambda > \lambda_3 := \max\left\{1,\ \frac{1}{3\rho}C_1\right\} \tag{3.19}$$

and define

$$\Omega_5 = \left\{u \in X \ :\ \|u\| < \frac{4\lambda\rho}{\delta} + \|\gamma\|\right\}.$$

Then, for $u \in K \cap \Omega_5$, we have

$$u(n) - v(n) = u(n) - \lambda\phi(n) \ge \delta\|u\| - \lambda\rho \ge 3\lambda\rho > C_1.$$

Thus, from (3.7) and (3.19), we obtain that

$$\begin{aligned}
Tu(n) &= \lambda \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j)[f(n_{ij}, u(n_{ij}^{\tau}) - v(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) + M(n_{ij})] \\
&\ge \lambda\delta_1 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} [f(n_{ij}, u(n_{ij}^{\tau}) - v(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) + M(n_{ij})] \\
&= \lambda\delta_1\omega \sum_{i=1}^{\omega} [f(i, u(i-\tau(i)) - v(i-\tau(i)) + \gamma(i-\tau(i))) + M(i)] \\
&\ge \lambda\delta_1\omega f(n_2, u(n_2-\tau(n_2)) - v(n_2-\tau(n_2)) + \gamma(n_2-\tau(n_2))) \\
&\ge \lambda\delta_1\omega B_1 \ge \frac{4\lambda\rho}{\delta} + \|\gamma\| = \|u\|.
\end{aligned}$$

Hence,

$$\|Tu\| \ge \|u\| \quad \text{for } u \in K \cap \partial\Omega_5. \tag{3.20}$$

Let

$$F(x) = \max_{n\in[1,\omega]}[f(n, x+\gamma(n-\tau(n))) + M(n)] \quad \text{for } x \in [0,\infty).$$

Then, (H5) implies that $F(x)$ is unbounded on $[0,\infty)$. Choose $\eta > 0$ small enough so that

$$\lambda\delta_2\omega^2\eta \le 1.$$

From (H6), we see that

$$\lim_{x\to\infty}\frac{F(x)}{x} = 0.$$

Then, there exists $E > 0$ such that

$$F(x) < \eta x \quad \text{for } x \in (E,\infty).$$

Since $F(x)$ is unbounded on $[0,\infty)$, there exists $R_1 > \max\{4\lambda\rho/\delta + \|\gamma\|,\ E\}$ such that

$$F(x) \le F(R_1) \quad \text{for } x \in [0, R_1].$$

Let

$$\Omega_6 = \{u \in X \ : \ \|u\| < R_1\}.$$

Then, for $u \in K \cap \partial\Omega_6$, we have $0 \le [u(n) - v(n)]^* \le u(n) \le R_1$ on $\mathbb{Z}$. Thus, we have

$$\begin{aligned}
Tu(n) &\le \lambda\delta_2\sum_{i=1}^{\omega}\sum_{j=1}^{\omega}[f(n_{ij}, [u(n_{ij}) - v(n_{ij})]^* + \gamma(n_{ij})) + M(n_{ij})]\\
&= \lambda\delta_2\omega\sum_{i=1}^{\omega}[f(i, [u(i-\tau(i)) - v(i-\tau(i))]^* + \gamma(i)) + M(i)]\\
&\le \lambda\delta_2\omega^2 F(R_1)\\
&\le \lambda\delta_2\omega^2\eta R_1\\
&\le R_1 = \|u\|.
\end{aligned}$$

Hence,

$$\|Tu\| \le \|u\| \quad \text{for } u \in K \cap \partial\Omega_6. \tag{3.21}$$

In view of (3.20) and (3.21), T has a fixed point $u \in K \cap (\overline{\Omega}_6 \setminus \Omega_5)$ by Lemma 3.4. From Lemma 3.2 (a), we have

$$u(n) \ge 4\lambda\rho/\delta + \|\gamma\| > \lambda\rho \ge \lambda\phi(n) = v(n) \quad \text{for } n \in \mathbb{Z}.$$

Then, by Lemma 3.3, (1.1) has a solution $y(n) = u(n) - v(n) + \gamma(n) > \gamma(n)$ on $\mathbb{Z}$. This completes the proof of the theorem. □

Remark 3.2. For the solution $y(n)$ of (1.1) given in the proof of Theorem 2.3, we have

$$\|y\| \ge \|u - v\| \ge \|u\| - \|v\| \ge 4\lambda\rho/\delta + \|\gamma\| - \lambda\rho > 3\lambda\rho/\delta + \|\gamma\|.$$

Proof of Theorem 2.4. For λ_3 defined in (3.19), by Theorem 2.3 and Remark 3.2, we know that, for $\lambda \in (\lambda_3, \infty)$, (1.1) has a positive periodic solution $y_1(n)$ satisfying $y_1(n) > \gamma(n)$ on $\mathbb{Z}$ and

$$\|y_1\| > 3\lambda\rho/\delta + \|\gamma\|. \tag{3.22}$$

In the following, we show the existence of a second positive periodic solution of (1.1). Consider the following equation

$$Lu(n) = \lambda f(n, u(n-\tau(n)) + \gamma(n-\tau(n))), \quad n \in \mathbb{Z}. \tag{3.23}$$

Then, as in Lemma 3.5, we can prove the following claim:
Claim 1: If (3.23) has a solution $u(n)$, then $y(n) = u(n) + \gamma(n)$ is a solution of (1.1).

