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Coarse Structures and Higson Compactification

Christian Stuart Hoffland
University of Tennessee, Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by Christian Stuart Hoffland entitled "Coarse Structures and Higson Compactification." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Jurek Dydak, Major Professor

We have read this thesis and recommend its acceptance:

Nikolay Brodskiy, James Conant

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Dean of Graduate Studies

Coarse Structures and Higson Compactification

A thesis presented for the Master of Science degree
University of Tennessee, Knoxville

Christian Stuart Hoffland
August 2006

To John,
for everything,

and to my parents,
for always believing in me

Acknowledgments

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Proofreaders for any thesis are hard to come by, and I must express my gratitude to John Rumple for helping with some of the text sections of this work. I can only hope that when proofreading his work that he will receive from me the same level of skill and patience he lavished on mine. Žiga Virk pointed out some errors in an early version, and his various suggestions proved extremely useful. Any remaining errors, however, are completely my responsibility.

Conversations with Brendon LaBuz and Atish Mitra were helpful in discussing possible counterexamples and eliminating potential errors: my thanks.

Making LaTeX do what one wants can sometimes be a hard task. Fortunately for me, Mike Saum has both the technical ability and saint-like patience to deal with the many, many queries and problems that came from me on what seemed like a daily basis. The look of the text is very much the better for his presence in the department.

Any graduate student in mathematics knows that the graduate secretary is the first person to go for assistance. Pam Armentrout has provided support and advice in that capacity before I even arrived at the department, and has always been available since. To her goes the warmest appreciation for her dedication to the graduate students she serves.

Finally, my special thanks go again to my partner, John Rumple, for, among countless other things, his advice and thoughts on the process of writing. This, as one discovers when one begins any serious writing, is a lot harder than it looks, and his generosity in sharing his experience—and the experience of others—in the written word has benefited this small book in more ways than I can mention here. ❦

Abstract

FOLLOWING John Roe in his *Lectures on Coarse Geometry*, we begin by describing the large-scale structure of metric spaces by means of *coarse maps* between them, those being maps which preserve distances at large scales. Using these techniques, we demonstrate that the real numbers and the integers have the same large scale structure—or are *coarsely equivalent*—but that the real line is coarsely equivalent to neither the Euclidean plane nor the set of positive real numbers. Following a generalization of these concepts for general topological spaces with the introduction of an abstract *coarse structure* on the space, we show, among other things, that the real line is not coarsely equivalent to the *long line* of the countable ordinals with the order topology.

We depart from Roe to describe a connection between locally compact Hausdorff spaces and a sub-class of Banach algebras known as C^*-algebras, where we find that every such algebra can be described as a set of continuous functions on a particular locally compact Hausdorff space. In particular, we see that there is a very strong relationship between the two: the *categories* of C^*-algebras and $*$-homomorphisms, and the opposite of locally compact Hausdorff spaces and continuous functions, are *dual*.

Returning to coarse structures, we examine *compactifications* of locally compact Hausdorff spaces with a view to construction of a *topological coarse structure*, one in which those maps which are coarse are precisely those which may be continuously extended to the boundary of the space. We complete our investigation by describing the inverse process: given a coarse structure, can we find a compactification possessing the same properties? We provide an partial answer to this question, called a *Higson compactification*, and end by calculating Higson compactifications of some familiar spaces. ❦

Abstract

Preface

THE typical study of metric or topological spaces involves examination of the *small-scale structure* of the space in varying ways. The quintessential property, *continuity*, depends on either functional values taken at small distances, or on the inverse image of a neighborhood: that is to say, small-scale properties.

To study the *large-scale structure* of a metric or topological space, one naturally becomes concerned with properties which hold at large scales, such as *boundedness*, *degrees of freedom*, and *restriction of movement*: continuity becomes correspondingly less important, since it has little impact on these qualities of a space. *Coarse geometry* provides a set of tools for discussion of large-scale structure by consideration of maps which preserve these properties.

This thesis examines the foundations of coarse geometry, with a particular emphasis in describing a certain *compactification* of a locally compact Hausdorff space, by following John Roe in [16]. Although it is assumed that the reader has a basic knowledge of geometry and topology, many of the terms used are defined either in the text itself, or in the Appendix. Care has been taken to ensure that any topological space or structure which might be unfamiliar is given at least a brief definition. Particular attention is given to descriptions of the *long line* and the *sphere at infinity* since these spaces are familiar probably only to those who have taken a course in topology.

Examples have been provided throughout to illustrate the ideas presented. Any background material which was thought necessary—but not central to the main arguments—has been included in an extensive appendix to the text, including a summary of some ideas from category theory.

The first two chapters outline the basic principles of coarse geometry, first by defining the required concepts for metric spaces, and then generalizing them for, primarily, topological spaces. Some knowledge of metric and topological spaces is assumed: [5] provides an excellent introduction to the former, while the first few chapters of [14] would provide the rest of what is required.

Familiarity with C^*-algebras is *not* assumed, and since this material was deemed important enough for inclusion, an entire chapter on it is justifiable. There were two reasons for

providing the material. The first is that C^*-algebras are generally studied in the context of functional analysis, and as a result, geometers or topologists may have had no cause to become acquainted with the subject. Since this text is primarily of interest to them, an account has been given of the basic facts connected with C^*-algebras. The second reason is the very strong connection between the compactifications of locally compact Hausdorff spaces and C^*-algebras which Roe assumes in [16], and is vital for an understanding of most of the fourth chapter. The third chapter can be skipped if the reader is conversant with the main concepts involved.

In the fourth chapter some familiarity with locally compact Hausdorff spaces is assumed, although introductory material on these is again included in the Appendix.

All proofs and figures are mine unless otherwise stated. When a proof is not original, or if it is an adaptation of another author's proof, the reference is given in the text.

August 1, 2006 *Christian Stuart Hoffland*
Knoxville, Tennessee

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List of Symbols

$\sim$	is close to; is related to
$\cong$	is $*$-isomorphic to
$\approxeq$	is coarsely equivalent to
$\frown$	is M-scale connected to
$\\|f\\|_\infty$	the supremum norm of f; $\sup_{x \in X} \lvert f(x)\rvert$
$[0,\pi)^n$	$[0,\pi) \times \cdots \times [0,\pi)$, an n-dimensional subset of $\mathbb{R}^n$
A^*	the dual space of a C*-algebra A; the set $C(A,\mathbb{C})$
$\aleph_0$	the cardinality of $\mathbf{Z}$
$B_d(x,r)$, $B(x,r)$	a metric ball of radius r; the set $\{y \mid d(x,y) < r\}$
$\mathcal{B}(X,Y)$	the set of bounded functions from X to Y
$B_0(X)$	the set of bounded functions which tend to a constant at infinity
$B_h(X)$	the set of bounded functions $f : X \longrightarrow \mathbb{C}$ with the Higson property
βX	the Stone–Čech compactification of X
$\mathcal{C}_0$	the bounded coarse structure corresponding to a metric d
$C(X)$	the set of complex-valued continuous functions on X; the set $C(X,\mathbb{C})$
$C_0(X)$	the set of complex-valued continuous bounded functions on X which vanish at infinity
$C_b(X)$	the set of complex-valued continuous bounded functions on X
$C_h(X)$	the set of Higson functions on X
$C(X,Y)$	the set of continuous functions from X to Y; a subset of Y^X
$\mathbf{C}^{\mathrm{op}}$	the opposite category of $\mathbf{C}$
Δ	the diagonal of $X \times X$
$d_X(x,y)$, $d(x,y)$	the value of the metric d_X applied to (x,y); the distance between $x, y \in X$
$d_{X\times Y}$	the max metric on X and Y
d_∞	the extended standard metric on either $\mathbb{R}_\infty$ or $\mathbb{R}^+_\infty$
$\mathbf{d}f$	the function $\mathbf{d}f : X \times X \longrightarrow \mathbb{C}$ given by $\mathbf{d}f(x,y) = f(y) - f(x)$ for $f \in C(X)$

∂X	the boundary of $\overline{X}$; the set $\overline{X} \setminus X$
$E \circ F$	the product of $E, F \subset X \times X$; the set $\{(x, y) \mid \exists z \in X : (x, z) \in E, (z, y) \in F\}$
E^{-1}	the inverse of $E \subset X \times X$; the set $\{(y, x) \mid (x, y) \in E\}$
$E[K]$	the set $\{x \in X \mid \exists y \in K : (x, y) \in E\}$
E^x	the x-section of E; the set $E^{-1}[\{x\}]$
E_y	the y-section of E; the set $E[\{y\}]$
$f \times j$	the product map of maps f and j
$f\|_A$	the restriction of the map f to a set A
hX	the Higson compactification of X
$\mathrm{Hom}_{\mathbf{C}}(A, B)$	the set of morphisms from an object A to an object B in the category $\mathbf{C}$
id_X	the identity map on X; the map $id_X(x) = x$
$\mathrm{Im}\, f$	the image of $f : X \longrightarrow Y$; the set $\{f(x) \mid x \in X\}$
$\mathrm{Ker}\, f$	the kernel of $f : X \longrightarrow Y$; the set $\{x \in X \mid f(x) = 0_Y\}$
$\mathbb{N}$	the set of natural numbers; $\{0, 1, 2, \dots\}$
ω	the first infinite ordinal
$\Omega(A)$	the dual of a C^*-algebra A; the set $A^* \setminus \{0 : A \longrightarrow \mathbb{C}\}$
π_i	the ith coordinate projection
$\mathfrak{P}(X)$	the power set of X; the set $\{A \mid A \subset X\}$
$\mathbb{Q}$	the set of rational numbers
ρ	the standard bounded metric on $\mathbb{R}$; $\rho(x, y) = \min\{d(x, y), 1\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of non-negative real numbers
$\mathbb{R}^-$	the set of non-positive real numbers
$\mathbb{R}_\infty$	the extended real numbers; $\mathbb{R} \cup \{-\infty, \infty\}$
$\mathbb{R}^+_\infty$	the positive extended real numbers; $\mathbb{R}^+ \cup \{\infty\}$
S^{n-1}	the unit $(n-1)$-sphere; the set $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$
S^{n-1}_∞	the sphere at infinity
S_Ω	the minimal well-ordered uncountable set
νX	the Higson corona of hX; the set $\nu X = hX \setminus X$
$\overline{X}$	the compactification of X
(X, d)	a metric space X with metric d
Y^X	the set of all functions from X into Y; $\{f : X \longrightarrow Y\}$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers

CHAPTER 1 ∼

Elementary Coarse Geometry

We begin by stating explicitly how large-scale structures on metric spaces are compared, by introducing the idea of *coarseness*. A *coarse map* is one which preserves distances at large scales: it sends an unbounded set to an unbounded set, and does not allow finite distances to become infinite upon mapping. We also define *closeness* of maps, a term which indicates that two maps are uniformly bounded. The central idea in the chapter is that of a *coarse equivalence*—the term used when metric spaces have the same large-scale structure—which is comprised of two coarse maps whose compositions, in whichever order, are close to the respective identity maps on each space.

These terms give rise to an assortment of examples and consequences of the definitions. The main theorem of the chapter proves that coarseness is indeed an equivalence relation on the class of metric spaces. We end by demonstrating coarse equivalence between some familiar metric spaces, such as between $\mathbb{Z}$ and $\mathbb{R}$. We shall also show that $\mathbb{R}$ is not coarsely equivalent to $\mathbb{R}^2$, and neither is $\mathbb{R}$ coarsely equivalent to $\mathbb{R}^+$.

1.1 CHOICE OF METRIC

As with small-scale structure, the large-scale properties of a metric space (X, d) depend not only on the set X, but on the choice of metric $d : X \times X \longrightarrow \mathbb{R}$. One large-scale property which can be dramatically affected by the choice of metric is *boundedness*. Even a metric which preserves small-scale structure can alter the large-scale structure.

EXAMPLE 1.1. If we take $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, then the metric space $(\mathbb{R}, d)$ is unbounded. We define the *standard bounded metric* $\rho : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\rho(x, y) = \min\{d(x, y), 1\}.$$

That ρ actually is a metric is shown in [5]. The metric space $(\mathbb{R}, \rho)$ is homeomorphic to $(\mathbb{R}, d)$ since $B_\rho(x, r) = B_d(x, r)$ for any $r < 1$, so these spaces therefore have the same small-scale structure. But ρ is bounded whereas d is not, so they do not have the same large-scale structure.

In general we shall assume that all metrics are the usual ones unless otherwise stated. One important property which these metrics have is that they are *proper*.

Definition 1.2. A metric $d : X \times X \longrightarrow \mathbb{R}$ on a space X is *proper* if closed and bounded sets are compact. A *proper metric space* is a space with a proper metric.

In other words, a bounded set in a proper metric space is *relatively compact*: it can be made compact merely by taking its closure.

One reason for the greater ease with which we can compare large-scale structure between metric spaces—as opposed to, say, topological spaces—is that all of the familiar metric spaces are proper. It is a routine matter, for example, to show that the usual metric $d(x,y) = |x - y|$ on $\mathbb{R}$ is so. The concept of properness for a topological space, however, is not quite as immediate since we have, at the moment, no concept of what *bounded* means. A substantial part of the work we will do in the next chapter is to formulate a definition of boundedness for topological spaces.

1.2 Coarse maps

The following definition is due to Roe in [16].

Definition 1.3. Let X and Y be metric spaces, and let $f : X \longrightarrow Y$ be a map.

(*a*) The map f is (*metrically*) *proper* if the inverse image, under f, of each bounded subset of Y is a bounded subset of X.

(*b*) The map f is (*uniformly*) *bornologous* if for every $R > 0$ there is an $S > 0$ such that

$$d(x,y) < R \Rightarrow d(f(x), f(y)) < S.$$

(*c*) The map f is *coarse* if it is proper and bornologous.

It is important to note that a map f being bornologous does not imply that the map is continuous. Neither is the converse true.

Example 1.4. (*a*) Let $X = \mathbb{R}$ and $Y = \mathbf{Z}$, and suppose that $f(x) = \lfloor x \rfloor$ is the *integer part* of x. Then $f(x) \leq x < f(x) + 1$ for all $x \in X$. Thus for every $R > 0$ there is an $S = R + 1 > 0$ such that if $d(x,y) < R$ then

$$d(f(x), f(y)) = d(\lfloor x \rfloor, \lfloor y \rfloor) < R + 1 = S$$

so f is bornologous. But f is not continuous at each integer point in $\mathbb{R}$.

(*b*) Let X and Y each be the subspace $(0,1] \subset \mathbb{R}$ and let $f(x) = 1/x$ for each $x \in X$. Then f is continuous. But for $R = 1$ and $y = 1$, each $x \in X$ is such that $d(x,1) < 1$, but there does not exist an $S > 0$ such that

$$d(x,1) < 1 \Rightarrow d(f(x), f(1)) = |1/x - 1| < S$$

since we may let x become arbitrarily small. Thus f is not bornologous.

We explore some other examples of maps which are proper, bornologous, or both.

Example 1.5. (*a*) For any metric space X it is obvious that the *identity map* $id_X : X \longrightarrow X$ is always coarse. If X is a subspace of a metric space Y, then the *inclusion map* $x \longmapsto x$ from X into Y is coarse.

(*b*) That inclusion maps are coarse is a specific instance of the fact that restrictions of coarse maps are coarse, since the proper and bornologous conditions hold on taking subsets.

Example 1.6. Let $f : X \longrightarrow Y$ be a map between metric spaces. If X is bounded then f is proper. If Y is bounded then f is bornologous. It follows that if X and Y are both bounded, then f is coarse.

Example 1.7. If $X = \mathbb{R}$ and $Y = \mathbb{R}$ then any *linear map* $f(x) = ax+b$ for some $a, b \in \mathbb{R}$, with a non-zero, is coarse. The inverse image under f of any bounded subset of Y is translated by b and scaled by a factor of $1/a$, and so is bounded in X. It follows that f is proper. And if $R > 0$ and $d(x, y) < R$ then there is an $S = a \cdot R$ such that

$$d(f(x), f(y)) = d(ax + b, ay + b) = a \cdot d(x, y) < a \cdot R = S$$

so f is bornologous. It follows that f is coarse.

Example 1.8. The above example can be generalized. A map $f : X \longrightarrow Y$ between metric spaces X and Y is said to be *Lipschitz* if there is a $\lambda \in \mathbb{R}^+$ such that

$$d(f(x), f(y)) \le \lambda d(x, y)$$

for all $x, y \in X$. Clearly a Lipschitz map is bornologous: for any $R > 0$, set $S = \lambda R$. A *bi-Lipschitz* map satisfies the additional condition that there is a $\mu \in \mathbb{R}^+$ such that $\mu \le \lambda$ and

$$\mu d(x, y) \le d(f(x), f(y)) \le \lambda d(x, y).$$

Let $B \subset Y$ be bounded by M. Then for each $x, y \in f^{-1}(B)$ we have $d(x, y) \le M/\mu$. It follows that if f is bi-Lipschitz, then f is proper. Since bi-Lipschitz maps are Lipschitz by definition, they are therefore also coarse.

Example 1.9. If (X, d_X) and (Y, d_Y) are metric spaces then any map $f : X \longrightarrow Y$ which is an *isometry*, or that

$$d_Y(f(x), f(y)) = d_X(x, y)$$

for all $x, y \in X$, is coarse. Inverse images of bounded sets are obviously bounded, and for any $R > 0$, there is an $S = R$ such that

$$d_X(x, y) < R \Rightarrow d_Y(f(x), f(y)) = d(x, y) < R = S.$$

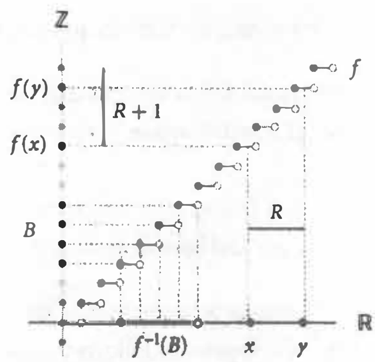

Figure 1.1: The inverse image of B under $f(x) = \lfloor x \rfloor$ is bounded, so f is proper. Points in $\mathbb{R}$ which are less than R apart are less than $R + 1$ apart upon mapping by f.

Example 1.10. Let $X = \mathbb{R}$, let $Y = \mathbf{Z}$, and let f again be the map $x \longmapsto \lfloor x \rfloor$ for all $x \in X$ (Figure 1.1). Then f is proper, for if $A \subset \mathbf{Z}$ is bounded then $A \subset \{-n, \dots, n\}$ for some $n \in \mathbf{Z}$. But

$$f^{-1}(A) \subset f^{-1}(\{-n, \dots, n\}) = [-n, n+1),$$

which is bounded. Example 1.4 showed that f is bornologous, so f is coarse.

It is clear that a coarse map $f : X \longrightarrow Y$ between unbounded metric spaces X and Y must map an unbounded subset $A \subset X$ to an unbounded subset of Y; that is to say, a coarse map is *unbounded*. Otherwise, if $f(A)$ is bounded, $f^{-1}(f(A))$ is also bounded, but

$$A \subset f^{-1}(f(A)),$$

which is a contradiction. It is also true that compositions of coarse maps are coarse.

Proposition 1.11. *Let X, Y, and Z be metric spaces, and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be coarse maps. Then $g \circ f : X \longrightarrow Z$ is coarse.*

Proof. If B is a bounded subset of Z then the properness of g ensures that $g^{-1}(B)$ is bounded, and that f is also proper means that $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is bounded, so $g \circ f$ is proper.

Let $R > 0$. Since f is bornologous there is an $S > 0$ such that $d(x, x') < R$ implies that $d(f(x), f(x')) < S$ for all $x, x' \in X$. Since g is bornologous there is a $T > 0$ such that if $d(y, y') < S$ then $d(g(y), (y')) < T$ for all $y, y' \in Y$, so this certainly applies to those $y, y' \in \operatorname{Im} f$. Letting $y = f(x)$ and $y' = f(x')$ means that there is a $T > 0$ such that $d(x, x') < R$ implies

$$d((g \circ f)(x), (g \circ f)(x')) = d(g(f(x)), g(f(x'))) < T$$

for all $x, x' \in X$, so $g \circ f$ is bornologous.

Since $g \circ f$ is proper and bornologous, the composition is coarse, as required. □

The converse, that if $g \circ f$ is coarse then both f and g are coarse, is not true, as the following example demonstrates.

Example 1.12. Let $f : \mathbb{R} \longrightarrow [0,1)$ be defined by $f(x) = x - \lfloor x \rfloor$, and let $g : [0,1) \longrightarrow \mathbb{R}$ be the inclusion map. Then $f \circ g : [0,1) \longrightarrow [0,1)$ is automatically coarse by Example 1.4 since $[0,1)$ is bounded for both the domain and co-domain. But f fails to be proper, since the inverse image of the bounded set $\{0\}$ under f is $f^{-1}(\{0\}) = \mathbb{Z}$, which is unbounded. Thus f is not coarse.

1.3 Closeness in metric spaces

Definition 1.13. We say that two maps f, g from a set X into a metric space (Y, d) are *close* [16], and write $f \sim g$, if $d(f(x), g(x))$ is bounded uniformly in X; that is, if

$$\sup\{d(f(x), g(x)) \mid x \in X\} < \infty.$$

Closeness, in geometric terms, means that the subset $E = \{(f(x), g(x))\}$ of $Y \times Y$ is "not too far" away from the diagonal $\Delta = \{(y, y) \mid y \in Y\}$, meaning that over the whole of X, g cannot map to values which are an arbitrarily large distance from f.

Example 1.14. A map from a set X into a metric space (Y, d) is always close to itself since $d(f(x), f(x)) = 0$ for all $x \in X$ by positive definiteness of d.

Example 1.15. Two constant maps k and l into a metric space (Y, d) are always close, for $d(k(x), l(x))$ is constant, and therefore uniformly bounded in X.

Closeness is an equivalence relation on the set of maps from X into Y.

Proposition 1.16. *Let $\mathcal{F}$ be the set of all maps from a set X into a metric space (Y, d). Let f, g be elements of $\mathcal{F}$, and say that $f \sim g$ if and only if f is close to g. Then $\sim$ is an equivalence relation on $\mathcal{F}$.*

Proof. That $\sim$ is reflexive follows from Example 1.14, since a map is always close to itself. Symmetry is clear by symmetry of the metric d. For transivity, if f is close to g, and g is close to h, then

$$\sup\{d(f(x), g(x)) \mid x \in X\} \leq M \quad \text{and} \quad \sup\{d(g(x), h(x)) \mid x \in X\} \leq N,$$

for some finite M and N. Then by the triangle inequality, for any $x \in X$,

$$d(f(x), h(x)) \leq d(f(x), g(x)) + d(g(x), h(x)) \leq M + N < \infty$$

so f is close to h. The result follows. □

Thus we have equivalence classes of maps from X into Y. Example 1.15 demonstrated that all constant maps are members of the same equivalence class. We can say a little more than this, however.

Proposition 1.17. *Let X be a set, and let (Y, d) be a metric space. Then the following are equivalent.*

(a) *Any two maps $f, g : X \longrightarrow Y$ are close.*
(b) *For each map $f : X \longrightarrow Y$ there is a constant map $k : X \longrightarrow Y$ to which f is close.*

Proof. That $(a) \Rightarrow (b)$ is obvious. For $(b) \Rightarrow (a)$, given any two maps f and g there are constant maps k_f and k_g to which f and g are close, respectively. Since any two constant maps are close, and closeness is an equivalence relation on the set of maps from X into Y, it follows that f and g are close, which completes the proof. □

Corollary 1.18. *Let X be a set, and let (Y, d) be a metric space. If Y is bounded, then any two maps from X into Y are close. If the cardinality of X is at a minimum countably infinite, then any two maps from X into Y being close implies that Y is bounded.*

Proof. If Y is bounded by, say, M, then for any two maps $f, g : X \longrightarrow Y$ we have

$$d(f(x), g(x)) \leq M$$

for all $x \in X$, whence f is close to g.

If we additionally require that $|X| \geq \aleph_0$, then the converse holds. Suppose that any two maps from X into Y are close, but that Y is unbounded. This implies that, on choosing a $y_0 \in Y$, we can find a sequence $(y_i)_{i \geq 1}$ such that $d(y_0, y_i) \longrightarrow \infty$. For each $i \geq 1$ choose an $x_i \in X$, which is possible since X is infinite, and let a subset $Z \subset X$ be given by $Z = \{x_i | i \in \mathbb{N}\}$. Define a map $f : X \longrightarrow Y$ by the rule $f(x) = y_i$ if $x = x_i$ and $f(x) = y_0$ otherwise. Then for any constant map $k : X \longrightarrow Y$ it follows that $\{d(f(x), k(x)) \mid x \in X\}$ is not uniformly bounded, whence f is close to no constant map. This contradicts our hypothesis by way of Proposition 1.17, which guarantees the existence of a constant map to which f is close. It follows that our assertion of Y being unbounded is false. □

The requirement that X must be at least countably infinite is demonstrated by the following example.

Example 1.19. Let X be a finite set, and let $Y = \mathbb{R}$ with the usual metric. For any map $f : X \longrightarrow Y$ it is obvious that $f(X)$ is finite, and hence bounded. Thus each map is close to some constant map (in fact, every constant map). But Y is unbounded.

1.4 Coarse equivalence

At the beginning of this chapter we claimed that two spaces have the same large-scale structure if we can find two maps, with certain properties, whose compositions are close to the identity maps on their respective domains. The properties we required were that the maps were coarse (proper and bornologous), but why do these maps imply that the spaces have the same large-scale structure? Essentially this has to do with *contractibility*. Spaces which

are the same on a large scale can be *scaled* so that points are not too far away from each other, but we are not concerned with any differences on a small scale that may arise.

The bornologous condition means that the distances between points are effectively scaled. The proper condition ensures that subsets of the space which are either bounded or unbounded stay that way upon mapping. And closeness of the compositions to the identities means that points can be moved by the compositions only by some (uniform) finite amount.

The definition following this makes explicit the requirements for similarity of large-scale structure. It is also natural that if two spaces are each, on a large scale, similar to a third, then all three should be similar to each other. This also is true, as the main theorem of this section demonstrates.

Definition 1.20. We say that metric spaces X and Y are *coarsely equivalent* [16] and write $X \approxeq Y$ if there are coarse maps $f : X \longrightarrow Y$ such that $f \circ g$ and $g \circ f$ are close to the identity maps id_Y and id_X on Y and X respectively.

We say also that the maps f and g with the above properties are the *corresponding coarse equivalence maps*.

Elsewhere, Roe presents an alternative definition [20]. A map $f : X \longrightarrow Y$ is a *coarse equivalence* if there exist constants $C > 0, A > 0$ such that

$$d(x,x') \le Cd(f(x), f(x')) + A, \quad \text{and} \quad d(f(x), f(x')) \le Cd(x,x') + A$$

for all $x, x' \in X$. We also say that f is *large-scale Lipschitz*. If we allow A to be equal to zero, then f actually is a bi-Lipschitz map with $\lambda = C$ and $\mu = 1/C$.

This definition is at least as strong as Definition 1.20. Suppose that $f : X \longrightarrow Y$ is a coarse equivalence. Given a bounded set $B \subset Y$, there is an $M \ge 0$ such that $d(x,x') \le M$ for all $x, x' \in B$. If $x, x' \in f^{-1}(B)$, then $f(x), f(x') \in B$, whence

$$d(x,x') \le Cd(f(x), f(x')) + A \le C \cdot M + A$$

and thus $f^{-1}(B)$ is bounded, so f is proper. If $R > 0$ and $d(x,x') < R$ then $S = C \cdot R + A > 0$ exists such that

$$d(f(x), f(x')) \le Cd(x,x') + A < C \cdot R + A = S$$

for all $x, x' \in A$, so f is bornologous. It follows that f is a coarse map.

Now we have to find a coarse map $g : Y \longrightarrow X$ satisfying the requirements of Definition 1.20. For the time being we shall find a map $g : f(X) \longrightarrow X$. For each $y \in f(X)$, choose an $x_y \in f^{-1}(\{y\}) \subset X$. Define $g(y) = x_y$ in this manner for all $y \in f(X)$, so g is a map by the axiom of choice. For any $R > 0$, set $S = C \cdot R + A$, and suppose that $d(y,y') < R$. Then

$$\begin{aligned} d(g(y), g(y')) &= d(x_y, x_{y'}) \\ &\le Cd(f(x_y), f(x_{y'})) + A \\ &= Cd(y,y') + A \\ &< C \cdot R + A = S, \end{aligned}$$

so g is bornologous. Let $B \subset X$ be bounded, so there is an M such that $d(x, x') \leq M$ for all $x, x' \in B$. Then

$$g^{-1}(B) = \{y \in Y \mid g(y) \in B\} = \{y \in Y \mid x_y \in B\},$$

so if $y, y' \in g^{-1}(B)$ we have

$$d(y, y') = d(f(x_y), f(x_{y'})) \leq Cd(x_y, x_{y'}) + A \leq C \cdot M + A,$$

whence $g^{-1}(B)$ is bounded.

We see immediately that

$$(f \circ g)(y) = f(g(y)) = f(x_y) = y$$

so $f \circ g$ actually is the identity on Y. We observe also that

$$\begin{aligned} d(x, (g \circ f)(x)) &= d(x, g(f(x))) \\ &\leq Cd(f(x), f(g(f(x)))) + A \\ &= Cd(f(x), f(x)) + A \\ &= A \end{aligned}$$

so $g \circ f \sim id_X$. It follows that $X \approxeq f(X)$ in the sense of Definition 1.20 as long as we restrict ourselves to consideration of images of the maps. No generality is lost in so doing, however, as a later result (Lemma 1.22) shows.

The term *coarse equivalence* suggests that this actually is an equivalence relation on the class of metric spaces. This is proved in the following theorem.

Theorem 1.21. *Let X and Y be metric spaces, and say that $X \approxeq Y$ if and only if X is coarsely equivalent to Y. Then $\approxeq$ is an equivalence relation.*

Proof. Clearly $X \approxeq X$ since letting f and g each be the identity map id_X on X (which Example 1.5 showed is coarse) means that both $f \circ g$ and $g \circ f$ actually *are* the identity maps from X to X, so $\approxeq$ is reflexive. If X is close to Y then by the symmetry of the definition of coarse equivalence it follows that Y is close to X. Thus symmetry of $\approxeq$ is satisfied.

