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Applications of Optimal Control

Katherine Renee Fister
University of Tennessee, Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Katherine Renee Fister entitled "Applications of Optimal Control." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Suzanne M. Lenhart, Major Professor

We have read this dissertation and recommend its acceptance:

J. D. Birdwell, Debra Polignone, Philip Schaefer

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Applications of Optimal Control

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Katherine Renee Fister

May 1996

DEDICATION

To the gift that God granted me, my precious husband Kenny.

Kenny deserves as much recognition for this degree as anyone. He provides encouragement and laughter. He listens and hears when my heart and mind are in a turmoil. He bestows his amazing love through his actions and not just by his words. In addition, he has patiently devoted his time to the computer for this all-consuming dissertation.

No one could be more blessed with such a devoted man. Indeed, my beloved Christian husband has already earned his Ph.D. in life. Thank you, my love.

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Throughout my graduate experience at the University of Tennessee, I will always remember the love and friendship that the support staff in the Mathematics Department provided. Specifically, I thank Pam Armentrout for her belief in me and Linda Silvers for her prayers.

In addition, my parents are a God-send since they listen to my complaints and love me through all the heartaches. Also, a genuine thanks to Kenny's parents who help me remember what is important.

Most assuredly, my husband, Kenny, is a remarkable blessing. With his everlasting faith and love for me, I am learning to "walk by faith and not by sight".

ABSTRACT

In this dissertation, we investigate optimal control of partial and ordinary differential equations. We prove the existence of an optimal control for which the objective functional is maximized. The goal is to characterize the optimal control in terms of the solution of the optimality system. The optimality system consists of the state equations coupled with the adjoint equations. To obtain the optimality system we differentiate the objective functional with respect to the control. This process is applied to harvesting in a predator-prey parabolic system, to analyzing surface runoff in a parabolic problem, and to controlling the effect of the HIV virus on T cells in an AIDS patient. In the predator-prey problem, the profit associated with harvesting is shown to be positive under certain constraints. In the runoff problem, the concentration of contaminants being deposited into a major river flow is modeled as point sources. To explicitly characterize the optimal controls, two choices of the revenue function are used. One revenue function is a Michaelis-Menton function and the other is a quadratic function. In the HIV problem, we control the effect that HIV has on the T cells in the immune system. We seek to maximize the number of T cells, minimize the free virus, and minimize the systemic cost to the body.

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Part I

Introduction

Optimal control theory evolved from the classical calculus of variations. Its originators, L. S. Pontryagin and his associates, developed the maximum principle for optimal control of finite dimensional problems. In these problems, the state system is governed by ordinary differential equations. In addition, the variables are separated into state variables and control variables. The trajectories of the state variables are influenced directly by the controls. The controls can also explicitly affect the objective function. In general, the number of control variables do not have to equal the number of state variables. [5]

To illustrate a simple version of Pontryagin's maximum principle, the following problem is considered. Given a piecewise continuous control vector $\vec{u}(t) = (u_1(t), u_2(t))$, there is an associated continuous and piecewise differentiable state vector $\vec{x}(t) = (x_1(t), x_2(t))$ defined on $[t_0, t_1]$, a finite time interval, that solves

$$x_i'(t) = g_i(t, \vec{x}(t), \vec{u}(t)), \quad i = 1, 2$$

with initial conditions

$$x_i(t_0) = x_0^i \quad i = 1, 2 \quad \text{where } x_0^i \text{ are fixed and}$$

$$x_i(t_1) \quad i = 1, 2, \quad \text{are free terminal conditions.}$$

The goal is to find $\vec{u}^*$ that maximizes the objective functional, i.e.

$$J(u_1^*, u_2^*) = \max_{(u_1, u_2)} J(u_1, u_2), \tag{1.1}$$

where

$$J(u_1, u_2) = \int_{t_0}^{t_1} f(t, x_1(t), x_2(t), u_1(t), u_2(t))\, dt.$$

Assume f, g_i, $\frac{\partial f}{\partial x_j}$, $\frac{\partial g_i}{\partial x_j}$ are continuous functions of each argument for $i, j = 1, 2$.

The following theorem of Pontryagin gives necessary conditions for an optimal control [11].

Theorem 1.1

If $\vec{x}^*(t), \vec{u}^*(t)$ are optimal in (1.1), then it is necessary that there exists a continuous function $\vec{\lambda} = (\lambda_1(t), \lambda_2(t))$ where for all $t_0 \leq t \leq t_1$, $\vec{\lambda}(t) \neq 0$, and

$$H(t, \vec{x}^*(t), \vec{u}(t), \vec{\lambda}(t)) \leq H(t, \vec{x}^*(t), \vec{u}^*(t), \vec{\lambda}(t))$$

where the Hamiltonian is defined by

$$H(t, \vec{x}, \vec{u}, \vec{\lambda}) = \lambda_1 g_1(t, \vec{x}, \vec{u}) + \lambda_2 g_2(t, \vec{x}, \vec{u}) + f(t, \vec{x}, \vec{u}).$$

Except at points of discontinuity of $\vec{u}^*(t)$,

$$\lambda_i'(t) = -\frac{\partial H}{\partial x_i}(t, \vec{x}^*(t), \vec{u}^*(t), \vec{\lambda}(t)) \quad i = 1, 2.$$

Furthermore, the following transversality conditions are satisfied

$$\lambda_i(t_1) = 0 \quad \text{for } i = 1, 2.$$

To better grasp the concept of an adjoint variable $\vec{\lambda}$, a similarity may be noted with the concept of the Lagrange multiplier method. However, since the constraining relations $x_i'(t) = g_i(t, \vec{x}, \vec{u}) \quad i = 1, 2$, hold for each t, multiplier functions

$\vec{\lambda}(t) = (\lambda_1(t), \lambda_2(t))$ must be utilized rather than a Lagrange multiplier value for each constraint. Furthermore, the adjoint variables play a significant role since they are chosen to remove the difficulty of determining the exact influence of modifying the control.

Another aspect about the adjoints is that the number of state equations must coincide with the number of adjoint equations. In addition, the adjoint is interpreted as the marginal valuation of the functional $J(\vec{u})$ with respect to the associated state variable at time t.

These properties of deterministic finite dimensional optimal control theory provided a foundation for the investigation of optimal control of systems governed by partial differential equations. The bridge to this new area of endeavor was built from the fundamental work of J. L. Lions [9]. Indeed, the ideas of Pontryagin's maximum principle are used to aid in characterizing an optimal control through an optimality system, which involves the state system coupled with the adjoint system. Explicitly, there are several connections between finite and infinite dimensional optimal control theory. Given the optimal controls and the corresponding state variables, there exist adjoint variables that satisfy systems in which the source terms of the adjoint partial differential equations equal the partial derivative of the integrand of the objective functional with respect to the state variables. In the case that the state equation is parabolic, then the adjoint equation is a backward parabolic equation and a transversality condition occurs at the final time. Even

though other similarities occur, there are no full generalizations of Pontryagin's maximum principle to nonlinear partial differential equations. For some specialized results, see [8].

Before the technique of solving these optimal control problems is discussed, the setting of the problem is illustrated as follows [9].

i) A control f belongs to some set U, the set of admissible controls.

ii) The state $u(f)$ of the system to be controlled, depending on f, is the solution of the equation

$$\mathcal{L}u = g(u, f)$$

where $\mathcal{L}$ is a known operator specifying the system to be controlled and g is a continuous function of u and f.

iii) The "objective functional" $J(f)$ is defined in terms of $u(f)$, f, or a combination of both and is to be maximized or minimized.

The definitive goal is to find $f^* \in U$ such that

$$J(f^*) = \inf J(f),$$

$$\text{or } J(f^*) = \sup J(f).$$

We outline the solution technique. Initially, possibly via an iteration scheme, existence and uniqueness of the solutions to the state system is proven. Through a maximizing or minimizing sequence argument, the existence of optimal control is obtained. The next step involves the derivation of the optimality system which

consists of the state system and the adjoint system coupled with the relationship between f^* and the state u and the adjoint p. To derive necessary conditions of the optimality system, the objective functional is differentiated with respect to the control. Explicitly, for $f, h \in U$, take the Gateaux derivative of J with respect to f in the direction h, i.e., $\lim_{\epsilon \to 0} \frac{J(f+\epsilon h)-J(f)}{\epsilon}$. Let this limit be denoted as $\frac{\partial J}{\partial f}(f; h)$ where h denotes the direction.

However, the objective functional may contain the state variable, u. Therefore, u must also be differentiated with respect to the control. The difference quotient $\frac{u(f+\epsilon h)-u(f)}{\epsilon}$ is shown to converge to ψ. A priori estimates are needed for existence of the state system and for convergence of this difference quotient to ψ. The "derivative" function ψ solves a linearized version of the state equation. Then the adjoint of the operator of the linearized ψ equation is found. Therefore to compute f^* from $\frac{\partial J}{\partial f}$, the adjoint system is introduced with a transversality condition and other appropriate boundary conditions. Moreover, upon analyzing the objective functional and utilizing the relationships between the state and adjoint equations, an explicit representation of an optimal control f is determined via standard optimality techniques. This representation characterizes the optimal control in terms of the state and adjoint variables. Combining this information produces the optimality system which consists of the state and adjoint equations associated with the optimal control through this representation. In the parabolic system case, uniqueness for the optimality system, which characterizes the unique optimal con-

trol, holds when the final time T is sufficiently small. Since the state system is a forward parabolic system and the adjoint system is a backward parabolic system, this uniqueness only holds for small time.

As a formal example, consider the solution u of the following problem in Ω, a domain in $\mathbb{R}^n$ where f represents a control:

$$
\begin{aligned}
u_t - \Delta u &= f - u^2 \quad \text{in } Q = \Omega \times (0,T) \\
u(x,0) &= u_0(x), \quad x \in \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T),
\end{aligned}
\tag{1.2}
$$

where $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ where ν is the outward normal vector on $\partial\Omega$.

For $f \in U$ where $U = \{f \in L^\infty(Q) | 0 \le f \le M\}$, the set of admissible controls, the cost functional is

$$
J(f) = \frac{1}{2}\int_Q (u-z)^2 \, dxdt + \frac{1}{2}\int_Q f^2 \, dxdt \tag{1.3}
$$

where z is the target function. The goal is to find f^* that minimizes the cost functional. After proving existence and uniqueness of the state system in $L^2\left((0,T); H^1(\Omega)\right)$, then existence of an optimal control through a minimizing sequence argument is found.

To derive the optimality system , first analyze $\frac{u(f+\epsilon h)-u(f)}{\epsilon}$ where $\epsilon \to 0^+$. Let $u^\epsilon = u(f+\epsilon h)$. Then system (1.2) becomes

$$
\begin{aligned}
\left(\frac{u^\epsilon - u}{\epsilon}\right)_t - \frac{1}{\epsilon}\Delta(u^\epsilon - u) &= h - \left(\frac{(u^\epsilon + u)(u^\epsilon - u)}{\epsilon}\right) \\
\left(\frac{u^\epsilon - u}{\epsilon}\right)(x,0) &= 0, \quad x \in \Omega \\
\frac{\partial\left(\frac{u^\epsilon - u}{\epsilon}\right)}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T).
\end{aligned}
$$

Assuming one shows that $\frac{u^\epsilon - u}{\epsilon} \rightharpoonup \psi$ in $L^2\left((0,T); H^1(\Omega)\right)$, then ψ satisfies the linearized system

$$\begin{aligned} \psi_t - \Delta\psi + 2u\psi &= h \quad \text{in } Q \\ \psi(x,0) &= 0, \quad x \in \Omega \\ \frac{\partial \psi}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{1.4}$$

Next, the adjoint of the linearized ψ operator is determined through integration by parts.

$$\begin{aligned} \int_Q p(\psi_t - \Delta\psi + 2u\psi)\,dxdt = &-\int_Q p_t\psi\,dxdt + \int_{\Omega\times\{T\}} p\psi(1)\,dx + \int_{\Omega\times\{0\}} p\psi(-1)\,dx \\ &+ 2\int_Q up\psi\,dxdt + \int_Q \nabla p \nabla\psi\,dxdt - \int_{\partial\Omega\times(0,T)} p\frac{\partial\psi}{\partial\nu}\,ds \\ = &-\int_Q p_t\psi\,dxdt + \int_{\Omega\times\{0\}} p\psi\,dx - \int_Q \psi\Delta p\,dxdt \\ &+ \int_{\partial\Omega\times(0,T)} \psi\frac{\partial p}{\partial\nu}\,ds + 2\int_Q up\psi\,dxdt. \end{aligned}$$

Similar to the ordinary differential equation case, the adjoint variable p is chosen to eliminate the difficulty in determining the exact influence of the control variable on the state variable. Also the source of the adjoint partial differential equation equals the derivative of the cost functional integrand with respect to the state. Hence,

$$\begin{aligned} -p_t - \Delta p + 2up &= u - z \quad \text{in } Q \\ p(x,T) &= 0, \quad x \in \Omega \\ \frac{\partial p}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{1.5}$$

where the transversality condition on the adjoint variable is at $t = T$. Now to determine the representation for f^*, suppose f^* is an optimal control with corresponding

solution u^* and consider another control $f^* + \epsilon h$ with associated solution u^ϵ. Since the minimum of the cost functional occurs at f^*, then

$$
\begin{aligned}
0 \leq & \lim_{\epsilon \to 0+} \frac{J(f^* + \epsilon h) - J(f^*)}{\epsilon} \\
= & \lim_{\epsilon \to 0+} \left[\frac{1}{2\epsilon} \int_Q \left[(u^\epsilon - z)^2 - (u^* - z)^2 \right] \, dxdt + \frac{1}{2\epsilon} \int_Q \left[(f^* + \epsilon h)^2 - (f^*)^2 \right] \, dxdt \right] \\
= & \lim_{\epsilon \to 0+} \left[\int_Q \left(\frac{u^\epsilon - u^*}{\epsilon} \right) \left(\frac{u_\epsilon - z + u^* - z}{2} \right) \, dxdt \right] + \lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_Q \left(2f^* \epsilon h + \epsilon^2 h^2 \right) \, dxdt \\
= & \int_Q \psi(u^* - z) \, dxdt + \int_Q f^* h \, dxdt \\
= & \int_Q \left[\psi(-p_t - \Delta p + 2up) + f^* h \right] \, dxdt \\
= & \int_Q \left[p(\psi_t - \Delta \psi + 2u\psi) + f^* h \right] \, dxdt \\
= & \int_Q h(f^* + p) \, dxdt.
\end{aligned}
$$

To determine f^*, standard optimality techniques are used. Consider the following three cases.

i) On the set $\{(x,t) \in Q | f^*(x,t) = 0\}$, one can choose nonnegative variations h with support on this set. Hence, $f^* + p \geq 0$ or $0 = f^* \geq -p$.

ii) On the set $\{(x,t) \in Q | 0 < f^*(x,t) < M\}$, one can choose h with arbitrary sign and support on this set. This implies that $f^* + p = 0$ or $f^* = -p$.

iii) On the set $\{(x,t) \in Q | f^*(x,t) = M\}$, one chooses nonpositive variations h with support on this set. Then $f^* + p \leq 0$ or $M = f^* \leq -p$.

Combining these three cases $f^* = \min((-p)^+, M)$, where $p^+ = 0$ if $p < 0$ or $p^+ = p$ if $p \geq 0$.

Using the relationship between the optimal control and the associated adjoint

variable, the optimality system is as follows (formally):

$$\left.\begin{aligned} u_t - \Delta u &= \min\big((-p)^+, M\big) - u^2 \\ -p_t - \Delta p + 2up &= u - z \end{aligned}\right\} \text{ in } Q$$

$$u(x,0) = u_0(x) \quad , \quad p(x,T) = 0 \quad , \quad x \in \Omega \tag{1.6}$$

$$\frac{\partial u}{\partial \nu} = 0 = \frac{\partial p}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T).$$

In Part II, we consider optimal control of harvesting in a predator-prey parabolic system. Given controls f_1, f_2 representing harvesting a proportion of the populations, the corresponding prey and predator state variables u and v satisfy the state system:

$$\begin{aligned} L_1 u &= u(a_1 - d_1 u) - c_1 uv - f_1 u \qquad \text{in } Q = \Omega \times (0,T) \\ L_2 v &= v(a_2 - d_2 v) + c_2 uv - f_2 v \end{aligned} \tag{1.7}$$

with suitable initial data and boundary conditions.

We use the convention

$$L_k u \equiv u_t - \sum_{i,j=1}^{n} \left(a_{ij}^k u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} b_i^k u_{x_i}.$$

The system (1.7) has been used to model capelin as prey and cod as predator [1]. The solutions u and v represent population densities of the prey and predator species. The coefficients of the standard Lotka-Volterra growth terms are a_1, a_2, d_1, d_2. The interaction terms, $-c_1 uv$ and $c_2 uv$, signify decay for the prey population and growth for the predator population respectively.

The objective is to maximize J with respect to controls f_1 and f_2 where

$$J(f_1\ f_2) = \int_Q \left\{K_1 u f_1 + K_2 v f_2 - M_1 f_1^2 - M_2 f_2^2\right\} dxdt.$$

We establish the existence of an optimal control pair and then characterize $(f_1^*,\ f_2^*)$ such that

$$J(f_1^*,\ f_2^*) = \max_{(f_1,\ f_2)\in A} J(f_1,\ f_2)$$

where A is our control set. Leung [7] considers optimal control of harvesting of two interacting populations. In [3], Cañada, Gámez, and Montero analyze the profitability of harvesting species in a diffusive Lotka-Volterra elliptic system. The objective functional's positivity depends on the negativity of the principle eigenvalues of their elliptic operators.

The solution technique, as previously mentioned, involves establishing the existence of the solution to the above system (1.7). This involves a method of monotone iterations which involve utilizing maximum or comparison principle arguments. The existence of an optimal control pair is shown via a maximizing sequence argument and a priori estimates. The derivation of the optimality system, which consists of the state system coupled with an adjoint system, emanates from taking the partial derivative of the objective functional J with respect to the controls. The optimality system is a nonlinear partial differential equation system and has opposite orientations due to the forward and backward in time nature of the state and adjoint equations. In addition, we analyze the positivity of the payoff functional under certain restrictions on the bounds of the optimal controls.

In Part III, we consider flow of chemicals or sediments along a major river flow. We assume that the entire watershed can be visualized as point sources flowing

into the major river from these smaller branches. Then the flow of contaminants is modeled in the advection-diffusion problem:

$$u_t - (\beta u_x)_x + \alpha u_x = -D(x)u + \sum_{i=1}^{k} \delta(x - x_i) S_i(t) \quad \text{in } Q$$

$$u(x, 0) = u_0(x), \quad x \in (a, b)$$

$$u_x(b, t) = 0, \quad u(a, t) = u_0 \quad \text{for } 0 < t < T,$$

where $Q = (a, b) \times (0, T)$.

The function u represents the concentration of contaminant at location x at a given time t. The $\delta(x - x_i)S_i(t)$ is a source term at $x = x_i$ with $\delta(x - x_i)$ being the Dirac delta measure at x_i and $S_i(t)$ being the level of contaminant loading representing one component of our control.

We desire to maximize the following functional

$$J(\vec{S}) = \int_0^T e^{-rt} \sum_{i=1}^{k} R_i(S_i(t))\, dt - \int_0^T \int_a^b e^{-rt} f(u(x, t))\, dxdt$$

which represents the sum of the agricultural revenues less the environmental costs of contaminants. Again we characterize $\vec{S}^*$ such that $J(\vec{S}^*) = \max_{\vec{S} \in A} J(\vec{S})$ where A is our control set.

Our solution technique is similar to Part II where we analyze the objective functional with a Michaelis-Menton and a quadratic revenue function. Also, care must be taken when estimating terms involving the Dirac delta measure, [10].

In Part IV, we examine an ordinary differential system modeling the interaction of the HIV virus and the immune system of the human body. The optimal

control represents a percentage of effect the chemotherapy has on the interaction of the $CD4^+$ T cells with the virus. We investigate the optimal scheme for treatment, i.e., when and how treatment should be initiated given that treatment can only be continued for a two year window of time.

