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Ultrametric Properties of Homotopy and Refinement Critical Values

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Section I: Introduction and Background Material

One of the main concerns of topology is distinguishing one space from another. This is not a simple problem. How can we as mathematicians prove that a sphere is not equivalent to a torus? Clearly one difference between the two spaces is that a torus has a hole in the middle of it while a sphere does not have a hole. In topology, paths can be a powerful tool to gain information about a given space. By trying to draw in a path that has been wrapped around a suspected hole, we can determine if there is actually a hole in our space. A path p from x to y is a continuous function from the closed interval $[0, 1]$ to a topological space X where $p(0) = x$ and $p(1) = y$. A good way to imagine a path is as a curve from the point x to the point y in X. There is no reason to think that this curve is anything like the nice curve that we can picture in our minds. A path can become complicated very easily. For example, we can have a path that is nowhere differentiable, or we can have a path that fills our entire space. This is one of the reasons that the discrete version of paths, an ε-chain, is a useful tool. However, before we can delve into some basic properties of ε-chains, we must first cover some background material.

In order to implement ε-chains, we must limit ourselves to metric spaces.

Definition 1. *A set X along with a distance function $d : X \times X \to \mathbb{R}$ called a metric is called a* ***metric space*** *(X, d) if and only if d satisfies the following properties:*

1. Positive Definite: For all $x, y \in X$, $d(x, y) \geq 0$ where $d(x, y) = 0$ if and only if $x = y$
2. Symmetric: For all $x, y \in X$, $d(x, y) = d(y, x)$
3. Triangle Inequality: For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

Note that when no confusion will arise, we will refer to a metric space (X, d) by X and assume that d is the metric for X. It is not difficult to provide some simple examples of metric spaces. The set of real numbers forms a metric space under the metric $d(x, y) = |x - y|$. Let $x, y \in \mathbb{R}$. By the properties of absolute value, we see that $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. Suppose $d(x, y) = 0$. Then $|x - y| = 0$; so $x - y = 0$ or $-(x - y) = 0$. Therefore we have $x = y$. For the symmetric property, $d(x, y) = |x - y| = |-(x - y)| = |y - x| = d(y, x)$. Finally, the triangle inequality follows from the familiar triangle inequality of the real numbers.

Notice that a single set can be used to define multiple metric spaces. Consider the subset T of $\mathbb{R}^2$ where $T = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$. Here we can use the Euclidean distance to define a metric on T: $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The proof that this is a metric follows in a similar fashion as for the metric on $\mathbb{R}$ above. This sense of distance is useful in most cases, but it is not always what we may want. For example, if T is taken to be an idealized road map of a city, the straight line distance we have given would not tell a cab driver how far he has to drive to get from point A to point B. The taxi cannot drive straight to his destination because buildings are in the way. In this case we would want to use a different metric to measure distances for the cab driver.

Example 2. *[Taxicab Metric Space]*

Define $d : T \times T \to \mathbb{R}$ by $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$. We call this metric the taxicab metric. Now all that is left is to show that this function satisfies the properties of a metric. Let $(x_1, y_1), (x_2, y_2) \in T$. Then $d((x_1, y_1), (x_2, y_2)) \geq 0$ since the absolute value of a number is never negative, and the sum of two non-negative numbers is non-negative. Suppose $d((x_1, y_1), (x_2, y_2)) = 0$. Then $|x_2 - x_1| + |y_2 - y_1| = 0$. Thus $|x_2 - x_1| = 0$ and $|y_2 - y_1| = 0$. So $x_2 = x_1$ and $y_2 = y_1$. Therefore $(x_1, y_1) = (x_2, y_2)$. Now suppose $(x_1, y_1) = (x_2, y_2)$. Then we have $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1| = 0 + 0 = 0$. Furthermore, notice that $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1| = | - (x_2 - x_1)| + | - (y_2 - y_1)| = |x_1 - x_2| + |y_1 - y_2| = d((x_2, y_2), (x_1, y_1))$. Let $(u, v) \in \mathbb{R}^2$. Then for any $(u, v) \in T$,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= |x_2 - x_1| + |y_2 - y_1| \\ &= |x_2 - u + u - x_1| + |y_2 - v + v - y_1| \\ &\leq |x_2 - u| + |y_2 - v| + |u - x_1| + |v - y_1| \\ &= d((x_2, y_2), (u, v)) + d((u, v), (x_1, y_1)) \end{aligned}$$

So we have the triangle inequality as well. Therefore, as promised, d is a metric on T.

This example brings us to a specialized version of a metric space where the distance between two points is the length traveling inside the space to get from one point to another. In many ways this matches our sense of distance in the real world. So first we must somehow define what we mean by traveling inside a space. To do this we return to paths described earlier.

Definition 3. *Let X be a topological space. A function $p : [0, 1] \to X$ is called a* ***path*** *from x to y if and only if p is continuous, $p(0) = x$, and $p(1) = y$.*

Definition 4. *If a path $p : [0, 1] \to X$ is a differentiable function, we may define the* ***length*** *of p to be $L(p) = \int_0^1 \|p'(t)\| \, dt$.*

Now we can use the length of the path between two points to give us our desired metric space.

Definition 5. *A space (X, d) is called an* ***intrinsic metric space*** *if and only if for all $x, y \in X$, $d(x, y) = \inf\{L(p) : p \text{ is a path from } x \text{ to } y\}$. Furthermore (X, d) is called a* ***geodesic space*** *if and only if for all $x, y \in X$, $d(x, y) = \min\{L(p) : p \text{ is a path from } x \text{ to } y\}$, i.e. the minimizing path between points x and y is always achievable within the space.*

Notice that the taxicab metric space is an example of a geodesic space. The distance between two points in the Taxicab metric space is the length of the "shortest path" between two points. While the definition of a geodesic space guarantees that a "shortest path" exists, this path is by no means unique in general. The surface of a sphere is a geodesic space, and there are uncountably many different "shortest paths" from the point at the north pole to the point at the south pole.

Section II: Basic Machinery of Chains and Chain Homotopies

Now that we are familiar with the concept of metric spaces, we can develop a tool called an ε-chain. The key motivation for an ε-chain is to find a discrete version of a path. Like a path, an ε-chain can be wrapped around certain portions of a space to try to detect "holes" within that space. Once we have a loop around a certain section of a space, if we somehow cannot "reel in" the chain while staying inside the space, we have found a "hole" in the space.

Definition 6. *Let (X, d) be a metric space and let $\varepsilon > 0$. A finite, ordered sequence of points $\gamma = \{x_0, x_1, ..., x_n\}$ is called an **ε-chain** if and only if for all $0 \leq i \leq n-1$, $d(x_i, x_{i+1}) < \varepsilon$. Furthermore if $x_0 = x_n$ then γ is called an **ε-loop**. When no confusion will arise, we will simply call an ε-chain a chain.*

We will also define some useful terms and basic properties that we will need later.

Definition 7. *The **reverse** of an ε-chain $\gamma = \{x_0, x_1, x_2, ..., x_n\}$, notated $\overline{\gamma}$, is the ε-chain $\overline{\gamma} = \{x_n, x_{n-1}, ..., x_1, x_0\}$.*

Definition 8. *Suppose $\gamma = \{x_0, x_1, ..., x_n\}$ and $\alpha = \{y_0, y_1, ..., y_m\}$ are ε-chains where $x_n = y_0$. Then the **concatenation** of γ and α, notated $\gamma * \alpha$ is the ε-chain $\gamma * \alpha = \{x_0, x_1, ..., x_n, y_1, y_2, ..., y_m\}$.*

Lemma 9. *Let $\alpha = \{a_0, a_1, a_2, ..., a_n\}$ and $\beta = \{b_0, b_1, ..., b_k\}$ be ε-chains where $a_n = b_0$. Then $\overline{\alpha * \beta} = \overline{\beta} * \overline{\alpha}$ and $\overline{\overline{\alpha}} = \alpha$.*

Proof. By direct computation, $\overline{\overline{\alpha}} = \overline{\{a_n, a_{n-1}, ..., a_2, a_1, a_0\}} = \{a_0, a_1, a_2, ..., a_n\} = \alpha$. Furthermore, $\overline{\alpha * \beta} = \overline{\{a_0, a_1, ..., a_n, b_1, ..., b_k\}} = \{b_k, b_{k-1}, ..., b_1, a_n, ..., a_1, a_0\} = \{b_k, b_{k-1}, ..., b_1, b_0\} * \{a_n, a_{n-1}, ..., a_1, a_0\} = \overline{\beta} * \overline{\alpha}$. ■

Definition 10. *The **max norm** of a chain $\alpha = \{x_0, ..., x_n\}$ is defined to be: $|\alpha|_{\max} = \max_{i=0,...,n-1}\{d(x_i, x_{i+1})\}$.*

Proposition 11. *[Basic Properties of the Max Norm] Let α be an ε-chain. Then the following properties for α hold:*

1. $|\alpha|_{\max} \geq 0$ and $|\alpha|_{\max} = 0$ if and only if $x_0 = x_1 = ... = x_n$.
2. $|\alpha|_{\max} = |\overline{\alpha}|_{\max}$
3. $|\alpha|_{\max} < \varepsilon$.

Proof. (1) By definition, $|\alpha|_{\max} \geq d(x_i, x_{i+1})$ for all i. By the positive definite property of the metric, we have $|\alpha|_{\max} \geq 0$. Suppose $|\alpha|_{\max} = 0$. Then $\max_{0 \leq i \leq n-1}\{d(x_i, x_{i+1})\} = 0$. So by positive definiteness, $d(x_i, x_{i+1}) = 0$ for all $i - 0, ..., n-1$. Thus $x_i = x_{i+1}$ for all $i = 0, ..., n-1$ hence $x_0 = x_1 = ... = x_n$. Now suppose $x_0 = x_1 = ... = x_n$. Then for all $i = 0, ..., n-1$, $d(x_i, x_{i+1}) = 0$. So $|\alpha|_{\max} = \max\{d(x_i, x_{i+1})\} = 0$.

(2) $|\alpha|_{\max} = \max\{d(x_i, x_{i+1})\} = \max\{d(x_{i+1}, x_i)\} = |\overline{\alpha}|_{\max}$ by the symmetric property of the metric.