It is slightly trickier to show transivity, if only due to the number of spaces and maps between them. Suppose that $X \approxeq Y$ and $Y \approxeq Z$. Then there are coarse maps $f_X : X \longrightarrow Y$, $g_Y : Y \longrightarrow X$ and $f_Y : Y \longrightarrow Z$, $g_Z : Z \longrightarrow Y$ such that $f_X \circ g_Y$ and $g_Y \circ f_X$ are close to the identity maps id_Y and id_X on Y and X, respectively, and $f_Y \circ g_Z$ and $g_Z \circ f_Y$ are close to the identity maps id_Z and id_Y on Z and Y, respectively.

The coarseness of all of the above maps together with Proposition 1.11 means that the compositions $f_Y \circ f_X : X \longrightarrow Z$ and $g_Y \circ g_Z : Z \longrightarrow X$ are coarse. Proposition 1.16 showed that closeness of maps is an equivalence relation, which we will denote again by $\sim$. By associativity of composition of functions, we then have

$$(f_Y \circ f_X) \circ (g_Y \circ g_Z) = f_Y \circ ((f_X \circ g_Y) \circ g_Z) \sim f_Y \circ (id_Y \circ g_Z) = f_Y \circ g_Z \sim id_Z$$

and

$$(g_Y \circ g_Z) \circ (f_Y \circ f_X) = g_Y \circ ((g_Z \circ f_Y) \circ f_X) \sim g_Y \circ (id_Y \circ f_X) = g_Y \circ f_X \sim id_X$$

which shows that there exist coarse maps $f = f_Y \circ f_X$ from X to Z and $g = g_Y \circ g_Z$ from Z to X such that $f \circ g \sim id_Z$ and $g \circ f \sim id_X$. We have thus shown, given $X \approxeq Y$ and $Y \approxeq Z$, that $X \approxeq Z$, so $\approxeq$ is transitive, completing the proof. □

It is a routine matter to show that if $X \approxeq Y$ then the image of each space under the appropriate coarse equivalence map is coarsely equivalent to its co-domain. This means that we can always restrict coarse equivalence maps to their images if necessary.

Lemma 1.22. *Suppose $X \approxeq Y$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be corresponding coarse equivalence maps. Then $f(X) \approxeq Y$ and $g(Y) \approxeq X$.*

Proof. Define maps $f' : f(X) \longrightarrow Y$ and $g' : Y \longrightarrow f(X)$ by letting f' be the inclusion map from $f(X)$ into Y, and letting $g' = f \circ g$. Then f' and g' are both coarse by Example 1.5 and by Proposition 1.11.

If $y \in Y$ then $f(g(y)) \in f(X)$, whence

$$(f' \circ g')(y) = f'(g'(y)) = f'(f(g(y))) = f(g(y))$$

which is uniformly bounded in Y since $f \circ g$ is also. On the other hand, if $f(x) \in f(X)$ then

$$(g' \circ f')(f(x)) = g'(f'(f(x))) = f(g(f(x)))$$

which is uniformly bounded since both $f \circ g$ and $g \circ f$ are. Thus $f(X) \approxeq Y$. The symmetry of the definition of coarse equivalence means that $g(Y) \approxeq X$ can be shown in the same way. □

Corollary 1.23. *Suppose $X \approxeq Y$, and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be the corresponding coarse equivalence maps. Then $f(X) \approxeq g(Y)$.*

Proof. From Lemma 1.22

$$f(X) \approxeq Y \approxeq X \approxeq g(Y)$$

and since $\approxeq$ is an equivalence relation by Theorem 1.21, the result follows. □

We present two other straightforward consequences of the definition.

Proposition 1.24. *Suppose $X \approxeq Y$ and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be corresponding coarse equivalence maps. If $A \subset X$ then $A \approxeq f(A)$. Similarly, if $B \subset Y$ then $B \approxeq g(B)$.*

Proof. The restrictions $f|_A$ and $g|_{f(A)}$ are coarse by Example 1.5. Since $d(f(g(y)), y)$ and $d(g(f(x)), x)$ are uniformly bounded in Y and X, respectively, it follows that

$$d(g|_{f(A)}(f|_A(x)), x) \quad \text{and} \quad d(f|_A(g|_{f(A)}(y)), y)$$

are as well. Thus $A \approxeq f(A)$. That $B \approxeq g(B)$ is proved almost identically. □

Any space X is coarsely equivalent to itself. It is intuitive that removal of a bounded subset from X will not alter this condition. We remind ourselves that the distance between a point $x \in X$ and a non-empty subset $B \subset X$ is defined as

$$d(x, B) = \inf\{d(x, b) \mid b \in B\},$$

which is well-defined and satisfies the triangle inequality.

Proposition 1.25. *Let X be a metric space, and let K be a bounded subset of X. Then $X \smallsetminus K$ is coarsely equivalent to X.*

Proof. Define $f : X \smallsetminus K \longrightarrow X$ by inclusion. Choose any $x_0 \in X \smallsetminus K$ and define

$$g : X \longrightarrow X \smallsetminus K, \quad g(x) = \begin{cases} x & \text{if } x \in X \smallsetminus K \\ x_0 & \text{if } x \in K \end{cases}$$

so g is a map. As usual, f is coarse since it is an inclusion map (Example 1.5). We show that g is coarse. Let B be a bounded subset of $X \smallsetminus K$. Then

$$\begin{aligned} g^{-1}(B) &= \{x \in X \mid g(x) \in B\} \\ &= \{x \in X \smallsetminus K \mid g(x) \in B\} \cup \{x \in K \mid g(x) \in B\} \\ &= (B \cap (X \smallsetminus K)) \cup (B \cap \{x_0\}) \subset B \end{aligned}$$

which is bounded, so g is proper. Let $R > 0$, and let

$$S = R + \sup\{d(x, x_0) \mid x \in K\},$$

which is finite since K is bounded. Suppose for some $x, x' \in X$ that $d(x, x') < R$. If $x, x' \in X \smallsetminus K$ then $d(g(x), g(x')) = d(x, x') < R \leq S$. If $x, x' \in K$ then

$$d(g(x), g(x')) = d(x_0, x_0) = 0 < S.$$

And if, say, $x' \in K$ while $x \in X \smallsetminus K$ then

$$d(g(x), g(x')) = d(x_0, x) < R + \sup\{d(x, x_0) \mid x \in K\} = S.$$

In any case it follows that $d(g(x), g(x')) < S$, so g is bornologous.

The above also shows that any point in X is moved by $f \circ g$ or $g \circ f$ a distance of at most $d(x_0, X \smallsetminus K)$, and since K is bounded, this is finite.

It follows that $X \approxeq X \smallsetminus K$ as required. □

1.5 Examples of coarse equivalence

We have argued informally that a point and a finite interval of the real line have a similar large scale structure, and that this structure is different from $\mathbb{R}$. The first two, when viewed

from a large distance, look essentially the same: that is, like a point. The real line $\mathbb{R}$, however, looks like itself from any distance and is therefore, on a large scale, dissimilar from both a point and a finite interval. This section proves that a point and a finite interval are in fact coarsely equivalent, and that neither is coarsely equivalent to $\mathbb{R}$.

The normal notion of the *dimension* of a space, however, is at variance with this, since a point is zero-dimensional and both a finite interval and $\mathbb{R}$ are one-dimensional. A detailed discussion of dimension is beyond the scope of this work, but it is clear that the usual formulation of dimension is not an invariant of large scale structure: we need instead the notion of *coarse dimension*, which is a measure of dimension which does hold at large scales [3].

We examine coarse equivalence for bounded and unbounded metric spaces. Finally, we end this section by proving that $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent.

Proposition 1.26. *Let X and Y be bounded metric spaces. Then X and Y are coarsely equivalent.*

Proof. For coarse equivalence to be defined, neither X nor Y is empty, so choose $x_0 \in X$ and $y_0 \in Y$, and define maps $f(x) = y_0$ for all $x \in X$ and $g(y) = x_0$ for all $y \in Y$. By Example 1.6, since X and Y are bounded, both f and g are coarse. Since Y is bounded, Corollary 1.18 guarantees that any two maps from X into Y are close, so $g \circ f$ is close to id_X. Applying Corollary 1.18 again, because X is also bounded, it follows that $f \circ g$ is close to id_Y. Thus $X \approxeq Y$. □

This shows that any finite interval of $\mathbb{R}$ is coarsely equivalent to any other finite interval, including a point $\{p\} = [p, p] \subset \mathbb{R}$.

Proposition 1.27. *A bounded metric space X and an unbounded metric space Y are not coarsely equivalent.*

Proof. By Proposition 1.26 and that coarseness is an equivalence relation it is sufficient to show that a one-point metric space $X = \{x_0\}$, which is bounded, is not coarsely equivalent to any unbounded metric space Y.

Suppose $X = \{x_0\}$ is coarsely equivalent to Y. The only choice of maps between X and Y is to let $f(x_0) = y_0$ for some $y_0 \in Y$ and $g(y) = x_0$ for all $y \in Y$. But $\{x_0\}$ is bounded whereas $g^{-1}(\{x_0\}) = Y$ is unbounded, so g fails to be proper, or coarse, which contradicts our assumption that $X \approxeq Y$. □

Thus no finite interval of $\mathbb{R}$ is coarsely equivalent to $\mathbb{R}$. While it is straightforward to show that bounded spaces are coarsely equivalent, and that bounded and unbounded spaces are not, it requires a little more effort to decide whether two unbounded spaces are coarsely equivalent.

Example 1.28. Let $f : \mathbb{Z} \longrightarrow \mathbb{R}$ be the inclusion map $n \longmapsto n$, and let $g : \mathbb{R} \longrightarrow \mathbb{Z}$ be the map $x \longmapsto \lfloor x \rfloor$. Example 1.5 showed that f is coarse since it is an inclusion map, and

Example 1.10 showed that g is coarse. But

$$(f \circ g)(x) = f(g(x)) = f(\lfloor x \rfloor) = \lfloor x \rfloor$$

for each $x \in \mathbb{R}$, and

$$(g \circ f)(n) = g(f(n)) = g(n) = \lfloor n \rfloor = n$$

for each $n \in \mathbb{Z}$. That $g \circ f$ is close to the identity map on $\mathbb{Z}$ is obvious (since it *is* the identity map!). That $f \circ g$ is close to the identity map on $\mathbb{R}$ is clear also, since $d(\lfloor x \rfloor, x) < 1$ for all $x \in \mathbb{R}$. It follows that $\mathbb{Z} \approxeq \mathbb{R}$.

The argument in the previous example can be repeated to prove that $\mathbb{Q}$ is coarsely equivalent to $\mathbb{Z}$, and hence to $\mathbb{R}$, since the map $x \longmapsto \lfloor x \rfloor$ is defined for all $x \in \mathbb{Q}$ also.

Lastly, we show that a subset of a metric space is always coarsely equivalent to its closure in the space.

Proposition 1.29. *Let X be a metric space and let A be a subset of X. Then A is coarsely equivalent to $\overline{A}$.*

Proof. Let $f : A \longrightarrow \overline{A}$ be defined by inclusion on A, so f is a coarse map (Example 1.5). To define $g : \overline{A} \longrightarrow A$, we observe that for each $x \in \overline{A}$ there is an $a_x \in A$ such that $d(x, a_x) < 1$. By letting $g(x) = a_x$ for each $x \in \overline{A}$, g is a map by the axiom of choice.

We show that g is a coarse map. Let $B \subset A$ be bounded. Then

$$f^{-1}(B) = \{x \in \overline{A} \mid f(x) \in B\}$$

is contained in the set $\{x \in X \mid d(x, B) < 1\}$, which is bounded since B is bounded. Thus g is proper. Let $R > 0$, and let $S = R + 2$. If $d(x, y) < R$ in $\overline{A}$, then $d(g(x), g(y)) < R + 2 = S$ in A by the triangle inequality and definition of g, so g is bornologous. Lastly,

$$d(f(g(x)), x) = d(g(x), x) < 1 \quad \text{and} \quad d(g(f(x)), x) = d(g(x), x) < 1$$

for all $x \in \overline{A}$ and all $x \in A$, respectively, so $f \circ g \sim id_{\overline{A}}$ and $g \circ f \sim id_A$.

It follows that $A \approxeq \overline{A}$. □

Example 1.30. Proposition 1.29 shows that any dense subset of a metric space X is coarsely equivalent to X. Although already shown directly, this means $\mathbb{Q} \approxeq \mathbb{R}$, but it also gives us $(\mathbb{R} \setminus \mathbb{Z}) \approxeq \mathbb{R}$, for instance.

1.6 Products of coarse equivalences

It is intuitive from the ideas of coarse geometry presented so far that $\mathbb{R}$ is not coarsely equivalent to $\mathbb{R}^2$. From any distance at all, a line and the plane do not look similar. Proving this, however, is not as straightforward as have been some of the results up to this point. Since product spaces are involved it seems natural to begin with them. We remind ourselves of some definitions.

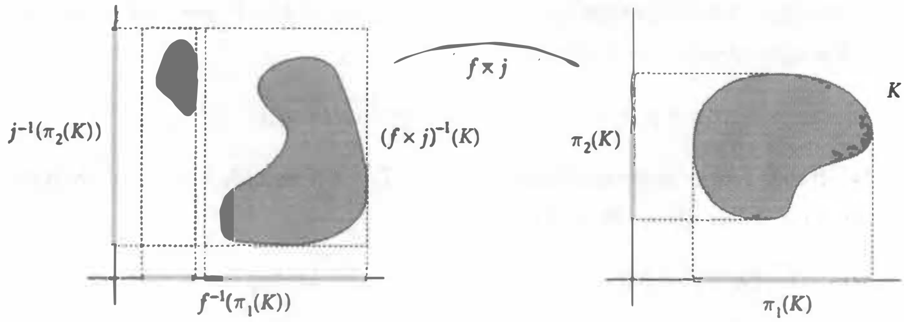

Figure 1.2: $(f \times j)^{-1}$ is a subset of the products of $f^{-1}(\pi_1(K))$ and $j^{-1}(\pi_2(K))$.

Definition 1.31. Given sets A, B, X, and Y, and two maps $f : A \longrightarrow X$ and $j : B \longrightarrow Y$, we define the *product of f and j* to be the map

$$f \times j : A \times B \longrightarrow X \times Y, \quad (f \times j)(a \times b) = f(a) \times j(b)$$

for each $a \times b \in A \times B$. We say that $f \times j$ is a *product map* , or $f \times j$ is *the product of f and j.*

This is the natural definition of a product map. The following technical result (see Figure 1.2) simplifies the proofs of some later propositions.

Lemma 1.32. *Let $f \times j : A \times B \longrightarrow X \times Y$ be a product map, and let K be a subset of $X \times Y$. Then*

$$(f \times j)^{-1}(K) \subset f^{-1}(\pi_1(K)) \times j^{-1}(\pi_2(K)).$$

Proof. Elementary set theory gives us

$$\begin{aligned}(f \times j)^{-1}(K) &= \{a \times b \mid (f \times j)(a \times b) \in K\} \\ &= \{a \times b \mid f(a) \times j(b) \in K\} \\ &\subset \{a \times b \mid f(a) \times j(b) \in \pi_1(K) \times \pi_2(K)\} \\ &= \{a \mid f(a) \in \pi_1(K)\} \times \{b \mid j(b) \in \pi_2(K)\}\end{aligned}$$

which is equal to $f^{-1}(\pi_1(K)) \times j^{-1}(\pi_2(K))$, completing the proof. □

Given two metric spaces (X, d_X) and (Y, d_Y), it is natural to ask what metric should be applied to $X \times Y$, since there is some choice in the matter. The following definition of the *max metric* on the product of two metric spaces is a good choice for the applications which follow it.

Definition 1.33. Let (X, d_X) and (Y, d_Y) be metric spaces. Define the *max metric* $d_{X \times Y} \longrightarrow \mathbb{R}$ on $X \times Y$ by

$$d_{X \times Y}(x \times y, x' \times y') = \max\{d_X(x, x'), d_Y(y, y')\}$$

for all $x \times y, x' \times y' \in X \times Y$.

It is an easy exercise to prove that $d_{X\times Y}$ is in fact a metric on $X \times Y$, and that this metric is topologically equivalent to the Euclidean metric

$$d(x \times y, x' \times y') = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$$

on $X \times Y$ [5]. If the identity maps of X and Y are $id_X : X \longrightarrow X$ and $id_Y : Y \longrightarrow Y$, then the identity map on $X \times Y$ is $id_{X\times Y}$, defined by

$$id_{X\times Y}(x \times y) = (id_X \times id_Y)(x \times y) = id_X(x) \times id_Y(y) = x \times y.$$

This is by definition of the product map, but produces the desired result, that $id_{X\times Y}(x \times y) = x \times y$, nonetheless.

We can show that a product of maps, from A into X and from B into Y, is coarse with respect to the max metrics on $A \times B$ and $X \times Y$ if and only if the maps individually are coarse.

Proposition 1.34. *Let $f : A \longrightarrow X$ and $j : B \longrightarrow Y$ be maps from metric spaces (A, d_A) to (X, d_X) and from (B, d_B) to (Y, d_Y), respectively. Then*

$$(f \times j) : A \times B \longrightarrow X \times Y$$

is a coarse map with respect to $d_{A\times B}$ and $d_{X\times Y}$ if and only if f and j are coarse maps.

Proof. Suppose that f and j are coarse maps. We need to show that $f \times j$ is proper and bornologous. Let $K \subset X \times Y$ be a bounded set with respect to $d_{X\times Y}$. The definition of the max metric implies that both $\pi_1(K)$ and $\pi_2(K)$ must be bounded, and by Lemma 1.32,

$$(f \times j)^{-1}(K) \subset f^{-1}(\pi_1(K)) \times j^{-1}(\pi_2(K)),$$

which is bounded in $d_{A\times B}$ since both f and j are proper, and projections and products of bounded sets are bounded.

Next we need to show that $f \times j$ is bornologous. Let $R > 0$. Since both f and j are bornologous, there are $S_1 > 0$ and $S_2 > 0$ such that

$$d_A(a_1, a_2) < R \Rightarrow d_X(f(a_1), f(a_2)) < S_1$$

and

$$d_B(b_1, b_2) < R \Rightarrow d_Y(j(b_1), j(b_2)) < S_2.$$

Let $S = \max\{S_1, S_2\} > 0$. Then for each $R > 0$ there is an $S > 0$ such that

$$\begin{aligned} d_{A\times B}(a_1 \times b_1, a_2 \times b_2) < R &\Rightarrow \max\{d_A(a_1,a_2), d_B(b_1,b_2)\} < R \\ &\Rightarrow d_X(f(a_1), f(a_2)) < S_1 \quad \text{and} \quad d_Y(j(b_1), j(b_2)) < S_2 \\ &\Rightarrow \max\{d_X(f(a_1), f(a_2)), d_Y(j(b_1), j(b_2))\} < S \\ &\Rightarrow d_{X\times Y}(f(a_1) \times j(b_1), f(a_2) \times j(b_2)) < S, \end{aligned}$$

so $f \times j$ is bornologous.

Conversely, suppose that $f \times j$ is a coarse map with respect to the max metrics on $A \times B$ and $X \times Y$ but that f, say, is not coarse: then f is either not proper, or not bornologous. If f is not proper, there is some bounded set $B \subset Y$ such that $f^{-1}(B)$ is unbounded. Let $y_0 \in Y$, so $B \times \{y_0\}$ is also bounded. Then by Lemma 1.32,

$$\begin{aligned}(f \times j)^{-1}(B \times \{y_0\}) &\subset f^{-1}(\pi_1(B \times \{y_0\})) \times f^{-1}(\pi_2(B \times \{y_0\})) \\ &= f^{-1}(B) \times j^{-1}(\{y_0\})\end{aligned}$$

which is then unbounded in $d_{A \times B}$, contradicting that $f \times j$ is proper.

On the other hand, if f is not bornologous, then there is some $R_0 > 0$ such that for each $n \geq 1$ there is a pair $a_n, a_n' \in A$ with the property that

$$d_A(a_n, a_n') < R_0 \quad \text{but} \quad d_X(f(a_n), f(a_n')) \geq n. \tag{*}$$

Choose any $b_0 \in B$. Then (*) contradicts that $f \times j$ is bornologous, since there is an $R_0 > 0$ such that no $S > 0$ exists to ensure

$$d_{A \times B}(a \times b, a' \times b') \Rightarrow d_{X \times Y}((f \times j)(a \times b), (f \times j)(a' \times b')) < S$$

because for any $S > 0$ we can choose $n > S$ so that

$$d_{A \times B}(a_n \times b_0, a_n' \times b_0) = \max\{d_A(a_n, a_n'), d_B(b_0, b_0)\} < R_0$$

but

$$\begin{aligned}d_{X \times Y}((f \times j)(a_n \times b_0), (f \times j)(a_n' \times b_0)) &= \max\{d_X(f(a_n), f(a_n')), d_Y(j(b_0), j(b_0))\} \\ &= \max\{d_X(f(a_n), f(a_n')), 0\} \geq n > S.\end{aligned}$$

Thus f must be coarse. That j must also be coarse is shown in the same way. With both implications shown, the result follows. □

The main theorem of this section is that coarse equivalence is preserved under taking products, with respect to the max metric. Before we prove this, we require a couple of lemmas. The first says that a product of compositions of maps is equal to the composition of the individual products; the second says that two maps are close, respectively, to two other maps if and only if the pairwise products are close.

Lemma 1.35. *Let $f : A \longrightarrow X$, $g : X \longrightarrow A'$, $j : B \longrightarrow Y$, $k : Y \longrightarrow B'$ be maps. Then*

$$(g \circ f) \times (k \circ j) = (g \times k) \circ (f \times j).$$

Proof. Let $a \times b \in A \times B$. Then

$$\begin{aligned}((g \circ f) \times (k \circ j))(a \times b) &= g(f(a)) \times k(j(b)) \\ &= (g \times k)(f(a) \times j(b)) \\ &= (g \times k)((f \times j)(a \times b)) \\ &= ((g \times k) \circ (f \times j))(a \times b)\end{aligned}$$

which completes the proof. □

Lemma 1.36. *Suppose that $f, g : A \longrightarrow X$ and $j, k : B \longrightarrow Y$ are maps. Say that $f \sim g$ if and only if f is close to g. Then the following are equivalent:*

(a) $f \sim g$ *and* $j \sim k$.
(b) $f \times j \sim g \times k$.

Proof. For (a) $\Rightarrow$ (b), we are given that $d_X(f(a), g(a))$ is bounded uniformly by M, and that $d_Y(j(b), k(b))$ is bounded uniformly by N, both of which are finite. It follows that

$$d_{X\times Y}((f \times j)(a \times b), (g \times k)(a \times b)) = \max\{d_X(f(a), g(a)), d_Y(j(b), k(b))\}$$

is bounded uniformly by $\max\{M, N\} < \infty$, so $f \times j \sim g \times k$.

Conversely, if $f \times j \sim g \times k$ then $d_{X\times Y}((f \times j)(a \times b), (g \times k)(a \times b))$ is bounded uniformly by M, which immediately implies that each of $d_X(f(a), g(a))$ and $d_Y(j(b), k(b))$ is also bounded uniformly by M. Thus $f \sim g$ and $j \sim k$. □

Theorem 1.37. *Let (A, d_A), (B, d_B), (X, d_X), and (Y, d_Y) be metric spaces. If A is coarsely equivalent to X and B is coarsely equivalent to Y then $A \times B$ is coarsely equivalent to $X \times Y$ with respect to the max metrics $d_{A\times B}$ and $d_{X\times Y}$.*

Proof. There are coarse maps $f : A \longrightarrow X$, $g : X \longrightarrow A$ such that $f \circ g \sim id_X$, $g \circ f \sim id_A$, and maps $j : B \longrightarrow Y$, $k : Y \longrightarrow B$, also coarse, such that $j \circ k \sim id_Y$, $k \circ j \sim id_B$.

By Proposition 1.34, the product maps $f \times j$ and $g \times k$ are both coarse with respect to $d_{A\times B}$ and $d_{X\times Y}$. It remains to show that

$$(f \times j) \circ (g \times k) \sim id_{X\times Y}$$

and

$$(g \times k) \circ (f \times j) \sim id_{A\times B},$$

but both of these follow from Lemmas 1.35 and 1.36. □

The converse to this, if it is true, would require an additional and reasonable condition. Before we provide it, we prove the following.

Proposition 1.38. *Let X and Y be metric spaces and suppose that X is bounded. Then $X \times Y$ is coarsely equivalent to Y.*

Proof. We know already that X is coarsely equivalent to any point $x_0 \in X$ by Proposition 1.26, so it remains to show that $(x_0, y) \longmapsto y$ is a coarse map whose inverse $y \longmapsto (x_0, y)$ is coarse. This, however, is clear since

$$d_{\{x_0\}\times Y}((x_0, y), (x_0, y')) = d_Y(y, y')$$

for all $y, y' \in Y$, making the map an isometry, which, by Example 1.9, makes both maps coarse. That the compositions are close to their respective identity maps is also clear since they are inverses of each other. □

The first of the trivialities we wish to avoid includes examples such as

$$\mathbb{Z} \times \mathbf{Z} \approxeq \mathbf{Z}^2 \times 0$$

but $\mathbf{Z}$ is not coarsely equivalent to $\mathbf{Z}^2$ and neither is $\mathbf{Z}$ coarsely equivalent to $\{0\}$. The reason for the failure is the presence of a bounded space $\{0\}$. Proposition 1.38 shows that the product of a bounded space X with Y is coarsely equivalent to Y itself. Thus we can always write a product space without any factors which are bounded: we say that a space X has *no bounded factors* if this is the case.

The second involves the example

$$\mathbf{Z}^2 \times \left[\bigoplus_{i=1}^{\infty} \mathbf{Z}\right] \approxeq \mathbf{Z} \times \left[\bigoplus_{i=1}^{\infty} \mathbf{Z}\right]$$

but $\mathbf{Z} \not\approxeq \mathbf{Z}^2$, as shown in the next section. The infinite direct sum has merely been written in two different ways, but there is no actual difference between the two. The reason for the failure in this instance is that $\bigoplus_{i=1}^{\infty} \mathbf{Z}$ can be written in more than one way as the product of its factors. That is to say, the direct sum is not *irreducible* .

The second condition is more general than the first. That X has a bounded factor means that $X = Y \times K \approxeq Y$ where K is bounded. This shows that X is not irreducible. Our second example, however, is irreducible but has no bounded factors. It is not known currently whether the converse of Theorem 1.37, with the addition of irreducibility, is true.

Conjecture 1.39. *Suppose A, B, and X are irreducible metric spaces. If $A \times X$ is coarsely equivalent to $B \times X$ then $A \approxeq B$.*

In Example 1.28 we showed that $\mathbf{Z}$ is coarsely equivalent to $\mathbb{R}$. Thus, by Theorem 1.37, which guarantees that products hold under coarse equivalence, $\mathbf{Z} \times \mathbf{Z}$ is coarsely equivalent to $\mathbb{R}^2$. What Theorem 1.37 cannot help us with, however, is showing that $\mathbb{R}$ and $\mathbb{R}^2$ are *not* coarsely equivalent, since the required converse direction is absent. However, if we can show that $\mathbf{Z}$ is not coarsely equivalent to $\mathbf{Z} \times \mathbf{Z}$, then by Theorem 1.21—which stated that coarseness is an equivalence relation—we also derive the fact that $\mathbb{R}$ and $\mathbb{R}^2$ cannot be coarsely equivalent either. This is the aim of the next section.

1.7 Direct proofs in coarse equivalence

Our approach in showing that a line—that is, $\mathbb{R}$—has a different large-scale structure than a plane—$\mathbb{R}^2$, or the product of two such lines—relies upon a fact from our intuitive understanding of metric spaces according to large-scale properties: that motion in a line is unrestricted in any direction along an axis (one degree of freedom), and motion in a plane is again unrestricted in any direction, but there are *two* degrees of freedom. The following proposition makes use of the additional degree of freedom to complete the proof. To simplify matters, we show directly that $\mathbf{Z}$ is not coarsely equivalent to $\mathbf{Z} \times \mathbf{Z}$. The desired result, that $\mathbb{R}$ is not coarsely equivalent to $\mathbb{R}^2$, follows from this.

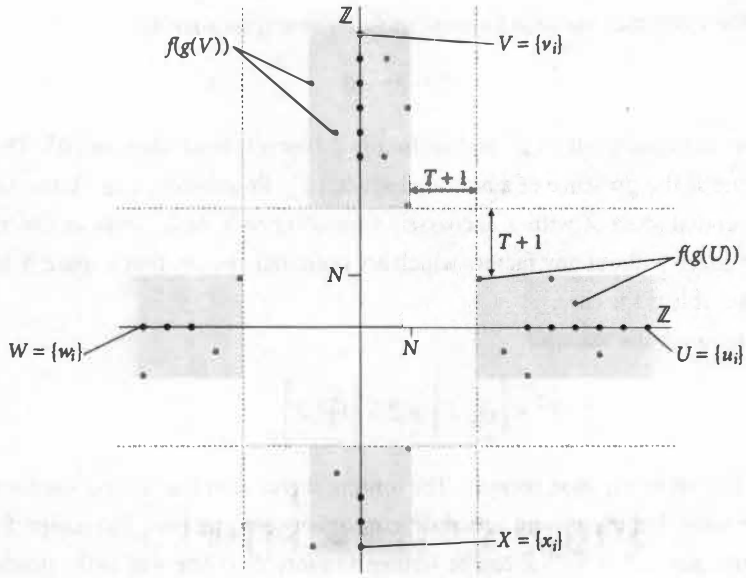

Figure 1.3: The sets U, V, W, and X and $f(g(U)), \ldots, f(g(X))$ of $\mathbb{Z} \times \mathbb{Z}$.

Proposition 1.40. *$\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are not coarsely equivalent.*

Proof. Suppose they are: then there are coarse maps $f : \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ and $g : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f \circ g$ and $g \circ f$ are close to the identity maps $id_{\mathbb{Z}\times\mathbb{Z}}$ and $id_{\mathbb{Z}}$ respectively.

