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Cohomological Dimension With Respect to Nonabelian Groups

Atish Jyoti Mitra
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Atish Jyoti Mitra entitled "Cohomological Dimension With Respect to Nonabelian Groups." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Jerzy Dydak, Major Professor

We have read this dissertation and recommend its acceptance:

Robert J. Daverman, Morwen Thistlehwaite, Pavlos Tzermias, John Nolt

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

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Cohomological dimension with respect to nonabelian groups

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Atish Jyoti Mitra

August 2006

Dedication

This dissertation is dedicated to my father Tarak Nath Mitra and to my mother Anjana Mitra, who have always encouraged me to study whatever I liked.

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I want to thank my advisor Prof. Jurek Dydak for everything he has done for me. When I first met him 6 years back I was an outsider to mathematics and he was one of the very few who actually encouraged me to get into graduate studies in mathematics. In the past few years I have got to know him closely, as a teacher, a mentor and a friend. He has redefined for me what teaching and learning mathematics really means. Most important of all, I have always felt that he treated me as a colleague in studying mathematics.

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Any effort to express my gratitude towards my parents will be insufficient. They have been very supportive of me all these years while I was enjoying the luxury of graduate studies.

Abstract

This dissertation addresses three aspects of cohomological dimension of metric spaces with respect to nonabelian groups.

In the first part we examine when the Eilenberg-Maclane space ($n = 1$) of the abelianization of a solvable group being an absolute extensor of a metric space implies the Eilenberg-Maclane space of the group itself is an absolute extensor. We also give an elementary approach to this problem in the case of nilpotent groups and 2-dimensional metric spaces.

The next part of the dissertation is devoted to generalizations of the Cencelj-Dranishnikov theorems relating extension properties of nilpotent CW complexes to its homology groups.

In the final part we extend the definition of Bockstein basis of abelian groups to nilpotent groups G, and prove a version of the First Bockstein Theorem for such groups.

Contents

Chapter 1

Introduction

One of the most basic geometric concepts is that of dimension. It is a concept that is as intuitive as it is difficult to define in a rigorous way.

One first encounters the notion of dimension in a purely algebraic setting in elementary linear algebra and gets the algebraic dimension of the line and plane.

The first non-algebraic ways of thinking about dimension comes from elementary point set topology. In general topology there are various ways of defining dimension (small inductive dimension, large inductive dimension, covering dimension, etc.), most of which coincide in nice spaces. There are nice expositions of these notions in [32].

Once homology and cohomology groups are defined it is useful to introduce some algebra into the geometric setting of dimension. Then we have the notions of homological and cohomological dimension with respect to $\mathbb{Z}$, and soon can generalize to homological and cohomological dimension with respect to arbitrary abelian groups. These questions have kept topologists busy for many years, maybe starting from Alexandroff's questions [1] during the First International Topology Conference in Moscow in 1935. Kuzminov's influential paper [41] of 1968 was a great contribution to the algebraic point of view in studying homological dimension. A return to the geometric roots of cohomological dimension followed in 1981 starting with Walsh's elegant paper [49]. Since then Dranishnikov solved many of the problems of homological and cohomological dimension theory [18, 19, 20]. Dydak has major contributions in cohomological dimension and extended many of the results from the case of compacta to arbitrary metric spaces [25, 28].

The first attempt to study cohomological dimension of compacta with respect to non-abelian groups was by Dranishnikov and Repovš in [23] - they considered the case of perfect groups and used geometric techniques. Some of the ideas were explored later in [11].

Later Cencelj and Dranishnikov looked at the case of nilpotent groups to answer some questions - specifically attempting to develop a Hurewicz-Serre type theorem for nilpotent complexes [5, 6, 7].

In this dissertation we look at some questions concerning cohomological dimension of metrizable spaces with respect to nilpotent and solvable groups. After some background on cohomological dimension and groups in the next two chapters, in chapter 4 we consider the problem of comparing cohomological dimension of spaces with respect to groups and their abelianizations. We provide generalizations to some existing results [5, 6, 7] of nilpotent groups and then move to the case of solvable groups. In chapter 5 we provide a version of Hurewicz-Serre theorem for nilpotent groups. In chapter 6 we prove a version of the Bockstein first theorem for nilpotent groups.

Chapter 2

Cohomological dimension theory

2.1 Covering dimension

Definition 2.1.1. Let X be a normal space. The covering dimension $\dim X$ (also known as the Čech-Lebesgue dimension) is defined as follows:

1. $\dim X \leq n$, where $0 \leq n < \infty$, if every finite open cover of X has a finite open refinement of order n (every $n + 2$ distinct elements of the cover have empty intersection)

2. $\dim X = n$,where $0 \leq n < \infty$, if $\dim X \leq n$ and $\dim X \nleq n - 1$

3. $\dim X = \infty$, if $\dim X \nleq n$ for all n.

There are various other ways of defining dimension (the large inductive dimension and small inductive dimension) and we refer the reader to [32] for a discussion of cases in which the various notions coincide. For example, they all coincide for seperable metrizable spaces. In Euclidean spaces the topological and algebraic dimensions coincide.

Theorem 2.1.2. *The algebraic and topological notions of dimension coincide for* $\mathbb{R}^n$.

2.2 Cohomological dimension

The following theorem of Alexandroff lets us think of covering dimension in an algebraic way.

Theorem 2.2.1. *If X is a finite dimensional compactum, then its dimension is the smallest integer n such that $\check{H}^{n+1}(X, A; \mathbb{Z}) = 0$ for all closed subsets A of X.*

In the above theorem one can replace $\mathbb{Z}$ by any abelian group $G \neq 0$ which leads to the notion of the cohomological dimension $\dim_G(X)$.

Definition 2.2.2. Given a paracompact space X and an abelian group G, one assigns to X the <u>cohomological dimension</u> $\dim_G(X)$ which is an integer larger than or equal to -1, or infinity (∞), by the following conditions:

1. $\dim_G(X) \leq n$, where $0 \leq n < \infty$, if $\check{H}^{n+1}(X, A; G) = 0$ for all closed subsets A of X,

3. $\dim_G(X) = n$, where $0 \leq n < \infty$, if $\dim_G(X) \leq n$ and $\dim_G(X) \leq n - 1$ does not hold,

4. $\dim_G(X) = \infty$, if $\dim_G(X) \leq n$ does not hold for all $n < \infty$.

The relevance of dimension with respect to arbitrary groups was underscored by Pontryagin [46] who realized that in order to construct compacta X and Y with $\dim(X \times Y) < \dim(X) + \dim(Y)$, one has to construct a compactum P such that $\dim(P) > \dim_{\mathbb{Z}/p}(P)$ for some prime p.

Theorem 2.2.3. *For each prime $\mathbf{p}$ there is a compactum $P_{\mathbf{p}}$ such that the following conditions hold:*

1. $dim(P_{\mathbf{p}}) = 2,\ dim_{\mathbb{Z}/\mathbf{q}}(P_{\mathbf{p}}) = 1$ if $\mathbf{q}$ is a prime and $\mathbf{q} \neq \mathbf{p}$,

2. $dim(P_{\mathbf{p}} \times P_{\mathbf{q}}) = 3$ if $\mathbf{q}$ is a prime and $\mathbf{p} \neq \mathbf{q}$.

2.3 Extension theory

Definition 2.3.1. K is an absolute extensor of X (notations: $K \in AE(X)$ or $X\tau K$) if every map $f : A \to K$ extends over X if A is closed in X.

We can use absolute extensors to characterize covering dimension.

Theorem 2.3.2. *If X is compact, then $dim(X) \leq n$ if and only if $S^n \in AE(X)$.*

A similar characterization holds in cohomological dimension.

Theorem 2.3.3. *[Cohen's Theorem] If X is locally compact, then $dim_G(X) \leq n$ if and only if $K(G, n) \in AE(X)$.*

When translating results from extension theory to cohomological dimension theory we find that the following theorem is of fundamental importance.

Theorem 2.3.4. *Suppose X is a metrizable space and K is a connected CW complex. Consider the following conditions:*

1. $K \in AE(X)$.
2. $SP^\infty(K) \in AE(X)$.
3. $dim_{H_m(K;\mathbb{Z})}(X) \leq m$ *for all* $m \geq 0$.
4. $dim_{\pi_m(K;\mathbb{Z})}(X) \leq m$ *for all* $m \geq 0$.

Then, 1 implies 2. If K is simply connected, then Conditions 2-3-4 are equivalent. If X is of finite dimension and K is simply connected, then Conditions 1-2-3-4 are equivalent.

Recall that $SP^\infty(K)$ is the infinite symmetric product of K (see [16]).

For X compact, Theorem 2.3.4 is due to A.Dranishnikov [19]. Subsequently, it was generalized to metrizable spaces by J.Dydak [25], [28].

Theorem 2.3.4 is reminiscent of the classical Hurewicz Theorem, and in fact part of its proof relies on the Serre version of the Hurewicz Theorem. Thus, it represents a point of overlap between extension theory and algebraic topology.

A geometric result repeatedly used in the next chapters is the following.

Theorem 2.3.5. *Let X be a metric space and let L, K be complexes. If F is the homotopy fiber of $L \to K$ and $X\tau F$, then $X\tau L$ is equivalent to $X\tau K$.*

Chapter 3

Nilpotent and solvable groups

A natural way to build groups out of abelian groups is by forming extensions. Nilpotent and solvable groups arise in this way.

3.1 Abelian series and solvable groups

Definition 3.1.1. 1. A normal series of a group is a finite sequence of subgroups $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that $G_i \lhd G$ for all i.

2. An abelian series is a normal series in which each factor G_{i+1}/G_i is abelian.

Definition 3.1.2. A group is called solvable if it has an abelian series.

Every abelian group is trivially solvable. The simplest example of a nonabelian solvable group is the symmetric group S_3.

Definition 3.1.3. 1. The length of a normal series is the number of nontrivial factor groups.

2. For any solvable group G, the length of the shortest abelian series is called the derived length.

Thus only the trivial group has derived length 0, and abelian groups are the groups of derived length 1.

We will now describe a particularly nice way of constructing an abelian series that realizes the derived length of a solvable group. For any group G we can construct the derived subgroup $G^{'} = [G, G]$, the subgroup generated by all commutators in G. By repeatedly forming derived subgroups we get a descending sequence $G = G^{(0)} > G^{(1)} > \cdots$, where $G^{(n+1)} = (G^{(n)})^{'}$. Clearly all the factors are abelian, but in general the series may not terminate, unless the group is solvable as in the next result.

Proposition 3.1.4. *If $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ is an abelian series of a solvable group, then $G^{(i)} < G_{n-i}$, and the derived length of the group is equal to the length of the derived series.*

The next fact shows that solvable groups form a nice class.

Proposition 3.1.5. *The class of solvable groups is closed with respect to formation of subgroups, images and extensions.*

Finite solvable groups have particularly nice abelian series.

Proposition 3.1.6. *A finite group is solvable if and only if it has a series with all factors cyclic of prime order.*

3.2 Central series and nilpotent groups

Definition 3.2.1. A central series is an abelian series in which each G_{i+1}/G_i is contained in the center of G/G_i.

Definition 3.2.2. A group is called nilpotent if it has a central series.

As in the case of solvable groups, the next result shows that nilpotent groups form a nice class.

Proposition 3.2.3. *The class of nilpotent groups is closed with respect to formation of subgroups, images and finite direct products.*

Finite nilpotent groups can be characterized as below.

Proposition 3.2.4. *A finite group is nilpotent if and only if it is the direct product of its Sylow subgroups.*

3.3 Abelianizations of nilpotent groups

This section develops a nice technique that lets us conclude facts about nilpotent groups by studying their abelianizations.

Recall that the lower central series of a group G is $G = \gamma_1 G \geqq \gamma_2 G \geqq \cdots$, where $\gamma_{i+1}G = [\gamma_i G, G]$. Notice that $\gamma_i G/\gamma_{i+1}G$ lies in the center of $G/\gamma_{i+1}G$ and that the nilpotent class of G equals the length of the lower central series of G. The following lemma shows that the first lower central factor G_{ab} exerts a strong influence on the subsequent lower central factors.

Lemma 3.3.1 (Robinson)**.** *Let G be a group and let $F_i = \gamma_i G/\gamma_{i+1}G$. Then the mapping $a(\gamma_{i+1}G) \otimes g[G,G] \mapsto [a,g](\gamma_{i+2}G)$ is a well defined module epimorphism from $F_i \otimes_{\mathbb{Z}} G_{ab}$ to F_{i+1}.*

Proof: Let $g \in G$ and $a \in G_i$, and consider the function $(a(\gamma_{i+1}G), g[G,G]) \mapsto [a,g](\gamma_{i+2}G)$. Check that the mapping is well-defined and bilinear. By the universal property of tensor products, we have an induced homomorphism. It is easily checked that it is an epimorphism.

The above lemma gives the following useful corollary.

Corollary 3.3.2. *Let $\mathcal{P}$ be a group theoretical property which is inherited by images of tensor products (of abelian groups) and by extensions. If G is a nilpotent group such that G_{ab} has $\mathcal{P}$, then G has $\mathcal{P}$.*

Proof: Let $F_i = \gamma_i G/\gamma_{i+1} G$, and note that $F_1 = G_{ab}$. Suppose F_i has $\mathcal{P}$: then the previous lemma implies that F_{i+1} has $\mathcal{P}$. As G is nilpotent, some $\gamma_{c+1} = 1$. Since $\mathcal{P}$ is closed under extensions, G has $\mathcal{P}$.