(3) By definition of an ε-chain, $d(x_i, x_{i+1}) < \varepsilon$ for all $i = 0, ..., n-1$. Thus $|\alpha|_{\max} = \max\{d(x_i, x_{i+1})\} < \varepsilon$. ■

Proposition 12. *Let α be an ε-chain where $\gamma_1 * \gamma_2 = \alpha$ for some ε-chain γ_1 and γ_2. Then for any ε-loop β where $\gamma_1 * \beta * \gamma_2$ is defined, $|\alpha|_{\max} \leq |\gamma_1 * \beta * \gamma_2|_{\max}$.*

Proof. Suppose $\gamma_1 * \beta * \gamma_2$ is defined where $\beta = \{y_0, ..., y_p\}$, $\gamma_1 = \{a_0, ..., a_k\}$, and $\gamma_2 = \{a_{k+1}, ..., a_n\}$. Then

$$\begin{aligned}|\gamma_1 * \beta * \gamma_2|_{\max} &= \max\{d(a_i, a_{i+1})_{i=0,...,k}, d(y_i, y_{i+1}), d(a_i, a_{i+1})_{i=k+1,...,n}\}\\ &= \max\{\max_{i=0,...,k}\{d(a_i, a_{i+1})\}, \max\{d(y_i, y_{i+1})\}, \max_{i=k+1,...,n}\{d(a_i, a_{i+1})\}\}\\ &= \max\{|\gamma_1|_{\max}, |\beta|_{\max}, |\gamma_2|_{\max}\}.\end{aligned}$$

So, since the max norm of α is the maximum of the set $\{d(a_i, a_{i+1})\} \subseteq \{d(a_i, a_{i+1})_{i=0,...,k}, d(y_i, y_{i+1}), d(a_i, a_{i+1})_{i=k+1,...,n}\}$, we have $|\alpha|_{\max} \leq |\gamma_1 * \beta * \gamma_2|_{\max}$. ■

Lemma 13. *Let α be an ε-chain. Then α is also a δ-chain for all $\delta > |\alpha|_{\max}$.*

Proof. The proof of this statement follows trivially from the definition of the max norm. Since $|\alpha|_{\max} = \max\{d(x_i, x_{i+1})\}$ we have $\delta > d(x_i, x_{i+1})$ for all i. Then α fits the definition of a δ-chain. ■

Now that we have a rigorous definition of our discrete version of a path, we must work to define what we mean when we want to "reel in" our chain around suspected holes in our spaces. We want to find a way to deform and shorten our chain where we cannot cross over holes in the space. To do this, we can define basic moves we can use to transform our chain in a chain homotopy.

Definition 14. *Suppose $\gamma = \{x_0, x_1, ..., x_n\}$ is an ε-chain. Then the following changes to γ are considered basic moves as long as the endpoints of γ do not change:*

1. *Removing a point: $\gamma = \{x_0, x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n\} \longrightarrow \gamma' = \{x_0, ..., x_{i-1}, x_{i+1}, ..., x_n\}$ as long as $d(x_{i-1}, x_{i+1}) < \varepsilon$.*
2. *Adding a point: $\gamma = \{x_0, x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n\} \longrightarrow \gamma' = \{x_0, ..., x_{i-1}, x_i, x_k, x_{i+1}, ..., x_n\}$ as long as $d(x_i, x_k) < \varepsilon$ and $d(x_k, x_{i+1}) < \varepsilon$*

Definition 15. *Let $\gamma = \{x_0, x_1, ..., x_n\}$ be an ε-chain in a metric space X. We say γ is **ε-homotopic** to α, notated $\gamma\widetilde{\varepsilon}\alpha$ if and only if α can be obtained from performing a finite number of basic moves to γ. The **ε-homotopy class** of an ε-chain γ, notated $[\gamma]_\varepsilon$, is the set of all ε-chains that are ε-homotopic to γ. A **chain homotopy** is the sequence of basic moves that convert γ to α.*

With this definition of a chain homotopy, we have a way of deforming our chains around a space. As you may have guessed, we have formed an equivalence relation.

Proposition 16. *Homotopies of ε-chains form an equivalence relation.*

Proof. Suppose γ is an ε-chain for some $\varepsilon > 0$. Then $\gamma\widetilde{\varepsilon}\gamma$ since γ can be formed by performing zero basic moves to γ. Thus homotopies of chains are reflexive. Next, suppose $\gamma\widetilde{\varepsilon}\alpha$. Then α can be formed by applying a finite number of basic moves to γ, say $\{\eta_1, \eta_2, ..., \eta_j\}$ where η_i represents the ith basic move applyed to γ to form α. Apply the following basic moves to α: $\{\overline{\eta_j}, \overline{\eta_{j-1}}, ..., \overline{\eta_1}\}$ where $\overline{\eta_i}$ represents the inverse of the basic move η_i. If η_i is a basic move to add a point x_m to a chain, then $\overline{\eta_i}$ is a basic move to remove the point x_m from a chain. Similarly, if η_i is a basic move to remove a point x_m from a chain, then $\overline{\eta_i}$ is a basic move to add the point x_m to a chain. Thus we can see that the basic moves $\{\overline{\eta_j}, \overline{\eta_{j-1}}, ..., \overline{\eta_1}\}$ form γ from the chain α. Thus homotopies of chains are symmetric. Suppose that $\alpha\widetilde{\varepsilon}\beta$ and $\beta\widetilde{\varepsilon}\gamma$. Then β may be formed by applying a finite number of basic moves to α, say $\{\eta_0, ..., \eta_n\}$, and γ may be formed by applying a finite number of basic moves to β, say $\{\kappa_0, ..., \kappa_j\}$. Thus if we apply the required basic moves to form β from α and then apply the basic moves to β to form γ, we can form γ by applying a finite number of basic moves to α, namely apply the moves $\{\eta_0, ..., \eta_n, \kappa_0, ..., \kappa_j\}$ to α to form γ. Thus homotopies of chains are transitive, and by definition homotopies of chains form an equivalence relation. ∎

Example 17. *Define an ε-chain in the complex plane by $\alpha = \{0, 1+i, 2+i, 3\}$. Show that it is ε-homotopic to $\beta = \{0, 1, 2, 3\}$ for $\varepsilon = 2$.*

First notice that α and β have the same endpoints. Without this fact, there is no chance for α to be homotopic to β. This homotopy will transform α to β:

$$\begin{aligned} \alpha &= \{0, 1+i, 2+i, 3\} \\ & \widetilde{\varepsilon}\{0, 1, 1+i, 2+i, 3\} \\ & \widetilde{\varepsilon}\{0, 1, 2+i, 3\} \\ & \widetilde{\varepsilon}\{0, 1, 2, 2+i, 3\} \\ & \widetilde{\varepsilon}\{0, 1, 2, 3\} \\ &= \beta \end{aligned}$$

We added the point 1 to our chain first, and then removed the point $1+i$. Then we added the point 2, and next we removed the point $2+i$. Throughout this process, we have not changed the endpoints of the chain, and we have not added or removed points that violate the fact that the distance between any two adjacent points is less than ε. So, we have now shown that α is homotopic to β.

Clearly walking through a chain homotopy step by step is a lengthy process. The above example used an ε value that was large compared to the max norm of the chains, and we still had to use several basic moves to transform our chain how we wanted it. It would be nice to be able to have a few shortcuts in this process.

Lemma 18. *Let $\alpha = \{a_0, ..., a_n\}$ be an ε-chain. Then $[\alpha * \overline{\alpha}]_\varepsilon = [\{a_0\}]_\varepsilon$.*

Proof. Let $\alpha = \{a_0, ..., a_n\}$ be an ε-chain. Then $\alpha * \overline{\alpha} = \{a_0, ..., a_{n-1}, a_n, a_{n-1}, ..., a_0\}$. Notice that we may remove the point a_n from this chain immediately since $d(a_{n-1}, a_{n-1}) = 0 < \varepsilon$. So we have $[\alpha * \overline{\alpha}]_\varepsilon = [\{a_0, ..., a_{n-2,}a_{n-1}, a_{n-1}, a_{n-2}, ..., a_0\}]_\varepsilon$. Since α is an ε-loop, we know that $d(a_{n-2}, a_{n-1}) < \varepsilon$. Then we may remove the point $a_{n-1}$from our chain. Thus $[\alpha * \overline{\alpha}]_\varepsilon = [\{a_0, ..., a_{n-2}, a_{n-1}, a_{n-2}, ..., a_0\}]_\varepsilon$, and we are back in the same situation that we started with in the beginning of this proof. However this time we have an ε-chain $\beta = \{a_0, ..., a_{n-1}\}$ that is one point shorter than the chain from which we began. Since we performed a chain homotopy on α to get to β, we have $[\alpha]_\varepsilon = [\beta]_\varepsilon$. Repeat this pattern to see that $[\alpha * \overline{\alpha}]_\varepsilon = [\{a_0, a_0\}]_\varepsilon$. Finally, we may remove one of these a_0 since $d(a_0, a_0) = 0 < \varepsilon$, and this will not change the endpoints of our chain. Therefore $[\alpha * \overline{\alpha}]_\varepsilon = [\{a_0\}]_\varepsilon$. ∎

Lemma 19. *Suppose α and β are ε-chains and $\alpha\widetilde{\varepsilon}\beta$. Then $\overline{\alpha}\widetilde{\varepsilon}\overline{\beta}$, i.e. the reverses of homotopic chains are homotopic.*

Proof. Let $\{\eta_1, ..., \eta_n\}$ be the sequence of basic moves used to transform $\alpha = \{a_0, ..., a_k\}$ to $\beta = \{b_0, ..., b_j\}$. We have $\overline{\alpha} = \{a_k, ..., a_0\}$ and $\overline{\beta} = \{b_j, ..., b_0\}$. Notice that η_1 is still a valid basic move to apply to $\overline{\alpha}$ since we are dealing with the same constituent points and the points are still adjacent to the same points that they were adjacent to in α. This arguement also applies to the basic move η_{i+1} after we have applied the basic moves $\{\eta_1, ..., \eta_i\}$. Then if we apply all of the basic moves $\{\eta_1, ..., \eta_n\}$ to $\overline{\alpha}$, we are left with $\overline{\beta}$. So, we have found a homotopy from $\overline{\alpha}$ to $\overline{\beta}$, namely $\{\eta_1, ..., \eta_n\}$. Therefore $\overline{\alpha}\widetilde{\varepsilon}\overline{\beta}$. // ■

Lemma 20. *Suppose $\alpha\widetilde{\varepsilon}\beta$ and $\gamma\widetilde{\varepsilon}\lambda$ where $\alpha * \gamma$ is defined. Then $(\alpha * \gamma)\widetilde{\varepsilon}(\beta * \lambda)$.*

