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Fractions of Numerical Semigroups

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To the Graduate Council:

I am submitting herewith a dissertation written by Harold Justin Smith entitled "Fractions of Numerical Semigroups." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

David E. Dobbs, Major Professor

We have read this dissertation and recommend its acceptance:

David F. Anderson, Pavlos Tzermias, Michael W. Berry

Accepted for the Council:
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Fractions of Numerical Semigroups

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Harold Justin Smith
May 2010

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Abstract

Let S and T be numerical semigroups and let k be a positive integer. We say that S is the *quotient of* T *by* k if an integer x belongs to S if and only if kx belongs to T. Given any integer k larger than 1 (resp., larger than 2), every numerical semigroup S is the quotient T/k of infinitely many symmetric (resp., pseudo-symmetric) numerical semigroups T by k. Related examples, probabilistic results, and applications to ring theory are shown.

Given an arbitrary positive integer k, it is not true in general that every numerical semigroup S is the quotient of infinitely many numerical semigroups of maximal embedding dimension by k. In fact, a numerical semigroup S is the quotient of infinitely many numerical semigroups of maximal embedding dimension by each positive integer k larger than 1 if and only if S is itself of maximal embedding dimension. Nevertheless, for each numerical semigroup S, for all sufficiently large positive integers k, S is the quotient of a numerical semigroup of maximal embedding dimension by k. Related results and examples are also given.

Contents

Chapter 1

Introduction

1.1 Numerical Semigroups

A **semigroup** is a set S with an associative binary operation $S \times S \to S$. In this dissertation, this binary operation will always be assumed to be commutative and will be denoted using additive notation. A **monoid** is a semigroup S which contains an identity element, denoted 0, having the property that $s + 0 = 0 + s = s$ for all $s \in S$. Note that the set of nonnegative integers $\{0, 1, 2, \ldots\}$ under addition is a monoid. In this dissertation, the symbol $\mathbb{N}$ will be used to denote the set of nonnegative integers. Moreover, the symbols $\subseteq$ and $\subset$ will denote, respectively, set-theoretic inclusion and proper inclusion.

Let S be a semigroup and let $T \subseteq S$. We say that T is a **subsemigroup** of S if T is closed under the binary operation defined on S. If S is a monoid with identity element 0, a **submonoid** of S is a subsemigroup T of S which contains 0. We adopt the conventions that the empty set, $\emptyset$, is a semigroup (typically called the trivial semigroup) and that the set $\{0\}$ is a monoid. Hence, it is easy to see that an arbitrary intersection of subsemigroups (resp., submonoids) of S is also a subsemigroup (resp., submonoid) of S.

Given two semigroups S and T, a **semigroup homomorphism** from S to T is a function $f : S \to T$ such that for all $a, b \in S$, $f(a + b) = f(a) + f(b)$. If S and T are monoids, then f is a **monoid homomorphism** if, in addition, $f(0) = 0$. The map f is a **monomorphism** (resp., **epimorphism**; resp., **isomorphism**) if f is injective (resp., surjective; resp., bijective). More information on commutative semigroups may be found in [10] and [4].

This dissertation will be primarily concerned with a special class of monoids called numerical semigroups. A **numerical semigroup** is a submonoid S of $\mathbb{N}$ such that $\mathbb{N} \backslash S$ is finite. Numerical semigroups first arose in problems from classical number theory posed by J. J. Sylvester and Ferdinand Georg Frobenius in the late 19th century. Because of the number of open problems and because of applications to other areas such as electrical engineering, differential equations and algebraic geometry, numerical semigroups have remained the focus of extensive study. In recent years, the study of factorization in integral domains has brought attention to cancellative commutative monoids and presentations of monoids. Numerical semigroups, which are themselves cancellative commutative monoids whose presentations have been extensively researched, serve as a convenient source of examples for this study.

Let S be a numerical semigroup. If T is a submonoid of S which is also a numerical semigroup, we say that T is a **subnumerical semigroup** of S and that S is an **overnumerical semigroup** of T. If S is a numerical semigroup, there exist infinitely many subnumerical semigroups of S and only finitely many overnumerical semigroups of S. Moreover, it is easy to see that a finite intersection of subnumerical semigroups of S is also a subnumerical semigroup of S. Note that an arbitrary intersection of numerical semigroups may not be a numerical semigroup. Indeed, if for each $n \in \mathbb{N}$ we let $S_n := \{x \in \mathbb{N} \mid x \geq n\} \cup \{0\}$ and define $T := \cap_{n\in\mathbb{N}} S_n$, then $T = \{0\}$ does not have finite complement in $\mathbb{N}$ and therefore is not a numerical semigroup.

Let S be a numerical semigroup and let $A \subseteq S$ such that $\gcd(A) = 1$ (that is, the greatest common divisor of the elements of A is 1). Then $\langle A\rangle$ is defined to be the intersection of all subnumerical semigroups of S which contain A. It is easy to verify that $\langle A\rangle$ is itself a numerical semigroup. In particular, $\langle A\rangle$ is a subnumerical semigroup of S called **the subnumerical semigroup of S generated by** A. Note that $\langle A\rangle = \{\sum_{i=1}^{m} n_i a_i \mid 1 \leq m \in \mathbb{N}, n_i \in \mathbb{N}, a_i \in A \text{ for all } i\}$.

We now introduce some of the basic invariants of numerical semigroups and some common notation used in numerical semigroup theory. Let S be a numerical semigroup. The elements of $\mathbb{N}\backslash S$ are called the **gaps of** S. Since $\mathbb{N}\backslash S$ is finite, there exists an integer $F(S) := \max(\mathbb{Z}\backslash S)$. We call $F(S)$ the **Frobenius number** of S. Note that $F(S) = -1$ if and only if $S = \mathbb{N}$. If $S \neq \mathbb{N}$, then $F(S)$ is always a positive integer.

The Frobenius number is one of the oldest known invariants in numerical semigroup theory. The first result below, due to J.J. Sylvester, reportedly dates back to 1884. Four different proofs of this famous result may be found in [9].

Theorem 1.1.1 (Sylvester)**.** *Let a and b be relatively prime positive integers. Then $\langle a, b\rangle$ is a numerical semigroup with Frobenius number $ab - a - b$.*

If S is a numerical semigroup and $A \subseteq S$, one can easily show that $\langle A\rangle$ is a numerical semigroup if and only if the greatest common divisor of the elements of A is 1. See Lemma 2.1 of [18] for a proof. If $\langle A\rangle$ is a numerical semigroup, we say that A is a **system of generators** for $\langle A\rangle$ or that A is a **generating set** of $\langle A\rangle$. It is well known that every numerical semigroup is finitely generated and, in fact, has a unique minimal generating set. See [2] or [18] for simple proofs of these statements.

Finally, we note that the class of all numerical semigroups is a denumerable set.

1.2 Maximality Properties of Numerical Semigroups

Let S be a numerical semigroup. Define $N(S) := S \cap \{0, 1, \ldots, F(S)\}$; that is, $N(S) = \{x \in S \mid x < F(S)\}$. Note that $N(S) = \emptyset$ if and only if $S = \mathbb{N}$. In this dissertation, if A is a set, then $|A|$ will denote the cardinality of A. If $n := |N(S)|$, it is often convenient to index the elements of S with $\mathbb{N}$ so that $S = \{s_i\}_{i\in\mathbb{N}}$ and $s_i < s_j$ whenever $i < j$. Thus, $S = \{0 = s_0, s_1, \ldots, s_{n-1}, s_n, \rightarrow\}$ where the notation "$\rightarrow$" will be used to indicate that every nonnegative integer greater than s_n belongs to S. Note that $F(S) = s_n - 1$ and, for all integers $i \geq 0$, $s_{n+i} = s_n + i$.

Let $g \in \mathbb{N}$. Consider the set $\mathcal{S}_g := \{S \mid S \text{ is a numerical semigroup and } F(S) = g\}$. Note that the set $\mathcal{S}_g$ is partially ordered under set-theoretic inclusion. Since each numerical semigroup has only finitely many overnumerical semigroups, it follows that $\mathcal{S}_g$ has a maximal element. If g is odd and if a numerical semigroup S is a maximal element of $\mathcal{S}_g$, we say that S is a **symmetric** numerical semigroup. If g is even and if a numerical semigroup S is a maximal element of $\mathcal{S}_g$, we say that S is a **pseudo-symmetric** numerical semigroup.

Symmetric numerical semigroups exist in abundance. In fact, it is well known that every numerical semigroup generated by two relatively prime positive integers, as in Theorem 1.1.1, is symmetric (see Corollary 4.7 of [18] for a proof). As a numerical semigroup, $\mathbb{N}$ is considered to be symmetric but not pseudo-symmetric. The next two results offer multiple characterizations of symmetric and pseudo-symmetric numerical semigroups. For proofs, see [2], [1] or [9].

Proposition 1.2.1. *Let $g \in \mathbb{N}$ be odd and let S be a numerical semigroup with Frobenius number g. The following are equivalent:*

(1) *S is symmetric;*
(2) *For all $z \in \mathbb{Z}$, either $z \in S$ or $g - z \in S$;*
(3) *The map $S \cap \{0, 1, \ldots, g\} \to (\mathbb{N} \backslash S) \cap \{0, 1, \ldots, g\}$, defined by $s \mapsto g - s$, is a bijection;*
(4) *If x and y are integers and $x + y = g$, then exactly one of x and y belongs to S;*
(5) *$|N(S)| = \frac{g+1}{2}$.*

Proposition 1.2.2. *Let $g \in \mathbb{N}$ be even and let S be a numerical semigroup with Frobenius number g. The following are equivalent:*

(1) *S is pseudo-symmetric;*
(2) *For all $z \in \mathbb{Z}$, either $z \in S$, $g - z \in S$, or $z = \frac{g}{2}$;*
(3) *The map $S \cap \{0, 1, \ldots, \frac{g}{2} - 1, \frac{g}{2} + 1, \ldots, g\} \to (\mathbb{N} \backslash S) \cap \{0, 1, \ldots, \frac{g}{2} - 1, \frac{g}{2} + 1, \ldots, g\}$, defined by $s \mapsto g - s$, is a bijection;*
(4) *If x and y are integers such that $x, y \neq \frac{g}{2}$ and $x + y = g$, then exactly one of x and y belongs to S;*
(5) *$|N(S)| = \frac{g}{2}$.*

Note that condition (2) of Proposition 1.2.1 explains the choice of terminology for symmetric numerical semigroups. Indeed, a symmetric numerical semigroup S with Frobenius number $F(S)$ can be seen as being "symmetric" about the rational number $\frac{F(S)}{2}$; that is, given any two integers a and b equidistant from $\frac{F(S)}{2}$ on the real number line, exactly one of a and b lies in S.