The state system is governed by the following

$$\begin{aligned}
\frac{dT}{dt} &= \frac{s}{1+V} - \mu_T T + rT\left(1 - \frac{T+T^*+T^{**}}{T_{MAX}}\right) - u(t)K_1 VT \\
\frac{dT^*}{dt} &= u(t)K_1 VT - \mu_T T^* - K_2 T^* \\
\frac{dT^{**}}{dt} &= K_2 T^* - \mu_b T^{**} \\
\frac{dV}{dt} &= N\mu_b T^{**} - K_1 VT - \mu_v V
\end{aligned}$$

with suitable initial conditions, where $u(t)$ is our control. T, T^*, T^{**}, and V represent concentrations of uninfected $CD4^+$ T cells, latently infected T cells, actively infected T cells, and free virus.

Our goal is to maximize the functional

$$J(u) = \int_{t_0}^{t_1} \left[B_1 T(t) - B_2 V(t) - B_3(1-u) - \frac{1}{2}B_4(1-u)^2\right] dt.$$

We maximize the benefit based on the T cell count, minimize the free virus, and minimize the systemic cost based on the percentage of chemotherapy given. Again our goal is to characterize u^* such that $J(u^*) = \max_{0 \le u(t) \le 1} J(u)$.

The solution technique requires some work involving the existence of the optimal control. For example, the concavity of the integrand of our functional on U

is needed [4]. Then we utilize Pontryagin's Maximum Principle on our constrained problem to obtain our optimality system and an explicit characterization of our optimal control [5]. As in the other two problems, the uniqueness of the optimality system holds for sufficiently small final time due to being a boundary value problem. Kirschner, Lenhart, and Serbin [6], analyzed a similar system. However, they controlled the effect chemotherapy has on viral production. Butler, Kirschner, and Lenhart [2] analyzed a similar system except they had no differential equation for the latently infected T cells. Both references analyzed treatment numerically.

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Part II

Parabolic System for a Predator-Prey Model

1. Introduction

We consider optimal control of a parabolic system with Neumann boundary conditions. The parabolic system is posed in a bounded, multi-dimensional domain Q. Solutions of the system represent population densities of the prey and the predator species. The system has Lotka-Volterra type growth terms with local interaction terms representing the predator-prey situation. The controls are a proportion of the species populations to be harvested.

Given controls f_1, f_2, the corresponding prey and predator state variables, $u = u(f_1, f_2)$ and $v = v(f_1, f_2)$ satisfy the state system:

$$
\begin{aligned}
&L_1 u = u(a_1 - d_1 u) - c_1 uv - f_1 u \qquad \text{in } Q = \Omega \times (0,T)\\
&L_2 v = v(a_2 - d_2 v) + c_2 uv - f_2 v
\end{aligned}
\tag{2.1.1}
$$

$$
u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \text{ for } x \in \Omega \subset \mathbb{R}^n
$$

$$
\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T),
$$

where we use the notation

$$
\begin{aligned}
L_1 u &\equiv u_t - \sum_{i,j=1}^{n} (a^1_{ij} u_{x_j})_{x_i} + \sum_{i=1}^{n} b^1_i u_{x_i}\\
L_2 v &\equiv v_t - \sum_{i,j=1}^{n} (a^2_{ij} v_{x_j})_{x_i} + \sum_{i=1}^{n} b^2_i v_{x_i}
\end{aligned}
$$

and $\nu_i = \sum_{j=1}^{n} a_{ij} n_j$ for the ith component of the conormal vector.

The functions f_1 and f_2 are controls that represent harvesting a proportion of the population. The coefficients of the standard Lotka-Volterra (logistic) growth terms are a_1, a_2, d_1, d_2, with a_1, a_2 denoting growth and d_1, d_2 signifying crowding. The terms c_1uv, c_2uv represent interaction effects. The $-c_1uv$ term is a decay term for the prey population; whereas, the c_2uv term is a growth term for the predator population.

We define the class of admissible controls,

$$A = A(\Gamma_1,\ \Gamma_2) \equiv \{(f_1,\ f_2) \mid 0 \le f_i \le \Gamma_i \text{ a.e. in } Q,\ i = 1,2\}.$$

Our payoff functional is

$$J(f_1\ f_2) = \int_Q \left\{K_1uf_1 + K_2vf_2 - M_1f_1^2 - M_2f_2^2\right\} dxdt \qquad (2.1.2)$$

where K_1uf_1, K_2vf_2 represent the revenue of harvesting and $M_1f_1^2$, $M_2f_2^2$ denote the cost of the controls. This functional combines the controls and their effects on the populations. We desire to maximize the functional over the admissible class of controls; i.e. to characterize f_1^*, f_2^* such that

$$J(f_1^*,\ f_2^*) = \max_{(f_1,\ f_2)\in A} J(f_1,\ f_2).$$

In this part, we develop a control model to understand the dynamics in a diffusive Lotka-Volterra equation. Because of the dispersive nature of the predator and prey species, the population dynamics of these species is dependent on space and time. This dependence is evident by the use of a parabolic diffusive Lotka-Volterra PDE. However, we assume that a given species does not diffuse across

the boundary, due to possibly unsuitable environmental conditions outside of our region. The functions $u_0(x)$ and $v_0(x)$ give the initial density distributions of the prey and predator species inside our spatial area.

When thinking of harvesting in a predator-prey system, one can think of the simple (fish) example with capelin as the prey and cod as the predator (Akenhead et. al. [1]). See Mesterton-Gibbons [11] for an optimal control problem with combined harvesting in a predator-prey system in the ordinary differential equations case. Our paper treats the predator-prey situation with spatial diffusion, yielding a PDE system.

For background in control of PDE's, see Lions [10]. In addition Waltman [15] and Murray [12] provide insight into competitive and predator prey models. Leung [6,7] considers optimal control of the harvesting of two interacting populations. His species concentrations satisfy a predator-prey system, and they are assumed to be in steady state under diffusion. Goh, Leitmann, and Vincent [4] study the predator-prey system in the ordinary differential equation case with removal of pest prey species by insecticides. In addition, Cañada, Gámez, and Montero [2] analyze the profitability of harvesting species modeled by a diffusive Lotka-Volterra elliptic system. They prove that the payoff functional is positive under certain conditions. Specifically, the payoff functional is positive if the principal eigenvalue of the elliptic operator modeling the species is negative.

Section 2 gives the existence of solutions of the system (2.1.1) and existence

of an optimal control. In section 3, an optimal control is characterized in terms of the optimality system, which is the state system coupled with the adjoint system. We derive the optimality system by differentiating the payoff functional with respect to the control and evaluate the result at an optimal control. Uniqueness of solution to the optimality system is established. Hence, we obtain a precise characterization of the unique optimal control.

2. Existence of an Optimal Control

The following assumptions are made throughout this part:

Ω is a smooth bounded domain in $\mathbb{R}^n$.

$u_0(x),\ v_0(x) \in L^\infty(\Omega)$.

$0 < u_0(x) < B,\ 0 < v_0(x) < B$ for some $B \in \mathbb{R}$.

$a_i,\ c_i,\ d_i,\ f_i \in L^\infty(Q),\quad i = 1, 2.$

$a_{ij}^k \in C^2(\overline{Q}),\ a_{ij}^k = a_{ji}^k,\ b_i^k \in C^1(\overline{Q})$ for $k = 1, 2$, and $i, j = 1, \ldots, n$.

$\vec{b} \cdot \vec{n} \geq 0,\ \vec{b} = (b_i)\ \ i = 1, \ldots, n$, where $\vec{n}$ is the outward unit normal on

$\partial\Omega \times (0, T)$.

$\sum_{i,j=1}^n a_{ij}^k \xi_i \xi_j \geq \theta \sum_{i=1}^n \xi_i^2$, $k = 1, 2$, where $\theta > 0$ for all $\xi \in \mathbb{R}^n$.

This ellipticity condition guarantees that the conormal direction ν is outward. The underlying state space for system (2.1.1)

$$V = L^2((0, T); H^1(\Omega)).$$

Using repeated indices summation convention, we define the bilinear forms:

$$a^k(t,u,\phi) = \int_\Omega a_{ij}^k u_{x_j} \phi_{x_i}\, dx + \int_\Omega b_i^k u_{x_i} \phi\, dx$$

for $u, \phi \in V$ and $k = 1, 2$. We are interested in weak solutions, $u, v \in V$, in the following sense:

$$\begin{aligned}
\int_0^T \langle u_t, \phi\rangle\, dt + \int_0^T a^1(t,u,\phi)\, dt &= -\int_Q c_1 uv\phi\, dxdt - \int_Q d_1 u^2 \phi\, dxdt \\
&\quad + \int_Q (a_1 - f_1) u\phi\, dxdt \\
\int_0^T \langle v_t, \psi\rangle\, dt + \int_0^T a^2(t,v,\psi)\, dt &= \int_Q c_2 uv\psi\, dxdt - \int_Q d_2 v^2 \psi\, dxdt \\
&\quad + \int_Q (a_2 - f_2) v\psi\, dxdt
\end{aligned}$$

for all $\phi, \psi \in V$ where the inner product $\langle\ ,\ \rangle$ is the duality between $(H^1(\Omega))^*$ and $H^1(\Omega)$.

In the proofs below, we refer to the following inequality as "Cauchy's inequality": $ab \le \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ for $a,\ b \ge 0,\ \epsilon > 0$. First, we prove an existence and uniqueness result for the state system (2.1.1). Also, we note for convenience that we omit the differentials of the integrals in the proofs.

Theorem 2.1

Given $(f_1, f_2) \in A$, there exists a unique solution (u, v) in $V \times V$ solving system (2.1.1).

Proof

We will prove existence of solutions of (2.1.1) by an iteration method. We will construct supersolutions for the u, v iterates. Let u^1 and v^2 be solutions to the following:

$$\begin{aligned} L_1 u^1 &= a_1 u^1 \qquad \text{in } Q \\ u^1(x,0) &= u_0(x), \qquad x \in \Omega \\ \frac{\partial u^1}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{2.2.1}$$

and

$$\begin{aligned} L_2 v^2 &= a_2 v^2 + c_2 u^1 v^2 \qquad \text{in } Q \\ v^2(x,0) &= v_0(x), \qquad x \in \Omega \\ \frac{\partial v^2}{\partial \nu} &= 0 \qquad \text{in } \partial\Omega \times (0,T). \end{aligned} \tag{2.2.2}$$

Note u^1 and v^2 are our supersolutions. By the parabolic maximum principle [13], u^1 and v^2 are L^∞ bounded in Q.

Also define $u^0 = 0$ and $v^1 = 0$.

Let u^k, v^k be solutions to the following iteration scheme, where v^k is found first and used in the u^k equation.

$$\begin{aligned} L_1 u^k + R u^k &= f(u^{k-2}, v^k) \qquad \text{in } Q \\ u^k(x,0) &= u_0(x), \qquad x \in \Omega \\ \frac{\partial u^k}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{2.2.3}$$

$$L_2 v^k + R v^k = g(u^{k-1}, v^{k-2}) \qquad \text{in } Q$$

$$v^k(x,0) = v_0(x), \qquad x \in \Omega$$

$$\frac{\partial v^k}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{2.2.4}$$

where $f(u^{k-2}, v^k) = Ru^{k-2} + a_1 u^{k-2} - d_1(u^{k-2})^2 - c_1 u^{k-2} v^k - f_1 u^{k-2}$ and $g(u^{k-1}, v^{k-2}) = Rv^{k-2} + a_2 v^{k-2} - d_2(v^{k-2})^2 + c_2 u^{k-1} v^{k-2} - f_2 v^{k-2}$. R is chosen such that the right-hand side of the u^k equation is increasing in u^{k-2} for iterates in the range $0 \leq u^k \leq U$ and the right-hand side of the v^k equation is increasing in v^{k-2} for iterates in the range $0 \leq v^k \leq V$. The right-hand side of the u^k equation decreases in v^k, and the right-hand side of the v^k equation increases in u^{k-1}. See Leung [8] for further information on iteration schemes.

Comparing right-hand sides of (2.2.3) and (2.2.4) [6],[7], we obtain the monotone convergence of odd/even iterates. There exist u,$\overline{u}$,v,$\overline{v}$ such that

$$u^{2k} \nearrow \overline{u}, \; u^{2k+1} \searrow u, \; v^{2k} \searrow \overline{v}, \; v^{2k+1} \nearrow v \text{ pointwise.}$$

Since the right-hand sides of (2.2.3) and (2.2.4) are bounded on $L^\infty(Q)$, then u^{2k}, u^{2k+1}, v^{2k}, v^{2k+1} are uniformly bounded in V. Hence,

$$u^{2k} \rightharpoonup \overline{u}, \; u^{2k+1} \rightharpoonup u, \; v^{2k} \rightharpoonup \overline{v}, \; v^{2k+1} \rightharpoonup v \text{ in } V.$$

Using the bound in our solution space V on u^{2k}, u^{2k+1}, v^{2k}, v^{2k+1} coupled with the PDEs gives

$$\|u_t^{2k}\|, \; \|u_t^{2k+1}\|, \; \|v_t^{2k}\|, \; \|v_t^{2k+1}\| \leq C \text{ in } L^2((0,T), \; (H^1(\Omega))^*).$$

Hence, $u_t^{2k} \rightharpoonup \overline{u}_t$, $u_t^{2k+1} \rightharpoonup u_t$, $v_t^{2k} \rightharpoonup \overline{v}_t$, $v_t^{2k+1} \rightharpoonup v_t$ in $L^2((0,T),\ (H^1(\Omega))^*)$.

Applying a compactness result ([9], Proposition 4.2, Chapter 4) we have

$$u^{2k} \to \overline{u},\ u^{2k+1} \to u,\ v^{2k} \to \overline{v},\ v^{2k+1} \to v \text{ in } L^2(Q).$$

This strong convergence is needed in the quadratic terms.

Passing to the limit in the u^{2k}, u^{2k+1}, v^{2k}, v^{2k+1} PDEs, we obtain

$$\begin{cases} L_1\overline{u} = a_1\overline{u} - d_1(\overline{u})^2 - c_1\overline{uv} - f_1\overline{u} & \text{in } Q \\ \overline{u}(x,0) = u_0(x), & x \in \Omega \\ \frac{\partial \overline{u}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{2.2.5}$$

$$\begin{cases} L_1 u = a_1 u - d_1 u^2 - c_1 uv - f_1 u & \text{in } Q \\ u(x,0) = u_0(x), & x \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{2.2.6}$$

$$\begin{cases} L_2\overline{v} = a_2\overline{v} - d_2(\overline{v})^2 + c_2\overline{uv} - f_2\overline{v} & \text{in } Q \\ \overline{v}(x,0) = v_0(x), & x \in \Omega \\ \frac{\partial \overline{v}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{2.2.7}$$

$$\begin{cases} L_2 v = a_2 v - d_2 v^2 + c_2 uv - f_2 v & \text{in } Q \\ v(x,0) = v_0(x), & x \in \Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.2.8}$$

To show $\overline{u} = u$, $v = \overline{v}$, let $u = e^{\lambda t}w$, $\overline{u} = e^{\lambda t}\overline{w}$, $v = e^{\lambda t}z$, $\overline{v} = e^{\lambda t}\overline{z}$ where $\lambda > 0$ is to be chosen.

We give as an example the transformed PDE problem (2.2.6) for $u = e^{\lambda t}w$.

$$L_1 w + \lambda w = a_1 w - d_1 e^{\lambda t} w^2 - c_1 e^{\lambda t} wz - f_1 w \quad \text{in } Q$$

$$w(x,0) = u_0(x), \quad x \in \Omega \tag{2.2.9}$$

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T).$$

We consider the weak formulation of the $w-\overline{w}$ problem and the $z-\overline{z}$ problem. Then we add these two weak formulations together utilizing the test functions $w-\overline{w}$ and $z-\overline{z}$ respectively. We have

$$
\begin{aligned}
&\int_Q [(w-\overline{w})_t(w-\overline{w}) + (z-\overline{z})_t(z-\overline{z})] + \lambda \int_Q [(w-\overline{w})^2 + (z-\overline{z})^2] \\
&\quad + \int_Q \left[a^1_{ij}(w-\overline{w})_{x_i}(w-\overline{w})_{x_j} + a^2_{ij}(z-\overline{z})_{x_i}(z-\overline{z})_{x_j}\right] \\
&\quad = -\int_Q \left[b^1_i(w-\overline{w})_{x_i}(w-\overline{w}) + b^2_i(z-\overline{z})_{x_i}(z-\overline{z})\right] + \int_Q (a_1 - f_1)(w-\overline{w})^2 \\
&\quad + \int_Q (a_2 - f_2)(z-\overline{z})^2 - \int_Q c_1 e^{\lambda t}(wz - \overline{wz})(w-\overline{w}) \\
&\quad - \int_Q d_1 e^{\lambda t}(w^2 - \overline{w}^2)(w-\overline{w}) + \int_Q c_2 e^{\lambda t}(wz - \overline{wz})(z-\overline{z}) \\
&\quad - \int_Q d_2 e^{\lambda t}(z^2 - \overline{z}^2)(z-\overline{z}).
\end{aligned}
\tag{2.2.9a}
$$

We use that

$$
\begin{aligned}
wz - \overline{wz} &= wz - \overline{w}z + \overline{w}z - \overline{wz} \\
&= z(w-\overline{w}) + \overline{w}(z-\overline{z}).
\end{aligned}
$$

Also we recognize that

$$
\begin{aligned}
\int_Q (w-\overline{w})_t(w-\overline{w}) &= \frac{1}{2}\int_Q ((w-\overline{w})^2)_t \\
&= \frac{1}{2}\int_\Omega (w(x,T) - \overline{w}(x,T))^2
\end{aligned}
\tag{2.2.9b}
$$

utilizing $w(x,0) = u_0(x) = \overline{w}(x,0)$. We also use the ellipticity condition

$$
\sum_{i,j=1}^n a^k_{ij}\xi_i\xi_j \geq \theta \sum_{i=1}^n \xi_i^2 \quad \text{for } k = 1, 2.
$$

We then use (2.2.9b) and ellipticity condition in (2.2.9a) to obtain the following

$$\begin{aligned}
&\frac{1}{2}\int_\Omega ([w(x,T)-\overline{w}(x,T)]^2+[z(x,T)-\overline{z}(x,T)]^2) \\
&\qquad +\theta\int_Q (|\nabla(w-\overline{w})|^2+|\nabla(z-\overline{z})|^2)+\lambda\int_Q [(w-\overline{w})^2+(z-\overline{z})^2] \\
&\le e^{\lambda T}\int_Q ((w-\overline{w})^2[-c_1 z-d_1(\overline{w}+w)]+(z-\overline{z})^2[c_2\overline{w}-d_2(\overline{z}+z)]) \\
&\qquad +e^{\lambda T}\int_Q (w-\overline{w})(z-\overline{z})\,[-c_1\overline{w}+c_2 z]-\int_Q\sum_{i=1}^n b_i^1(w-\overline{w})_{x_i}(w-\overline{w}) \\
&\qquad -\int_Q\sum_{i=1}^n b_i^2(z-\overline{z})_{x_i}(z-\overline{z})+\int_Q a_1(w-\overline{w})^2+\int_Q a_2(z-\overline{z})^2.
\end{aligned}$$

Further estimation utilizing Cauchy's inequality produces the following

$$\begin{aligned}
&\frac{1}{2}\int_\Omega ([w(x,T)-\overline{w}(x,T)]^2+[z(x,T)-\overline{z}(x,T)]^2) \\
&\qquad +\frac{3\theta}{4}\int_Q (|\nabla(w-\overline{w})|^2+|\nabla(z-\overline{z})|^2) \\
&\qquad +(\lambda-C)\int_Q (w-\overline{w})^2+(z-\overline{z})^2\le 0
\end{aligned} \tag{2.2.10}$$

where C depends on the coefficients, T, and $\|z\|_{L^\infty},\ \|w\|_{L^\infty}$.

If $\lambda > C$, then the inequality (2.2.10) holds if and only if

$$w=\overline{w} \text{ and } z=\overline{z} \text{ a.e. in } Q.$$

Therefore $u=\overline{u}$ and $v=\overline{v}$ a.e. in Q, and u,v solve (2.1.1). The uniqueness of u,v follow as in the above argument. $\square$

Next, a maximizing sequence argument is used to prove the existence of an optimal control.

Theorem 2.2

There exists a pair of optimal controls in A that maximizes the functional $J(f_1, f_2)$ in (2.1.2) subject to our problem (2.1.1).