Proof. Let $\alpha = \{a_0, ..., a_k\}$, $\beta = \{b_0, ..., b_m\}$, $\gamma = \{c_0, ..., c_n\}$, $\lambda = \{l_0, ..., l_p\}$. Since $\alpha * \gamma$ is defined, $a_k = \gamma_0$. So we have $\alpha * \gamma = \{a_0, ..., a_k, c_1, ..., c_n\}$. Notice that performing basic moves on a chain only depends on the points in the chain directly adjacent to the point being added or removed. Furthermore the endpoints of a chain do not change when performing a basic move. Thus the same basic moves required to convert the chain $\alpha = \{a_0, ..., a_k\}$ to the chain $\beta = \{b_0, ..., b_m\}$ also convert the chain $\alpha * \gamma = \{a_0, ..., a_k\} * \gamma$ to the chain $\beta * \gamma = \{b_0, ..., b_n\} * \gamma$. By the same arguement, the same basic moves that convert the chain $\gamma = \{c_0, ..., c_n\}$ to the chain $\lambda = \{l_0, ..., l_p\}$ convert the chain $\beta * \gamma = \beta * \{c_0, ..., c_n\}$ to the chain $\beta * \lambda = \beta * \{l_0, ..., l_p\}$. Thus $(\alpha * \gamma)\widetilde{\varepsilon}(\beta * \lambda)$. ■

Using the above lemma we may extend the concatenation operation to homotopy classes of chains. Define $[\alpha]_\varepsilon * [\beta]_\varepsilon$ to be $[\alpha * \beta]_\varepsilon$. This is a well-defined operation. If $[\alpha]_\varepsilon = [\alpha']_\varepsilon$ and $[\beta]_\varepsilon = [\beta']_\varepsilon$ then $[\alpha * \beta]_\varepsilon = [\alpha' * \beta']_\varepsilon$ by the above lemma as long as $\alpha * \beta$ is defined. Notice that the concatenation operation is defined for all loops based at the same point. Furthermore, once we concatenate a loop with another loop, we get a third loop that is based at the same point. This means that concatenation is a closed operation on homotopy classes of loops.

Definition 21. *Let X be a metric space and $\varepsilon > 0$. Choose a base point $\bigstar \in X$. Then X_ε is the set of all homotopy classes of ε-chains beginning at $\bigstar$ and $\Pi_\varepsilon(X)$ is the set of all homotopy classes of ε-loops beginning at $\bigstar$.*

Proposition 22. *Let X be a metric space with a base point $\bigstar \in X$. Then the space $\Pi_\varepsilon(X)$ is a group under the concatenation operation.*

Proof. Let $[\alpha]_\varepsilon = [\{a_0, a_1, ..., a_n\}]_\varepsilon, [\beta]_\varepsilon = [\{b_0, b_1, ..., b_m\}]_\varepsilon$, and $[\gamma]_\varepsilon = [\{c_0, c_1, ..., c_k\}]_\varepsilon$ be homotopy classes of ε-loops in $\Pi_\varepsilon(X)$ with base point $\bigstar$. Then $a_0 = a_n = b_0 = b_m = c_0 = c_k = \bigstar$. So

$$\begin{aligned}
([\alpha]_\varepsilon * [\beta]_\varepsilon) * [\gamma]_\varepsilon &= [\{a_0, a_1, ..., a_n = b_0, b_1, ..., b_m\}]_\varepsilon * [\{c_0, c_1, ..., c_k\}]_\varepsilon \\
&= [\{a_0, a_1, ..., a_n = b_0, b_1, ..., b_m = c_0, c_1, ..., c_k\}]_\varepsilon \\
&= [\{a_0, a_1, ..., a_n\}]_\varepsilon * [\{b_0, b_1, ..., b_m = c_0, c_1, ..., c_k\}]_\varepsilon \\
&= [\alpha]_\varepsilon * ([\beta]_\varepsilon * [\gamma]_\varepsilon)
\end{aligned}$$

Thus concatenation is associative. Notice that since $[\alpha]_\varepsilon$,$[\beta]_\varepsilon \in \Pi_\varepsilon(X)$, the homotopy class $[\alpha*\beta]_\varepsilon$ is an equivalence class of loops that are homotopic to the ε-loop $\alpha*\beta$. Thus $[\alpha*\beta]_\varepsilon \in \Pi_\varepsilon(X)$. Therefore $\Pi_\varepsilon(X)$ is closed under the concatenation operation. $[\{\bigstar\}]_\varepsilon$ is the identity element of $\Pi_\varepsilon(X)$. Notice that $[\{\bigstar\}]_\varepsilon * [\lambda]_\varepsilon = [\lambda]_\varepsilon * [\{\bigstar\}]_\varepsilon = [\lambda]_\varepsilon$ for any $[\lambda]_\varepsilon \in \Pi_\varepsilon(X)$ since concatenation of loops is always defined and we may use a single basic move to remove the redundant $\bigstar$ at either the end or the beginning of a loop. Finally, for any $[\alpha]_\varepsilon \in \Pi_\varepsilon(X)$, the inverse of $[\alpha]_\varepsilon$ is $[\overline{\alpha}]_\varepsilon$. Using our earlier lemma, we can see that $[a]_\varepsilon * [\overline{\alpha}]_\varepsilon = [\overline{\alpha}]_\varepsilon * [\alpha]_\varepsilon = [\{\bigstar\}]_\varepsilon$. Therefore $\Pi_\varepsilon(X)$ with the concatenation operation is a group. ■

Notice that $\Pi_\varepsilon(X)$ is somewhat analogous to the fundamental group of a topological space X which is constructed using paths. As with the fundamental group, we are most interested in whether or not an ε-loop is homotopic to the trivial loop $\{\bigstar\}$. For certain values of ε, important changes occur to the homotopy classes of our loops. We call these values of ε critical values.

Section III: Homotopy Critical Values of Loops

As we become more familiar with ε-loops and their equivalence classes, we see that there are certain values of ε where the characteristics of our equivalence class change drastically. For example, let's examine the ε-loop $\lambda = \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$ on the unit circle where each of the points of λ represents an angle in the usual positions. Since $|\lambda|_{\max} = \pi/2$, we need to have $\varepsilon > \pi/2$ for this loop to even make sense in the first place (or add more points to refine our chain so that the max norm of the loop is smaller). What happens when $\varepsilon = \pi$? Then we have:

$$\begin{aligned}
[\lambda]_\varepsilon &= [\{0, \pi/2, \pi, 3\pi/2, 0\}]_\varepsilon \\
&= [\{0, 2\pi/3, \pi/2, \pi, 3\pi/2, 0\}]_\varepsilon \\
&= [\{0, 2\pi/3, \pi, 3\pi/2, 0\}]_\varepsilon \\
&= [\{0, 2\pi/3, \pi, 3\pi/2, 4\pi/3, 0\}]_\varepsilon \\
&= [\{0, 2\pi/3, \pi, 4\pi/3, 0\}]_\varepsilon \\
&= [\{0, 2\pi/3, 4\pi/3, 0\}]_\varepsilon \quad \text{(A)} \\
&= [\{0, 2\pi/3, 0\}]_\varepsilon \quad \text{(B)} \\
&= [\{0, 0\}]_\varepsilon \\
&= [\{0\}]_\varepsilon
\end{aligned}$$

So for $\varepsilon = \pi$, we have $[\lambda]_\varepsilon = [\{0\}]_\varepsilon$. Notice the basic move between steps (A) and (B). This part of our chain homotopy is interesting. We get a sudden change in the way we measure the distance from the second to last and the last points in our loop. In step (A), we are measuring the counterclockwise angle from $4\pi/3$ to 0. In step (B), we measure the clockwise angle from $2\pi/3$ to 0. This trick allows us to show that λ is homotopic to the trivial chain. In fact, this type of trick works for all $\varepsilon > 2\pi/3$, i.e. one-third of the circumference of the circle. For $\varepsilon \leq 2\pi/3$, this is no longer the case. It is possible to show that for $\varepsilon \leq 2\pi/3$, $[\lambda]_\varepsilon \neq [\{0\}]_\varepsilon$. For λ , $2\pi/3$ is a critical value that we mentioned earlier. We can now define this term rigorously.

Definition 23. *Let λ be a loop beginning at $\bigstar$ in a metric space X. Then*

1. λ is called ε-critical if and only if λ is not ε-homotopic to $\{\bigstar\}$ and λ is δ-homotopic to $\{\bigstar\}$ for all $\delta > \varepsilon$. In this case we say ε is the **homotopy critical value** of λ, notated $hc(\lambda) = \varepsilon$.

2. λ is called ε-stable if and only if λ is not ε-critical.

3. The **homotopy critical spectrum** of X is defined to be: $hc(X) := \{r \in \mathbb{R} : nc(\gamma) = r$ *for some loop* γ *in* $X\}$.

For the case where we have a loop that is ε-homotopic to the trivial loop for any value of ε, we use the convention that the homotopy critical value of that loop is 0. We can also define the homotopy critical value on the homotopy class of an ε-loop by $hc([\lambda]_\varepsilon) = hc(\lambda)$. Now we must make sure this new definition is well-defined.

Proposition 24. *Let λ be a loop beginning at $\bigstar$ in a metrix space X. Suppose $\varepsilon = hc(\lambda)$. If λ' is ε-homotopic to λ, then $hc(\lambda) = hc(\lambda')$.*

Proof. Suppose not, i.e. there exists a loop λ' homotopic to λ where $hc(\lambda') \neq hc(\lambda)$. Let $\delta = hc(\lambda')$. Then for all $\eta > \delta$, $[\lambda']_\eta = [\{\bigstar\}]_\eta$. and $[\lambda']_\delta \neq [\{\bigstar\}]_\delta$.

Case 1: $hc(\lambda) < hc(\lambda')$

Since $hc(\lambda) < hc(\lambda')$, there exists an η such that $hc(\lambda) < \eta < hc(\lambda')$. So $[\lambda]_\eta = [\{\bigstar\}]_\eta$ since $\eta > hc(\lambda)$. Since $\varepsilon = hc(\lambda) < \eta$, we have $[\lambda]_\eta = [\lambda']_\eta$. However since $\eta < hc(\lambda')$, $[\lambda]_\eta = [\lambda']_\eta \neq [\{\bigstar\}]_\eta$. So we have a contradicton. Thus $hc(\lambda) = hc(\lambda')$.