Let S be a numerical semigroup. If S cannot be expressed as an intersection of two numerical semigroups which properly contain S, then S is said to be an **irreducible** numerical semigroup. For example, the numerical semigroup $\langle 3, 8, 10 \rangle = \{0, 3, 6, 8, \rightarrow\}$ is not irreducible since $\langle 3, 8, 10 \rangle = \langle 3, 4 \rangle \cap \langle 3, 5 \rangle$ where $\langle 3, 4 \rangle = \{0, 3, 4, 6, \rightarrow\}$ and $\langle 3, 5 \rangle = \{0, 3, 5, 6, 8, \rightarrow\}$. In [13], Rosales and Branco show that a numerical semigroup S with odd Frobenius number (resp., even Frobenius number) is irreducible if and only if S is symmetric (resp., pseudo-symmetric).

Symmetric numerical semigroups are of particular interest in algebraic geometry. Let K be a field and let a, b and c be positive integers such that $\gcd(a, b, c) = 1$. In [6], Herzog shows that the affine space curve $\{(t^a, t^b, t^c) \mid t \in K\}$ is a global idealtheoretic complete intersection if and only if the associated numerical semigroup $\langle a, b, c \rangle$ is symmetric. The following classic result, due to Kunz, appears in [7] and suggests that some classes of local rings may be studied via associated numerical semigroups.

Theorem 1.2.3 (Kunz). *Let R be a one-dimensional analytically irreducible Noetherian local ring with associated value semigroup S. Then R is a Gorenstein ring if and only if S is symmetric.*

In [1], a pseudo-symmetric analog of Theorem 1.2.3 is developed. Let V be a rank 1 discrete valuation domain with maximal ideal M, residue field K and discrete valuation v. In other words, v is a surjective map $v : K\backslash\{0\} \to \mathbb{Z}$ such that $v(xy) = v(x) + v(y)$ for all $x, y \in K\backslash\{0\}$ and $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K\backslash\{0\}$ such that $x + y \neq 0$. If R is a Noetherian subring of V with integral closure $R' = V$, then (excluding the trivial case of a field) R is a one-dimensional analytically irreducible local domain. It follows that $v(R)$ is a numerical semigroup called **the value semigroup associated to** R. The reader is directed to Chapter II, Section 1 of [1] for additional details.

The following is given as Proposition II.1.12 of [1].

Theorem 1.2.4 (Barucci - Dobbs - Fontana). *Let V be a rank 1 discrete valuation domain with maximal ideal M, residue field K and valuation v. Let R be a Noetherian conducive domain with integral closure V such that $R/(R \cap M) \cong K$. Then R is a Kunz domain if and only if $v(R)$ is pseudo-symmetric.*

Let S be a numerical semigroup. In this dissertation, the notation S^* will be used to denote the set of nonzero elements of S. For any integer n, we let $S \pm n := \{s \pm n \mid s \in S\}$. Given subsets A, B of S, we let $A + B := \{a + b \mid a \in A, b \in B\}$. Given a subset $A \subseteq S$ and positive integer k, we let $kA := \{ka \mid a \in A\}$.

The smallest nonzero element of a numerical semigroup S is called the **multiplicity** of S and will be denoted $\mu(S)$. Recall that every numerical semigroup is finitely generated and has a unique minimal generating set. The cardinality of this unique minimal generating set is called the **embedding dimension** of S and will be denoted $e(S)$. It is easy to see that $e(S) = |S^*\backslash(S^* + S^*)|$.

For any numerical semigroup S, it is always the case that $e(S) \leq \mu(S)$. To see this, suppose that $\{s_1, \ldots, s_e\}$ is the minimal generating set for S. Then for any $s \in S^*$ and any $i, j \in \{1, \ldots, e\}$, we must have that s_i is not congruent to $s_j \pmod{s}$ whenever $i \neq j$. If $e(S) = \mu(S)$, we say that S is of **maximal embedding dimension** or is an **MED numerical semigroup**. Numerous characterizations of numerical semigroups of maximal embedding dimension may be found in Proposition I.2.9 of [1]. One particular characterization from the aforementioned proposition, which will be used several times in this dissertation, is the following.

Proposition 1.2.5 (Barucci - Dobbs - Fontana). *Let S be a numerical semigroup with multiplicity m. Then S is of maximal embedding dimension if and only if $S^* - m$ is a numerical semigroup.*

1.3 The Quotient of a Numerical Semigroup by a Positive Integer

Let T be a numerical semigroup and let k be a positive integer. Define $\frac{T}{k} := \{x \in \mathbb{N} \mid kx \in T\}$. We call $\frac{T}{k}$ the **quotient of** T **by** k. It is easy to verify that $\frac{T}{k}$ is a numerical semigroup and, in fact, is an overnumerical semigroup of T. Indeed, if $x \in T$, then $kx \in T$

and, by definition, $x \in \frac{T}{k}$. Numerical semigroups of the form $\frac{T}{k}$ first appeared in [20], where it is shown that they often appear as solution sets of proportionally modular diophantine inequalities.

Let S be a numerical semigroup and let $k \geq 2$ be an integer. It is known that there exist infinitely many subnumerical semigroups T of S which satisfy $S = \frac{T}{k}$. Given a positive integer F, we find in Chapter 2 an upper bound on the number of numerical semigroups T with Frobenius number $F(T) \leq F$ which satisfy $S = \frac{T}{k}$. Using this result, we establish that, although they are infinite in number, the numerical semigroups T which satisfy $S = \frac{T}{k}$ are "rare" among all subnumerical semigroups of S.

In [17] it is shown that every numerical semigroup S can be expressed as $S = \frac{T}{2}$ for infinitely many symmetric numerical semigroups T. This result was later generalized by Swanson in [21]. In Chapter 3, we offer another proof of the generalization of this result. More precisely, we show that for any integer $k \geq 2$, every numerical semigroup S can be expressed as $S = \frac{T}{k}$ for infinitely many symmetric numerical semigroups T.

In [14] it is shown that a numerical semigroup S can be expressed as $S = \frac{T}{2}$ for a pseudo-symmetric numerical semigroup T if and only if S is irreducible. From this, it follows that every numerical semigroup S can be expressed as $S = \frac{T}{4}$ for infinitely many pseudo-symmetric numerical semigroups T. This result was also generalized by Swanson in [21]. In Chapter 4, we offer another proof of this generalization. More precisely, we prove that for any integer $k \geq 3$, every numerical semigroup S can be expressed as $S = \frac{T}{k}$ for infinitely many pseudo-symmetric numerical semigroups T.

In Chapter 5 we study the extent to which analogous results hold when the "numerators" T are drawn from the class of numerical semigroups of maximal embedding dimension. Theorem 5.1.7 and Example 5.1.8 show that numerical semigroups of maximal embedding dimension behave in a way that is qualitatively different from the behavior of the symmetric or pseudo-symmetric numerical semigroups that was described above. In fact, it is shown that if k is a positive integer, then it is not true in general that a numerical semigroup S can be expressed as $S = \frac{T}{k}$ for some numerical semigroup T of maximal embedding dimension. Given a positive integer $k \geq 2$, we describe in Example 5.1.8 an infinite family of numerical semigroups S which cannot be expressed as $S = \frac{T}{k}$ for any MED numerical semigroup T. Moreover, Theorem 5.1.7 shows that S can, for each positive integer $k \geq 2$, be expressed as $S = \frac{T}{k}$ for infinitely many numerical semigroups T of maximal embedding dimension if and only if S is itself of maximal embedding dimension.

Nevertheless, numerical semigroups of maximal embedding dimension do support a realization theorem of the kind that were given for symmetric and pseudo-symmetric numerical semigroups in [15], [14] and [21]. More precisely, Theorem 5.2.11, in conjunction with Theorem 5.1.7, shows that for all sufficiently large integers k, each numerical semigroup S can be expressed as $S = \frac{T}{k}$ for some corresponding numerical semigroup T of maximal embedding dimension. In order to streamline the presentation, we devote the first section of Chapter 5 to the case of Theorem 5.2.11 where k is a (sufficiently large) prime number. The second section of Chapter 5 will then expand upon this case to obtain the realization theorem described above.

Chapter 2

Fractions of Numerical Semigroups

2.1 On Fractions of Numerical Semigroups

Let T be an additive submonoid of $\mathbb{N}$ (for instance, a numerical semigroup) and let r be a positive rational number. Define $r \cdot T := \{x \in \mathbb{N} \mid x = rt \text{ for some } t \in T\}$. **The quotient of T by r**, denoted $\frac{T}{r}$, is defined as $\{x \in \mathbb{N} \mid xr \in T\}$. An equivalent definition of the quotient is given by $\frac{T}{r} = \frac{1}{r} \cdot T$. It is easy to see that if T is a numerical semigroup and k is a positive integer, then $\frac{T}{k}$ is an overnumerical semigroup of T (see Proposition 2.1.4 which follows) and we say that $\frac{T}{k}$ is a **fraction** of the numerical semigroup T.

Let T be a numerical semigroup and let k be a positive integer. Let a, b, and c be positive integers. An expression of the form $ax \pmod{b} \leq cx$ is called a **proportionally modular diophantine inequality**. Numerical semigroups of the form $\frac{T}{k}$ first appeared in [20], where it is shown that they often appear as solution sets of proportionally modular diophantine inequalities. Further results involving fractions of numerical semigroups may be found in [15], [17], [14], [11], [16], [21] and [12].

Our first goal is to show that fractions of numerical semigroups are themselves numerical semigroups only when the "denominator" is a positive integer.

Proposition 2.1.1. *If $0 < r \in \mathbb{Q}$ and $T \subseteq S$ are numerical semigroups such that $S = r \cdot T$, then $\frac{1}{r} \in \mathbb{N}$.*

Proof. Let $r = \frac{m}{n}$ where $0 < n \in \mathbb{Z}$, $m \in \mathbb{N}$, and $\gcd(m, n) = 1$. Choose $s \in S$ such that $s > F(S)$. Then $s + 1 \in S$, and so $s = r \cdot t_1$ and $s + 1 = r \cdot t_2$ for some $t_1, t_2 \in T$. Note that $1 = s + 1 - s = rt_2 - rt_1 = r(t_2 - t_1) = \frac{m}{n}(t_2 - t_1)$. Hence, $n = m(t_2 - t_1)$ and so m divides n in $\mathbb{N}$. By hypothesis, $\gcd(m, n) = 1$, so $m = 1$ and $\frac{1}{r} = n \in \mathbb{N}$. □

Note that the above proposition, while true for numerical semigroups, does not extend to arbitrary additive submonoids of $\mathbb{N}$. If S is an additive submonoid of $\mathbb{N}$ (possibly a numerical semigroup), r is a positive rational number and $T := r \cdot S = \{x \in \mathbb{N} \mid x = rs \text{ for some } s \in S\}$, then $S = \frac{T}{r}$ by the above definition, even though r need not be of the form $\frac{1}{k}$ for some positive integer k (for example, if $1 \leq n \in \mathbb{N}, S := n\mathbb{N}, r := \frac{m}{n}$ for some $1 < m \in \mathbb{N}$ and $T := r \cdot S$, then $T = m\mathbb{N}$). Thus, the above humble proposition identifies one of many subtle ways in which numerical semigroups differ from arbitrary additive submonoids of $\mathbb{N}$.

