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Blackbody Radiation Within a Closed Cylindrical Capsule:
An Analysis of the Environment Within a SEE-Type Experimental Capsule

A Senior Honors Project
In Partial Fulfillment of
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in Physics
The University of Tennessee, Knoxville

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Department of Physics and Astronomy
Blackbody Radiation Within a Closed Cylindrical Capsule:  
An Analysis of the Environment Within a SEE-Type Experimental Capsule

Introduction

Over the course of the previous semester and this summer, we have constructed a series of computer simulations in order to examine the effects the spatial variation of emissivity upon of blackbody radiation field within the experimental chamber of a SEE-type observatory.

Project SEE (Satellite Energy Exchange) is designed to use an orbital experimental platform to make a number of fundamental tests in gravitation including measurements of the rate of change of Newton’s gravitational constant, \( \dot{G} \), over time, as well as \( G \) itself. It will also test aspects of General Relativity and aid in the search for new large-scale forces in addition to the four forces of nature already known to physics. By making these measurements in the environment of space, the experiment is potentially isolated from vibration and other forms of interference that are unavoidable on the earth’s surface, allowing measurements of \( G \) with an error of less than one part per million and a measurement of \( \dot{G}/G \) with an error of less than one part in \( 10^{14} \) per year (1).

SEE-type measurements also hold the special advantage that the method for measuring \( \dot{G} \) is not to measure \( G \) to one part in \( 10^{14} \) at different times and then take the difference. It is certainly not possible at this time to test for \( \dot{G} \) to one part per million in \( G \) per year, much less \( 10^{14} \). Instead, \( \dot{G} \), if it exists can be measured by examining the change in the orbital period of a large test body, a 200 kg “shepherd”, over several years. The other test body is a lighter 100 g “particle”.

Blackbody radiation could be a significant source of error on account of the sensitivity required of any apparatus designed to make fine measurements of gravity.
(2 and 3). It is possible that variations in the thermal emissivity of the interior of the experimental capsule could cause apparent "hot spots" that would produce more radiation than surrounding areas, leading to a significant net force from radiation pressure on the test masses. A time-averaged force upon the shepherd in the axial direction as small as \( 1.5 \times 10^{-16} \text{ N} \) (2) would introduce enough error in the measurement of \( \dot{G} \) to negate the advantage gained by taking measurements in space instead of on the earth's surface.

**Specific Investigations**

In order to predict the effects of such radiation on the experiment, we constructed several programs to simulate the conditions within the experimental capsule:

1) A program designed to find the total force due to blackbody radiation pressure upon a test mass situated on the central axis of the observatory. This program assumes that there are 314,000 domains of different emissivity \( \pm \Delta \epsilon \) distributed randomly on the cylinder's inner surface. It is the simplest of the three complete programs in that it does not take account of reflections. The temperature of the radiating regions will not vary spatially nor with time. This is equivalent to assuming that the capsule is thermally coupled to a constant temperature bath. This simplification will lead to an overestimation of the radiation pressure, since regions of higher emissivity would in fact cool as they radiate heat, and so produce less radiation as the capsule approaches thermal equilibrium.

2) A two-dimensional Monte Carlo simulation that maps the momentum exchange due to radiation from a single source across the whole area of a rectangular box 10 m long and 2 m tall. This simulation takes account of reflected radiation.

3) Another two-dimensional Monte Carlo simulation that examines a more computationally complicated two-dimensional circular box in the same manner as the
second program.

4) A three-dimensional Monte Carlo simulation is nearly complete at submission time. It will carry out a more thorough calculation of the momentum exchange experienced by bodies at all points within the experimental capsule due to numerous radiating regions of varying emissivity. It will use techniques developed for programs 2 and 3 in order to take account of reflected radiation. It will also assume that the capsule is thermally coupled to a constant temperature bath. Since the shepherd will block or scatter approximately 10% of the cylinder’s cross section, the three-dimensional simulation will also take into account reflections and absorptions due to the shepherd.

**First Simulation**

As stated above, the first simulation disregards reflections, considering only the effects of the initial emissions, greatly simplifying the calculation. This permitted us to find a “ballpark” figure for the forces caused by variations in thermal emissivity. The program calculates the force upon the test mass by dividing up the interior of the cylindrical chamber into quasi-rectangles of equal area, randomly distributing variations in emissivity which were equal in magnitude but varied in sign (positive or negative) among the areas, then calculating the sum of intensities of radiation falling on the test mass from each individual region. If we assume that the test masses are a perfectly light-absorbing black, then the force upon the test mass is exactly equal to the energy of this radiation.