Case 2: $hc(\lambda') < hc(\lambda)$

Since $hc(\lambda') < \varepsilon$, we have $[\lambda']_\varepsilon = [\{\bigstar\}]_\varepsilon$. Since $hc(\lambda) = \varepsilon$, we have the fact that $[\lambda]_\varepsilon \neq [\{\bigstar\}]_\varepsilon$. However, $[\lambda']_\varepsilon = [\lambda]_\varepsilon$; so we have a contradiction. Therefore $hc(\lambda) = hc(\lambda')$. ■

Proposition 25. *[HC Slide Rule] Let λ be an ε-loop beginning at $\bigstar$ in a metric space X. Then $\varepsilon > hc(\lambda)$ if and only if $[\lambda]_\varepsilon = [\bigstar]_\varepsilon$.*

Proof. " $\Rightarrow$ " Suppose $\varepsilon > hc(\lambda)$. Then it follows from the definition of nullity critical value that $[\lambda]_\varepsilon = [\bigstar]_\varepsilon$.

" $\Leftarrow$ " Suppose $[\lambda]_\varepsilon = [\bigstar]_\varepsilon$. Then by definition, $\varepsilon \neq hc(\lambda)$. So either $\varepsilon < hc(\lambda)$ or $\varepsilon > hc(\lambda)$. Suppose $\varepsilon < hc(\lambda)$. Since $[\lambda]_\varepsilon = [\bigstar]_\varepsilon$, there exists a finite sequence of basic moves which will transform λ to $\{\bigstar\}$. Notice that these basic moves are still viable for any $\delta \geq \varepsilon$, specifically for $\delta = hc(\lambda)$. So using the same sequence of basic moves we have that $[\lambda]_{hc(\lambda)} = [\bigstar]_{hc(\lambda)}$, a contradiction to the definition of $hc(\lambda)$. Therefore $\varepsilon > hc(\lambda)$. ■

We will use this result often in our later work. It is most useful in elliminating possible values for the homotopy critical value of a particular loop.

Proposition 26. *Let α be an ε-loop. Then $hc(\alpha) = hc(\overline{\alpha})$.*

Proof. From the HC Slide Rule, we have that α is δ-homotopic to $\{\bigstar\}$ for any $\delta > hc(\alpha)$. Then from an earlier lemma, we have that $\overline{\alpha}$ is δ-homotopic to $\overline{\{\bigstar\}} = \{\bigstar\}$ for any $\delta > hc(\alpha)$. Now if we suppose that $\overline{\alpha}$ is δ-homotopic to $\{\bigstar\}$ for some $\delta \leq hc(\alpha)$, we can use the same lemma to show that α is δ-homotopic to $\{\bigstar\}$, a contradiction of the fact that $\delta \leq hc(\alpha)$. Thus $hc(\overline{\alpha}) = hc(\alpha)$. ■

Now that we are more comfortable with using the homotopy critical value of a loop, we can start to gain information about metric spaces by examining the homotopy values of different chains in our space. The homotopy critical value of a chain can be used to detect the "holes" mentioned earlier under certain circumstances.

Recall our example earlier in this section where we were working with the unit circle. We saw that for ε values that were under a certain threshold, namely $\varepsilon = 2\pi/3$, we could not begin with a general ε-loop and be guaranteed to be able to transform that loop to the trivial chain with basic moves. For the loop $\lambda = \{0, \pi/2, \pi, 3\pi/2, 0\}$, the homotopy critical value is $hc(\lambda) = 2\pi/3$. This is exactly what we wanted to happen. Intuition tells us that there is a hole in the middle of the circle. Using the homotopy critical value, we have in a sense measured the size of this hole. Whenever we can find a loop with a nonzero homotopy critical value, we have found a hole in our space.

However, using ε-loops is not without its own flavor of difficulties. Define X to be the punctured plane, i.e. $X = \mathbb{R}^2 \backslash \{\maltese\}$ where $\maltese$ is the origin. We would expect to find a hole where we removed the origin in X, but for any $\varepsilon > 0$, any ε-loop is ε-homotopic to the trivial chain. All we have to do is use basic moves to bring our chain to lie on a circle centered around the origin of circumference 2ε (If needed, we can add a "tail" to our loop for when our basepoint is not on the desired circle. Just return back along the tail after the loop has gone around the circle once). Then if we restrict ourselves to this circle, we have an ε-loop on a circle of circumference 2ε. Then we can copy the rescaled version of our earlier example. This chain is ε-homotopic to the trivial chain since $\varepsilon > 2\varepsilon/3$, i.e. two-thirds of the circumference of the circle. If the chain is homotopic to the trivial chain when we restrict ourselves to only this circle, it must be trivial when we can use the whole of X as well. In this case, the single point that we removed is too small to register as a hole using ε-loops.

We would also like information on how the homotopy critical values of loops act under the concatenation operation.

Proposition 27. *Let α and β be loops beginning at $\bigstar$ in a metric space X. Then $0 \leq hc(\alpha * \beta) \leq \max\{hc(\alpha), hc(\beta)\}$.*

Proof. Notice that the homotopy critical value for any loop has zero as a lower bound by definition. So $0 \leq hc(\alpha * \beta)$. If we take $\varepsilon > \max\{hc(\alpha), hc(\beta)\}$, we get $[\alpha * \beta]_\varepsilon = [\bigstar * \beta]_\varepsilon = [\beta]_\varepsilon = [\{\bigstar\}]_\varepsilon$ using the HC Slide Rule. So, $hc(\ \alpha * \beta) \leq \max\{hc(\alpha), hc(\beta)\}$. ■

It is not possible to get a better interval for the homotopy critical value under concatenation in general. We know that on unit circle for $\lambda = \{0, \pi/2, \pi, 3\pi/2, 0\}$, $hc(\lambda) = 2\pi/3$. We also know that $hc(\overline{\lambda}) = 2\pi/3$ as well. We have already shown that $\lambda * \overline{\lambda}$ is homotopic to the trivial loop for any ε. So $hc(\lambda * \overline{\lambda}) = 0$ which is strictly less than the homotopy critical value of either of our original loops. On the other hand, $hc(\lambda * \{0\}) = 2\pi/3$ is equal to the maximum of the homotopy critical values of its constituent loops.

If $x = -x$ in the real numbers, we have $x = 0$. It is tempting to believe that something of this nature holds for ε-loops, say if $[\alpha]_\varepsilon = [\overline{\alpha}]_\varepsilon$, then $[\alpha]_\varepsilon = [\{\bigstar\}]_\varepsilon$. This would be a helpful fact to have when trying to calculate the homotopy critical value of a loop. Instead of trying to find a homotopy to only the trivial chain, we also have the option of trying to find a homotopy to the reverse of our chain. However, this statement is not true. We can construct an example where $[\alpha]_\varepsilon = [\overline{\alpha}]_\varepsilon$ but α is not ε-homotopic to the trivial loop.

For our example, we use the real projective plane. To construct the projective plane, we begin with a unit sphere $S^2 = \{x \in \mathbb{R}^3 : d(x, 0) = 1\}$. Each point x on S^2 has an antipodal point $-x$. To get the projective plane, we introduce an equivalence relation on S^2 where each point x is equivalent to its antipodal point $-x$. We will use the notation $\overline{x} = \{x, -x\}$ for the equivalence class of the point x. For the metric on the real projective plane, we can use the $d(\overline{x}, \overline{y}) = \min\{L(p) : p$ is a path from x or $-x$ to y or $-y$ in $S^2\}$. Define a chain $\beta = \{\pi, \frac{5\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, 0, \frac{-\pi}{6}, \frac{-\pi}{3}, \frac{-\pi}{2}, \frac{-2\pi}{3}, \frac{-5\pi}{6}, -\pi\}$ on S^2. Each of these points represents the angle on the xz-plane of the point in β with the positive x-axis. So β runs directly along a "meridian of the globe" from the "north pole" to the "south pole". Now we can define a chain α in the projective plane by, $\alpha = \{\overline{\pi}, \overline{\frac{5\pi}{6}}, \overline{\frac{2\pi}{3}}, \overline{\frac{\pi}{2}}, \overline{\frac{\pi}{3}}, \overline{\frac{\pi}{6}}, \overline{0}, \overline{\frac{-\pi}{6}}, \overline{\frac{-\pi}{3}}, \overline{\frac{-\pi}{2}}, \overline{\frac{-2\pi}{3}}, \overline{\frac{-5\pi}{6}}, \overline{-\pi}\}$. So, α is a loop in the projective plane since $\overline{\pi} = \overline{-\pi}$. The homotopy critical value of α is $\pi/3$, i.e. $hc(\alpha) = c/6$ where c is the length of the equator of the sphere (The homotopy critical value is $\frac{1}{3}$ of the length of the "half-equator"). The loop α has the following property.

Proposition 28. *Let α be an ε-loop in the projective plane as defined above. Then $[\alpha]_\varepsilon = [\overline{\alpha}]_\varepsilon$ where $\overline{\alpha}$ represents the reverse of α.*

Proof. Let $\alpha = \{\overline{\pi}, \overline{\frac{5\pi}{6}}, \overline{\frac{2\pi}{3}}, \overline{\frac{\pi}{2}}, \overline{\frac{\pi}{3}}, \overline{\frac{\pi}{6}}, \overline{0}, \overline{\frac{-\pi}{6}}, \overline{\frac{-\pi}{3}}, \overline{\frac{-\pi}{2}}, \overline{\frac{-2\pi}{3}}, \overline{\frac{-5\pi}{6}}, \overline{-\pi}\}$ where $c\pi$ represents the angle of the point in the xz-plane with the positive x-axis on the sphere. Then the reverse of α is:

$$\overline{\alpha} = \{\overline{-\pi}, \overline{\frac{-5\pi}{6}}, \overline{\frac{-2\pi}{3}}, \overline{\frac{-\pi}{2}}, \overline{\frac{-\pi}{3}}, \overline{\frac{-\pi}{6}}, \overline{0}, \overline{\frac{\pi}{6}}, \overline{\frac{\pi}{3}}, \overline{\frac{\pi}{2}}, \overline{\frac{2\pi}{3}}, \overline{\frac{5\pi}{6}}, \overline{\pi}\}$$

However in the projective plane, x is equivalent to $-x$. So we have $\overline{x} = \overline{-x}$ for any x in the projective plane. Thus

$$\begin{aligned}
\overline{\alpha} &= \{\overline{-\pi}, \overline{\frac{-5\pi}{6}}, \overline{\frac{-2\pi}{3}}, \overline{\frac{-\pi}{2}}, \overline{\frac{-\pi}{3}}, \overline{\frac{-\pi}{6}}, \overline{0}, \overline{\frac{\pi}{6}}, \overline{\frac{\pi}{3}}, \overline{\frac{\pi}{2}}, \overline{\frac{2\pi}{3}}, \overline{\frac{5\pi}{6}}, \overline{\pi}\} \\
&= \{\overline{\pi}, \overline{\frac{5\pi}{6}}, \overline{\frac{2\pi}{3}}, \overline{\frac{\pi}{2}}, \overline{\frac{\pi}{3}}, \overline{\frac{\pi}{6}}, \overline{0}, \overline{\frac{-\pi}{6}}, \overline{\frac{-\pi}{3}}, \overline{\frac{-\pi}{2}}, \overline{\frac{-2\pi}{3}}, \overline{\frac{-5\pi}{6}}, \overline{-\pi}\} \\
&= \alpha
\end{aligned}$$

Therefore, $[\alpha]_\varepsilon = [\overline{\alpha}]_\varepsilon$.