Let S be a numerical semigroup and let k be a positive integer. It is noted above that the quotient $\frac{S}{k}$ is an overnumerical semigroup of S. The next simple result shows precisely when this inclusion is proper.

Proposition 2.1.2. *Let S be a numerical semigroup and let $2 \leq k \in \mathbb{N}$. Then $S = \frac{S}{k}$ if and only if $S = \mathbb{N}$.*

Proof. Since each $n \in \mathbb{N}$ satisfies $n = \frac{nk}{k}$, it follows that $\mathbb{N} \subseteq \frac{\mathbb{N}}{k}$, while the reverse inclusion is clear. This proves the "if" assertion.

We offer two proofs of the converse. First, suppose the statement is not true. In other words, suppose there exists a numerical semigroup $S \neq \mathbb{N}$ and $S = \frac{S}{k}$ for some positive integer $k \geq 2$. Since $F(S) > 0$, $kF(S) > F(S)$ and therefore $kF(S) \in S$. Note that since $S = \frac{S}{k}$ and $kF(S) \in S$, we must have $F(S) \in S$, which is a contradiction.

For the second proof of the converse, we will suppose that $\frac{S}{k} \subseteq S$ and then show that $\mathbb{N} \subseteq S$. Since $\mathbb{N} \setminus S$ is finite and $\lim_{m \to \infty} k^m = \infty$, there exists a nonnegative integer m such that $k^m \in S$. Choose m minimal with this property. It is enough to show that $m = 0$ (for then $1 \in S$ and, necessarily, $S = \mathbb{N}$). If the assertion fails, then $k^{m-1} = \frac{k^m}{k} \in \frac{S}{k} \subseteq S$, contradicting the minimality of m. □

Let S and T be numerical semigroups and let k be a positive integer. We now present some properties associated with the condition $S = \frac{T}{k}$. Proposition 2.1.3 appears in [14] and is easy to prove. It will be used several times throughout this dissertation. Proposition 2.1.4, which follows, offers equivalent descriptions of the condition $S = \frac{T}{k}$.

Proposition 2.1.3. *Let S be a numerical semigroup and let a and b be positive integers. Then $\frac{S/a}{b} = \frac{S}{ab}$.*

Proposition 2.1.4. *Let S and T be numerical semigroups and let k be a positive integer. Then the following are equivalent:*

(1) *$S = \frac{T}{k}$;*
(2) *$S = \{x \in \mathbb{N} \mid kx \in T\}$;*
(3) *$S = \mathbb{N} \cap \frac{1}{k}T$;*
(4) *$T \subseteq S$, and a nonnegative integer $x \in S$ if and only if $kx \in T$.*

Proof. Note that (1) and (2) are equivalent by definition. We will complete the proof by showing that (2) ⇒ (3) ⇒ (4) ⇒ (2).

(2) ⇒ (3): We will show that both inclusions hold. Suppose that $S = \{x \in \mathbb{N} \mid kx \in T\}$. It's clear that $S \subseteq \mathbb{N}$. We must show that $S \subseteq \frac{1}{k}T$. Suppose that $x \in S$. By hypothesis, $kx \in T$. Therefore, $x = \frac{1}{k}kx \in \frac{1}{k}T$ and we have that $S \subseteq \frac{1}{k}T$. Hence, $S \subseteq \mathbb{N} \cap \frac{1}{k}T$. We now prove the reverse inclusion. Let $y \in \mathbb{N} \cap \frac{1}{k}T$. Then $y \in \mathbb{N}$ and we must show that $ky \in T$. Note that $y = \frac{t}{k}$ for some $t \in T$. Therefore, $ky = t \in T$ and, by (2), $y \in S$. Thus, the reverse inclusion holds and we have that $\mathbb{N} \cap \frac{1}{k}T = S$.

(3) ⇒ (4): Suppose that $S = \mathbb{N} \cap \frac{1}{k}T$. We first show that S is a numerical semigroup. Since T is a numerical semigroup, $0 \in T$. Thus, it is clear that $0 \in \mathbb{N} \cap \frac{1}{k}T = S$. We show that S is closed under addition. Let $a, b \in S$. Since $S \subseteq \mathbb{N}$, it's clear that $a + b \in \mathbb{N}$. We must show that $a + b \in \frac{1}{k}T$. By (3), $a = \frac{t_1}{k}$ and $b = \frac{t_2}{k}$ for some $t_1, t_2 \in T$. Hence, $ka, kb \in T$. Since T is a numerical semigroup, $ka + kb \in T$. Thus, $a + b = \frac{1}{k}(ka + kb) \in \frac{1}{k}T$ and so S is closed under addition. Finally, we must show that $\mathbb{N} \backslash S$ is finite. Since T is a numerical semigroup, $\mathbb{N} \backslash T$ is finite. Thus, for some $m \in \mathbb{N}$, we have that $kz \in T$ for all positive integers $k > m$. Hence, $z \in \mathbb{N}$ and $z = \frac{1}{k}kz \in \frac{1}{k}T$ for all $z > m$. Thus, by (3), $z \in S$ for all $z > m$. This implies that $\mathbb{N} \backslash S$ is finite and S is a numerical semigroup.

We show that S is an overnumerical semigroup of T. Let $t \in T$. Then, since T is a numerical semigroup, $t \in \mathbb{N}$ and $kt \in T$. Thus, $t = \frac{1}{k}kt \in \frac{1}{k}T$ so $t \in \mathbb{N} \cap \frac{1}{k}T = S$. Therefore, $T \subseteq S$ and S is an overnumerical semigroup of T.

Let $x \in \mathbb{N}$. We show that $x \in S$ if and only if $kx \in T$. Suppose $x \in S$. By (3), $x = \frac{t}{k}$ for some $t \in T$. Thus, $kx = t \in T$. For the converse, suppose $kx \in T$. By hypothesis, $x \in \mathbb{N}$. Moreover, $x = \frac{1}{k}kx \in \frac{1}{k}T$. Hence, $x \in \frac{1}{k}T \cap \mathbb{N} = S$.

(4) $\Rightarrow$ (2): We show both inclusions hold. Let $z \in S$. Since S is a numerical semigroup, $z \in \mathbb{N}$. Moreover, by (4), $kz \in T$. Thus, $S \subseteq \{x \in \mathbb{N} \mid kx \in T\}$. To see the reverse inclusion, let $y \in \{x \in \mathbb{N} \mid kx \in T\}$. Then $y \in \mathbb{N}$ and $ky \in T$. Therefore, by (4), $y \in S$ and the reverse inclusion holds. □

Suppose that T is a numerical semigroup, k is a positive integer and $S = \frac{T}{k}$. We now show that statement (4) of Proposition 2.1.4 can be used to prove the existence of an interesting property of T, namely, if $S = \frac{T}{k}$, then an isomorphic copy of the numerical semigroup S is "embedded" in the numerical semigroup T as a submonoid.

Proposition 2.1.5. *Let S and T be numerical semigroups and let k be a positive integer. If $S = \frac{T}{k}$, then there exists a submonoid of T that is isomorphic to S.*

Proof. Define $A := \{x \in T \mid x \equiv 0 \pmod{k}\}$. We show that A is a monoid. Clearly, $0 \in A$. If $a, b \in A$, then $a + b \in T$ (since T is a numerical semigroup). Moreover, $a = kx$ and $b = ky$ for some $x, y \in \mathbb{N}$. Thus, $a + b = k(x + y) \in A$ and therefore, by definition, A is a monoid.

Note that if $a \in A$, then $a = kx$ for some $x \in \mathbb{N}$. Thus, since $A \subseteq T$ and $S = \frac{T}{k}$, $x \in S$ by statement (4) of Proposition 2.1.4. It follows that $\frac{a}{k} = x \in S$ and the assignment $\phi : A \to S$ defined by $\phi(a) := \frac{a}{k}$ is well-defined. We show that ϕ is a monoid homomorphism. Let $a, b \in A$. Then $a = kx$ and $b = ky$ for some $x, y \in \mathbb{N}$. Thus, $\phi(a) + \phi(b) = x + y = \frac{1}{k}(kx + ky) = \frac{1}{k}(a + b) = \phi(a + b)$ so ϕ is a monoid homomorphism. It is clear that ϕ is injective. We show that ϕ is surjective. Let $s \in S$. Since $S = \frac{T}{k}$, we have that $ks \in T$ by statement (4) of Proposition 2.1.4. Furthermore, note that $ks \in A$ and $\phi(ks) = s$. Thus, ϕ is surjective. □

2.2 Associated Probabilities

Let S be a numerical semigroup and let $k \geq 2$ be a positive integer. It is easy to see that there exist infinitely many numerical semigroups T such that $S = \frac{T}{k}$. Indeed, if n is any positive integer greater than $kF(S)$, then $T_n := kS \cup \{n + 1, \rightarrow\}$ is a numerical semigroup with Frobenius number n which satisfies $S = \frac{T_n}{k}$.

The goal of this section is to establish that, although a given numerical semigroup can be expressed as a fraction with given denominator in infinitely many ways, the relevant "numerators" of those fractions are "rare" among all subnumerical semigroups. In particular, we show that if S is a numerical semigroup and F is a positive integer, then the number of subnumerical semigroups (resp., symmetric; resp., pseudo-symmetric; resp., MED subnumerical semigroups) of S with Frobenius number less than F grows exponentially with F. However, the number of subnumerical semigroups T of S with Frobenius number less than F satisfying $S = \frac{T}{k}$ for a given $2 \leq k \in \mathbb{N}$ grows slower than exponentially with F. In fact, we next establish polynomial growth by giving an upper bound on the number of ways to express a given numerical semigroup as a fraction. We first establish in Proposition 2.2.1 an upper bound on the number of ways to express a given numerical semigroup as a fraction.

In the results which follow, we will use the notation "$\lfloor\ \rfloor$" to denote the **greatest integer function**. If x is a real number then $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. It will be beneficial to recall at this time some basic properties of this greatest integer function. In particular, for any real number x, recall that $x - 1 < \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. See Chapter 4 of [8] for additional properties of the greatest integer function.

Note that the bound $F > kF(S)$ in Proposition 2.2.1 is harmless in view of the following observation. If S and T are numerical semigroups and $S = \frac{T}{k}$ for some positive integer k, then by Proposition 2.1.4 a positive integer $x \in S$ if and only if $kx \in T$. Therefore, $kF(S) \notin T$ and we must have that $F(T) \geq kF(S)$.