Since $f \circ g \sim id_{\mathbb{Z}\times\mathbb{Z}}$ then there is an N such that $d_{\mathbb{Z}\times\mathbb{Z}}(f(g(m \times n)), m \times n) \le N$ for all $m \times n \in \mathbb{Z} \times \mathbb{Z}$. Since g is bornologous, for $R = \frac{3}{2}$ there is an $S > 0$ such that

$$d_{\mathbb{Z}\times\mathbb{Z}}(m \times n, m' \times n') < \tfrac{3}{2} \Rightarrow d_{\mathbb{Z}}(g(m \times n), g(m' \times n')) < S,$$

and since $S > 0$ and f is bornologous there is a $T > 0$ such that

$$d_{\mathbb{Z}}(m, n) < S \Rightarrow d_{\mathbb{Z}\times\mathbb{Z}}(f(m), f(n)) < T.$$

Next, define sequences (u_i), (v_i), (w_i), and (x_i) in $\mathbb{Z} \times \mathbb{Z}$ for each $i \ge 1$ by letting

$$u_i = (2N + T + i) \times 0, \quad v_i = 0 \times (2N + T + i),$$
$$w_i = -(2N + T + i) \times 0, \quad x_i = 0 \times -(2N + T + i),$$

and let $U = \{u_i\}$, $V = \{v_i\}$, $W = \{w_i\}$, and $X = \{x_i\}$. By the above arguments, each of $f(g(U))$, $f(g(V))$, $f(g(W))$, and $f(g(X))$ must be a distance of at least $T + 1$ from any of the other three, since each point in any of U, V, W, or X is at least a distance $2N + T + 1$ from any point in one of the other sets (see Figure 1.3).

Since g is proper and $\mathbb{Z}$ is unbounded, then each of $g(U)$, $g(V)$, $g(W)$, and $g(X)$ is unbounded; if not, for example, the inverse image of $g(U)$ under g, or $g^{-1}(g(U))$, contains U and is therefore unbounded, contradicting the properness of g.

The interval $[m, n]$ will be understood as being a subset of $\mathbb{Z}$; we will interpret $[m, n)$ and $(m, n]$ similarly. Again taking $g(U)$ as an example, if $\max g(U)$ exists (i.e., is finite) then $g(U) \subset (-\infty, \max g(U)]$; if $\min g(U)$ exists, then $g(U) \subset [\min g(U), \infty)$; if neither exists, $g(U) \subset (-\infty, \infty)$. These containments are the smallest containments for $g(U)$ by an interval of $\mathbb{Z}$ since $g(U)$ is unbounded. It follows that each of $g(U), g(V), g(W)$, and $g(X)$ can be contained in a set of the form $[a, \infty)$, $(-\infty, b]$, or $(-\infty, \infty)$ for some $a, b \in \mathbb{Z}$. By the pigeonhole principle, at least two of them must have containments of the same form.

For each $i \geq 1$, $d_{\mathbb{Z}\times\mathbb{Z}}(u_i, u_{i+1})$, $d_{\mathbb{Z}\times\mathbb{Z}}(v_i, v_{i+1})$, $d_{\mathbb{Z}\times\mathbb{Z}}(w_i, w_{i+1})$, and $d_{\mathbb{Z}\times\mathbb{Z}}(x_i, x_{i+1})$ are all equal to 1, and hence the distance of those points in $\mathbb{Z} \times \mathbb{Z}$ is less than $R = \frac{3}{2}$, whence

$$d_{\mathbb{Z}}(g(u_i), g(u_{i+1})) < S, \quad d_{\mathbb{Z}}(g(v_i), g(v_{i+1})) < S$$
$$d_{\mathbb{Z}}(g(w_i), g(w_{i+1})) < S, \quad d_{\mathbb{Z}}(g(x_i), g(x_{s+1})) < S.$$

This means that in any interval of length S in the images $\{g(U), \ldots, g(X)\}$, there is at least one point of each set. But two of the images have containments which have infinite intersection, and thus we are guaranteed that there is at least one such interval of length S containing a point of two different images. This provides the contradiction we sought, since then f maps those points to within T of each other in $\mathbb{Z} \times \mathbb{Z}$, but $f(g(U))$, $f(g(V))$, $f(g(W))$, and $f(g(X))$ each have a minimum distance apart of $T + 1$.

Our assumption that $\mathbb{Z} \approxeq \mathbb{Z}\times\mathbb{Z}$ is therefore incorrect, and thus $\mathbb{Z}$ is not coarsely equivalent to $\mathbb{Z} \times \mathbb{Z}$. □

We are now in position to prove the last goal of this section.

Corollary 1.41. *$\mathbb{R}$ is not coarsely equivalent to $\mathbb{R}^2$.*

Proof. By Example 1.28 $\mathbb{Z} \approxeq \mathbb{R}$, by Proposition 1.40 $\mathbb{Z} \not\approxeq \mathbb{Z} \times \mathbb{Z}$, and by Theorem 1.37 coarse equivalence holds under taking products. Combining these we see that

$$\mathbb{R} \approxeq \mathbb{Z} \not\approxeq \mathbb{Z} \times \mathbb{Z} \approxeq \mathbb{R} \times \mathbb{R},$$

which, on application of Theorem 1.21 (which showed that coarseness is an equivalence relation), shows that $\mathbb{R} \not\approxeq \mathbb{R}^2$, completing the proof. □

Our intuition about the large-scale structure of a line and a plane is therefore accurate: $\mathbb{R}$ and $\mathbb{R}^2$ are not coarsely equivalent, and hence do not have the same large-scale structure.

1.8 Subspaces of $\mathbb{R}$

We might ask what are necessary and sufficient conditions for a subset $X \subset \mathbb{R}$ to be coarsely equivalent to $\mathbb{R}$. They turn out to be quite straightforward. We first define some new terms.

Definition 1.42. Let M be a positive real number. We say that a metric space X is *M-scale connected* at x and y [4] if there is a finite chain of points $x = x_1, x_2, \ldots, x_k = y$ such

that $d(x_i, x_{i+1}) \leq M$ for all $i < k$. If the above condition holds for each $x, y \in X$ then we say that X is *M-scale connected.*

It is trivial that if X is M-scale connected then it is also M'-scale connected for any M' greater than M. Images of M-scale connected sets, under coarse maps, are N-scale connected for some positive real number N.

Proposition 1.43. *Let $f : X \longrightarrow Y$ be a bornologous map of metric spaces, and let $A \subset X$ be M-scale connected. Then $f(A)$ is N-scale connected for some positive real number N.*

Proof. For $M > 0$, since f is bornologous, there is an $N > 0$ such that

$$d(x, y) < M + 1 \Rightarrow d(f(x), f(y)) < N + 1.$$

Choose any $f(x), f(y) \in f(X)$. There is a chain $x = x_1, \ldots, x_k = y$ in X such that $d(x_i, x_{i+1})$ is at most M for all $i < k$. It follows that there is a chain $f(x) = f(x_1), \ldots, f(x_k) = f(y)$ such that $d(f(x_i), f(x_{i+1}) \leq N$ for all $i < k$, whence Y is N-scale connected at $f(x)$ and $f(y)$. It follows that $f(A)$ is N-scale connected. □

We note that Proposition 1.43 applies to all coarse maps as well. We note also that Definition 1.42 implies that X can be partitioned into subsets, each of which is M-scale connected.

Proposition 1.44. *Let X be a metric space, and M a positive real number. Say, for points $x, y \in X$, that $x \frown y$ if and only if X is M-scale connected at x and y. Then $\frown$ is an equivalence relation on X.*

Proof. Clearly $x \frown x$ vacuously, satisfying reflexivity. Symmetry of the distance function means that $\frown$ is symmetric. If $x \frown y$ and $y \frown z$, then the existence of chains $x = x_1, \ldots, x_k = y$ and $y = y_1, \ldots, y_m = z$ with $d(x_i, x_{i+1}) \leq M$ and $d(y_j, y_{j+1}) \leq M$ for all $i < k$ and $j < m$ means that there is a chain $x = x_1, \ldots, x_k = y_1, \ldots, y_m = z$ with the required property, so $x \frown z$, which makes $\frown$ transitive. Thus $\frown$ is an equivalence relation on X. □

This means that X can be partitioned into *M-scale connected components.*

Example 1.45. The real line $\mathbb{R}$ has exactly one M-scale connected component for every $M > 0$, that is, $\mathbb{R}$ itself, and the same applies to the positive real numbers $\mathbb{R}^+$. The set of integers $\mathbb{Z}$, on the other hand, has one M-scale connected component for every $M \geq 1$ and infinitely many M-scale connected components for every $0 < M < 1$. The set $\{n^2 \mid n \in \mathbb{N}\}$ has no *unbounded* M-scale connected components for any $M > 0$.

The next result shows that unbounded M-scale connected components, under coarse maps, are mapped into unbounded N-scale connected components (for some $N > 0$) in the co-domain.

Lemma 1.46. *Let X and Y be unbounded metric spaces, and let $f : X \longrightarrow Y$ be a coarse map. If A is an unbounded M-scale connected component of X then there is some $N > 0$ such that f maps A entirely into an unbounded N-scale connected component of Y.*

Proof. By Proposition 1.43, there is an $N > 0$ such that $f(A)$ is N-scale connected. Since f is coarse then $f(A)$ is unbounded, and thus lies entirely in the N-scale component containing $f(A)$, making this component unbounded also. $\square$

We are now in a position to prove the first of the three main results of this section. Before we do, we recall that the distance between two non-empty subsets $A, B \subset X$ is defined as

$$d(A, B) = \inf\{d(x, B) \mid x \in A\}.$$

Theorem 1.47. *Let X be a metric space. If $X \approxeq \mathbb{R}^+$ then there exists $M > 0$ such that X contains exactly one unbounded M-scale connected component.*

Proof. We first show existence of an unbounded M-scale connected component in X. Since $X \approxeq \mathbb{R}^+$ then X is unbounded. Let g be a corresponding coarse equivalence map $\mathbb{R}^+ \longrightarrow X$, so g is unbounded. By Lemma 1.46, since $\mathbb{R}^+$ is an unbounded 1-scale connected component of itself, it follows that $g(\mathbb{R}^+)$ is mapped entirely into an unbounded M-scale connected component $A \subset X$ for some $M > 0$.

We now show uniqueness. Given $X \approxeq \mathbb{R}^+$, with $f : X \longrightarrow \mathbb{R}^+$ and $g : \mathbb{R}^+ \longrightarrow X$ being the corresponding coarse equivalence maps, suppose that for every integer N we can find an $M > N$ such that there are least two distinct unbounded M-scale connected components A, B contained in X. Then there is an N satisfying this condition, that $d(g(f(x)), x) \leq M/2$ for all $x \in X$, and that $\mathbb{R}^+$ is mapped by g into an M-scale connected component of X. It follows that $d(A, B) > M$ since otherwise A and B are the same component. The map $f : X \longrightarrow \mathbb{R}^+$ maps both A and B into $\mathbb{R}^+$, but g maps the whole of $\mathbb{R}^+$, including $f(A)$ and $f(B)$, into one unbounded M-scale connected component, which must therefore be either A or B. This contradicts the choice of M so that $d(f(g(x)), x) \leq M/2$ for all $x \in X$. Thus our assumption is false, and it follows that there exists N such that X contains fewer than two unbounded M-scale connected components for every $M > N$. This completes the proof since existence of one unbounded M-scale connected component has already been shown. $\square$

If X is also a *subspace* of $\mathbb{R}^+$ then the converse is true.

Theorem 1.48. *Suppose X is a subspace of $\mathbb{R}^+$. Then $X \approxeq \mathbb{R}^+$ if and only if there is an $M > 0$ such that X contains exactly one unbounded M-scale connected component.*

Proof. For the necessity we appeal to Theorem 1.47. For the sufficiency, we may assume that X is closed, since by Proposition 1.29, $X \approxeq \overline{X}$. Suppose that there is an $M > 0$ such that X contains exactly one unbounded M-scale connected component. Since X is automatically non-empty (being unbounded) and closed, for each $x \in \mathbb{R}^+$ the set

$$Y_x = \{y_x \in X \mid d(x, y_x) = d(x, X)\}$$

contains at most two elements. Let $f : \mathbb{R}^+ \longrightarrow X$ be defined by letting $f(x) = \min Y_x$ for each $x \in \mathbb{R}^+$, so f is indeed a map.

We show that f is coarse. Let $B \subset X$ be bounded. Since X is unbounded there is an $x_0 \in X$ such that $x_0 > \sup B$. Then $f^{-1}(B) \subset [0, x_0)$, which is bounded, so f is proper.

Let the unbounded M-scale connected component of X be A, let $R > 0$, and set

$$S = R + M + \inf A.$$

Let $x, y \in \mathbb{R}^+$ such that $d(x, y) < R$. There are three cases. If both x and y are greater than $\inf A$, then they can be at most a distance of $M/2$ from a point in the unbounded M-scale connected component, so $d(f(x), f(y)) < R + M$ by the triangle inequality. If either of x or y is less than $\inf A$ then either of x or y could be moved, on mapping, by at most a distance of $M + \inf A$, so $d(f(x), f(y)) < R + M + \inf A$. It follows that in either case,

$$d(f(x), f(y)) < R + M + \inf A = S,$$

so f is bornologous.

Let $g : X \longrightarrow \mathbb{R}^+$ be the inclusion map. Then g is coarse by Example 1.5, and the argument which showed that f is bornologous shows that each of $f \circ g$ and $g \circ f$ is uniformly bounded by $M + \inf A$, so $f \circ g \sim i_X$ and $g \circ f \sim i_{\mathbb{R}^+}$.

It follows that $X \approxeq \mathbb{R}^+$ as required. □

A trivial modification of the last theorem is the following, which we shall state without proof.

Corollary 1.49. *Suppose X is an unbounded subspace of $\mathbb{R}^-$. Then $X \approxeq \mathbb{R}^-$ if and only if there is an $M > 0$ such that X contains exactly one unbounded M-scale connected component.*

Example 1.50. From Theorem 1.48 it is clear both $\mathbf{Z}^+$ (or $\mathbf{Z}^-$) is coarsely equivalent to either $\mathbb{R}^+$ or $\mathbb{R}^-$ since it has exactly one 1-scale connected component.

Lemma 1.51. *Let X be a subspace of $\mathbb{R}$ and let $g : \mathbb{R} \longrightarrow X$ be a coarse map. Then $g(\mathbb{R}^+)$ is either unbounded above and bounded below, or is bounded above and unbounded below.*

Proof. Since $\mathbb{R}^+$ is coarsely equivalent to $\mathbf{Z}^+$ by Example 1.50, and because compositions of coarse maps are coarse, it is sufficient to prove the corresponding result for $\mathbf{Z}^+$. Certainly $g(\mathbf{Z}^+)$ is unbounded since g is coarse and $\mathbf{Z}^+$ is unbounded. If $g(\mathbf{Z}^+)$ is unbounded above then each $x \in X$ has the property that

$$\text{there is an } N \in \mathbf{Z}^+ \text{ such that } g(n) \geq x \text{ for all } n \geq N. \qquad (*)$$

If not, there is an $x \in X$ such that for all N there is an $n \geq N$ with $g(n) < x$. But because $\mathbf{Z}^+$ is 1-scale connected, there is an $M > 0$ for which $g(\mathbf{Z}^+)$ is M-scale connected, and because $g(\mathbf{Z}^+)$ is unbounded, this implies that $[x - M, x + M] \cap X$ must contain $g(n)$ for infinitely many n. Thus the inverse image $g^{-1}([x - M, x + M] \cap X)$ of a bounded set is unbounded, contradicting our assumption that g is coarse.

It follows that for each x there is an N_x with property (*). Since $\mathbf{Z}^+$ is well-ordered, there is a minimum N_x in the set $\{N_x \mid x \in X\}$. We set

$$a = \min\{g(0), g(1), \ldots, g(N_x - 1), x\}.$$

If $n \geq N_x$ then $g(n) \geq x \geq a$. If $0 \leq n \leq N_x - 1$ then $g(n)$ is one of $g(0), g(1), \ldots, g(N_x - 1)$, and therefore $g(n) \geq a$. Thus for all $n \in \mathbf{Z}^+$ we have $g(n) \geq a$ and hence $g(\mathbf{Z}^+)$ is bounded below.

On the other hand, if $g(\mathbf{Z}^+)$ is unbounded below, then

$$-g(\mathbf{Z}^+) = \{-g(n) \mid n \in \mathbf{Z}^+\}$$

is unbounded above. By that previously shown, since $-g : \mathbb{R} \longrightarrow -X$ is clearly coarse, it follows that $-g(\mathbf{Z}^+)$ is bounded below, whence $g(\mathbf{Z}^+)$ is bounded above.

With both cases shown, the result follows. □

The last of the three main results of this section is the following theorem, which allows us to classify subspaces of $\mathbb{R}$. The statement of the theorem is slightly different to the one which applies to $\mathbb{R}^+$, however. If X is coarsely equivalent to $\mathbb{R}$, then for a sufficiently large M the two unbounded components we seek in X might be a distance of less than M apart, making only one component. Since we wish to be able to distinguish between $\mathbb{R}$ and $\mathbb{R}^+$, we remove a certain bounded set from X to ensure the separation.

Theorem 1.52. *Let X be a subspace of $\mathbb{R}$. Then X is coarsely equivalent to $\mathbb{R}$ if and only if there is an $M > 0$ such that $X \setminus [-M, M]$ contains exactly two unbounded M-scale connected components.*

Before we provide the proof, we note that we have indeed found a categorization of unbounded subsets of $\mathbb{R}$ which is able to distinguish between $\mathbb{R}$ and $\mathbb{R}^+$. Intuitively, $\mathbb{R}$ and $\mathbb{R}^+$ are in different categories of coarse space because motion along one axis in the first space is unrestricted in two directions, whereas in the second space motion is unrestricted in only one direction. This corresponds to the presence of two unbounded M-scale connected components in $\mathbb{R}$, but only one in $\mathbb{R}^+$.

Proof of Theorem 1.52. Suppose $X \cong \mathbb{R}$. Let $f : X \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow X$ be the corresponding coarse equivalence maps. Since $\mathbb{R}^+$ and $\mathbb{R}^-$ are both unbounded and 1-scale connected, each of $g(\mathbb{R}^+)$ and $g(\mathbb{R}^-)$ is an unbounded M-scale connected subset of X by choosing M sufficiently large. We also know by Lemma 1.51 that each of $g(\mathbb{R}^+)$ and $g(\mathbb{R}^-)$ is either bounded above or bounded below.

We show that one must be bounded above while the other is bounded below. Suppose that both are bounded below (the other case, that both are bounded above, is proved similarly). Both, then, are contained in $[a, \infty) \cap X$ for some finite $a \in \mathbb{R}$. But

$$([a, \infty) \cap X) \cong f(X \cap [a, \infty)),$$

by Proposition 1.24, so both $f(g(\mathbb{R}^+))$ and $f(g(\mathbb{R}^-))$ are contained in a subset of $\mathbb{R}$ which is bounded below, which contradicts that $f \circ g \sim i_{\mathbb{R}}$ since $\mathbb{R}$ is neither bounded above or below. It follows that one of $g(\mathbb{R}^+)$ and $g(\mathbb{R}^-)$ is bounded above while the other is bounded below.

Thus $g(\mathbb{R}^+)$ and $g(\mathbb{R}^-)$ are contained in at most two unbounded M-scale connected components, the union of which is neither bounded above nor below. It follows immediately that

$$(g(\mathbb{R}^+) \cup g(\mathbb{R}^-)) \smallsetminus [-M, M] \subset X \smallsetminus [-M, M]$$

contains exactly two unbounded M-scale components.

For the converse, suppose that there is an $M > 0$ such that $X \smallsetminus [-M, M]$ contains exactly two unbounded M-scale connected components. Then each of $X \cap \mathbb{R}^+$ and $X \cap \mathbb{R}^-$ must contain exactly one unbounded M-scale connected component, making them coarsely equivalent to, respectively, $\mathbb{R}^+$ and $\mathbb{R}^-$ by Theorem 1.48 and Corollary 1.49. □

We can now decide easily whether a subspace of $\mathbb{R}$ or $\mathbb{R}^+$ is coarsely equivalent to its parent space.

Example 1.53. Although shown already by a direct method in Example 1.28, the set of integers $\mathbb{Z}$ is coarsely equivalent to $\mathbb{R}$ since $\mathbb{Z} \smallsetminus [1, 1]$ contains exactly two unbounded 1-scale connected components. The set of natural numbers $\mathbb{N}$ is coarsely equivalent to $\mathbb{R}^+$ because $\mathbb{N}$ contains exactly one 1-scale connected component.

Example 1.54. The space $X = \{n^2 \mid n \in \mathbb{N}\}$ is not coarsely equivalent to $\mathbb{R}^+$, since for any $M > 0$ there is no unbounded M-scale connected component at all.

Showing directly that two spaces are not coarsely equivalent is, in general, not trivial, as we saw in this section and the one previous to it. Even showing an obvious case (that a line, $\mathbb{R}$, is not coarsely equivalent to a plane, $\mathbb{R}^2$) required an extended argument which depended upon properties specific to the spaces involved. We also saw that $\mathbb{R}$ is not coarsely equivalent to $\mathbb{R}^+$, as our initial classification suggested. Our argument in the first case relied upon $\mathbb{R}^2$ having "too great a dimension" to fit into $\mathbb{R}$. In the second case, the connection between the number of unbounded M-scale connected components in $\mathbb{R}$ and $\mathbb{R}^+$ with the restriction on motion in each space was discussed in the previous section. But finding the exact large-scale property which distinguishes one class of coarsely equivalent spaces from another might take some ingenuity in more complex cases.

By generalizing the idea of coarse equivalence, we attempt in the next chapter to find ways of comparing the large-scale structure of topological spaces. Eventually, by considering certain structures on functions in those spaces, we will have enough apparatus to make comparisons of large-scale structure without requiring the same level of detail or an argument which depends directly upon specific properties of the space itself. ❦

Chapter 2

Abstract Coarse Structures and Spaces

In the the previous chapter we considered whether two metric spaces had the same large-scale structure, by defining a pair of maps between the spaces with certain properties. We also saw that these properties depended on the metrics in question. However, in order to compare spaces which are not metrizable, we require a generalization which can establish whether points are "close," or whether sets are bounded, without reference to a metric. For example, the work done so far cannot tell us whether $\mathbf{Z}^+$ is coarsely equivalent to S_Ω (the minimal uncountable well-ordered set), because the latter is not metrizable.

One approach in describing the idea of closeness—in a way which is quite direct—is to specify neighborhoods of points explicitly, rather than by using a metric: in other words, by giving the space a *topology*. But this description of closeness applies only to small scales, and does not take account of the large-scale structure.

Another approach is analogous to the original definition of distance. For a metric space X, the distance function d is defined on the product $X \times X$. If two points $x, y \in X$ are close with respect to d, then $d(x, y)$ is relatively small. Put another way, x and y are close to each other if the point $x \times y$ is not too far away from the *diagonal* Δ of $X \times X$ defined by $\{(x,x) \mid x \in X\}$ (Figure 2.1). Of course, what "not too far away" means has to be specified in terms which do not involve a metric.

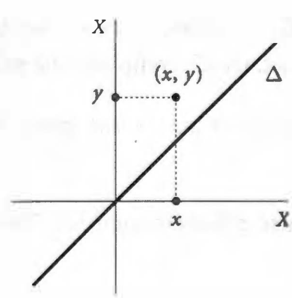

Figure 2.1: The diagonal Δ of $X \times X$.

The absence of a distance function means that we must find some way of redefining the concepts of *bounded* and *close*—and thereby *coarse maps* and *coarse equivalences*—for a more general class of spaces. We also require that the generalized concepts may agree with our previous definitions when restricted to metric spaces. The context in which we redefine these terms is, at first, very general (sets). We later allow a little more structure, and consider these concepts when X is a topological space. We therefore begin with a discussion of *relative compactness*, a concept required for an important definition in this chapter. Included in Appendix B are some reminders of other basic definitions from topology such as local compactness, Hausdorff spaces, and paracompactness.

The central idea in this chapter, however, is that of a *coarse structure* on X, which consists of the diagonal together with other subsets of $X \times X$, and which defines explicitly what it means for points to be close. This will lead us to redefinitions of the terms used in the previous chapter, and will allow us to construct coarse equivalences between topological spaces. In particular, we show that $\mathbf{Z}^+$ and S_Ω are not coarsely equivalent in a certain coarse structure.

2.1 Topological preliminaries

Of the definitions and propositions concerning topology required for later discussion, most of what is needed which may not be familiar centers on the idea of *relative compactness*. We assume throughout that all topological spaces are Hausdorff.

Definition 2.1. Let X be a topological space. A subset A of X is *relatively compact* if its closure $\overline{A}$ in X is compact.

We note that any compact set is automatically relatively compact. An example of relative compactness in a familiar setting is in Euclidean space $X = \mathbb{R}^n$, where any bounded set is relatively compact by the Heine–Borel Theorem.

Proposition 2.2. *Let X be a topological space, and let A be a relatively compact subset of X. Then any subset of A is relatively compact.*

Proof. If $B \subset A$, it follows that $\overline{B} \subset \overline{A}$. Since $\overline{A}$ is compact by hypothesis, and $\overline{B}$ is a closed subset of $\overline{A}$, then $\overline{B}$ is also compact, which completes the proof. □

Corollary 2.3. *Let X be a compact topological space. Then any subset of X is relatively compact.*

Proof. Since X is compact, it is also relatively compact. The proof follows from Proposition 2.2 with $A = X$. □

Proposition 2.4. *Let X be a topological space. Then a finite union of relatively compact subsets of X is relatively compact.*

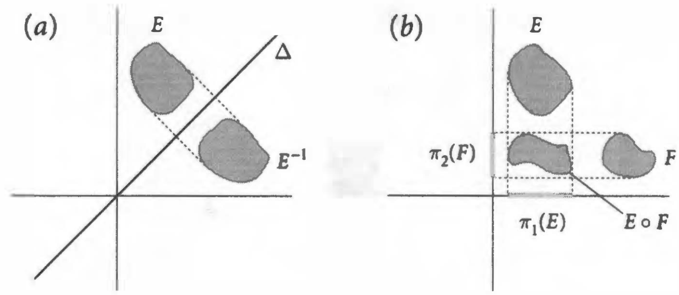

Figure 2.2: Inverses and products of subsets of $X \times X$. (*a*) The inverse E^{-1} of a subset E of $X \times X$. (*b*) The product $E \circ F$ is a subset of projections of E and F.

Proof. Let A and B be relatively compact subsets of X, so $\overline{A}$ and $\overline{B}$ are compact. We note that the union of two compact subsets of X is itself compact [14]. But $\overline{A \cup B} = \overline{A} \cup \overline{B}$, so $A \cup B$ is relatively compact. We have proved that a union of two relatively compact subsets is relatively compact; the result follows by induction on the number of relatively compact sets in the union. □

Although a finite union of relatively compact sets is relatively compact, even a countable union of such sets may not be, as the following example shows.

Example 2.5. Take X to be the set of real numbers $\mathbb{R}$, and let U_n be the interval $(n, n+1)$ for each integer n. Then each U_n is relatively compact since its closure is the closed and bounded (and hence compact) set $\overline{U_n} = [n, n+1]$. But $\bigcup U_n = \mathbb{R} - \mathbf{Z}$, whose closure is $\mathbb{R}$, which is not compact.

The reason the preceding example failed is because $\mathbb{R}$ is neither compact nor relatively compact; Proposition 2.2 showed that even infinite unions are relatively compact if X is at least relatively compact.

2.2 Products and inverses

Let X be a set. We define two operations on subsets of $X \times X$ [16]:

Definition 2.6. (*a*) If E is a subset of $X \times X$ then E^{-1} denotes the set $\{(y,x) | (x,y) \in E\}$, called the *inverse of E*.

(*b*) If E and F are subsets of $X \times X$, then $E \circ F$ denotes the set

$$\{(x,y) \mid \exists z \in X : (x,z) \in E, (z,y) \in F\}$$

called the *product of E and F*.

In geometrical terms, the inverse of E—shown in Figure 2.2 (*a*)—is the reflection of E in the diagonal $\Delta = \{(x,x) \mid x \in X\}$. If $E = E^{-1}$ then we say that E is *symmetric*. The

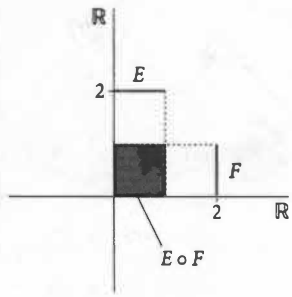

Figure 2.3: The product of $E = \{2\} \times I$ with $F = I \times \{2\}$ is the set $E \circ F = I \times I$.

interpretation of the product of E and F is more involved; we note, however, that at least $E \circ F$ is a subset of $\pi_1(E) \times \pi_2(F)$, where $\pi_1, \pi_2 : X \times X \longrightarrow X$ are the standard coordinate projections, shown in Figure 2.2 (*b*). A few examples show how products behave in specific circumstances.

Example 2.7. A concrete example is obtained by letting $X = \mathbb{R}$ and taking as subsets

$$E = I \times \{2\} \quad \text{and} \quad F = \{2\} \times I,$$

where I is the *unit interval* (Figure 2.3). Then $E \circ F$ is equal to $I \times I$, because for every (x, y) in $I \times I$, there is a $2 \in \mathbb{R}$ such that $(x, 2) \in E$ and $(2, y) \in F$, but no other point in $\mathbb{R}^2$ satisfies such a condition. But if we take $G = \{3\} \times I$ then $E \circ G = \varnothing$, because for each (x, y) in $\mathbb{R}^2$ there is no $z \in \mathbb{R}$ which will simultaneously allow (x, z) to be in E while (z, y) is in G, since z would have to be equal to both 2 and 3.

Example 2.8. The above example can be generalized to the result that, for any subset K of X, and any element $p \in X$,

$$(K \times \{p\}) \circ (\{p\} \times K) = K \times K.$$

The proof is similar to the explanation given in Example 2.7.

Example 2.9. If E is any subset of $X \times X$ then

$$\begin{aligned} E \circ (X \times X) &= \{(x, y) \mid \exists z \in X : (x, z) \in E, (z, y) \in X \times X\} \\ &= \pi_1(E) \times X \end{aligned}$$

and

$$\begin{aligned} (X \times X) \circ E &= \{(x, y) \mid \exists z \in X : (x, z) \in X \times X, (z, y) \in E\} \\ &= X \times \pi_2(E) \end{aligned}$$

whereas

$$E \times \varnothing = \varnothing = \varnothing \times E,$$

since for each point $(x, y) \in X \times X$ no such $z \in X$ exists that $(z, y) \in \varnothing$ in the first case, or that $(x, z) \in \varnothing$ in the second.