■

We must be careful when attempting to use analogies to the real numbers on our chains.

Section IV: Refinement Critical Values of Chains

In the beginning of the previous section, we mentioned adding points to a chain so that its max norm would be smaller. Let's take a 2.5-chain $\alpha = \{0, 2, 4, 6, 8\}$ in $\mathbb{R}$. Then $|\alpha|_{\max} = 2$. We can add points to α one at a time to get a chain $\beta = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Then $|\beta|_{\max} = 1$. Really, in this space we can refine α to be an ε-chain for any $\varepsilon > 0$. We can just keep adding points until we are satisfied with how close our points are to their neighbors. This is not the case for every space. Instead of using $\mathbb{R}$, let's use the same points for a new chain γ in $(-\infty, 2] \cup [4, \infty)$. Can we use basic moves on γ to get some ε-chain for $\varepsilon = 1$? Clearly we cannot. Basic moves cannot change the endpoints of a chain, and at least once, the distance between two points in the chain must be at least 2. There is a gap in our space that our chain must cross, and we cannot refine our chain to get any finer than the size of our gap. This is the motivation behind the refinement critical value of a chain.

Definition 29. *Let λ be a δ-chain beginning at ★ in a metric space X. Then*

1. λ is said to be ε-**refinable** if and only if $[\lambda]_\delta = [\lambda']_\delta$ for some ε-chain λ'.

2. λ is called ε-refinement critical if and only if λ is not ε-refinable, but for all $\eta > \varepsilon$, λ is η-refinable. In this case we say ε is the **refinement critical value** of λ, notated $rc(\lambda) = \varepsilon$.

3. λ is called ε-refinement stable if and only if λ is not ε-refinement critical.

4. The **refinement critical spectrum** of X is the set $\{\delta : rc(\lambda) = \delta \ for\ some\ chain\ \lambda\ in\ X\}$.

If a chain can be arbitrarily refined, we use the convention that the refinement critical value of the chain is 0. The refinement critical value of a chain α is the infimum of the max norms of all the chains homotopic to α.

Proposition 30. *[RC Slide Rule] Let α be an ε-chain beginning at ★ in a metric space X. Then $\eta > rc(\alpha)$ if and only if $[\alpha]_\varepsilon = [\alpha']_\varepsilon$ for some δ-chain α' such that $\delta < \eta$.*

Proof. " $\Rightarrow$ " Suppose $\eta > rc(\alpha)$. Then α is η-refinable. So $[\alpha]_\varepsilon = [\alpha']_\varepsilon$ for some η-chain α'. From an earlier lemma, since α' is an η-chain, α' is also a δ-chain for $\delta \in (|\alpha'|_{\max}, \eta)$. Thus $[\alpha]_\varepsilon = [\alpha']_\varepsilon$ for some δ-chain where $\delta < \eta$.

" $\Leftarrow$ " Suppose $[\alpha]_\varepsilon = [\alpha']_\varepsilon$ for some δ-chain α' such that $\delta < \eta$. So α is δ-refinable. Since α' is also a κ-chain for all $\kappa \geq \delta$, α is trivially κ-refinable for all $\kappa > \delta$ as well. So $rc(\alpha) < \delta < \eta$. ■

Like with the homotopy critical value, we can define the refinement critical value on homotopy classes of chains: $rc([\alpha]_\varepsilon) = rc(\alpha)$. We must prove that this is well-defined.

Proposition 31. *Let α be an ε-chain. Then if $[\alpha]_\varepsilon = [\alpha']_\varepsilon$, $rc(a) = rc(\alpha')$.*

Proof. Suppose not, i.e. there exists an α' homotopic to α such that $rc(\alpha') \neq rc(\alpha)$.

Case 1: $rc(\alpha') < rc(\alpha)$

Since $rc(\alpha') < rc(\alpha)$ using the RC Slide Rule, $[\alpha']_\varepsilon = [\beta]_\varepsilon$ for some δ-chain β where $\delta < rc(\alpha)$. So $[\alpha]_\varepsilon = [\alpha']_\varepsilon = [\beta]_\varepsilon$, i.e. α is δ-refinable. This is a contradiction since $\delta < rc(\alpha)$.

Case 2: $rc(\alpha) < rc(\alpha')$

Since $rc(\alpha) < rc(\alpha')$ using the RC Slide Rule, $[\alpha]_\varepsilon = [\beta]_\varepsilon$ for some δ-chain β where $\delta < rc(\alpha')$. So $[\alpha']_\varepsilon = [\alpha]_\varepsilon = [\beta]_\varepsilon$, i.e. α' is δ-refinable. This is a contradiction since $\delta < rc(a')$.

Therefore, $rc(\alpha) = rc(\alpha')$. ■

The refinement critical value has the same property as the homotopy critical value in regards to the concatenation operation.

Proposition 32. *Let α and β be ε-chains beginning at $\bigstar$ in a metric space X where $\alpha * \beta$ is defined. Then $0 \le rc(\alpha * \beta) \le \max\{rc(\alpha), rc(\beta)\}$.*

Proof. By definition, 0 is a lower bound for the refinement critical value. Suppose $\delta > \max\{rc(\alpha), rc(\beta)\}$. Then both α and β are δ-refinable, i.e. $[\alpha]_\varepsilon = [\gamma]_\varepsilon$ for some δ-chain γ and $[\beta]_\varepsilon = [\gamma']_\varepsilon$ for some δ-chain γ'. Using our earlier lemma, we have $[\alpha * \beta]_\varepsilon = [\gamma * \gamma']_\varepsilon$. Notice that $\gamma * \gamma'$ is a δ-chain. We have shown that $\alpha * \beta$ is δ-refinable for all $\delta > \max\{rc(\alpha), rc(\beta)\}$. So by the RC Slide Rule, $rc(\alpha * \beta) \le \max\{rc(\alpha), rc(\beta)\}$. ■

In general it is not possible to get a better interval on the refinement critical values of chains. For any chain α beginning at $\bigstar$, $rc(\{\bigstar\}) = 0$, and $rc(\{\bigstar\} * \alpha) = rc(\alpha)$. This is an example where the refinement critical value is equal to the maximum of the refinement critical values of the constituent chains. On the other hand, $rc(\alpha * \overline{\alpha}) = rc(\{\bigstar\}) = 0$.

It is also possible to relate the refinement critical values and homotopy critical values of loops.

Proposition 33. *Let λ be an ε-loop beginning at $\bigstar$ in a metric space X. Then $0 \le rc(\lambda) \le hc(\lambda)$ with $rc(\lambda) = hc(\lambda)$ if and only if $rc(\lambda) = hc(\lambda) = 0$.*

Proof. First we will show that $rc(\lambda) \le hc(\lambda)$. If $\varepsilon > hc(\lambda)$, then λ is ε-homotopic to $\{\bigstar\}$. So since the trivial loop is arbitrarily refinable, $rc(\lambda) = rc(\{\bigstar\}) = 0 \le hc(\lambda)$. What happens when $\varepsilon \le hc(\lambda)$? Since λ is already an ε-loop, it is trivially ε-refinable. So $rc(\lambda) \le \varepsilon \le hc(\lambda)$. Now we can focus on the second part of the statement.

" $\Leftarrow$ " This follows trivially from our statement.

" $\Rightarrow$ " Suppose $rc(\lambda) = hc(\lambda)$. If $\varepsilon > hc(\lambda)$, λ is ε-homotopic to the trivial loop. So $rc(\lambda) = 0$. If $0 < \varepsilon \le hc(\lambda)$, λ is not ε-homotopic to the trivial loop. Since λ is an ε-loop, $|\lambda|_{\max} < \varepsilon$. Recall that λ is also an η-loop for any $\eta > |\lambda|_{\max}$, including $\eta < \varepsilon$. So $rc(\lambda) \le \eta < \varepsilon \le hc(\lambda)$, which is a contradiction to our hypotheses.

Therefore, $0 \le rc(\lambda) \le hc(\lambda)$ with $rc(\lambda) = hc(\lambda)$ if and only if $rc(\lambda) = hc(\lambda) = 0$.

■

While the example of removing a part of the plane was useful in motivating the refinement critical value, path connected spaces are much more interesting for our purposes. Consider the space X where X is the portion of the unit circle remaining after the points within an angle of $\delta/2$ from the positive x-axis are removed. We choose δ to be small compared to the circumference of the circle. This space is known as the cut circle. We can see that we have a gap in our space where we removed part of S'. If we have an ε-loop that wraps directly around the circle once, then we know that the refinement critical value of our chain is equal to δ. Notice that when we use this loop around the cut circle, we are considering X as a subspace of S', and we are using the metric from S'. The distance between the "endpoints" of the cut circle are the same as the distance that they were before we removed the points. What happens when we consider a different metric for the cut circle? For example, if we take the distance between two points to be the intrinsic distance, i.e. the length of the shortest path within the space between two points, then any chain on the cut circle is arbitrarily refinable. Somehow by using a different metric, we have changed what we mean when we say a space has a gap.