Corollary 3.3.3. *1. If G is a nilpotent group and G_{ab} is finite , then so is G.*

2. If G is a nilpotent group and G_{ab} is p-torsion , then so is G.

Chapter 4

Dimension with respect to groups and their abelianizations

In chapter 2 we gave a brief outline of cohomological dimension of metric spaces. In this chapter we will concentrate on the more geometric viewpoint of looking at cohomological dimension in terms of extensions of maps. Recall that for any compactum X and abelian group G, $\dim_G X \leq n \Leftrightarrow X \tau K(G, n)$, where τ is Kuratowski's notation described in chapter 2 and $K(G, n)$ is the Eilenberg-Maclane space.

4.1 Dimension with respect to nonabelian groups

Trying to extend the notion of cohomological dimension to non-abelian groups G makes sense only for $n = 1$, as the corresponding Eilenberg-Maclane spaces are defined only for $n = 1$.

Dranishnikov and Řepovs studied cohomological dimension with respect to perfect groups in [23], using techniques of a geometric flavour.

Later Cencelj and Dranishnikov studied the case of nilpotent groups in a series of three papers [5, 6, 7].

Generalizing Dranishnikov's theorem 2.3.4 about extension of maps to simply-connected complexes [19], they obtained the following result.

Theorem 4.1.1. *For any nilpotent CW-complex M and finite-dimensional metric compactum X, the following are equivalent:*

1. $X \tau M$

2. $X \tau SP^{\infty} M$

3. $dim_{H_i(M)} X \leq i$ *for every* $i > 0$

4. $dim_{\pi_i(M)} X \leq i$ *for every* $i > 0$

Recall that $\mathrm{SP}^i X = X^i/(\Sigma_i)$, where Σ_i is the symmetric group on i letters. If X is pointed, there is a natural embedding $\mathrm{SP}^i X \to \mathrm{SP}^{i+1} X$, and $\mathrm{SP}^{\infty} X$ is defined as direct limit of the $\mathrm{SP}^i X$.

In course of the proof of 4.1.1, they obtained the following relation between dimension with respect to a nilpotent group and its abelianization.

Theorem 4.1.2. *For a nilpotent group N and every metric compactum X the following equivalence holds :* $dim_N X \leq 1 \Leftrightarrow dim_{N_{ab}} X \leq 1$ *provided N has one of the following properties:*

1. *N is a torsion group.*

2. *for every prime p such that* $Tor_p N \neq 1$

    (a) *N is not p-divisible, or*

    (b) $Tor_p N_{ab} \neq 0$.

In view of 4.1.2 it seems natural to ask the following question.

Question 4.1.3. Let $\mathcal{C}$ be a class of spaces. Describe groups for which the following are equivalent for any $X \in \mathcal{C}$:

1. $X \tau K(G,1)$

2. $X \tau K(G_{ab},1)$

The following theorem gives one of the implications of question 4.1.3 for any group G.

Theorem 4.1.4. *Let X be a metric space and G a group. Then $X\tau K(G,1) \Rightarrow X\tau K(G_{ab},1)$.*

Theorem 4.1.4 was proven for the case of all compacta by Dranishnikov and Cencelj in [5].

The proof of theorem 4.1.4 in its present form is essentially contained in a paper by Dydak [29]. The proof depends on two results, the first of which is due to Dydak and the second is a version of the celebrated Dold-Thom theorem.

Lemma 4.1.5. *Suppose X is metrizable, K is a pointed CW complex, and $X\tau K$. Given a closed subset A of X and $g : A \to SP^k(K)$, there is an extension $g' : X \to SP^{k.k!}(K)$.*

Theorem 4.1.6. *[16] Let K be a CW complex. Then $SP^\infty K$ has the weak homotopy type of a product of Eilenberg-Maclane spaces $\prod K(H_n(K),n)$.*

Proof. (of theorem 4.1.4) Replacing $\mathrm{SP}^\infty(K(G,1))$ by the telescope $\bigcup \mathrm{SP}^k(K(G,1)) \times [k-1,k]$ and using 4.1.5 we have $X\tau\mathrm{SP}^\infty K(G,1)$. Then use 4.1.6 to see that $X\tau K(H_1(K(G,1)),1)$, which is equivalent to $X\tau K(G_{ab},1)$. ■

In the next section (4.2) we list well known results which will be useful later in the chapter. In section 4.3 we show that if $\mathcal{C}$ is the class of all 2-dimensional compacta, then nilpotent groups satisfy question 4.1.3. In section 4.4 we show that if $\mathcal{C}$ is the class of all 2-dimensional compacta, then finite solvable groups satisfy question 4.1.3. Finally in section 4.6 we give an example showing that infinite solvable groups do not satisfy question 4.1.3 even when $\mathcal{C}$ is the class of all 2-dimensional compacta.

4.2 Preliminaries

We collect below some classical results which we will use in the rest of the chapter.

Proposition 4.2.1. *Let X and Y be metric spaces and let A be closed in X. A map $f : A \to Y$ has an extension to X iff some map $g : A \to Y$ homotopic to f has an extension.*

Recall that a map $f : X \to Y$ between CW complexes is called cellular if $f(X^{(n)}) \subset Y^{(n)}$ for all n.

Theorem 4.2.2 (Cellular Approximation Theorem). *[42] Every map $f : X \to Y$ between CW complexes is homotopic to a cellular map.*

Theorem 4.2.3 (Seifert - Van Kampen). *Let X_1, X_2 be path connected open subsets of X, and the intersection $X_0 = X_1 \cap X_2$ be path connected. Then the following commutative diagram*

$$\begin{array}{ccc} \pi_1(X_0) & \xrightarrow{i_1} & \pi_1(X_1) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \pi_1(X_2) & \xrightarrow{j_2} & \pi_1(X) \end{array}$$

of fundamental groups (based at some $x_0 \in X_0$) where all maps are induced by inclusions is a pushout diagram.

Theorem 4.2.4 (Bockstein exact sequence). *[25] If $1 \to G \to E \to \Pi \to 1$ is a short exact sequence of abelian groups, then there is a natural exact sequence*

$$\cdots \to \check{H}^n(X, A; G) \to \check{H}^n(X, A; E) \to \check{H}^n(X, A; \Pi) \to \check{H}^{n+1}(X, A; G) \to \cdots \quad (4.1)$$

for any paracompact space X and its closed subspace A. Here $\check{H}^n(X, A; G)$ is the n-th Čech cohomology group of the pair (X, A) with coefficients in G.

4.3 Nilpotent groups

We present a minor generalization of theorem 4.1.2 for 2-dimensional metrizable spaces, with an elementary proof.

Theorem 4.3.1. *Let G be a nilpotent group and X be a 2-dimensional metric space. Then $X \tau K(G, 1) \Leftrightarrow X \tau K(G_{ab}, 1)$.*

Recall from chapter 3 the useful device 3.3.2 of proving statements about nilpotent groups from their abelianizations: if $\mathcal{P}$ be a group theoretical property which is inherited by images of tensor products (of abelian groups) and by extensions, and if G is a nilpotent group such that G_{ab} has $\mathcal{P}$, then G has $\mathcal{P}$.

Proof. (of theorem 4.3.1) Let $\mathcal{P}$ be the group theoretical property of "belonging to $\mathcal{G}$", the class of groups with $X \tau K(G, 1)$.

We know that if $1 \to \Gamma \to E \to \Pi \to 1$ is a short exact sequence and $X \tau K(\Gamma, 1)$ and $X \tau K(\Pi, 1)$, then $X \tau K(E, 1)$[25]. We also know that if X is metrizable, then $X \tau K(\Gamma, 1) \Rightarrow X \tau K(\Gamma \otimes \Pi, 1)$ for any abelian groups Γ, Π [25].

Finally we consider the Bockstein exact sequence 4.2.4 for the short exact sequence $1 \to \ker \phi \to \Gamma \to \phi(\Gamma)$ for any homomorphism ϕ and any abelian group Γ . As X is 2-dimensional, $\check{H}^3(X, A; \ker \phi) = 0$. Now if $X \tau K(\Gamma, 1)$ then $\check{H}^2(X, A; \Gamma) = 0$, so the Bockstein sequence reduces to $0 \to \check{H}^2(X, A; \phi(\Gamma)) \to 0$, which shows $X \tau (K(\phi(\Gamma), 1))$. Then we apply 3.3.2 to complete the proof.

■

Remark 4.3.2. In the above that proof we only need X to be a metric space of cohomological dimension 2.

4.4 Finite solvable groups

Our effort to answer question 4.1.3 for solvable groups gives the next theorem. The nice trick 3.3.2 that we used for nilpotent groups does not work for solvable groups, hence we have to use geometric methods for proving 4.4.1.

Theorem 4.4.1. *Let G be a finite solvable group and X be a 2-dimensional metrizable space. Then $X \tau K(G,1) \Leftrightarrow X \tau K(G_{ab},1)$.*

For proving the above theorem, we need some preliminaries. The following result from extension theory must have been well known. As we couldn't locate a proof in the usual texts, we outline a proof.

Proposition 4.4.2. *Let X be a metrizable space and K_1, K_2 be subcomplexes of the CW complex $K_1 \cup K_2$. Then:*

1. *If K_1, K_2 and $K_0 = K_1 \cap K_2$ are absolute extensors of X, then so is $K_1 \cup K_2$.*
2. *If $K_1 \cup K_2$ and $K_0 = K_1 \cap K_2$ are absolute extensors of X, then so are K_1, K_2.*

Proof. (1) Let $f : A \to K_1 \cup K_2$ be a map from a closed subset A of X. Define $C_i = f^{-1}(K_i)$ and note that $A = C_1 \cup C_2$ and $f(C_1 \cap C_2) \subset K_1 \cap K_2$. As the closure of any one of the sets $C_1 - C_2$ and $C_2 - C_1$ misses the other we can find U open in X such that $C_1 - C_2 \subset U \subset \overline{U} \subset X - (C_2 - C_1)$. Define $D_1 = \overline{U} \cup (C_1 \cap C_2)$ and $D_2 = (X - U) \cup (C_1 \cap C_2)$. Then we have $D_1 \cap A = C_1$, $D_2 \cap A = C_2$ and $D_1 \cup D_2 = X$. As $C_1 \cap C_2$ is closed in $D_1 \cap D_2$ (which is closed in X) and as $K_1 \cap K_2$ is absolute extensor of X, we can extend $f \mid C_1 \cap C_2$ to $\overline{f} : D_1 \cap D_2 \to K_1 \cap K_2$. Now we have a map from $C_1 \cup (D_1 \cap D_2)$ to K_1 and a map from $C_2 \cup (D_1 \cap D_2)$ to K_2, both of which we can extend and then paste the resulting maps to get the desired extension.

(2) Start with $f : A \to K_1$ and extend to $\overline{f} : X \to K_1 \cup K_2$. Calling $C_i = (\overline{f})^{-1}(K_i)$, we have $X = C_1 \cup C_2$, $A \subset C_1$ and $\overline{f}(C_1 \cap C_2) \subset K_1 \cap K_2$. Extend $\overline{f} \mid C_1 \cap C_2$ to $\widetilde{f} : C_2 \to K_1 \cap K_2$. Paste $\overline{f} \mid C_1$ and $\widetilde{f} \mid C_2$ to get the desired extension.

$\blacksquare$

Definition 4.4.3. Let G_1, G_2 and A be groups, and $f_i : A \to G_i$ be homomorphisms. Then the amalgamated product $G_1 *_A G_2$ is defined as $\dfrac{G_1 * G_2}{\langle f_1(a) = f_2(a), \forall a \in A \rangle}$ (Here $\langle\rangle$ denotes the normal closure). Notice that the amalgamated product is a pushout of the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f_1} & G_1 \\ {\scriptstyle f_2}\downarrow & & \\ G_2 & & \end{array}$$

The following proposition is of interest in itself. We use it later to get a useful lemma.

Proposition 4.4.4. *Let X be a 2-dimensional metrizable space and define $\mathcal{H}_X = \{G : X\tau K(G,1)\}$.*

1. *If G_1,G_2 and A are in $\mathcal{H}_X$, then so is $G_1 *_A G_2$.*

2. *If $G_1 *_A G_2$ and A are in $\mathcal{H}_X$, then so are G_1 and G_2.*

Proof. Note that a homomorphism $f_i : A \to G_i, i = 1, 2$ induces a map $f_{*i} : K(A,1)^{(2)} \to K(G_i,1)^{(2)}$: we start by sending the generators to the corresponding loops in $K(G_i,1)^{(1)}$, and then use the relators of A to extend the map to the 2-skeleton. Let M_{f_*i} be the mapping cylinder of f_{*i}: then we have $K(A,1)^{(2)} \hookrightarrow M_{f_*i}$. Finally we apply the previous theorem with $K_1 = M_{f_*1}$, $K_2 = M_{f_*2}$ and $K_0 = K(A,1)^{(2)}$.

$\blacksquare$

We can now prove the following lemma, which we use later to prove theorem 4.4.1.

Lemma 4.4.5. *Let X be a 2-dimensional metrizable space with $\dim_{\mathbb{Z}/p} X = 1$ and let K be a CW complex. Let $\alpha \in \pi_1(K)$ with $\alpha^p = 1$, for some prime p. Then $X\tau K \Leftrightarrow X\tau K \cup_\alpha D^2$, where D^2 is a 2-cell.*

Proof.