Proposition 2.2.1. *Let S be a numerical semigroup with Frobenius number $F(S)$, let $c = |\mathbb{N}\backslash S|$, let $1 \leq k \in \mathbb{N}$ and let $F > kF(S)$ be a positive integer. Then the number of numerical semigroups T which satisfy $S = \frac{T}{k}$ and have Frobenius number $F(T) \leq F$ is at most $(2^c(\lfloor \frac{F}{k} \rfloor + 2))^{k-1}$.*

Proof. Consider any such T. Since $S = \frac{T}{k}$, by Proposition 2.1.4 any integer $n \in S$ if and only if $kn \in T$. Thus, $kS \subseteq T$ and, in fact, kS constitutes the only elements of T which are congruent to 0 modulo k. For each $i \in \{1, \ldots, k-1\}$, let q_i be the smallest element of T congruent to i modulo k. For each such i, note that q_i must lie in the set $\{i, i+k, i+2k, \ldots, i + (\lfloor \frac{F}{k} \rfloor + 1)k\}$ (since $i + (\lfloor \frac{F}{k} \rfloor + 1)k > i + (\frac{F}{k})k > F$). Hence, for each i, there exist at most $\lfloor \frac{F}{k} \rfloor + 2$ possibilities for q_i.

Since q_i is the smallest element of T congruent to i modulo k, every element of T congruent to i modulo k is of the form $q_i + kn$ for some nonnegative integer n. In particular, since $kS \subseteq T$ and T is a numerical semigroup, $q_i + ks \in T$ for all $s \in S$ and $q_i + kn \in T$ for all integers $n > F(S)$. Therefore, apart from these elements, it follows that, given i and q_i, the only other integers congruent to i modulo k that may be elements of T are $q_i + kg_1, \ldots, q_i + kg_c$, where $g_1, \ldots, g_c$ are the gaps of S.

Since the set $\{q_i + kg_1, \ldots, q_i + kg_c\}$ has cardinality c, there are at most 2^c sets that can play the role of the as-yet unidentified part of $T \cap (q_i + k\mathbb{N})$. Therefore, for each $i \in \{1, \ldots, k-1\}$, there are at most $\lfloor \frac{F}{k} \rfloor + 2$ possibilities for q_i and, for each such possible q_i, there exist at most 2^c ways to find the "additional" sets of elements of T that are congruent to i modulo k. Hence, the number of possible sets of elements of T that are congruent to a given i modulo k is at most $2^c(\lfloor \frac{F}{k} \rfloor + 2)$. Thus, the number of possibilities for T is at most $(2^c(\lfloor \frac{F}{k} \rfloor + 2))^{k-1}$. ☐

We next establish lower bounds on the number of subnumerical semigroups, symmetric subnumerical semigroups, pseudo-symmetric subnumerical semigroups, and subnumerical semigroups of maximal embedding dimension of a numerical semigroup that have bounded Frobenius number. Readers familiar with the literature on numerical semigroups will note that the proof of Proposition 2.2.2 utilizes the notion of "fundamental gap" developed in [19].

Proposition 2.2.2. *Let S be a numerical semigroup and let $F \in \mathbb{N}$ such that $F > \max\{2F(S) + 1, 4\}$. Then the number of subnumerical semigroups T of S with Frobenius number $F(T) = F$ is at least $2^{\lfloor (F-1)/2 \rfloor - 1}$.*

Proof. Since $F \geq 5$, note that $F+5 \leq 2F$. Hence, $F \geq \frac{F+5}{2} = \frac{F+1}{2} + 2 \geq \lfloor \frac{F+1}{2} \rfloor + 2$. Thus, $\lfloor \frac{F+1}{2} \rfloor + 1 \leq F - 1$. Also, since $F > 2F(S) + 1$, we have that $\lfloor \frac{F+1}{2} \rfloor \geq \lfloor F(S) + 1 \rfloor > F(S)$. Thus, $A := \{\lfloor \frac{F+1}{2} \rfloor + 1, \lfloor \frac{F+1}{2} \rfloor + 2, \ldots, F-1\} \subseteq S$. Next, define $U := \{0, F+1, \rightarrow\}$.

Note that for all $x, y \in A$, $x + y \geq 2\min(A) = 2(\lfloor \frac{F+1}{2} \rfloor + 1) \geq F + 1 > F$. Thus, for each $B \subseteq A$, we have that $T := T_B := U \cup B$ is a subnumerical semigroup of S such that $F(T) = F$. Since different choices of B result in different numerical semigroups T and there are $2^{\lfloor (F-1)/2 \rfloor - 1}$ possibilities for B, the proof is complete. □

In Proposition 5 of [2], it is shown that given a fixed odd number F, the number of symmetric numerical semigroups with Frobenius number F is at least $2^{\lfloor F/8 \rfloor}$. The proof used in [2] may be modified slightly to prove the next result.

Proposition 2.2.3. *Let S be a numerical semigroup and consider a fixed odd integer $F > 4F(S) + 4$. Then the number of symmetric subnumerical semigroups T of S with Frobenius number $F(T) = F$ is at least $2^{\lfloor F/8 \rfloor}$.*

Proof. Since $F \geq 1$ and $F > 4F(S)+4$, note that $F > F(S)$ and $\lfloor \frac{F}{4} \rfloor \geq \lfloor F(S)+1 \rfloor > F(S)$. Hence, $T_0 := \langle F+1, F+2, \ldots, 2F+1 \rangle$ is a subnumerical semigroup of S with $F(T_0) = F$. In fact, $T_0 = \{0, F+1, \rightarrow\}$. We show that T_0 can be embedded in at least $2^{\lfloor F/8 \rfloor}$ different symmetric subnumerical semigroups of S, each with Frobenius number F.

First, T_0 may be extended to a subnumerical semigroup T_1 of S by adjoining as generators any set E of even numbers between $\lfloor \frac{F}{4} \rfloor + 1$ and $\frac{F}{2}$. Any semigroup T_1 that arises in this way has $F(T_1) = F$ (since F is odd) and it is easy to see that different sets E always lead to different numerical semigroups T_1 (since the elements of E act as generators). Define $H(T_1) := \{z \in \mathbb{Z} \backslash T_1 \mid F - z \notin T_1\}$. Note that by Proposition 1.2.1, $H(T_1) = \emptyset$ if and only if T_1 is symmetric; and $x \in H(T_1)$ if and only if $F - x \in H(T_1)$. Thus, if $H(T_1) \neq \emptyset$, then $h(T_1) := \max H(T_1) > \frac{F}{2}$. If T_1 is not symmetric, extend T_1 to a numerical semigroup T_2 with $F(T_2) = F$ by adjoining as a generator the element $h(T_1)$. We now repeat this procedure, obtaining a finite list $T_1, T_2, \ldots, T_n$ such that $T := T_n$ is a symmetric numerical semigroup.

Note that different choices of the set E always yield different symmetric numerical semigroups T (again, since the elements of E act as generators). Moreover, for each such T, we have that $F(T) = F$ (since $h(T_i) > \frac{F}{2}$ for each i). Since E can be chosen in at least $2^{\lfloor F/8 \rfloor}$ ways, the proof is complete. □

We now give a pseudo-symmetric analog of the previous result. The proof also follows closely the proof of Proposition 5 of [2].

Proposition 2.2.4. *Let S be a numerical semigroup and consider a fixed even integer $F > \max\{3F(S) + 3, 6\}$. Then the number of pseudo-symmetric subnumerical semigroups T of S with Frobenius number $F(T) = F$ is at least $2^{\lfloor (F-6)/6 \rfloor}$.*

Proof. Since $F > 3F(S) + 3$, note that $F > F(S)$ and $\lfloor \frac{F}{3} \rfloor \geq \lfloor F(S) + 1 \rfloor > F(S)$. Hence, $T_0 := \langle F+1, F+2, \ldots, 2F+1 \rangle$ is a subnumerical semigroup of S with $F(T_0) = F$. In fact, $T_0 = \{0, F+1, \rightarrow\}$.

We show that T_0 can be embedded in at least $2^{\lfloor (F-6)/6 \rfloor}$ different pseudo-symmetric subnumerical semigroups of S, each with Frobenius number F. Since $F > 6$ and $F > 3F(S) + 3$, note that $A := \{\lfloor \frac{F}{3} \rfloor + 1, \lfloor \frac{F}{3} \rfloor + 2, \ldots, \frac{F}{2} - 1\}$ is a subset of S of cardinality at least $\lfloor \frac{F-6}{6} \rfloor$. To see this, note that since $F > 6$, we have that $|A| = (\frac{F}{2}-1)-(\lfloor\frac{F}{3}\rfloor+1)+1 = \frac{F}{2}-1-\lfloor\frac{F}{3}\rfloor \geq \frac{F}{2}-1-\frac{F}{3} = \frac{F-6}{6} \geq \lfloor\frac{F-6}{6}\rfloor$. Extend T_0 to a numerical semigroup T_1 by adjoining as generators any subset E of A.

We claim that $F(T_1) = F$. Since $T_0 \subseteq T_1$ and $F(T_0) = F$, it is clear that $F(T_1) \leq F$. Note that if $F(T_1) < F$, then there must exist some positive integer r and elements $a_1, \ldots, a_r \in A$ such that $a_1 + \cdots + a_r = F$. Let $x, y, z \in A$. Note that $x + y \leq 2\max(A) = 2(\frac{F}{2} - 1) = F - 2 < F$ and $x + y + z \geq 3\min(A) = 3(\lfloor\frac{F}{3}\rfloor + 1) > 3(\frac{F}{3}) = F$. Therefore, the sum of any two elements of A is less than F and the sum of any three elements of A is greater than F. This shows that $F \notin T_1$, which completes the proof of the claim that $F(T_1) = F$.

Define $H'(T_1) := \{z \in \mathbb{Z}\backslash(T_1 \cup \{\frac{F}{2}\}) \mid F - z \notin T_1\}$. Note that by Proposition 1.2.2, $H'(T_1) = \emptyset$ if and only if T_1 is pseudo-symmetric; and $x \in H'(T_1)$ if and only if $F - x \in H'(T_1)$. Thus, if $H'(T_1) \neq \emptyset$, then $h'(T_1) := \max H'(T_1) > \frac{F}{2}$. If T_1 is not pseudo-symmetric, we may extend T_1 to a numerical semigroup T_2 with $F(T_2) = F$ by adjoining as a generator the element $h'(T_1)$. We now repeat this procedure until we have a pseudo-symmetric numerical semigroup T.