Proposition 34. *Suppose X is a geodesic space, and α is an ε-chain in X. Then $rc(\alpha) = 0$.*

Proof. It is sufficient to show that for any two adjacent points m and n of α, we can arbitrarily refine the ε-chain $\{m, n\}$. So, let m, n be adjacent points in the ε-chain α. Since X is a geodesic space, the distance between m and n is the length of the minimal path between m and n. Suppose p is this minimal path (parameterized to have constant speed). Then the image of p is contained in X. Let $z := p(1/2)$. Then $d(m, z) = \int_0^{1/2} ||p'(t)||dt = \frac{1}{2}\int_0^1 ||p'(t)||dt < \varepsilon$, and $d(z, n) = \int_{1/2}^1 ||p'(t)||dt = \frac{1}{2}\int_0^1 ||p'(t)||dt < \varepsilon$. So adding z to the center of our chain is a valid basic move. Therefore we have $[\{m, n\}]_\varepsilon = [\{m, z, n\}]_\varepsilon$. Notice that the max norm of $\{m, z, n\}$ is exactly one half of the max norm of $\{m, n\}$. Now we can repeat this process on each of the sub-chains $\{m, z\}$ and $\{z, n\}$ to get a max norm that is arbitrarily small and homotopic to the chain $\{m, n\}$. Since we can use basic moves to "shrink" our max norm however small we need, we know that for any $\delta > 0$, $\{m, n\}$ is ε-homotopic to a chain β where β is a δ-chain. Then by definition, $rc(\{m, n\}) = 0$. So, the chain between any two adjacent points in α is arbitrarily refinable. Then α as a whole is arbitrarily refinable. Therefore, $rc(\alpha) = 0$. ■

So, if we take our definition of a "gap" in the space to be wherever there is a chain with a nonzero refinement critical value, no geodesic space has a gap. Returning to our example with the cut circle, we see that when we used the intrinsic metric we considered the cut circle as a geodesic space. This changed how gaps appeared (or in this case did not appear) in our space.

Section V: Chain Homotopies in Products of Metric Spaces

In this section we would like to understand homotopy and refinement critical values in regards to taking Cartesian products of various metric spaces. In order to make sense of chains in products of metric spaces, we must have a metric. Consider the following where (X_i, d_i) is a metric space for all i.

$$X = \Pi_{i=1}^n (X_i, d_i)$$

Then we may define a metric D on X as follows:

$$D(x, y) = \sqrt{\Sigma_{i=1}^n d_i(x_i, y_i)^2}$$

While this is not the only way to define a metric on X, we will always use this metric in the product of metric spaces unless stated otherwise. The fact that this is a valid metric follows easily since all of the constituent d_i's are metrics. Now that we have the structure of a Cartesian product of metric, we would like to know how chain homotopies work on products. Before we can get to that, we need some way to connect a chain in one of the X_i's to a chain in the product X.

Definition 35. *Let $\alpha = \{x_0, ..., x_n\}$ be an ε-chain in a metric space X, and let $\beta = \{y_0, ..., y_n\}$ be a δ-chain in a metric space Y. For $\eta \geq \sqrt{\varepsilon^2 + \delta^2}$ the **product** of α and β is an η-chain $\alpha \times \beta = \{(x_0, y_0), ..., (x_n, y_n)\}$.*

Notice that to be strictly legal, we can only form the product of chains that have the same number of points. Even if α and β do not have the same number of points, repeating points in the shorter chain to make their sizes match does not typically pose any serious problems. Adding these repeated points still gives us a chain that is homotopic to our shorter chain, and typically this is enough for our purposes.

Another issue to notice is that even if α and β are both ε-chains, $\alpha \times \beta$ may not be an ε-chain. For example, suppose $\varepsilon = 1.1$. Then $\alpha = \{0, 1, 2, 3, 4\}$ and $\beta = \{0, -1, -2, -3, -4\}$ are ε-chains in $\mathbb{R}$. However in $\mathbb{R}^2$ with the Euclidean metric, $\{(0, 0), (1, -1), (2, -2), (3, -3), (4, -4)\}$ is not an ε-chain because the distance between each point in the chain is $\sqrt{2} \approx 1.414 > 1.1$. To guarantee that the product of two ε-chains α and β is actually an ε-chain, we also need the condition that $|\alpha|^2_{\max} + |\beta|^2_{\max} < \varepsilon^2$.

Now that we have products of chains in our arsenal, we would like to be able to retain chain homotopies in products of chains. It turns out that we can get close to this conclusion.

Proposition 36. *Let α_1, α_2 be ε-chains with the same number of points in a metric space X and let $\alpha_1 \widetilde{\varepsilon} \alpha_2$. Let β be a δ-chain in a metric space Y with the same number of points as α_1 and α_2. Suppose $\sqrt{\varepsilon^2 + \delta^2} \leq \eta$. Then in the metric space $X \times Y$ with the product metric $d((x,y),(c,d)) = \sqrt{d_X(x,c)^2 + d_Y(y,d)^2}$, $(\alpha_1 \times \beta)\widetilde{\eta}(\alpha_2' \times \beta')$ where α_2' and β' differ from α_2 and β respectively by having possibly repeated points. Likewise in the metric space $Y \times X$ with the same product metric, $(\beta \times \alpha_1)\widetilde{\eta}(\beta' \times \alpha_2')$ where α_2' and β' differ from α_2 and β respectively by having possibly repeated points.*

Proof. It suffices to show that the statement holds when α_1 differs from α_2 by a single basic move. Then we have two cases: adding a point and removing a point.

Case 1: Adding a point

Let $\alpha_1 = \{x_{0,...,}x_{i-1}, x_{i+1}, ..., x_k\}$ and let $\alpha_2 = \{x_0, ..., x_{i-1}, x_i, x_{i+1}, ..., x_k\}$. Since α_1 and α_2 differ by a single basic move, it is clear that that move is adding the point x_i. By definition of basic move we can conclude $d_X(x_{i-1}, x_i) < \varepsilon$, $d_X(x_{i-1}, x_{i+1}) < \varepsilon$, and $d_X(x_i, x_{i+1}) < \varepsilon$. Now we have:

$$(\alpha_1 \times \beta) = \{(x_0, \beta_0), ..., (x_{i-1}, \beta_{i-1}), (x_{i+1}, \beta_{i+1}), ..., (x_k, \beta_k)\}$$

Now notice:

$$d((x_i, \beta_{i-1}), (x_{i-1}, \beta_{i-1})) = \sqrt{d_X(x_i, x_{i-1})^2 + d_Y(\beta_{i-1}, \beta_{i-1})^2} < \sqrt{\varepsilon^2 + 0^2} = \varepsilon \leq \eta$$

$$d((x_i, \beta_{i-1}), (x_{i+1}, \beta_{i+1})) = \sqrt{d_X(x_i, x_{i+1})^2 + d_Y(\beta_{i-1}, \beta_{i+1})^2} < \sqrt{\varepsilon^2 + \delta^2} \leq \eta$$

So, adding the point (x_i, β_{i-1}) is a valid basic move. Now after adding this point we have:

$$(\alpha_2 \times \beta') = \{(x_0, \beta_0), ..., (x_{i-1}, \beta_{i-1}), (x_i, \beta_{i-1}), (x_{i+1}, \beta_{i+1}), ..., (x_k, \beta_k)\}$$

As promised, we have $(\alpha_1 \times \beta)\widetilde{\eta}(\alpha_2' \times \beta')$ where α_2 and β' differ from α_2 and β respectively by possibly having repeated points.

Case 2: Removing a point

Let $\alpha_1 = \{x_0, ..., x_{i-1}, x_i, x_{i+1}, ..., x_k\}$ and let $\alpha_2 = \{x_0, ..., x_{i-1}, x_{i+1}, ..., x_k\}$. So by definition of a basic move we have $d_X(x_{i-1}, x_i) < \varepsilon$, $d_X(x_i, x_{i+1}) < \varepsilon$, and $d_X(x_{i-1}, x_{i+1}) < \varepsilon$. Thus we have:

$$(\alpha_1 \times \beta) = \{(x_0, \beta_0), ..., (x_{i-1}, \beta_{i-1}), (x_i, \beta_i), (x_{i+1}, \beta_{i+1}), ..., (x_k, \beta_k)\}$$

Now we will add two points individually: (x_{i-1}, β_i) first, and then (x_{i+1}, β_i). Notice that adding (x_{i-1}, β_i) is a valid basic move. Likewise, adding (x_{i+1}, β_i) is a valid basic move after adding the first point.

$$\begin{aligned} d((x_{i-1}, \beta_i), (x_{i-1}, \beta_{i-1})) &= \sqrt{d_X(x_{i-1}, x_{i-1})^2 + d_Y(\beta_i, \beta_{i-1})^2} < \sqrt{0^2 + \delta^2} \leq \eta \\ d((x_{i-1}, \beta_i), (x_i, \beta_i)) &= \sqrt{d_X(x_{i-1}, x_i)^2 + d_Y(\beta_i, \beta_i)^2} < \sqrt{\varepsilon^2 + 0^2} \leq \eta \end{aligned}$$

$$(\alpha_1 \times \beta)\widetilde{\eta}\{(x_0, \beta_0), ..., (x_{i-1}, \beta_{i-1}), (x_{i-1}, \beta_i), (x_i, \beta_i), (x_{i+1}, \beta_{i+1}), ..., (x_k, \beta_k)\}$$

$$d((x_{i+1},\beta_i),(x_i,\beta_i)) = \sqrt{d(x_{i+1},x_i)^2+d(\beta_i,\beta_i)^2} < \sqrt{\varepsilon^2+0^2} \leq \eta$$
$$d((x_{i+1},\beta_i),(x_{i+1},\beta_{i+1})) = \sqrt{d(x_{i+1},x_{i+1})^2+d(\beta_i,\beta_{i+1})^2} < \sqrt{0^2+\delta^2} \leq \eta$$

$$(\alpha_1\times\beta)\widetilde{\eta}\{(x_0,\beta_0),...,(x_{i-1},\beta_{i-1}),(x_{i-1},\beta_i),(x_i,\beta_i),(x_{i+1},\beta_i),(x_{i+1},\beta_{i+1}),...,(x_k,\beta_k)\}$$

Now remove the point (x_i,β_i)

$$d((x_{i-1},\beta_i),(x_{i+1},\beta_i)) = \sqrt{d_X(x_{i-1},x_{i+1})^2+d_Y(\beta_i,\beta_i)^2} < \sqrt{\varepsilon^2+0^2} \leq \eta$$

$$(\alpha_1\times\beta)\widetilde{\eta}\{(x_0,\beta_0),...,(x_{i-1},\beta_{i-1}),(x_{i-1},\beta_i),(x_{i+1},\beta_i),(x_{i+1},\beta_{i+1}),...,(x_k,\beta_k)\}$$

Now remove the point (x_{i+1},β_i)

$$d((x_{i-1},\beta_i),(x_{i+1},\beta_{i+1})) = \sqrt{d_X(x_{i-1},x_{i+1})^2+d_Y(\beta_i,\beta_{i+1})^2} < \sqrt{\varepsilon^2+\delta^2} \leq \eta$$

$$(\alpha_1\times\beta)\widetilde{\eta}\{(x_0,\beta_0),...,(x_{i-1},\beta_{i-1}),(x_{i-1},\beta_i),(x_{i+1},\beta_{i+1}),...,(x_k,\beta_k)\}$$

Clearly we have our promised $(\alpha_1\times\beta)\widetilde{\eta}(\alpha_2'\times\beta')$.