Note that $\alpha : S^1 \to K$ induces a homomorphism $\alpha_* : \mathbb{Z} \to \pi_1(K)$ with $\alpha_*(1)^p = 1$, i.e. $\alpha_*(p) = 1$. Then $\pi_1(K \cup_\alpha D^2)$ is the pushout of

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\alpha_1} & \pi_1(K) \\ \downarrow & & \\ \{1\} & & \end{array}$$

But in this case it is also the pushout of

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{\alpha_1|} & \pi_1(K) \\ \downarrow & & \\ \{1\} & & \end{array}$$

Then we can use proposition 4.4.4.

■

The previous lemma has the following generalization:

Lemma 4.4.6. *Let X be a 2-dimensional metrizable space, $\mathcal{P} = \{p : \dim_{\mathbb{Z}/p} X = 1\}$ and let K be a CW complex. Let $\alpha_i \in \pi_1(K)$ with ${\alpha_i}^{{p_i}^{m_i}} = 1$, for some prime $p_i \in \mathcal{P}$ and positive integer m_i. Then $X\tau K \Leftrightarrow X\tau K \cup_{\alpha_i} {D_i}^2$, where ${D_i}^2$ is a 2-cell.*

Proof.

The proof uses induction on the number of primes and the same technique used in 4.4.5.

■

Proof. (of theorem 4.4.1)

$A = G_{ab}$ is a finite abelian group and hence of the form $\mathbb{Z}/{q_1}^{n_1} \oplus \cdots \oplus \mathbb{Z}/{q_l}^{n_l}$. Using the previous lemma, we can "kill" all elements of G of order products of q_i: we create the quotient of G by the normal subgroups generated by those elements. As G is finite , we can repeat this process finitely many times to get a solvable group $\tilde{G}$ such that $\tilde{G}$ has no elements of order products of primes appearing in the decomposition of $\tilde{G}_{ab}$ into cyclic groups, and $X\tau K(G,1) \Leftrightarrow X\tau K(\tilde{G},1)$ for 2-dimensional compacta X.

We claim that $\tilde{G}$ has trivial abelianization: in the exact sequence $1 \to [\tilde{G},\tilde{G}] \to \tilde{G} \to \tilde{G}_{ab} \to 1$ let y be any generator of $\tilde{G}_{ab}$ and consider its preimage y'. As $y^q = 1$ for some $q \in \{q_1, \cdots, q_l\}$, y'^q is mapped to 1 in $\tilde{G}_{ab}$, and hence $y'^q = k$ for some element $k \in [\tilde{G},\tilde{G}]$. Every element in $[\tilde{G},\tilde{G}]$ has order relatively prime to the q_i(s), so suppose k has order p, relatively prime to the q_i(s). Thus we have $y'^{pq} = 1$ which implies $y'^p = 1$, which in turn implies $y^p = 1$. Thus $y = 1$, and our claim is proved.

■

4.5 Extension properties of the Pontryagin Disk

We will use in this section a 2-dimensional compactum called the mod-2 Pontryagin disk $\mathbb{P}D^2$, which is a variant of a classical construction of L.S. Pontryagin [46].

The space $\mathbb{P}D^2$ is constructed as the inverse limit of an inverse sequence of spaces. As the first stage of the construction, we start with the usual 2-cell embedded in $\mathbb{R}^3$ and take some triangulation of it with mesh less than 1. To get the $k+1$-th stage from the k-th stage we take each 2-simplex, remove its interior and glue a disk along the degree 2 map, then give a triangulation of mesh less than $1/(k+1)$ to the resulting space. For the bonding map from $k+1$-th stage to the k-th stage, we send the interior 2-simplices of the attached disks to the barycenter of the removed 2-simplices.

The following theorem is very useful in constructing compacta with special extension properties. For explanation of terms and proof see [23]. Note that mesh$\{\lambda_i\} \to 0$ means that for every k,

$\lim_{i\to\infty}\{\text{mesh}(q_k^{k+i}(\lambda)_{k+i})\} = 0$. This is not the same as $\lim_{i\to\infty}\text{mesh}\lambda_i = 0$.

Theorem 4.5.1. *Suppose that K is a countable CW complex and that X is a compactum such that $X = \lim_{\leftarrow}(X_i, f_i^{i+1})$, where (L_i, f_i^{i+1}) is a K-resolvable inverse system of compact polyhedra L_i with triangulations λ_i such that mesh$\{\lambda_i\} \to 0$. Then $X\tau K$.*

Theorem 4.5.2. *Let $\mathbb{P}D^2$ be the Pontryagin disk. Then $\mathbb{P}D^2\tau K(\mathbb{Z}/2,1)$.*

Proof. Let $A \subset \mathbb{P}D^2$ be a closed and consider the map $f : A \to K(\mathbb{Z}/2,1)$. Represent $(\mathbb{P}D^2, A) = \lim_{\leftarrow}\{((\mathbb{P}D^2)_i, A_i), q_i^{i+1}\}$ in the usual way as the inverse limit of polyhedral pairs (all lying in I^∞). Find i_0 such that f can be extended over A_{i_0} to give a map $\overline{f_{i_0}} : A_{i_0} \to K(\mathbb{Z}/2,1)$ with $\overline{f_{i_0}}\circ(q_{i_0}^\infty) \simeq f$. Extend $\overline{f_{i_0}}$ over the 1-skeleton of $((\mathbb{P}D^2)_i$ and get an induced extension up to homotopy $\overline{G} : A\cup(\mathbb{P}D^2)^{(1)} \to K(\mathbb{Z}/2,1)$. Pick any 2-simplex $\sigma_j \in ((\mathbb{P}D^2)_{i_0})^{(1)} - (A_{i_0})^{(1)}$ and consider $\gamma_{\sigma_j} = g_{i_0}(\partial\sigma_j) \in \mathbb{Z}/2$. Consider $(q_{i_0}^\infty)^{-1}(\sigma_j) \subset \mathbb{P}D^2$ and note that we have an extension (of f) over $A \cup (q_{i_0}^\infty)^{-1}(\partial\sigma_j)$. We want to extend over $A \cup (q_{i_0}^\infty)^{-1}(\sigma_j)$: if γ_{σ_j} is trivial there is no obstruction. If not, find the smallest index $i_1 > i_0$ such that $(p_{i_0}^{i_1})^{-1}$ has an $(\mathbb{R}P)^2$. Clearly the extension over $A\cup(q_{i_0}^\infty)^{-1}$ is possible. Do the same for all such 2-simplices σ_j.

■

4.6 Infinite solvable groups

Theorem 4.6.1. *Let $\mathbb{P}D^2$ be the Pontryagin disk and Γ be a torsion free group with abelianization being a non-trivial 2-group. Then $K(\Gamma_{ab},1)$ is an absolute extensor of $\mathbb{P}D^2$, but $K(\Gamma,1)$ is not an absolute extensor of $\mathbb{P}D^2$.*

Proof. Let σ be any 2-simplex in the first stage of construction of $\mathbb{P}D^2$ and identify $\partial\sigma$ and its image in X under $(f_0^\infty)^{-1}\partial\sigma$. Consider the map $f : \partial\sigma \to K(\Gamma, 1)$ that sends $\partial\sigma$ to the generator x along the identity map $S^1 \to S^1$.

We claim that this map does not extend: suppose there is an extension $g : \mathbb{P}D^2 \to K(\Gamma, 1)$. Then there is some integer i_0 such that there is an extension up to homotopy $g_{i_0} : \mathbb{P}D^2{}_{i_0} \to K(\Gamma, 1)$. In that case, x could be expressed as products of elements of Γ, each of order 2. But Γ is torsion-free, and we have a contradiction.

■

We give 2 examples of solvable groups satisfying 4.6.1, with Γ_{ab} being a 2-group.

Example 4.6.2. Consider the following group, which was first studied by Hirsch:
$\Gamma_1 = \langle x, y, z | x^z = x^{-1}, y^z = y^{-1}, [x, y] = z^4 \rangle$.
This example of Hirsch is a torsion free polycyclic group which is not poly infinite cyclic. The folowing lemma shows that G is not nilpotent.

Lemma 4.6.3 (Robinson, 5.2.20)**.** *A finitely generated torsion-free nilpotent group has a central series with infinite cyclic factors.*

Example 4.6.4. [1] It is known [51] theorem 3.5.5 that there are just 6 affine diffeomorphism classes of compact connected orientable flat 3-dimensional Riemannian manifolds, represented by manifolds $\mathbb{R}^3/G$, where G is one of 6 known groups. One of them is our next example

$\Gamma_2 = \langle \alpha, \beta, \gamma x, y, z | \gamma\beta\alpha = xy, \alpha^2 = x, \alpha y \alpha^{-1} = y^{-1}, \alpha z \alpha^{-1} = z^{-1},$
$\beta x \beta^{-1} = x^{-1}, \beta^2 = y, \beta z \beta^{-1} = z^{-1}, \gamma x \gamma^{-1} = x^{-1},$
$\gamma y \gamma^{-1} = y^{-1}, \gamma^2 = z \rangle.$

Its abelianization is $\mathbb{Z}/4 \oplus \mathbb{Z}/4$. It is torsion free and solvable but not nilpotent.

[1] The author thanks Prof. Thomas Farrell for pointing out this example.

Remark 4.6.5. Even solvable groups with some elements of infinite order may not be enough to serve as an example of groups in 4.6.1.

As an example, consider D_∞, which is defined to be the semi-direct product $\mathbb{Z} \rtimes_\theta \mathbb{Z}/2$, where $\theta : \mathbb{Z}/2 \to \mathrm{Aut}(Z)$ is the "t^{-1}" map. A presentation of D_∞ is $\langle x, y \mid y^2 = 1, yxy^{-1} = x^{-1} \rangle$. Note that the relations imply $(yx)^2 = 1$, so $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ and therefore, the abelianization is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. As $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$, we have $\mathbb{P}D^2 \tau K(D_\infty, 1)$.

Chapter 5

Hurewicz-Serre theorem for nilpotent groups

5.1 Introduction

Given a metrizable space X and a connected CW complex L consider the following conditions:

1. $X\tau L$.

2. $X\tau K(\pi_n(L), n)$ for all $n \geq 1$.

3. $X\tau K(H_n(L), n)$ for all $n \geq 1$.

In chapter 2 we pointed out that (1) implies (3) was proved by Dranishnikov [19] for X compact and for arbitrary X in [[25], Theorem 3.4]. The difficulty in generalizing results in cohomological dimension theory from compact spaces to arbitrary metrizable spaces usually lies in the fact that the First Bockstein Theorem does not hold for metric spaces.

[0]This chapter is based on joint work with Cencelj,Dydak and Vavpetič [8]

By a Hurewicz-Serre Theorem in Extension Theory we mean any result showing (3) implies (2).

Here is the main problem for this chapter:

Problem 5.1.1. Suppose X is a metrizable space such that $X\tau K(H_n(L),n)$ for all $n \geq 1$. If L is nilpotent, does $X\tau K(\pi_n(L),n)$ hold for all $n \geq 1$?

Even a specialized version of 5.1.1 is open:

Problem 5.1.2. Suppose L is a nilpotent CW complex. If X is a metrizable space such that $X\tau K(H_k(L),k)$ for all $k \geq 1$, does $X\tau K(\pi_1(L),1)$ hold?

Notice that it is not sufficient to assume $X\tau K(H_1(L),1)$ in 5.1.2. Namely, take the group G from [50] whose abelianization is $\mathbb{Q} \oplus \mathbb{Q}$ and whose commutator group is $\mathbb{Z}/p^\infty$. Pick a compactum X so that $\dim_{\mathbb{Q}}(X) = 1$ and $\dim_{\mathbb{Z}/p^\infty}(X) = 2$. The complex $L = K(G,1)$ is nilpotent and $X\tau K(H_1(L),1)$ but $X\tau L$ does not hold. Indeed, as $\pi_1(L)$ is not $\bar{p}$-local and $H_1(L;\mathbb{Z}/p^\infty) = 0$, 5.5.4 says $H_2(L;\mathbb{Z}/p^\infty) \neq 0$ which means $H_2(L)/\mathrm{Tor}(H_2(L))$ is not p-divisible. If $X\tau L$, then $X\tau K(H_2(L),2)$ and $\dim_{\mathbb{Z}_{(p)}}(X) \leq 2$ and that, in combination with $\dim_{\mathbb{Q}}(X) = 1$, implies $\dim_{\mathbb{Z}/p^\infty}(X) \leq 1$, a contradiction.

However, if $H_1(L)$ is a torsion group, then the answer to 5.1.2 is positive.

Lemma 5.1.3. *Suppose N is a nilpotent group. If $X\tau K(Ab(N),1)$ for some metrizable space X and $Ab(N)$ is a torsion group, then $X\tau K(N,1)$.*

Proof. We will prove 5.1.3 by induction on the nilpotency class n of N. Let $\Gamma^n N = \Gamma^n$. Notice N/Γ^n is a nilpotent group of class $(n-1)$ whose abelianization is an image of $\mathrm{Ab}(N)$. Thus $X\tau K(N/\Gamma^n,1)$. The epimorphism

$$\otimes^n \mathrm{Ab}N \longrightarrow \Gamma^n N = \Gamma^n$$

implies $X\tau K(\Gamma^n, 1)$, so the fact that N is a central extension

$$1 \to \Gamma^n \to N \to N/\Gamma^n \to 1$$

concludes the proof. ■

For the sake of completeness let us show (2) is always stronger than (3).