We claim that different subsets E of A give rise to different pseudo-symmetric numerical semigroups T. Let E_1 and E_2 be distinct subsets of A. Let $W_1 := \langle E_1, T_0\rangle$ and let $V_1 := \langle E_2, T_0\rangle$. Let W and V (respectively) be the pseudo-symmetric numerical semigroups obtained from W_1 and V_1 by following the procedure in the previous paragraph. Without loss of generality, we may suppose that there exists $a \in E_1\backslash E_2$. Note that if $x, y \in A$, then $x + y \geq 2(\lfloor\frac{F}{3}\rfloor + 1) > 2(\frac{F}{3}) > \frac{F}{2} - 1 = \max(A)$. In particular, since $E_2 \subseteq A$, the sum of any two elements of E_2 is not an element of E_2. This proves that $a \in W_1\backslash V_1$. Since each such generator is greater than $\frac{F}{2} > \frac{F}{2} - 1 = \max(A)$, it follows that $a \in W\backslash V$. This completes the proof of the claim that different subsets E of A give rise to different pseudo-symmetric numerical semigroups T.

To complete the proof of the proposition, simply note that there exist at least $2^{\lfloor (F-6)/6\rfloor}$ subsets E of A. □

Let S be a numerical semigroup with multiplicity $\mu(S)$. Recall Proposition 1.2.5, which says that S is of maximal embedding dimension if and only if $S^* - \mu(S)$ is a numerical semigroup. It follows that if S is a numerical semigroup S and $0 < n \in S$, then $(S+n)\cup\{0\}$ is a subnumerical semigroup of S of maximal embedding dimension with Frobenius number $F(S) + n$. Using this observation and the basic argument of Proposition 5 of [2], we next give a result similar to Propositions 2.2.3 and 2.2.4 for subnumerical semigroups of maximal embedding dimension.

Proposition 2.2.5. *Let S be a numerical semigroup and let F be a positive integer such that $F > \max\{4F(S) + 1, 4\}$. Then the number of MED subnumerical semigroups T of S with Frobenius number $F(T) \leq F$ is at least $2^{\lfloor (F-4)/4\rfloor} - 1$.*

Proof. Let $G = \lfloor\frac{F+1}{2}\rfloor$. Since $F > 4$, note that $G \geq 2$. Moreover, since $F > 4F(S)+1$, note that $G = \lfloor\frac{F+1}{2}\rfloor > \lfloor 2F(S) + 1\rfloor = 2F(S) + 1$. Therefore, $T_0 := \langle G + 1, G + 2, \ldots, 2G + 1\rangle$ is a subnumerical semigroup of S with $F(T_0) = G$. In fact, $T_0 = \{0, G + 1, \rightarrow\}$.

Let $A := \{\lfloor\frac{G+1}{2}\rfloor+1, \lfloor\frac{G+1}{2}\rfloor+2, \ldots, G-1\}$. Since $G > 2F(S)+1$, note that $\lfloor\frac{G+1}{2}\rfloor+1 \geq \lfloor F(S)+1\rfloor+1 > F(S)$. Hence, A is a subset of S. Moreover, the cardinality of A is $\lfloor\frac{G-2}{2}\rfloor$. To see this, note that if $G = 2k$ for some $k \in \mathbb{N}$, then $A = \{k + 1, k + 2, \ldots, 2k - 1\}$; and if

$G = 2k+1$ for some $k \in \mathbb{N}$, then $A = \{k+2, k+3, \ldots, 2k\}$. In either case, A has cardinality $k-1$ and a simple calculation shows that $\lfloor \frac{G-2}{2} \rfloor = k-1$.

Note that for all $x, y \in A$, $x + y \geq 2\min(A) = 2(\lfloor \frac{G+1}{2} \rfloor + 1) \geq 2(\frac{G+1}{2}) = G+1 > F(S)$. Hence, if B is any nonempty subset of A, then $T_0 \cup B$ is a subnumerical semigroup of S with Frobenius number G. Define $T_B := ((T_0 \cup B) + \mu(T_0 \cup B)) \cup \{0\}$. By the observation above, T_B is a subnumerical semigroup of T_0 (hence of S) of maximal embedding dimension. Furthermore, since $B \neq \emptyset$, $F(T_B) = F(T_0 \cup B) + \mu(T_0 \cup B) = G + \min(B) \leq G + G - 1 \leq F$. Note that different nonempty subsets B of A yield different numerical semigroups T_B. (Indeed, if $b = \min(B_1) = \min(B_2)$ and $x \in B_1 \backslash B_2$, then $x + b \in T_{B_1} = T_{B_2}$ leads to a contradiction.) Hence, there exist at least $2^{\lfloor (G-2)/2 \rfloor} - 1$ such subnumerical semigroups T_B of S. We now have only to show that $\lfloor \frac{G-2}{2} \rfloor \geq \lfloor \frac{F-4}{4} \rfloor$.

We consider separately the cases where $F = 2n$ and $F = 2n+1$ for some $n \in \mathbb{N}$. If $F = 2n$ for some $n \in \mathbb{N}$, then $\lfloor \frac{G-2}{2} \rfloor = \lfloor \frac{\lfloor \frac{2n+1}{2} \rfloor - 2}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{F-4}{4} \rfloor$. If $F = 2n+1$ for some $n \in \mathbb{N}$, then $\lfloor \frac{G-2}{2} \rfloor = \lfloor \frac{\lfloor \frac{2n+2}{2} \rfloor - 2}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{F-3}{4} \rfloor \geq \lfloor \frac{F-4}{4} \rfloor$. This completes the proof. □

Let A be a set of real numbers and let n be a positive integer. As in [8], let $A(n)$ denote the number of positive integers in A that are less than or equal to n. If the sequence $\frac{A(n)}{n}$ has a limit, we say that A has **natural density** $\lim_{n\to\infty} \frac{A(n)}{n}$. Natural density may be used to formalize probabilistic intuition, as in [5, pp. 268-269 and Theorems 332 and 333]. In the case of numerical semigroups, we use the notion of natural density in like manner to give the final result of this section.

Theorem 2.2.6. *Let S be a numerical semigroup and let $2 \leq k \in \mathbb{N}$. The probability that a randomly chosen subnumerical semigroup T satisfies $S = \frac{T}{k}$ exists and is zero. Furthermore, the probability that a randomly chosen symmetric (resp., pseudo-symmetric; resp., MED) subnumerical semigroup T of S satisfies $S = \frac{T}{k}$ also exists and is zero.*

Proof. We prove the first statement. The others follow similarly by using Propositions 2.2.3, 2.2.4 or 2.2.5 in place of Proposition 2.2.2 in the proof given below.

Let F be a positive integer such that $F > kF(S)$. Let $N_F :=$ {numerical semigroups $T \subseteq S \mid (kF(S) \leq) F(T) \leq F$ and $S = \frac{T}{k}$} and let $D_F :=$ {numerical semigroups $T \subseteq S \mid (F(S) \leq) F(T) \leq F$}. Note that both N_F and D_F are finite sets. Furthermore, both N_F and D_F are nonempty since $kS \cup \{F+1, \rightarrow\} \in N_F$ and $\{0, F+1, \rightarrow\} \in D_F$.

Let $c := |\mathbb{N} \backslash S|$. By Propositions 2.2.1 and 2.2.2, the probability in question is (by definition)

$$\lim_{F\to\infty} \frac{|N_F|}{|D_F|} \leq \lim_{F\to\infty} \frac{(2^c(\lfloor F/k \rfloor + 2))^{k-1}}{2^{\lfloor (F-1)/2 \rfloor - 1}}.$$

Apply L'Hôpital's Rule $k-1$ times and it follows that the limit is 0. □

Chapter 3

Fractions of Symmetric Numerical Semigroups

In [15], Rosales and García-Sánchez showed that every numerical semigroup S can be expressed as $S = \frac{T}{2}$ for some symmetric numerical semigroup T. This result was later expanded upon in [17] and [12] as follows.

Theorem 3.0.7 (Rosales - García-Sánchez)**.** *If S is a numerical semigroup, then there exist infinitely many symmetric numerical semigroups T such that $S = \frac{T}{2}$.*

In Theorem 5 of [21], Theorem 3.0.7 is generalized. Our goal in this section is to offer another proof of the generalized result. In other words, we will show that if S is a numerical semigroup and $k \geq 2$ is a positive integer, then S can be expressed as $S = \frac{T}{k}$ for infinitely many symmetric numerical semigroups T. In light of Proposition 2.1.3 and Theorem 3.0.7, we have only to show that if S is a numerical semigroup and p is an odd prime number, then S can be expressed as $S = \frac{T}{p}$ for infinitely many symmetric numerical semigroups T.

Let S be a numerical semigroup with Frobenius number $F(S)$. Fix an odd prime number p. Define $A(S) := \{x \in \mathbb{N} \mid F(S) - x \notin S\}$ and $H(S) := \{x \in \mathbb{N} \backslash S \mid F(S) - x \notin S\}$. Elements of $H(S)$ are often referred to in the literature as the **gaps of S of type 2**.

First, we describe the relationship between the sets $A(S)$ and $H(S)$.

Proposition 3.0.8. *Let S be a numerical semigroup. Then $A(S) = S \cup H(S)$. Moreover, this union is disjoint.*

Proof. If $x \in A(S)$ and $x \notin S$, then it is easy to see that $x \in H(S)$. Hence, $A(S) \subseteq S \cup H(S)$. We show the reverse inclusion. It is clear that $H(S) \subseteq A(S)$. To see that $S \subseteq A(S)$, note that if $s \in S$ and $F(S) - s = s'$ for some $s' \in S$, then $F(S) = s + s' \in S$, a contradiction. Thus, the reverse inclusion holds and $A(S) = S \cup H(S)$. It is clear that the union is disjoint since $H(S) \subseteq \mathbb{N} \backslash S$. □

Let S be a numerical semigroup. We will now define our basic construction from which we will obtain the symmetric numerical semigroups we seek. For each $n \in \mathbb{N}$, define $m_n := pF(S) + 2pn + p + 1$ and let $F_n := pF(S) + m_n + p - 2$. Note that $m_n \equiv 1 \pmod{p}$. Let $A := A(S)$. Define $S_n := pS \cup (pA + m_n) \cup (pA + m_n + 1) \cup \cdots \cup (pA + m_n + p - 2)$.

Our first goal is to establish that S_n is a subnumerical semigroup of S with Frobenius number $F(S_n) = F_n$ which satisfies $S = \frac{S_n}{p}$. To improve the clarity of the exposition, we present the proof in two separate propositions.

Proposition 3.0.9. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then S_n contains every positive integer greater than F_n.*

Proof. Let $z \in \mathbb{N}$ and suppose that $z > F_n$. Let $i \in \{0, \ldots, p-1\}$ such that $z \equiv i \pmod{p}$. We consider three separate cases below.

First, consider the case when $i = 0$. Then $z = px$ for some $x \in \mathbb{N}$. Note that $px > F_n = pF(S) + m_n + p - 2$ by hypothesis. Because $m_n \geq 1$, we have that $px > pF(S) + p - 1$. Therefore, $px \geq p(F(S) + 1)$ and so $x > F(S)$. Since $x > F(S)$, we must have that $x \in S$ and so $z = px \in pS \subseteq S_n$.