Parameter choices for the simulations are as follows: The temperature is assumed to be 78 K in all simulations. The emissivity deviation is assigned the conservatively high nominal value of $\Delta \varepsilon = \pm 0.01$ in each domain, with the sign chosen randomly. Thus, the RMS deviation of emissivity is ipso facto $\Delta \varepsilon$. We choose the domain size $\sim 1$ square
centimeter in the form of quasi-squares ~1 cm on a side. To expedite the simulations, we group the squares into rings around the (cylindrical) wall of the experimental chamber. Thus, each ring is 1 cm wide in the axial direction, has a length of \(2\pi r = 3.14159 \text{ m}\), and is divided into 314 quasi-squares of width in the azimuthal direction \(\frac{2\pi}{314} = 1.0005\) cm. The rotation of the cylinder effectively averages over all the domains in each ring, so that the net emissivity deviation for the entire ring is \(\Delta\epsilon = \pm 0.000565\). This also allows us to treat 1,000 rings instead of 314,000 separate square domains.

The power of the radiation emitted from each ring is calculated from the Stefan-Boltzmann law:

\[
P = \Delta\epsilon \sigma T^4
\]

where \(\sigma = 5.67033 \times 10^{-8} \text{ W/(m}^2\text{K}^4)\) is the Stefan-Boltzmann constant. The area \(A\) of each ring is \(0.031416\) \(\text{m}^2\) and the net variation in emissivity is \(\Delta\epsilon = \pm 0.000565\). We note that this treatment of the force due to the rings of emissivity domains is very closely analogous to the treatment of force due to rings of mass defects, as described in Ref. (1).

Figures 2 through 5 in the present paper are analogous to Fig. 3 of Ref. (1), bold curve.

To find the power emitted by an infinitesimal unit of area toward an object at angle \(\varphi\):

\[
dp = \frac{1}{2} \Delta\epsilon \sigma T^4 \, dA \cos \varphi \, d\varphi
\]

Since the radius of the test mass will be small compared to \(r\), \(dl\) can be estimated as:

\[
dl \approx \frac{P}{\pi r^2} = \frac{p}{2\pi (R^2 + x^2)} = \Delta\epsilon \sigma T^4 \, dA \cos \varphi \, d\varphi/4 \pi (R^2 + x^2)
\]

The radiation pressure is:

\[
dP_{\text{radiation}} = dl/c = \Delta\epsilon \sigma T^4 \, dA \cos \varphi \, d\varphi/4 \pi c (R^2 + x^2)
\]
The force exerted by the pressure is proportional to the cross sectional area of the test mass:

\[ dF = (dP_{\text{radiation}})(A_{\text{cross section}}) \]  

Where a test mass, the "particle", with a radius of 2 cm has a cross sectional area of:

\[ A_{\text{cross section}} = 4\pi \times 10^{-4} \text{ m} \]

So the total force from an infinitesimal unit of area is:

\[ dF = A_{\text{cs}} \Delta \varepsilon A_0 T^4 dA \cos \phi \frac{d\phi}{4\pi c(R^2 + x^2)} \]

where \( x \) is the x-axis separation between the domain and the test mass. Now we resolve the force into its component vectors, both along and orthogonal to axial direction along the cylinder:

\[ dF = (dF_{\text{parallel}}^2 + dF_{\text{orthogonal}}^2)^{1/2} \]

Where:

\[ dF_{\text{parallel}} = dF (x/r) = dF \frac{x}{(R^2 + x^2)^{1/2}} \]

Since the radiating area of the cylinder is orbiting about the test mass in a smooth, periodic manner due to the rotation of the capsule, then on average we can say:

\[ dF_{\text{orthogonal}} = 0 \]

Clearly:

\[ dF_{\text{avg}} = dF_{\text{parallel}} \]

So that the total force on the test mass from an infinitesimal unit of area is:

\[ dF_{\text{avg}} = x A_{\text{cs}} \Delta \varepsilon A_0 T^4 dA \cos \phi \frac{d\phi}{4\pi c(R^2 + x^2)^{3/2}} \]

Each domain under consideration on the cylinder will exert a force on the test mass in the
manner described by this equation. In order to get the total force on the test mass, we sum over all the domains of varying emissivity along the surface of the cylinder. If we wanted to find the total force upon the larger 18 cm radius "shepherd" test mass, we would need to multiply the final answer by 77.3.

Figure 1: Size and composition information for both "particle" and "shepherd" test masses.