Proof

Since the state variables and the controls are uniformly bounded, $\sup\{J(f_1, f_2) | (f_1, f_2) \in A\} < \infty$. Then there exists a maximizing sequence $(f_1^n, f_2^n) \in A$ such that

$$\lim_{n\to\infty} J(f_1^n, f_2^n) = \sup\{J(f_1, f_2) | (f_1, f_2) \in A\}.$$

By the existence and uniqueness of solutions to the state system (2.1.1), we define

$$u^n = u(f_1^n, f_2^n) \text{ and } v^n = v(f_1^n, f_2^n) \text{ for each } n.$$

We consider for $0 \le t \le T$,

$$\begin{aligned}
\int_{\Omega\times(0,t)} u_t^n u^n &+ \int_{\Omega\times(0,t)} a_{ij}^1 u_{x_i}^n u_{x_j}^n + \int_{\Omega\times(0,t)} b_i^1 u_{x_i}^n u^n - \int_{\Omega\times(0,t)} (a_1 - f_1^n)(u^n)^2 \\
&+ \int_{\Omega\times(0,t)} v_t^n v^n + \int_{\Omega\times(0,t)} a_{ij}^2 v_{x_i}^n v_{x_j}^n + \int_{\Omega\times(0,t)} b_i^2 v_{x_i}^n v^n \\
&- \int_{\Omega\times(0,t)} (a_2 - f_2^n)(v^n)^2 \\
&= - \int_{\Omega\times(0,t)} c_1 (u^n)^2 v^n - \int_{\Omega\times(0,t)} d_1 (u^n)^2 u^n + \int_{\Omega\times(0,t)} c_2 u^n (v^n)^2 \\
&- \int_{\Omega\times(0,t)} d_2 (v^n)^2 v^n. \qquad (2.2.11)
\end{aligned}$$

Using Cauchy's inequality, the ellipticity condition, and that the state vari-

ables are uniformly bounded, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega}\left([u^n(x,t)]^2+[v^n(x,t)]^2\right)+\frac{\theta}{2}\int_{\Omega\times(0,t)}\left(|\nabla u^n|^2+|\nabla v^n|^2\right)\\
&\qquad\le\frac{1}{2}\int_{\Omega}\left[u_0^2(x)+v_0^2(x)\right]+C\int_{\Omega\times(0,t)}\left[(u^n)^2+(v^n)^2\right]
\end{aligned}
\tag{2.2.12}
$$

where C depends on the coefficients in (2.1.1) and n.

In particular,

$$
\begin{aligned}
\int_{\Omega}\left[(u^n(x,t))^2+(v^n(x,t))^2\right]\le\int_{\Omega}&\left[u_0^2(x)+v_0^2(x)\right]\\
&+2C\int_{\Omega\times(0,t)}\left[(u^n)^2+(v^n)^2\right].
\end{aligned}
\tag{2.2.13}
$$

Applying Gronwall's inequality to (2.2.13),

$$
\int_{\Omega}\left[(u^n(x,t))^2+(v^n(x,t))^2\right]\le e^{2CT}\int_{\Omega}\left[u_0^2(x)+v_0^2(x)\right].
$$

Using this estimate with (2.2.11),

$$
\begin{aligned}
&\sup_{0\le t\le T}\left\{\int_{\Omega}\left[(u^n(x,t))^2+(v^n(x,t))^2\right]\right\}+\int_Q\left(|\nabla u^n|^2+|\nabla v^n|^2\right)\\
&\le e^{2CT}\int_{\Omega}\left[u_0^2(x)+v_0^2(x)\right].
\end{aligned}
\tag{2.2.14}
$$

On a subsequence, $f_i^n \rightharpoonup f_i^*$ for $i=1,2$ in $L^2(Q)$. As in the proof of the existence of solutions to the state system (2.1.1), we obtain

$$
u^n\to u,\quad v^n\to v\quad \text{strongly in } L^2(Q) \text{ and weakly in } V.
$$

Passing to the limit in the u^n, v^n PDE problems and using the convergences like in Theorem 2.1, we have that (u,v) is a weak solution associated with (f_1^*,f_2^*). We have

$$
J(f_1^*,f_2^*)\le\sup\{J(f_1^n,f_2^n)|(f_1^n,f_2^n)\in A\}.
$$

For the weakly convergent control sequences [3], we have

$$\int_Q (f_i^*)^2 \leq \underline{\lim}_{n\to\infty} \int_Q (f_i^n)^2 \quad i = 1, 2.$$

Since the payoff functional is upper semi-continuous with respect to weak convergences, then

$$\max_{(f_1, f_2)\in A} J(f_1, f_2) = J(f_1^*, f_2^*).$$

Therefore (f_1^*, f_2^*) is a pair of optimal controls that maximizes the payoff functional. $\square$

3. Derivation of the Optimality System

We now derive the optimality system which consists of the state system coupled with the adjoint system. We differentiate the payoff functional with respect to the controls to obtain necessary conditions for the optimality system. Since the payoff functional contains u and v, we must differentiate u and v with respect to the controls.

Theorem 3.1

The mapping $(f_1, f_2) \in A \to (u, v) \in V \times V$ is differentiable in the following

sense:

$$\frac{u(f_1+\varepsilon h, f_2)-u(f_1,f_2)}{\varepsilon} \rightharpoonup \psi_1 \quad \text{in } V$$
$$\frac{v(f_1+\varepsilon h, f_2)-v(f_1,f_2)}{\varepsilon} \rightharpoonup \psi_2$$
$$\frac{u(f_1, f_2+\varepsilon h)-u(f_1,f_2)}{\varepsilon} \rightharpoonup \phi_1$$
$$\frac{v(f_1, f_2+\varepsilon h)-v(f_1,f_2)}{\varepsilon} \rightharpoonup \phi_2$$

as $\varepsilon \to 0$ for any $(f_1, f_2) \in A$ and $h \in L^\infty(Q)$ such that $(f_1+\varepsilon h, f_2+\varepsilon h) \in A$ for ε small. Also ψ_1, ψ_2 satisfy

$$L_1\psi_1 = a_1\psi_1 - 2d_1\psi_1 u - c_1 u\psi_2 - c_1 v\psi_1 - f_1\psi_1 - h_1 u \quad \text{in } V$$

$$L_2\psi_2 = a_2\psi_2 - 2d_2\psi_2 v + c_2 u\psi_2 + c_2 v\psi_1 - f_2\psi_2$$

$$\psi_1(x,0) = 0 = \psi_2(x,0) \quad \text{for } x \in \Omega \tag{2.3.1}$$

$$\frac{\partial \psi_1}{\partial \nu} = 0 = \frac{\partial \psi_2}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T).$$

and ϕ_1, ϕ_2 satisfy

$$L_1\phi_1 = a_1\phi_1 - 2d_1\phi_1 u - c_1 u\phi_2 - c_1 v\phi_1 - f_1\phi_1 \quad \text{in } V$$

$$L_2\phi_2 = a_2\phi_2 - 2d_2\phi_2 v + c_2 u\phi_2 + c_2 v\phi_1 - f_2\phi_2 - h_2 v$$

$$\phi_1(x,0) = 0 = \phi_2(x,0) \quad \text{for } x \in \Omega \tag{2.3.2}$$

$$\frac{\partial \phi_1}{\partial \nu} = 0 = \frac{\partial \phi_2}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T).$$

Proof

Define $u^\varepsilon = u(f_1 + \varepsilon h, f_2)$, $v^\varepsilon = v(f_1 + \varepsilon h, f_2)$. Let $u^\varepsilon = e^{\lambda t} w^\varepsilon$, $u = e^{\lambda t} w$, $v^\varepsilon = e^{\lambda t} z^\varepsilon$, $v = e^{\lambda t} z$, where $\lambda > 0$ is to be chosen below.

Estimating from the PDE's satisfied by $\frac{w^\varepsilon - w}{\varepsilon}$, $\frac{z^\varepsilon - z}{\varepsilon}$, we obtain

$$\begin{aligned}
&\int_\Omega \left[\left(\frac{w^\varepsilon(x,T) - w(x,T)}{\varepsilon} \right)^2 + \left(\frac{z^\varepsilon(x,T) - z(x,T)}{\varepsilon} \right)^2 \right] \\
&\quad + \frac{\theta}{4} \int_Q \left[\left| \nabla \left(\frac{w^\varepsilon - w}{\varepsilon} \right) \right|^2 + \left| \nabla \left(\frac{z^\varepsilon - z}{\varepsilon} \right) \right|^2 \right] \\
&\quad + \lambda \int_Q \left[\left(\frac{w^\varepsilon - w}{\varepsilon} \right)^2 + \left(\frac{z^\varepsilon - z}{\varepsilon} \right)^2 \right] \\
&\leq C_1 e^{\lambda T} \int_Q \left[\left(\frac{w^\varepsilon - w}{\varepsilon} \right)^2 (1 + z^\varepsilon) + \left(\frac{z^\varepsilon - z}{\varepsilon} \right)^2 (1 + w) \right. \\
&\quad \left. + \left(\frac{w^\varepsilon - w}{\varepsilon} \right) \left(\frac{z^\varepsilon - z}{\varepsilon} \right) (w + z^\varepsilon) \right] \\
&\quad - \int_Q h w^\varepsilon \left(\frac{w^\varepsilon - w}{\varepsilon} \right)
\end{aligned}$$

where C_1 depends on the coefficients.

Continuing to estimate, using L^∞ bounds on w^ε, w, z^ε, we have

$$\begin{aligned}
&\int_\Omega \left[\left(\frac{w^\varepsilon(x,T) - w(x,T)}{\varepsilon} \right)^2 + \left(\frac{z^\varepsilon(x,T) - z(x,T)}{\varepsilon} \right)^2 \right] \\
&\quad + \frac{\theta}{4} \int_Q \left[\left| \nabla \left(\frac{w^\varepsilon - w}{\varepsilon} \right) \right|^2 + \left| \nabla \left(\frac{z^\varepsilon - z}{\varepsilon} \right) \right|^2 \right] \\
&\quad + (\lambda - C_2) \int_Q \left[\left(\frac{w^\varepsilon - w}{\varepsilon} \right)^2 + \left(\frac{z^\varepsilon - z}{\varepsilon} \right)^2 \right] \leq C_3 \int_Q h^2.
\end{aligned}$$

For $\lambda > C_2$, we conclude

$$\left\| \frac{w^\varepsilon - w}{\varepsilon} \right\|_V^2 + \left\| \frac{z^\varepsilon - z}{\varepsilon} \right\|_V^2 \leq C_4 \int_Q h^2.$$

This estimate justifies the convergence of w, z quotients, and hence

$$\frac{u^{\varepsilon}-u}{\varepsilon} \rightharpoonup \psi_1 \text{ and } \frac{v^{\varepsilon}-v}{\varepsilon} \rightharpoonup \psi_2 \text{ in } V.$$

Similar to Theorem 2.1,

$$\left(\frac{u^{\varepsilon}-u}{\varepsilon}\right)_t \rightharpoonup (\psi_1)_t \text{ and } \left(\frac{v^{\varepsilon}-v}{\varepsilon}\right)_t \rightharpoonup (\psi_2)_t \text{ in } L^2\left((0,T),(H^1(\Omega))^*\right)$$

and $\frac{u^{\varepsilon}-u}{\varepsilon} \to \psi_1$, $\frac{v^{\varepsilon}-v}{\varepsilon} \to \psi_2$ in $L^2(Q)$. The above convergences justify that ψ_1, ψ_2 solve (2.3.1). Similarly, ϕ_1, ϕ_2 solve (2.3.2). □

To derive the optimality system and to characterize the pair of optimal controls, we need adjoint variables and the adjoint operator associated with ψ_1, ψ_2. We rewrite the ψ_1, ψ_2 PDEs as

$$\mathcal{L}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}-h_1 u\\ 0\end{pmatrix}, \text{ where } \mathcal{L}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}L_1\psi_1\\ L_2\psi_2\end{pmatrix} + M\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

and

$$M = \begin{pmatrix}-a_1+2d_1u+c_1v+f_1 & c_1u\\ -c_2v & -a_2+2d_2v-c_2u+f_2\end{pmatrix}.$$

Similarly, we can rewrite the ϕ_1, ϕ_2 PDEs as

$$\mathcal{L}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}0\\ -h_2v\end{pmatrix}.$$

We define the adjoint PDEs as

$$\mathcal{L}^*\begin{pmatrix}p\\ q\end{pmatrix} = \begin{pmatrix}K_1f_1\\ K_2f_2\end{pmatrix}$$

where

$$\mathcal{L}^*\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} L_1^* p \\ L_2^* q \end{pmatrix} + M^T \begin{pmatrix} p \\ q \end{pmatrix}$$

and

$$L_1^* p = -p_t - \sum_{i,j=1}^{n} (a_{ij}^1 p_{x_i})_{x_j} - \sum_{i=1}^{n} (b_i^1 p)_{x_i}$$

$$L_2^* q = -q_t - \sum_{i,j=1}^{n} (a_{ij}^2 q_{x_i})_{x_j} - \sum_{i=1}^{n} (b_i^2 q)_{x_i}.$$

When we write

$$\int_Q (\psi_1, \psi_2)\mathcal{L}^*\begin{pmatrix} p \\ q \end{pmatrix} = \int_Q (p, q)\mathcal{L}^*\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

we mean it in the appropriate V weak sense. To illustrate

$$\begin{aligned}\int_Q (\psi_1, \psi_2)\mathcal{L}^*\begin{pmatrix} p \\ q \end{pmatrix} &= \int_0^T \langle -p_t, \psi_1 \rangle + \langle -q_t, \psi_2 \rangle \\ &+ \int_Q a_{ij}^1 p_{x_i} (\psi_1)_{x_j} + a_{ij}^2 q_{x_i} (\psi_2)_{x_j} - \left(b_i^1 p\right)_{x_i} \psi_1 - \left(b_i^2 q\right)_{x_i} \psi_2 \\ &+ \int_Q (\psi_1, \psi_2) M^T \begin{pmatrix} p \\ q \end{pmatrix} + \int_{\partial\Omega \times (0,t)} b^1 \vec{\eta} p \psi_1 + b^2 \vec{\eta} q \psi_2.\end{aligned}$$

For the adjoint system, we must attach the appropriate boundary conditions.

Theorem 3.2

Given an optimal pair (f_1, f_2) and corresponding solutions u, v, there exists $(p, q) \in V \times V$ satisfying the adjoint system

$$L_1^* p = K_1 f_1 + a_1 p - 2d_1 up - c_1 vp - f_1 p + c_2 vq \text{ in } Q$$

$$p(x, T) = 0 \text{ for } x \in \Omega$$

$$\frac{\partial p}{\partial \nu} + (b^1 \cdot \vec{n})p = 0 \text{ on } \partial\Omega \times (0, T) \tag{2.3.3}$$

$$L_2^* q = K_2 f_2 + a_2 q - 2 d_2 v q + c_1 u p - f_2 q + c_2 u q \text{ in } Q$$

$$q(x,T) = 0 \text{ for } x \in \Omega$$

$$\frac{\partial q}{\partial \nu} + (b^2 \cdot \vec{n}) q = 0 \text{ on } \partial\Omega \times (0,T) \tag{2.3.4}$$

and furthermore

$$f_1 = \min\left(\left(\frac{(K_1 - p)u}{2M_1}\right)^+, \Gamma_1\right)$$

$$f_2 = \min\left(\left(\frac{(K_2 - q)v}{2M_2}\right)^+, \Gamma_2\right).$$

Remark: Note the transversality conditions on the adjoint variables are at $t = T$.

Proof

Suppose (f_1, f_2) is an optimal pair and (u, v) are its corresponding solutions. Consider $(f_1 + \varepsilon h_1, f_2) \in A$ with associated solutions $u^\varepsilon, v^\varepsilon$. Since the adjoint equations are linear, there exists p, q satisfying (2.3.1), (2.3.2) [5]. Since the maximum of the payoff functional is attained at (f_1, f_2),

$$\begin{aligned}
0 &\geq \lim_{\varepsilon \to 0+} \frac{J(f_1 + \varepsilon h, f_2) - J(f_1, f_2)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0+} \int_Q \left(K_1 f_1 \left(\frac{u^\varepsilon - u}{\varepsilon}\right) + K_1 h_1 u^\varepsilon + K_2 f_2 \left(\frac{v^\varepsilon - v}{\varepsilon}\right) - 2M_1 f_1 h_1 - \varepsilon M_1 h_1^2\right) \\
&= \int_Q (K_1 f_1 \psi_1 + K_2 f_2 \psi_2 + h_1 K_1 u - 2M_1 f_1 h_1) \\
&= \int_Q (\psi_1, \psi_2) \begin{pmatrix} K_1 f_1 \\ K_2 f_2 \end{pmatrix} + h_1 (K_1 u - 2M_1 f_1) \\
&= \int_Q (\psi_1, \psi_2) \mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} + h_1 (K_1 u - 2M_1 f_1) \\
&= \int_Q (\psi_1, \psi_2) \left(\begin{pmatrix} L_1^* p \\ L_2^* q \end{pmatrix} + M^T \begin{pmatrix} p \\ q \end{pmatrix}\right) + h_1 (K_1 u - 2M_1 f_1)
\end{aligned}$$

$$= \int_Q (p,q)\mathcal{L}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + h_1(K_1u - 2M_1f_1)$$

$$= \int_Q (p,q)\begin{pmatrix}-h_1u\\ 0\end{pmatrix} + h_1(K_1u - 2M_1f_1)$$

$$= \int_Q h_1(-pu + K_1u - 2M_1f_1).$$

Hence,

$$\int_Q h_1(-pu + K_1u - 2M_1f_1) \leq 0. \tag{2.3.5}$$

By standard analysis, $f_1 = \min\left(\left(\frac{(K_1-p)u}{2M_1}\right)^+, \Gamma_1\right)$. Similarly, $f_2 = \min\left(\left(\frac{(K_2-q)v}{2M_2}\right)^+, \Gamma_2\right)$. □

Using the relationship between our optimal control pair and the associated adjoint variables from this theorem, we now form the following optimality system:

$$
\begin{aligned}
L_1u &= u(a_1 - d_1u) - c_1uv - \min\left(\left(\frac{(K_1-p)u}{2M_1}\right)^+, \Gamma_1\right)u \quad \text{in } Q\\
L_2v &= v(a_2 - d_2v) + c_2uv - \min\left(\left(\frac{(K_2-q)v}{2M_2}\right)^+, \Gamma_2\right)v\\
L_1^*p &= K_1\min\left(\left(\frac{(K_1-p)u}{2M_1}\right)^+, \Gamma_1\right) + a_1p - 2d_1up - c_1vp\\
&\quad - \min\left(\left(\frac{(K_1-p)u}{2M_1}\right)^+, \Gamma_1\right)p + c_2vq\\
L_2^*q &= K_2\min\left(\left(\frac{(K_2-q)v}{2M_2}\right)^+, \Gamma_2\right) + a_2q - 2d_2vq + c_1up\\
&\quad - \min\left(\left(\frac{(K_2-q)v}{2M_2}\right)^+, \Gamma_2\right)q + c_2uq
\end{aligned}
\tag{2.3.6}
$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x)$$

$$p(x,T) = 0 = q(x,T) \qquad x \in \Omega$$

$$\frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu}, \quad \frac{\partial p}{\partial \nu} + (b^1 \cdot \vec{n})p = 0, \quad \frac{\partial q}{\partial \nu} + (b^2 \cdot \vec{n})q = 0, \quad \partial\Omega \times (0, T).$$

Since an optimal pair (f_1, f_2) exists by Theorem 2.2, u, v exist as solutions to the state system (2.1.1), and p, q exist as solutions to (2.3.3) and (2.3.4). Therefore, a solution to the optimality system exists.

Now, we prove the uniqueness result for the optimality system which characterizes the unique optimal control pair.

Theorem 3.3

When T is sufficiently small, the solution of the optimality system (2.3.6) is unique.

Proof

Suppose (u, v, p, q) and $(\overline{u}, \overline{v}, \overline{p}, \overline{q})$ are two different solutions of the optimality system (2.3.6). Let $u = e^{\lambda t}w$, $\overline{u} = e^{\lambda t}\overline{w}$, $v = e^{\lambda t}z$, $\overline{v} = e^{\lambda t}\overline{z}$, $p = e^{-\lambda t}y$, $\overline{p} = e^{-\lambda t}\overline{y}$, $q = e^{-\lambda t}\xi$, $\overline{q} = e^{-\lambda t}\overline{\xi}$,

$$f_1 = \min\left(\left(\frac{(K_1 - p)u}{2M_1}\right)^+, \Gamma_1\right), \quad \overline{f}_1 = \min\left(\left(\frac{(K_1 - \overline{p})\overline{u}}{2M_1}\right)^+, \Gamma_1\right),$$
$$f_2 = \min\left(\left(\frac{(K_2 - q)v}{2M_2}\right)^+, \Gamma_2\right), \quad \overline{f}_2 = \min\left(\left(\frac{(K_2 - \overline{q})\overline{v}}{2M_2}\right)^+, \Gamma_2\right)$$

where $\lambda > 0$ is to be chosen.