Proposition 5.1.4. *Suppose X is a metrizable space and L is a connected CW complex. If $X\tau K(\pi_n(L), n)$ hold for all $n \geq 1$, then $X\tau K(H_n(L), n)$ for all $n \geq 1$.*

Proof. Let L_n be the CW complex obtained from L by killing all homotopy groups higher than n. Since L_n is obtained from L by attaching k-cells for $k > n+1$, $H_n(L_n) = H_n(L)$. Also, one has $X\tau L_n$ as $X\tau K(\pi_i(L_n), i)$ holds for all i and only finitely many homotopy groups of L_n are non-trivial (see Theorem G of [25]). Therefore $X\tau K(H_n(L_n), n)$. ■

Definition 5.1.5. X is called a *Knoxville space* if it is metrizable and for any connected CW complex L the conditions $X\tau K(\pi_n(L), n)$ for all $n \geq 1$ imply $X\tau L$.

Problem 5.1.6. Characterize Knoxville spaces.

It follows from Theorem G of [25] that any finite dimensional X or any $X \in ANR$ is a Knoxville space. Also, it is easy to see that any countable union of closed Knoxville subspaces is a Knoxville space.

5.2 Properties of the homotopy fiber of $L \to SP(L)$

Notice that condition (3) of Section 5.1 is equivalent to $X\tau SP(L)$ as $SP(L)$ is the weak product of $K(H_n(L), n)$ for all $n \geq 1$ according to the famous theorem of Dold and Thom [16]. Since we are interested in deriving $X\tau L$ it makes sense to ponder the stronger condition $X\tau F$, where F is the homotopy fiber of the inclusion $L \to SP(L)$. In this section we concentrate on basic properties of F and its homotopy groups.

Proposition 5.2.1. *Suppose L is a CW complex and F is the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If L is nilpotent, then F is nilpotent.*

Proof. The homotopy sequence

$$\cdots \to \pi_n(F) \xrightarrow{j_*} \pi_n(L) \xrightarrow{i_*} \pi_n(SP(L)) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots$$

of the fibration $F \xrightarrow{j} L \xrightarrow{i} SP(L)$ is a sequence of $\pi_1(L)$-modules [43, Proposition 8^{bis}.2]. For the action of $\pi_1(L)$ on $\pi_n(F)$ which is described in the proof of [43, Proposition 8^{bis}.2] holds $g \cdot \alpha = j_*(g) \cdot \alpha$ for $\alpha \in \pi_n(F)$ and $g \in \pi_1(F)$.

Let I_F and I_L be the augmentation ideals of group rings $\mathbb{Z}[\pi_1(F)]$ and $\mathbb{Z}[\pi_1(L)]$, respectively. Because L is a nilpotent, there is an integer c, such that $(I_L)^c\pi_n(L) = 0$. Let $\eta \in (I_F)^c$ and $\alpha \in \pi_n(F)$. Then $j_*(\eta\alpha) = j_*(\eta)j_*(\alpha) = 0$, because $j_*(\eta) \in (I_L)^c$. Thus there exists $\beta \in \pi_{n+1}(SP(L))$, such that $\partial\beta = \eta\alpha$. Let $g \in \pi_1(F)$. Then $(j_*(g) - 1) \in I_L$ and

$$\partial((j_*(g) - 1)\beta) = (j_*(g) - 1)\partial\beta = (j_*(g) - 1)\eta\alpha(g - 1)\eta\alpha.$$

The action of $\pi_1(L)$ on $\pi_n(SP(L))$ is defined as $l\gamma = i_*(l)\gamma$ for $l \in \pi_1(L)$ and $\gamma \in \pi_n(SP(L))$. Hence

$$(j_*(g) - 1)\beta = (i_*j_*(g) - 1)\beta = (1 - 1)\beta = 0,$$

therefore $(g - 1)\eta\alpha = 0$. This shows that $(I_F)^{c+1}\pi_n(F) = 0$, so the space F is nilpotent. ■

Proposition 5.2.2. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If $\mathcal{P}$ is a set of primes such that $H_k(L)$ is a $\mathcal{P}$-torsion group for all $k \leq n$, where $n \geq 1$ is given,* *then $\pi_k(F)$ is a $\mathcal{P}$-torsion group for all $1 \le k \le n+1$.*

Proof. Let $\mathcal{P}'$ be the complement of $\mathcal{P}$ in the set of all primes. Consider the localization $L_{(\mathcal{P}')}$ of L at $\mathcal{P}'$. Notice that $L_{(\mathcal{P}')}$ is n-connected, so the Hurewicz homomorphism $\phi_k : \pi_k(L_{(\mathcal{P}')}) \to H_k(L_{(\mathcal{P}')})$ is an isomorphism for $k = n+1$ and an epimorphism for $k = n+2$. Let us split the exact sequence $\ldots \to \pi_k(F) \to \pi_k(L) \to H_k(L) \to \ldots$ into $\ldots \to \pi_2(F) \to \pi_2(L) \to H_2(L) \to A \to 0$ and $1 \to A \to \pi_1(F) \to B \to 1$, where B is the commutator subgroup of $\pi_1(L)$. Localizing the first sequence at $\mathcal{P}'$ yields A being a $\mathcal{P}$-torsion group and $\pi_k(F)$ being $\mathcal{P}$-torsion for $2 \le k \le n+1$. Since B is $\mathcal{P}$-torsion, 5.2.2 follows. ■

Corollary 5.2.3. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If $n > 1$ is a number such that $H_k(L)$ is a torsion group for all $k < n$, then for any metrizable space X the conditions $X\tau K(H_k(L), k)$ for all $k \le n$ imply $X\tau K(\pi_k(F), k)$ for all $1 \le k \le n$.*

Proof. The case $k = 1$ is taken care of by 5.1.3. If p-torsion of $\pi_k(F)$ is not trivial, then 5.2.2 implies that p-torsion of $H_m(L)$ is not trivial for some $m < k$. Therefore $X\tau K(\mathbb{Z}/p^\infty, m)$ and $X\tau K(\mathbb{Z}/p, m+1)$. This implies $X\tau K(G, k)$ for all G in the Bockstein basis of $\pi_k(F)$ resulting in $X\tau K(\pi_k(F), k)$. ■

5.3 Homotopy groups with coefficients

Given a countable Abelian group G consider a pointed compactum $P_n(G)$ such that its integral cohomology is concentrated in dimension n and equals G. The n-th homotopy group $\pi_n(L; G)$ of a pointed CW complex L is defined in [45] to be the set $[P_n(G), L]$ of pointed homotopy classes from $P_n(G)$ to L. If $P_{n-1}(G)$ exists (that is always true if $n > 2$ or G is torsion free and $n \ge 2$), then $P_n(G)$ could be taken as the suspension $\Sigma P_{n-1}(G)$ of $P_{n-1}(G)$ with the resulting group structure on $\pi_n(L, G)$.

If one puts $D = P_2(G)$ (or $D = P_1(G)$ if G is torsion-free), then one can analyze homotopy groups of $L^D = \mathrm{Map}(D, L)$ and realize that $\pi_n(L^D) = \pi_{n+2}(L; G)$ (respectively, $\pi_n(L^D) = \pi_{n+1}(L; G)$). Therefore, given a Hurewicz fibration $F \to E \to B$, one concludes there is a long exact sequence $\ldots \to \pi_n(F; G) \to \pi_n(E; G) \to \pi_n(B; G) \to \pi_{n-1}(F; G) \to \ldots$ (see [45] for the special case of $G = \mathbb{Z}/m$) because $F^D \to E^D \to B^D$ is a Serre fibration.

In the special case of $G = \mathbb{Z}/m$ one can pick the Moore space $D = M(\mathbb{Z}/m, 1)$ for $P_2(G)$. In that case one has a Serre fibration (that follows from the Homotopy Extension Theorem) $\mathrm{Map}(S^2, L) \to \mathrm{Map}(D, L) \to \mathrm{Map}(S^1, L)$ where S^1 is the 1-skeleton of D and $S^2 = D/S^1$. The map $\mathrm{Map}(D, L) \to \mathrm{Map}(S^1, L)$ is simply restriction induced. Since the boundary homomorphism $\pi_{n+1}(B) \to \pi_n(F)$ in that case amounts to multiplication by m from $\pi_{n+1}(\mathrm{Map}(S^1, L)) = \pi_{n+2}(L)$ to $\pi_n(\mathrm{Map}(S^2, L)) = \pi_{n+2}(L)$, one concludes the following (see [45] for another way of deriving an equivalent result):

Lemma 5.3.1. *Let $D = M(\mathbb{Z}/m, 1)$ for some $m \geq 2$. For each pointed CW complex L and each $n \geq 0$ one has a natural exact sequence*

$$0 \to \pi_{n+2}(L) \otimes \mathbb{Z}/m \to \pi_n(L^D) \to \pi_{n+1}(L) * \mathbb{Z}/m \to 0,$$

where $\pi_1(L) * \mathbb{Z}/m$ is $\{x \in \pi_1(L) | x^m = 1\}$.

We are interested in homotopy groups with coefficients in $\mathbb{Z}/p^\infty$, the direct limit of $\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \ldots$. Notice that one can construct $P_2(\mathbb{Z}/p^\infty)$ as the inverse limit of $\ldots \to M(\mathbb{Z}/p^{n+1}, 1) \to M(\mathbb{Z}/p^n, 1) \to \ldots \to M(\mathbb{Z}/p, 1)$ which can be viewed as $M(\hat{Z}_p, 1)$, the Moore space for the p-adic integers $\hat{Z}_p$ in terms of Steenrod homology. In that case 5.3.1 becomes

Lemma 5.3.2. *Let p be prime. For each pointed CW complex L and each $n \geq 0$ one* *has a natural exact sequence*

$$0 \to \pi_{n+2}(L) \otimes \mathbb{Z}/p^\infty \to \pi_{n+2}(L; \mathbb{Z}/p^\infty) \to \pi_{n+1}(L) * \mathbb{Z}/p^\infty \to 0,$$

where $\pi_1(L) * \mathbb{Z}/p^\infty$ *is* $\{x \in \pi_1(L) | x^{p^k} = 1$ *for some* $k \geq 1\}$.

As a consequence of 5.3.1, 5.3.2, and Dold-Thom Theorem [16] ($\pi_n(SP(L)) = H_n(L)$) one can get that $\pi_n(SP(L); G) = H_n(L; G)$ for all n and $G = \mathbb{Z}/p$ or $G = \mathbb{Z}/p^\infty$.

Proposition 5.3.3. *Suppose* L *is a nilpotent CW complex whose fundamental group is* $\bar{p}$*-local for some prime* p*. Let* F *be the homotopy fiber of the inclusion* $i : L \to SP(L)$ *of* L *into its infinite symmetric product. If* $H_k(L, \mathbb{Z}/p) = 0$ *for* $k \leq n$*, where* $n \geq 1$*, then* $\pi_k(F, \mathbb{Z}/p) = 0$ *for* $2 \leq k \leq n+1$ *and* $\pi_k(L, \mathbb{Z}/p) = 0$ *for* $2 \leq k \leq n$*.*

Proof. Let $\tilde{L}$ be the universal cover of L and let $\pi : \tilde{L} \to L$ be the covering projection. Recall that every nilpotent CW complex L has the p-completion L_p with the map $L \to L_p$ inducing isomorphisms of all homology groups with coefficients in $\mathbb{Z}/p$ such that one has a natural exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n(L)) \to \pi_n(L_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(L)) \to 0$$

for all $n \geq 1$ (see [[34], Theorem 3.7 on p.416]). By 5.5.2 and 5.5.3 one has $\mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_1(L)) = \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_1(L)) = 0$, so the induced map $\hat{\tilde{L}}_p \to \hat{L}_p$ is a homotopy equivalence. Therefore it induces isomorphism of homology mod p and π induces isomorphism of homology mod p. However, $H_2(\tilde{L}; \mathbb{Z}/p) = \pi_2(\tilde{L}; \mathbb{Z}/p) = \pi_2(L; \mathbb{Z}/p)$ and $\pi_3(L) \to H_3(\tilde{L})$ is an epimorphism resulting in $\pi_3(L; \mathbb{Z}/p) \to H_3(L; \mathbb{Z}/p)$ being an epimorphism. By exactness of mod p groups of a fibration one gets $\pi_2(F, \mathbb{Z}/p) = 0$. That proves 5.3.3 for $n = 1$.