<table>
<thead>
<tr>
<th>Density (g/cc)</th>
<th>Radius (cm)</th>
<th>Mass (kg)</th>
<th>Cross sectional area (m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shepherd</td>
<td>7.9</td>
<td>18.2</td>
<td>200</td>
</tr>
<tr>
<td>Particle</td>
<td>2.7</td>
<td>20.7</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Figures 2 through 5 contain graphs of the estimated force on the "particle" in the axial direction for typical distributions of domains of varying emissivity using the described methods. In Figure 6, we see that the momentum transfer, in the case where reflections are ignored, approaches the range where it may cause problems with the experiment.

The relationship between the RMS force upon the shepherd along the axis, $\Delta\varepsilon$, and the domain area can be approximated by the equation:

$$F_{RMS} \approx 4.19 \times 10^{-14} \times \Delta\varepsilon \times \sqrt{A}$$

where $A$ is in cm². For example, $F_{RMS} = 419 \times 10^{-18}$ for $A = 1$ cm² and $\Delta\varepsilon = 0.01$, as shown in Figure 6. As illustrated in Figure 7, we find that larger domain sizes can be permitted on the interior of the capsule if the emissivity variation is small enough. Thus, if $\Delta\varepsilon = 0.0001$, a domain size of 600 cm² would result in an RMS force on the Shepherd of only $10^{-16}$ N, which is acceptable for a measurement of $\dot{G}/G$ with an accuracy of $10^{-14}$/yr (2).
Figures 2 - 5: Typical force on "particle" test mass along axis according to the first simulation program (in units of $10^{-18}$ N). The X displacement is the distance from one end of the cylinder, between 0 and 10 m.

Figure 6: Estimated Root Mean Squared Force on test masses lying on the central axis, as calculated by first simulation program

<table>
<thead>
<tr>
<th>Run</th>
<th>On Particle</th>
<th>On Shepherd</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.92</td>
<td>458</td>
</tr>
<tr>
<td>2</td>
<td>4.05</td>
<td>313</td>
</tr>
<tr>
<td>3</td>
<td>5.53</td>
<td>427</td>
</tr>
<tr>
<td>4</td>
<td>6.16</td>
<td>476</td>
</tr>
<tr>
<td>Average</td>
<td>5.414</td>
<td>418.5</td>
</tr>
</tbody>
</table>
Second and Third Simulations

Like the first program, the two-dimensional simulations were constructed in C/C++. First, a series of several tens of thousands of light rays are emitted from a radiation source which obeys Lambert's cosine law (Note: I think this is OK, since this statement describes the program instead of the real physical system), which states that the intensity of emission from a "Lambertian" source varies as the cosine of the viewing angle with respect to a line normal to the emitting surface. This distribution was produced by first choosing a number in the range (0, 1) using the ran2 random number generator function from Numerical Recipes in C++ (2). This function chooses numbers according to a uniform distribution, so that every number between 0 and 1 has an equal probability of being chosen. Using simple arithmetic we map these numbers onto the range (-1, 1), and then choose the emission angle by finding the inverse of the antiderivative of our desired distribution, cos θ, where θ is the emission angle with respect to the surface normal. For reasons that will be clear shortly, we will measure our emission angle from the positive x-axis instead of the normal, as would usually
be done. Let $\phi$ represent this angle. Then our desired distribution is

\[(14) \quad f(\phi) = \sin \phi = \cos \theta\]

which should equal the output of ran2, and the emission angles are

\[(15) \quad \phi = F^{-1}(x) = \arccos(x)\]

where $x$ is the output of the ran2 function mapped onto the range (-1, 1).

Each ray carries an equal fraction of the total energy emitted by the source, and each ray is reflected multiple tens of times, losing energy with each reflection. We approximate the variation of $R$ with the incidence angle by the equation:

\[(16) \quad R = R_0 + (1 - R_0) \sin^a \theta\]

This equation gives a good estimate for the combined reflection of light in both the incident and perpendicular planes from a high-reflectivity surface as given by Fresnel. Here $R$ is the percentage of the light ray’s energy reflected each impact, $R_0$ is the “base” reflectivity at $\theta = 0$ equal to 1 minus the emissivity, and the value of the parameter $n$ is chosen to correspond to a particular material.