Next, consider the case when $1 \leq i \leq p-2$. As $z > F_n = pF(S) + m_n + p - 2$, we have $z \geq pF(S)+m_n+p-1$. Suppose that $b \geq 0$ is the integer such that $z = pF(S)+m_n+p-1+b$. Write $b = pq + r$ where q and r are integers and $0 \leq r < p$. Then $z = pF(S) + m_n + p - 1 + pq + r = p(F(S) + 1 + q) + m_n + r - 1$. Since $m_n \equiv 1 \pmod{p}$, we have that $r = i$. Therefore, $z = p(F(S) + 1 + q) + m_n + i - 1 \in pS + m_n + i - 1 \subseteq pA(S) + m_n + i - 1 \subseteq S_n$.

Finally, consider the case when $i = p - 1$. Then $z \equiv F_n \pmod{p}$. Since $z > F_n$, we have that $z = F_n + pc$ for some positive integer c. Thus, $z = pF(S) + m_n + p - 2 + pc = p(F(S) + c) + m_n + p - 2$. Since $c > 0$, note that $F(S) + c \in S \subseteq A(S)$. Therefore, $z \in pA(S) + m_n + p - 2 \subseteq S_n$. □

Proposition 3.0.10. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then S_n is a subnumerical semigroup of S, $F(S_n) = F_n$ and $S = \frac{S_n}{p}$.*

Proof. We first show that S_n is a numerical semigroup. Since S is a numerical semigroup, $0 \in S$. Thus, $0 = p0 \in pS \subseteq S_n$. By Proposition 3.0.9, $\mathbb{N} \backslash S_n$ is finite. Hence, we have only to show that S_n is closed under addition.

Let $A := A(S)$. Since S is a numerical semigroup, it is clear that $pS + pS \subseteq pS \subseteq S_n$. Moreover, for each $i \in \{0, \ldots, p-2\}$, we claim that $pS + pA + m_n + i \subseteq pA + m_n + i \subseteq S_n$. To prove this, it is enough to show that $S + A \subseteq A$. Let $s \in S$ and let $a \in A$. If $F(S) - (s + a) = s'$ for some $s' \in S$, then $F(S) - a = s + s' \in S$, contradicting $a \in A$. Thus, $F(S) - (s + a) \notin S$ and hence $s + a \in A$ by definition. Therefore, $S + A \subseteq A$, completing the proof of the claim.

Finally, we show that for each $i, j \in \{0, \ldots, p-2\}$, we have that $pA+m_n+i+pA+m_n+j \subseteq S_n$. Let $i, j \in \{0, \ldots, p-2\}$. Note that since every element of $pA + m_n + i$ is greater than or equal to m_n, it suffices by Proposition 3.0.9 to show that $2m_n > F_n$. Indeed, $m_n + m_n = pF(S) + 2pn + p + 1 + m_n = pF(S) + m_n + p - 2 + 2pn + 3 = F_n + 2pn + 3 > F_n$. Thus, S_n is a numerical semigroup.

We now show that $S_n \subseteq S$. Clearly $pS \subseteq S$. Since $m_n = pF(S) + 2pn + p + 1 \geq p(F(S) + 1) + 1 > F(S)$, we have that $\mathbb{N} + m_n \subseteq S$. Therefore, $pA + m_n + i \subseteq S$ for each $i \in \{0, \ldots, p-2\}$. Hence, the proof to this point allows us to conclude that $S_n \subseteq S$.

Next, we show that $F(S_n) = F_n$. By Proposition 3.0.9, it suffices to show that $F_n \notin S_n$. Since $F_n \equiv p - 1 \pmod{p}$ and $m_n \equiv 1 \pmod{p}$, if $F_n \in S_n$ we must have that $F_n \in pA + m_n + p - 2$. Therefore, since $F_n = pF(S) + m_n + p - 2 \in pA + m_n + p - 2$, we have that $F(S) \in A$. But this is a contradiction since $F(S) - F(S) = 0 \in S$. Thus, $F_n \notin S_n$ and $F(S_n) = F_n$.

Finally, we show that $S = \frac{S_n}{p}$. By Proposition 2.1.4, we must show that a natural number $x \in S$ if and only if $px \in S_n$. Clearly if $x \in S$, then $px \in pS \subseteq S_n$. For the converse, note that because $m_n \equiv 1 \pmod{p}$, by the construction of S_n we have that pS

must contain all of the elements of S_n which are congruent to 0 modulo p. Therefore, if $px \in S_n$ then px must lie in pS and so $x \in S$. □

The numerical semigroup S_n defined above is the construction from Lemma 3 of [12] generalized to an arbitrary odd prime. It is shown in Proposition 13 of [12] that when $p = 2$, the numerical semigroup S_n is symmetric for any $n \in \mathbb{N}$. We shall see that if p is an odd prime, then S_n is symmetric in only one special case. In particular, we shall see that S_n is usually "too small" and must have additional elements adjoined to it, without changing its Frobenius number, in order to obtain a symmetric numerical semigroup. We shall identify the precise elements that may be adjoined to S_n in order to make it symmetric and we shall show that all of these are elements of S.

Let S be a numerical semigroup and let $B \subseteq S$. Define $N(B, S) := \{x \in B \mid x < F(S)\}$. In other words, if $B \subseteq S$, then $N(B, S) = B \cap \{0, 1, \ldots, F(S)\}$. If $B = S$, we shall use the familiar notation $N(S)$ instead of $N(S, S)$. Recall that $F(\mathbb{N}) = -1$. Thus, for convenience, we shall say that $N(\mathbb{N}) = \emptyset$. Therefore, we have that $N(S) = \emptyset$ if and only if $S = \mathbb{N}$.

Our next goal is to establish two useful formulas for $|N(S_n)|$.

Proposition 3.0.11. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $|N(S_n)| = p|N(S)| + (p-1)|H(S)| + F(S) + 2n + 1$.*

Proof. Let $A := A(S)$ and $H := H(S)$. Recall that $S_n := pS \cup (pA + m_n) \cup (pA + m_n + 1) \cup \cdots \cup (pA + m_n + p - 2)$. Note that this union is disjoint because each element of pS is congruent to 0 modulo p and, since $m_n \equiv 1 \pmod{p}$, each element of $pA + m_n + i$ is congruent to $i + 1$ modulo p for each $i \in \{0, \ldots, p-2\}$. Thus, $|N(S_n)| = |N(pS, S_n)| + |N(pA + m_n, S_n)| + |N(pA + m_n + 1, S_n)| + \cdots + |N(pA + m_n + p - 2, S_n)|$. We consider each of these p disjoint sets separately.

By definition, $N(pS, S_n) = \{px \in pS \mid px < F(S_n)\}$. By Proposition 3.0.10, $F(S_n) = F_n = pF(S) + m_n + p - 2$. Therefore, $N(pS, S_n) = \{px \in pS \mid px < pF(S) + m_n + p - 2\}$. Note that $\{px \in pS \mid px < pF(S) + m_n + p - 2\}$ is the disjoint union $\{px \in pS \mid px \leq pF(S)\} \cup \{px \in pS \mid pF(S) < px < pF(S) + m_n + p - 2\}$. Clearly, $\{px \in pS \mid px \leq pF(S)\} = p\{x \in S \mid x \leq F(S)\} = pN(S)$ and $|pN(S)| = |N(S)|$. By definition, $m_n = pF(S) + 2pn + p + 1$. Hence, $pF(S) + m_n + p - 2 = p(2F(S) + 2n + 2) - 1$ and so $\{px \in pS \mid pF(S) < px < pF(S) + m_n + p - 2\} = \{px \in pS \mid pF(S) < px < p(2F(S) + 2n + 2) - 1\} = \{p(F(S) + 1), p(F(S) + 2), \ldots, p(2F(S) + 2n + 1)\}$. Therefore, $|N(pS, S_n)| = |N(S)| + F(S) + 2n + 1$.

Now let $i \in \{0, \ldots, p-2\}$ and consider $N(pA + m_n + i, S_n)$. By definition, $N(pA + m_n + i, S_n) = \{x \in pA + m_n + i \mid x < F(S_n)\}$. Since $F(S_n) = F_n = pF(S) + m_n + p - 2$, we have that $N(pA + m_n + i, S_n) = \{x \in pA + m_n + i \mid x < pF(S) + m_n + p - 2\}$. Note that by Proposition 3.0.8, A is the disjoint union $S \cup H$. Thus, $N(pA + m_n + i, S_n)$ is the disjoint union $\{px + m_n + i \in pS + m_n + i \mid px + m_n + i < pF(S) + m_n + p - 2\} \cup \{py + m_n + i \in pH + m_n + i \mid py + m_n + i < pF(S) + m_n + p - 2\}$. We consider these two disjoint sets separately.

Let $k := |N(S)|$. Then $S = \{0 = s_0, s_1, \ldots, s_{k-1}, s_k, \rightarrow\}$. Since $s_k = F(S) + 1$ and $0 \leq i < p$, note that $ps_k + m_n + i = pF(S) + p + m_n + i \geq pF(S) + p + m_n > pF(S) + m_n + i$. Therefore, $\{px + m_n + i \in pS + m_n + i \mid px + m_n + i < pF(S) + m_n + p - 2\} = \{ps_0 + m_n + i, ps_1 + m_n + i, \ldots, ps_{k-1} + m_n + i\}$, which has cardinality $N(S)$. Now consider the set $\{py + m_n + i \in pH + m_n + i \mid py + m_n + i < pF(S) + m_n + p - 2\}$. Note that since $F(S) \notin H$.

Thus, if $y \in H$, then $y < F(S)$. Hence, $py+m_n+i < pF(S)+m_n+p-2$ for all $y \in H$ and so $\{py+m_n+i \in pH+m_n+i \mid py+m_n+i < pF(S)+m_n+p-2\}$ has cardinality $|H|$. Hence, combining this with the previous result, we have that $|N(pA+m_n+i, S_n)| = |N(S)| + |H|$ for each $i \in \{0, \ldots, p-2\}$.

Finally, by all of the above observations, we have that $|N(S_n)| = |N(pS, S_n)| + |N(pA+m_n, S_n)| + |N(pA+m_n+1, S_n)| + \cdots + |N(pA+m_n+p-2, S_n)| = |N(S)| + F(S) + 2n + 1 + (p-1)(|N(S)| + |H|) = p|N(S)| + (p-1)|H(S)| + F(S) + 2n + 1$. □

The following result appears as Lemma 11 of [12].

Proposition 3.0.12. *Let S be a numerical semigroup. Then $|H(S)| = 2|\mathbb{N}\backslash S| - F(S) - 1$.*

Combining Propositions 3.0.11 and 3.0.12, we establish the following formula for $|N(S_n)|$.