The proof for $(\beta\times\alpha_1)\widetilde{\eta}(\beta'\times\alpha_2')$ is similar. ■

Notice that this result generalizes easily from the product of only two metric spaces to the product of n metric spaces for any positive integer n. Just take β to be a product of chains itself. Unfortunately, this result is not quite everything that we were hoping for. The chains with repeated points bring complications that we would rather avoid. However, there is a silver lining. We can use this result to prove that the product of loops that are each homotopic to the trivial loop is also homotopic to the trivial loop. Since the homotopy with the trivial loop is one of the most important homotopies with which we involve ourselves, we can salvage our earlier work.

Proposition 37. *Suppose that α is an ε-loop beginning at $\bigstar$ in a metric space X and β is a δ-loop beginning at ✠ in a metric space Y where α and β have the same number of points and $\sqrt{\varepsilon^2+\delta^2}\leq\eta$. Using the product metric $d((x,y),(c,d))=\sqrt{d_X(x,c)^2+d_Y(y,d)^2}$, if $\alpha\widetilde{\varepsilon}\{\bigstar\}$ and $\beta\widetilde{\delta}\{$✠$\}$, then $(\alpha\times\beta)\,\widetilde{\eta}(\{\bigstar\}\times\{$✠$\})$.*

Proof. By the above proposition, we have $(\alpha\times\beta)\widetilde{\eta}(\{\bigstar\}'\times\beta')$. Now we can the proposition again to show $(\alpha\times\beta)\widetilde{\eta}(\{\bigstar\}'\times\beta')\widetilde{\eta}(\{\bigstar\}'\times\{$✠$\}')$. Notice however that both $\{\bigstar\}'$and $\{$✠$\}'$ are at worst repeated points of $\bigstar$ and ✠ respectively. So $(\{\bigstar\}'\times\{$✠$\}')=\{(\bigstar,$✠$)\}'$ and our work is done, or $(\{\bigstar\}'\times\{$✠$\}')=\{(\bigstar,$✠$),...,(\bigstar,$✠$)\}$. The later however can be easily brought to the trivial loop by removing the needed number of points one by one. Therefore, $(\alpha\times\beta)\widetilde{\eta}(\{\bigstar\}\times\{$✠$\})$. ■

Proposition 38. *Let $\alpha=\{x_0,...,x_n\}$ be an ε_1-loop beginning at $\{\bigstar\}$ in a metric space X and let $\beta=\{y_0,...,y_n\}$ be an ε_2-loop beginning at $\{$✠$\}$ in a metric space Y. Then $hc(\alpha\times\beta)\leq\sqrt{hc(\alpha)^2+hc(\beta)^2}$ and $rc(\alpha\times\beta)\leq\sqrt{rc(\alpha)^2+rc(\beta)^2}$*

Proof. (1) Let $\eta>\sqrt{hc(\alpha)^2+hc(\beta)^2}$. We will show that $\alpha\times\beta$ is η-homotopic to the trivial loop. Since $\eta>\sqrt{hc(\alpha)^2+hc(\beta)^2}$, we may choose $\delta_1,\delta_2>0$ such that $(\delta_1)^2+(\delta_2)^2=\eta^2$ and $\delta_1>hc(\alpha)$ and $\delta_2>hc(\beta)$. By the HC Slide Rule, α is δ_1-homotopic to $\{\bigstar\}$ and β is δ_2-homotopic to $\{$✠$\}$. So by the above proposition, $\alpha\times\beta$ is η-homotopic to the trivial loop. Thus $\eta>hc(\alpha\times\beta)$.

(2) Let $\kappa = \sqrt{(\varepsilon_1)^2 + (\varepsilon_2)^2}$. Then $\alpha \times \beta$ is a κ-chain (We will not use the fact that $\alpha \times \beta$ is a loop). Let $\eta > \sqrt{rc(\alpha)^2 + rc(\beta)^2}$. We will show that $\alpha \times \beta$ is κ-homotopic to an η-chain. Since $\eta > \sqrt{rc(\alpha)^2 + rc(\beta)^2}$, we may choose $\delta_3, \delta_4 > 0$ such that $\eta = \sqrt{(\delta_3)^2 + (\delta_4)^2}$ and $\delta_3 > rc(\alpha)$ and $\delta_4 > rc(\beta)$. By the RC Slide Rule, α is ε_1-homotopic to a δ_3-chain $\widehat{\alpha}$, and β is ε_2-homotopic to a δ_4-chain $\widehat{\beta}$. Therefore, we have that $\alpha \times \beta$ is κ-homotopic to the chain $\widehat{\alpha}' \times \widehat{\beta}'$ where $\widehat{\alpha}'$ is the same chain as $\widehat{\alpha}$ except with possibly repeated points, and $\widehat{\beta}'$ is the same chain as $\widehat{\beta}$ except with possibly repeated points. Notice that even if we do have repeated points, $\widehat{\alpha}'$ and $\widehat{\beta}'$ are still δ_3 and δ_4 chains respectively. Therefore, $\widehat{\alpha}' \times \widehat{\beta}'$ is an η-chain. Thus $\eta > rc(\alpha \times \beta)$. ■

We can also relate the homotopy critical spectrum of a single X_k to the homotopy critical spectrum of the product of the X_i's.

Proposition 39. *Suppose $\lambda = \{x_0, ..., x_n\}$ is an ε-loop beginning at $\bigstar$ in a metric space X. Then if Y is a metric space, there exists a chain Λ in $X \times Y$ such that $hc(\lambda) = hc(\Lambda)$.*

Proof. Let ✠ be the base point of Y. Again, we use the notation that $\{✠\}' = \{✠, ..., ✠\}$ as needed to match the number of points in λ. Then $\Lambda := \lambda \times \{✠\}'$ is an ε-loop in $X \times Y$ (We have the condition that $|\lambda|^2_{\max} + |\{✠\}|^2_{\max} < \varepsilon^2$). We claim that $hc(\Lambda) = hc(\lambda)$.

Suppose not, i.e. $hc(\Lambda) \neq hc(\lambda)$. Then we have two cases.

Case 1: $hc(\Lambda) > hc(\lambda)$

Let $\delta \in \mathbb{R}$ such that $hc(\lambda) < \delta < hc(\Lambda)$. Then by the HC Slide Rule, $[\lambda]_\delta = [\{\bigstar\}]_\delta$ where $\bigstar$ is the base point of λ. Thus by the above proposition, $[\Lambda]_\delta = [\{(\bigstar, ✠)\}]_\delta$. Therefore by the HC Slide Rule $hc(\Lambda) < \delta$, a contradiction of our hypothesis.

Case 2: $hc(\lambda) < hc(\Lambda)$

Let $\delta \in \mathbb{R}$ such that $hc(\Lambda) < \delta < hc(\lambda)$. Then by the HC Slide Rule, $[\Lambda]_\delta = [\{(\bigstar, ✠)\}]_\delta$. Examine the homotopy from $\Lambda = \{(x_0, ✠), ..., (x_n, ✠)\}$ to $\{(\bigstar, ✠)\}$. To go from Λ to the trivial loop, we must first either add a point (x_k, y_k) to Λ or remove a point from Λ. First we will consider when we add the point (x_k, y_k).

Since we are able to add the point (x_k, y_k) to our loop Λ, we know that $D((x_i, y_i), (x_k, y_k)) = \sqrt{d_X(x_i, x_k)^2 + d_Y(y_i, y_k)^2} < \delta$ and $D((x_{i+1}, y_{i+1}), (x_k, y_k)) = \sqrt{d_X(x_i, x_k)^2 + d_Y(y_i, y_k)^2} < \delta$. Notice that $d_X(x_i, x_k) < \delta$ and $d_X(x_{i+1}, x_k) < \delta$ from this statement since we are adding a non-negative number with our metric inside the radical. So adding the point x_k to the loop λ in X is a valid basic move.

Now suppose we had to remove the point (x_k, y_k) from our loop Λ.

Then $D((x_{k-1}, y_{k-1}), (x_{k+1}, y_{k+1})) = \sqrt{d_X(x_{k-1}, x_{k+1})^2 + d_Y(y_{k-1}, y_{k+1})^2} < \delta$. By the same arguement as above, we see that $d_X(x_{k-1}, x_{k+1}) < \delta$. So removing the point x_k is a valid basic move for λ in X.

From this, we can conclude that any basic move on Λ is also a basic move for λ when we project to X. So we have found a δ-homotopy where λ is transformed to $\{\bigstar\}$. Then $\delta > hc(\lambda)$, a contradiction.

Therefore $hc(\lambda) = hc(\Lambda)$. ■

Corollary 40. *Let X and Y be metric spaces with base points $\bigstar$ and ✠ respectively. Then $hc(X) \subseteq hc(X \times Y)$ and $hc(Y) \subseteq hc(X \times Y)$.*

With this fact, we can greatly simplify our work when we study products of metric spaces. Once we know the homotopy critical spectrum of our constituent spaces, we have at least a part of the homotopy critical spectrum of the product.

Section VI: Ultrametric Properties of Homotopy and Refinement Critical Values

In the first section, we examined metric spaces. All metric spaces satisfy certain conditions. If we weaken or strengthen these conditions, we get some interesting geometry in our spaces. First, what could happen if we weaken one of our conditions?

Definition 41. *A* ***pseudometric space*** (X, d) *is a set* X *along with a function* $d : X \times X \to \mathbb{R}$ *that satisfies the following properties for all* $x, y, z \in X$*:*

1. $d(x, y) = d(y, x)$
2. $d(x, z) \leq d(x, y) + d(y, z)$
3. $d(x, x) = 0$

The difference between a pseudometric space and a metric space is the fact that we are permitted to have "zero distance" between two distinct points. In $\mathbb{R}^2$, the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that $d(x, y) = |x|$ is a pseudometric. The points $(0, 1)$ and $(0, 0)$ are distinct points, but their distance from each other is 0. Now, what if we strengthen our conditions?