This continues until the light ray is degraded to one percent of its original strength, at which point a new ray is emitted and traced by the program. The ray itself is stored as a combination of the coefficients of the linear equation

\[(17) \quad y = ax + b\]

where $a$ is the slope and $b$ is the $y$ intercept, as well as a two-dimensional unit vector that stores the direction of travel. When the ray strikes a flat wall, as in the rectangular model, reflections can be modeled by simply negating the value of the slope $a$, then solving for the new $y$ intercept $b$ and negating the appropriate coordinate of the direction vector. Figures 8 through 11 describe the simulated momentum field due to radiation from a single source inside of a rectangular cavity.
Figures 8-9: These graphs contain data representing the x momentum transfer field due to a radiating area, coordinates (0, 0.5), centered at the top of a rectangular cavity having reflectivity of 0 and 0.99. The vertical axis represents the x-momentum transfer in arbitrary units. The horizontal axis represents the location in the "radial" y direction of the test point within a rectangular cavity 1 m high. The "axial" x axis location is denoted by color and can be found using the legend to the right of the chart. Note that the radiation field becomes very smooth at all points past 0.5 m in the axial direction. The high reflectivity graph has 6th degree fits past 0.22 m due to roughness of data.

X-Momentum Density versus Y-axis position for selected X-axis positions:
A simulation using 1000000 photon paths in a rectangular cavity

R = 0.0

X-Momentum Density versus Y-axis position for selected X-axis positions:
A simulation using 100000 photon paths in a rectangular cavity

R = 0.99
Figures 10-11: These graphs compare the average momentum transfer across 251 planes placed parallel with the y-axis to their x-axis positions. Without reflection, the momentum transfer from the radiation source decreases approximately as an inverse power of distance in the x direction, whereas the high-reflectivity surface decreases linearly with distance.

Average Momentum transfer through planes across X-axis
versus X-axis location
10k Emissions, 0.0 R

Average Momentum transfer through planes across X-axis
versus X-axis location
10k Emissions, 0.99 R

X-axis position (meters)
Reflections on the curved surface of a circle are a little more complicated. First, we solve the system of equations containing the equation defining the circle

\[(18)\quad x^2 + y^2 = r^2\]

(where \(r\) is the radius of the circular cavity) and the linear equation \((8)\) that defines the current light ray, which gives two points in Cartesian \((x, y)\) coordinates. The point that is not the previous emission or reflection point is now the current reflection point. We consider this point as a vector, and then find the angle it makes with the positive \(x\) axis. Let us call this new angle \(\alpha\). We then rotate both intersection points via the coordinate transformations

\[(19)\quad X' = X \cos \alpha + Y \sin \alpha\]
\[(20)\quad Y' = -X \sin \alpha + Y \cos \alpha\]

From these two points we can determine a new linear equation

\[(21)\quad y = a'x + b'\]

where \(a'\) and \(b'\) are the slope and \(y\) intercept of the current ray in the rotated coordinate system. The advantage of this new equation is that we can find its reflection in the rotated coordinate system by negating \(a'\) and solving for \(b'\) in the same way as was done in creating the original light ray. This is possible because we chose a coordinate system that places the reflection point, and thus the normal line of the surface, along the positive \(x\) axis. We choose an arbitrary point along the line so that we have two points on the new reflected line, then we rotate the coordinates back into the original coordinate systems using the reverse translations

\[(22)\quad X = X' \cos \alpha - Y' \sin \alpha\]
(23) \[ Y = X' \sin \alpha + Y' \cos \alpha \]

It is a simple matter to find the new \( a \) and \( b \) in the unrotated coordinate system. We can find the direction vector from these two points (the reflection point and the arbitrary point) by normalizing the differences in the \( x \) and \( y \) coordinates.

Once the parameters that define the current reflection of the light ray have been found, we must input their values into the histogram. The histogram of the rectangular model is a simple grid, and each bin holds three values: the \( x \) momentum, \( y \) momentum, and total scalar momentum. A hit is counted whenever the line intersects the border of a rectangular bin, at which point the momentum of the line is added to the values stored in that bin. The circular model, though, is again more complicated. The bin system is designed with the three-dimensional model in mind, so that a simple combination of the rectangular and circular models will yield the full simulation. The histogram bins for the circular model were devised by Dr. Alvin Sanders so that each one would have an equal area. They come in annular sets, where the outer radius of the \( n^{th} \) annular set of bins is

(24) \[ r_n = \sqrt{(1 + 2n(n + 1))r_{\text{circle}}/(1 + 2N(N + 1))} \]

where \( N \) is the total number of annuli, and the central disk is represented by \( n = 0 \). The bins in the two dimensional case, however, are not two dimensional, but one dimensional, so that the walls of the “bricks” are the actual bins. This change was made is because of problems arising from the line leaving and then reentering the curved 2D bins. This will not happen in three dimensions, however, since the intersection of the line and the 2D bin will be defined by a point, and not the whole line as in the 2D case. This system has the added bonus of allowing us to find the average total momentum transfer at a particular radius, which is the same as the time averaged momentum transfer at that point. Figure
12 shows some values for a run of 100,000 emissions within a cavity of radius 1 m and with reflectivity of 0.95. Note that the program outputs are in arbitrary units, not SI, on account of the dimensionality of the simulation.