Example 2.10. The product of the diagonal $\Delta = \{(x,x) \mid x \in X\}$ with any other subset of $X \times X$ leaves the subset unchanged. If $E \subset X \times X$, we see that

$$\begin{aligned} E \circ \Delta &= \{(x,y) \mid \exists\, z \in X : (x,z) \in E, (z,y) \in \Delta\} \\ &= \{(x,y) \mid \exists\, y \in X : (x,y) \in E, (y,y) \in \Delta\} \\ &= E \end{aligned}$$

and

$$\begin{aligned} \Delta \circ E &= \{(x,y) \mid \exists\, z \in X : (x,z) \in \Delta, (z,y) \in E\} \\ &= \{(x,y) \mid \exists\, y \in X : (y,y) \in \Delta, (x,y) \in E\} \\ &= E. \end{aligned}$$

The inverse of a product is the product of the individual inverses taken in reverse, as one might expect.

Proposition 2.11. *Let E and F be subsets of $X \times X$. Then $(E \circ F)^{-1} = F^{-1} \circ E^{-1}$.*

Proof. It is easy to see that

$$\begin{aligned} E \circ F &= \{(x,y) \mid \exists z \in X : (x,z) \in E, (z,y) \in F\} \\ &= \{(y,x) \mid \exists z \in X : (y,z) \in F^{-1}, (z,x) \in E^{-1}\} \\ &= F^{-1} \circ E^{-1} \end{aligned}$$

which completes the proof. □

The presence of a binary operation $\circ$ on a collection $\mathcal{E}$ of subsets of $X \times X$ which is closed on taking products and inverses, together with associativity—which we shall not show—and the existence of an identity Δ, makes the collection a *paired groupoid* [16].

Before we resolve the issue of geometric interpretation of the product in $X \times X$, it would be helpful to list a few definitions which will allow us to be more specific about subsets of $X \times X$.

Definition 2.12. Let X be a set. If E is a subset of $X \times X$ and K is a subset of X then define [16]

$$E[K] = \{x \in X \mid \exists\, z \in K : (x,z) \in E\}.$$

Example 2.13. If K is the whole set, that is, $K = X$, then $E[X] = \pi_1(E)$; in fact, all that is required for this to be true is that $\pi_2(E) \subset K$. Similarly, $E^{-1}[X] = \pi_2(E)$. It is also clear that $E[\varnothing] = \varnothing$. Considering the diagonal Δ,

$$\Delta[K] = \{x \in X \mid \exists\, y \in K, (x,y) \in \Delta\} = \{x \in X \mid x \in K\} = K.$$

So if K is relatively compact, then $\Delta[K]$ is also.

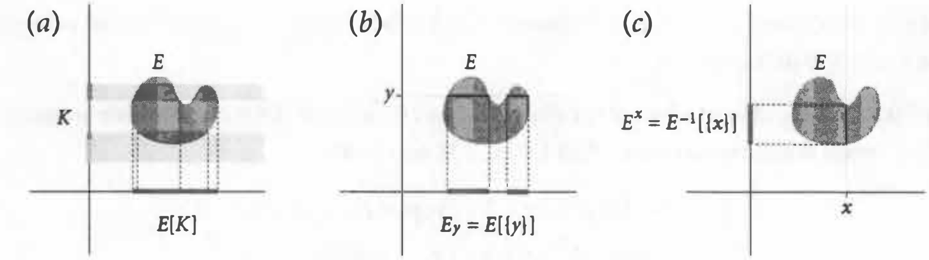

Figure 2.4: Sections of subsets $E \subset X \times X$. Shown are (*a*) the set $E[K]$, (*b*) the section $E_y = E[\{y\}]$, and (c) the section $E^x = E^{-1}[\{x\}]$.

Figure 2.4 (*a*) illustrates the idea behind $E[K]$, which implies that the set is the result of projecting, onto the first coordinate in $X \times X$, the intersection of sets $X \times K$ and E. The following proposition confirms that this is indeed the case.

Proposition 2.14. *Let X be a set; let $E \subset X \times X$; let $K \subset X$. Then*

$$E[K] = \pi_1((X \times K) \cap E).$$

Proof. If $E[K]$ is non-empty, let $x \in E[K]$, which is true if and only if there is a $z \in K$ such that $(x, z) \in E$, which is equivalent to $(x, z) \in (X \times K) \cap E$, which in turn is true if and only if $x \in \pi_1((X \times K) \cap E)$. This proves both $E[K] \subset \pi_1((X \times K) \cap E)$ and the reverse inclusion.

On the other hand, $E[K]$ is empty if and only if for each $x \in X$ there is no $z \in K$ such that $(x, z) \in E$; this is equivalent to $(X \times K) \cap E = \varnothing$, which is true if and only if $\pi_1((X \times K) \cap E)$ is empty.

In either case, $E[K] = \pi_1((X \times K) \cap E)$; the result follows. □

That $E^{-1}[K] = \pi_2((K \times X) \cap E)$ is proved almost identically. If K is a singleton set $\{x\}$ or $\{y\}$ then we use the notation E_y and E^x for the *sections* $E[\{y\}]$ and $E^{-1}[\{x\}]$ respectively, illustrated in Figure 2.4 (*b*) and (*c*). We observe that

$$E[K] = \bigcup_{y \in K} E_y \quad \text{and} \quad E^{-1}[K] = \bigcup_{x \in K} E^x$$

so $E[K]$ and $E^{-1}[K]$ are, respectively, the union of the E_y and E^x sections over K.

These definitions allow us to formulate a way of looking at the product of $E \circ F$ with a slightly more geometric interpretation.

Proposition 2.15. *Let E and F be subsets of $X \times X$. Then*

$$E \circ F = \{(x, y) \mid E^x \cap F_y \neq \varnothing\}.$$

The proof is straightforward; before we do so, however, this characterization of the product of E and F makes it clear that the product is in some sense a measurement of how "close"

the sets E and F are with respect to the diagonal Δ, in the sense that the first coordinates of E match with the second coordinates of F. The closer E and F are (that is, the greater the intersection in X of the x-sections of E and the y-sections of F) the more points of $X \times X$ will appear in the product.

Proof. Let $(x, y) \in E \circ F$, which is equivalent to there being a $z \in X$ such that $(x, z) \in E$ and $(z, y) \in F$. This is true if and only if $z \in E^x$ and $z \in F_y$, or that $z \in E^x \cap F_y$, which is to say that $E^x \cap F_y$ is non-empty. This proves both inclusions; the result follows. □

The following lemma is technical, but greatly simplifies the proof of Proposition 2.18.

Lemma 2.16. *Let E and F be subsets of $X \times X$. If $K \subset X$ then $(E \circ F)[K] = E[F^{-1}[K]]$.*

Proof. We see that

$$\begin{aligned} E[F^{-1}[K]] &= \{x \in X \mid \exists z \in F^{-1}[K] : (x, z) \in E\} \\ &= \{x \in X \mid \exists z \in \{z \in X \mid \exists y \in K : (z, y) \in F^{-1}\} : (x, z) \in E\} \\ &= \{x \in X \mid \exists y \in K, \exists z \in X : (x, z) \in E, (y, z) \in F\} \\ &= \{x \in X \mid \exists y \in K : (x, y) \in E \circ F\} \\ &= (E \circ F)[K], \end{aligned}$$

completing the proof. □

With the basic definitions in place, we now consider some outcomes of letting X have more structure than a set.

Definition 2.17. Suppose X is a topological space. A subset E of $X \times X$ is said to be *proper* if both $E[K]$ and $E^{-1}[K]$ are relatively compact whenever $K \subset X$ is relatively compact.

A proper subset of $X \times X$ in this sense will be distinguished from the set-theoretical sense of *proper* (that is A is a proper subset of B if there is an $x \in B$ such that $x \notin A$) by writing $A \subsetneq B$ for the latter if there is any danger of confusion.

For reference (see also Section 2.1) a subset K of a topological space is said to be *relatively compact* if its closure is compact. Our definition of a proper subset of $X \times X$, together with Lemma 2.16, makes the following proposition uncomplicated.

Proposition 2.18. *Let E, F be proper subsets of $X \times X$. Then*

(*a*) *E^{-1} is proper.*
(*b*) *A subset of E is proper.*
(*c*) *$E \circ F$ is proper.*

Proof. The symmetry of Definition 2.17 makes it clear that the inverse E^{-1} of a proper set E is proper, which takes care of (*a*). For (*b*), let F be a subset of E, and let K be any relatively compact subset of X. The result follows from the fact that $F[K] \subset E[K]$. Finally for (*c*),

let K be a relatively compact subset of X. Then by hypothesis, $F^{-1}[K]$ and $E[F^{-1}[K]]$ are relatively compact, and $E^{-1}[K]$ and $F[E^{-1}[K]]$ are also relatively compact. But by Lemma 2.16,

$$E[F^{-1}[K]] = (E \circ F)[K]$$

and additionally by Proposition 2.11,

$$F^{-1}[E[K]] = F^{-1}[(E^{-1})^{-1}[K]] = (F^{-1} \circ E^{-1})[K] = (E \circ F)^{-1}[K],$$

so $(E \circ F)[K]$ and $(E \circ F)^{-1}[K]$ are relatively compact, whence $E \circ F$ is proper. □

Example 2.19. We showed in Example 2.13 that $\Delta[K] = K$ for any subset K of X. Thus, if K is relatively compact, so is $\Delta[K]$; it follows that Δ is always proper for any X.

The geometric interpretation of properness is that a subset E of $X \times X$ is proper if, when taking the y-sections of a reasonably well-behaved (that is, relatively compact) subset K of X, and intersecting those sections with E, then the x-projection of this intersection produces a well-behaved subset of X.

2.3 Coarse structures

The main definition of this chapter, due to Roe in [16], is the following.

Definition 2.20. A *coarse structure* on a set X is a collection $\mathcal{E}$ of subsets of $X \times X$—called the *controlled sets* for the coarse structure—which contains the diagonal and is closed under the formation of subsets, inverses, products, and finite unions. A set X equipped with a coarse structure is called a *coarse space*.

The controlled sets are also called *entourages*. In a similar manner to that used in discussions of topological spaces, we sometimes refer to a coarse space by its set X alone in place of the pair $(X, \mathcal{E})$. We leave a discussion of the interpretation of the axioms for a coarse structure until Section 2.5.

In order to check whether a collection $\mathcal{E}$ of subsets of a set X is a coarse structure, it is usually straightforward to show that subsets, inverses, and finite unions of controlled sets are also contained in $\mathcal{E}$. For unions, in fact, it is enough to show that $\mathcal{E}$ is closed under taking unions of just two subsets; finite unions are then satisfied by induction on the number of subsets in the union. The difficult part is often to show that $\mathcal{E}$ is closed under taking products. We begin with an easy example.

Example 2.21. The *maximal coarse structure* on X has $\mathcal{E} = \mathfrak{P}(X \times X)$.

Clearly the diagonal Δ is contained in $\mathcal{E}$; subsets, inverses, and finite unions also present no problem; and in this case, since products are also subsets of $X \times X$, we are guaranteed that $\mathcal{E}$ is also closed under this operation.

That X has a maximal coarse structure is equivalent to $X \times X$ being contained in $\mathcal{E}$ since $\mathcal{E}$ is closed under taking subsets.

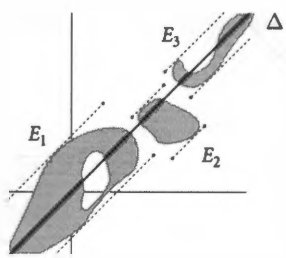

Figure 2.5: The bounded coarse structure on X. Each set E_1, E_2, and E_3 may have a different bound from the diagonal Δ.

Example 2.22. The *trivial coarse structure* on X consists only of subsets of the diagonal.

The diagonal is automatically controlled, and subsets, inverses and even infinite unions are controlled as long as we can show that $E \circ F \subset \Delta$ whenever $E, F \subset \Delta$. But

$$\begin{aligned} E \circ F &= \{(x, y) \mid \exists z \in X : (x, z) \in E, (z, y) \in F\} \\ &= \{(x, x) \mid (x, x) \in E \cap F\} \\ &= E \cap F, \end{aligned}$$

which is controlled since it is a subset of Δ.

Example 2.23. Let (X, d) be a metric space and let $\mathcal{E}$ be the collection of all subsets E of $X \times X$ for which the coordinate projection maps $\pi_1, \pi_2 : E \longrightarrow X$ are close; that is to say,

$$d(\pi_1(x \times y), \pi_2(x \times y)) = d(x, y)$$

is uniformly bounded on E, which is equivalent to $\sup\{d(x, y) | (x, y) \in E\} < \infty$ (see Figure 2.5). In this case we will say that E has a *finite supremum* . Then $\mathcal{E}$ is a coarse structure, called the *bounded coarse structure* associated to the given metric.

In geometric terms, the collection $\mathcal{E}$ consists of all those sets E which are not "too far" away from the diagonal. To check that $\mathcal{E}$ actually is a coarse structure, we see that $\Delta \in \mathcal{E}$ by the fact that $d(x, x) = 0$ for every $(x, x) \in \Delta$. The collection $\mathcal{E}$ is closed under inverses by the symmetry of the metric d, and if $E \in \mathcal{E}$ has a finite supremum then any subset of E will also have a finite supremum. For unions, let E and F be elements of $\mathcal{E}$, and suppose that $s_E = \sup\{d(x, y) | (x, y) \in E\}$ and $s_F = \sup\{d(x, y) | (x, y) \in F\}$, so both s_E and s_F are finite. Let $s_{E \cup F}$ be the larger of s_E and s_F; then $s_{E \cup F}$ is finite and $\sup\{d(x, y) \mid (x, y) \in E \cup F\} \leq s_{E \cup F} < \infty$, so $E \cup F \in \mathcal{E}$.

For products, we take subsets E and F with suprema s_E and s_F respectively. If $(x, y) \in E \circ F$ then there is a $z \in X$ such that $(x, z) \in E$ and $(z, y) \in F$. By the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) \leq s_E + s_F$$

and so $s_E + s_F$ serves as a bound on $E \circ F$. It follows that $\mathcal{E}$ is a coarse structure.

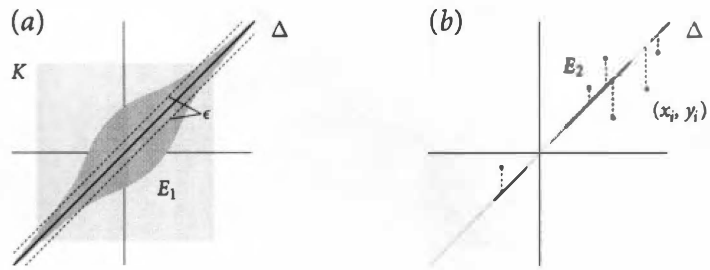

Figure 2.6: The C_0 and discrete coarse structures. (*a*) A set E_1 in the C_0 coarse structure has $d(x,y) \longrightarrow 0$ as $(x,y) \longrightarrow \infty$. (*b*) A set E_2 in the discrete coarse structure on X.

Example 2.24. Let (X,d) be a metric space and let $\mathcal{E}$ be the collection of all those subsets $E \subset X \times X$ for which the distance function d, when restricted to E, tends to zero at infinity, shown in Figure 2.6 (*a*). In other words, E is controlled if for any $\epsilon > 0$ there is a compact subset K of X such that $d(x,y) < \epsilon$ whenever $(x,y) \in E \smallsetminus K \times K$. Then $\mathcal{E}$ is a coarse structure on X, called the C_0 *coarse structure* associated to the metric d.

Proof. For subsets, let $E \in \mathcal{E}$, let $F \subset E$, and let $\epsilon > 0$. There is a compact $K \subset X$ such that $d(x,y) < \epsilon$ on $E \smallsetminus K \times K$, so this is certainly true on $F \smallsetminus K \times K \subset E \smallsetminus K \times K$.

Since the distance function d is symmetric, $E^{-1} \in \mathcal{E}$ whenever $E \in \mathcal{E}$.

For unions, let E and E be controlled subsets, and let $\epsilon > 0$. There exist compact sets K and L such that $d(x,y) < \epsilon$ on both $E \smallsetminus K \times K$ and $F \smallsetminus L \times L$. Since $K \cup L$ is compact, and

$$(E \cup F) \smallsetminus (K \cup L) \times (K \cup L) \subset (E \smallsetminus K \times K) \cup (F \smallsetminus L \times L),$$

it follows that $d(x,y) < \epsilon$ on $(E \cup F) \smallsetminus (K \cup L) \times (K \cup L)$, so $E \cup F$ is a controlled subset.

Finally, for products, let E, F be controlled subsets; let $\epsilon > 0$; let K, L be compact subsets of X such that $d(x,y) < \epsilon/2$ on both $E \smallsetminus K \times K$ and $F \smallsetminus L \times L$. We show that the compact set $M = K \cup L$ is again sufficient for $d(x,y) < \epsilon$ on $E \circ F$, and thus $E \circ F$ is a controlled subset. Suppose $(x,y) \in E \circ F \smallsetminus M \times M$. Then there is a $z \in X$ such that $(x,z) \in E$ and $(z,y) \in F$, but (x,y) is not an element of $M \times M$, which is to say, neither x nor y are elements of either K or L. Thus

$$(x,z) \in E \smallsetminus K \times K \quad \text{and} \quad (z,y) \in F \smallsetminus L \times L,$$

whence

$$d(x,y) \leq d(x,z) + d(z,y) < \epsilon/2 + \epsilon/2 = \epsilon$$

by the triangle inequality.

Thus $\mathcal{E}$ is a coarse structure, which completes the proof. □

Example 2.25. Let X be a set and let $\mathcal{E}$ be the collection of all subsets of $X \times X$ that contain only finitely many points off the diagonal, shown in Figure 2.6 (*b*). Then $\mathcal{E}$ is a coarse structure, called the *discrete coarse structure* [16] on X.

Clearly $\Delta \in \mathcal{E}$ since it has no points (a finite number) off the diagonal at all; taking the inverse of a subset $E \subset X \times X$ preserves the number of points off the diagonal; taking subsets will not increase the number of points off the diagonal, so a finite number of points off the diagonal remains finite; a finite union of sets with finitely many points off the diagonal will still have only finitely many points off the diagonal.

Slightly harder is showing that the collection is closed under products. Let E and F each have finitely many points off the diagonal. Then $E \circ F \setminus \Delta = \{(x_\alpha, y_\alpha) \mid \alpha \in J\}$ for some indexing set J. For each $\alpha \in J$ there is a $z_\alpha \in X$ such that $(x_\alpha, z_\alpha) \in E$ and $(z_\alpha, y_\alpha) \in F$; for all but a finite number of α, $z_\alpha = x_\alpha$ and $z_\alpha = y_\alpha$ by hypothesis, so letting $K = \{i \in J \mid z_i \neq x_i\}$, and $L = \{i \in J \mid z_i \neq y_i\}$ it follows that K and L are finite indexing subsets of J. Thus for all $\alpha \in J \setminus (K \cup L)$, $x_\alpha = z_\alpha = y_\alpha$, leaving only a finite number of points of $E \circ F$ off the diagonal. It follows that $E \circ F \in \mathcal{E}$ and thus $\mathcal{E}$ is a coarse structure.

Example 2.26. Let X be a topological space, and let $\mathcal{E}$ be the collection of all sets E of the form $C \cup \Delta_Y$ where $C \subset X \times X$ is relatively compact, Y is a subset of X, and

$$\Delta_Y = \{(x, x) \in X \times X \mid x \in Y\}.$$

Then $\mathcal{E}$ is a coarse structure, called the *relatively compact coarse structure* .

The diagonal $\Delta = \varnothing \cup \Delta_X$ is in the collection $\mathcal{E}$ since $\varnothing$ is relatively compact. If $E = C \cup \Delta_Y$ is controlled then C^{-1} is relatively compact, and $(\Delta_Y)^{-1} = \Delta_Y$, whence

$$E^{-1} = (C \cup \Delta_Y)^{-1} = C^{-1} \cup \Delta_Y \in \mathcal{E}.$$

Let $E = C \cup \Delta_Y \in \mathcal{E}$. Then if $F \subset E$,

$$F = F \cap E = F \cap (C \cup \Delta_Y) = (F \cap C) \cup (F \cap \Delta_Y) = (F \cap C) \cup \Delta_{\pi_1(F) \cap Y} \in \mathcal{E}$$

since subsets of relatively compact sets are relatively compact. If $E = C \cup \Delta_Y$ and $F = D \cup \Delta_Z$ for $C, D \subset X \times X$ relatively compact and $Y, Z \subset X$ then

$$E \cup F = (C \cup \Delta_Y) \cup (D \cup \Delta_Y) = (C \cup D) \cup (\Delta_Y \cup \Delta_Z)$$

which is a member of $\mathcal{E}$ since finite unions of relatively compact sets are relatively compact, and $\Delta_Y \cup \Delta_Z = \Delta_{Y \cup Z}$. Finally, if $E = C \cup \Delta_Y$ and $F = D \cup \Delta_Z$ are members of $\mathcal{E}$, then

$$\begin{aligned} E \circ F &= (C \cup \Delta_Y) \circ (D \cup \Delta_Z) \\ &= (C \circ D) \cup (\Delta_Y \circ D) \cup (C \circ \Delta_Z) \cup (\Delta_Y \circ \Delta_Z) \\ &= K \cup \Delta_{Y \cap Z} \end{aligned}$$

where $K = (C \circ D) \cup (\Delta_Y \circ D) \cup (C \circ \Delta_Z)$ is relatively compact. Thus $\mathcal{E}$ is a coarse structure.

Example 2.27. Let X be an topological space, and let $\mathcal{E}$ be the collection of all proper (in the sense of Definition 2.17) subsets of $X \times X$. Then $\mathcal{E}$ is a coarse structure, called the *indiscrete coarse structure* on X.

Example 2.19 showed that the diagonal Δ is proper (since for any relatively compact subset $K \subset X$, $\Delta[K] = K$). Proposition 2.18 showed that inverses, subsets, and products of proper sets are themselves proper. Thus $\mathcal{E}$ is indeed a coarse structure.

Definition 2.28. A coarse structure on X is *connected* if each point of $X \times X$ belongs to some controlled set.

It is clear that if X has the maximal coarse structure $\mathcal{E} = \mathfrak{P}(X \times X)$ then $\mathcal{E}$ is connected. Some other coarse structures are always connected.

Example 2.29. The discrete, bounded, indiscrete, and relatively compact coarse structures on X are connected.

For any of these coarse structures, a singleton ordered pair $\{(x, y)\} \subset X \times X$ is controlled. In the discrete case this is true because it is either contained in the diagonal, or is one point (a finite number) off the diagonal, depending on whether $x = y$. For the bounded coarse structure on a metric space X it is clear that $d(x, y)$ is finite. In the indiscrete case, $\{(x, y)\}[K]$ is equal to either $\{x\}$ or $\varnothing$, both of which are relatively compact, depending on whether $y \in K$. And for the relatively compact coarse structure, $\{(x, y)\}$ (which is relatively compact) is equal to either $\{(x, y)\} \cup \Delta_{\varnothing}$ or $\varnothing \cup \Delta_{\{x\}}$ depending on whether $x = y$.

In a way similar to the comparison of one topology on X with another, we can compare coarse structures on X.

Definition 2.30. Let $\mathcal{E}$ and $\mathcal{F}$ be coarse structures on X. If every $\mathcal{E}$-controlled set is also $\mathcal{F}$-controlled, that is, if $\mathcal{E} \subset \mathcal{F}$, then we say that $\mathcal{E}$ is *finer* than $\mathcal{F}$.

Note that the terminology used here is the opposite to its use in discussion of topological spaces, where a *topology* $\mathcal{T}$ on X is *finer* than S if $S \subset \mathcal{T}$, because $\mathcal{T}$ has more open sets than S. For coarse structures, the finer of the two is the one with the *fewer* controlled subsets.

Proposition 2.31. *The C_0 coarse structure on a non-compact metric space X is strictly finer than the indiscrete coarse structure on X.*

Proof. Let E be controlled in the C_0 coarse structure, and let L be a relatively compact subset of X. For $\epsilon = \frac{1}{2}$ there is a compact $K \subset X$ such that $d(x, y) < \frac{1}{2}$ for all $(x, y) \in E \setminus K \times K$, so $E \subset K \times K \cup D(\frac{1}{2})$ where $D(\epsilon)$ is defined as $\{(x, y) \in X \times X \mid d(x, y) < \epsilon\}$. Then

$$E[L] \subset (K \times K \cup D(\tfrac{1}{2}))[L] \subset K \cup B(L, 1)$$

whether L intersects K or not. Since L is relatively compact and K is compact it follows that $K \cup B(L, 1)$ is relatively compact, which implies that $E[L]$ is relatively compact. By symmetry

$E^{-1}[L]$ is also relatively compact, and thus it follows that E is proper, showing that the C_0 coarse structure is finer than the indiscrete coarse structure.

To show that the containment is strict, we note that $D(1)$ is proper, but that $D(1)$ is not contained in the C_0 coarse structure because for any $\epsilon < 1$ there is no compact set $K \subset X$ such that $d(x, y) < \epsilon$ on $D(1) \setminus K \times K$ since X is not compact by hypothesis. □

Since we are always concerned with large-scale structure, we lose nothing by considering only topological spaces with are not compact. In these cases, the relatively compact coarse structure is strictly contained in the indiscrete coarse structure.

Proposition 2.32. *Let X be a non-compact topological space. Then the relatively compact coarse structure on X is strictly finer than the indiscrete coarse structure on X.*

Proof. Let $E = C \cup \Delta_Y$ be a member of the relatively compact coarse structure $\mathcal{T}$ on X, and let K be a relatively compact subset of X. Then

$$\begin{aligned} E[K] &= (C \cup \Delta_Y)[K] \\ &= \{x \in X \mid \exists y \in K, (x, y) \in (C \cup \Delta_Y)\} \\ &\subset K \cup \pi_1(C) \end{aligned}$$

which is relatively compact, so E is proper, whence $E \in \mathcal{E}$ (the indiscrete coarse structure on X) and thus $\mathcal{T} \subset \mathcal{E}$. On the other hand, in $\mathcal{E}$ there is a neighborhood $U \subset X \times X$ of the diagonal which is also controlled (see Section 4.2), so $U \in \mathcal{E}$. But $U \setminus \Delta_X$ is not even relatively compact, for if it were, $U[X]$ would be relatively compact, but $U[X] = X$, which is not by hypothesis. □

Proposition 2.33. *The intersection of a family of coarse structures is a coarse structure.*

Proof. Let $\{\mathcal{E}_\alpha\}$ be a family of coarse structures indexed by some set J. Clearly $\Delta \in \bigcap\{\mathcal{E}_\alpha\}$, since $\Delta \in \mathcal{E}_\alpha$ for each $\alpha \in J$. If $E \in \bigcap\{\mathcal{E}_\alpha\}$ then $E \in \mathcal{E}_\alpha$ for each α, and hence any subset of E is in each $\mathcal{E}_\alpha$, and hence in $\bigcap\{\mathcal{E}_\alpha\}$. Inverses are proved similarly. If $E, F \in \bigcap\{\mathcal{E}_\alpha\}$ then $E, F \in \mathcal{E}_\alpha$ for each α; it follows that both $E \cup F$ and $E \circ F$ are contained in $\mathcal{E}_\alpha$ for each α, and thus $E \cup F$ and $E \circ F$ are in $\bigcap\{\mathcal{E}_\alpha\}$. With all of the axioms demonstrated, it follows that $\bigcap\{\mathcal{E}_\alpha\}$ is a coarse structure. □

Corollary 2.34. *Let S be a collection of subsets of $X \times X$. Then there is a unique coarse structure $\mathcal{E}$ on X that contains S and is finer than any other coarse structure on X.*

Proof. Since $\mathfrak{P}(X \times X)$ contains S, then the family of coarse structures $\{\mathcal{E}_\alpha\}$ which contain S is non-empty. It follows by Proposition 2.33 that $\mathcal{E} = \bigcap\{\mathcal{E}_\alpha\}$ is a coarse structure containing $\mathcal{E}$, and is finer than any other such coarse structure by definition. □

We say that the coarse structure $\mathcal{E}$ in Corollary 2.34 is *generated* by S, denoted $\mathcal{E} = \langle S \rangle$. To end, we note that the discrete coarse structure is not the finest non-trivial coarse structure

on a set X containing at least two points, since for $x \neq y$ in X there is the collection

$$\langle\{(x,y)\}\rangle = \{\Delta_Y, \{(x,y)\} \cup \Delta_Y, \{(y,x)\} \cup \Delta_Y, \{(x,y),(y,x)\} \cup \Delta_Y \mid Y \subset X\};$$

that is, there are coarse structures whose controlled sets consist of subsets of Δ together with one point (x,y) off the diagonal, its inverse (y,x), or both. The discrete coarse structure, however, is the finest connected structure on a set X.

2.4 Coarse structures and closeness

In the previous chapter we declared that two maps $f, g : X \longrightarrow Y$ between metric spaces X and Y were *close* if $d(f(x), f(y))$ was uniformly bounded. Since we do not necessarily have a metric on Y, we need to reformulate our definition for a general coarse space.

Definition 2.35. Let X be a coarse space, and let S be a set. Two maps $f, g : S \longrightarrow X$ are *close* if the subset $\{(f(s), g(s)) \mid s \in S\}$ of $X \times X$ is controlled.

Example 2.36. If we let $\mathcal{E}$ be the bounded coarse structure on X from Example 2.23 then the two definitions of closeness coincide, since the controlled sets of $\mathcal{E}$ all have finite suprema.

Closeness again is an equivalence relation on the set of maps from S into X.