Definition 42. *An* ***ultrametric space*** (X, d) *is a set* X *along with a function* $d : X \times X \to \mathbb{R}$ *that satisfies the following properties for all* $x, y, z \in X$*:*

1. Positive Definite: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. Symmetric: $d(x, y) = d(y, x)$
3. Ultrametric: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

The third property of the ultrametric implies the triangle inequality of an ordinary metric space since $\max\{d(x, y), d(y, z)\} \leq d(x, y) + d(y, z)$. Let X be a set with the discrete metric $d : X \times X \to \mathbb{R}$ where:

$$d(x, y) = \{^{0 \text{ if } x=y}_{1 \text{ if } x \neq y}$$

is an ultrametric space. The positive definite property follows immediately from the definition of the metric. Notice $d(x, y) = 1 = d(y, x)$ if $x \neq y$, and if $x = y$, $d(x, y) = d(x, x) = d(y, x)$. So we have the symmetric property. Suppose $x, y, z \in X$. Then $d(x, z) = 0$ or $d(x, z) = 1$. Likewise $d(y, z)$ and $d(x, y)$ are either equal to 0 or equal to 1 depending on if $y = x$ or $y = z$. If $d(x, z) = 0$, then the positive definite property gives us the ultrametric property. If $d(x, z) = 1$, then either $d(x, y)$ or $d(y, z)$ is also equal to 1 because y cannot be simultaneously equal to both the distinct points x and z. So since 1 is the maximum distance that can be obtained in X, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. Therefore the ultrametric property is satisfied.

The following is a fun fact in an ultrametric space if you define a triangle as any three distinct non-linear points.

Proposition 43. *All triangles are isosceles in an ultrametric space.*

Proof. Let a, b, and c be the vertices of a triangle. Then from the ultrametric property we have:

$$\begin{aligned} (1) \quad d(a, b) &\leq \max\{d(a, c), d(b, c)\} \\ (2) \quad d(b, c) &\leq \max\{d(a, b), d(a, c)\} \\ (3) \quad d(a, c) &\leq \max\{d(a, b), d(b, c)\} \end{aligned}$$

Without loss of generality, we can assume that the length of the side between vertices b and c is either less than or equal to the length of both of the other sides, i.e. no side is shorter than side

bc. From statement (1) we have $d(a,b) \leq d(a,c)$ since $d(b,c) \leq d(a,b)$. From statement (3) we have $d(a,c) \leq d(a,b)$ since $d(b,c) \leq d(a,b)$. Therefore, $d(a,b) = d(a,c)$. So the lengths of two sides of the triangle are equal. By definition, we must have an isosceles triangle. ■

Thoughout our work with ε-chains, we have been coming across properties that closely resemble the ultrametric property. We can use these properties to define ultrametrics on the abstract spaces constructed with chains.

Proposition 44. *Let C be the set of all ε-chains beginning at a point $\bigstar$ in a metric space X. Then $d : C \times C \to \mathbb{R}$ where $d(\alpha, \beta) = |\overline{\alpha} * \beta|_{\max}$ is an ultrametric.*

Proof. From the properties of the max norm, we have that d is positive definite. Let α,$\beta \in C$. Then $d(\alpha, \beta) = |\overline{\alpha} * \beta|_{\max}$. Again from the basic properties of the max norm, we have that $|\overline{\alpha} * \beta|_{\max} = |\overline{\overline{\alpha} * \beta}|_{\max} = |\overline{\beta} * \alpha|_{\max} = d(\beta, \alpha)$. Thus d is symmetric. For the triangle inequality, let $\alpha, \beta, \gamma \in C$. We claim that $|\overline{\alpha} * \beta|_{\max} \leq |\overline{\alpha} * \gamma * \overline{\gamma} * \beta|_{\max}$. Notice that $|\overline{\alpha} * \beta|_{\max} = \max\{|\alpha|_{\max}, |\beta|_{\max}\}$. Likewise $|\overline{\alpha} * \gamma * \overline{\gamma} * \beta|_{\max} = \max\{|\overline{\alpha} * \gamma|_{\max}, |\overline{\gamma} * \beta|_{\max}\} = \max\{|\alpha|_{\max}, |\gamma|_{\max}, |\beta|_{\max}\}$. Therefore $|\overline{\alpha} * \beta|_{\max} \leq |\overline{\alpha} * \gamma * \overline{\gamma} * \beta|_{\max}$. Then $d(\alpha, \beta) = |\overline{\alpha} * \beta|_{\max} \leq |\overline{\alpha} * \gamma * \overline{\gamma} * \beta|_{\max} \leq \max\{|\overline{\alpha} * \gamma|_{\max}, |\overline{\gamma} * \beta|_{\max}\} = \max\{d(\alpha, \gamma), d(\gamma, \beta)\}$. Thus, d is an ultrametric. ■

Proposition 45. *Let X be a pointed metric space with base point $\bigstar$. Define a function $d : \Pi_\varepsilon(X) \times \Pi_\varepsilon(X) \to \mathbb{R}$ by $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = hc([\overline{\alpha} * \beta]_\varepsilon)$. Then d is an ultrametric.*

Proof. Let $[\alpha]_\varepsilon, [\beta]_\varepsilon \in \Pi_\varepsilon(X)$ where $[\alpha]_\varepsilon \neq [\beta]_\varepsilon$. Then $d([\alpha]_\varepsilon, [\alpha]_\varepsilon) = hc([\overline{\alpha} * \alpha]_\varepsilon) = hc([\bigstar]_\varepsilon) = 0$. Furthermore suppose not, i.e. $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = hc([\overline{\alpha} * \beta]_\varepsilon) = 0$. By the HC Slide Rule, $[\overline{\alpha} * \beta]_\varepsilon = [\bigstar]_\varepsilon$, hence $[\overline{\alpha}]_\varepsilon * [\beta]_\varepsilon = [\bigstar]_\varepsilon \Rightarrow [\alpha]_\varepsilon * [\overline{\alpha}]_\varepsilon * [\beta]_\varepsilon = [\alpha]_\varepsilon * [\bigstar]_\varepsilon \Rightarrow [\bigstar]_\varepsilon * [\beta]_\varepsilon = [\alpha]_\varepsilon \Rightarrow [\beta]_\varepsilon = [\alpha]_\varepsilon$. So we have a contradiction. Therefore, d has the positive definite property. Notice that $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = hc([\overline{\alpha} * \beta]_\varepsilon) = hc([\overline{\overline{\alpha} * \beta}]_\varepsilon) = hc([\overline{\beta} * \alpha]_\varepsilon) = d([\beta]_\varepsilon, [\alpha]_\varepsilon)$. So d has the symmetric property. Finally, $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = hc([\overline{\alpha} * \beta]_\varepsilon) \leq \max(hc([\overline{\alpha}]_\varepsilon, hc([\beta]_\varepsilon)\}$. Thus since $hc([\alpha]_\varepsilon) = hc([\overline{\alpha}]_\varepsilon)$, d has the ultrametric property. Therefore, d is an ultrametric. ■

Proposition 46. *Let X be a pointed metric space with base point $\bigstar$. Define a function $d : X_\varepsilon \times X_\varepsilon \to \mathbb{R}$ by $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = rc([\overline{\alpha} * \beta]_\varepsilon)$. Then d is a pseudometric with the property $d([\alpha]_\varepsilon, [\beta]_\varepsilon) \leq \max\{d([\alpha]_\varepsilon, [\gamma]_\varepsilon), d([\gamma]_\varepsilon, [\beta]_\varepsilon)\}$ for any $[\alpha]_\varepsilon, [\beta]_\varepsilon, [\gamma]_\varepsilon \in \Pi_\varepsilon(X)$.*

Proof. Let $[\alpha]_\varepsilon, [\beta]_\varepsilon, [\gamma]_\varepsilon \in \Pi_\varepsilon(X)$. Then $d([\alpha]_\varepsilon, [\alpha]_\varepsilon) = rc([\overline{\alpha} * \alpha]_\varepsilon) = rc([\{\bigstar\}]_\varepsilon) = 0$. Next, notice

$$\begin{aligned} d([\alpha]_\varepsilon, [\beta]_\varepsilon) &= rc([\overline{\alpha} * \beta]_\varepsilon) \\ &= rc([\overline{\overline{\alpha} * \beta}]_\varepsilon) \\ &= rc([\overline{\beta} * \alpha]_\varepsilon) \\ &= d([\beta]_\varepsilon, [\alpha]_\varepsilon) \end{aligned}$$

Finally, from our earlier work, we have $d([\alpha]_\varepsilon, [\beta]_\varepsilon) = rc([\overline{\alpha} * \beta]_\varepsilon) = rc([\overline{\alpha} * \gamma * \overline{\gamma} * \beta]_\varepsilon) \leq \max\{rc([\overline{\alpha} * \gamma]_\varepsilon), rc([\overline{\gamma} * \beta]_\varepsilon)\} = \max\{d([\alpha]_\varepsilon, [\gamma]_\varepsilon), d([\gamma]_\varepsilon, [\beta]_\varepsilon)\}$. Recall that the ultrametric property implies the triangle inequality, so d is a pseudometric. ■

Notice that using the refinement critical value, we do not in general get to assume the positive definite property of a metric space. This is because even if we have two chains with refinement critical values of 0, these chains are not forced to be homotopic to one another. For example in $\mathbb{R}$, $\alpha = \{0, 1, 2, 3\}$ and $\beta = \{0, -1, -2, -3\}$ are both elements of $\mathbb{R}_\varepsilon$ for $\varepsilon = 1.5$. However, α and β do not have the same endpoints. It is impossible for α to be homotopic to β even though the distance between $[\alpha]_\varepsilon$ and $[\beta]_\varepsilon$ is 0 using the refinement critical value pseudometric.

With two chains that have 0 as a homotopy critical value, these chains are both homotopic to the trivial loop, hence homotopic to each other. This fact gives us the positive definite property using the homotopy critical value, but the refinement critical value does not have a similar result.

In conclusion, there is still much work to be done regarding ε-chains and the homotopy and refinement critical values. A pseudometric space with the ultrametric property like the one formed with the refinement critical value can be made into an ultrametric using quotients of spaces. Furthermore, proving that two chains are not homotopic to one another is difficult at best. It would be helpful to have a simple method for determining if two chains are not homotopic. One way to attempt this is to develop theory in regards to ε-chains for something analogous to covering spaces when using paths. Easily determining when chains are and are not homotopic to each other will greatly enhance our ability to calculate the critical values of our chains. Regardless, it is clear that ε-chains are a useful tool in addition to paths on metric spaces.

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