If $n > 1$ we apply mod p Hurewicz Theorem of [45] to $\tilde{L}$ to conclude $\pi_{n+1}(\tilde{L}; \mathbb{Z}/p) \to$

$H_{n+1}(\tilde{L};\mathbb{Z}/p)$ is an isomorphism and $\pi_{n+2}(\tilde{L};\mathbb{Z}/p) \to H_{n+2}(\tilde{L};\mathbb{Z}/p)$ is an epimorphism. Consequently, $\pi_{n+1}(L;\mathbb{Z}/p) \to H_{n+1}(L;\mathbb{Z}/p)$ is an isomorphism and $\pi_{n+2}(L;\mathbb{Z}/p) \to H_{n+2}(L;\mathbb{Z}/p)$ is an epimorphism. Thus $\pi_{n+1}(F;\mathbb{Z}/p) = 0$. ■

Corollary 5.3.4. *Suppose L is a nilpotent CW complex whose fundamental group is $\bar{p}$-local for some prime p. Let F be the homotopy fiber of the inclusion $i : L \to SP(L)$ of L into its infinite symmetric product. If $H_k(L,\mathbb{Z}/p^\infty) = 0$ for $k \leq n$, where $n \geq 2$, then $\pi_k(L,\mathbb{Z}/p^\infty) = \pi_k(F,\mathbb{Z}/p^\infty) = 0$ for $2 \leq k \leq n$.*

Proof. Case 1: $n > 2$. Notice $H_k(L,\mathbb{Z}/p) = 0$ for $k \leq n-1$ resulting in $\pi_k(F,\mathbb{Z}/p) = 0$ for $2 \leq k \leq n$. Hence $\pi_k(F,\mathbb{Z}/p^\infty) = 0$ for $2 \leq k \leq n$ and from an exact sequence we get $\pi_k(L,\mathbb{Z}/p^\infty) = 0$ for $2 \leq k \leq n$.

Case 2: $n = 2$. Let $\tilde{L}$ be the universal cover of L and let $\pi : \tilde{L} \to L$ be the covering projection. Notice that the induced map $\hat{\tilde{L}}_p \to \hat{L}_p$ is a homotopy equivalence. Therefore it induces isomorphism of homology mod p and π induces isomorphism of homology mod p. However, $H_2(\tilde{L};\mathbb{Z}/p^\infty) = \pi_2(L;\mathbb{Z}/p^\infty)$ and $\pi_3(L) \to H_3(\tilde{L})$ is an epimorphism resulting in $\pi_3(L;\mathbb{Z}/p^\infty) \to H_3(L;\mathbb{Z}/p^\infty)$ being an epimorphism. By exactness of mod p groups of a fibration one gets $\pi_2(F,\mathbb{Z}/p^\infty) = \pi_2(L,\mathbb{Z}/p^\infty) = 0$. ■

Proposition 5.3.5. *Suppose L is a nilpotent CW complex, F is the homotopy fiber of the inclusion $i : L \to SP(L)$ of L into its infinite symmetric product, and $n \geq 2$. If $H_k(L,\mathbb{Z}_{(\mathcal{P})}) = 0$ for $k \leq n$, then $\pi_k(F,\mathbb{Z}_{(\mathcal{P})}) = 0$ for $2 \leq k \leq n+1$ and $\pi_k(L,\mathbb{Z}_{(\mathcal{P})}) = 0$ for $2 \leq k \leq n$.*

Proof. Let $\mathcal{P}'$ be the complement of $\mathcal{P}$ in the set of all primes. If $H_k(L,\mathbb{Z}_{(\mathcal{P})}) = 0$ for $k \leq n$, then $H_1(L)$ is a $\mathcal{P}'$-torsion group resulting in $\pi_1(L)$ being a $\mathcal{P}'$-torsion group. Let $L_{(\mathcal{P})}$ be the $\mathcal{P}$-localization of L. It is n-connected, so by the classical Hurewicz Theorem $\pi_k(L_{(\mathcal{P})}) \to H_k(L_{(\mathcal{P})})$ is an isomorphism for $k \leq n+1$ and an epimorphism for $k = n+2$. That corresponds to $\pi_k(L;\mathbb{Z}_{(\mathcal{P})}) \to H_k(L;\mathbb{Z}_{(\mathcal{P})})$ being an isomorphism for $k \leq n+1$ and an epimorphism for $k = n+2$. In view of exactness of $\ldots \to \pi_k(F;\mathbb{Z}_{(\mathcal{P})}) \to \pi_k(L;\mathbb{Z}_{(\mathcal{P})}) \to H_k(L;\mathbb{Z}_{(\mathcal{P})}) \to \ldots$, 5.3.5 follows. ■

5.4 Main results

Lemma 5.4.1. *Suppose X is a metrizable space, G is an Abelian group, and $n \geq 1$. If $dim_G(X) > n$, then one of the following conditions holds:*

a. $dim_{\mathbb{Q}}(X) \geq n+1$ and G is not a torsion group.

b. There is a prime p such that $G \otimes \mathbb{Z}/p^\infty \neq 0$ and $dim_{\mathbb{Q}}(X) \leq n$, $dim_{\mathbb{Z}/p^\infty}(X) \geq n$.

c. There is a prime p such that G is p-divisible, $Tor_p(G) \neq 0$, and $dim_{\mathbb{Z}/p^\infty}(X) \geq n+1$.

d. There is a prime p such that $Tor_p(G)$ is not p-divisible and $dim_{\mathbb{Z}/p}(X) \geq n+1$.

Proof. Let $\dim_G(X) = m$. Suppose none of (a)-(d) holds. According to Part (b) of Theorem B of [25] one has $m = \dim_{G/\mathrm{Tor}(G)}(X)$ or $m = \dim_{\mathrm{Tor}(G)}(X)$. If $\dim_{\mathrm{Tor}(G)}(X) \geq n+1$, then, according to Part (a) of Theorem B of [25], there is a prime p such that either $\mathrm{Tor}(G)$ is p-divisible, $\mathrm{Tor}_p(G) \neq 0$, and $\dim_{\mathrm{Tor}(G)}(X) = \dim_{\mathbb{Z}/p^\infty}(X)$ (in which case (c) holds) or $\mathrm{Tor}(G)$ is not p-divisible, $\mathrm{Tor}_p(G) \neq 0$, and $\dim_{\mathrm{Tor}(G)}(X) = \dim_{\mathbb{Z}/p}(X)$ (in which case (d) holds). Therefore $m = \dim_{G/\mathrm{Tor}(G)}(X)$ and $\dim_{\mathrm{Tor}(G)}(X) \leq n$. In particular G is not a torsion group, so $\dim_{\mathbb{Q}}(X) \leq n$ as (a) fails to hold.

Consider $\mathcal{P} = \{p | G \otimes \mathbb{Z}/p^\infty \neq 0\}$, the set of primes p such that $G/\mathrm{Tor}(G)$ is not p-divisible. It is shown in [25] (Part (f) of Theorem B) that $\dim_{\mathbb{Z}_{(\mathcal{P})}}(X) \geq \dim_{G/\mathrm{Tor}(G)}(X)$, so $\dim_{\mathbb{Z}_{(\mathcal{P})}}(X) \geq m$. As (b) does not hold, one has $\dim_{\mathbb{Z}/p^\infty}(X) \leq n-1$ for all $p \in \mathcal{P}$. From the exact sequence

$$0 \to \mathbb{Z}_{(\mathcal{P})} \to \mathbb{Q} \to \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p^\infty \to 0$$

one concludes that the homotopy fiber of $K(\mathbb{Z}_{(\mathcal{P})}, m-1) \to K(\mathbb{Q}, m-1)$ is $K(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p^\infty, m-2)$. Since $m-2 \geq n-1$, $X\tau K(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p^\infty, m-2)$ which implies $X\tau K(\mathbb{Z}_{(\mathcal{P})}, m-1)$ as $X\tau K(\mathbb{Q}, m-1)$ is true. Thus $\dim_{\mathbb{Z}_{(\mathcal{P})}}(X) \leq m-1$, a contradiction. ■

Theorem 5.4.2. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If X is a metrizable space such that $X\tau K(H_k(L), k)$ for all $k \geq 1$, then $X\tau K(\pi_k(F), k)$ and $X\tau K(\pi_k(L), k)$ for all $k \geq 2$.*

Proof. Suppose $n \geq 2$ is the smallest natural number such that $X\tau K(\pi_k(F), k)$ fails (similar argument in case $X\tau K(\pi_k(L), k)$ fails). By 5.4.1 one of the following cases holds for $G = \pi_n(F)$:

a. $\dim_{\mathbb{Q}}(X) \geq n+1$ and G is not a torsion group.

b. There is a prime p such that $G \otimes \mathbb{Z}/p^\infty \neq 0$ and $\dim_{\mathbb{Q}}(X) \leq n$, $\dim_{\mathbb{Z}/p^\infty}(X) \geq n$.

c. There is a prime p such that G is p-divisible, $\mathrm{Tor}_p(G) \neq 0$, and $\dim_{\mathbb{Z}/p^\infty}(X) \geq n+1$.

d. There is a prime p such that $\mathrm{Tor}_p(G)$ is not p-divisible and $\dim_{\mathbb{Z}/p}(X) \geq n+1$.

Case 1: $\dim_{\mathbb{Q}}(X) \leq n-1$. Now only (b)-(d) are possible. Let p be the prime from one of those cases. Notice $H_k(L)$ is $\bar{p}$-local for $k \leq n-1$ as otherwise $\dim_{\mathbb{Z}/p^\infty}(X) \leq n-1$ and $\dim_{\mathbb{Z}/p}(X) \leq n$ so none of (b)-(d) would be valid. Another observation is $H_n(L) \otimes \mathbb{Z}/p^\infty = 0$. Indeed, $H_n(L) \otimes \mathbb{Z}/p^\infty \neq 0$ leads to $H_n(L)/\mathrm{Tor}(H_n(L))$ not being p-divisible in which case [[25], Part (d) of Theorem B] implies $\dim_{\hat{\mathbb{Z}}_p}(X) \leq n$ as $\dim_{H_n(L)}(X) \leq n$. Therefore $\dim_{\mathbb{Z}_{(p)}}(X) \leq n$ (see Part (e) of Theorem B in [25]) and $\dim_{\mathbb{Z}/p^\infty}(X) \leq \max(\dim_{\mathbb{Q}}(X), \dim_{\mathbb{Z}_{(p)}}(X) - 1) \leq n-1$, a contradiction.

$\pi_1(L)$ is $\bar{p}$-local by 5.5.4 and $G \otimes \mathbb{Z}/p^\infty = 0$ by 5.3.4. That means (b) is not possible. If $H_n(L)$ is not p-divisible, then $\dim_{\mathbb{Z}/p}(X) \leq n$ and neither (c) nor (d) would be possible. Thus $H_k(L; \mathbb{Z}/p) = 0$ for $k \leq n$ resulting in G being p-divisible by 5.3.3. That means only (c) is possible. In addition, $\mathrm{Tor}_p(H_n(L)) = 0$. Also $H_{n+1}(L) \otimes \mathbb{Z}/p^\infty = 0$ (otherwise $\dim_{\mathbb{Z}_{(p)}}(X) \leq n+1$ and $\dim_{\mathbb{Z}/p^\infty}(X) \leq \max(\dim_{\mathbb{Q}}(X), \dim_{\mathbb{Z}_{(p)}}(X) - 1) \leq n$). Thus $H_k(L; \mathbb{Z}/p^\infty) = 0$ for $k \leq n+1$. By 5.3.4 $\mathrm{Tor}_p(G) = 0$, a contradiction.

Case 2: $\dim_{\mathbb{Q}}(X) > n-1$. By 5.2.2 the group G is $\mathcal{P}$-torsion such that $\dim_{\mathbb{Z}/p^\infty}(X) \leq n-1$ for all $p \in \mathcal{P}$ which implies $\dim_G(X) \leq n$, a contradiction. ■

Corollary 5.4.3. *Suppose L is a nilpotent CW complex such that $\pi_n(L) = \pi_{n+1}(L) = 0$ for some $n \geq 1$. If $X \tau SP(L)$ for some metrizable space X, then $X \tau K(H_{n+1}(L), n)$.*

Proof. If $n = 1$, then $H_{n+1}(L) = 0$, so assume $n \geq 2$. Notice that $\pi_n(F) = H_{n+1}(L)$, where F is the homotopy fiber of $i : L \to SP(L)$. ■

Lemma 5.4.4. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If $X \tau K(H_1(L), 1)$ for some metrizable space X and $H_1(L)$ is finitely generated, then $X \tau K(\pi_1(F), 1)$*

Proof. If $H_1(L)$ is finitely generated and non-torsion, then X is at most 1-dimensional in which case $X \tau L$ for all connected CW complexes. Therefore assume $H_1(L)$ is a torsion group and (see 5.2.2) there is an exact sequence $1 \to A \to \pi_1(F) \to B \to 1$ such that A and B are $\mathcal{P}$-torsion groups, where $\mathcal{P} = \{p | \mathrm{Tor}_p(H_1(L)) \neq 0\}$. Notice $H_1(L)$ does not contain $\mathbb{Z}/p^\infty$ for any prime p, so $X \tau K(A, 1)$ and $X \tau K(B, 1)$ which implies $X \tau K(\pi_1(F), 1)$. ■

Theorem 5.4.5. *Let X be a metrizable space such that dim$(X) < \infty$ or $X \in ANR$. Suppose L is a nilpotent CW complex and $SP(L)$ is its infinite symmetric product. If $X \tau SP(L)$, then $X \tau L$ in the following cases:*

a. $H_1(L)$ is finitely generated.

b. $H_1(L)$ is a torsion group.