Figure 12: Typical values of momentum exchange (in arbitrary units) inside of a circular cavity

<table>
<thead>
<tr>
<th>Reflectivity = 0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>R(n) (meters)</td>
</tr>
<tr>
<td>0.007886</td>
</tr>
<tr>
<td>0.052428</td>
</tr>
<tr>
<td>0.127206</td>
</tr>
<tr>
<td>0.251882</td>
</tr>
<tr>
<td>0.376566</td>
</tr>
<tr>
<td>0.501252</td>
</tr>
<tr>
<td>0.625938</td>
</tr>
<tr>
<td>0.750625</td>
</tr>
<tr>
<td>0.875313</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

The final step is to produce a visualization of the data. The drawing routine was made using OpenGL, which is a standard graphical protocol used by modern computers. We assigned pixels to each bin in the histogram, and then gave them a color code that varies as the sine of the total momentum transfer in that bin normalized to $2\pi$. In addition, we implemented code to allow the user to zoom into and out of the picture in order to bring out additional detail. Figures 13 through 15 display a sampling of visualizations of outputs from both the 2D rectangular and circular models.

For the simulation of a circular cavity, light will be focused in certain areas of the cavity, leading to bright focal points. In addition, spherical aberration will cause heart-
shaped frustums where radiation density will be higher than surrounding regions. The frustums lead into the focal points. The first focus will be located at a distance of $1/3$ of the circle’s radius from the center according to the mirror equation for concave mirrors:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

where $f$ is the focal length measured from the mirror’s center, $d_o$ is the distance from the mirror to the object, and $d_i$ is the distance from the mirror to the object’s image. We see in Figures 14 and 15 that each focal point acts as another source, producing further focal points as far as the simulation’s resolution would allow. These occur at distances of $r/5$, $r/7$, $r/9$, $r/11$, and so forth, from the center. This effect can be seen in Figures 14 and 15.

Figure 13: Visualization of x-axis momentum transfer due to a single radiation source. Blue indicates momentum transfer in negative x direction, red indicates momentum transfer in positive x direction. Reflectivity is 0.9. Simulation tracked 50,000 emissions.
Figure 14: Simulation of circular cavity with reflectivity of 0.8 and 10,000 emissions. Red indicates regions of highest momentum transfer, blue denotes regions of lower momentum transfer. The cusps of the heart-shaped curves (cardioids) are the focal points of a concave spherical mirror. In addition to the focal point for the source, each focal point also acts as a source, producing its own focal point.

Figure 15: Higher resolution version of above simulation using reversed color scheme.
Conclusions

The results of the two dimensional cases give us considerable insight into how the real three-dimensional physical system should behave. According to the rectangular model, we find that the effects of a given region of varying emissivity drops off to nearly zero once the distance from the radiation source grows much larger than about half a meter, which is the radius of the chamber. This means that an even distribution of defects in emissivity would reduce the total momentum transfer on the test masses even more than would normally be expected, since only the nearest ones will have a large effect.

Also important is the result described by equation (13) and Figure 7, which describe the relationship between magnitude of $\Delta \varepsilon$ and the size of the domains of variation.

The rectangular simulation raised as many questions as it answered. Previous publications on SEE expected that high reflectivities would tend to homogenize the radiation field, leading to a reduction in the total force on the test masses. Our simulations find the opposite, however. Higher reflectivities generally lead to stronger forces on the test bodies at significant distances (~ 0.5 m), from the radiation source although at short distances reflections seem to have little effect. Also puzzling is the change (1) to a linear drop-off of momentum transfer from (2) a drop-off that goes approximately with an inverse power of increasing distance along the x-axis when reflectivity approaches 1.

In the circular case, we can clearly see the foci of the circular mirror. There was some concern that these foci would cause regions of unusually high momentum transfer,
but our findings show that these average out completely almost everywhere over the course of a capsule rotation except for at the center (\(< 0.2\) m from the center), where there is a significant peak.

At the time of the submission of this paper, the three dimensional model is nearly complete, requiring only the completion of the histogram and graphics routines. It will give a mapping of momentum transfer in SI units for the whole interior of the cylinder, and should answer once and for all the question of whether or not blackbody radiation will disrupt an attempt at a space-based measurement of \(G\) and related quantities using a SEE-type apparatus.
Works Cited


Bibliography