We subtract the bilinear forms of w and $\overline{w}$, z and $\overline{z}$, y and $\overline{y}$, ξ and $\overline{\xi}$. The

form for w is illustrated below.

$$\int_Q (w-\overline{w})_t\phi + \int_Q a^1_{ij}(w-\overline{w})_{x_j}\phi_{x_i} + \int_Q b^1_i(w-\overline{w})_{x_i}\phi + \lambda\int_Q (w-\overline{w})\phi$$
$$= \int_Q a_1(w-\overline{w})\phi - \int_Q (f_1 w - \overline{f}_1\overline{w})\phi - \int_Q c_1 e^{\lambda t}(wz - \overline{wz})\phi - \int_Q d_1 e^{\lambda t}(w^2-\overline{w}^2)\phi$$

where $\phi \in V$.

By standard estimation techniques, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Big([w(x,T)-\overline{w}(x,T)]^2 + [z(x,T)-\overline{z}(x,T)]^2 + [y(x,0)-\overline{y}(x,0)]^2 \\
&\qquad + \big[\xi(x,0)-\overline{\xi}(x,0)\big]^2\Big) \\
&+\theta\int_Q \Big(|\nabla(w-\overline{w})|^2 + |\nabla(z-\overline{z})|^2 + |\nabla(y-\overline{y})|^2 + \big|\nabla(\xi-\overline{\xi})\big|^2\Big) \\
&+\lambda\int_Q \left[(w-\overline{w})^2 + (z-\overline{z})^2 + (y-\overline{y})^2 + (\xi-\overline{\xi})^2\right] \\
&\le \int_Q a_1\left[(w-\overline{w})^2 + (y-\overline{y})^2\right] + a_2\left[(z-\overline{z})^2 + (\xi-\overline{\xi})^2\right] \\
&+\int_Q (w-\overline{w})\left[(\overline{f}_1\overline{w} - f_1 w) - (c_1 e^{\lambda t}(wz-\overline{wz}))\right] \\
&+\int_Q (y-\overline{y})\left[(\overline{f}_1\overline{y} - f_1 y) + K_1 e^{\lambda t}(f_1-\overline{f}_1) - 2d_1 e^{\lambda t}(wy-\overline{wy})\right. \\
&\left. - c_1 e^{\lambda t}(zy-\overline{zy}) + c_2 e^{\lambda t}(z\xi-\overline{z\xi})\right] \\
&+\int_Q (z-\overline{z})\left[(\overline{f}_2\overline{z} - f_2 z) + c_2 e^{\lambda t}(wz-\overline{wz})\right] \\
&+\int_Q (\xi-\overline{\xi})\Big[(\overline{f}_2\overline{\xi} - f_2\xi) + K_2 e^{\lambda t}(f_2-\overline{f}_2) - 2d_2 e^{\lambda t}(z\xi-\overline{z\xi}) \\
&+ c_1 e^{\lambda t}(wy-\overline{wy}) + c_2 e^{\lambda t}(w\xi-\overline{w\xi})\Big].
\end{aligned} \tag{2.3.7}$$

We illustrate estimating a specific term from the right-hand side, using that

z, w, y, ξ are L^∞ bounded,

$$\int_Q c_1 e^{\lambda t}(wz - \overline{wz})(w - \overline{w}) \le C_1 e^{\lambda T} \int_Q |wz - \overline{w}z + \overline{w}z - \overline{wz}||w - \overline{w}|$$
$$\le C_2 e^{\lambda T} \int_Q \left[(w - \overline{w})^2 + (z - \overline{z})^2\right].$$

One can show

$$|f_1 - \overline{f}_1|^2 \le \left(\frac{(K_1 - e^{-\lambda t} y) e^{\lambda t} w}{2M_1} - \frac{(K_1 - e^{-\lambda t}\overline{y}) e^{\lambda t}\overline{w}}{2M_1}\right)^2 \quad \text{and}$$
$$|f_2 - \overline{f}_2|^2 \le \left(\frac{(K_2 - e^{-\lambda t}\xi) e^{\lambda t} z}{2M_2} - \frac{(K_2 - e^{-\lambda t}\overline{\xi}) e^{\lambda t}\overline{z}}{2M_2}\right)^2.$$

By further estimation, (2.3.7) becomes

$$\frac{1}{2}\int_\Omega \Big([w(x,T) - \overline{w}(x,T)]^2 + [z(x,T) - \overline{z}(x,T)]^2$$
$$+ [y(x,0) - \overline{y}(x,0)]^2 + \left[\xi(x,0) - \overline{\xi}(x,0)\right]^2\Big)$$
$$+ \frac{3\theta}{4}\int_Q \left(|\nabla(w - \overline{w})|^2 + |\nabla(z - \overline{z})|^2 + |\nabla(y - \overline{y})|^2 + |\nabla(\xi - \overline{\xi})|^2\right)$$
$$+ (\lambda - C_1 - C_2 e^{2\lambda T})\int_Q \left[(w - \overline{w})^2 + (z - \overline{z})^2 + (y - \overline{y})^2 + (\xi - \overline{\xi})^2\right] \le 0$$

where C_1, C_2 depend on the coefficients and the L^∞ bounds of $z, w, y,$ and ξ.

If we choose λ such that $\lambda > C_1 + C_2$ and $T < \frac{1}{2\lambda}\ln\left(\frac{\lambda - C_2}{C_1}\right)$, then $w = \overline{w}$, $z = \overline{z}$, $y = \overline{y}$, and $\xi = \overline{\xi}$. Therefore, $u = \overline{u}$, $v = \overline{v}$, $p = \overline{p}$, and $q = \overline{q}$. $\square$

This T sufficiently small condition in the uniqueness result is due to opposite time orientation of the state and adjoint equations.

In conclusion, we have that the optimal control pair

$$f_1 = \min\left(\left(\frac{(K_1 - p)u}{2M_1}\right)^+, \Gamma_1\right), \quad f_2 = \min\left(\left(\frac{(K_2 - q)v}{2M_2}\right)^+, \Gamma_2\right)$$

is characterized in terms of (u, v, p, q), the unique solution of the optimality system.

4. Positivity of the Payoff Functional

To analyze the positivity of the payoff functional, we determine the positivity of its integrand. However, on the set where $f_1 = 0$ and $f_2 = 0$, the integrand of the payoff functional is zero. Hence, we determine conditions on Γ_1 and Γ_2 so that this cannot occur.

Theorem 4.1

For

$$\Gamma_1 \leq \min(e^{-cT}, e^{-cT}\frac{K_2}{K_1}) \tag{2.4.1}$$

and

$$\Gamma_2 \leq \min(e^{-cT}, e^{-cT}\frac{K_1}{K_2}) \tag{2.4.2}$$

where $c = \max(1+\|a_1\|+\|c_2\|\|u_0\|e^{\tilde{\eta}T}, 1+\|a_2\|+(\|c_1\|+\|c_2\|)\|u_0\|e^{\gamma T})$ and $\gamma > a_1$, and $\tilde{\eta} > a_2 + c_2\|u_0\|e^{\gamma T}$, $\max_{(f_1,f_2)\in A} J(f_1, f_2) > 0$, where $\|\cdot\|$ denotes L^∞ norms.

Proof

Denote the L^∞ norm by $\|\cdot\|$. Let U and Z be supersolutions for u and v respectively. We determine the upper bounds on U and Z via a parabolic maximum

principle. U and Z satisfy the following

$$\begin{aligned} L_1 U &= a_1 U \quad \text{in } Q \\ U(x,0) &= u_0(x), \quad \text{for } x \in \Omega \\ \frac{\partial U}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{2.4.3}$$

$$\begin{aligned} L_2 Z &= a_2 Z + c_2 U Z \quad \text{in } Q \\ Z(x,0) &= v_0(x), \quad \text{for } x \in \Omega \\ \frac{\partial Z}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{2.4.4}$$

Let $U = e^{\gamma t} W$ where γ is to be chosen. Then substitution yields

$$\begin{aligned} L_1 W + (\gamma - a_1) W &= 0 \quad \text{in } Q \\ W(x,0) &= u_0(x), \quad \text{for } x \in \Omega \\ \frac{\partial W}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{2.4.5}$$

where $\gamma > a_1$.

Via the parabolic maximum principle, the maximum of W must occur on the boundary, so $W \leq \|u_0\|$. Moreover,

$$U \leq \|u_0\| e^{\gamma T}.$$

Using the bound on U, we find a bound on Z in (2.4.4), rewritten as

$$\begin{aligned} L_2 Z - (a_2 + c_2 U) Z &= 0 \quad \text{in } Q \\ Z(x,0) &= v_0(x), \quad \text{for } x \in \Omega \\ \frac{\partial Z}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned}$$

Let $Z = e^{\tilde{\eta}T}Y$ where $\tilde{\eta}$ is to be chosen. Then

$$L_2 Y + (\tilde{\eta} - a_2 - c_2 U)Y = 0 \quad \text{in } Q$$

$$Y(x,0) = v_0(x), \qquad \text{for } x \in \Omega \tag{2.4.6}$$

$$\frac{\partial Y}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T).$$

Since $U \leq \|u_0\|e^{\gamma T}$, then $\tilde{\eta}$ is chosen such that

$$\tilde{\eta} - a_2 - c_2 U \geq \tilde{\eta} - a_2 - c_2\|u_0\|e^{\gamma T},$$

$$\tilde{\eta} - a_2 - c_2\|u_0\|e^{\gamma T} > 0 \quad \text{or} \quad \tilde{\eta} > a_2 + c_2\|u_0\|e^{a_1 T}.$$

By the parabolic maximum principle, the maximum of Y occurs on the boundary

$$Y \leq \|u_0(x)\| \quad \text{or}$$

$$Z \leq \|u_0(x)\|e^{\tilde{\eta}T}.$$

If we consider our adjoint system in (2.3.6) then we can obtain conditions so that the maximum principle for weakly coupled systems [13] can be applied. The adjoint system is

$$\begin{pmatrix} L_1^* p \\ L_2^* q \end{pmatrix} + \begin{pmatrix} -a_1 + 2d_1 u + c_1 v + f_1^* & -c_2 v \\ -c_1 u & -a_2 + 2d_2 v - c_2 u + f_2^* \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} K_1 f_1^* \\ K_2 f_2^* \end{pmatrix}$$

$$p(x,T) = 0 = q(x,T) \quad \text{for } x \in \Omega$$

$$\frac{\partial p}{\partial \nu} + (b^1 \cdot \vec{\eta})p = 0 = \frac{\partial q}{\partial \nu} + (b^2 \cdot \vec{\eta})q \quad \text{on } \partial\Omega \times (0,T). \tag{2.4.7}$$

First we let $p = e^{ct}\hat{p}$ and $q = e^{ct}\hat{q}$ where c is chosen later. Then we have

$$\begin{pmatrix} L_1^*\hat{p} \\ L_2^*\hat{q} \end{pmatrix} + \begin{pmatrix} -a_1 + 2d_1u + c_1v + f_1^* + c & -c_2v \\ -c_1u & -a_2 + 2d_2v - c_2u + f_2^* + c \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}$$
$$= e^{-ct} \begin{pmatrix} K_1 f_1^* \\ K_2 f_2^* \end{pmatrix}$$

$$\hat{p}(x,T) = 0 = \hat{q}(x,T) \quad \text{for } x \in \Omega$$
$$\frac{\partial \hat{p}}{\partial v} + (b^1 \cdot \vec{\eta})p = 0 = \frac{\partial \hat{q}}{\partial v} + (b^2 \cdot \vec{\eta})q \quad \text{on } \partial\Omega \times (0,T). \tag{2.4.8}$$

Using the techniques as in Protter and Weinberger [13], we require that off-diagonal terms of the coupling are non-positive, i.e., $-c_2v \leq 0$ and $-c_1u \leq 0$ which are true since $v > 0$ and $u > 0$. Also we require the positive row sum condition in the coupling matrix

$$-a_1 + 2d_1u + c_1v + f_1^* + c - c_2v > 0 \quad \text{and}$$
$$-a_2 + 2d_2v - c_2u + f_2^* + c - c_1u > 0.$$

Hence, we must choose c such that

$$c \geq 1 + a_1 + c_2\|u_0\|e^{\bar{\eta}T} > a_1 + c_2v \quad \text{and}$$
$$c \geq 1 + a_2 + (c_1 + c_2)\|u_0\|e^{\gamma T} > a_2 + (c_1 + c_2)u. \tag{2.4.9}$$

We choose $c = \max\left(1 + \|a_1\| + \|c_2\|\|u_0\|e^{\bar{\eta}T}, 1 + \|a_2\| + (\|c_1\| + \|c_2\|)\|u_0\|e^{\gamma T}\right)$.

To utilize Protter and Weinberger's maximum principle [13] for systems, we will choose $M > 0$ such that

$$\begin{pmatrix} L_1^*M \\ L_2^*M \end{pmatrix} + \begin{pmatrix} -a_1 + 2d_1u + c_1v + f_1^* + c & -c_2v \\ -c_1u & -a_2 + 2d_2v - c_2u + f_2^* + c \end{pmatrix} \begin{pmatrix} M \\ M \end{pmatrix}$$
$$\geq \begin{pmatrix} \|e^{-ct}K_1 f_1^*\| \\ \|e^{-ct}K_2 f_2^*\| \end{pmatrix}$$
$$\geq e^{-ct} \begin{pmatrix} K_1 f_1^* \\ K_2 f_2^* \end{pmatrix}. \tag{2.4.10}$$

So we want

$$(-a_1 + 2d_1u + c_1v + f_1^* + c - c_2v)M > K_1\|f_1^*\| \tag{2.4.11}$$

$$(-c_1u - a_2 + 2d_2v - c_2u + f_2^* + c)M > K_2\|f_2^*\| \tag{2.4.12}$$

where c is defined above.

Using $c \geq 1 + \|a_1\| + \|c_2\|\|u_0\|e^{\bar{\eta}T}$, we have that

$$\begin{aligned}
-a_1 + 2d_1u + c_1v + f_1^* + c - c_2v &\geq -a_1 + f_1^* + 1 + a_1 + c_2\|u_0\|e^{\bar{\eta}T} - c_2\|u_0\|e^{\bar{\eta}T} \\
&= 1 + f_1^* \\
&\geq 1.
\end{aligned}$$

So in (2.4.11), we want

$$M \geq K_1\|f_1^*\|.$$

Using $c \geq 1 + \|a_2\| + (\|c_1\| + \|c_2\|)\|u_0\|e^{\gamma T}$, we have

$$-a_2 + 2d_2v - c_2u - c_1u + f_2^* + c \geq 1.$$

So in (2.4.12), we want

$$M \geq K_2\|f_2^*\|.$$

Hence, we choose $M = \max(K_1\|f_1^*\|, K_2\|f_2^*\|)$. Then by the weakly coupled maximum principle [13], we determine that $p \leq e^{ct}M$ and $q \leq e^{ct}M$.

We want $p < K_1$ and $q < K_2$ so that the optimal pair f_1^* and f_2^* are nonzero. By assumption (2.4.1) for $M = K_1\|f_1^*\|$, we have the following:

i) Since $\Gamma_1 \le e^{-cT}$, then $\|f_1^*\| \le e^{-cT} \le e^{-ct}$ for $t \in (0,T)$ or $e^{ct}K_1\|f_1^*\| < K_1$.

This means that $p < K_1$.

ii) Since $\Gamma_1 \le e^{-cT}\frac{K_2}{K_1}$, then $K_1 e^{ct}\|f_1^*\| < K_2$. Hence, $q < K_2$.

By assumption (2.4.2) using $M = K_2\|f_2^*\|$, we have the following:

i) Since $\Gamma_2 \le e^{-cT}$, then $\|f_2^*\| \le e^{-cT} \le e^{-ct}$ for $t \in (0,T)$ or $e^{ct}K_2\|f_2^*\| < K_2$.

Hence $q < K_2$.

ii) Since $\Gamma_2 \le e^{-cT}\frac{K_1}{K_2}$, then $e^{ct}K_2\|f_2^*\| < K_1$. Hence, $p < K_1$.

So for $\Gamma_1 \le \min\left(e^{-cT}, e^{-cT}\frac{K_2}{K_1}\right)$ and $\Gamma_2 \le \min\left(e^{-cT}, e^{-cT}\frac{K_1}{K_2}\right)$, then the optimal controls f_1 and f_2 are positive. To understand why $\max\limits_{(f_1,f_2)\in A} J(f_1,f_2) > 0$, we illustrate a few cases.

On the set where $f_1 = \frac{(K_1-p)u}{2M_1}$ and $f_2 = \frac{(K_2-q)v}{2M_2}$, we estimate the integrand of J:

$$
\begin{aligned}
&K_1uf_1 + K_2vf_2 - M_1f_1^2 - M_2f_2^2 \\
&\quad = \frac{(K_1-p)u}{2M_1}\left(K_1u - M_1\left(\frac{(K_1-p)u}{2M_1}\right)\right) + \frac{(K_2-q)v}{2M_2}\left(K_2v - M_2\left(\frac{(K_2-q)v}{2M_2}\right)\right) \\
&\quad = \frac{(K_1-p)u}{2M_1}\frac{u}{2}(K_1+p) + \frac{(K_2-q)v}{2M_2}\frac{v}{2}(K_2+q) > 0.
\end{aligned}
$$

The integrand of the payoff functional is positive since $u > 0$, $v > 0$, $p > 0$, $K_1 - p > 0$, $q > 0$, and $K_2 - q > 0$ by a parabolic maximum principle.

On the set where $f_1 = \Gamma_1$ and $f_2 = \frac{(K_2-q)v}{2M_2}$ the integrand becomes

$$
\begin{aligned}
&\Gamma_1(K_1u - M_1\Gamma_1) + \frac{(K_2-q)v}{2M_2}\left(K_2v - M_2\left(\frac{(K_2-q)v}{2M_2}\right)\right) \\
&\quad > \Gamma_1\left(K_1u - M_1\left(\frac{(K_1-p)u}{2M_1}\right)\right) + \frac{(K_2-q)v}{2M_2}\frac{v}{2}(K_2+q) > 0
\end{aligned}
$$

since $\Gamma_1 \leq \frac{(K_1-p)u}{2M_1}$ in this case.

We can show analogous cases for $f_1 = \frac{(K_1-p)u}{2M_1}$, $f_2 = \Gamma_2$ and $f_1 = \Gamma_1$, $f_2 = \Gamma_2$.

Since the integrand of the payoff functional is positive for the optimal control pair and we integrate over a set of positive measure, then

$$\max_{(f_1,f_2)\in A} J(f_1, f_2) > 0.$$

$\square$

Under appropriate conditions, we have characterized the unique optimal harvesting stategy and have shown that the payoff functional is positive.

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Part III

Parabolic Problem Modeling Surface Run-off in a Watershed

1. Introduction

We consider the flow of chemicals or sediment along a major river flow in a given watershed. We assume that this entire watershed can be divided into more than one smaller segments, each comprising of a major water canal emptying into the one major river flow in the watershed. Thus, the chemical and sediment loading into the major river can be treated as several point inflows of contaminants. Then the flow of contaminants in the major river can be modeled by the following advection-diffusion problem:

$$\begin{aligned} u_t - (\beta u_x)_x + \alpha u_x &= -D(x)u + \sum_{i=1}^{k} \delta(x - x_i)S_i(t) \quad \text{in } Q \\ u(x,0) &= u_0(x), \quad x \in (a,b) \\ u_x(b,t) = 0, &\quad u(a,t) = u_a \quad \text{for } 0 < t < T, \end{aligned} \tag{3.1.1}$$

where $Q = \Omega \times (0,T)$ and $\Omega = (a,b)$. The solution $u(x,t)$ represents the concentration of contaminant at location x at a given time t. The terms involving α, β represent advection and diffusion, respectively. The $D(x)$ is the rate of natural degrading a proportion of the contaminant. The $\delta(x - x_i)S_i(t)$ is a source term at $x = x_i$ location with $\delta(x - x_i)$ being the Dirac delta measure at x_i, and $S_i(t)$ being the level of contaminant loading resulting from pollution source material (e.g. fertilizers, pesticides, soil eroded).