Proof. a. By 5.4.2 and 5.4.4 one concludes $X\tau K(\pi_n(F), n)$ for all $n \geq 1$. Theorem G of [25] gives $X\tau F$ which implies $X\tau L$.

b. By 5.4.2 and 5.1.3 one concludes $X\tau K(\pi_n(L), n)$ for all $n \geq 1$. Theorem G of [25] yields $X\tau L$. ■

5.5 Appendix on nilpotent groups

Lemma 5.5.1. *Suppose N is a nilpotent group and p is a prime. $Ab(N)$ is p-divisible if and only if N is p-divisible.*

Proof. If N is p-divisible clearly so is its abelianization. We will prove the converse by induction on the nilpotency class n of N. Let $\Gamma^n N = \Gamma^n$. Notice N/Γ^n is a nilpotent group of class $(n-1)$ whose abelianization is p-divisible. Thus it is p-divisible. The epimorphism

$$\otimes^n \mathrm{Ab}N \longrightarrow \Gamma^n N = \Gamma^n$$

implies Γ^n is p-divisible, so the fact that N is a central extension

$$1 \to \Gamma^n \to N \to N/\Gamma^n \to 1$$

concludes the proof. ■

Lemma 5.5.2. *Suppose N is a nilpotent group and p is a prime. The following conditions are equivalent:*

a. $Ext(\mathbb{Z}/p^\infty, N) = 0$,

b. $Ext(\mathbb{Z}/p^\infty, N)$ is p-divisible,

c. N is p-divisible.

Proof. (a) $\Longrightarrow$ (b) is obvious. For (c) $\Longrightarrow$ (a) let N be p-divisible. Then so is its abelianization and Proposition 3 of [6] implies $\mathrm{Ext}(\mathbb{Z}/p^\infty, N) = 0$.

(b) $\Longrightarrow$ (c) If N is not p-divisible neither is its abelianization by 5.5.1. Therefore, by Proposition 3 of [6] $\mathrm{Ext}(\mathbb{Z}_{p^\infty}, \mathrm{Ab}N)$ is not p-divisible. Then the six-term exact sequence of *Hom* and *Ext* [[4], p.170] implies that $\mathrm{Ext}(\mathbb{Z}/p^\infty, N)$ is not p-divisible. ■

Lemma 5.5.3. *Suppose G is a nilpotent group and p is a prime. The following conditions are equivalent:*

a. $Hom(\mathbb{Z}/p^\infty, G) = 0$,

b. $Hom(\mathbb{Z}/p^\infty, G)$ *is p-divisible,*

c. $Hom(\mathbb{Z}/p^\infty, G) \otimes \mathbb{Z}/p^\infty = 0$

d. G *does not contain* $\mathbb{Z}/p^\infty$.

Proof. Note that albeit Bousfield and Kan [4] defined *Hom* as a space, they showed that it is also the set of the respective homomorphisms.

(a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c) are obvious.

(c) $\Longrightarrow$ (b). Notice the p-torsion of $\mathrm{Hom}(\mathbb{Z}/p^\infty, G)$ is trivial. Indeed, if $i : \mathbb{Z}/p^\infty \to G$ and $i^p = 1$, then for any $a \in \mathbb{Z}/p^\infty$ we find $b \in \mathbb{Z}/p^\infty$ satisfying $b^p = a$. Now, $i(a) = i(b^p) = i^p(b) = 1$. If an Abelian group A has no p-torsion and $A \otimes \mathbb{Z}/p^\infty = 0$, then A is p-divisible.

(b) $\Longrightarrow$ (d) Suppose $i : \mathbb{Z}/p^\infty \to G$ is a monomorphism. Given $a \in \mathbb{Z}/p^\infty$ find $k \geq 1$ such that $a^{p^k} = 1$ and choose $\phi : \mathbb{Z}/p^\infty \to G$ so that $i = \phi^{p^k}$. Now $i(a) = \phi^{p^k}(a) = (\phi(a))^{p^k} = \phi(a^{p^k}) = \phi(1) = 1$, a contradiction.

(d) $\Longrightarrow$ (a). Given a non-trivial $i : \mathbb{Z}/p^\infty \to G$ its image is a direct sum of copies of $\mathbb{Z}/p^\infty$, a contradiction. ■

Lemma 5.5.4. *Suppose L is a nilpotent CW complex and p is a prime. If $H_1(L;\mathbb{Z}/p^\infty) = H_2(L;\mathbb{Z}/p^\infty) = 0$, then $\pi_1(L)$ is $\bar{p}$-local.*

Proof. In view of $H_2(L;\mathbb{Z}/p^\infty) = 0$, $H_1(L)$ has trivial p-torsion and $H_1(L;\mathbb{Z}/p^\infty) = 0$ implies $H_1(L)$ is p-divisible. So is $\pi_1(L)$ (see 5.5.1). Consider the p-completion $\hat{L}_p$ of L. As $\pi_1(\hat{L}_p) = \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_1(L)) = 0$ and $H_2(\hat{L}_p;\mathbb{Z}/p^\infty) = H_2(L;\mathbb{Z}/p^\infty) = 0$ one gets $\pi_2(\hat{L}_p) \otimes \mathbb{Z}/p^\infty = 0$ by the Hurewicz Theorem. The exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_2(L)) \to \pi_2(\hat{L}_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_1(L)) \to 0$$

implies $\mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_1(L)) \otimes \mathbb{Z}/p^\infty = 0$. By 5.5.3 $\pi_1(L)$ is $\bar{p}$-local. ■

Chapter 6

Bockstein Theorem for nilpotent groups

6.1 Introduction

Problem 6.1.1. Alexandroff's Problem: Is there a countable family $\mathcal{G}$ of abelian groups such that for any compactum X and any abelian group G, the dimension $\dim_G(X)$ can be expressed in terms of $\dim_H(X)$, $H \in \mathcal{G}$?

Problem 6.1.1 was solved by Bockstein [2] in 1956, who singled out the following groups (known now as *Bockstein groups*):

Definition 6.1.2. The set $\mathcal{B}$ of Bockstein groups is

$$\{\mathbb{Q}\} \cup \bigcup_{\mathbf{p}\ prime} \{\mathbb{Z}/\mathbf{p}, \mathbb{Z}/\mathbf{p}^\infty, \mathbb{Z}_{(\mathbf{p})}\},$$

where $\mathbb{Z}/\mathbf{p}^\infty$ is the $\mathbf{p}$-torsion of $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{Z}_{(\mathbf{p})}$ consists of the rationals whose denominator is not divisible by $\mathbf{p}$.

Theorem 6.1.3. *If X is compact, then*

[0] This chapter is based on joint work with Cencelj,Dydak and Vavpetič [9]

$$dim_G(X) = \max\{dim_H(X) \mid H \in \sigma(G)\}.$$

The set $\sigma(G)$ appearing in the theorem above is known as the *Bockstein basis* of G, and is defined as follows:

Definition 6.1.4. Given an abelian group G its Bockstein basis $\sigma(G)$ is the subset of $\mathcal{B}$ defined as follows:

1. $\mathbb{Q} \in \sigma(G)$ if and only if $\mathbb{Q} \otimes G \neq 0$,

2. $\mathbb{Z}/\mathbf{p} \in \sigma(G)$ if and only if $(\mathbb{Z}/\mathbf{p}) \otimes G \neq 0$,

3. $\mathbb{Z}_{(\mathbf{p})} \in \sigma(G)$ if and only if $(\mathbb{Z}/\mathbf{p}^\infty) \otimes G \neq 0$,

4. $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$ if and only if $(\mathbb{Z}/\mathbf{p}^\infty) * G \neq 0$ (here $H * G$ is the torsion product of groups H and G).

Since Pontryagin's example (see theorem 2.2.3) demonstrated that the logarithmic law $\dim(X \times Y) = \dim(X) + \dim(Y)$, known to be true in the case of euclidean spaces, does not hold for compacta, it was natural to seek formulae describing $\dim_G(X \times Y)$ in terms of the cohomological dimensions of X and Y. This problem was completely solved by Bockstein as follows:

Theorem 6.1.5. *Suppose X and Y are compact. Then,*

1. $dim_{\mathbb{Z}/\mathbf{p}}(X \times Y) = dim_{\mathbb{Z}/\mathbf{p}}(X) + dim_{\mathbb{Z}/\mathbf{p}}(Y)$,

2. $dim_{\mathbb{Q}}(X \times Y) = dim_{\mathbb{Q}}(X) + dim_{\mathbb{Q}}(Y)$,

3. $dim_{\mathbb{Z}/\mathbf{p}^\infty}(X \times Y) = \max\{dim_{\mathbb{Z}/\mathbf{p}^\infty}(X) + dim_{\mathbb{Z}/\mathbf{p}^\infty}(Y), dim_{\mathbb{Z}/\mathbf{p}}(X) + dim_{\mathbb{Z}/\mathbf{p}}(Y) - 1\}$,

4. $dim_{\mathbb{Z}_{(\mathbf{p})}}(X \times Y) = dim_{\mathbb{Z}_{(\mathbf{p})}}(X) + dim_{\mathbb{Z}_{(\mathbf{p})}}(Y)$ *if*

   $dim_{\mathbb{Z}_{(\mathbf{p})}}(X) = dim_{\mathbb{Z}/\mathbf{p}^\infty}(X)$ *or* $dim_{\mathbb{Z}_{(\mathbf{p})}}(Y) = dim_{\mathbb{Z}/\mathbf{p}^\infty}(Y)$, *and*

5. $dim_{\mathbb{Z}_{(\mathbf{p})}}(X \times Y) = \max\{dim_{\mathbb{Z}/\mathbf{p}^\infty}(X \times Y) + 1, dim_{\mathbb{Q}}(X) + dim_{\mathbb{Q}}(Y)\}$ *if* $dim_{\mathbb{Z}_{(\mathbf{p})}}(X) > dim_{\mathbb{Z}/\mathbf{p}^\infty}(X)$ *and* $dim_{\mathbb{Z}_{(\mathbf{p})}}(Y) > dim_{\mathbb{Z}/\mathbf{p}^\infty}(Y)$.

6.2 Exact sequences

As we have seen earlier, there is no Eilenberg-MacLane space $K(G, n)$, $1 < n < \infty$, for non-Abelian groups G, so $\dim_G X \in \{0, 1, \infty\}$. Since $\dim_G X = 0$ if and only if X is a discrete space, the only interesting question is if $X \tau K(G, 1)$ holds.

Lemma 6.2.1. *If X is metrizable space and $dim_G(X) = 0$ for some group $G \neq 1$, then $dim_H(X) = 0$ for any group H.*

Proof. If $\dim_G(X) = 0$, then $K(G, 0) \in Æ(X)$. Because $K(G, 0)$ is a discrete space, any discrete space belongs to $Æ(X)$, in particular $K(H, 0) \in Æ(X)$ for any group H, so $\dim_H(X) = 0$ for every group H. ■

Lemma 6.2.2. *Let X be a metrizable space. If $1 \to K \to G \to I \to 1$ is an exact sequence of groups and $dim_K X, dim_G X \leq 1$, then* $\max\{dim_K(X), dim_I(X)\} = dim_G(X)$.

Proof. In view of 6.2.1 the only interesting case is that of $\dim_K X = \dim_G X = 1$. ■

Lemma 6.2.3. *Let X be a metrizable space, then $dim_{Ab(G)}(X) \leq dim_G X$ for any group G.*

Proof. In view of 6.2.1 the only interesting case is that of G non-Abelian and $\dim_G X = 1$. ■

6.3 Nilpotent groups

Notation: If G is a group, then $\mathrm{Ab}(G)$ is its *abelianization*. If $A, B \subset G$ are subgroups, then the *commutator subgroup* $[A, B]$ is a group generated with all commutators $[a, b]$, $a \in A$ and $b \in B$.

The *lower central series* $\{\Gamma_n(G)\}$ for a group G is defined as follows: $\Gamma_1(G) = G$, and $\Gamma_{n+1}(G) = [\Gamma_n(G), G]$. If a group G is nilpotent, then there exists an integer c such that $\Gamma_c(G) \neq \{1\}$ but $\Gamma_{c+1}(G) = \{1\}$. The number c (denoted by $h(G)$) is called the *nilpotency class* of the nilpotent group G or its *Hirsch length*. Abelian groups are nilpotent of Hirsch length 1. By [50, Theorem 3.1], for every n exists an epimorphism

$$\otimes^n \mathrm{Ab}(G) \to \Gamma_n(G)/\Gamma_{n+1}(G).$$

In particular, there is an epimorphism $\otimes^c \mathrm{Ab}(G) \to \Gamma_c(G)$.

A central extension $K \to G \to I$ of nilpotent group, for which exists an epimorphism $\otimes^n \mathrm{Ab}(G) \to K$ for some n, is called a *nilpotent central extension*. Thus, for every (nonabelian) nilpotent group G, there exists a nilpotent central extension $K \to G \to I$ such that Hirsch length of I is less than Hirsch length of G.

Lemma 6.3.1. *Let $1 \to K \to G \xrightarrow{\pi} I \to 1$ be a central extension of nilpotent groups.*

a. If K and I are p-divisible then G is p-divisible.

b. If the extension is a nilpotent central extension and G is p-divisible, then K and I are p-divisible.

Proof. Suppose K and I are p-divisible. Let $g \in G$. Then $\pi(g) = i^p$ for some $i \in I$. Let $\bar{g} \in G$ be such that $\pi(\bar{g}) = i$. Then $\pi(\bar{g}^p g^{-1}) = 1$, so $\bar{g}^p g^{-1} = k^p$ for some $k \in K$. Because $k \in K \subset C(G)$, $g = (\bar{g}k^{-1})^p$, so G is p-divisible.