Proposition 3.0.13. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $|N(S_n)| = pF(S) + 2n + p - (p-2)|N(S)|$.*

Proof. Since S is a numerical semigroup, $|\mathbb{N}\backslash S| = F(S) + 1 - |N(S)|$. By Proposition 3.0.12, $|H(S)| = 2|\mathbb{N}\backslash S| - F(S) - 1$. Combining these results with Proposition 3.0.11 gives $|N(S_n)| = p|N(S)| + (p-1)|H(S)| + F(S) + 2n + 1 = p|N(S)| + (p-1)(2|\mathbb{N}\backslash S| - F(S) - 1) + F(S) + 2n + 1 = p|N(S)| + (p-1)(2(F(S)+1-|N(S)|) - F(S) - 1) + F(S) + 2n + 1 = pF(S) + 2n + p - (p-2)|N(S)|$. □

Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Recall Proposition 1.2.1, which states that a numerical semigroup T with odd Frobenius number $F(T)$ is symmetric if and only if $|N(T)| = \frac{F(T)+1}{2}$. Recall also that $F(S_n) = F_n = pF(S) + m_n + p - 2$ and $m_n = pF(S) + 2pn + p + 1$. It is easy to verify that $F(S_n)$ is odd and that $\frac{F(S_n)+1}{2}$ is an integer.

We now define $k_n := \frac{F(S_n)+1}{2} - |N(S_n)|$. In light of Proposition 1.2.1, k_n may be viewed as the number of elements that would have to be "adjoined" to S_n in order to obtain a symmetric numerical semigroup with Frobenius number F_n. The next result combines Propositions 3.0.10 and 3.0.13 to give a useful formula for k_n.

Proposition 3.0.14. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $k_n = (p-2)(n + |N(S)|)$.*

Proof. By Proposition 3.0.10, $k_n = \frac{F(S_n)+1}{2} - |N(S_n)| = \frac{F_n+1}{2} - |N(S_n)| = \frac{pF(S)+m_n+p-2+1}{2} - |N(S_n)|$. Recall that $m_n = pF(S) + 2pn + p + 1$. Therefore, $k_n = \frac{2pF(S)+2pn+2p}{2} - |N(S_n)| = pF(S) + pn + p - |N(S_n)|$. By Proposition 3.0.13, $k_n = pF(S) + pn + p - (pF(S) + 2n + p - (p-2)|N(S)|) = (p-2)(n + |N(S)|)$. □

We can now demonstrate that when $p > 2$, the construction from Lemma 3 of [12] fails to be symmetric except in a very special case.

Proposition 3.0.15. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. The following are equivalent:*

(1) *S_n is symmetric;*
(2) *$k_n = 0$;*
(3) *$n = 0$ and $S = \mathbb{N}$.*

Proof. We prove (1)⇒(2)⇒(3)⇒(1).

Note that (1)⇒(2) by Proposition 1.2.1.

We show that (2)⇒(3). By Proposition 3.0.14, If $k_n = 0$, then since p is an odd prime, we must have that $n + |N(S)| = 0$. Note that $n, |N(S)| \in \mathbb{N}$. Therefore, both n and $|N(S)|$ must be 0. Furthermore, since $|N(S)| = 0$, we must have $S = \mathbb{N}$.

Finally, we show that (3)⇒(1). If $n = 0$ and $S = \mathbb{N}$, then $k_n = 0$ by Proposition 3.0.14. Therefore, by definition of k_n, we have that $\frac{F(S_n)+1}{2} = |N(S_n)|$. Hence, S_n is symmetric by Proposition 1.2.1. □

Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. In light of Proposition 3.0.15, from now on we will always assume that S_n is not symmetric. Our goal now is to adjoin exactly k_n elements of $S \backslash S_n$ to S_n so that the resulting set, which will be denoted T_n, is a numerical semigroup with Frobenius number F_n which satisfies $S = \frac{T_n}{p}$.

Let $v := |N(S)|$. It will be convenient to index the elements $\{s_i\}_{i \in \mathbb{N}}$ of S so that $s_i < s_j$ whenever $i < j$. Then $S = \{0 = s_0, s_1, \ldots, s_{v-1}, s_v, \rightarrow\}$. Note that $F(S) = s_v - 1$ and, for each integer $i \geq 0$, $s_{v+i} = s_v + i$.

We now define the k_n elements of $S \backslash S_n$ which can be adjoined to S_n in order to obtain the desired result. For each integer $i \in \{0, \ldots, v+n-1\}$ and for each integer $j \in \{1, \ldots, p-2\}$, define $d_{i,j} := F_n - j - ps_i$. Let $D(n) := \{d_{i,j}\}_{i,j}$.

We will now show that if we adjoin the set $D(n)$ to S_n, we obtain a symmetric numerical semigroup that satisfies our objectives. For clarity, we will provide the proof in separate installments. Our first goal is to show that $D(n) \subseteq S$.

Proposition 3.0.16. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $d_{i,j} > pF(S)$ for each $i \in \{0, \ldots, v+n-1\}$ and for each $j \in \{1, \ldots, p-2\}$. In particular, $D(n) \subseteq S$.*

Proof. Let $D := D(n)$. Note that $d_{v+n-1,p-2} = \min(D)$. Therefore, it suffices to show that $d_{v+n-1,p-2} > pF(S)$. By definition, $d_{v+n-1,p-2} = F_n - p + 2 - ps_{v+n-1}$. Since $F_n = pF(S) + m_n + p - 2$ and $m_n = pF(S) + 2pn + p + 1$, we have that $d_{v+n-1,p-2} = 2pF(S) + 2np + p + 1 - ps_{v+n-1}$.

If $n = 0$, then $s_{v+n-1} = s_{v-1} < F(S)$, and so $d_{v+n-1,p-2} > pF(S)+p+1 > pF(S)$. Thus, without loss of generality, $n \geq 1$. Then $s_{v+n-1} = s_v+n-1 = F(S)+n$ (since $F(S) = s_v - 1$). Therefore, $d_{v+n-1,p-2} = 2pF(S)+2np+p+1-ps_{v+n-1} = 2pF(S)+2np+p+1-p(F(S)+n) = pF(S) + np + p + 1 > pF(S)$. □

Next, we show that $S_n \cap D(n) = \emptyset$.

Proposition 3.0.17. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $d_{i,j} \notin S_n$ for each $i \in \{0, \ldots, v+n-1\}$ and for each $j \in \{1, \ldots, p-2\}$.*

Proof. Suppose the assertion is not true and there exists $i \in \{0, \ldots, v+n-1\}$ and $j \in \{1, \ldots, p-2\}$ such that $d_{i,j} \in S_n$. By definition, $d_{i,j} = F_n - j - ps_i$. Since $F_n = pF(S) + m_n+p-2$ and $m_n = pF(S)+2pn+p+1$, we have that $d_{i,j} = 2pF(S)+2pn+2p-1-j-ps_i$. Hence, $d_{i,j} \equiv p-(1+j) \pmod{p}$.

Let $A := A(S)$ and let $H := H(S)$. Recall that $m_n \equiv 1 \pmod{p}$ and note that $p-(1+j) \in \{1, \ldots, p-2\}$. Thus, if $d_{i,j} \in S_n$, then we must have that $d_{i,j} \in pA + m_n +$

$p-(j+2)$. Moreover, by Proposition 3.0.8, we have that $d_{i,j} \in pS + m_n + p - (j+2)$ or $d_{i,j} \in pH + m_n + p - (j+2)$. We consider these two cases separately.

First, consider the case when $d_{i,j} \in pS + m_n + p - (j+2)$. Note that by the definitions of $d_{i,j}$ and F_n, we have that $d_{i,j} = F_n - j - ps_i = pF(S) + m_n + p - 2 - j - ps_i = p(F(S)-s_i)+m_n+p-(j+2)$. Therefore, $F(S)-s_i \in S$. Note that if $F(S)-s_i = s'$ for some $s' \in S$, then $F(S) = s_i + s' \in S$, which is a contradiction. Hence, $d_{i,j} \notin pS+m_n+p-(j+2)$.

Next, consider the case when $d_{i,j} \in pA + m_n + p - (j+2)$. By a similar argument, since $d_{i,j} = p(F(S) - s_i) + m_n + p - (j+2)$, we must have that $F(S) - s_i = a$ for some $a \in A$. Note that we now have that $F(S) - a = s_i \in S$, contradicting $a \in A$. Hence, $d_{i,j} \notin pA + m_n + p - (j+2)$ and it follows that $d_{i,j} \notin S_n$. □

Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Define $T_n := S_n \cup D(n)$. We will show that T_n is a symmetric subnumerical semigroup of S such that $F(T_n) = F_n$ and $S = \frac{T_n}{p}$. To improve the clarity of the exposition, we will separate the proof into the next two propositions.

Proposition 3.0.18. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then T_n is a subnumerical semigroup of S with Frobenius number F_n.*

Proof. We claim that T_n is a subnumerical semigroup of S. Let $A := A(S)$, let $H := H(S)$, let $D := D(n)$ and let $v := |N(S)|$. Then $S = \{0 = s_0, s_1, \dots, s_{v-1}, s_v, \rightarrow\}$. Since $S_n \subseteq S$ by Proposition 3.0.10 and $D \subseteq S$ by Proposition 3.0.16, the claim will follow if we show that T_n is a numerical semigroup. Clearly $0 \in T_n$. Since S_n is a numerical semigroup, $\mathbb{N}\backslash S_n$ is finite. Moreover, since $S_n \subseteq T_n$, note that $\mathbb{N}\backslash T_n \subseteq \mathbb{N}\backslash S_n$. Therefore, $\mathbb{N}\backslash T_n$ is finite.

It remains to be shown that T_n is closed under addition. Since S_n is a numerical semigroup contained in T_n, it is clear that $S_n + S_n \subseteq T_n$. Therefore, we must show that $D + D \subseteq T_n$ and that $D + S_n \subseteq T_n$.

We show that $D + D \subseteq T_n$. Note that $d_{v+n-1,p-2} = \min(D)$. Thus, since $F(S_n) = F_n$ by Proposition 3.0.10, it suffices to show that $2d_{v+n-1,p-2} > F_n$, for then $D + D \subseteq S_n \subseteq T_n$. We consider separately the cases $n = 0$ and $n \geq 1$.

First, consider the case when $n = 0$. Then $s_{v+n-1} = s_{v-1} < F(S)$. Hence, $2d_{v-1,p-2} = 2(F_0 - p + 2 - ps_{v-1}) > 2(F_0 - p + 2 - pF(S))$. By the definitions of F_n and m_n, note that $2(F_0 - p + 2 - pF(S)) = 2F_0 - 2p + 4 - 2pF(S) = F_0 + pF(S) + m_0 + p - 2 - 2p + 4 - 2pF(S) = F_0 + 3 > F_0$. Hence, $2d_{v-1,p-2} \in S_n$.