The contamination loading $\vec{S} = (S_1(t), S_2(t), \ldots, S_k(t))$ is chosen as our control. We define our class of admissible controls

$$A = \{\vec{S} = (S_1, \ldots, S_k) \in (L^\infty(0,T))^k \,|0 \le S_i \le \Gamma_i, \ 1 \le i \le k\},$$

and we consider the following management decision functional:

$$J(\vec{S}) = \int_0^T e^{-rt} \sum_{i=1}^{k} R_i(S_i(t))\, dt - \int_0^T \int_a^b e^{-rt} f(u(x,t))\, dxdt \tag{3.1.2}$$

which represents the sum of net agricultural revenues R_i less the environment costs, f, of contaminants in the river system. We seek to maximize this functional and to characterize the optimal control, $\vec{S}^*$, i.e., $J(\vec{S}^*) = \max_{\vec{S}\in A} J(\vec{S})$. See Lions [6] for a similar control problem.

In section 2, we state our assumptions, define our solution space, and prove the existence of an optimal control. In section 3, we derive a characterization of the optimal control, in terms of an optimality system, which is the state problem (3.1.1) coupled with an adjoint problem.

2. Existence of an Optimal Control

When considering the state problem (3.1.1) and the desired functional to be maximized (3.1.2), the following assumptions are made:

$$u_0(x) \in L^\infty(\Omega), \quad u_0(x) > 0 \quad \text{on } \Omega. \tag{3.2.1}$$

$$\alpha, \beta \in C^1(\overline{\Omega}) \tag{3.2.2}$$

$$\beta(x) \geq \nu > 0 \quad \text{in } \Omega \tag{3.2.3}$$

$$u_a > 0, \quad \text{a positive constant} \tag{3.2.4}$$

$$D(x) \in L^\infty(\Omega) \tag{3.2.5}$$

$$\exists M > 0 \text{ s.t. } |f'(u) - f'(\widetilde{u})| \leq M|u - \hat{u}| \text{ for } u, \widetilde{u} \in \mathbb{R}$$

and $f'(u) > 0$ for $u \in \mathbb{R}$ (3.2.6)

$R_i(S_i(t))$ is upper semicontinuous with respect to weak L^2 convergence.

We shall consider two classes of revenue functions

$$R_i(S_i) = \frac{a_i S_i}{b_i + S_i}, \quad \text{where } a_i,\ b_i \text{ are positive constants}$$

and

$$R_i S_i(t) = a_i S_i - b_i (S_i)^2 \quad \text{where } a_i,\ b_i \text{ satisyfy } \frac{a_i}{2b_i} \geq \Gamma_i.$$

To obtain zero boundary conditions at "a", let $w = u - u_a$ where u solves (3.1.1). Then w solves

$$w_t - (\beta w_x)_x + \alpha w_x = -D(x)(u_a + w) + \sum_{i=1}^{k} \delta(x - x_i) S_i(t) \text{ in } Q \tag{3.2.7}$$

$$w(x,0) = u_0(x) - u_a, \quad x \in \Omega$$

$$w_x(b,t) = 0, \quad w(a,t) = 0, \quad 0 < t < T. \tag{3.2.8}$$

Our solution space for problem (3.2.7), (3.2.8) is

$$W = L^2\left((0,T); H^1_{\{a\}}(\Omega)\right),$$

where the subscript $\{a\}$ indicates the zero boundary condition at $x = a$ with $\|\cdot\|_W = \|\cdot\|_{L^2((0,T);H^1(\Omega))}$ [5]. We define the bilinear form

$$a(t, w, \phi) = \int_\Omega \beta w_x \phi_x \, dx + \int_\Omega \alpha w_x \phi \, dx + \int_\Omega D w \phi \, dx \text{ for } \phi, w \in W.$$

We work with the w problem (3.2.7), (3.2.8) for the convenience of having the zero boundary condition at "a".

We define our weak solution $w \in W$ to (3.2.7), (3.2.8) in the following sense:

$$\int_0^T \langle w_t, \phi \rangle \, dt + \int_0^T a(t, w, \phi) \, dt$$
$$= -u_a \int_Q D(x)\phi \, dxdt + \int_Q \sum_{i=1}^{k} S_i(t)\delta(x - x_i)\phi \, dxdt$$

for all $\phi \in W$, where $\langle \, , \, \rangle$ denotes the duality between $\left(H^1_{\{a\}}(\Omega)\right)^*$ and $H^1_{\{a\}}(\Omega)$.

Given $\vec{S} \in A$, there exists a unique solution $w = w(\vec{S})$ to (3.2.7), (3.2.8) in W using a result from Renardy and Rogers ([8], Theorem 10.3, Chapter 10). Moreover, Renardy and Rogers [8] shows that the solution $w \in C\left((0,T); L^2(\Omega)\right)$.

Note for convenience we prefer to omit the differentials in the integrals.

Theorem 2.1

There exists an optimal control in A that maximizes the functional $J(\vec{S})$ in (3.1.2) subject to the PDE problem (3.2.7), (3.2.8).

Proof

Since the state variable is a priori bounded on $L^2(Q)$ and the controls are uniformly bounded in $L^2(Q)$,

$$\sup_{\vec{S} \in A} J(\vec{S}) < \infty.$$

Then there exists a maximizing sequence $\{\vec{S}^n\}$ in A such that

$$\lim_{n \to \infty} J(\vec{S}^n) = \sup_{\vec{S} \in A} J(\vec{S}).$$

We define $w^n = w(\vec{S}^n)$ for each n by existence and uniqueness of solutions to (3.2.7), (3.2.8). We consider the weak formulation of the w^n PDE problem with the test function w^n

$$\int_{\Omega\times(0,t)} w_t^n w^n + \int_{\Omega\times(0,t)} (\beta w_x^n) w_x^n + \int_{\Omega\times(0,t)} \alpha w_x^n w^n$$
$$= -\int_{\Omega\times(0,t)} D(x)(u_a + w^n) w^n + \int_{\Omega\times(0,t)} \sum_{i=1}^{k} S_i^n(s)\delta(x - x_i) w^n.$$

Upon simplification (using $w_t^n w^n = \frac{1}{2}\frac{d(w^n)^2}{dt}$) , we have

$$\frac{1}{2}\int_\Omega [w^n(x,t)]^2 + \int_{\Omega\times(0,t)} \beta (w_x^n)^2 + \int_{\Omega\times(0,t)} \alpha w_x^n w^n$$
$$= \frac{1}{2}\int_\Omega [u_0(x) - u_a]^2 - \int_{\Omega\times(0,t)} D(x)(u_a + w^n) w^n + \int_{\{x_i\}\times(0,t)} \sum_{i=1}^{k} S_i^n(s) w^n.$$

By Cauchy's inequality with $\theta > 0$ on the $\alpha w_x^n w_x$ term moved to the right-hand side, we have

$$\frac{1}{2}\int_\Omega [w^n(x,t)]^2 + \nu \int_{\Omega\times(0,t)} (w_x^n)^2 \leq \frac{1}{2}\int_\Omega [u_0(x) - u_a]^2 + \frac{\theta}{2}\int_{\Omega\times(0,t)} (w_x^n)^2$$
$$+ A\int_{\Omega\times(0,t)} (w^n)^2 + D_a \int_{\Omega\times(0,t)} [|w^n| + (w^n)^2]$$
$$+ \int_{\{x_i\}\times(0,t)} \left(\sum_{i=1}^{k} S_i^n(s)\right)^2 + C\int_{\{x_i\}\times(0,t)} (w^n)^2$$

where A, D_a, C depend on α, $||D(x)||$, θ, and u_a. Now we have

$$\frac{1}{2}\int_\Omega [w^n(x,t)]^2 + \left(\nu - \frac{\theta}{2}\right)\int_{\Omega\times(0,t)} (w_x^n)^2 \leq \frac{1}{2}\int_\Omega [u_0(x) - u_a]^2 + \widetilde{C}\int_{\{x_i\}\times(0,t)} (w^n)^2$$
$$+ C_2 \int_{\Omega\times(0,t)} (w^n)^2$$
$$+ \int_{\{x_i\}\times(0,t)} \left(\sum_{i=1}^{k} S_i^n(s)\right)^2 + C_1$$

where C_2, $\widetilde{C}$, and C_1 depend on A and D_a. By using a trace estimate on the $(w^n)^2$ term and absorbing the $(w_x^n)^2$ term on the left-hand side for each slice $\{x_i\} \times (0,t)$, we have

$$\int_\Omega [w^n(x,t)]^2 + 2\left(\nu - \frac{3\theta}{4}\right)\int_{\Omega\times(0,t)} (w_x^n)^2 \le \int_\Omega [u_0(x) - u_a]^2 + C_1 + C_3 \int_{\{x_i\}\times(0,t)} \sum_{i=1}^{k} (S_i^n(s))^2 \; ds + C_4 \int_{\Omega\times(0,t)} (w^n)^2. \tag{3.2.9}$$

Via Gronwall's inequality, we have

$$\int_\Omega [w^n(x,t)]^2 \le e^{C_4 T}\left[\int_\Omega [u_0(x) - u_a]^2 + C_3 \int_{\{x_i\}\times(0,t)} \left(\sum_{i=1}^{k} S_i^n(s)\right)^2 ds + C_1\right] \tag{3.2.10}$$

where C_1 depends on the coefficients. Replacing (3.2.10) into (3.2.9) and choosing θ such that $\nu = \frac{3\theta}{4} + \frac{1}{2}$, we obtain

$$\sup_{0\le t\le T}\left(\int_\Omega [w^n(x,t)]^2\right) + \int_Q (w_x^n)^2 \le (C_4 T e^{C_4 T} + 1)\left\{\int_\Omega [u_0(x) - u_a]^2 + C_3 \int_{\{x_i\}\times(0,t)} \left(\sum_{i=1}^{k} S_i^n(s)\right)^2 ds + C_1\right\}.$$

Since the right hand side is bounded, then on a subsequence, $w^n \rightharpoonup w$ in W and $S_i^n(t) \rightharpoonup S_i^*(t)$ in $L^2(0,T)$ since S_i^n are L^∞ bounded for $0 \le t \le T$. Using the PDE and $\|w^n\|_W \le \hat{C}$, then $\|w_t^n\| \le C_5$ in $L^2((0,T);(H^1_{\{a\}}(\Omega))^*)$ where $\hat{C}, C_5$ depend on the coefficients. Hence, $w_t^n \rightharpoonup w_t$ in $L^2((0,T);(H^1_{\{a\}}(\Omega))^*)$, using a compactness interpolation result from ([6], Theorem 4.2, Chapter 4). Passing to the limit in the PDE, we obtain $w = w(\vec{S}^*)$.

Using the upper semi-continuity of the payoff functional with respect to weak convergences [1], then

$$\begin{aligned} J(\vec{S}^*) &\geq \sup_n \left(\int_0^T e^{-rt} \sum_{i=1}^k R_i(S_i^n(t))\, dt - \int_Q e^{-rt} f(u^n)\, dx dt \right) \\ &\geq \lim_{n\to\infty} J(\vec{S}^n) = J(\vec{S}^*) \quad \text{where } u^n = w^n + u_a. \end{aligned}$$

Therefore, $\max_{\vec{S}\in A} J(\vec{S}) = J(\vec{S}^*)$. Hence, $\vec{S}^*$ is an optimal control.

3. Characterization of the Optimal Control

To characterize the optimal control, we must differentiate the functional $J(\vec{S})$ with respect to $\vec{S}$ at an optimal control. Since the functional involves w via u in (3.1.2) and $w = u - u_a$, which depends on $\vec{S}$, we must first calculate the derivative of w with respect to $\vec{S}$. After the characterization of an optimal control, then we show the uniqueness of the optimal control.

Theorem 3.1

The mapping $\vec{S} = (S_1, \dots, S_k) \in A \to w \in W$ is differentiable in the following sense:

$$\frac{w(S_1, \dots, S_i + \varepsilon h, \dots, S_k) - w(\vec{S})}{\varepsilon} \rightharpoonup \psi^i \quad \text{in } W$$

as $\varepsilon \to 0$ for any $(S_1, \dots, S_k) \in A$, $h \in L^\infty(0,T)$ such that $(S_1+\varepsilon h, \dots, S_k+\varepsilon h) \in A$

for ε small for $i = 1, \dots, k$. Also ψ^i, $i = 1, \dots, k$ satisfies

$$\begin{aligned}
&\psi^i_t - [\beta\psi^i_x]_x + \alpha\psi^i_x + D(x)\psi^i = \delta(x - x_i)h \quad \text{in } Q \\
&\psi^i(x,0) = 0 \quad \text{for } x \in \Omega \qquad\qquad (3.3.1)\\
&\psi^i_x(b,t) = 0, \quad \psi^i(a,t) = 0, \quad 0 < t < T.
\end{aligned}$$

Proof

We suppose that $w = w(\vec{S})$ and $w^\varepsilon = w(S_1, \dots, S_i + \varepsilon h, \dots, S_k)$ are solutions to (3.2.7), (3.2.8). Now we consider the following with our choices for w, w^ε:

$$\begin{aligned}
&\left(\frac{w^\epsilon - w}{\varepsilon}\right)_t - \left[\beta\left(\frac{w^\epsilon - w}{\varepsilon}\right)_x\right]_x + \alpha\left(\frac{w^\epsilon - w}{\varepsilon}\right)_x + D(x)\left(\frac{w^\epsilon - w}{\varepsilon}\right) = \delta(x - x_i)h(t) \\
&\left(\frac{w^\epsilon - w}{\varepsilon}\right)(x,0) = 0 \quad \text{for } x \in \Omega \qquad\qquad (3.3.2)\\
&\left(\frac{w^\epsilon - w}{\varepsilon}\right)_x (b,t) = 0, \quad \left(\frac{w^\epsilon - w}{\varepsilon}\right)(a,t) = 0, \quad 0 < t < T.
\end{aligned}$$

Then let $w = e^{\lambda t}v$ and $w^\varepsilon = e^{\lambda t}v^\varepsilon$ where $\lambda > 0$ is to be chosen.

Estimating the PDE satisfied by $\frac{v^\epsilon - v}{\varepsilon}$, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left(\frac{v^\varepsilon(x,T) - v(x,T)}{\varepsilon}\right)^2 + \int_{\Omega\times(0,T)} \beta\left[\left(\frac{v^\varepsilon - v}{\varepsilon}\right)_x\right]^2 + \lambda\int_{\Omega\times(0,T)}\left(\frac{v^\varepsilon - v}{\varepsilon}\right)^2 \\
&\qquad \le D_\alpha \int_{\Omega\times(0,T)}\left(\frac{v^\varepsilon - v}{\varepsilon}\right)^2 + \frac{\nu}{4}\int_{\Omega\times(0,T)}\left[\left(\frac{v^\varepsilon - v}{\varepsilon}\right)_x\right]^2 \\
&\qquad\qquad + e^{-\lambda T}\int_{\{x_i\}\times(0,T)} h(t)\left(\frac{v^\varepsilon - v}{\varepsilon}\right)(x,t). \qquad\qquad (3.3.3)
\end{aligned}$$

where D_α depends on α and $\|D(x)\|$.

To estimate the last term on the right-hand side of (3.3.3) we use a trace

estimate as follows:

$$
\begin{aligned}
\int_{\{x_i\}\times(0,T)} h(t)\left(\frac{v^\varepsilon - v}{\varepsilon}\right)(x,t)\,dt &\leq \int_0^T h^2 + \int_{\{x_i\}\times(0,T)}\left(\frac{v^\varepsilon - v}{\varepsilon}\right)^2 \\
&\leq \int_0^T h^2 + C_1\int_{\Omega\times(0,T)}\left(\frac{v^\varepsilon - v}{\varepsilon}\right)^2 \\
&\quad + \frac{\nu}{4}\int_{\Omega\times(0,T)}\left[\left(\frac{v^\varepsilon - v}{\varepsilon}\right)_x\right]^2 .
\end{aligned}
\tag{3.3.4}
$$

Replacing (3.3.4) into (3.3.3) and using (3.2.3), we obtain

$$
\begin{aligned}
\frac{1}{2}\int_\Omega\left(\frac{v^\varepsilon(x,T) - v(x,T)}{\varepsilon}\right)^2 &+ \frac{\nu}{2}\int_Q\left[\left(\frac{v^\varepsilon - v}{\varepsilon}\right)_x\right]^2 \\
&+ (\lambda - C_3)\int_Q\left(\frac{v^\varepsilon - v}{\varepsilon}\right)^2 \leq C_2
\end{aligned}
\tag{3.3.5}
$$

where C_3 depends on the coefficients of problem (3.3.2).

For $\lambda > C_3$, we conclude that

$$
\|\frac{v^\varepsilon - v}{\varepsilon}\|_W^2 \leq C_2\|h\|^2_{L^2(0,T)}.
$$

This estimate justifies the convergence of the v quotient, and hence, $\frac{w^\varepsilon - w}{\varepsilon} \rightharpoonup \psi^i$ in W. Using $\|\frac{w^\varepsilon - w}{\varepsilon}\|_W \leq C_4$ and the PDE in (3.3.2), we have $\|\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t\| \leq C_5$ in $L^2((0,T);(H^1_{\{a\}}(\Omega))^*)$. Moreover, $\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t \rightharpoonup (\psi^i)_t$ in $L^2((0,T);(H^1_{\{a\}}(\Omega))^*)$. We use the convergences above to justify that ψ^i solves (3.3.1). $\square$

We explicitly characterize our optimal control under two particular choices of the revenue functions, $R_i(S_i)$. The two cases are:

$$
R_i(S_i) = \frac{a_i S_i}{b_i + S_i}, \quad a_i, b_i \text{ positive constants}
$$

and

$$
R_i(S_i) = a_i S_i - b_i (S_i)^2 \quad \text{where } a_i, b_i \text{ are positive constants and } \frac{a_i}{2b_i} \geq \Gamma_i .
$$

This condition that $\Gamma_i \leq \frac{a_i}{2b_i}$ guarantees that the revenue function is increasing in S_i.

Next the adjoint operator in the ψ^i equation is determined. We use the weak form of the ψ^i problem with the test function $e^{-rt}p$ and integrate by parts on two terms.

$$\int_Q e^{-rt} p\psi_t^i + \int_Q e^{-rt}\beta\psi_x^i p_x + \int_Q \alpha\psi_x^i e^{-rt}p + \int_Q D(x)e^{-rt}p\psi^i - \int_{\partial\Omega\times(0,T)} e^{-rt}\beta\psi_x^i p$$

$$= \int_Q e^{-rt}\psi^i\left[-p_t - (\alpha p)_x + (r + D(x))p\right] + \int_Q e^{-rt}\beta\psi_x^i p_x + \int_{\Omega\times\{T\}} e^{-rt}p\psi$$

$$- \int_{\{a\}\times(0,T)} e^{-rt}\beta\psi_x^i p + \int_{\{b\}\times(0,T)} e^{-rt}\alpha\psi^i p + \int_{\{b\}\times(0,T)} e^{-rt}\beta\psi^i p_x.$$

When

$$\mathcal{L}\psi = \psi_t^i - \left(\beta\psi_x^i\right)_x + \alpha\psi_x^i + D(x)\psi^i, \quad \text{we conclude}$$

$$\mathcal{L}p^* = -p_t - \left(\beta p_x\right)_x - (\alpha p)_x + D(x)p + rp \quad \text{in the weak sense.}$$

We choose p to eliminate the difficulty in determining the exact influence of the control on the state variable. Also recall that the adjoint operator applied to p equals the derivative of the integrand of the payoff functional with respect to the state variable, $f'(u)$. Refer to (3.3.6) for the adjoint problem.