If G is p-divisible, then any epimorphic image of G is p-divisible. Thus both I and $\mathrm{Ab}(G)$ are p-divisible. As there is an epimorphism $\otimes^n \mathrm{Ab}(G) \to K$, K is also

p-divisible. ■

Lemma 6.3.2. *Suppose $\mathcal{P}_i$, $i = 1, 2$, are two classes of nilpotent groups such that for any nilpotent central extension $1 \to K \to G \to I \to 1$ where $h(I) < h(G)$ the following conditions hold*

a. K and I belong to $\mathcal{P}_1$ if $G \in \mathcal{P}_1$,

b. $G \in \mathcal{P}_2$ if $K, I \in \mathcal{P}_2$,

are equivalent. If $A \in \mathcal{P}_1 \implies A \in \mathcal{P}_2$ for all Abelian groups A, then $\mathcal{P}_1 \subset \mathcal{P}_2$.

Proof. We prove the implication $G \in \mathcal{P}_1 \implies G \in \mathcal{P}_2$ by induction on Hirsch length of G. Suppose $1 \to K \to G \to I \to 1$ is a nilpotent central extension of groups and I is of lower Hirsch length than G. Notice $K, I \in \mathcal{P}_1$. By inductive assumption $K, I \in \mathcal{P}_2$. Therefore $G \in \mathcal{P}_2$. ■

Corollary 6.3.3. *Let A be an Abelian group and $1 \leq n \leq 2$. Consider the following statements:*

1. $\widetilde{H}_*(G; A) = 0$,

2. $\widetilde{H}_i(G; A) = 0$ for all $i \leq n$.

If (1) is equivalent to (2) for all Abelian groups G, then the two statements are equivalent for all nilpotent groups G.

Proof. Let $\mathcal{P}_1^r$ (respectively, $\mathcal{P}_2^r$) be the class of all nilpotent groups G such that $\widetilde{H}_i(G; A) = 0$ for all $i \leq n$ (respectively, $\widetilde{H}_i(G; A) = 0$ for all i) and $h(G) \leq r$. Our goal is to prove, by induction on r, that $\mathcal{P}_1^r = \mathcal{P}_2^r$. It is clearly so for $r = 1$. Assume $\mathcal{P}_1^m = \mathcal{P}_2^m$ for all $m < r$.

Suppose $1 \to K \to G \to I \to 1$ is a nilpotent central extension of groups such that $h(I) < h(G) = r$. If $G \in \mathcal{P}_j^r$, then $H_1(K; A) = 0$ which implies $H_i(G; A) \to H_i(I; A)$

is an epimorphism for $i \leq 2$, so $H_i(I; A) = 0$ for $1 \leq i \leq n$. By inductive assumption $H_i(I; A) = 0$ for all $i \geq 1$. If $\widetilde{H}_i(K; A)$ is the first non-trivial reduced homology group of K, then Leray-Serre spectral sequence implies $H_{i+1}(I; A) \neq 0$, a contradiction. Thus both K and I belong to $\mathcal{P}_j^r$. Conversely, if $K, I \in \mathcal{P}_j^{r-1}$, then (by inductive assumption) they have trivial homology with coefficients in A resulting in G having trivial homology with coefficients in A and $G \in \mathcal{P}_j^r$. Applying 6.3.2 one gets $\mathcal{P}_1^r = \mathcal{P}_2^r$.
■

6.4 Bockstein basis

Notation: If G is a group, then $\mathrm{Tor}(G)$ is the subgroup generated by torsion elements of G, $\mathrm{Tor}_p(G)$ is the subgroup generated by all elements of G whose order is a power of p, $F_p(G) = G/\mathrm{Tor}_p(G)$, and $F(G) = G/Tor(G)$.

The *Bockstein groups* are: rationals $\mathbb{Q}$, cyclic groups $\mathbb{Z}/p$ of p elements, p-adic circles $\mathbb{Z}/p^\infty$, and p-localizations of integers $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is not divisible by } p\}$, where p is a prime number.

We gave the classical definition of the Bockstein basis of an Abelian group G earlier in this section. Here is a definition for nilpotent groups which is more convenient in this paper as it allows using localization of short exact sequences of nilpotent groups.

Definition 6.4.1. Let G be a nilpotent group, then the Bockstein basis $\sigma(G)$ is defined as follows:

1. $\mathbb{Q} \notin \sigma(G)$ if and only if $G = \mathrm{Tor}(G)$.
2. $\mathbb{Z}/p^\infty \notin \sigma(G)$ if and only if G is $\bar{p}$-local.
3. $\mathbb{Z}/p \notin \sigma(G)$ if and only if G is divisible by p.
4. $\mathbb{Z}_{(p)} \notin \sigma(G)$ if and only if $F(G)$ is $\bar{p}$-local.

Remark 6.4.2. Note that according to the above definition we have

$$\mathbb{Z}_{(p)} \in \sigma(G) \Rightarrow \mathbb{Z}/p \in \sigma(G) \Rightarrow \mathbb{Z}/p^{\infty} \in \sigma(G).$$

Corollary 6.4.3. *For a nilpotent group G the following statements are equivalent:*

1. $\mathbb{Q} \notin \sigma(G)$,

2. $\widetilde{H}_*(G;\mathbb{Q}) = 0$, *and*

3. $H_1(G;\mathbb{Q}) = 0$.

Proof. The equivalence of (2) and (3) follows from 6.3.3. Let $\mathcal{P}_1$ be the class of torsion nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G;\mathbb{Q}) = 0$ (i.e., $\mathrm{Ab}(G)$ is torsion). Use 6.3.2 to conclude $\mathcal{P}_1 = \mathcal{P}_2$. ■

Corollary 6.4.4. *For a nilpotent group G, $\mathbb{Z}/p \notin \sigma(G)$ if and only if $H_1(G;\mathbb{Z}/p) = 0$.*

Proof. Let $\mathcal{P}_1$ be the class of p-divisible nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G;\mathbb{Z}/p) = 0$. Use 6.3.2 and 6.3.1 to conclude $\mathcal{P}_1 = \mathcal{P}_2$. ■

Corollary 6.4.5. *For a nilpotent group G the following statements are equivalent:*

1. $\mathbb{Z}/p^{\infty} \notin \sigma(G)$,

2. $\widetilde{H}_*(G;\mathbb{Z}/p^{\infty}) = 0$, *and*

3. $H_1(G;\mathbb{Z}/p^{\infty}) = H_2(G;\mathbb{Z}/p^{\infty}) = 0$.

Proof. If $\mathbb{Z}/p^{\infty} \notin \sigma(G)$, then G is $\bar{p}$-local, so all its integral homology groups are $\bar{p}$-local and $H_i(G;\mathbb{Z}/p^{\infty}) = 0$ for all $i \geq 1$, which proves the implication (1) $\Longrightarrow$ (2).

Notice (2) $\iff$ (3) by 6.3.3. Indeed, if A is Abelian and $H_1(A;\mathbb{Z}/p^{\infty}) = H_2(A;\mathbb{Z}/p^{\infty}) = 0$, then A must be $\bar{p}$-local.

Let $\mathcal{P}_1$ be the class of $\bar{p}$-local nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups such that $H_1(G;\mathbb{Z}/p^\infty) = H_2(G;\mathbb{Z}/p^\infty) = 0$. Use 6.3.2 to conclude $\mathcal{P}_1 = \mathcal{P}_2$. ■

Corollary 6.4.6. *For a nilpotent group G the following statements are equivalent:*

1. $\mathbb{Z}_{(p)} \notin \sigma(G)$,
2. $\widetilde{H}_*(F(G);\mathbb{Z}/p^\infty) = 0$, *and*
3. $H_1(F(G);\mathbb{Z}/p^\infty) = H_2(F(G);\mathbb{Z}/p^\infty) = 0$.

It is obvious that $\mathbb{Z}/p^\infty \notin \sigma(G)$ if and only if $G \to G_{(\bar{p})}$ is an isomorphism. The following lemma characterizes $\mathbb{Z}_{(p)} \notin \sigma(G)$ via localizations.

Lemma 6.4.7. *For a nilpotent group G the following statements are equivalent:*

1. $\mathbb{Z}_{(p)} \notin \sigma(G)$,
2. $G \to G_{(\bar{p})}$ *is an epimorphism.*

Proof. Observe that (1) $\iff$ (2) in case G is an Abelian group. Indeed, in that case $G_{(\bar{p})} = G \otimes \mathbb{Z}_{(\bar{p})}$ and $G \to G_{(\bar{p})}$ is an epimorphism if and only if $G \otimes \mathbb{Z}/p^\infty = 0$. Since $G \otimes \mathbb{Z}/p^\infty = F(G) \otimes \mathbb{Z}/p^\infty$, $G \otimes \mathbb{Z}/p^\infty = 0$ if and only if $F(G)$ is $\bar{p}$-local.

In view of 6.3.2 it remains to show that both classes $\mathcal{P}_1 = \{G \mid \mathbb{Z}_{(p)} \notin \sigma(G)\}$ and $\mathcal{P}_2 = \{G \mid G \to G_{(\bar{p})}$ is an epimorphism$\}$ are EXACT. ■

Definition 6.4.8. The *torsion-divisible Bockstein basis* $\sigma_{TD}(G)$ of G consists of all $\mathbb{Z}/p^\infty$ belonging to $\sigma(G)$. We set $\sigma_{NTD}(G) = \sigma(G)\backslash\sigma_{TD}(G)$.

Lemma 6.4.9. *If $G \to I$ is an epimorphism of nilpotent groups, then $\sigma_{NTD}(I) \subset \sigma_{NTD}(G)$.*

Proof. Suppose $\mathbb{Q} \notin \sigma(G)$, then G is a torsion group. So I is a torsion group, hence $\mathbb{Q} \notin \sigma(I)$.

Let $\mathbb{Z}/p \notin \sigma(G)$, then G is p-divisible and then also I is p-divisible.

Let $\mathbb{Z}_{(p)} \notin \sigma(G)$, then $G \to G_{(\bar{p})}$ is an epimorphism. Because $\bar{p}$-localization is an exact functor, the map $G_{(\bar{p})} \to I_{(\bar{p})}$ is an epimorphism and hence $I \to I_{(\bar{p})}$ is an epimorphism. ■

Lemma 6.4.10. *Let* $1 \to K \to G \to I \to 1$ *be a central extension of nilpotent groups, then* $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.

Proof. Let $\mathbb{Q} \notin \sigma(K) \cup \sigma(I)$, then K and I are torsion groups. Hence G is torsion, so $\mathbb{Q} \notin \sigma(G)$.

Let $\mathbb{Z}/p \notin \sigma(K) \cup \sigma(I)$, then K and I are p-divisible. By Lemma 6.3.1, also G is p-divisible.

Let $\mathbb{Z}/p^\infty \notin \sigma(K) \cup \sigma(I)$, then $K \to K_{(\bar{p})}$ and $I \to I_{(\bar{p})}$ are isomorphisms. Using Five Lemma and the fact that $\bar{p}$-localization is an exact functor, we conclude that also $G \to G_{(\bar{p})}$ is an isomorphism.

Let $\mathbb{Z}_{(p)} \notin \sigma(K) \cup \sigma(I)$, then $K \to K_{(\bar{p})}$ and $I \to I_{(\bar{p})}$ are epimorphisms. By the three-lemma $G \to G_{(\bar{p})}$ is also an epimorphism. ■

Lemma 6.4.11. *Let* $1 \to K \to G \to I \to 1$ *be a nilpotent central extension of groups. If* $\mathbb{Z}_{(p)} \notin \sigma(Ab(G))$ *for some prime* p, *then* $\mathbb{Z}_{(p)} \notin \sigma(K)$ *and* $\mathbb{Z}_{(p)} \notin \sigma(Ab(I))$.

Proof. Assume $\mathbb{Z}_{(p)} \notin \sigma(\mathrm{Ab}(G))$. That means the map $F(\mathrm{Ab}(G)) \to F(\mathrm{Ab}(G))_{(\bar{p})}$ is an isomorphism. Because $\mathrm{Ab}(G) \to \mathrm{Ab}(I)$ is an epimorphism and F is a right exact functor, the map $F(\mathrm{Ab}(G)) \to F(\mathrm{Ab}(I))$ is an epimorphism. Hence the map $F(\mathrm{Ab}(I)) \to F(\mathrm{Ab}(I))_{(\bar{p})}$ is an epimorphism. Its kernel is a p-torsion group, so the kernel is trivial and the map $F(\mathrm{Ab}(I)) \to F(\mathrm{Ab}(I))_{(\bar{p})}$ is an isomorphism. That means $\mathbb{Z}_{(p)} \notin \sigma(\mathrm{Ab}(I))$.

There exists an epimorphism $\otimes^n \mathrm{Ab}(G) \to K$ for some integer n. Because $F(\mathrm{Ab}(\otimes^n G)) \to F(\mathrm{Ab}(\otimes^n G))_{(\bar{p})}$ is an isomorphism, in the same way as in the previous paragraph we can prove that $F(K) \to F(K)_{(\bar{p})}$ is an isomorphism. Hence $\mathbb{Z}_{(p)} \notin \sigma(K)$. ■

Lemma 6.4.12. *Let G be a nilpotent group, then $\sigma_{NTD}(G) = \sigma_{NTD}(Ab(G))$ and $\sigma(Ab(G)) \subset \sigma(G)$.*

Proof. The inclusion $\sigma_{NTD}(\mathrm{Ab}(G)) \subset \sigma_{NTD}(G)$ follows from Lemma 6.4.9.