Proposition 2.37. *Let S be a set, and let X be a coarse space. For two maps $f, g : S \longrightarrow X$, say that $f \sim g$ if and only if f is close to g in the sense of Definition 2.35. Then $\sim$ is an equivalence relation on the set of maps from S to X.*

Proof. A map f is clearly close to itself because $\{(f(s), f(s)) \mid s \in S\} \subset \Delta$ and subsets of controlled sets are controlled: this satisfies reflexivity. Symmetry is shown by supposing that f is close to g: then $E = \{(f(s), g(s)) | s \in S\}$ is controlled, and so $E^{-1} = \{(g(s), f(s)) | s \in S\}$ is also controlled, meaning that g is close to f. Finally, suppose that f and g are close, and g and h are close. Then $E = \{(f(s), g(s)) \mid s \in S\}$ and $F = \{(g(s), h(s)) \mid s \in S\}$ are controlled. But

$$\begin{aligned} \{(f(s), h(s)) \mid s \in S\} &= \{(f(s), h(s)) \mid \exists s \in S : (f(s), g(s)) \in E, (g(s), h(s)) \in F\} \\ &= E \circ F \end{aligned}$$

which is controlled, so f is close to h, satisfying transivity. It follows that $\sim$ is an equivalence relation, as required. □

Suppose a set $E \subset X \times X$ is controlled. This is equivalent to saying

$$\begin{aligned} E &= \{(x,y) \mid (x,y) \in E\} \\ &= \{(\pi_1(x,y), \pi_2(x,y)) \mid (x,y) \in E\} \end{aligned}$$

or that π_1 and π_2 are close on E. As noted in [16], the closeness relation determines the coarse structure, which the following proposition—the notation of which is modified slightly from [16] for greater clarity—makes clear.

Figure 2.7: The extended real line R_∞.

Proposition 2.38. *Let $(X, \mathcal{E})$ be a coarse space. Say that $p \sim q$ if a map $p : S \longrightarrow X$ is close to a map $q : S \longrightarrow X$. Then the relation $\sim$ determined by $\mathcal{E}$ has the following properties:*

(a) *if $p_1, p_2 : S \longrightarrow X$ are close and $q : T \longrightarrow S$ is any map, then $p_1 \circ q \sim p_2 \circ q$.*

(b) *if $S = S_1 \cup S_2$ and if $p_1, p_2 : S \longrightarrow X$ are maps whose restrictions to both S_1 and S_2 are close, then $p_1 \sim p_2$.*

Conversely, suppose for each set S there is an equivalence relation $\sim_S$ on the maps $S \longrightarrow X$ which satisfies (a) and (b) above. Then the collection $\{\sim_S\}$ of relations for a specific S is a collection of closeness relations arising from some uniquely determined coarse structure on X. Connectedness of the coarse structure is equivalent to the additional condition that

(c) *any two constant maps $S \longrightarrow X$ are close to each other.*

Before we prove this, we should note that Example 1.15, which showed that any two constant maps $k, l : X \longrightarrow Y$ between metric spaces X and Y were close, does not necessarily apply to (c): the closeness relation here is dependent on some coarse structure, which may have different properties from the bounded coarse structure (this coarse structure was shown in Example 2.36 to have the same notion of closeness as for metric spaces). If distances are finite, then the coarse structures defined thus far are connected, and (c) therefore applies.

Example 2.39. For a counter-example we require a coarse structure which is not connected: a metric space where the distance function is allowed to become infinite is sufficient. We define the *extended real numbers* $\mathbb{R}_\infty$ as $\mathbb{R} \cup \{-\infty, \infty\}$; we let $d_\infty : \mathbb{R}_\infty \times \mathbb{R}_\infty \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ be an extension of a metric $d : \mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$, so that

$$d_\infty(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in \mathbb{R} \\ \infty & \text{if } x = \infty, y \neq \infty \\ \infty & \text{if } x = -\infty, y \neq -\infty. \end{cases}$$

Then d_∞ is an *extended metric* on $\mathbb{R}_\infty$, the extension arising from the mapping into $\mathbb{R} \cup \{\infty\}$ rather than $\mathbb{R}$ (see Figure 2.7).

Giving $\mathbb{R}_\infty$ the bounded coarse structure—that is, all subsets of $\mathbb{R}_\infty \times \mathbb{R}_\infty$ which have a bounded supremum—obviously means that we have two points $\{-\infty, \infty\}$ which are not contained in any controlled set, so $\mathbb{R}_\infty$ is not connected. But there are constant maps

$$k(x) = -\infty \quad \text{and} \quad l(x) = \infty$$

which are not close in the sense of Definition 2.35.

We now prove the proposition.

Proof of Proposition 2.38. We begin with the sufficiency. In (a), we have by hypothesis that $E = \{p_1(s) \times p_2(s) \mid s \in S\}$ is controlled. But

$$\begin{aligned}\{(p_1 \circ q)(t) \times (p_2 \circ q)(t) \mid t \in T\} &= \{p_1(q(t)) \times p_2(q(t)) \mid t \in T\} \\ &= \{p_1(s) \times p_2(s) \mid s \in \operatorname{Im} q\} \\ &\subset E\end{aligned}$$

and since E is controlled, so are each of the other sets in the above equation. It follows that $p_1 \circ q \sim p_2 \circ q$. In (b) we have

$$\begin{aligned}\{p_1(s) \times p_2(s) \mid s \in S\} &= \{p_1(s) \times p_2(s) \mid s \in S_1\} \cup \{p_1(s) \times p_2(s) \mid s \in S_2\} \\ &= \{p_1|_{S_1}(s) \times p_2|_{S_1}(s) \mid s \in S_1\} \cup \{p_1|_{S_2}(s) \times p_2|_{S_2}(s) \mid s \in S_2\}\end{aligned}$$

which is controlled since each set in the latter union is controlled.

For the necessity we have for each set S an equivalence relation $\sim_S$ on the maps $S \longrightarrow X$, and this collection satisfies (a) and (b). As in [16], the controlled sets $E \subset X \times X$ are those for which the two coordinate projections $\pi_1|_E, \pi_2|_E$ are close. We then need to check that the family of controlled sets $\mathcal{E}$ is closed under taking subsets, inverses, unions, and products.

For subsets, if E is controlled then $\pi_1|_E \sim \pi_2|_E$. It follows from (a) applied to the inclusion $F \longmapsto E$ that if $F \subset E$ then $\pi_1|_F \sim \pi_2|_F$. For inverses we note that if $E \subset X \times X$ then $\pi_1|_E : E \longrightarrow X$ and $\pi_2|_{E^{-1}} : E^{-1} \longrightarrow X$ are the same map, for

$$\pi_1(\{x \times y\}) = \{x\} = \pi_2(\{y \times x\}) = \pi_2(\{(x \times y)^{-1}\}).$$

It follows from (a) applied to the bijection $E^{-1} \longrightarrow E$ that if $\pi_1|_E$ and $\pi_2|_E$ are close then so are $\pi_2|_{E^{-1}}$ and $\pi_1|_{E^{-1}}$.

For unions, if E and F are controlled, then $\pi_1, \pi_2 : E \longrightarrow X$ are close, and $\pi_1, \pi_2 : F \longrightarrow X$ are close. Thus the union of the controlled sets

$$\begin{aligned}\{(\pi_1(x \times y), \pi_2(x \times y)) \mid (x \times y) \in E\} &\cup \{(\pi_1(x \times y), \pi_2(x \times y)) \mid (x \times y) \in F\} \\ &= \{(\pi_1(x \times y), \pi_2(x \times y)) \mid (x \times y) \in E \cup F\}\end{aligned}$$

is controlled by (b).

The proof that a product of two controlled sets E and F is controlled is given in [16].

For (c), let $k, l : S \longrightarrow X$ be constant maps, so $k(s) = x$ and $l(s) = y$ for some $x, y \in X$. Then $\{k(s) \times l(s) \mid s \in S\} = \{x \times y\}$ is controlled if and only if X is connected, which completes the proof. $\square$

We have therefore redefined what it means for two maps to be close. Previously, when we had a metric to work with, two maps were close if they were uniformly bounded. The redefinition of closeness makes it clear that closeness is dependent on a coarse structure: two maps $f, g : S \longrightarrow X$ are close if the set of points $\{f(s) \times g(s) \mid s \in S\}$ is controlled.

But that a closeness relation actually *determines* a coarse structure (via coordinate projection) means that the structure and closeness are effectively saying the same thing, since a coarse structure also determines closeness.

2.5 Interpretation of the axioms

Now that we have defined closeness of maps in terms of controlled sets, we can provide an interpretation of the axioms for a coarse structure: the diagonal Δ is controlled, and subsets, finite unions, and products of controlled sets are controlled.

The set $X \times X$ can be regarded as the complete set of *relations* on X, so a controlled set is a certain subset of these relations. In particular, that a point $(x, y) \in X \times X$ is contained in a controlled set E—in view of the comments on closeness—says that "x is close to y." That is, the controlled sets of a coarse structure $\mathcal{E}$ determine which points in a space X are said to be close together. That the diagonal is controlled—

$$\Delta = \{(x,x) \mid x \in X\} \in \mathcal{E}$$

—means that points are close to themselves; the inverses axiom

$$\forall E \in \mathcal{E}(E^{-1} = \{(y,x) \mid (x,y) \in E\} \in \mathcal{E})$$

states that closeness is symmetric; and the products axiom

$$\forall E, F \in \mathcal{E}(E \circ F = \{(x,y) \mid \exists z \in X : (x,z) \in E, (z,y) \in F\} \in \mathcal{E})$$

is a version of the triangle inequality.

The axioms specifying closure under taking subsets and finite unions—

$$\forall E \in \mathcal{E}(F \subset E \Rightarrow F \in \mathcal{E}), \quad \forall E, F \in \mathcal{E}(E \cup F \in \mathcal{E}),$$

—however, are responsible for the ability of coarse spaces to describe large-scale structure. Taking subsets corresponds to "zooming in." A subset F of a controlled set E excludes some (x, y) points that were in E from the new controlled set F. This means, essentially, that we are throwing away some relations of closeness, but since we are interested in large-scale structure this is not a problem. Formation of finite unions, on the other hand, corresponds to "zooming out." In the union $E \cup F$, two sets of relations which previously specified closeness separately now specify closeness jointly. Points x and y which are unrelated—that is, (x, y) remains outside of our union—are therefore further apart. If we wish to include them we can, by taking larger unions, which in essence is zooming out: looking at the space from a larger distance makes points which seemed further apart appear somewhat closer. We cannot, however, in a finite number of unions include relations between points which are arbitrarily far apart, for this would violate our understanding of large-scale structure.

Thus a coarse structure describes the usual notions of points being close to themselves, that closeness is symmetric, and that points x and y should not be any further apart than

are a combination of x and z, with z and y. However, it also describes large-scale structure in terms of an arbitrary throwing away of small-scale relations, and a careful inclusion of large-scale relations.

2.6 Boundedness in coarse spaces

The work up to this point has prepared us to be able to define what a *bounded* set should be if one does not have a metric to determine distance. We have redefined what it means for two maps to be close: previously we defined this concept in terms of uniform boundedness, but in Section 2.4 we saw that this can now be framed in terms of a coarse structure.

We saw in Section 1.5 that any metrically-bounded subset B is coarsely equivalent to a point. This, effectively, says that an inclusion map on B is close to a constant map, and this gives us one of the most straightforward definitions of boundedness in terms of a coarse structure. With the redefinition of closeness in terms of the coarse structure, however, close does not necessarily mean "near" in the previous sense.

Since a coarse structure and closeness are related by coordinate projections, we can also say that a subset $B \subset X$ is bounded if $B \times B$ is controlled, since then the coordinate projections are in fact equal. We do not even require this, however: we can also say that B is bounded if $B \times \{p\}$ is controlled for some $p \in X$. This says that the first projection is close to a fixed point, which seems natural. There is another reformulation which is useful to us, given in the first proposition of the section.

Proposition 2.40. *Let X be a coarse space, and let B be a subset of X. Then the following are equivalent* [16]*:*

(*a*) *The inclusion map $B \longrightarrow X$ is close to a constant map.*
(*b*) *$B \times \{p\}$ is controlled for some $p \in X$.*
(*c*) *$B \times B$ is controlled.*
(*d*) *$B = E_p$ for some controlled set E and some $p \in X$.*

Proof. That (*a*) holds means $i : B \hookrightarrow X$ is close to a constant map $k(b) = p$ for some $p \in X$, which is equivalent to $\{i(b) \times k(b) \mid b \in B\}$ being controlled. This set is equal to

$$\{(b, p) \mid b \in B\} = B \times \{p\},$$

so (*a*) holds if and only if (*b*) does. Starting this time with (*b*), we have that $B \times \{p\}$ is controlled for some $p \in X$, and hence $\{p\} \times B$ is controlled since coarse structures are closed under taking inverses. But by Example 2.8, $B \times \{p\} \circ \{p\} \times B = B \times B$, so $B \times B$ is controlled, making (*c*) true. Conversely, given (*c*), or that $B \times B$ is controlled, any $p \in B$ is sufficient for $B \times \{p\} \subset B \times B$, whence $B \times \{p\}$ is controlled, and thus (*b*) holds.

Finally, beginning with (*c*), so that $B \times B$ is controlled, letting $E = B \times B$ means

$$B = \{x \in X \mid \exists p \in \{p\}, (x, p) \in E\} = E[\{p\}] = E_p,$$

so (d) is true. Conversely, if $B = E_p$ for some controlled set E and $p \in X$, then $B \times \{p\} \subset E$ is controlled, giving (b) and hence (d). $\square$

Definition 2.41. Let $B \subset X$ be a set satisfying the conditions of Proposition 2.40. Then we say that B is *bounded* [16].

If there is any confusion between this definition and the usual sense of bounded, we shall refer to the latter as *d-boundedness*, or that the set is *metrically bounded.*

Example 2.42. Recall that the indiscrete coarse structure on a topological space X consists of all proper subsets of $X \times X$, that is, all sets E such that $E[K]$ and $E^{-1}[K]$ are relatively compact whenever K is relatively compact. The bounded sets in X are those that have compact closure [16].

Proof. If B is a bounded subset of X, this is equivalent, by Proposition 2.40, to $B \times B$ being controlled, which means that $(B \times B)[K]$ is relatively compact whenever K is. But

$$\begin{aligned}(B \times B)[K] &= \{x \in X \mid \exists y \in K : (x, y) \in B \times B\} \\ &= \{x \in B \mid \exists y \in K : y \in B\}\end{aligned}$$

which is equal to B if $B \cap K$ is non-empty, and $\varnothing$ if $B \cap K$ is empty. This amounts to saying that B is bounded if and only if B is relatively compact, that is, when B has compact closure. $\square$

Example 2.43. In the discrete coarse structure, every bounded set has only finitely many points.

Proof. Suppose that $B = \{x_\alpha \mid \alpha \in J\}$ has infinitely many points: it follows that the product $B \times B = \{x_\alpha \times x_\beta \mid \alpha, \beta \in J\}$ also has infinitely many points. Fix any $\beta \in J$. We can allow α to be distinct from β for infinitely many $\alpha \in J \setminus \{\beta\}$, which means that $B \times B$ has infinitely many points off the diagonal and is therefore not controlled in the discrete coarse structure. This implies that B is not bounded. The contrapositive to this, that if B is bounded, then B must be finite, therefore holds. $\square$

Proposition 2.44. *Let X be a coarse space* [16].

(a) *If B is a bounded subset of X, and E is controlled, then $E[B]$ is bounded.*
(b) *If X is coarsely connected, then the union of any two bounded subsets of X is bounded.*

Proof. For (a), we note that $B \times B$ is controlled since B is bounded. Choose any $p \in B$. We will show that $E[B] \times \{p\} \subset E \circ (B \times B)$, which completes this part of the proof since $E \circ (B \times B)$ is controlled. Let $x \in E[B]$, so there is a $z \in B$ such that $(x, z) \in E$. Then there is a $z \in X$ such that $(x, z) \in E$ and $(z, p) \in B \times B$, whence $(x, y) \in E \circ (B \times B)$.

For (b), we modify the scheme used in [16]. Let B_1 and B_2 be bounded. We then have inclusion maps $i_1 : B_1 \hookrightarrow X$ and $i_2 : B_2 \hookrightarrow X$ being close, respectively, to some constant maps by Proposition 2.40. Because X is coarsely connected, applying Proposition 2.38 (c) to these

constant maps means that they are close, which, because closeness is an equivalence relation (Proposition 2.37), implies that i_1 and i_2 are close. Then, applying Proposition 2.38 (b) to the inclusion map $i : B_1 \cup B_2 \longrightarrow X$ means that this is close to a constant map, whence $B_1 \cup B_2$ is bounded by Proposition 2.40. □

With the ideas of closeness and boundedness now re-defined, we are in a position to generalize the concepts of *closeness*, *properness*, *coarse*, and *coarse equivalence* for a coarse space.

Definition 2.45. Let X and Y be coarse spaces, and let $f : X \longrightarrow Y$ be a map [16].

(a) The map f is *proper* if the inverse image under f of each bounded subset of Y is a bounded subset of X.
(b) The map f is *bornologous* if for each controlled subset E of $X \times X$ the set $(f \times f)(E)$ is a controlled subset of $Y \times Y$.
(c) The map f is *coarse* if it is proper and bornologous.
(d) The spaces X and Y are *coarsely equivalent* if there exist coarse maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and on X respectively.

Example 2.46. The identity map id_X on a coarse space X is again coarse with the generalization given in Definition 2.45.

Later we require that coarse equivalence is again actually an equivalence relation. Most of the work has been done for us by Theorem 1.21. First we need to show that compositions of coarse maps are again coarse.

Proposition 2.47. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be coarse maps between coarse spaces X, Y, and Z. Then $g \circ f : X \longrightarrow Z$ is a coarse map.*

Proof. Let B be a bounded subset of Z. Then $g^{-1}(B)$ is a bounded subset of Y, and thus

$$f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$$

is a bounded subset of X. It follows that $g \circ f$ is proper. Let E be a controlled subset of $X \times X$. Then $(f \times f)(E)$ is controlled in $Y \times Y$ since f is coarse; since g is also coarse, it follows by Lemma 1.35 that

$$(g \times g)((f \times f)(E)) = ((g \times g) \circ (f \times f))(E) = ((g \circ f) \times (g \circ f))(E)$$

is controlled in $Z \times Z$, whence $g \circ f$ is bornologous.

Thus $g \circ f$ is coarse as required. □

Theorem 2.48. *Let X and Y be coarse spaces, and say that $X \approxeq Y$ if and only if X is coarsely equivalent to Y. Then $\approxeq$ is an equivalence relation.*

Proof. The method by which coarse equivalence was shown to be an equivalence relation in Theorem 1.21 relied upon only the facts that an identity map is coarse, that closeness of maps is equivalence relation, and that compositions of coarse maps are coarse. Since these are shown by Example 2.46 and Propositions 2.37 and 2.47 respectively, the result follows. □

2.7 Are coarse structures generalizations?

We would prefer if the definition of coarse equivalence just given is in fact a generalization of the concept with the same name given in Definition 1.20. This question is of course relative to the coarse structure we choose for the metric space X, but is also dependent on properties concerning the metric space.

The bounded coarse structure on X is the structure in which we would expect to find a natural equivalence between the generalized definitions presented here, and the notions specific to metric spaces given in Section 1.4.

Proposition 2.49. *Let (X, d) be a metric space, and let $\mathcal{E}$ be the bounded coarse structure on X. Then the bounded sets are the sets which are d-bounded* [16].

Proof. To see this, suppose B is a bounded subset in X, which implies that $B \times B$ is controlled. By definition of the course bounded structure this means that $B \times B$ has finite supremum $R = \sup\{d(x, y) \mid (x, y) \in B \times B\}$. Choose any $b_0 \in B$. Then the metric ball $B(b_0, R+1)$ contains B, for if $b \in B$ then $(b_0, b) \in B \times B$, whence $d(b_0, b) \leq R < R+1$, which implies that $b \in B(b_0, R+1)$. On the other hand, if B is a d-bounded subset of X then there is a $b_0 \in X$ and an $R > 0$ such that $B \subset B(b_0, R)$. Without loss of generality, $b_0 \in B$. If $b, b' \in B$ then $d(b, b') \leq d(b, b_0) + d(b_0, b') \leq 2R$ by the triangle inequality, so $B \times B$ has a finite supremum $2R$, and hence $B \times B$ is controlled, implying that B is bounded. □

Thus the term *bounded* signifies the same idea in a metric space and the bounded coarse structure on a metric space.

Proposition 2.50. *Let X and Y be metric spaces. Let $\mathcal{E}$ and $\mathcal{F}$ be the bounded coarse structures on X and Y, respectively, and let $f : X \longrightarrow Y$ be a map. Then the following are equivalent.*

(a) *For all $R > 0$ there is an $S > 0$ such that $d(x, y) < R$ implies $d(f(x), f(y)) < S$.*
(b) *$(f \times f)(E)$ is controlled whenever E is controlled.*

Proof. Given (a), let E be a controlled subset of $X \times X$, which means that

$$\sup\{d(x, y) \mid (x, y) \in E\} \leq R - 1$$

for some $R > 1$. Then there is an $S > 0$ such that $d(f(x), f(y)) < S$ for all $(x, y) \in E$. It follows that $(f \times f)(E) = \{(f(x), f(y)) \mid (x, y) \in E\}$ is controlled in $\mathcal{F}$.

Conversely, let $R > 0$. Let E be a controlled set, so $E = \{(x, y) \mid d(x, y) < R\}$ for some $R > 0$. It follows that $(f \times f)(E)$ is controlled, meaning that there is some $S > 0$ such that $\sup(f \times f)(E) < S$, which is to say, $d(f(x), f(y)) < S$ for all $x, y \in X$, implying (a). □

Proposition 2.51. *Let (X, d_X) and (Y, d_Y) be proper metric spaces, and let $\mathcal{E}$ and $\mathcal{F}$ be the bounded coarse structures on X and Y relative to d_X and d_Y, respectively. Then $X \cong Y$ in the sense of Definition 1.20 if and only if $(X, \mathcal{E}) \cong (Y, \mathcal{F})$ in the sense of Definition 2.45.*

Proof. Since bounded sets in $\mathcal{E}$ and $\mathcal{F}$ are simply d_X- and d_Y-bounded sets in X and Y by Proposition 2.49, the definition of properness in metric spaces and in the bounded coarse structure is equivalent. Proposition 2.50 shows that the definition of bornologous is also equivalent. If $f \circ g \sim id_Y$ in the sense of Definition 1.20, this is equivalent to

$$\{f(g(y)) \times y \mid y \in Y\}$$

being controlled, which in turn is the same as saying that $d(f(g(y)), y) \leq N$ for all $y \in Y$. That is, $f \circ g \sim id_Y$ in the sense of Definition 2.45. The same is true of $g \circ f \sim id_X$.

Since the definitions of coarse and closeness therefore coincide, the result follows. □

2.8 An application of coarse spaces and structures

At the beginning of the chapter we noted that the definition of coarse equivalence from Section 1.4 required that both spaces in question possessed a metric. With the reformulation of these concepts for (in particular) topological spaces, we are now in a position to find coarse equivalences between spaces which are not metrizable.

One question of particular interest is whether $\mathbb{R}^+$—a space which in large-scale terms can be categorized as having unlimited motion in one direction only along one degree of freedom—is coarsely equivalent to the minimal uncountable well-ordered set S_Ω with the order topology. At first glance they seem to share the same coarse properties: in S_Ω (see Appendix A) one can move "to the right" indefinitely, but there is an "end-point" to the left of the each space, namely the element 0.

We examine the possibility that $\mathbb{R}^+$ and S_Ω are coarsely equivalent.

Lemma 2.52. *Let X be a coarse space. Then one-point subsets of X are bounded.*

Proof. If $\{x\} \subset X$, then $\{x\} \times \{x\} \subset \Delta$, which shows that $\{x\} \times \{x\}$ is controlled. It follows that $\{x\}$ is bounded. □

We only need the preceding lemma for the proofs ahead, but the following is a consequence if X is coarsely connected.

Corollary 2.53. *If X is coarsely connected, then finite subsets of X are bounded.*

Proof. Let $A = \{x_1, \ldots, x_n\}$ be a subset of X. For each $1 \leq i, j \leq n$, (x_i, x_j) belongs to some subset E_{ij} of $X \times X$. Then

$$A \times A \subset \bigcup_{i,j} E_{ij},$$

and since $\bigcup E_{ij}$ is a finite union of controlled subsets of $X \times X$, so is $A \times A$. Thus A is bounded. □

Example 2.54. It is necessary that X is coarsely connected for finite subsets to be bounded. Suppose $X = \mathbf{Z}$ with the trivial coarse structure consisting of only subsets of the diagonal, which is not coarsely connected since 0×1 is not contained in any controlled set. Then $A = \{0, 1\}$ is finite but not bounded, since $A \times A = \{0 \times 0, 0 \times 1, 1 \times 0, 1 \times 1\}$ is not controlled.

We intend to explore whether $\mathbb{R}^+$ is coarsely equivalent to S_Ω by exploiting a variation of the previously-shown coarse equivalence between $\mathbb{R}^+$ and $\mathbf{Z}^+$ given in Example 1.53. However, this showed only that these spaces are *metrically* coarsely equivalent—which is the same as their being coarsely equivalent in the bounded coarse structure on both spaces by Proposition 2.51. Because S_Ω is not metrizable, if we want to use the similarity in large-scale structure of $\mathbb{R}^+$ and $\mathbf{Z}^+$ then we need to show that coarse equivalence between them also holds in some coarse structure which does not depend on a metric. The one we shall choose is the indiscrete coarse structure.

Since $\mathbf{Z}^+$ is a subspace of $\mathbb{R}^+$ we begin this task by outlining how a subset of a coarse space may inherit its coarse structure from the parent space.

Definition 2.55. Let X be a coarse space and let Y be a subset of X. We declare that the controlled subsets of $Y \times Y$ are those which are controlled when considered as subsets of $X \times X$, and we say that Y is a *coarse subspace* of X [16].

Lemma 2.56. *Let Y be a coarse subspace of a coarse space X. Then the inclusion map from Y into X is always coarse.*

Proof. Let $g : Y \longrightarrow X$ be the inclusion map, and let E be a controlled subset of $Y \times Y$. It follows that $(g \times g)(E) = E$ is controlled in $X \times X$ by Definition 2.55, so g is bornologous. Let B be bounded in X, so $B \times B$ is controlled. But $g^{-1}(B) \subset B$ in Y. Since

$$g^{-1}(B) \times g^{-1}(B) \subset B \times B,$$

it follows that the former set is controlled in Y, whence $g^{-1}(B)$ is bounded, making g proper. It follows that g is coarse. □

Proposition 2.57. *$\mathbb{R}^+$ is coarsely equivalent to $\mathbf{Z}^+$ in the indiscrete coarse structure.*

Proof. Let $f : \mathbb{R}^+ \longrightarrow \mathbf{Z}^+$ be given by the usual integer part map $f(x) = \lfloor x \rfloor$ for each $x \in X$, and let $g : \mathbf{Z}^+ \longrightarrow \mathbb{R}^+$ be given by inclusion. Since inclusion maps are always coarse in any coarse structure by Proposition 2.56, it remains to show that f is coarse with respect to the indiscrete coarse structure on $\mathbb{R}^+$.

Let E be a controlled subset of $\mathbb{R}^+ \times \mathbb{R}^+$, so E is proper. Since $(f \times f)(E) \subset E$ it follows from the definition of a coarse subspace that $(f \times f)(E)$ is a controlled subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$, so f is bornologous. Let B be a bounded subset of $\mathbf{Z}^+$, so B is relatively compact by Example 2.42 and is thus contained in a finite interval $[0, n]$ for some integer n. By definition of f, $f^{-1}(B) \subset [0, n+1)$, which is relatively compact in $\mathbb{R}^+$ and therefore bounded. It follows that f is proper, and thus coarse.

We note that $f(g(n)) = n$ and $g(f(x)) = \lfloor x \rfloor$ for all $n \in \mathbf{Z}^+$ and all $x \in \mathbb{R}^+$. Then

$$\{(f \circ g)(n) \times id_{\mathbf{Z}^+}(n) \mid n \in \mathbf{Z}^+\} = \{n \times n \mid n \in \mathbf{Z}^+\} \subset \Delta_{\mathbf{Z}^+}$$

the last of which, being the diagonal of $\mathbf{Z}^+$, is proper by Example 2.19, and

$$\{(g \circ f)(x) \times id_{\mathbb{R}^+(x)} \mid x \in \mathbb{R}^+\} = \{\lfloor x \rfloor \times x \mid x \in \mathbb{R}^+\} \subset \{[x, x+1) \times x \mid x \in \mathbb{R}^+\}$$

which is clearly proper. We have thus shown that $f \circ g \sim id_{\mathbf{Z}^+}$ and $g \circ f \sim id_{\mathbb{R}^+}$. It follows that $\mathbb{R}^+ \cong \mathbf{Z}^+$ in the indiscrete coarse structure. □

Theorem 2.58. *$\mathbf{Z}^+$ is not coarsely equivalent to S_Ω in either the indiscrete or discrete coarse structures on S_Ω.*

Proof. Suppose $\mathbf{Z}^+ \cong S_\Omega$, so let $g : S_\Omega \longrightarrow \mathbf{Z}^+$ be a corresponding coarse equivalence map. Since $\mathbf{Z}^+$ is countable, but S_Ω is uncountable, there is an $n \in \mathbf{Z}^+$ such that $S = g^{-1}(\{n\})$ is uncountable. Since $\{n\}$ is bounded and g is a coarse map it follows that S is also bounded, which by Example 2.42 means that S is relatively compact. Cover $\overline{S}$ (which we have just shown is compact) with open sets of the form $[0, \alpha + 1)$ for every $\alpha \in \overline{S}$, so there is a finite subcover $[0, \alpha_1 + 1), \ldots, [0, \alpha_k + 1)$ of $\overline{S}$. Among the finite number of α_i there is a maximum α_m, whence $\overline{S} \subset [0, \alpha_m + 1)$. But $[0, \alpha_m + 1)$, being a section $\{\alpha \in S_\Omega \mid \alpha < \alpha_m + 1\}$ of S_Ω, is countable (see Appendix A), while $\overline{S}$ is uncountable (since S is uncountable and $S \subset \overline{S}$). Thus there is a $\beta \in \overline{S}$ which is not in $[0, \alpha_m + 1)$, contradicting that $\overline{S} \subset [0, \alpha_m + 1)$.

On the other hand, in the discrete coarse structure on S_Ω, $S \times \{\rho\}$ is controlled for some $\rho \in S_\Omega$, whence $S \times \{\rho\}$ contains finitely many points off the diagonal. This is an immediate contradiction since $S \times \{\rho\}$ is uncountable.