Theorem 3.2

Given an optimal control $S = (S_1, \ldots, S_k)$ and corresponding solution w, there exists $p \in W$ satisfying the adjoint problem

$$-p_t - (\beta p_x)_x - (\alpha p)_x + D(x)p + rp = f'(u) \text{ in } Q$$

$$p(x,T) = 0 \quad \text{for } x \in (a,b) \tag{3.3.6}$$

$$(\beta p_x + \alpha p)(b,t) = 0, \quad p(a,t) = 0 \quad \text{for } 0 < t < T$$

and

$$S_i = \min\left(\left(\sqrt{\frac{a_i b_i}{p(x_i,t)}} - b_i\right)^+, \Gamma_i\right) \quad \text{if } R_i(S_i) = \frac{a_i S_i}{b_i + S_i} \tag{3.3.6a}$$

and

$$S_i = \left(\frac{a_i - p(x_i,t)}{2b_i}\right)^+ \quad \text{if } R_i(S_i) = a_i S_i - b_i S_i^2 \quad \text{for } 1 \leq i \leq k. \tag{3.3.6b}$$

Proof

Suppose $\vec{S} = (S_1, \ldots, S_k)$ is an optimal control and w is its corresponding solution. For i fixed in $\{1, \ldots, k\}$, consider $S_i + \varepsilon h \in A$ with associated solution $w^\varepsilon = w(S_1, \ldots, S_i + \varepsilon h, \ldots, S_k)$ for $1 \leq i \leq k$ where $h \in L^\infty(0,T)$. Since the adjoint equation is linear, there exists p satisfying (3.3.6) ([3]). Since the maximum of the payoff functional is attained at S,

$$\begin{aligned}
0 &\geq \lim_{\varepsilon \to 0+} \frac{J(S_1, \ldots, S_i + \varepsilon h, \ldots, S_k) - J(S_1, \ldots, S_k)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^T e^{-rt} \left[R_i(S_i + \varepsilon h) - R_i(S_i)\right] - \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_Q e^{-rt} \left[f(w^\varepsilon) - f(w)\right] \\
&= \int_0^T e^{-rt} \frac{\partial R_i}{\partial S_i} h - \int_Q e^{-rt} f'(w) \psi^i \\
&= \int_0^T e^{-rt} \frac{\partial R_i}{\partial S_i} h - \int_Q e^{-rt} \left[-\psi^i p_t + (\psi^i)_x \beta p_x - \psi^i (\alpha p)_x + (D(x) + r) p \psi^i\right] \\
&\qquad - \int_0^T e^{-rt} \alpha p \psi^i (b,t) \\
&= \int_0^T e^{-rt} \frac{\partial R_i}{\partial S_i} h - \int_Q e^{-rt} \left[\left(\psi_t^i\right) p - r \psi^i p + \left(\psi^i\right)_x \beta p_x + \alpha p \left(\psi^i\right)_x \right. \\
&\qquad \left. + (D(x) + r) p \psi^i\right]
\end{aligned}$$

$$+\int_{\Omega\times\{T\}} e^{-rt}\psi^i p - \int_{\Omega\times\{0\}} e^{-rt} p\psi^i$$
$$= \int_0^T e^{-rt}\frac{\partial R_i}{\partial S_i}h - \int_Q e^{-rt}\left[\left(\psi^i_t\right)p + \beta p_x\left(\psi^i\right)_x + \alpha p\left(\psi^i\right)_x + D(x)\psi^i p\right]$$
$$= \int_0^T e^{-rt}\frac{\partial R_i}{\partial S_i}h - \int_0^T e^{-rt}hp(x_i,t).$$

Considering the form of the state equation and the adjoint equation, the parabolic maximum principle [2] implies $u > 0$ and $p > 0$.

From the above work, we have

$$0 \geq \int_0^T e^{-rt}h\left[\frac{\partial R_i}{\partial S_i} - p(x_i,t)\right].$$

In case 1, when

$$R_i(S_i) = \frac{a_i S_i}{b_i + S_i}, \quad \frac{\partial R_i}{\partial S_i} = \frac{(b_i+S_i)a_i - a_i S_i}{(b_i+S_i)^2} = \frac{a_i b_i}{(b_i+S_i)^2}.$$

Then

$$0 \geq \int_0^T e^{-rt}h\left[\frac{a_i b_i}{(b_i+S_i)^2} - p(x_i,t)\right].$$

On the set $\{(x,t) \in Q \mid S_i = 0\}$, we can choose non-negative variations h with support on this set. Then

$$\frac{a_i b_i}{(b_i+S_i)^2} - p(x_i,t) \leq 0 \Rightarrow \frac{a_i b_i}{(b_i+S_i)^2} \leq p(x_i,t) \Rightarrow \frac{a_i b_i}{p(x_i,t)} \leq (b_i+S_i)^2.$$

Since $S_i = 0$, then $\sqrt{\frac{a_i b_i}{p(x_i,t)}} \leq b_i$.

On the set $\{(x,t) \in Q \mid 0 < S_i < \Gamma_i\}$, we can choose variations h with arbitrary sign and with support on this set. Then

$$\frac{a_i b_i}{(b_i+S_i)^2} - p(x_i,t) = 0 \Rightarrow \frac{a_i b_i}{p(x_i,t)} = (b_i+S_i)^2 \Rightarrow S_i = -b_i + \sqrt{\frac{a_i b_i}{p(x_i,t)}}.$$

On the set $\{(x,t) \in Q \mid S_i = \Gamma_i\}$, we can choose nonpositive variations h with support on this set. Then

$$\frac{a_i b_i}{(b_i + S_i)^2} - p(x_i,t) \geq 0 \;\Rightarrow\; \frac{a_i b_i}{p(x_i,t)} - (b_i + S_i)^2 \geq 0$$
$$\Rightarrow \left(\sqrt{\frac{a_i b_i}{p(x_i,t)}} - (b_i + S_i)\right)\left(\sqrt{\frac{a_i b_i}{p(x_i,t)}} + (b_i + S_i)\right) \geq 0.$$

Since a_i, b_i, $p(x_i,t)$, and S_i are positive, we require that $\sqrt{\frac{a_i b_i}{p(x_i,t)}} - (b_i + S_i) \geq 0$. Then $\sqrt{\frac{a_i b_i}{p(x_i,t)}} \geq b_i + S_i$ or $\sqrt{\frac{a_i b_i}{p(x_i,t)}} - b_i \geq S_i = \Gamma_i$. Combining these three cases, we have

$$S_i = \min\left(\left(\sqrt{\frac{a_i b_i}{p(x_i,t)}} - b_i\right)^+, \Gamma_i\right).$$

In case 2, when

$$R_i = a_i S_i - b_i S_i^2, \quad \frac{\partial R_i}{\partial S_i} = a_i - 2b_i S_i$$

we have

$$0 \geq \int_0^T e^{-rt} h\left[(a_i - 2b_i S_i) - p(x_i,t)\right].$$

On the set $\{(x,t) \in Q \mid S_i = 0\}$, we can choose non-negative variations h with support on this set. Then

$$a_i - 2b_i S_i - p(x_i,t) \leq 0 \quad \Rightarrow \quad a_i - p(x_i,t) \leq 0.$$

On the set $\{(x,t) \in Q \mid 0 < S_i < \Gamma_i\}$, we can choose variations h with arbitrary sign and with support on this set. Then

$$a_i - 2b_i S_i - p(x_i,t) = 0 \quad \Rightarrow \quad S_i = \frac{a_i - p(x_i,t)}{2b_i}.$$

On the set $\{(x,t) \in Q \mid S_i = \Gamma_i\}$, we can choose non-positive variations h with support on this set. Then

$$a_i - 2b_iS_i - p(x_i,t) \geq 0 \quad \Rightarrow \quad \frac{a_i - p(x_i,t)}{2b_i} \geq S_i = \Gamma_i.$$

Combining these three cases,

$$S_i = \min\left(\left(\frac{a_i - p(x_i,t)}{2b_i}\right)^+, \Gamma_i\right).$$

We show $S_i = \min\left(\left(\frac{a_i-p(x_i,t)}{2b_i}\right)^+, \Gamma_i\right)$ is bounded independent of Γ_i. Notice by a parabolic maximum principle that $p > 0$ (from the adjoint problem) since $f'(u) > 0$. Moreover for any strip x_i, $-p(x_i,t) < 0$. Therefore $\left(\frac{a_i-p(x_i,t)}{2b_i}\right)^+ \leq \frac{a_i-p(x_i,t)}{2b_i} < \frac{a_i}{2b_i} = M_1$ where M_1 is independent of Γ_i, [4]. For $\Gamma_i > M_1$ for $1 \leq i \leq k$, our control representation becomes $S_i = \left(\frac{a_i-p(x_i,t)}{2b_i}\right)^+$. For the other case, we use $S_i = \min\left(\left(\sqrt{\frac{a_ib_i}{p(x_i,t)}} - b_i\right)^+, \Gamma_i\right)$ because we do not necessarily know if the solution to the adjoint equation is bounded below by a constant. $\square$

Using $R_i(S_i) = \frac{a_iS_i}{b_i+S_i}$, we can form an optimality system using the relationship between our optimal controls and the associated adjoint variables from

Theorem 3.2 as follows

$$\begin{aligned}
&w_t - (\beta w_x)_x + \alpha w_x + D(x)(u_a + w) = \\
&\qquad \sum_{i=1}^{k} \delta(x - x_i) \min\left(\left(\sqrt{\frac{a_i b_i}{p(x_i,t)}} - b_i\right)^+, \Gamma_i\right) \text{ in } Q \\
&\qquad - p_t - (\beta p_x)_x - (\alpha p)_x + D(x)p + rp = f'(u_a + w) \text{ in } Q \\
&\qquad w(x,0) = u_0(x) - u_a, \quad p(x,T) = 0, \quad x \in \Omega \\
&\qquad w_x(b,t) = 0, \quad (\beta p_x + \alpha p)(b,t) = 0, \quad 0 < t < T \\
&\qquad w(a,t) = 0, \quad p(a,t) = 0, \quad 0 < t < T.
\end{aligned} \tag{3.3.7}$$

By the theorem above, there exists a solution to the optimality system.

We now consider uniqueness for the optimality system (3.3.7).

Theorem 3.3

When T is sufficiently small and $R_i(S_i) = \frac{a_i S_i}{b_i + S_i}$, the solution to the optimality system (3.3.7) is unique.

Proof

Suppose (w,p) and (v,q) are two different solutions of the optimality system. Let $w = e^{\lambda t}\xi$, $v = e^{\lambda t}\overline{\xi}$, $p = e^{-\lambda t}y$, and $q = e^{-\lambda t}\overline{y}$. Also suppose $S_i = \min\left(\left(\sqrt{\frac{a_i b_i e^{\lambda t}}{y(x_i,t)}} - b_i\right)^+, \Gamma_i\right)$ and $\overline{S}_i = \min\left(\left(\sqrt{\frac{a_i b_i e^{\lambda t}}{\overline{y}(x_i,t)}} - b_i\right)^+, \Gamma_i\right)$ where $\lambda > 0$ is to be chosen. We subtract the weak PDE forms of w and v and p and q.

$$\begin{aligned}
&\int_Q (\xi - \overline{\xi})_t \phi \, dxdt + \int_Q \beta(\xi - \overline{\xi})_x \phi_x \, dxdt + \int_Q \alpha(\xi - \overline{\xi})_x \phi \, dxdt \\
&\qquad + \int_Q (D(x) + \lambda)(\xi - \overline{\xi})\phi \, dxdt \\
&= \int_0^T e^{-\lambda t} \sum_{i=1}^{k} (S_i - \overline{S}_i)\phi(x_i,t) \, dt
\end{aligned}$$

and

$$
\begin{aligned}
-\int_Q (y-\overline{y})_t \phi \, dxdt + \int_Q \beta (y-\overline{y})_x \phi_x \, dxdt + \int_Q \alpha (y-\overline{y}) \phi_x \, dxdt \\
+ \int_Q (D(x) + r + \lambda)(y-\overline{y}) \phi \, dxdt \\
= \int_Q e^{\lambda t} \left[f'(e^{\lambda t}\xi) - f'(e^{\lambda t}\overline{\xi}) \right] \phi \, dxdt
\end{aligned}
$$

where $\phi \in W$. By the standard estimation techniques, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_Q \left\{ \left[\xi(x,T) - \overline{\xi}(x,T)\right]^2 + [y(x,0) - \overline{y}(x,0)]^2 \right\} \\
&\quad + \nu \int_Q [(\xi-\overline{\xi})_x^2 + (y-\overline{y})_x^2] + \lambda \int_Q [(\xi-\overline{\xi})^2 + (y-\overline{y})^2] \\
&\le -\int_Q \alpha (\xi-\overline{\xi})_x (\xi-\overline{\xi}) - \int_Q \alpha (y-\overline{y})_x (y-\overline{y}) - \int_Q D(x)(\xi-\overline{\xi})^2 \\
&\quad - \int_Q (D(x)+r)(y-\overline{y})^2 + \int_0^T e^{-\lambda t} \sum_{i=1}^k (S_i - \overline{S}_i)(\xi-\overline{\xi})(x_i,t) \\
&\quad + \int_Q e^{\lambda t} \left[f'(e^{\lambda t}\xi) - f'(e^{\lambda t}\overline{\xi}) \right] (y-\overline{y}).
\end{aligned}
$$

By Cauchy's inequality and by f' being a Lipschitz function, we have

$$
\begin{aligned}
&\frac{1}{2}\int_Q \left\{ \left[\xi(x,T) - \overline{\xi}(x,T)\right]^2 + [y(x,0) - \overline{y}(x,0)]^2 \right\} \\
&\quad + \frac{3\nu}{4} \int_Q [(\xi-\overline{\xi})_x^2 + (y-\overline{y})_x^2] + \lambda \int_Q [(\xi-\overline{\xi})^2 + (y-\overline{y})^2] \\
&\le (C_1 + D_r) \int_Q [(\xi-\overline{\xi})^2 + (y-\overline{y})^2] + \int_{\{x_i\}\times(0,T)} e^{-\lambda t} \left(\sum_{i=1}^k |S_i - \overline{S}_i| \right)^2 \\
&\quad + \int_{\{x_i\}\times(0,T)} e^{-\lambda t} (\xi-\overline{\xi})^2 + \int_Q M e^{2\lambda t} (\xi-\overline{\xi})(y-\overline{y}).
\end{aligned}
\tag{3.3.8}
$$

Also, we have

$$
\begin{aligned}
|S_i - \overline{S}_i| &= \left| \min\left(\left(\sqrt{\frac{a_i b_i e^{\lambda t}}{y(x_i,t)}} - b_i\right)^+, \Gamma_i\right) - \min\left(\left(\sqrt{\frac{a_i b_i e^{\lambda t}}{\overline{y}(x_i,t)}} - b_i\right)^+, \Gamma_i\right)\right| \\
&\leq \left| \sqrt{\frac{a_i b_i e^{\lambda t}}{y(x_i,t)}} - \sqrt{\frac{a_i b_i e^{\lambda t}}{\overline{y}(x_i,t)}} \right|^2 \\
&\leq e^{\lambda t} a_i b_i \left| \frac{1}{\sqrt{y(x_i,t)}} - \frac{1}{\sqrt{\overline{y}(x_i,t)}} \right|.
\end{aligned}
$$

First, let us notice that $y(x_i,t) \geq \frac{\inf f'(u_a + e^{\lambda t}\xi)}{r}$ by a parabolic maximum principle. A similar estimate holds for $\overline{y}$. Also if we let $\delta = \min_{x\in\Omega}\{u_a, \inf u_0\}$, then we have a lower bound, $\inf u \geq \delta > 0$. In addition,

$$
\begin{aligned}
\left| \frac{1}{\sqrt{y(x_i,t)}} - \frac{1}{\sqrt{\overline{y}(x_i,t)}} \right| &= \left| \frac{\sqrt{\overline{y}(x_i,t)} - \sqrt{y(x_i,t)}}{\sqrt{y(x_i,t)}\sqrt{\overline{y}(x_i,t)}} \right| \\
&= \left| \frac{\overline{y}(x_i,t) - y(x_i,t)}{\sqrt{y(x_i,t)}\sqrt{\overline{y}(x_i,t)}\left(\sqrt{y(x_i,t)} + \sqrt{\overline{y}(x_i,t)}\right)} \right|.
\end{aligned}
$$

Using that $y(x_i,t) \geq \frac{\inf f'(u)}{r}$, [2], we have

$$
\left| \frac{1}{\sqrt{y(x_i,t)}} - \frac{1}{\sqrt{\overline{y}(x_i,t)}} \right| \leq \frac{r^{\frac{3}{2}} |\overline{y}(x_i,t) - y(x_i,t)|}{(\inf f'(u))^{\frac{3}{2}}}.
$$

Notice that $(\sum_{i=1}^k |S_i - \overline{S}_i|)^2 \leq C_2 \sum_{i=1}^k |S_i - \overline{S}_i|^2$, where C_2 is a constant. Upon further estimation in (3.3.8), we have

$$
\begin{aligned}
&\frac{1}{2}\int_Q \left\{ \left[\xi(x,T) - \overline{\xi}(x,T)\right]^2 + [y(x,0) - \overline{y}(x,0)]^2 \right\} + \frac{\nu}{2}\int_Q [(\xi - \overline{\xi})_x^2 + (y - \overline{y})_x^2] \\
&\qquad + (\lambda - C_3 - Me^{2\lambda T}) \int_Q [(\xi - \overline{\xi})^2 + (y - \overline{y})^2] \\
&\quad \leq C_2 \int_0^T \frac{r^3}{[\inf f'(u)]^3} \sum_{i=1}^k |y(x_i,t) - \overline{y}(x_i,t)|^2 .
\end{aligned}
\tag{3.3.9}
$$

By using a trace estimate on the right-hand side, we have

$$\frac{1}{2}\int_Q \left\{\left[\xi(x,T)-\overline{\xi}(x,T)\right]^2+\left[y(x,0)-\overline{y}(x,0)\right]^2\right\}+\frac{\nu}{2}\int_Q \left[(\xi-\overline{\xi})_x^2+(y-\overline{y})_x^2\right]$$
$$+(\lambda-C_3-\widetilde{C}_2-Me^{2\lambda T})\int_Q \left[(\xi-\overline{\xi})^2+(y-\overline{y})^2\right]\leq 0 \tag{3.3.10}$$

where $\widetilde{C}_2$, C_3, M depend on the coefficients, b_i, r, $\inf f'(u)$. Let $C_4 = C_3+\widetilde{C}_2$. We need to choose λ such that

$$\lambda - C_4 - Me^{2\lambda T} > 0.$$

So we have $\frac{\lambda - C_4}{M} > e^{2\lambda T}$, which implies

$$\ln\left(\frac{\lambda - C_4}{M}\right) > 2\lambda T$$

and

$$\frac{1}{2\lambda}\ln\left(\frac{\lambda - C_4}{M}\right) > T.$$

We choose λ such that $\frac{1}{2\lambda}\ln\left(\frac{\lambda-C_4}{M}\right) > T$ which can be satisfied if λ is sufficiently large, (i.e. $\lambda > C_4 + M$) and T is sufficiently small. Therefore, $\xi = \overline{\xi}$ and $y = \overline{y}$. Hence, $w = v$ and $p = q$. $\square$

Remark. We can prove the analogous uniqueness result if $R_i(S_i) = a_iS_i - b_iS_i^2$.

Examples of the cost function f that satisfies our assumptions are

$$f(u) = k_1u^2 + k_2u + k_3$$

or

$$f(u) = \frac{k_4 u}{k_5 + u}.$$

In conclusion, we have characterized the unique optimal control in terms of the unique solution of the optimality system. This control gives the optimal contaminant loading strategy in a river flow.

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Part IV

Optimizing Chemotherapy in an HIV Model

1. Introduction

Various chemotherapies for patients with human immunodeficiency virus (HIV) are being examined to determine the optimal scheme for treatment. We analyze the optimal chemotherapy strategy, given that the treatment can only be used for a finite interval. We utilize an ordinary differential equation model which describes the interaction of HIV in the immune system. We explore optimal control of this ordinary differential equation model. We assume that our treatment control affects the ability of the virus to further infect the patient.

Let T represent the concentration of the uninfected $CD4^+$ T cells (a type of human immune cell), and let T^* and T^{**} denote the concentrations of latently infected and actively infected $CD4^+$ T cells. Let V denote the concentration of free infectious virus particles. We assume that the populations evolve as follows [4]:

$$\frac{dT}{dt} = \frac{s}{1+V} - \mu_T T + rT\left(1 - \frac{T+T^*+T^{**}}{T_{MAX}}\right) - K_1 VT \tag{4.1.1}$$

$$\frac{dT^*}{dt} = K_1 VT - \mu_T T^* - K_2 T^* \tag{4.1.2}$$

$$\frac{dT^{**}}{dt} = K_2 T^* - \mu_b T^{**} \tag{4.1.3}$$

$$\frac{dV}{dt} = N\mu_b T^{**} - K_1 VT - \mu_v V \tag{4.1.4}$$

with initial conditions $T(0) = T_0$, $T^*(0) = T_0^*$, $T^{**}(0) = T_0^{**}$, and $V(0) = V_0$.