Let us prove $\sigma_{NTD}(G) \subset \sigma_{NTD}(\mathrm{Ab}(G))$. Suppose $\mathbb{Q} \notin \sigma_{NTD}(\mathrm{Ab}(G))$. Because G is a torsion group if and only if $\mathrm{Ab}(G)$ is a torsion group, $\mathbb{Q} \notin \sigma_{NTD}(G)$.

Suppose $\mathbb{Z}/p \notin \sigma_{NTD}(\mathrm{Ab}(G))$. Because G is p-divisible if and only if $\mathrm{Ab}(G)$ is p-divisible, $\mathbb{Z}/p \notin \sigma_{NTD}(G)$.

Consider the class $\mathcal{P}$ of all nilpotent groups such that $\mathbb{Z}_{(p)} \notin \sigma_{NTD}(\mathrm{Ab}(G))$ implies $\mathbb{Z}_{(p)} \notin \sigma_{NTD}(G)$. $\mathcal{P}$ clearly contains all Abelian groups. To show $\mathcal{P}$ equals the class $\mathcal{N}$ of all nilpotent groups it suffices to show (see 6.3.2) that for any nilpotent central extension $A \to G \to G'$ such that Hirsch length of G' is less than $h(G)$, $A, G' \in \mathcal{P}$ implies $G \in \mathcal{P}$. Assume $\mathbb{Z}_{(p)} \notin \sigma_{NTD}(\mathrm{Ab}(G))$. By 6.4.11 we conclude $\mathbb{Z}_{(p)} \notin \sigma_{NTD}(G')$ as $G' \in \mathcal{P}$ and $\mathbb{Z}_{(p)} \notin \sigma_{NTD}(A)$. By 6.4.10 $\mathbb{Z}_{(p)} \notin \sigma_{NTD}(G)$.

Let us prove now that $\sigma(\mathrm{Ab}(G)) \subset \sigma(G)$. By Lemma 6.4.9, $\sigma_{NTD}(\mathrm{Ab}(G)) \subset \sigma_{NTD}(G)$. Suppose $\mathbb{Z}/p^\infty \notin \sigma(G)$. Then G is uniquely p-divisible, so $H_*(G;\mathbb{Z})$ is uniquely p-divisible. In particular $\mathrm{Ab}(G) = H_1(G;\mathbb{Z})$ is uniquely p-divisible, hence $\mathbb{Z}/p^\infty \notin \sigma(\mathrm{Ab}(G))$. ■

Theorem 6.4.13. *If $1 \to K \to G \to I \to 1$ is a nilpotent central extension, then $\sigma(G) = \sigma(K) \cup \sigma(I)$.*

Proof. By Lemma 6.4.10, $\sigma(G) \subset \sigma(K) \cup \sigma(I)$.

Let us prove that $\sigma(K) \cup \sigma(I) \subset \sigma(G)$. Suppose $\mathbb{Q} \notin \sigma(G)$, then G is a torsion group. Therefore K and I are also torsion groups, so $\mathbb{Q} \notin \sigma(K) \cup \sigma(I)$.

Suppose $\mathbb{Z}/p \notin \sigma(G)$, then G is p-divisible. By Lemma 6.3.1, K and I are p-divisible, so $\mathbb{Z}/p \notin \sigma(K) \cup \sigma(I)$.

Suppose $\mathbb{Z}/p^\infty \notin \sigma(G)$, then $G \to G_{(\bar{p})}$ is an isomorphism. Because $\bar{p}$-localization is an exact functor, the map $K \to K_{(\bar{p})}$ is a monomorphism and the map $\mathrm{Ab}(G) \to (\mathrm{Ab}(G))_{(\bar{p})}$ is an epimorphism. Because there exists an epimorphism $\otimes^n \mathrm{Ab} G \to K$, the map $K \to K_{(\bar{p})}$ is also an epimorphism, hence it is an isomorphism. By Five Lemma, also the map $I \to I_{(\bar{p})}$ is an isomorphism, so $\mathbb{Z}/p^\infty \notin \sigma(K) \cup \sigma(I)$.

If $\mathbb{Z}_{(p)} \notin \sigma(G)$, then $\mathbb{Z}_{(p)} \notin \sigma(K)$ and $\mathbb{Z}_{(p)} \notin \sigma(I)$ by 6.4.11 and 6.4.12. ■

6.5 Bockstein spaces

Definition 6.5.1. A metrizable space X is called a *Bockstein space* if $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all Abelian groups G.

Remark 6.5.2. In the above definition observe $\dim_G X$ is an element of $\mathbf{N} \cup \{0, \infty\}$ and not only in $\{0, 1, \infty\}$ as in the case of non-Abelian groups G.

Dranishnikov-Repovš-Shchepin [24] showed the existence of a separable metric space X of dimension 2 such that $\dim_{\mathbb{Z}_{(p)}} X = 1$ for all primes p. Thus, X is not a Bockstein space as $\dim_{\mathbb{Z}} X = 2 > 1 = \sup\{\dim_H X \mid H \in \sigma(\mathbb{Z})\}$.

Problem 6.5.3. Is every metric ANR a Bockstein space?

Theorem 6.5.4. *A metrizable space X is a Bockstein space if and only if $\dim_{\mathbb{Z}_l} X = \dim_{\hat{\mathbb{Z}}_l} X$ for all subsets $l \subset \mathbb{P}$ of the set of prime numbers.*

Proof. Since $\sigma(\hat{\mathbb{Z}}_l) = \sigma(\mathbb{Z}_l)$ for all $l \subset \mathbb{P}$, $\dim_{\mathbb{Z}_l} X = \dim_{\hat{\mathbb{Z}}_l} X$ holds for any Bockstein space X.

Assume $\dim_{\mathbb{Z}_l} X = \dim_{\hat{\mathbb{Z}}_l} X$ for all subsets $l \subset \mathbb{P}$ of the set of prime numbers. Suppose G is a torsion-free Abelian group G. If $\mathbb{Z}_{(p)} \in \sigma(G)$, then Theorem B(d) of [25] says $\dim_{\hat{\mathbb{Z}}_{(p)}} X \leq \dim_G X$. Therefore $\dim_G X \geq \sup\{\dim_H X \mid H \in \sigma(G)\}$.

Suppose $\sup\{\dim_H X \mid H \in \sigma(G)\} = n$ and consider $l = \{p \mid p \cdot G \neq G\}$. Theorem B(f) of [25] says $\dim_G(X) \leq \dim_{\mathbb{Z}_l} X$. Since $\sigma(G) = \sigma(\mathbb{Z}_l)$, $\dim_G X \leq \text{dimdim}_{\mathbb{Z}_l} X = \sup\{\dim_H X \mid H \in \sigma(\mathbb{Z}_l)\} = \sup\{\dim_H X \mid H \in \sigma(G)\} = n$. That proves $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ for all torsion-free Abelian groups. The same equality holds for all torsion Abelian groups by Theorem B(a) of [25]. In the case of arbitrary Abelian groups G, as $\sigma(G) = \sigma(F(G)) \cup \sigma(\text{Tor}(G))$ and $\dim_G = \max(\dim_{F(G)} X, \dim_{\text{Tor}(G)} X)$ (see Theorem B(b) of [25]) one gets $\dim_G X = \sup\{\dim_H X \mid H \in \sigma(G)\}$ as well. ■

Remark 6.5.5. Notice it is not sufficient to assume $\dim_{\mathbb{Z}_{(p)}} X = \dim_{\hat{\mathbb{Z}}_{(p)}} X$ for all primes p in 6.5.4. Indeed, the space X in [24] has that property as $1 = \dim_{\mathbb{Z}_{(p)}} X \geq \dim_{\hat{\mathbb{Z}}_{(p)}} X \geq 1$ for all primes p.

Theorem 6.5.6. *Let X be a Bockstein space. If G is nilpotent, then $\dim_G(X) \leq 1$ if and only if $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$.*

Proof. 6.2.3 Let $\mathcal{P}_1$ be the class of all nilpotent groups and let $\mathcal{P}_2$ be the class of nilpotent groups G such that $\dim_G(X) \leq 1$ if and only if $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$. Since $\mathcal{P}_2$ contains all Abelian groups, in view of 6.3.2 it suffices to show that for any nilpotent central extension $K \to G \to I$ the conditions $K, I \in \mathcal{P}_2$ imply $G \in \mathcal{P}_2$. It is so if G is Abelian, so assume G is not Abelian. Moreover, as $\sigma(\text{Ab}(G)) \subset \sigma(G)$ by 6.4.12 and $\dim_G X \leq 1$ implies $\dim_{\text{Ab}(G)} X \leq 1$ (see 6.2.3), each $\dim_G(X) \leq 1$ and $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$ imply $\dim_{\text{Ab}(G)} X \leq 1$, so we may as well assume $\dim_{\text{Ab}(G)} X \leq 1$.

In view of Lemma 6.2.1 and the fact $\text{Ab}(G) = 1$ implies $G = 1$, the equivalence of conditions $\dim_G(X) \leq 1$ and $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$ may fail only if $\dim_{\text{Ab}(G)} X = 1$, so assume $\dim_{\text{Ab}(G)} X = 1$.

Suppose $\dim_H(X) = n > 1$ for some $H \in \sigma(K)$. If $H \neq \mathbb{Z}/p^\infty$, then $H \in \sigma_{NTD}(G) = \sigma_{NTD}(\text{Ab}(G))$ (Theorem 6.4.13 and Lemma 6.4.12), a contradiction. So $H = \mathbb{Z}/p^\infty$ for some prime p and $\mathbb{Z}/p^\infty \notin \sigma(\text{Ab}(G))$. Hence $\text{Ab}(G)$ is p-divisible and this is equivalent to G being p-divisible [8, Lemma 5.1]. Because $\mathbb{Z}/p^\infty \notin \sigma(\mathrm{Ab}(G))$, by Lemma 6.4.5 $H_1(\mathrm{Ab}(G);\mathbb{Z}/p^\infty) = 0$, so also $H_1(G;\mathbb{Z}/p^\infty) = 0$. By Lemma 6.4.5, $H_2(G;\mathbb{Z}/p^\infty) \neq 0$ as $\mathbb{Z}/p^\infty \in \sigma(G)$. This implies $F(H_2(G;\mathbb{Z}))$ is not p-divisible, so $\mathbb{Z}_{(p)} \in \sigma(H_2(G;\mathbb{Z}))$. If G is not a torsion group, then $\mathrm{Ab}(G)$ is not a torsion group, hence by definition $\mathbb{Q} \in \sigma(\mathrm{Ab}(G))$. Therefore $\dim_{\mathbb{Q}}(X) \leq 1$. Using that fact and Bockstein Inequalities (BI5, BI6 [41]), we get $\dim_{\mathbb{Z}/p^\infty}(X) = \dim_{\mathbb{Z}_{(p)}}(X) - 1$. Because $\mathbb{Z}_{(p)} \in \sigma(H_2(G;\mathbb{Z}))$, the dimension $\dim_{\mathbb{Z}_{(p)}}(X) \leq \dim_{H_2(G;\mathbb{Z})}(X) \leq 2$ as X is a Bockstein space and then $\dim_{\mathbb{Z}/p^\infty}(X) = \dim_{\mathbb{Z}_{(p)}}(X) - 1 \leq 1$, a contradiction.

Thus G is a torsion group and is a product of q-groups $G = \Pi_{q\in\mathbb{P}} G_q$. Hence $\mathrm{Ab}(G) = \Pi_{q\in\mathbb{P}}\mathrm{Ab}(G_q)$. Because G is not $\bar{p}$-local, $G_p \neq 1$, but $\mathrm{Ab}(G)$ is uniquely p-divisible, so $\mathrm{Ab}(G_p) = 1$. Therefore G_p is a perfect nilpotent group, but such group is trivial, a contradiction.

Thus $\dim_H X \leq 1$ for all $H \in \sigma(K)$ and $\dim_K X = \sup\{\dim_H(X) | H \in \sigma(K)\} \leq 1$ as K is Abelian and X is a Bockstein space.

Suppose $\sup\{\dim_H(X) | H \in \sigma(G)\} \leq 1$. By Theorem 6.4.13, $\sigma(G) = \sigma(K) \cup \sigma(I)$. Therefore $\sup\{\dim_H(X) | H \in \sigma(I)\} \leq 1$ and $\dim_I X \leq 1$ as $I \in \mathcal{P}_2$. Consequently, $\dim_G X \leq 1$ by 6.2.2.

Suppose $\dim_G X \leq 1$. By 6.2.2 one gets $\dim_I X \leq 1$ and $\sup\{\dim_H(X) | H \in \sigma(I)\} \leq 1$ as $I \in \mathcal{P}_2$. By Theorem 6.4.13, $\sigma(G) = \sigma(K) \cup \sigma(I)$, hence

$$\sup\{\dim_H(X) | H \in \sigma(G)\} = \sup\{\dim_H(X) | H \in \sigma(K) \cup \sigma(I)\} \leq 1.$$

■

Corollary 6.5.7. *If X is a Bockstein space and $X\tau L$ for some nilpotent CW complex L, then dim$_{\pi_n(L)} X \leq n$ for all $n \geq 1$.*

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