In either case we have a contradiction. It follows that our assumption that $\mathbf{Z}^+ \cong S_\Omega$ is false, completing the proof. □

Corollary 2.59. *$\mathbb{R}^+$ is not coarsely equivalent to S_Ω in the indiscrete coarse structure on each space.*

Proof. By considering the indiscrete coarse structure on each space, $\mathbb{R}^+$ is coarsely equivalent to $\mathbf{Z}^+$ by Proposition 2.57, but $\mathbf{Z}^+$ is *not* coarsely equivalent to S_Ω. Since coarse equivalence is an equivalence relation by Theorem 2.48 it follows that $\mathbb{R}^+$ is not coarsely equivalent to S_Ω in the indiscrete coarse structure on each space. □

The explanation, in terms of intuitive ideas of coarse geometry, of why $\mathbb{R}^+$ and S_Ω are not coarsely equivalent is that while motion in both spaces is unlimited in one direction with one degree of freedom, motion in $\mathbb{R}^+$ (or $\mathbf{Z}^+$) to the *left* (if we view larger elements of S_Ω as being on the right of smaller elements) is never prevented except by the element 0, while in S_Ω every limit point

$$\omega, 2\omega, \ldots, \omega^2, \omega^2 + \omega, \ldots, \omega^3, \ldots, \omega^\omega, \ldots$$

Figure 2.8: The positive integers $\mathbf{Z}^+$ alongside the set of countable ordinals S_Ω.

to name a few of the uncountable number of limit points of S_Ω, prevents further motion to the left, since there are no predecessors to these elements (see Figure 2.8).

By this understanding, it should also be the case that there is a coarse structure in which ω is not coarsely equivalent to 2ω, for the same intuitive reasons as given above: $\omega \in 2\omega$ does not have a predecessor. In this case, however, we cannot rely upon uncountability as we did for S_Ω, and so the proof, which is left to the reader, would require more work. ❦

CHAPTER 3 ∼

Basic Theory of C^*-Algebras

THIS chapter highlights an important link between locally compact Hausdorff (topological) spaces and a sub-class of Banach spaces known as *C^*-algebras*. It turns out that the *categories* (see Appendix C) of locally compact Hausdorff spaces and continuous functions, and of commutative C^*-algebras and $*$-homomorphisms, are very closely related. At the heart of this is a particular set of continuous functions on a locally compact Hausdorff space X; namely, those which vanish at infinity, denoted by $C_0(X)$.

To be precise, what turns out to be true is that the opposite of the category of locally compact Hausdorff spaces and the category of commutative C^*-algebras are *dual*, meaning that each locally compact Hausdorff space can be completely described by considering the C^*-algebra of the continuous, complex-valued, infinity-vanishing functions on it.

It is not our intention to provide a full account of the theory of C^*-algebras. We develop some terminology and provide the examples that we will require, and make explicit the link between the categories outlined above.

3.1 ELEMENTARY PROPERTIES AND EXAMPLES

The idea of an *algebra* is intuitive. It consists of a *vector space* in which one can also perform a multiplicative operation which is compatible with the addition given in the vector space. We begin with definitions [15, 21] of this and an *involution*, an operation which, among others, has the property that it is its own inverse.

DEFINITION 3.1. An *algebra* over K is a vector space A over a field K together with a *bilinear map* $A \times A \longrightarrow A$ called *multiplication*, given by $(a, b) \longmapsto a \cdot b$. In this text we will say that multiplication is associative, so $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Thus

$$(a+b)\cdot c = a\cdot c + b\cdot c, \quad (\lambda a)\cdot b = \lambda(a\cdot b)$$
$$a\cdot(b+c) = a\cdot b + a\cdot c, \quad a\cdot(\lambda b) = \lambda(a\cdot b)$$

for all $a, b, c \in A$ and $\lambda \in K$.

A *subalgebra* of A is a subspace $B \subset A$ which is closed with respect to multiplication.

Definition 3.2. An *involution* on an algebra A is a conjugate-linear map $A \longrightarrow A$ given by $a \longmapsto a^*$ such that $a^{**} = a$ and $(a \cdot b)^* = b^* \cdot a^*$ for all $a, b \in A$.

If A is a *commutative* algebra then $(a \cdot b)^* = a^* \cdot b^*$ for all $a, b \in A$.

Definition 3.3. A *C^*-algebra* is an algebra over $\mathbb{C}$ with a *norm* $\| \ \| : A \longrightarrow \mathbb{R}$ and an involution $a \longmapsto a^*$ such that A is complete with respect to the norm, and such that

$$\|a \cdot b\| \leq \|a\|\|b\| \quad \text{and} \quad \|a^* \cdot a\| = \|a\|^2$$

for all $a, b \in A$. Define a *sub-C^*-algebra* of A to be a subalgebra $B \subset A$ which is also a C^*-algebra with respect to the operations given on A, that is, B is also complete. We say that a C^*-algebra A is *unital* if there is a non-zero *unit* $1_A \in A$ such that $a \cdot 1_A = a = 1_A \cdot a$ for all $a \in A$.

A unital C^*-algebra contains the subset $\mathbb{C} \cdot 1$, so we understand λ as being identified with $\lambda \cdot 1$ in the context of operations on A.

Example 3.4. The set of complex numbers $\mathbb{C}$ is obviously a unital C^*-algebra. The involution is *complex conjugation* $z \longmapsto \bar{z}$, the norm is the *modulus function* $z \longmapsto |z|$, and the unit is simply $1_{\mathbb{C}} = 1 + 0i$.

The axioms for C^*-algebra imply that $0_A{}^* = 0_A$ and if A is unital, that $1_A{}^* = 1_A$. They also imply that the involution is *isometric* .

Proposition 3.5. *Let A be a C^*-algebra. Then $0_A{}^* = 0_A$. If A is unital then $1_A{}^* = 1_A$ and $\|1_A\| = 1$.*

Proof. The element $0_A \in A$ is the additive identity from the vector space properties of A. Thus $0_A = a \cdot 0_A$ for all $a \in A$, including the case when we let $a = 0_A{}^*$. It follows that

$$0_A{}^* = (a \cdot 0_A)^* = (0_A{}^* \cdot 0_A)^* = 0_A{}^* \cdot 0_A{}^{**} = 0_A{}^* \cdot 0_A = 0_A.$$

If A is unital then for any $a \in A$ we have $1_A{}^* \cdot a = (a^* \cdot 1_A)^* = a^{**} = a$. Thus, letting $a = 1_A$, $1_A{}^* \cdot 1_A = 1_A$, which shows that $1_A{}^* = 1_A$. With this proved, we can then write

$$\|1_A\| = \|1_A{}^* \cdot 1_A\| = \|1_A\|^2.$$

Since $\|a\| = 0$ if and only if $a = 0_A$ by the properties of a normed vector space [10], this implies $\|1_A\| = 1$. □

Proposition 3.6. *The involution $a \longmapsto a^*$ on a C^*-algebra A is isometric, that is to say, $\|a\| = \|a^*\|$ for every $a \in A$.*

Proof. If $a = 0_A$ then

$$\|a\| = \|0_A\| = 0 = \|0_A{}^*\| = \|a^*\|.$$

Otherwise, we see that $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, whence $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|$ since $\|a\| \neq 0$. □

We usually omit the operation of multiplication and write ab for $a \cdot b$. We also write 0 for 0_A and 1 for 1_A when it is clear to which C^*-algebra we refer. It is also the case the multiplication in a C^*-algebra is *jointly continuous*, shown by the inequality

$$\|ab - a'b'\| \leq \|a\|\|b - b'\| + \|a - a'\|\|b'\|.$$

All of our attention will be devoted to C^*-algebras of continuous functions. We recall a definition and give some examples.

Definition 3.7. If X is set and $f : X \longrightarrow \mathbb{C}$ is a map, then the *supremum norm of f over X* is defined as

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

If f is bounded then the supremum norm over X is finite; otherwise, $\|f\|_\infty = \infty$. If the functions under consideration are all bounded, $\|f\|_\infty : \mathcal{B}(X, \mathbb{C}) \longrightarrow \mathbb{R}$ actually is a norm, for

$$\|fg\|_\infty = \sup_{x \in X} |f(x)g(x)| \leq \sup_{x \in X} |f(x)||g(x)| \leq \sup_{x \in X} |f(x)| \cdot \sup_{x \in X} |g(x)| = \|f\|_\infty \|g\|_\infty.$$

If X is a locally compact Hausdorff space, denote by $C(X)$ the set of all complex-valued continuous functions on X. We define addition, multiplication, and scalar multiplication on functions in $C(X)$ pointwise. Explicitly, we write

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(\lambda f)(x) = \lambda f(x)$$

for all $f \in C(X)$, $\lambda \in \mathbb{C}$ and $x \in X$. Then $C(X)$ is a commutative algebra. Denote by $C_b(X)$ the set of all bounded continuous functions on X, and by $C_0(X)$ the set of all bounded continuous functions which vanish at infinity, that is, for all $\epsilon > 0$ there is a compact $K \subset X$ such that $|f(x)| < \epsilon$ on $X \setminus K$. We observe that

$$C_0(X) \subset C_b(X) \subset C(X).$$

It follows that the supremum norm over X for every $f \in C_b(X)$, and therefore for every $f \in C_0(X)$, is finite because f is bounded and continuous in each case, making the supremum norm an actual norm.

We therefore restrict our attention to $C_b(X)$. If a map $C_b(X) \longrightarrow C_b(X)$ is given by $f \longmapsto f^*$, where $f^*(x) = \overline{f(x)}$ for each $x \in X$, then this operation is obviously an involution. Because $\mathbb{C}$ is complete and $\| \ \|_\infty$ is a uniform metric [14], it follows that $C_b(X)$ is complete with respect to $\| \ \|_\infty$. Finally,

$$\|f^* f\|_\infty = \sup_{x \in X} |(f^* f)(x)| = \sup_{x \in X} |\overline{f(x)} f(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|_\infty^2.$$

We have thus shown the following.

PROPOSITION 3.8. *The set $C_b(X)$ is a commutative C^*-algebra with respect to the involution $f \mapsto f^*$ and the supremum norm $\| \; \|_\infty$.*

COROLLARY 3.9. *The set $C_0(X)$ is a sub-C^*-algebra of $C_b(X)$.*

Proof. First, $C_0(X)$ is a sub-algebra of $C_b(X)$ since it is clearly closed under addition and multiplication. If $f \in C_0(X)$ then f vanishes at infinity; complex conjugation does not change this, so $f^* \in C_0(X)$.

The set $C_0(X)$ can be characterized by saying that a map $f \in C_0(X)$ if and only if for all $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ when $x \in X \setminus K$. Let f be a limit point of $C_0(X)$. Then there is a sequence (f_n) of functions converging to it in $\| \; \|_\infty$. Let $\epsilon > 0$, and let N be large enough so that $\|f - f_N\|_\infty < \epsilon/2$. There is a compact K such that $|f_N(x)| < \epsilon/2$ for all $x \in X \setminus K$. Thus

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \le |f(x) - f_N(x)| + |f_N(x)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

for all $x \in X \setminus K$, so $f \in C_0(X)$, which shows that $C_0(X)$ is a closed subset of $C_b(X)$. Since $\mathbb{C}$ is complete, it follows that $C_0(X)$ is complete with respect to $\| \; \|_\infty$. Thus $C_0(X)$ is a sub-C^*-algebra of $C_b(X)$. □

3.2 C^*-ALGEBRAS AND *-HOMOMORPHISMS

We take some liberties with terminology in this section: a number of the terms which follow are also defined for Banach algebras [15], but since we do not need to consider such a general case we limit their definitions to C^*-algebras.

DEFINITION 3.10. A **-homomorphism* is a linear and multiplicative map $\varphi : A \longrightarrow B$ of C^*-algebras that respects involution. Explicitly,

$$\varphi(\lambda a + b) = \lambda\varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(a^*) = \varphi(a)^*$$

for all $a, b \in A$, and $\lambda \in \mathbb{C}$.

That $\varphi(0_A) = 0_B$ for any *-homomorphism $\varphi : A \longrightarrow B$ is true from the vector space properties of A and B. We say that φ is *unital* if A and B are unital and $\varphi(1_A) = 1_B$. Clearly if a is *invertible*—that is, a^{-1} exists, or $a \in \mathrm{Inv}(A)$—then

$$\varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(1_A) = 1_B$$

whence $\varphi(a^{-1}) = \varphi(a)^{-1}$. The following is a central property of *-homomorphisms, the proof of which is given in [17]. Because it involves the definition of terms which are not central to our purpose in this chapter, we shall omit it.

PROPOSITION 3.11. *Let A and B be C^*-algebras, and let $\varphi : A \longrightarrow B$ be a *-homomorphism. Then φ is continuous* [17].

A *character* on a commutative C^*-algebra A is a continuous non-zero *-homomorphism $\alpha : A \longrightarrow \mathbb{C}$. This leads to an important definition.

Definition 3.12. The *dual* of a commutative C^*-algebra A is the set

$$\Omega(A) = \{\alpha : A \longrightarrow \mathbb{C}\}$$

of characters on A.

Note that $\Omega(A)$ may be empty. For example, if $A = \{0_A\}$, then there are no characters $A \longrightarrow \mathbb{C}$ at all.

We recall that the *topology of pointwise convergence* on a function space Y^X has as a sub-basis sets of the form

$$S(x, U) = \{f \in Y^X \mid f(x) \in U\},$$

where $x \in X$ and U is an open subset of Y. A general basis element for this topology is therefore a finite intersection of sub-basis elements $S(x, U)$ [14].

The *dual space* A^* of a normed vector space A consists of all continuous functions $C(A, \mathbb{C})$. Since we are interested specifically in $\Omega(A)$, being a subset of the dual space A^*, then a sub-basis element of $\Omega(A)$ is

$$S(a, \Lambda) = \{\gamma \in \Omega(A) \mid \gamma(a) \in \Lambda\}$$

for an open subset $\Lambda \subset \mathbb{C}$. With this as a sub-basis, the resulting topology in the context of C^*-algebras is called the *weak topology* .

Before we prove the next proposition, we should state the result of the *Banach–Alaoglu theorem*, which is that the closed unit ball S of the dual space A^* of a normed vector space A is compact in the weak topology. Since a C^*-algebra A is also a normed vector space, the theorem applies to its dual space A^*. We also require the following, that the norm of any character α is equal to 1, which we state without proof [15].

Lemma 3.13. *Let A be a unital C^*-algebra. If $\alpha \in \Omega(A)$ then $\|\alpha\| = 1$.*

Proposition 3.14. *The dual $\Omega(A)$ of a C^*-algebra A is a locally compact Hausdorff space in the weak topology.*

Proof. We first show that $\Omega(A)$ is Hausdorff. Let α, β be distinct elements of $\Omega(A)$. Then there is an $a \in A$ such that $\alpha(a) \neq \beta(a)$. Since $\mathbb{C}$ is a metric space choose $\epsilon > 0$ and $\delta > 0$ such that the sets

$$\Lambda = \{z \in \mathbb{C} \mid d(z, \alpha(a)) < \epsilon\} \quad \text{and} \quad \Theta = \{z \in \mathbb{C} \mid d(z, \beta(a)) < \delta\}$$

are disjoint. Then $\alpha \in S(a, \Lambda)$ and $\beta \in S(a, \Theta)$. Suppose that there is a $\gamma \in S(a, \Lambda) \cap S(a, \Theta)$. But then $\gamma(a) \in \Lambda \cap \Theta$, which is a contradiction since these open sets are disjoint, so $S(a, \Lambda)$ and $S(a, \Theta)$ are disjoint open sets containing α and β, respectively.

That $\Omega(A)$ is locally compact is a consequence of the fact that $\Omega(A) \cup \{0\}$ is a closed subset contained, by Lemma 3.13, in the closed unit ball S of A^*. It follows that $\Omega(A) \cup \{0\}$ is also compact, making $\Omega(A)$ locally compact. $\square$

3.3 The Gelfand representation theorem

Having developed most of the terminology we require, we begin to explore the relationship between locally compact Hausdorff spaces and C^*-algebras.

Definition 3.15. Let A be a C^*-algebra. For each $a \in A$ we define a map $\widehat{a} : \Omega(A) \longrightarrow \mathbb{C}$ by

$$\widehat{a}(\alpha) = \alpha(a)$$

which we call the [11].

In other words, the Gelfand transform takes an element a in a C^*-algebras A and associates to it a function $\widehat{a}$ which evaluates characters in $\Omega(A)$ at a.

In the previous section we considered the C^*-algebras $C_b(X)$ and $C_0(X)$, where X was a locally compact Hausdorff space. Since $\Omega(A)$ is also a locally compact Hausdorff space, we can form $C_0(\Omega(A))$, the C^*-algebra of all characters (non-zero continuous $*$-homomorphisms $A \longrightarrow \mathbb{C}$) which vanish at infinity. We might ask if we lose any C^*-algebras by considering only those locally compact Hausdorff spaces which are duals of C^*-algebras. The following remarkable result by Gelfand—the proof of which is beyond the scope of this work—shows that in fact no C^*-algebras are lost: every C^*-algebra is of the form $C_0(\Omega(A))$ for some C^*-algebra A.

Theorem 3.16 (Gelfand). *If A is a non-zero commutative C^*-algebra, then the Gelfand representation*

$$\Phi : A \longrightarrow C_0(\Omega(A)), \quad a \longmapsto \widehat{a}$$

is an isometric $*$-isomorphism [15].

This theorem is known as the *Gelfand representation theorem*. For compact spaces, there is an analogous result, one which shows that we can perform the complimentary process.

Theorem 3.17. *Let X be a compact Hausdorff space, and for each $x \in X$ let φ_x be the character on $C(X)$ given by evaluation at x, that is, $\varphi_x(f) = f(x)$. Then the map*

$$\Psi : X \longrightarrow \Omega(C(X)), \quad x \longmapsto \varphi_x$$

is a homeomorphism [15].

In other words, for every C^*-algebra A there is a locally compact Hausdorff space $X = \Omega(A)$ such that A and $C_0(X)$ are equal up to an isometric $*$-isomorphism. Restricting X to be a compact Hausdorff space means that $C_0(X) = C(X)$ vacuously. So if X is compact, it follows not only that there is precisely one C^*-algebra A corresponding to it, but that X is homeomorphic to $\Omega(C(X))$. This restriction on X is, however, not as great as it might seem, as we shall show later. For the rest of this section, however, we limit our attention to compact Hausdorff spaces.

Theorem 3.18. *Let A and B be unital C^*-algebras, and X and Y be compact Hausdorff spaces. Then*

(a) *Each *-homomorphism $\varphi : A \longrightarrow B$ induces a continuous function*

$$\widehat{\varphi} : \Omega(B) \longrightarrow \Omega(A), \quad \widehat{\varphi}(\alpha) = \alpha \circ \varphi$$

with respect to the weak topologies on $\Omega(A)$ and $\Omega(B)$.

(b) *Each continuous function $f : Y \longrightarrow X$ induces a *-homomorphism*

$$f_* : C(X) \longrightarrow C(Y), \quad f_*(g) = g \circ f.$$

Proof. For (a), since φ is continuous by Proposition 3.11 and $\alpha : B \longrightarrow \mathbb{C}$ is a character, then $\alpha \circ \varphi : A \longrightarrow \mathbb{C}$ is well-defined, is continuous, and is non-zero; hence $\alpha \circ \varphi$ is a character in $\Omega(B)$. To show that $\widehat{\varphi}$ is continuous, take a sub-basis element $S(a, \Lambda)$ in the weak topology on $\Omega(A)$. Then

$$\begin{aligned}\widehat{\varphi}^{-1}(S(a,\Lambda)) &= \{\beta \in \Omega(B) \mid \widehat{\varphi}(\beta) \in S(a,\Lambda)\} \\ &= \{\beta \in \Omega(B) \mid \beta \circ \varphi \in \{\gamma \in \Omega(A) \mid \gamma(a) \in \Lambda\}\} \\ &= \{\beta \in \Omega(B) \mid \beta(\varphi(a)) \in \Lambda\} \\ &= S(\varphi(a), \Lambda)\end{aligned}$$

which is a sub-basic set in the weak topology on $\Omega(B)$ and is thus open. It follows that $\widehat{\varphi}$ is continuous.

For (b), if g is a continuous function $X \longrightarrow \mathbb{C}$, it follows that $g \circ f : Y \longrightarrow \mathbb{C}$ is well-defined and continuous, and therefore is an element in $C(Y)$. We then need to show that f_* is a *-homomorphism. Let $g, h \in C(X)$, and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}f_*(\lambda g + h)(y) &= ((\lambda g + h) \circ f)(y) = \lambda g(f(y)) + h(f(y)) \\ &= (\lambda g \circ f)(y) + (h \circ f)(y) = \lambda f_*(g)(y) + f_*(h)(y)\end{aligned}$$

for all $y \in Y$, showing that $f_*(\lambda g + h) = \lambda f_*(g) + f_*(h)$. Also,

$$\begin{aligned}f_*(g \cdot h)(y) &= ((g \cdot h) \circ f)(y) = g(f(y)) \cdot h(f(y)) \\ &= (g \circ f)(y) \cdot (h \circ f)(y) = f_*(g)(y) \cdot f_*(h)(y)\end{aligned}$$

for all $y \in Y$, which shows that $f_*(g \cdot h) = f_*(g) \cdot f_*(h)$. Finally, for involution,

$$\begin{aligned}(f_*)(g^*)(y) &= (g^* \circ f)(y) = g^*(f(y)) \\ &= \overline{g(f(y))} = \overline{f_*(g)(y)} \\ &= f_*(g)(y)^*\end{aligned}$$

proving that $f_*(g^*) = f_*(g)^*$. It follows that f_* is a *-homomorphism. □

3.4 Equivalence of categories

That the maps

$$\Phi : A \longrightarrow C_0(\Omega(A)), \quad a \longmapsto \widehat{a}$$
$$\Psi : X \longrightarrow \Omega(C(X)), \quad x \longmapsto \varphi_x$$

are, respectively, an isometric *-isomorphism and a homeomorphism, means that we can move between the categories of unital C^*-algebras and compact Hausdorff spaces. This section demonstrates that this is indeed possible.

Proposition 3.19. *X is a compact Hausdorff space if and only if $C_0(X)$ is unital.*

Proof. If X is compact then $C_0(X) = C(X)$. Let $1_{C(X)} : X \longrightarrow \mathbb{C}$ be given by $1_{C(X)}(x) = 1$ for all $x \in X$. Then

$$(f \cdot 1_{C(X)})(x) = f(x) \cdot 1_{C(X)}(x) = f(x) = 1_{C(X)}(x) \cdot f(x) = (1_{C(X)} \cdot f)(x)$$

for all $x \in X$. And $1_{C(X)}$ is clearly continuous, whence $C(X)$ is unital.

Suppose X is not compact, but suppose that there is a unit $1_{C_0(X)} \in C_0(X)$. Then clearly $1_{C_0(X)}(x) = 1$ for every $x \in X$, for otherwise there is an $x \in X$ such that $1_{C_0(X)}(x) \neq 1$, but $(f \cdot 1_{C_0(X)})(x) = f(x) \cdot 1_{C_0(X)}(x) \neq f(x)$. But $1_{C_0(X)}$ does not vanish at infinity. Thus if X is not compact, then $C_0(X)$ does not have a unit. The contrapositive, that if $C_0(X)$ has a unit, then X is compact, is therefore shown. □

We therefore restrict our attention briefly to commutative, unital C^*-algebras and compact Hausdorff spaces, and thus $C_0(X) = C(X)$.

Theorem 3.20. *The category of commutative, unital C^*-algebras (and unital *-homomorphisms)* **Alg** *is equivalent to the opposite of the category of compact Hausdorff spaces and continuous maps* $\mathbf{Comp}^{\mathrm{OP}}$.

Proof. To write out explicitly the categories involved, **Alg** consists of objects A, B being C^*-algebras, and morphisms $\mathrm{Hom}_{\mathbf{Alg}}(A, B)$ of unital *-homomorphisms $\varphi : A \longrightarrow B$. The category $\mathbf{Comp}^{\mathrm{OP}}$ consists of objects X, Y being compact Hausdorff spaces, and morphisms $\mathrm{Hom}_{\mathbf{Comp}^{\mathrm{OP}}}(X, Y)$ of continuous functions $f : Y \longrightarrow X$.

Let $\mathcal{F}$ be such that for every A in **Alg** we have $\mathcal{F}(A) = \Omega(A)$ and for every *-homomorphism $\varphi \in \mathrm{Hom}_{\mathbf{Alg}}(A, B)$ we have $\mathcal{F}(\varphi) = \widehat{\varphi}$, where $\widehat{\varphi} : \Omega(B) \longrightarrow \Omega(A)$ is defined by $\widehat{\varphi}(\alpha) = \alpha \circ \varphi$ for every character $\alpha \in \Omega(B)$, that is, every $\alpha : B \longrightarrow \mathbb{C}$. Then $\mathcal{F}$ is a contravariant functor from **Alg** to $\mathbf{Comp}^{\mathrm{OP}}$.

Now let $\mathcal{G}$ be such that for every object X in $\mathbf{Comp}^{\mathrm{OP}}$ we have $\mathcal{G}(X) = C(X)$ and for every morphism $f \in \mathrm{Hom}_{\mathbf{Comp}^{\mathrm{OP}}}(X, Y)$ we have $\mathcal{G}(f) = f_*$, where $f_* : C(X) \longrightarrow C(Y)$ is defined by $f_*(g) = g \circ f$ for every $g \in C(X)$. Then $\mathcal{G}$ is a contravariant functor from $\mathbf{Comp}^{\mathrm{OP}}$ to **Alg**. We defer the proofs that these are in fact contravariant functors to the final section of this chapter, since they are technical in nature.

We consider the diagrams

$$\begin{tikzcd} 1_{\mathbf{Alg}}(A) \arrow[r, "\Phi_A"] \arrow[d, "1_{\mathbf{Alg}}(\alpha)"'] & \mathcal{G}\mathcal{F}(A) \arrow[d, "\mathcal{G}\mathcal{F}(\alpha)"] \\ 1_{\mathbf{Alg}}(B) \arrow[r, "\Phi_B"'] & \mathcal{G}\mathcal{F}(B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1_{\mathbf{Comp}^{\mathrm{op}}}(Y) \arrow[r, "\Psi_Y"] \arrow[d, "1_{\mathbf{Comp}^{\mathrm{op}}}(f)"'] & \mathcal{F}\mathcal{G}(Y) \arrow[d, "\mathcal{F}\mathcal{G}(f)"] \\ 1_{\mathbf{Comp}^{\mathrm{op}}}(X) \arrow[r, "\Psi_X"'] & \mathcal{F}\mathcal{G}(X) \end{tikzcd}$$

which, after evaluation, become

$$\begin{tikzcd} A \arrow[r, "\Phi_A"] \arrow[d, "\alpha"'] & C(\Omega(A)) \arrow[d, "\widehat{\alpha}_*"] \\ B \arrow[r, "\Phi_B"'] & C(\Omega(B)) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, "\Psi_Y"] \arrow[d, "f"'] & \Omega(C(Y)) \arrow[d, "\widehat{f}_*"] \\ X \arrow[r, "\Psi_X"'] & \Omega(C(X)) \end{tikzcd}$$

However, letting Φ_A be the isometric $*$-isomorphism from Theorem 3.16 for every A in **Alg** and letting Ψ_X be the homeomorphism from Theorem 3.17 for every X in $\mathbf{Comp}^{\mathrm{op}}$, it follows that all of the diagrams commute. Thus **Alg** is equivalent to $\mathbf{Comp}^{\mathrm{op}}$, which completes the proof. $\square$

If A is a C^*-algebra which does not possess a unit then we can uniquely define $\widetilde{A} = A \oplus \mathbb{C}$ as a vector space [15]. By declaring that multiplication is given by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu),$$

and involution by $(a, \lambda)^* = (a^*, \overline{\lambda})$ it follows that $\widetilde{A}$ is a C^*-algebra with unit $(0_A, 1)$. We call $\widetilde{A}$ the *unitalization* of A. There is an injective $*$-homomorphism

$$\varphi : A \longrightarrow \widetilde{A}, \quad a \longmapsto (a, 0)$$

which implies that A can be identified with an ideal of $\widetilde{A}$. Thus

$$\widetilde{A}/A = (A \times \mathbb{C})/(A \times 0) \cong \mathbb{C}.$$

The addition of a unit to A corresponds to the *compactification* $\overline{X}$ of a locally compact space X by the addition of an element at infinity [11]. Thus if $A = C_0(X)$ then $\widetilde{A} \cong C(\overline{X})$. By defining the category of locally compact Hausdorff spaces so that morphisms from Y to X are those which preserve the basepoint from $\overline{Y}$ to $\overline{X}$ we have then shown the following [11].

Theorem 3.21. *The category of commutative C^*-algebras and $*$-homomorphisms is equivalent to the opposite of the category of locally compact Hausdorff topological spaces.*

3.5 Proofs of functorial properties

We finish by proving that $\mathcal{F} : \mathbf{Alg} \longrightarrow \mathbf{Comp}^{\mathrm{OP}}$ and $\mathcal{G} : \mathbf{Comp}^{\mathrm{OP}} \longrightarrow \mathbf{Alg}$ have functorial properties.