In (4.1.1), $\frac{s}{1+V}$ is a source term from the thymus and represents the rate of generation of new $CD4^+$ T cells. The T cells have a finite life span with a death rate μ_T per cell. In (4.1.2), latently infected T cells are assumed to have a natural

death rate, μ_T, even though other factors can change the natural death rate. In (4.1.1), r is the coefficient of the growth rate of T cells, which is a logistic-type growth. This growth ensures that the T cells never grow larger than T_{MAX}.

In (4.1.1) and (4.1.2), the term K_1VT models the rate that free virus infects $CD4^+$ T cells. After a T cell becomes infected, it becomes a latently infected T cell. Hence the K_1VT term is subtracted from (4.1.1) and added to (4.1.2).

Equation (4.1.3) describes the actively infected $CD4^+$ T cells. At the rate K_2, latently infected T cells become actively infected. The actively infected T cells manufacture virus and die at a rate per cell μ_b. Equation (4.1.4) models the free virus population. We assume that when an actively infected $CD4^+$ T cell becomes stimulated by antigen exposure, replication of the virus begins. Further, N viruses are formed before the host cell dies. We assume that free virus is lost by connecting to $CD4^+$ T cells at a rate K_1. The term $-\mu_v V$ takes into account loss of infectivity or removal from the body.

See the references in [1],[4] for similar models of HIV infection. We deal with an optimal chemotherapy treatment with our control affecting the interaction term K_1VT. In section 2, we introduce the optimal control problem. We also discuss a constraint on T_{MAX} and a reason for $T(t)$ to be less than T_{MAX}. In section 3, we show the existence of the optimal control. In section 4, we seek to maximize the objective functional which is based on the benefit of the T cell less the cost of the virus, cost of the drugs, and the cost of the damage to the patient's body. The

optimal control is characterized using Pontryagin's Maximum Principle. In section 5, uniqueness of the optimality system, which is the state system coupled with the adjoint system, is determined.

2. Formulation of the Problem

Our control represents the percentage of effect the chemotherapy has on interaction of T cells with the virus. The control for the chemotherapy, $u(t)$, multiplies the interaction term K_1VT in equations (4.1.1) and (4.1.2). Therefore, we choose as our control class, measurable functions defined on $[t_0, t_1]$, with the condition $0 \leq u(t) \leq 1$. The interval of treatment is necessary since we assume that chemotherapy only has a certain designated time for allowable treatment. After some finite time frame, HIV is able to build up resistance to the treatment due to its mutation ability. Also, chemotherapy has potentially hazardous side effects. Therefore, the length of treatment is restricted. Hence, for $t_0 \leq t \leq t_1$ where $t_1 - t_0 < 2$ years, the state system is

$$\frac{dT}{dt} = \frac{s}{1+V} - \mu_T T + rT\left(1 - \frac{T + T^* + T^{**}}{T_{MAX}}\right) - u(t)K_1VT \tag{4.2.1}$$

$$\frac{dT^*}{dt} = u(t)K_1VT - \mu_T T^* - K_2T^* \tag{4.2.2}$$

$$\frac{dT^{**}}{dt} = K_2T^* - \mu_b T^{**} \tag{4.2.3}$$

$$\frac{dV}{dt} = N\mu_b T^{**} - K_1VT - \mu_v V \tag{4.2.4}$$

with given initial values for T, T^*, T^{**}, and V at t_0.

Define the objective functional

$$J(u) = \int_{t_0}^{t_1} \left[B_1 T(t) - B_2 V(t) - B_3(1-u) - \frac{1}{2} B_4 (1-u)^2 \right] dt. \qquad (4.2.4a)$$

The parameters B_1, B_2, B_3, and B_4 represent the "weight" on the benefit and cost. We are maximizing the benefit based on the T cell count, minimizing the level of virus, and minimizing the systemic cost based on the percentage effect of the chemotherapy given. If $u(t) = 0$ represents maximal use of chemotherapy, then the maximal cost is represented by $(1-u)$ and $(1-u)^2$. Our goal is to characterize the optimal control u^* satisfying $\max_{0 \leq u \leq 1} J(u) = J(u^*)$.

There are certain parameter restrictions that we impose to ensure that this model is realistic. The steady state population size T_{equil} should be below T_{MAX} in order for the T cell population to expand when stimulated by the infection of HIV. Furthermore if the population ever gets near T_{MAX}, it's growth should slow. For the death rate at T_{MAX} to be greater than the supply rate, s, we assume that

$$\mu_T T_{MAX} > s, \qquad (4.2.5)$$

which is also assumed in [6].

In the absence of virus, T cell population has a steady state value which comes from setting $\frac{dT}{dt}$, $\frac{dT^*}{dt}$, and $\frac{dT^{**}}{dt}$ equal to zero. See [6] for analysis of the steady state behavior of (4.2.1) - (4.2.4). Consequently, we obtain, $T^* = 0$, $T^{**} = 0$,

and $s - (\mu_T - r)T - \frac{r}{T_{MAX}}T^2 = 0$ at equilibrium. Moreover,

$$
\begin{aligned}
T_{equil} &= \frac{(\mu_T - r) \pm \sqrt{(\mu_T - r)^2 + 4\frac{r}{T_{MAX}}s}}{-2\frac{r}{T_{MAX}}} \\
&= \frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r} \pm \left(\frac{\sqrt{\mu_T^2 - 2r\mu_T + r^2 + \frac{4sr}{T_{MAX}}}}{\sqrt{r^2}}\right)\right] \\
&= \frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r} \pm \sqrt{\left(1 - \frac{\mu_T}{r}\right)^2 + \frac{4s}{rT_{MAX}}}\right].
\end{aligned}
$$

Since we desire positive concentration of T cells, we consider

$$
T_{equil} = \frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r} + \sqrt{\left(1 - \frac{\mu_T}{r}\right)^2 + \frac{4s}{rT_{MAX}}}\right]. \tag{4.2.6}
$$

Utilizing (4.2.5), we see that $T_{equil} < T_{MAX}$ as shown below:

$$
\begin{aligned}
T_{equil} &< \frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r} + \sqrt{\left(1 - \frac{\mu_T}{r}\right)^2 + \frac{4\mu_T T_{MAX}}{rT_{MAX}}}\right] \\
&= \frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r} + \sqrt{1 + \frac{2\mu_T}{r} + \frac{\mu_T^2}{r^2}}\right]
\end{aligned}
$$

or $T_{equil} < T_{MAX}$.

We now show that $T(t) < T_{MAX}$ for all t. We consider (4.2.1) with $T^* = T^{**} = V = 0$

$$
\frac{d\widehat{T}}{dt} = s - \mu_T\widehat{T} + r\widehat{T}\left(1 - \frac{\widehat{T}}{T_{MAX}}\right) = f(\widehat{T}). \tag{4.2.7}
$$

Let $\widetilde{T}_0 = \frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r}\right] < T_{equil}$. Then

$$
\begin{aligned}
f(\widetilde{T}_0) &= s - (\mu_T - r)\frac{T_{MAX}}{2}\left[1 - \frac{\mu_T}{r}\right] - \frac{r}{T_{MAX}}\left(\frac{T_{MAX}}{2r}[r - \mu_T]\right)^2 \\
&= s + (r - \mu_T)^2\frac{T_{MAX}}{2r} - \frac{T_{MAX}}{4r}(r - \mu_T)^2 \\
&= s + \frac{T_{MAX}}{4r}(r - \mu_T)^2.
\end{aligned}
$$

Now let $\widetilde{T}_1 = \frac{T_{MAX}}{2}\left[1+\sqrt{(1-\frac{\mu_T}{r})^2+\frac{4s}{rT_{MAX}}}\right] > T_{equil}$. Then

$$\begin{aligned}
f(\widetilde{T}_1) &= s + (r-\mu_T)\frac{T_{MAX}}{2}\left[1+\sqrt{\left(1-\frac{\mu_T}{r}\right)^2+\frac{4s}{rT_{MAX}}}\right] \\
&\quad - \frac{rT_{MAX}}{4}\left[1+\sqrt{\left(1-\frac{\mu_T}{r}\right)^2+\frac{4s}{rT_{MAX}}}\right]^2 \\
&= s + (r-\mu_T)\frac{T_{MAX}}{2}\left[1+\sqrt{\left(1-\frac{\mu_T}{r}\right)^2+\frac{4s}{rT_{MAX}}}\right] \\
&\quad - \frac{rT_{MAX}}{4} - \frac{rT_{MAX}}{2}\left[\sqrt{\left(1-\frac{\mu_T}{r}\right)^2+\frac{4s}{rT_{MAX}}}\right] \\
&\quad - \frac{T_{MAX}}{4r}(r-\mu_T)^2 - s \\
&= \frac{-\mu_T T_{MAX}}{2}\left[\sqrt{\left(1-\frac{\mu_T}{r}\right)^2+\frac{4s}{rT_{MAX}}}\right] - \frac{\mu_T^2 T_{MAX}}{4r} \\
&< 0.
\end{aligned}$$

By examining the phase portrait, $\widehat{T}(t) < T_{MAX}$. Now if we compare (4.1.1) and (4.2.7) with the same initial conditions, then $T(t) < \widehat{T}(t)$. Hence, $T(t) < T_{MAX}$. Refer to [6].

3. Existence of an Optimal Control

To determine existence of an optimal control to our problem, we use a result from ([2], Theorem 4.1, pg. 68-69).

Theorem 3.1

Consider a control problem with system equations

$$\vec{T}' = \vec{f}\left(t, \vec{T}(t), u(t)\right), t_0 \le t \le t_1$$
$$\vec{T}(t_0) = \vec{T_0} \tag{4.3.1}$$

where $\vec{T} = (T, T^*, T^{**}, V)$, $u \in U$ and

$$U = \{u \text{ measurable} \mid 0 \le u(t) \le 1, t \in [t_0, t_1]\}$$

is the control class. The objective functional is defined by

$$J(u) = \int_{t_0}^{t_1} L\left(t, \vec{T}(t), u(t)\right)\ dt$$

where $L\left(t, \vec{T}(t), u(t)\right)$ is continuous. The following assumptions must hold

i) F', the class of all $(\vec{T}_o, u)$ such that u is a Lebesgue-integable function on $[t_0, t_1]$ with values in U and $\vec{T}$ satisfies (4.3.1) with initial condition $\vec{T_0}$, is not empty.

ii) U is closed and convex.

iii) $\vec{f}$ is continuous, $|\vec{f}(t, \vec{T}, u)| \le C_1\left(1 + |\vec{T}| + |u|\right)$, $\vec{f}(t, \vec{T}, u) = \vec{\alpha}(t, \vec{T}) + \vec{\beta}(t, \vec{T})u$ for $t_0 \le t \le t_1$, $\vec{T} \in \mathbb{R}^n$, $u \in \mathbb{R}$.

iv) $L\left(t, \vec{T}, \cdot\right)$ is concave on U.

v) $L\left(t, \vec{T}, u\right) \le c_2 - c_1|u|^\beta$, with $c_1 > 0$, $\beta > 1$.

Then there exists $u^* \in U$ such that

$$\max_{u \in U} J(u) = J(u^*).$$

To verify the assumptions of Theorem 3.1, first, we use a result in Lukes ([5], Theorem 9.2.1) to give existence of solutions to our ODE system (4.2.1) - (4.2.4) with bounded coefficients. Hence assumption (i) holds. Then note, U is closed and convex and (iii) holds, since our state system is bilinear in u.

Next, we need that $L(t, \vec{T}, \cdot)$ is concave on U, where

$$L(t, \vec{T}, u) = B_1 T(t) - B_2 V(t) - B_3(1-u) - \frac{1}{2} B_4 (1-u)^2.$$

We show that

$$L\left[t, \vec{T}, (1-\epsilon)u_1 + \epsilon u_2\right] \geq (1-\epsilon) L[t, \vec{T}, u_1] + \epsilon L[t, \vec{T}, u_2] \quad \text{for } 0 < \epsilon < 1.$$

We see that

$$\begin{aligned} L\left[t, \vec{T}, (1-\epsilon)u_1 + \epsilon u_2\right] &= B_1 T(t) - B_2 V(t) - B_3 \left(1 - [(1-\epsilon)u_1 + \epsilon u_2]\right) \\ &\quad - \frac{1}{2} B_4 \left(1 - [(1-\epsilon)u_1 + \epsilon u_2]\right)^2. \end{aligned}$$

We have

$$\begin{aligned} (1-\epsilon) L[t, \vec{T}, u_1] + \epsilon L[t, \vec{T}, u_2] &= B_1 T(t) - B_2 V(t) - B_3 (1-\epsilon)(1-u_1) \\ &\quad - (1-\epsilon)\frac{1}{2} B_4 (1-u_1)^2 - \epsilon B_3 (1-u_2) - \frac{\epsilon}{2} B_4 (1-u_2)^2 \\ &= B_1 T(t) - B_2 V(t) - B_3 \left(1 - ((1-\epsilon)u_1 + \epsilon u_2)\right) \\ &\quad - \frac{1}{2} B_4 \left[(1-\epsilon)(1-u_1)^2 + \epsilon (1-u_2)^2\right]. \end{aligned}$$

To show that $L(t, \vec{T}, \cdot)$ is concave on U, we note the inequality below is true.

$$\left(1 - [(1-\epsilon)u_1 + \epsilon u_2]\right)^2 \leq (1-\epsilon)(1-u_1)^2 + \epsilon (1-u_2)^2.$$

So

$$L\left[t,\vec{T},(1-\epsilon)u_1+\epsilon u_2\right] \geq (1-\epsilon)L[t,\vec{T},u_1]+\epsilon L[t,\vec{T},u_2]$$

for $0<\epsilon<1$.

To complete the existence of an optimal control, we need to show that $L\left(t,\vec{T},u\right) \leq c_2 - c_1|u|^{\beta}$, where $c_1>0$, and $\beta>1$. So, in our case, we have

$$\begin{aligned}
L\left(t,\vec{T},u\right) &= B_1T(t) - B_2V(t) - B_3(1-u) - \frac{1}{2}B_4(1-u)^2 \\
&\leq D_1(B_3+B_4)u - \frac{1}{2}B_4u^2 \\
&\leq C_2 + \frac{B_5}{2}u^2 - B_5u^2 \\
&= C_2 - \frac{B_5}{2}u^2
\end{aligned}$$

where B_5, D_1, C_1, C_2 depend on B_1, B_2, B_3, B_4, $||T||$, $||V||$ and $B_5>0$.

4. Optimality System

Since an optimal control exists for maximizing the functional (4.2.4a) subject to (4.2.1)-(4.2.4), then we use Pontryagin's Maximum Principle to derive necessary conditions on that optimal control [3].

Theorem 4.1

Given an optimal control u^* and solutions of the corresponding state system

(4.2.1) - (4.2.4), there exist adjoint variables λ_i, $i = 1, \ldots, 4$ satisfying

$$\begin{aligned}
\lambda_1' = -\frac{\partial L}{\partial T} &= -\Bigg\{\Bigg[B_1 + \lambda_1\left(-\mu_T + r\left(1 - \frac{T + T^* + T^{**}}{T_{MAX}}\right) + rT\left(-\frac{1}{T_{MAX}}\right)\right.\\
&\qquad \left.- u(t)K_1V\right)\Bigg] + \lambda_2 u(t)K_1V - \lambda_4 K_1 V\Bigg\}\\
\lambda_2' = -\frac{\partial L}{\partial T^*} &= -\left(-\frac{\lambda_1 rT}{T_{MAX}} - \lambda_2(\mu_T + K_2) + \lambda_3 K_2\right) \qquad (4.4.1)\\
\lambda_3' = -\frac{\partial L}{\partial T^{**}} &= -\left(-\frac{\lambda_1 rT}{T_{MAX}} - \lambda_3\mu_b + \lambda_4 N\mu_b\right)\\
\lambda_4' = -\frac{\partial L}{\partial V} &= -\left(\frac{-s\lambda_1}{(1+V)^2} - \lambda_1 u(t)K_1V + \lambda_2 u(t)K_1V - \lambda_4(K_1V + \mu_v) - B_2\right)
\end{aligned}$$

with $\lambda_1(t_1) = \lambda_2(t_1) = \lambda_3(t_1) = \lambda_4(t_1) = 0$ transversality conditions. Further, u^* is represented by

$$u^* = \min\left(1, \left(\frac{(\lambda_2 - \lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right).$$

Proof

We define the Lagrangian as the following:

$$\begin{aligned}
&\mathcal{L}(T, T^*, T^{**}, V, u, \lambda_1, \lambda_2, \lambda_3, \lambda_4)\\
&\quad = B_1T - B_2V - B_3(1-u) - \frac{1}{2}B_4(1-u)^2\\
&\qquad + \lambda_1\left(\frac{s}{1+V} - \mu_T T + rT\left(1 - \frac{T + T^* + T^{**}}{T_{MAX}}\right) - u(t)K_1VT\right)\\
&\qquad + \lambda_2\left(u(t)K_1VT - \mu_T T^* - K_2T^*\right)\\
&\qquad + \lambda_3\left(K_2T^* - \mu_b T^{**}\right) + \lambda_4\left(N\mu_b T^{**} - K_1VT - \mu_v V\right)\\
&\qquad + w_1(t)u(t) + w_2(t)(1 - u(t))
\end{aligned} \qquad (4.4.2)$$

where $w_1(t) \geq 0$, $w_1(t) \geq 0$ are penalty multipliers satisfying

$$w_1(t)u(t) = 0, \quad w_2(t)(1 - u(t)) = 0 \quad \text{at the optimal } u^*.$$

In [3], the maximum principle gives the existence of adjoint variables satisfying (4.4.1).

Since

$$\mathcal{L} = -\frac{1}{2}B_4(1 - u(t))^2 - \lambda_1 u(t)K_1VT + \lambda_2 u(t)K_1VT + w_1(t)u(t) - B_3(1 - u(t))$$
$$+ w_2(t)(1 - u(t)) + \text{ other terms without } u,$$

then by differentiating this expression for L with respect to u, we have

$$\frac{\partial \mathcal{L}}{\partial u} = B_4(1 - u(t)) - \lambda_1 K_1VT + \lambda_2 K_1VT + w_1(t) - w_2(t) + B_3 = 0.$$

Solving for the optimal control yields

$$u^*(t) = \frac{(\lambda_2 - \lambda_1)K_1VT + w_1(t) - w_2(t) + B_4 + B_3}{B_4}.$$

To determine an explicit expression for the optimal control (without w_1 and w_2), we utilize a standard optimality technique. We consider three cases:

(i) On the set $\{t|0 < u^*(t) < 1\}$, we have $w_1(t) = 0 = w_2(t)$. Hence the optimal control is

$$u^*(t) = \frac{(\lambda_2 - \lambda_1)K_1VT + B_4 + B_3}{B_4}.$$

(ii) On the set $\{t|u^*(t) = 1\}$, we have $w_1(t) = 0$. Hence,

$$1 = u^*(t) = \frac{(\lambda_2 - \lambda_1)K_1VT - w_2(t) + B_3}{B_4} + 1.$$

This implies that $0 \leq w_2(t) = \frac{(\lambda_2-\lambda_1)K_1VT+B_3}{B_4}$ and $1 = u^*(t) \leq \frac{(\lambda_2-\lambda_1)K_1VT+B_4+B_3}{B_4}$.

(iii) On the set $\{t|u^*(t) = 0\}$, we have $w_2(t) = 0$. Hence,

$$0 = u^*(t) = \frac{(\lambda_2 - \lambda_1)K_1VT + w_1(t) + B_4 + B_3}{B_4}.$$

Since $w_1(t) \geq 0$, then $\frac{(\lambda_2-\lambda_1)K_1VT+B_4+B_3}{B_4} \leq 0$. Notice $\left(\frac{(\lambda_2-\lambda_1)K_1VT+B_4+B_3}{B_4}\right)^+ = 0 = u^*(t)$ in this case.