Let the cone K be defined by (3.4). Define a completely continuous operator $W : X \to K$ by

$$Wu(n) = \lambda \sum_{i=1}^{\omega} \sum_{j=1}^{\omega} G(i,j) f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})).$$

By Lemmas 3.1 and the above Claim 1, we have
Claim 2: If $u \in X$ is a fixed point of W, then $y(n) = u(n) + \gamma(n)$ is a periodic solution of (1.1).

Let

$$\Omega_7 = \left\{ u \in X \ : \ \|u\| < \frac{2\lambda\rho}{\delta} \right\}$$

and

$$B_2 = \frac{2\rho}{\delta\delta_1\omega}.$$

Then, (H5) implies that there exists $C_2 > 0$ such that

$$f(n_2, x + \gamma(n_2 - \tau(n_2))) \geq B_2 \quad \text{for } x \in (C_2, \infty). \tag{3.24}$$

Let

$$\lambda > \lambda_4 := \max\left\{ \lambda_3, \ \frac{C_2}{\rho} \right\}.$$

Then, for $u \in K \cap \partial\Omega_7$, we see that

$$u(n) - v(n) = u(n) - \lambda\phi(n) \geq \delta\|u\| - \lambda\rho = 2\lambda\rho - \lambda\rho = \lambda\rho > C_2.$$

Thus, from (3.24), we have

$$\begin{aligned} Wu(n) &\geq \lambda\delta_1 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \\ &= \lambda\delta_1\omega \sum_{i=1}^{\omega} f(i, u(i-\tau(i)) + \gamma(i-\tau(i))) \\ &\geq \lambda\delta_1\omega f(n_2, u(n_2-\tau(n_2)) + \gamma(n_2-\tau(n_2))) \\ &\geq \lambda\delta_1\omega B_2 = \frac{2\lambda\rho}{\delta} = \|u\|. \end{aligned}$$

This shows that

$$\|Wu\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_7. \tag{3.25}$$

By (H7), there exists $R_2 > 0$ such that $R_2 < 2\lambda\rho/\delta$ and

$$f(n, x + \gamma(n-\tau(n))) < \zeta x \quad \text{for } (n, x) \in \mathbb{Z} \times [0, R_2],$$

where ζ is small enough so that

$$\lambda\delta_2\omega^2\zeta \leq 1.$$

Define

$$\Omega_8 = \{u \in X \ : \ \|u\| < R_2\}.$$

Then, for $u \in K \cap \partial\Omega_8$, we have

$$\begin{aligned} Wu(n) &\leq \lambda\delta_2 \sum_{i=1}^{\omega}\sum_{i=1}^{\omega} f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \\ &= \lambda\delta_2\omega \sum_{i=1}^{\omega} f(i, u(i-\tau(i)) + \gamma(i-\tau(i))) \\ &\leq \lambda\delta_2\omega\zeta \sum_{i=1}^{\omega} u(i-\tau(i)) \\ &\leq \lambda\delta_2\omega^2\zeta\|u\| \leq \|u\|. \end{aligned}$$

Hence, we obtain that

$$\|Wu\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_8. \tag{3.26}$$

In view of (3.25) and (3.26), W has a fixed point $u \in K \cap (\overline{\Omega}_7 \setminus \Omega_8)$ by Lemma 3.4. Clearly, $R_2 \leq \|u\| \leq 2\lambda\rho/(\delta)$. By the above Claim 2, $y_2(n) = u(n) + \gamma(n) > \gamma(n)$ is a positive periodic solution of (1.1). Note that

$$\|y_2\| \leq \|u\| + \|\gamma\| \leq 2\lambda\rho/\delta + \|\gamma\|.$$

Then, from (3.22), we see that $y_2(n) \not\equiv y_1(n)$ on $\mathbb{Z}$, i.e., $y_2(n)$ is a second positive periodic solution of (1.1). This completes the proof of the theorem. $\square$

Finally, with $f(n,x) = c(n)x^{\mu(x)} - d(n)$, it is easy to see that parts (a)–(d) of Corollary 2.1 follow directly from Theorems 2.1–2.4, respectively. The details are omitted.

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