Proposition 3.22. *$\mathcal{F} : \mathbf{Alg} \longrightarrow \mathbf{Comp}^{\mathrm{OP}}$ and $\mathcal{G} : \mathbf{Comp}^{\mathrm{OP}} \longrightarrow \mathbf{Alg}$ are contravariant functors.*

Proof. It is clear that $\mathcal{F}(A) = \Omega(A)$ is an object in $\mathbf{Comp}^{\mathrm{OP}}$ for each A in $\mathbf{Alg}$ since $\Omega(A)$ is a compact Hausdorff space, and that $\mathcal{F}(\varphi) = \widehat{\varphi}$ is in $\mathrm{Hom}_{\mathbf{Comp}^{\mathrm{OP}}}(\Omega(A), \Omega(B))$ for each $\varphi \in \mathrm{Hom}_{\mathbf{Alg}}(A, B)$ by Theorem 3.18. We note that

$$\mathcal{F}(\psi \circ \varphi) = \widehat{\psi \circ \varphi}$$

where $\widehat{\psi \circ \varphi}(\alpha) = \alpha \circ (\psi \circ \varphi)$ for all $\alpha \in \mathrm{Hom}_{\mathbf{Alg}}(A, B)$ and

$$\mathcal{F}(\varphi) \circ \mathcal{F}(\psi) = \widehat{\varphi} \circ \widehat{\psi}$$

where $(\widehat{\varphi} \circ \widehat{\psi})(\alpha) = \widehat{\varphi}(\widehat{\psi}(\alpha)) = \widehat{\varphi}(\alpha \circ \psi) = (\alpha \circ \psi) \circ \varphi$ are equal by associativity of composition of functions. Finally we observe that applying $\mathcal{F}$ to map 1_A given by $1_A(a) = a$ results in $\mathcal{F}(1_A) = \widehat{1}_A$ where

$$\widehat{1}_A(\alpha) = \alpha \circ 1_A = \alpha = 1_{\Omega(A)}(\alpha) = 1_{\mathcal{F}(A)}(\alpha),$$

whence $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

It is similarly clear that $\mathcal{G}(X) = C(X)$ is an object in $\mathbf{Alg}$ for every X in $\mathbf{Comp}^{\mathrm{OP}}$, and that $\mathcal{G}(f) = f_*$ is a morphism in $\mathrm{Hom}_{\mathbf{Alg}}(C(X), C(Y))$ for each $f \in \mathrm{Hom}_{\mathbf{Comp}^{\mathrm{OP}}}(X, Y)$ again by Theorem 3.18. We note that

$$\mathcal{G}(g \circ f) = (g \circ f)_*$$

where $(g \circ f)_*(h) = h \circ (g \circ f)$ for every $h \in \mathrm{Hom}_{\mathbf{Alg}}(C(X), C(Y))$, and

$$\mathcal{G}(f) \circ \mathcal{G}(g) = f_* \circ g_*$$

where $(f_* \circ g_*)(h) = f_*(g_*(h)) = f_*(h \circ g) = (h \circ g) \circ f$ are equal again by associativity of composition of functions. Last, we note that applying $\mathcal{G}$ to a map 1_Y given by $1_Y(y) = y$ we have $\mathcal{G}(1_Y) = (1_Y)_*$ where

$$(1_Y)_*(h) = h \circ 1_Y = h = 1_{C(X)}(h) = 1_{\mathcal{G}(X)}(h),$$

whence $\mathcal{G}(1_X) = 1_{\mathcal{G}(X)}$.

It follows that $\mathcal{F}$ and $\mathcal{G}$ are contravariant functors as required. ❦ □

CHAPTER 4

Topological Coarse Structure and Higson Compactification

A COARSE structure which is compatible with the topological space on which it is defined is naturally concerned with the large-scale behavior of continuous functions on the space. In particular, we shall be asking whether a continuous function is continuous *at infinity*. To extend a function in this way, we are required to examine the "edge" of the space in some sense.

Spaces for which this idea can be described precisely include locally compact Hausdorff spaces, where the concept of *compactification* can be defined. The *boundary* of the compactification can be thought of as adding "infinity" to the space, and in some compactifications there is more than one such infinity.

We will first define what a compactification is, and then proceed to create a *topological coarse structure*, in which the concept of "going to infinity" in the space—something which is a large-scale action—can be described in such a way as to agree with the topology on the space. There is a connection between the compactification of a space and the C^*-algebra of those continuous functions which may be extended to the boundary, which we shall also discuss in relation to various coarse structures on those spaces.

Last, we provide a partial inverse to the task of finding coarse structures of compactifications by describing *Higson compactification*, a construction which, on taking a coarse structure, gives a compactification in which the coarse structure is *continuously controlled*. We end by describing a Higson compactification of a familiar space.

4.1 COMPACTIFICATION OF SPACES

In this chapter X is always taken to be a locally compact Hausdorff space.

DEFINITION 4.1. A *compactification* of X is a compact space $\overline{X}$ containing X as a dense open subset. The compact subspace $\partial X = \overline{X} \smallsetminus X$ is called the *boundary* of the compactification.

Example 4.2. The *trivial compactification* occurs when X is itself compact, so that $\overline{X} = X$ (note that X must be both open and closed in this case) and thus the boundary ∂X is empty.

The trivial compactification is of limited usefulness, and is only provided for completeness. We note that the concept of "tending to infinity" has no meaning in the trivial compactification: that is to say, a continuous function $f : X \longrightarrow \mathbb{C}$ cannot tend to any value on ∂X since the boundary is empty.

Example 4.3. The *one-point compactification* of X is $X = X \cup \{\infty\}$, and thus $\partial X = \{\infty\}$. An instance of this compactification is the identification of the limit of every unbounded sequence in $\mathbb{R}^n$ with the point ∞, the result being homeomorphic to the unit n-sphere S^n.

Example 4.4. The *Stone–Čech compactification* βX is the largest Hausdorff compactification of X and has the property that any continuous function $f : X \longrightarrow K$, where K is a compact Hausdorff space, extends uniquely to a continuous function $\beta f : \beta X \longrightarrow K$.

A compactification $\overline{X}$ can be completely described by a C^*-algebra $C(\overline{X})$ of (bounded) complex-valued continuous functions on it by Theorem 3.21; a function in $C(\overline{X})$ is uniquely determined by its restriction to X [16], and so in turn each X has a C^*-algebra of continuous functions associated with it. To view this from the opposite direction, we can think of extending a certain sub-C^*-algebra of continuous functions on X so that they are continuous "at infinity," meaning that for each $\omega \in \partial X$ we require a function continuous on X to be continuous at ω also.

Example 4.5. If we consider the Stone–Čech compactification βX, then by definition *every* continuous function $f : X \longrightarrow \mathbb{C}$ can be so extended, so the C^*-algebra associated with this compactification is the algebra $C(X)$ of all continuous functions on X. For the one-point compactification, in order to be continuous at infinity (comprised of only one point) a function must tend to a constant, so the C^*-algebra of extendable functions is $C_0(X)$. Finally, for consistency, in the trivial compactification there are *no* points in the boundary ∂X on which functions $f : X \longrightarrow \mathbb{C}$ can converge, so the C^*-algebra of the trivial compactification is in fact empty.

The relationship between a compactification and the continuous functions which may be extended to the boundary thus works in reverse: if we take a sub-C^*-algebra of continuous functions and declare that these are extended to the boundary, then a compactification results. This fact will be used in later sections.

4.2 Proper coarse structures

In Section 2.7 we saw that in the bounded coarse structure the concepts of *bounded*, *proper*, and *bornologous* previously defined for metric spaces coincided. For a topological space X, we can generalize this by requiring that there is some compatibility between the topology and a given coarse structure on X, which the following definition provides.

Definition 4.6. Let X be a paracompact Hausdorff topological space. We say that a coarse structure on X is *proper* if:

(*a*) there is a controlled neighborhood of the diagonal; and
(*b*) every bounded subset of X is relatively compact.

From (*a*) and (*b*) we see that X must be locally compact. If not, there is an $x \in X$ having the property that none of its neighborhoods can be contained in a compact subspace of X: that is to say, if $x \in U$, an open subset of X, then U is not relatively compact. By (*a*) there is a controlled neighborhood E of the diagonal in $X \times X$, and thus E contains (x, x). Since E is open in $X \times X$ there is a neighborhood $U \times V$ of (x, x) contained in E. Thus $x \in U \cap V$, and $(U \cap V) \times (U \cap V)$ is a subset of E, which implies that $(U \cap V) \times (U \cap V)$ is controlled. It follows that $U \cap V$ is bounded in the sense of Definition 2.41, which is a contradiction since then by (*b*) $U \cap V$ is a relatively compact subset of X which contains x.

We note that all metric spaces are paracompact (Appendix B) and Hausdorff. We shall again distinguish between bounded sets (in the sense of Definition 2.41) and the metric sense by referring to the latter as d-bounded.

Example 4.7. Let (X, d) be a metric space. The bounded coarse structure on X is proper if and only if X is a proper metric space (that is, d-bounded sets are relatively compact).

Let $\mathcal{E}$ be the bounded coarse structure on X. Suppose X is a proper metric space, and let $B \subset X$ be bounded. Thus B is d-bounded by Proposition 2.49, and is therefore relatively compact. To show that there is a controlled neighborhood of the diagonal, for each $x \in X$ there is a neighborhood $B(x, \epsilon_x) \subset X$ where $\epsilon_x < 1$. Then

$$U = \bigcup_{x \in X} B(x, \epsilon_x) \times B(x, \epsilon_x)$$

is open and contains Δ. But U is controlled, for

$$\sup\{d(x, y) \mid (x, y) \in U\} \leq 1.$$

Conversely, suppose the coarse bounded structure on X is proper. Let B be d-bounded. Then B is bounded, and hence relatively compact.

Example 4.8. The C_0 coarse structure on a proper metric space X is proper.

If a bounded set B in the C_0 coarse structure is not relatively compact, then $B \setminus K$ is not relatively compact for any compact K, and thus $B \setminus K$ is neither empty nor d-bounded. There is therefore an unbounded sequence (x_n) in $B \setminus K$, whence the sequence $(x_1 \times x_n)$ in $B \setminus K \times B \setminus K$ goes to infinity. This contradicts that there exists for any $\epsilon > 0$ a compact set K such that $d(x, y) < \epsilon$ for all $(x, y) \in B \times B \setminus K \times K$. It follows that B must be relatively compact.

To find a neighborhood of the diagonal, choose any $x_0 \in X$. For each $x \in X$ let r_x be equal to $1/(2d(x, x_0))$, and let $U_x = B(x, r_x) \times B(x, r_x)$ for each $x \in X$. Thus $x \times x \in U_x$

for each $x \in X$, and by taking the union $U = \bigcup_{x \in X} U_x$ it is clear that $\Delta \subset U$ and U is open in $X \times X$. Let $\epsilon > 0$. We note that, because X is proper, letting K be the closure of the metric ball $B(x_0, 1/\epsilon)$ means that K is compact. If (x, y) lies in $U \setminus K \times K$ then (x, y) lies in U_z such that $d(z, x_0) > 1/\epsilon$. It follows that

$$r_z = \frac{1}{2d(z, x_0)} < \frac{1}{2(1/\epsilon)} = \epsilon/2$$

and thus $d(x, y) < 2(\epsilon/2) = \epsilon$ as required.

Example 4.9. It is also true that the indiscrete coarse structure on a locally compact paracompact Hausdorff topological space X is proper. Let B be bounded. Then $B \times B$ is controlled, and thus is proper. If K is any relatively compact subset of X, $(B \times B)[K] = B$ or $(B \times B)[K] = \varnothing$ depending on whether K intersects B. It follows that B must be relatively compact.

For each $x \in X$, since X is locally compact, there is a relatively compact neighborhood $U_x \subset X$ of x. Let

$$U = \bigcup_{x \in X} U_x \times U_x,$$

and U is a neighborhood of Δ. Let K be relatively compact, so $\overline{K}$ is compact. But

$$U[K] \subset U[\overline{K}] = \{x \in X \mid \exists y \in \overline{K} : (x, y) \in U\} = \bigcup_{x \in \overline{K}} U_x.$$

Since $\{U_x \mid x \in \overline{K}\}$ is an open cover of a compact set K, there is a finite subcover $U_{x_1}, \ldots, U_{x_n}$ of $\overline{K}$ for some $x_1, \ldots, x_n \in \overline{K}$. Thus

$$U[K] \subset \bigcup_{1 \leq i \leq n} U_{x_i}$$

the latter of which is relatively compact by Proposition 2.4 because it is a finite union of relatively compact subsets of X. It follows that $U[K]$, being a subset of this, is also relatively compact by Proposition 2.2, whence U is proper and therefore controlled in the indiscrete coarse structure. Thus U is a controlled neighborhood of the diagonal.

In [16] Roe proves a useful result of which we require only one part, shown here.

Proposition 4.10. *Let X be a topological space provided with a proper coarse structure $\mathcal{E}$. Then a subset of X is bounded if and only if it is relatively compact.*

Proof. That a bounded set is relatively compact is given by Definition 4.6. Conversely, for any $x \in X$ and any open controlled set $E \in \mathcal{E}$ the section E_x is a bounded neighborhood of x. Any compact subset of X is covered by finitely many such E_x, and therefore must be bounded. □

4.3 Topological coarse structures

The following is adapted from [16].

Theorem 4.11. *Let X be a paracompact and locally compact Hausdorff space, with a compactification $\overline{X}$. Let $E \subset X \times X$. Then the following conditions are equivalent.*

(a) *The closure $\overline{E}$ of E in $\overline{X} \times \overline{X}$ meets the complement of $X \times X$ only in the diagonal $\Delta_{\partial X} = \{(\omega, \omega) \mid \omega \in \partial X\}$.*
(b) *E is proper in the sense of Definition 2.17, and for every net (x_λ) in E, if (x_λ) converges to a point $\omega \in \partial X$, then (y_λ) converges to ω.*

Moreover, the sets E satisfying these conditions form the controlled sets for a proper connected coarse structure on X.

Proof. Given (a), let (x_λ, y_λ) be a net in E, and suppose $x_\lambda \longrightarrow \omega \in \partial X$. Suppose that y_λ converges to a point $y \in X$. Since X is locally compact there is a relatively compact neighborhood U of y. Then y_λ is eventually in U, for all $\lambda \geq \beta$, say. But then $E[U]$ contains all x_λ for $\lambda \geq \beta$, which is a contradiction since $E[U]$ is relatively compact, but $\{x_\lambda \mid \lambda \geq \beta\}$ is not. This shows that E is proper.

Thus y_λ converges to a point $\omega' \in \partial X$. Since $\overline{E}$ meets the boundary of $X \times X$ only in its diagonal, this shows that $\omega = \omega'$, and thus $y_\lambda \longrightarrow \omega$ as required, proving (b).

For the converse, if (b) holds, then suppose $\overline{E}$ meets the complement of $X \times X$ in (x, y). If either x or y is a point in X, then E cannot be proper, so $x = \omega_x$ and $y = \omega_y$ for some $\omega_x, \omega_y \in \partial X$. There is then a net (x_λ, y_λ) in E converging to ω_x, ω_y, which implies $\omega_x = \omega_y$ and proves (a).

Let $\mathcal{E}$ be the collection of sets with the properties given in the statement of the theorem. Using (a) it is clear that $\mathcal{E}$ is closed under the formation of inverses and subsets. For finite unions, again using (a), let $\{E_i\}$ be a finite collection of subsets of $\mathcal{E}$, and let $E = \bigcup E_i$. That $\overline{E} = \bigcup \overline{E_i}$ shows that $E \in \mathcal{E}$. For products, we use (b). Let (x_λ, y_λ) be a net in $E \circ F$ for sets $E, F \in \mathcal{E}$, and suppose that (x_λ) converges to $\omega \in \partial X$. For each λ there is a z_λ such that $(x_\lambda, z_\lambda) \in E$ and $(z_\lambda, y_\lambda) \in F$. Since $E \in \mathcal{E}$ it follows that (z_λ) converges to ω; since $F \in \mathcal{F}$ it follows that (y_λ) converges to ω also. Thus $\mathcal{E}$ is a coarse structure.

Compact subsets of $X \times X$ satisfy either condition vacuously, so the coarse structure $\mathcal{E}$ is connected.

Let B be bounded, so $B \times B$ is controlled and thus proper. It follows that $(B \times B)[K]$ is either B or $\varnothing$ depending on whether K intersects B, and hence B is relatively compact. Thus, to show that $\mathcal{E}$ is a proper coarse structure we need only find a controlled neighborhood of the diagonal, which we do here by reproducing an argument by Roe [16]. By the Urysohn lemma [14], we can choose continuous functions $f, g : \overline{X} \longrightarrow \mathbb{R}^{+}$ such that f vanishes on the diagonal of $X \times X$ and g vanishes only at infinity (that is, on $\overline{X} \times \partial X \cup \partial X \times \overline{X}$), each of which we note is closed. Let

$$E = \{(x, y) \in X \times X \mid f(x, y) < g(x, y)\}.$$

Then E is open and contains the diagonal of $X \times X$. If (x_λ, y_λ) is a net in E and $x_\lambda \longrightarrow \omega \in \partial X$, then $g(x_\lambda, y_\lambda) \longrightarrow 0$ and thus $f(x_\lambda, y_\lambda) \longrightarrow 0$ also. It follows that y_λ converges to ω, so E is controlled. □

This provides us with a new coarse structure.

Definition 4.12. Let X be a paracompact and locally compact Hausdorff space with a compactification $\overline{X}$. The coarse structure on X satisfying the conditions of Theorem 4.11 is called the *topological coarse structure* associated to X.

The topological coarse structure on X is sometimes referred to as the *continuously controlled coarse structure*, so that we may say that a controlled set of the topological coarse structure is *continuously controlled*. Note that every topological coarse structure is finer than the indiscrete coarse structure since every continuously controlled subset of $X \times X$ is proper.

Each compactification $\overline{X}$ of a locally compact Hausdorff space X has associated to it a C^*-algebra of continuous bounded functions. There is also associated to each compactification a topological coarse structure.

Example 4.13. The topological coarse structure associated to X if X is compact (that is, $\overline{X} = X$) is the maximal coarse structure $\mathfrak{P}(X \times X)$.

Since $\partial X = \varnothing$ then the conditions of Theorem 4.11 are satisfied for *any* subset of $X \times X$ trivially, making every set controlled.

Example 4.14. The topological coarse structure associated with the one-point compactification of a space X is the indiscrete coarse structure.

Let $\mathcal{E}$ be the topological coarse structure associated to the one-point compactification of X, which is $X \cup \{\infty\}$. Since every continuously controlled set is proper by definition, we must show that every proper subset of $X \times X$ is continuously controlled. This, however, is obvious since if a net (x_λ, y_λ) in E had (x_λ) converging to ∞ while (y_λ) converges to some $y \in X$ (there is no other point at infinity for y to converge to) then E would not be proper.

Example 4.15. The bounded coarse structure associated to the usual metric on $X = \mathbb{R}^n$ is strictly finer than the topological coarse structure associated to its compactification by the sphere at infinity S^{n-1}_∞ (Appendix D).

Let E be controlled in the bounded coarse structure, and let (x_λ, y_λ) be a net in E, so there is an $M > 0$ such that $d(x_\lambda, y_\lambda) \leq M$ for all λ. If (x_λ) converges to a point ω at infinity in S^{n-1}_∞, then (y_λ) also converges to ω, since any other point in either $\mathbb{R}^n$ or in S^{n-1}_∞ is an infinite distance from ω. Thus E is continuously controlled, so the bounded coarse structure is contained in the associated topological coarse structure.

To show that the containment is strict, let (x_μ) and (y_μ) be sequences in $\mathbb{R}^n$ defined by

$$x_\mu = (\mu, 0, 0, \ldots, 0) \quad \text{and} \quad y_\mu = (\mu^2 + \mu, 0, 0, \ldots, 0)$$

for each $\mu \in \mathbb{R}^+$ (so x_μ and y_μ are each unbounded in only the first coordinate), and let $F = \{(x_\mu, y_\mu) \mid \mu \in \mathbb{R}^+\} \subset \mathbb{R}^n \times \mathbb{R}^n$. If (x_λ, y_λ) is a net in F and (x_λ) converges to a point

in S^{n-1}_∞ then the only choice is the point at an angle of 0 to each of the first $n-1$ axes in $\mathbb{R}^n$; that is $\omega = (0, \ldots, 0) \in [0, \pi)^{n-1}$. If this occurs, then (y_λ) also converges to ω since the first coordinate of y_λ is similarly unbounded. It follows that F is continuously controlled. However, F is not controlled in the bounded coarse structure on $\mathbb{R}^n$, since $d(x_\lambda, y_\lambda) = \lambda^2$ which is unbounded since λ is unbounded.

Example 4.16. The topological coarse structure associated to the Stone–Čech compactification βX is the discrete coarse structure, which Roe shows in [16].

The trivial compactification $\overline{X} = X$, that is, when X is itself compact, has the maximal coarse structure as its topological coarse structure. The next-smallest compactification, the one-point compactification $\overline{X} = X \cup \{\infty\}$ of a locally compact Hausdorff space X, is associated with the indiscrete coarse structure. On the other hand, the Stone–Čech compactification $\overline{X} = \beta X$ of X is the largest Hausdorff compactification of X, and the topological coarse structure with which it is associated is the discrete coarse structure, which is the finest connected coarse structure.

Similarly, the C^*-algebras associated with the compactifications have a connection with the coarse structures outlined above. The C^*-algebra of continuous functions associated with the trivial compactification is empty; the algebra associated with the one-point compactification is $C_0(X)$; and the algebra associated with the Stone–Čech compactification is the algebra of all continuous (bounded) functions $C(X)$.

There is a relationship between the topological coarse structure and the C^*-algebra of continuous functions associated with a compactification $\overline{X}$ which the following theorem makes clear.

Theorem 4.17. *Let X and Y be locally compact Hausdorff spaces with compactifications $\overline{X}$ and $\overline{Y}$, and assume that X and Y are metrizable. Then a continuous and proper map $f : X \longrightarrow Y$ is coarse with respect to the topological coarse structures on X and Y if and only if it can be extended continuously to a map $\overline{X} \longrightarrow \overline{Y}$* [16].

Proof. We follow closely the structure of Roe's proof* in [16], expanding a few details therein as necessary. Suppose that f can be extended continuously to the compactifications, and let E be a controlled subset of $X \times X$ in the topological coarse structure. Since f is proper by hypothesis, it remains to show that f is bornologous. If $(f(x_\lambda) \times f(x'_\lambda))$ is a net in $(f \times f)(E)$ with $f(x_\lambda) \longrightarrow y$ and $f(x'_\lambda) \longrightarrow y'$ then we may assume that $y \in \partial Y$ since otherwise there is nothing to show. Because $\overline{X}$ is compact we may pass to a convergent subnet $(x_\lambda \times x'_\lambda) \longrightarrow x \times x'$ which, by continuity of f, means that $f(x) = y$ and $f(x') = y'$. Since f is proper it follows that $x \in \partial X$. But E is controlled, which implies, by Theorem 4.11, that $x = x'$. Thus

$$y = f(x) = f(x') = y',$$

*There is a minor typographical error in Roe's version of this proof, which states that the net $f(x_\lambda) \times f(x'_\lambda)$ is in $(f \times f)(E \times E)$.

which means that $(f \times f)(E)$ is controlled in the topological coarse structure on $Y \times Y$. Thus f is bornologous.

For the converse direction, we suppose that f is coarse. We will define the extension of f to the boundary ∂X. Let $x \in \partial X$. Since X is metrizable, choose a sequence (x_n) in X which converges to x, and define $y = f(x)$ to be any limit point in $\{f(x_n)\}$ in $\overline{Y}$. Since f is proper and $x \in \partial X$ it follows that $y \in \partial Y$. We must first show that f is well-defined. Suppose that we have two sequences (x_n) and (x_n') converging to $x \in \partial X$, with y and y' being limit points in $\{f(x_n)\}$ and $\{f(x_n')\}$. Since we are again working in compact spaces we may pass to convergent subsequences, and so we may assume that $f(x_n) \longrightarrow y$ and $f(x_n') \longrightarrow y'$. The set $\{x_n \times x_n'\}$ has only $x \times x$ as a limit point, and so is controlled in the topological coarse structure. Because f is coarse, it follows that $\{f(x_n) \times f(x_n')\}$ is also controlled. The point $y \times y'$ is in the closure of this set, and so it follows that $y = y'$, showing that f is well-defined.

It remains to show that f is continuous. Since X and Y are metrizable, then f is continuous if and only if it preserves convergence of sequences. Let $x \in \partial X$. By construction and the continuity of f on X it follows that $f : X \cup \{x\} \longrightarrow Y \cup \{f(x)\}$ is continuous since any sequence (x_n) converging to x has $f(x_n)$ converging to $f(x)$. That X is dense in $\overline{X}$ completes the proof. □

4.4 Higson compactification

We have seen that, for a compactification $\overline{X}$ of a locally compact Hausdorff space, we can find a coarse structure—the topological coarse structure—in which the continuous proper functions are the same as the coarse maps. The obvious question is then to find a converse situation; namely, given a coarse structure, can we find a compactification of X so that the original coarse structure is continuously controlled? As we have seen, if the coarse structure is one of those which we have already encountered, the answer is yes. Higson outlines a construction which generalizes the situation to any proper coarse structure.

Definition 4.18. Let X be a proper coarse space with coarse structure $\mathcal{E}$. Let $f : X \longrightarrow \mathbb{C}$ be a bounded continuous function. Define the function $\mathbf{d}f : X \times X \longrightarrow \mathbb{C}$ by

$$\mathbf{d}f(x, y) = f(y) - f(x), \quad (x, y) \in X \times X.$$

We say that f is a *Higson function* if for each controlled set $E \in \mathcal{E}$ we have $\mathbf{d}f \in C_0(E)$, that is to say, the restriction of $\mathbf{d}f$ to E vanishes at infinity.

We denote by $C_h(X)$ the set of Higson functions, for reasons which are outlined below (Proposition 4.20).

Example 4.19. If the coarse structure on a metric space X is the C_0 coarse structure, then *every* bounded function $f : X \longrightarrow \mathbb{C}$ is a Higson function.

Obviously $C_h(X) \subset C(X)$. For the other inclusion, let $f \in C(X)$. Let $\epsilon > 0$; since f is continuous, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(y) - f(x)| < \epsilon$. If E is controlled

then there is a compact $K \subset X$ such that if $(x, y) \in E \smallsetminus K \times K$ then $d(x, y) < \delta$, whence

$$|\mathbf{d}f(x, y)| = |f(y) - f(x)| < \epsilon,$$

showing that $\mathbf{d}f \in C_0(E)$. Thus $f \in C_h(X)$, and $C(X) \subset C_h(X)$, completing the proof.

Proposition 4.20. *The Higson functions on a proper coarse space X form a unital C^*-algebra of bounded continuous functions on X.*

Proof. We will say throughout that f and g are Higson functions, so for any controlled set E we have $\mathbf{d}f, \mathbf{d}g \in C_0(E)$. It follows immediately that $\mathbf{d}(f + g)$ is a Higson function, for

$$\begin{aligned} \mathbf{d}(f+g)(x,y) &= (f+g)(y) - (f+g)(x) \\ &= f(y) - f(x) + g(y) - g(x) \\ &= \mathbf{d}f(x,y) + \mathbf{d}g(x,y) \end{aligned}$$

will also vanish at infinity. In addition,

$$(\mathbf{d}f)^*(x,y) = \overline{f(y) - f(x)} = \overline{f}(y) - \overline{f}(x) = f^*(y) - f^*(x) = \mathbf{d}f^*(x,y)$$

so $(\mathbf{d}f)^*$ vanishes at infinity if and only if $\mathbf{d}f^*$ does. For products,

$$\begin{aligned} \mathbf{d}(f \cdot g)(x,y) &= (f \cdot g)(y) - (f \cdot g)(x) \\ &= f(y)g(y) - f(x)g(x) \\ &= f(y)g(y) - f(y)g(x) + f(y)g(x) - f(x)g(x) \\ &= f(y)(g(y) - g(x)) + (f(y) - f(x))g(x) \\ &= f(y)\,\mathbf{d}g(x,y) + \mathbf{d}f(x,y)g(x) \end{aligned}$$

which vanishes at infinity since both $\mathbf{d}f$ and $\mathbf{d}g$ do. Thus the Higson functions $C_h(X)$ form a subalgebra of $C(X)$.

The function $1_X : X \longrightarrow \mathbb{C}$ defined by $1_X(x) = 1$ for all $x \in X$ is such that $f \cdot 1_X = f = 1_X \cdot f$ for any f, and

$$\mathbf{d}1_X(x,y) = 1_X(y) - 1_X(x) = 1 - 1 = 0$$

which shows that $\mathbf{d}1_X \in C_0(E)$ for any (controlled) set E. Thus 1_X is a Higson function.

That Higson functions are bounded is given in Definition 4.18.

It remains to show that $C_h(X)$ is complete with respect to the supremum norm $\| \ \|_\infty$ of $C(X)$. Let f be a limit point of $C_h(X)$, so there is a sequence (f_n) of functions converging to f in $\| \ \|_\infty$. Let E be a controlled subset of $X \times X$, let $\epsilon > 0$ and let N be sufficiently large for $\|f - f_N\|_\infty < \epsilon/3$. That $f_N \in C_h(X)$ means $\mathbf{d}f_N \in C_0(E)$ and thus there is a compact $K \subset X$ such that $|f_N(y) - f_N(x)| < \epsilon/3$ for all $(x,y) \in E \smallsetminus K \times K$. So, for all $(x,y) \in E \smallsetminus K \times K$ we have

$$\begin{aligned} |\mathbf{d}f(x,y)| &= |f(y) - f(x)| \\ &\le |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< 3(\epsilon/3) = \epsilon \end{aligned}$$

by the triangle inequality, whence $\mathbf{d}f \in C_0(E)$. This means that $f \in C_h(X)$, which shows that $C_h(X)$ is complete.

It follows that the Higson functions constitute a unital C^*-algebra as required. □

The notation $C_h(X)$ for the C^*-algebra of Higson functions is therefore justified. It follows from Theorem 3.21 that $C_h(X)$ is the algebra of continuous functions on some compactification of X.

Definition 4.21. The compactification hX of X given by the property that

$$C(hX) = C_h(X)$$

is called the *Higson compactification* of X. The boundary $\partial(hX) = hX \setminus X$ is denoted by νX, and is called the *Higson corona* of X.

Example 4.22. The Higson compactification of a space X with the discrete coarse structure is the Stone–Čech compactification.

We recall that a controlled set E in the discrete coarse structure has only finitely many points off the diagonal. For a continuous function $f : X \longrightarrow \mathbb{C}$ to have the Higson property, $\mathbf{d}f$ when restricted to E must tend to zero at infinity. Infinity can be approached in E only along the diagonal, on which $\mathbf{d}f$ vanishes. It follows that $C_h(X) = C(X)$, which implies that $hX = \beta X$.