Combining these three cases, the optimal control is characterized as

$$u^*(t) = \min\left(1, \left(\frac{(\lambda_2 - \lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right)$$

where

$$s^+ = \begin{cases} s, & \text{if } s > 0 \\ 0, & \text{if } s \leq 0. \end{cases}$$

If $\lambda_2 - \lambda_1 < 0$ for some t, then $u^*(t) \neq 1$. Hence, $0 \leq u^*(t) < 1$ for such t. $\lambda_2 < \lambda_1$ means the marginal valuation of the benefit functional with respect to the T cells is greater than the marginal valuation of the benefit functional with respect to the T^* cells. $\square$

The optimality system consists of the state system coupled with the adjoint system with the initial and transversality conditions together with relationship

$$u^*(t) = \min\left(1, \left(\frac{(\lambda_2 - \lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right). \tag{4.4.3}$$

Utilizing (4.4.3), we have the following optimality system which characterizes the

optimal control.

$$\begin{aligned}
\frac{dT}{dt} &= \frac{s}{1+V} - \mu_T T + rT\left(1 - \frac{T+T^*+T^{**}}{T_{MAX}}\right)\\
&\quad - \min\left(1, \left(\frac{(\lambda_2-\lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right)K_1VT\\
\frac{dT^*}{dt} &= \min\left(1, \left(\frac{(\lambda_2-\lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right)K_1VT - \mu_T T^* - K_2T^*\\
\frac{dT^{**}}{dt} &= K_2T^* - \mu_b T^{**}\\
\frac{dV}{dt} &= N\mu_b T^{**} - K_1VT - \mu_v V
\end{aligned}$$

$$T = T_0,\ T^* = T_0^*,\ T^{**} = T_0^{**},\ \text{and } V = V_0 \text{ at } t_0.$$

$$\begin{aligned}
\lambda_1' &= -\Bigg\{\Bigg[B_1 + \lambda_1\Bigg(-\mu_T + r\left(1 - \frac{T+T^*+T^{**}}{T_{MAX}}\right) - \frac{rT}{T_{MAX}}\\
&\qquad - \min\left(1, \left(\frac{(\lambda_2-\lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right)K_1V\Bigg)\Bigg]\\
&\qquad + \lambda_2 \min\left(1, \left(\frac{(\lambda_2-\lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right)K_1V - \lambda_4K_1V\Bigg\}
\end{aligned} \tag{4.4.4}$$

$$\begin{aligned}
\lambda_2' &= -\left(-\frac{\lambda_1 rT}{T_{MAX}} - \lambda_2(\mu_T + K_2) + \lambda_3K_2\right)\\
\lambda_3' &= -\left(-\frac{\lambda_1 rT}{T_{MAX}} - \lambda_3\mu_b + \lambda_4N\mu_b\right)\\
\lambda_4' &= -\Bigg(\frac{-s\lambda_1}{(1+V)^2} + (\lambda_2-\lambda_1)K_1T\min\left(1, \left(\frac{(\lambda_2-\lambda_1)K_1VT + B_4 + B_3}{B_4}\right)^+\right)\\
&\qquad - \lambda_4(K_1V + \mu_V) - B_2\Bigg)
\end{aligned}$$

$$\lambda_i(t_1) = 0 \quad \text{for } i = 1,2,3,4.$$

5. Uniqueness of Optimality System

Using $T(t) < T_{MAX}$, we analyze our state system and adjoint system to discuss upper bounds on our solutions. These bounds are needed in the uniqueness proof of the optimality system.

With $T(t) < T_{MAX}$, we can now obtain an upper bound on solutions of the state system.

$$\frac{d\widehat{T}^*}{dt} = K_1 \widehat{V} T_{MAX} \qquad \widehat{T}^*(t_0) = \widehat{T}_0^*$$
$$\frac{d\widehat{T}^{**}}{dt} = K_2 \widehat{T}^* \qquad \widehat{T}^{**}(t_0) = T_0^{**}$$
$$\frac{d\widehat{V}}{dt} = N\mu_b \widehat{T}^{**} \qquad \widehat{V}(t_0) = V_0$$

or

$$\begin{pmatrix} \widehat{T}^* \\ \widehat{T}^{**} \\ \widehat{V} \end{pmatrix}' = \begin{pmatrix} 0 & 0 & T_{MAX}K_1 \\ K_2 & 0 & 0 \\ 0 & N\mu_b & 0 \end{pmatrix} \begin{pmatrix} \widehat{T}^* \\ \widehat{T}^{**} \\ \widehat{V} \end{pmatrix}.$$

Since we have a linear system in finite time with bounded coefficients, then the supersolutions $\widehat{T}^*$, $\widehat{T}^{**}$, $\widehat{V}$ are uniformly bounded. Using these bounds, the adjoint system (4.4.1) is linear in λ_i with bounded coefficients. Hence, the solutions of the adjoint system are bounded.

Theorem 5.1

For t_1 sufficiently small, the solution to the optimality system is unique.

Proof

Suppose $(T, T^*, T^{**}, V, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\overline{T}, \overline{T}^*, \overline{T}^{**}, \overline{V}, \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \overline{\lambda}_4)$ are two different solutions of the optimality system (4.4.4). Let $T = e^{\lambda t}p$, $T^* = e^{\lambda t}p^*$, $T^{**} = e^{\lambda t}p^{**}$, $V = e^{\lambda t}q$, $\lambda_1 = e^{-\lambda t}w$, $\lambda_2 = e^{-\lambda t}z$, $\lambda_3 = e^{-\lambda t}v$, $\lambda_4 = e^{-\lambda t}y$ where positive λ is to be chosen below. Similarly let $\overline{T} = e^{\lambda t}\overline{p}$, $\overline{T}^* = e^{\lambda t}\overline{p}^*$, and so forth.

Let

$$u = \min\left(1, \left(\frac{K_1 e^{\lambda t}(z-w)qp + B_4 + B_3}{B_4}\right)^+\right)$$

and

$$\overline{u} = \min\left(1, \left(\frac{K_1 e^{\lambda t}(\overline{z}-\overline{w})\overline{q}\overline{p} + B_4 + B_3}{B_4}\right)^+\right).$$

Substituting $T = e^{\lambda t}p$ into our first ODE (4.4.4), we obtain

$$\begin{aligned} e^{\lambda t}\dot{p} + \lambda e^{\lambda t}p = {} & \frac{s}{1+e^{\lambda t}q} - \mu_T e^{\lambda t}p + r e^{\lambda t}p\left(1 - \frac{e^{\lambda t}(p+p^*+p^{**})}{T_{MAX}}\right) \\ & - \min\left(1, \left(\frac{K_1 e^{\lambda t}(z-w)qp + B_4 + B_3}{B_4}\right)^+\right) K_1 e^{2\lambda t}qp \end{aligned}$$

or

$$\begin{aligned} \dot{p} + \lambda p = {} & \frac{se^{-\lambda t}}{1+e^{\lambda t}q} - \mu_T p + rp\left(1 - \frac{e^{\lambda t}(2p+p^*+p^{**})}{T_{MAX}}\right) \\ & - \min\left(1, \left(\frac{K_1 e^{\lambda t}(z-w)qp + B_4 + B_3}{B_4}\right)^+\right) K_1 e^{\lambda t}qp \end{aligned} \tag{4.5.1}$$

where $\dot{p} = \frac{dp}{dt}$.

Similarly, for $\lambda_1 = e^{-\lambda t}w$, we obtain

$$\begin{aligned} -\dot{w} + \lambda w = {} & B_1 e^{\lambda t} - \mu_T w + rw\left(1 - \frac{e^{\lambda t}(2p+p^*+p^{**})}{T_{MAX}}\right) - K_1 e^{\lambda t}yq \\ & + \left\{K_1 e^{\lambda t}q \min\left(1, \left(\frac{K_1 e^{\lambda t}(z-w)qp + B_4 + B_3}{B_4}\right)^+\right)(-w+z)\right\}. \end{aligned} \tag{4.5.2}$$

We subtract the equations for T and $\overline{T}$, T^* and $\overline{T}^*$, T^{**} and $\overline{T}^{**}$, V and $\overline{V}$, λ_1 and $\overline{\lambda}_1$, λ_2 and $\overline{\lambda}_2$, λ_3 and $\overline{\lambda}_3$, λ_4 and $\overline{\lambda}_4$. Then we multiply each equation by an appropriate function and integrate from t_0 to t_1. Next, we add all eight "integral equations" and use estimates to obtain our result. Some of the "integral equations" are listed below for illustration.

$$\begin{aligned}\frac{1}{2}\left(p(t_1)-\overline{p}(t_1)\right)^2+\lambda\int_{t_0}^{t_1}(p-\overline{p})^2\,dt &= \int_{t_0}^{t_1}\left(\frac{s}{1+e^{\lambda t}q}-\frac{s}{1+e^{\lambda t}\overline{q}}\right)e^{-\lambda t}(p-\overline{p})\,dt\\ &+(r-\mu_T)\int_{t_0}^{t_1}(p-\overline{p})^2\,dt\\ &-\frac{r}{T_{MAX}}\int_{t_0}^{t_1}e^{\lambda t}\big[(p^2-\overline{p}^2)+(pp^*-\overline{pp}^*)\\ &\qquad+(pp^{**}-\overline{pp}^{**})\big](p-\overline{p})\,dt\\ &-K_1\int_{t_0}^{t_1}e^{\lambda t}(uqp-\overline{uqp})(p-\overline{p})\,dt.\end{aligned} \tag{4.5.3}$$

$$\begin{aligned}\frac{1}{2}\left(q(t_1)-\overline{q}(t_1)\right)^2+\lambda\int_{t_0}^{t_1}(q-\overline{q})^2\,dt &= \int_{t_0}^{t_1}N\mu_b(p^{**}-\overline{p}^{**})(q-\overline{q})\,dt\\ &-\int_{t_0}^{t_1}K_1e^{\lambda t}(qp-\overline{qp})(q-\overline{q})\,dt-\mu_v\int_{t_0}^{t_1}(q-\overline{q})^2\,dt.\end{aligned} \tag{4.5.4}$$

$$\begin{aligned}\frac{1}{2}\left(w(t_0)-\overline{w}(t_0)\right)^2+\lambda\int_{t_0}^{t_1}(w-\overline{w})^2\,dt &= B_1\int_{t_0}^{t_1}e^{\lambda t}\,dt+(r-\mu_T)\int_{t_0}^{t_1}(w-\overline{w})^2\,dt\\ &-\frac{r}{T_{MAX}}\int_{t_0}^{t_1}\Big[e^{\lambda t}\{2(wp-\overline{wp})+(wp^*-\overline{wp}^*)\\ &\qquad+(wp^{**}-\overline{wp}^{**})\}\Big](w-\overline{w})\,dt\\ &-K_1\int_{t_0}^{t_1}e^{\lambda t}\left\{(uwq-\overline{uwq})-(uzq-\overline{uzq})\right\}(w-\overline{w})\,dt\\ &-K_1\int_{t_0}^{t_1}e^{\lambda t}(yq-y\overline{q})(w-\overline{w})\,dt.\end{aligned} \tag{4.5.5}$$

$$\frac{1}{2}\left(y(t_0)-\overline{y}(t_0)\right)^2+\lambda\int_{t_0}^{t_1}(y-\overline{y})^2\,dt=\int_{t_0}^{t_1}\left(\frac{s\overline{w}}{\left(1+e^{\lambda t}\overline{q}\right)^2}-\frac{sw}{\left(1+e^{\lambda t}q\right)^2}\right)(y-\overline{y})\,dt$$
$$-K_1\int_{t_0}^{t_1}\left\{e^{\lambda t}\left[(uwp-\overline{uwp})-(uzp-\overline{uzp})\right]\right\}(y-\overline{y})\,dt$$
$$-K_1\int_{t_0}^{t_1}e^{\lambda t}(yp-\overline{yp})(y-\overline{y})\,dt$$
$$-\mu_v\int_{t_0}^{t_1}(y-\overline{y})^2\,dt-B_2\int_{t_0}^{t_1}e^{\lambda t}\,dt. \tag{4.5.6}$$

We show how to estimate several terms in these eight equations. First, notice

$$\left|\frac{1}{1+e^{\lambda t}q}-\frac{1}{1+e^{\lambda t}\overline{q}}\right|=\left|\frac{e^{\lambda t}(\overline{q}-q)}{(1+e^{\lambda t}q)(1+e^{\lambda t}\overline{q})}\right|<e^{\lambda t_1}\left|\overline{q}-q\right|.$$

Hence

$$\int_{t_0}^{t_1}s\left(\frac{1}{1+e^{\lambda t}q}-\frac{1}{1+e^{\lambda t}\overline{q}}\right)(p-\overline{p})\,dt\le C_1e^{\lambda t_1}\int_{t_0}^{t_1}(q-\overline{q})^2+(p-\overline{p})^2\,dt,$$

$$\frac{r}{T_{MAX}}\int_{t_0}^{t_1}e^{\lambda t}\left(p^2-\overline{p}^2\right)(p-\overline{p})\,dt\le\frac{r}{T_{MAX}}\int_{t_0}^{t_1}e^{\lambda t}(p-\overline{p})^2 2e^{-\lambda t}T_{MAX}\,dt$$
$$=2r\int_{t_0}^{t_1}(p-\overline{p})^2\,dt, \tag{4.5.7}$$

$$\frac{r}{T_{MAX}}\int_{t_0}^{t_1}e^{\lambda t}(pp^*-\overline{pp}^*)(p-\overline{p})\,dt=\frac{r}{T_{MAX}}\int_{t_0}^{t_1}e^{\lambda t}(pp^*-\overline{p}p^*+\overline{p}p^*$$
$$-\overline{pp}^*)(p-\overline{p})\,dt$$
$$\le C_2e^{\lambda t_1}\int_{t_0}^{t_1}(p-\overline{p})^2+(p^*-\overline{p}^*)^2\,dt, \tag{4.5.8}$$

$$
\begin{aligned}
\int_{t_0}^{t_1} (u-\overline{u})^2\ dt \le \int_{t_0}^{t_1} \Bigg(& \frac{K_1 e^{\lambda t}(z-w)qp + B_4 + B_3}{B_4} \\
& - \frac{K_1 e^{\lambda t}(\overline{z}-\overline{w})\overline{qp} + B_4 + B_3}{B_4} \Bigg)^2\ dt \\
= \frac{K_1^2 e^{2\lambda t_1}}{B_4^2} & \int_{t_0}^{t_1} \left[(z-w)qp - (\overline{z}-\overline{w})\overline{qp}\right]^2\ dt \\
\le \left(\frac{K_1 e^{\lambda t_1}}{B_4}\right)^2 & \int_{t_0}^{t_1} \Big|(z-w)^2 q^2 p^2 - 2(\overline{z}-\overline{w})\overline{qp}(z-w)qp \\
& + (\overline{z}-\overline{w})^2 \overline{q}^2 \overline{p}^2\Big|\ dt \\
\le \left(\frac{K_1 e^{\lambda t_1}}{B_4}\right)^2 & C_4 \int_{t_0}^{t_1} \Big|z^2 + \overline{z}^2 - 2zw + 2\overline{z}w - 2\overline{z}w + 2\overline{zw} + w^2 \\
& + \overline{w}^2 - 2w\overline{w} - 2\overline{z}z\Big|\ dt \\
= C_4 \left(\frac{K_1 e^{\lambda t_1}}{B_4}\right)^2 & \int_{t_0}^{t_1} (z-\overline{z})^2 + (w-\overline{w})^2 \\
& + \left|2w(\overline{z}-z)\right| + \left|2\overline{z}(\overline{w}-w)\right|\ dt \\
\le \widetilde{C}_4 \left(\frac{K_1 e^{\lambda t_1}}{B_4}\right)^2 & \int_{t_0}^{t_1} \left[(z-\overline{z})^2 + (w-\overline{w})^2\right]\ dt,
\end{aligned}
\tag{4.5.9}
$$

$$
\begin{aligned}
\int_{t_0}^{t_1} e^{\lambda t}(uqp - \overline{uqp})(p-\overline{p})\ dt &= \int_{t_0}^{t_1} e^{\lambda t}\left((u-\overline{u})qp + \overline{u}(qp - \overline{qp})\right)(p-\overline{p})\ dt \\
&\le C_5 e^{\lambda t_1} \int_{t_0}^{t_1} \left[(u-\overline{u})^2 + (p-\overline{p})^2 + (q-\overline{q})^2\right]\ dt \\
&\le \widetilde{C}_5 e^{3\lambda t_1} \int_{t_0}^{t_1} \Big[(p-\overline{p})^2 + (q-\overline{q})^2 + (z-\overline{z})^2 \\
&\qquad + (w-\overline{w})^2\Big]\ dt,
\end{aligned}
\tag{4.5.10}
$$

and

$$\int_{t_0}^{t_1} s\left(\frac{\overline{w}}{(1+e^{\lambda t}\overline{q})^2} - \frac{w}{(1+e^{\lambda t}q)^2}\right)(y-\overline{y})\,dt \le s\int_{t_0}^{t_1}\Big|[(w-\overline{w}) + 2e^{\lambda t}(qw-\overline{qw}) + e^{2\lambda t}(q^2 w - \overline{q}^2\overline{w})](y-\overline{y})\Big|\,dt$$
$$\le C_6 e^{3\lambda t}\int_{t_0}^{t_1}(w-\overline{w})^2 + (q-\overline{q})^2 + (y-\overline{y})^2\,dt. \tag{4.5.11}$$

To show uniqueness, we need to add all eight integral equations together and use the estimates. So, we have

$$\begin{aligned}
&\frac{1}{2}(p-\overline{p})^2(t_1) + \frac{1}{2}(p^*-\overline{p}^*)^2(t_1) + \frac{1}{2}(p^{**}-\overline{p}^{**})^2(t_1) + \frac{1}{2}(q-\overline{q})^2(t_1)\\
&\quad + \frac{1}{2}(w-\overline{w})^2(t_o) + \frac{1}{2}(z-\overline{z})^2(t_o) + \frac{1}{2}(v-\overline{v})^2(t_o) + \frac{1}{2}(y-\overline{y})^2(t_o)\\
&\quad + (\lambda+\mu_T+r+K_2)\int_{t_0}^{t_1}\left[(p-\overline{p})^2 + (w-\overline{w})^2 + (z-\overline{z})^2\right]\,dt\\
&\quad + (\lambda+\mu_T+\mu_v)\int_{t_0}^{t_1}\left[(q-\overline{q})^2 + (y-\overline{y})^2 + (p^{**}-\overline{p}^{**})^2\right]\,dt\\
&\quad + (\lambda-K_2+\mu_v-N\mu_b)\int_{t_0}^{t_1}(v-\overline{v})^2\,dt\\
&\le C_1 e^{3\lambda t_1}\int_{t_0}^{t_1}\Big[(q-\overline{q})^2 + (p-\overline{p})^2 + (p^*-\overline{p}^*)^2 + (p^{**}-\overline{p}^{**})^2\\
&\qquad + (z-\overline{z})^2 + (w-\overline{w})^2 + (y-\overline{y})^2 + (v-\overline{v})^2\Big]\,dt\\
&\quad + C_3\int_{t_0}^{t_1}\left[(p^*-\overline{p}^*)^2 + (p^{**}-\overline{p}^{**})^2 + (q-\overline{q})^2\right]\,dt.
\end{aligned} \tag{4.5.12}$$

From (4.5.12) we have

$$\left(\lambda - \widetilde{C}_1 - \widetilde{C}_2 e^{3\lambda t_1}\right)\int_{t_0}^{t_1}\left[(p-\overline{p})^2 + (p^*-\overline{p}^*)^2 + (p^{**}-\overline{p}^{**})^2 + (q-\overline{q})^2 + (w-\overline{w})^2 + (z-\overline{z})^2 + (v-\overline{v})^2 + (y-\overline{y})^2\right]\,dt \le 0$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ depend on the coefficients and the bounds of p, p^*, p^{**}, q, w, z, v, y.

If we choose λ such that $\lambda > \widetilde{C}_1 + \widetilde{C}_2$ and $t_1 < \frac{1}{3\lambda} \ln\left(\frac{\lambda - \widetilde{C}_2}{\widetilde{C}_1}\right)$, then $p = \overline{p}$, $p^* = \overline{p}^*$, $p^{**} = \overline{p}^{**}$, $q = \overline{q}$, $w = \overline{w}$, $z = \overline{z}$, $v = \overline{v}$, $y = \overline{y}$. $\square$

Uniqueness for a small time interval is not unusual in such a nonlinear boundary value problem. The unique optimal control u^* is characterized in terms of the unique solution of the optimality system. The optimal control u^* gives an optimal chemotherapy strategy for the HIV positive patient.

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VITA

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