Lemma 4.23. *Let X be a proper coarse space. Then $C_0(X)$ is an ideal of $C_h(X)$.*

Proof. To show that $C_0(X) \subset C_h(X)$, we let $f \in C_0(X)$. Then f converges to $\alpha \in \mathbb{C}$ at infinity. It follows that for any controlled set E, $\mathbf{d}f(x,y) = f(y) - f(x)$ converges to $\alpha - \alpha = 0$ at infinity, and thus $\mathbf{d}f \in C_0(X)$. This shows that $f \in C_h(X)$. We have already shown that $C_0(X)$ is a C^*-algebra, so it is closed under addition, multiplication, and involution, and in fact it is also a sub-C^*-algebra of $C_h(X)$ by an adaptation of Corollary 3.9. Let $c \in C_0(X)$, and let $f \in C_h(X)$. Then $c \cdot f \in C_0(X)$ since $c \cdot f$ vanishes at infinity. The result follows. □

By the construction of hX, the C^*-algebra of Higson functions is precisely the collection which can be continuously extended to the boundary νX. We therefore associate to each $f \in C_h(X)$ the continuous function $\varphi \in C(\nu X)$ to which f may be extended. It is immediate that the collection

$$\mathcal{F}_\varphi = \{f \in C_h(X) \mid f \text{ extends to } \varphi\}$$

partitions $C_h(X)$ because every Higson function may be so extended, and because if $\varphi \neq \psi$ then f and g may not be simultaneously extended continuously to the same function.

Say that $f \sim g$ defined in this fashion if and only if both $f, g \in \mathcal{F}_\varphi$. Then the partitioning of the set of Higson functions implies immediately that $\sim$ is an equivalence relation. The consequences of this equivalence relation are shown in the next proposition.

PROPOSITION 4.24. *Let X be a proper coarse space with compactification hX. Then $f \sim g$ if and only if $f - g \in C_0(X)$.*

Proof. That f and g may be continuously extended to agree with a $\varphi : \nu X \longrightarrow \mathbb{C}$ is equivalent to the existence, for any $\epsilon > 0$, of a compact $K \in X$ such that $|f(x) - g(x)| < \epsilon$ on $X \setminus K$, which means that $f - g \in C_0(X)$ as required. □

PROPOSITION 4.25. *Let X be a proper coarse space with Higson compactification hX, and let φ be a map in $C(\nu X)$. Then there is a surjective *-homomorphism $\Psi : C_h(X) \longrightarrow C(\nu X)$ with the property that $\varphi = \Psi(f)$ implies that f may be continuously extended to φ.*

Proof. By assigning $\Psi(f) = \varphi_f$ for each $f \in C_h(X)$ we see immediately that Ψ is surjective and has the required properties.

To show that Ψ is a *-homomorphism, let $f, g \in C_h(X)$, and let $\varphi = \Psi(f)$ and $\psi = \Psi(g)$. Then $\lambda f + g$, $f \cdot g$ and f^* extend continuously to $\lambda\varphi + \psi$, $\varphi \cdot \psi$, and φ^*, respectively, which completes the proof. □

PROPOSITION 4.26. *Let X be a proper coarse space. The C^*-algebra of the Higson corona νX is C^*-isomorphic to $C_h(X)/C_0(X)$.*

Proof. By Proposition 4.25 there is a surjective *-homomorphism $\Psi : C_h(X) \longrightarrow C(\nu X)$. Let $0_{C(\nu X)} : X \longrightarrow \mathbb{C}$ denote the zero function on νX. We note that the zero function $0_{C(X)}$ can be continuously extended to the zero function $0_{C(\nu X)}$. But by Proposition 4.24 it follows that

$$\begin{aligned}
\operatorname{Ker}\Psi &= \{f \in C_h(X) \mid \Psi(f) = 0_{C(\nu X)}\} \\
&= \{f \in C_h(X) \mid \varphi = 0_{C(\nu X)}\} \\
&= \{f \in C_h(X) \mid f \sim 0_{C(X)}\} \\
&= \{f \in C_h(X) \mid f - 0_{C(X)} \in C_0 X\} \\
&= C_0(X)
\end{aligned}$$

and thus $\operatorname{Ker}\Psi = C_0(X)$. Lemma 4.23 shows that $C_0(X)$ is an ideal of $C_h(X)$. Since $\operatorname{Im}\Psi = C(\nu X)$ it follows by the first isomorphism theorem† that $C(\nu X) \cong C_h(X)/C_0(X)$ as required. □

The Higson compactification hX is defined only for a proper coarse space. The Higson corona νX, however, is defined for all coarse spaces.

DEFINITION 4.27. Let X be a coarse space, and let $f : X \longrightarrow \mathbb{C}$ be a map. We say that f *tends to zero at infinity* and write $f \longrightarrow 0$ if for all $\epsilon > 0$ there is a bounded (as in Definition 2.41) set B such that $|f(x)| < \epsilon$ for all $x \notin B$.

†Let A be a C^*-algebra, and let $f : A \longrightarrow B$ be a surjective *-homomorphism of C^*-algebras. Then $\operatorname{Ker} f$ is an ideal of A, and $A/\operatorname{Ker} f \cong \operatorname{Im} f$.

When the space is a product, such as $X \times X$, we make a similar definition, this time saying that $f : X \times X \longrightarrow \mathbb{C}$ tends to zero at infinity if B is bounded in X.

Definition 4.28. Let X be a coarse space. Then denote by $B_h(X)$ the set of all bounded functions $X \longrightarrow \mathbb{C}$ such that $\mathbf{d}f \longrightarrow 0$ at infinity, and let $B_0(X)$ denote the ideal of all bounded functions that tend to zero at infinity.

It is clear that $B_h(X)$ is a C^*-algebra, and that $B_0(X)$ is a sub-C^*-algebra of $B_h(X)$.

Lemma 4.29. *Let X be a proper coarse space. Then* [16]

(*a*) $C_0(X) = C_h(X) \cap B_0(X)$;
(*b*) $B_h(X) = C_h(X) + B_0(X)$.

Proof. Since X is proper then by Proposition 4.10 the definition of bounded is equivalent to relatively compact, so the old and new definitions of "tending to zero at infinity" agree with each other, proving (*a*).

Since X is a proper coarse space, there is a controlled neighborhood E of the diagonal. Choose an open cover $\{U_\alpha\}$ of X such that $\bigcup_\alpha U_\alpha \times U_\alpha \subset E$ (which is possible since E is open) and then take a partition of unity ϕ_α with respect to $\{U_\alpha\}$. Choose an $x_\alpha \in U_\alpha$ for each α. Now let $f \in B_h(X)$, and set

$$g(x) = \sum_\alpha \phi_\alpha(x) f(x_\alpha)$$

Then g is continuous and bounded. We have

$$f(x) - g(x) = \sum_\alpha \phi_\alpha(x)(f(x) - f(x_\alpha))$$

and $(x, x_\alpha) \in E$ whenever $\phi_\alpha \neq 0$. Because f is a Higson function it follows that $f - g$ is in $B_0(X)$, and thus $g \in C_h(X)$ as required. $\square$

It follows from Theorem 4.26 and the second isomorphism theorem that

$$C(\nu X) = \frac{C_h(X)}{C_0(X)} = \frac{C_h(X)}{B_0(X) \cap C_h(X)} = \frac{B_0(X) + C_h(X)}{B_0(X)} = \frac{B_h(X)}{B_0(X)}.$$

We can use this identity as the *definition* of νX in the general case when X is not a proper coarse space. The Higson corona is *functorial.*

Theorem 4.30. *Let X and Y be proper coarse spaces. A coarse map $\phi : X \longrightarrow Y$ extends to a continuous map $\nu\phi : \nu X \longrightarrow \nu Y$. If $\phi, \psi : X \longrightarrow Y$ are close then $\nu\phi = \nu\psi$.*

Proof. We observe that $\phi : X \longrightarrow Y$ induces a map $\phi^* : B_h(Y) \longrightarrow B_h(X)$ by defining $\phi^*(f) = f \circ \phi$ for all functions $f \in B_h(Y)$, that is, for all f such that $\mathbf{d}f \in B_0(E)$ for all controlled sets E. Thus $f \circ \phi \in B_h(X)$ since f is bounded, $(\phi \times \phi)(E)$ is controlled for any controlled set $E \subset X \times X$, and

$$(f \circ \phi) \times (f \circ \phi) = (f \times f) \circ (\phi \times \phi),$$

so ϕ^* is well-defined. If $f \in B_0(Y)$ it is clear that $f \circ \phi \in B_0(X)$ because $\mathbf{d}f$ tends to zero at infinity, so ϕ also induces a map $\phi^* : B_0(Y) \longrightarrow B_0(X)$.

By taking quotients it then follows that we can construct a map

$$B_h(Y)/B_0(Y) \longrightarrow B_h(X)/B_0(X).$$

By the remarks following Theorem 4.29, $C(\nu X) \cong B_h(X)/B_0(X)$ so we can view this as a map $C(\nu Y) \longrightarrow C(\nu X)$. But since X and Y are proper coarse spaces it follows that they are locally compact Hausdorff, whence the subspaces $\nu X \subset hX$ and $\nu Y \subset hY$ are compact Hausdorff. The equivalence of the opposite of the category of compact Hausdorff spaces and continuous functions, and of C^*-algebras and *-homomorphisms (Theorem 3.20) implies that by taking the dual of each we have a continuous map

$$\Omega(C(\nu X)) \longrightarrow \Omega(C(\nu Y))$$

which, by Theorem 3.17 is equivalent to $\nu\phi : \nu X \longrightarrow \nu Y$ being continuous.

We observe that if ϕ and ψ are close, and if $f \in B_h(Y)$, then $\phi^*(f) - \psi^*(f) \in B_0(X)$. □

Corollary 4.31. *Coarsely equivalent spaces have homeomorphic Higson coronas.*

Proof. If X and Y are coarsely equivalent then there are coarse maps $\phi : X \longrightarrow Y$ and $\psi : Y \longrightarrow X$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are each close to the identity maps Y and on X, respectively. By Theorem 4.30 there are continuous maps $\nu\phi : \nu X \longrightarrow \nu Y$ and $\nu\psi : \nu Y \longrightarrow \nu X$. Again by Theorem 4.30 and the closeness of the maps to the identities, we have

$$\nu\phi \circ \nu\psi = \nu(\phi \circ \psi) = id_Y$$

and

$$\nu\psi \circ \nu\phi = \nu(\psi \circ \phi) = id_X$$

which implies that $(\nu\phi)^{-1} = \nu\psi$. The result follows. □

Example 4.32. Although already shown directly in Theorem 2.58, the minimal uncountable well-ordered set S_Ω cannot be coarsely equivalent to $\mathbf{Z}^+ = \mathbb{N}$ in the discrete coarse structure on each space since their Higson coronas

$$\nu(S_\Omega) = \beta S_\Omega \setminus S_\Omega = S_\Omega \cup \{\Omega\} = \{\Omega\}$$

and

$$\nu(\mathbb{N}) = \beta\mathbb{N} \setminus \mathbb{N} = \mathbb{N}^*$$

are not homeomorphic. The first corona is comprised of one point, but the second is the set of ultrafilters, and thus their cardinalities are different.

We have therefore achieved a method to determine whether two spaces are coarsely equivalent by examination of their Higson coronas, rather than by exploiting special large-scale properties of the spaces. In practice, of course, it may not be an easy task to calculate the coronas, but the principle by which they are found is somewhat more consistent.

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Appendix

APPENDIX A

Some Properties of S_Ω

This set, as in topology, provides counter-examples to some ideas in coarse geometry. We give a brief account of some of its properties.

DEFINITION. The *dictionary order* on a product $A \times B$ is given by $(\alpha, \beta) < (\alpha', \beta')$ if either $\alpha < \alpha'$, or if $\alpha = \alpha'$ and $\beta < \beta'$.

DEFINITION. A *section* of an ordered set A by $\beta \in A$ is the subset $A_\beta = \{\alpha \in A \mid \alpha < \beta\}$.

PROPOSITION. There is an minimal uncountable well-ordered set. [14]

Proof. By the axiom of choice there is an uncountable well-ordered set S. Then $\{0,1\} \times S$ is well-ordered in the dictionary order; it follows that some section of $\{0,1\} \times S$ is uncountable. Let Ω be the smallest element of $\{0,1\} \times S$ for which these sections are uncountable, which exists since S, and hence $\{0,1\} \times S$, is well-ordered. Then S_Ω has the properties required. $\square$

One of the most interesting properties of S_Ω is that every section $S_\beta = \{\alpha \in S_\Omega \mid \alpha < \beta\}$ is countable. The obvious choice of a topology on S_Ω is the order topology, so basic open sets are either of the form $[0, \alpha)$ or (α, β) for $\alpha, \beta \in S_\Omega$.

There is another interpretation of S_Ω which is sometimes useful to us. We can write elements in S_Ω explicitly as ordinals

$$0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, 2\omega, 2\omega + 1, 2\omega + 2, \dots$$
$$\dots, 3\omega, \dots, \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots$$

and so on.

Finally, the *Stone-Čech compactification* of S_Ω is $\beta S_\Omega = S_\Omega \cup \{\Omega\}$ and is sometimes denoted $\overline{S_\Omega}$.

APPENDIX B

Elementary Topology

The only topological spaces under consideration in this text are those which are *locally compact Hausdorff*. While most readers will be familiar with these terms, for completeness we give a brief reference and some examples.

DEFINITION. Let X be a topological space. If for each pair $x, y \in X$ there exist open disjoint neighborhoods U and V of x and y respectively, then we say that X is a *Hausdorff* space .

A Hausdorff space is also denoted as being a T_2 space [14]. Almost all topological spaces which one would wish to consider are Hausdorff. A trivial example of a class of spaces which is not Hausdorff is any space X containing two or more points and which has the *indiscrete topology* $\{X, \varnothing\}$. Another example is any space with the *finite complement topology*,[14] which is T_1 (one-point sets are closed) but not T_2. Metric spaces are always Hausdorff.

DEFINITION. A topological space X is *locally compact* if for each $x \in X$ there is a compact subspace C of X which contains a neighborhood of x.

Locally compact spaces are important because they admit *compactifications*, a term which is defined in Section 4.1.

EXAMPLE. Euclidean space $\mathbb{R}^n$ is locally compact, but not all metric spaces are.

For $X = \mathbb{R}^n$, for any $x \in X$ can be contained in $[a_1, b_1] \times \cdots \times [a_n, b_n]$, which contains the neighborhood $(a_1, b_1) \times \cdots \times (a_n, b_n)$. The space $Y = \mathbb{R}^\omega = \prod_{i \in \mathbb{N}} \mathbb{R}$ is not locally compact (see [14]).

In Section 4.2 we discuss a coarse structure defined on *paracompact* spaces, an outline of which is provided here.

DEFINITION. Let X be a set. A *refinement* of a collection $\mathcal{U}$ of subsets of X is a collection $\mathcal{V}$ such that for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ containing B. If X is a topological space and $\mathcal{U}$ a collection of open sets, then $\mathcal{V}$ is an *open refinement*.

A collection of open sets $\mathcal{V}$ of a topological space X is said to be *locally finite* if each $x \in X$ has a neighborhood which intersects only finitely many members of $\mathcal{V}$. A compact space is one where any open covering automatically has a finite subcover, or a cover which is, by analogy, *globally finite*. A space where we can can always find—given an open covering of the space—a (slightly modified) cover which is locally finite is a generalization of a compact space.

Definition. A space X is *paracompact* if every open covering $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ that covers X.

Any compact space is automatically paracompact, but the converse is not true, as might be expected. The argument for showing that the real line $\mathbb{R}$ is paracompact is adapted from [14].

Example. The real line $\mathbb{R}$ is paracompact but not compact.

Let $\mathcal{U}$ be any open cover of $\mathbb{R}$. For each natural number n, we can choose a finite number of open sets from $\mathcal{U}$ which cover $[-n, n]$. We let $\mathcal{V}_n$ be the collection whose open sets are each of the finite open sets covering $[-n, n]$, intersected with $(-\infty, -n) \cup (n, \infty)$. Then the collection $\mathcal{V} = \bigcup \mathcal{V}_n$ is obviously a refinement of $\mathcal{U}$. It is also locally finite, since the open interval $(-n, n)$ intersects only finitely many open sets from $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n$. Finally, $\mathcal{V}$ covers $\mathbb{R}$, for if $x \in \mathbb{R}$ then $x \in [-n, n]$ for some natural number n. There is an open set $U \in \mathcal{U}$ containing x, but then there is a set $V = U \cap (-\infty, -n) \cap (n, \infty)$ in $\mathcal{V}_n$ which contains x.

That $\mathbb{R}$ is not compact is obvious since it is not bounded.

In fact, every metric space is paracompact [14], because, given an open covering $\mathcal{U}$ of X, there is an open refinement that covers X and is countably locally finite, which implies that $\mathcal{U}$ has an open refinement that covers X and is locally finite.

One might image that every locally compact space is paracompact. This is, however, not true.

Example. Let X be a minimal uncountable well-ordered set S_Ω with the order topology, and denote the smallest element in S_Ω by 0. We show that $[0, \gamma] \subset S_\Omega$ is compact for every $\gamma \in S_\Omega$, which is equivalent to S_Ω being locally compact.

Let $\mathcal{U}$ be an open cover of $[0, \gamma]$, and let C be the set of all $\delta \in [0, \gamma]$ such that $[0, \gamma]$ can be covered by finitely many elements of $\mathcal{U}$. Clearly $0 \in C$, so C is non-empty. By the well-ordering of S_Ω, C has a greatest element μ, which we suppose is less than γ, for otherwise $[0, \gamma]$ is compact.

There is some $U \in \mathcal{U}$ containing μ, and thus μ is contained in an basis element $(\alpha, \beta) \subset U$ where $\beta < \gamma$. There is an $\eta \in (\alpha, \beta)$ such that $\alpha < \mu < \eta < \beta < \gamma$, and thus $[\mu, \eta] \subset U$. Since $[0, \eta] = [0, \mu] \cup [\mu, \eta]$, it follows that $[0, \eta]$ is also covered by finitely many elements of $\mathcal{U}$, whence $\eta \in C$. This contradicts that μ is the greatest element of C. Our premise, that $\mu < \gamma$ is therefore false, which implies that $[0, \gamma]$ is compact, and thus S_Ω is locally compact.

But S_Ω is not paracompact. We observe the the collection $\{[0,\alpha) \mid \alpha \in S_\Omega\}$ is an open covering of S_Ω, and that $\{[0,\alpha_x) \mid \alpha \in X\} = [0, \sup \alpha_x)$. But the supremum of a countable collection of countable ordinals is countable, so such a union can never cover S_Ω. This implies that S_Ω is not Lindelöf.

Because S_Ω is locally compact but not limit point compact, it is not metrizable [14]. This, together with that fact that it is not Lindelöf, implies that it is not paracompact. ❦

APPENDIX C

Categories and Opposites

Some basic terminology in this subject are given for completeness: Section 3.4 uses the *opposite of a category.*

DEFINITION. A *category* [8,12] C is a class of *objects* and sets of *morphisms* between those objects. For every ordered pair A, B of objects there is a set $\mathrm{Hom}_{\mathbf{C}}(A,B)$ of morphisms from A to B, and for every ordered triple A, B, C of objects there is a map

$$\mathrm{Hom}_{\mathbf{C}}(A,B) \times \mathrm{Hom}_{\mathbf{C}}(B,C) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A,C)$$

where $(f,g) \longmapsto g \circ f$ and $g \circ f$ is called the *composition of g with f*. We may also write gf in place of $g \circ f$. The objects and morphisms satisfy the two properties that (*a*) composition is associative, and (*b*) each object A has an identity $1_A \in \mathrm{Hom}_{\mathbf{C}}(A,A)$ such that $f \circ 1_A = f$ for every $f \in \mathrm{Hom}_{\mathbf{C}}(A,B)$ and $1_A \circ g = g$ for every $g \in \mathrm{Hom}_{\mathbf{C}}(B,A)$.

Examples of categories we are interested in include the category Haus consisting of objects X, Y which are Hausdorff spaces, and sets of morphisms $\mathrm{Hom}_{\mathbf{Haus}}(X,Y)$ which are sets of continuous functions $f : X \longrightarrow Y$. The categories **Loc** and **Comp** are subcategories of Haus whose objects are locally compact Hausdorff spaces, and compact Hausdorff spaces, respectively. The morphisms of these subcategories are continuous functions, and bounded continuous functions, respectively.

The category **Alg** is defined to consist of objects A, B which are C^*-algebras, and sets of morphisms $\mathrm{Hom}_{\mathbf{Alg}}(A,B)$ of *-homomorphisms $\alpha : A \longrightarrow B$ between them.

DEFINITION. Let C and D be categories. We say $\mathcal{F}$ is a *covariant functor* from C to D if for every object A in C, $\mathcal{F}(A)$ is an object in D, and for every $f \in \mathrm{Hom}_{\mathbf{C}}(A,B)$ we have $\mathcal{F}(f) \in \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(A), \mathcal{F}(B))$ satisfying the following properties.

(*a*) If gf is a composition of functions in C, $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$ as a composition of functions in D.

(*b*) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

We say $\mathcal{F}$ is a *contravariant functor* from **C** to **D** if for every object A in **C**, $\mathcal{F}(A)$ is an object in **D**, and for every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ we have $\mathcal{F}(f) \in \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(B), \mathcal{F}(A))$ satisfying the following properties.

(*a*) If gf is a composition of functions in **C**, $\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$ as a composition of functions in **D**.

(*b*) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

Definition. If **C** is a category, then the *opposite* category of **C**, denoted $\mathbf{C}^{\mathrm{op}}$, is defined as follows. The objects of $\mathbf{C}^{\mathrm{op}}$ are the same as the objects of **C**. The set $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(A, B)$ of morphisms of $\mathbf{C}^{\mathrm{op}}$ is defined to be the set $\mathrm{Hom}_{\mathbf{C}}(B, A)$ of morphisms in **C** from B to A. A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(B, A)$ is written as the morphism f^{op} in $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(A, B)$. Composition of morphisms in $\mathbf{C}^{\mathrm{op}}$ is defined by

$$g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}.$$

The opposite of the category **Comp**, for example, is the category $\mathbf{Comp}^{\mathrm{op}}$ whose sets of morphisms $\mathrm{Hom}_{\mathbf{Comp}^{\mathrm{op}}}(X, Y)$ are bounded continuous functions $f : Y \longrightarrow X$.

Definition. Let **C** and **D** be categories and $\mathcal{F} : \mathbf{C} \longrightarrow \mathbf{D}$ and $\mathcal{G} : \mathbf{C} \longrightarrow \mathbf{D}$ be covariant functors. A *natural transformation* $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a function that assigns to each object A of **C** a morphism $\Phi_A : \mathcal{F}(A) \longrightarrow \mathcal{G}(A)$ of **D** such that for every morphism $f : A \longrightarrow B$ of **C**, the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\Phi_A"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(A) \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(B) \arrow[r, "\Phi_B"'] & \mathcal{G}(B)
\end{tikzcd}$$

in **D** is commutative. If Φ_A is an equivalence for every A in **C** then Φ is a *natural isomorphism* of the functors $\mathcal{F}$ and $\mathcal{G}$.

Definition. Let **C** and **D** be categories. We say that an *equivalence of categories* consists of covariant functors $\mathcal{F} : \mathbf{C} \longrightarrow \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \longrightarrow \mathbf{C}$, and two natural isomorphisms $\Phi : \mathcal{F}\mathcal{G} \longrightarrow I_{\mathbf{D}}$ and $\Psi : I_{\mathbf{C}} \longrightarrow \mathcal{G}\mathcal{F}$, where $\mathcal{F}\mathcal{G} : \mathbf{D} \longrightarrow \mathbf{D}$ and $\mathcal{G}\mathcal{F} : \mathbf{C} \longrightarrow \mathbf{C}$ denote the respective compositions of $\mathcal{F}$ and $\mathcal{G}$, and $I_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{C}$ and $I_{\mathbf{D}} : \mathbf{D} \longrightarrow \mathbf{D}$ denote the respective identity functors on **C** and **D** which assign each object and morphism to itself. If $\mathcal{F}$ and $\mathcal{G}$ are instead contravariant functors then one speaks of a *duality of categories*. ❦

Appendix D ~

Spheres at Infinity

Compactification of topological spaces is a focus of Section 4.1. We outline here a compactification of $\mathbb{R}^n$ called the *sphere at infinity*, which for completeness we have defined here explicitly.

Recalling that a sequence (x_n) in $\mathbb{R}^+$ is said to be *unbounded* if for every $M > 0$ there is an n such that $x_n \geq M$, we declare that the symbol ∞ is the limit of any unbounded sequence in $\mathbb{R}^+$, (and thus ∞ is the limit of every such unbounded sequence) and say that $x_n \longrightarrow \infty$ if (x_n) is an unbounded sequence. We define the *positive extended real numbers* as $\mathbb{R}^+_\infty = \mathbb{R}^+ \cup \{\infty\}$, and extend the ordering on $\mathbb{R}$ so that $x < \infty$ for every real number x.

In Euclidean space $\mathbb{R}^n$ we may uniquely describe a *half-ray* ℓ through the origin as the set of *position vectors* $\{t\hat{\mathbf{v}} \mid t \in \mathbb{R}^+\}$ for some unit-length position vector $\hat{\mathbf{v}}$ in $\mathbb{R}^n$. In $\mathbb{R}$ there are two distinct unit-length position vectors $\mathbf{1}$ and $-\mathbf{1}$; in $\mathbb{R}^2$ there are two distinct unit-length position vectors $\hat{\mathbf{v}}$ and $-\hat{\mathbf{v}}$ for each $\theta \in [0, \pi)$;* and in general, for $n \geq 2$, there are two unit-length position vectors $\hat{\mathbf{v}}$ and $-\hat{\mathbf{v}}$ of $\mathbb{R}^n$ for each $(\theta_1, \theta_2, \ldots, \theta_{n-1})$, where $\theta_i \in [0, \pi)$ is the angle $\hat{\mathbf{v}}$ makes with the ith positive axis. Thus there are two distinct half-rays in $\mathbb{R}^n$ for every point in $[0, \pi)^{n-1}$ (where we assume that $[0, \pi)^0 = \{0\}$).

We produce a "point at infinity" for each half-ray ℓ through the origin if we instead allow t to range over the positive extended numbers $\mathbb{R}^+_\infty$ (i.e., we allow $t = \infty$).

Definition. Let ℓ be a half-ray through the origin of $\mathbb{R}^n$ with unit-length position vector $\hat{\mathbf{v}}$. Define the *end-point* of ℓ through the origin of $\mathbb{R}^n$ to be the point $\infty\hat{\mathbf{v}}$.

Equivalently, the end-point of a half-ray ℓ through the origin is $\lim_{t\to\infty} t\hat{\mathbf{v}}$.

Definition. The *sphere at infinity* S^{n-1}_∞ of $\mathbb{R}^n$ consists of the end-points of every half-ray through the origin.

The sphere at infinity is aptly named, for, if we allow distances to become infinite, our construction can also be described as

$$S^{n-1}_\infty = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| = \infty\}.$$

* Any half-open interval may be used equivalently, but $[0, \pi)$ is the natural choice. The half-interval $[0, 1)$ would also do.

It follows from the number of distinct half-rays through the origin of $\mathbb{R}^n$ that every point in S_∞^{n-1} can be identified with a point in $[0, \pi)^{n-1}$ or its "negative."

Let $\overline{\mathbb{R}^n}$ be defined as $\mathbb{R}^n \cup S_\infty^{n-1}$. We endow $\overline{\mathbb{R}^n}$ with a topology consisting of every open set $\mathbb{R}^n$ together with every *truncated cone*

$$C(\mathbf{w}, \epsilon) = \{\mathbf{v} \in \overline{\mathbb{R}^n} \mid \angle_i \mathbf{w} - \epsilon < \angle_i \mathbf{v} < \angle_i \mathbf{w} + \epsilon\}_{i=1\ldots n-1} \setminus K$$

where $\angle_i \mathbf{v}$ is the angle $\mathbf{v}$ makes with the ith axis, $0 < \epsilon < \pi/2$, and K is a compact subset of $\mathbb{R}^n$. Then $\overline{\mathbb{R}^n}$ is a compact Hausdorff space which contains $\mathbb{R}^n$ as a locally compact dense subset, and thus it follows that $\overline{\mathbb{R}^n}$ is a compactification of $\mathbb{R}^n$.

If two unit-length position vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are distinct—making their associated half-rays $\{t\mathbf{v}_1 \mid t \in \mathbb{R}^+\}$ and $\{t\mathbf{v}_2 \mid t \in \mathbb{R}^+\}$ in $\mathbb{R}^n$ distinct also—then $d(t\mathbf{v}_1, t\mathbf{v}_2) \longrightarrow \infty$ as $t \longrightarrow \infty$. In other words, the end-points of two distinct half-rays are infinitely far apart. We therefore extend the Euclidean metric d on $\mathbb{R}^n$ to an extended metric $d_\infty : \overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n} \longrightarrow \mathbb{R}_\infty^+$ by declaring that

$$d_\infty(x, y) = \begin{cases} 0 & \text{if } x = y \\ d(x, y) & \text{if } x, y \in \mathbb{R}^n \\ \infty & \text{if } x \neq y \text{ and at least one of } x, y \in S_\infty^{n-1}. \end{cases}$$

We observe that every point in the sphere at infinity is an infinite distance not only from every point in $\mathbb{R}^n$ but also from every other point in the sphere.

Example. The sphere at infinity S_∞^0 of $\mathbb{R} = \mathbb{R}^1$ consists of the points $\{-\infty, \infty\}$, so $\overline{\mathbb{R}}$ is the extended real line. The sphere at infinity S_∞^1 of $\mathbb{R}^2$ consists of a "circle" of points at infinity, two for each $\theta \in [0, \pi)$. ❦

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Vita

Christian Stuart Hoffland, originally a resident of Edinburgh, Scotland, attended the University of Edinburgh for his Bachelor of Science degree, graduating in 1997. During this time, his involvement with the University of Edinburgh's students' association led to him becoming its senior designer, with oversight in its publications wing, including the association's weekly newspaper.

After graduation, he received further training as a professional graphic designer, working for an Edinburgh publishing firm. He then returned to the University of Edinburgh for a Postgraduate Certificate of Education in Mathematics, graduating in 2000. Christian was subsequently invited to teach at one of Scotland's noted boarding schools, Loretto, where he taught both mathematics and graphic design.

After three years there, he traveled to the US for a Master of Science degree in Mathematics at the University of Tennessee, Knoxville, where he graduated in 2006. He is currently attending Indiana University, Bloomington for a PhD in pure mathematics, pursuing professional interests in topology and topological groups, geometry, algebra, and functional analysis. ❦