Next, consider the case when $n \geq 1$. Then $s_{v+n-1} = s_v + n - 1 = F(S) + n$. Hence, $2d_{v+n-1,p-2} = 2(F_n - p + 2 - ps_{v+n-1}) = 2(F_n - p + 2 - p(F(S) + n))$. By the definition of F_n, note that $2(F_n - p + 2 - p(F(S) + n)) = F_n + pF(S) + m_n - p + 2 - 2pF(S) - 2pn$. By the definition of m_n, we have that $F_n + pF(S) + m_n - p + 2 - 2pF(S) - 2pn = F_n + 3 > F_n$. This completes the proof that $D + D \subseteq S_n$.

We next show that $D + S_n \subseteq T_n$. To do this, it suffices to show that $D + pS \subseteq T_n$ and that $D + pA + m_n + b \subseteq T_n$ for each $b \in \{0, \dots p-2\}$.

We show that $D + pS \subseteq T_n$. Let $d_{i,j} \in D$ and let $ps \in pS$. If $s > s_i$, then $d_{i,j} + ps = F_n - j - ps_i + ps = p(F(S) + s - s_i) + m_n + p - 2 - j \in pS + m_n + p - 2 - j \subseteq pA(S) + m_n + p - 2 - j \subseteq S_n \subseteq T_n$. If $s = s_i$, then $d_{i,j} + ps = F_n - j = d_{0,j} \in D \subseteq T_n$. Therefore, it remains to show that if $s < s_i$, then $d_{i,j} + ps \in T_n$.

Suppose that $s < s_i$. Then $s = s_r$ for some $r \in \{0, \dots, i-1\}$ and $s_i - s_r \in \mathbb{N}$. If $s_i - s_r \in S$, then $s_i - s_r = s_u$ for some $u \in \{0, 1, \dots, v+n-1\}$ and it is clear that

$d_{i,j} + ps_r = F_n - j - p(s_i - s_r) = d_{u,j} \in D \subseteq T_n$. Therefore, we may suppose that $s_i - s_r \in \mathbb{N}\backslash S$. Then $d_{i,j} + ps_r = p(F(S) - (s_i - s_r)) + m_n + p - 2 - j$. If $s_i - s_r \notin H$, then $F(S) - (s_i - s_r) \in S$ and, since $j \in \{1, \ldots, p-2\}$, we have that $d_{i,j} + ps_r \in pS + m_n + p - 2 - j \subseteq pA + m_n + p - 2 - j \subseteq S_n \subseteq T_n$. Hence, we may suppose that $s_i - s_r \in H$. Then $F(S) - (s_i - s_r) \notin S$ and $F(S) - (F(S) - (s_i - s_r)) = s_i - s_r \in \mathbb{N}\backslash S$ and, by definition, $F(S) - (s_i - s_r) \in H$. Therefore, $d_{i,j} + ps_r = p(F(S) - (s_i - s_r)) + m_n + p - 2 - j \in pH + m_n + p - 2 - j \subseteq pA + m_n + p - 2 - j \subseteq S_n \subseteq T_n$. This completes the proof that $D + pS \subseteq T_n$.

We now have only to show that $D + pA + m_n + b \subseteq T_n$ for each $b \in \{0, \ldots p - 2\}$. Let $d_{i,j} \in D$, let $a \in A$ and let $b \in \{0, \ldots p-2\}$. Then $d_{i,j} + pa + m_n + b = F_n - j - ps_i + pa + m_n + b$. By definition of m_n, we have that $d_{i,j} + pa + m_n + b = F_n - j - ps_i + pa + pF(S) + 2pn + p + 1 + b$. Since $i \in \{0, 1, \ldots, v+n-1\}$, $s_i \leq s_{v+n-1}$. Suppose first that $n = 0$. Then $s_i \leq s_{v-1} < F(S)$. Therefore, $F_n - j - ps_i + pa + pF(S) + 2pn + p + 1 + b > F_n - j + pa + 2pn + p + 1 + b$. Since $j < p$, note that $d_{i,j} + pa + m_n + b > F_n + pa + 2pn + 1 + b \geq F_n + 1 > F_n = F(S_n)$. Hence, $d_{i,j} + pa + m_n + b \in S_n \subseteq T_n$. In the remaining case, $n \geq 1$. Then $s_i \leq s_{v+n-1} = s_v + n - 1$. Therefore, $F_n - j - ps_i + pa + pF(S) + 2pn + p + 1 + b \geq F_n - j - p(s_v + n - 1) + pa + pF(S) + 2pn + p + 1 + b = F_n - j - p(s_v - 1) - pn + pa + pF(S) + 2pn + p + 1 + b$. Note that $pF(S) = p(s_v - 1)$. Thus, $F_n - j - p(s_v - 1) - pn + pa + pF(S) + 2pn + p + 1 + b = F_n - j + pa + pn + p + 1 + b$. Since $1 \leq j \leq p - 2$, we have that $F_n - j + pa + pn + p + 1 + b \geq F_n + pa + pn + 3 + b$. Finally, since $a, b \geq 0$, it follows that $F_n + pa + pn + 3 + b > F_n = F(S_n)$ and $d_{i,j} + pa + m_n + b \in S_n \subseteq T_n$. This completes the proof that T_n is closed under addition and is therefore a numerical semigroup.

To see that $F(T_n) = F_n$, note that since $S_n \subseteq T_n$, it suffices to show that $F_n \notin T_n$. Suppose this assertion is not true and $F_n \in T_n$. It is clear by Proposition 3.0.10 that if $F_n \in T_n$, we must have that $F_n \in D$. Thus, $F_n = d_{i,j}$ for some $i \in \{0, \ldots, v + n - 1\}$ and some $j \in \{1, \ldots, p - 2\}$. Since $j > 0$, note that $F_n = F_n - j - ps_i < F_n$, which is a contradiction. Therefore, $F(T_n) = F_n$. □

Proposition 3.0.19. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then T_n is symmetric and $S = \frac{T_n}{p}$.*

Proof. Let $D := D(n)$. We show that $S = \frac{T_n}{p}$. For that purpose, it suffices, by Proposition 2.1.4, to show that if $x \in \mathbb{N}$, then $x \in S$ if and only if $px \in T_n$.

Let $x \in S$. Then clearly $px \in pS \subseteq S_n \subseteq T_n$. For the converse, let $x \in \mathbb{N}$ with $px \in T_n$. By definition, $T_n = S_n \cup D$. It is easy to check that $D \cap p\mathbb{N} = \emptyset$. Thus, $px \in S_n$. By Proposition 3.0.10, $S = \frac{S_n}{p}$. Hence, by Proposition 2.1.4, $x \in S$. This proves that $S = \frac{T_n}{p}$.

To show that T_n is symmetric, by Proposition 1.2.1 it suffices to show that $|N(T_n)| = \frac{F(T_n)+1}{2}$. Since $T_n = S_n \cup D$ and $S_n \cap D = \emptyset$ by Proposition 3.0.17, note that $|N(T_n)| = |N(S_n, T_n)| + |N(D, T_n)|$. By Propositions 3.0.18 and 3.0.10, $|N(S_n, T_n)| = |N(S_n)|$. Moreover, since $S_n \cap D = \emptyset$, we have that $d_{i,j} < F(S_n) = F_n = F(T_n)$ for each $i \in \{0, \ldots, v+n-1\}$ and for each $j \in \{1, \ldots, p-2\}$. Therefore, $|N(D, T_n)| = |D|$. Note that $|D| = (p-2)(v+n)$ so, by Proposition 3.0.14, $|D| = k_n$. By definition, $k_n = \frac{F(S_n)+1}{2} - |N(S_n)|$. Thus, $|N(T_n)| = |N(S_n)| + |D| = |N(S_n)| + \frac{F(S_n)+1}{2} - |N(S_n)| = \frac{F(S_n)+1}{2} = \frac{F(T_n)+1}{2}$, completing the proof. □

We now prove the main result of this section, which generalizes Theorem 3.0.7. As noted at the beginning of this chapter, the following also appears as Theorem 5 of [21].

Theorem 3.0.20. *Let S be a numerical semigroup and let $k \geq 2$ be a positive integer. Then $S = \frac{T}{k}$ for infinitely many symmetric numerical semigroups T.*

Proof. By Proposition 2.1.3, it suffices to prove the statement for the case when k is a prime number. If $k = 2$, then the result holds by Theorem 3.0.7. Therefore, we may suppose that k is an odd prime. By Proposition 3.0.18, note that if n_1 and n_2 are distinct nonnegative integers, then the symmetric numerical semigroups T_{n_1} and T_{n_2} satisfy $\frac{T_{n_1}}{k} = S = \frac{T_{n_2}}{k}$. Moreover, $T_{n_1} \neq T_{n_2}$ since $F(T_{n_1}) = F_{n_1} \neq F_{n_2} = F(T_{n_2})$. Thus, different nonnegative integers n give rise to different symmetric numerical semigroups T_n and the result follows. □

We now demonstrate that results involving fractions of numerical semigroups can have ring-theoretic applications. Let K be a field and let S be a numerical semigroup. The set $K[[S]]$ of all formal power series $\sum_{s \in S} k_s x^s \in K[[x]]$ is a subring of $K[[x]]$ known as the **semigroup ring associated to** S. For more information on commutative semigroup rings, see [3].

The following is a generalization of Theorem 10 of [17].

Proposition 3.0.21. *Let p be a prime number and let K be a field of characteristic p. Let S be a numerical semigroup. Then there exist infinitely many Gorenstein subrings R of $K[[S]]$ such that $K[[S]] = \{f \in K[[x]] \mid f^p \in R\}$.*

Proof. By Theorem 3.0.20, for each positive integer n, there exists a symmetric numerical semigroup T_n of S such that $S = \frac{T_n}{p}$. By Theorem 1.2.3, $K[[T_n]]$ is a Gorenstein ring for each n. Therefore, we have only to show that $K[[S]] = \{f \in K[[x]] \mid f^p \in K[[T_n]]\}$. We will show that both inclusions hold.

Suppose that $f \in K[[S]]$. Then $f = \sum_{s \in S} k_s x^s$ where each $k_s \in K$. Since K has characteristic p, we have that $f^p = (\sum_{s \in S} k_s x^s)^p = \sum_{s \in S} k_s^p x^{ps} \in K[[T_n]]$ (since each $ps \in T_n$ by Proposition 2.1.4).

We now prove the reverse inclusion. Let $f \in K[[x]]$ and suppose that $f^p \in K[[T_n]]$. Since $f \in K[[x]]$, we may consider $f = \sum_{i \geq 0} b_i x^i$ where each $b_i \in K$. Since $f^p \in K[[T_n]]$ and K has characteristic p, note that $\sum_{i \geq 0} b_i^p x^{ip} = (\sum_{i \geq 0} b_i x^i)^p = f^p \in K[[T_n]]$. Hence, $ip \in T_n$ for each i such that $b_i \neq 0$. Since $S = \frac{T_n}{p}$, by Proposition 2.1.4 it follows that $i \in S$ for each i such that $b_i \neq 0$. Therefore, $f \in K[[S]]$, completing the proof. □

In Theorem I.4.4 of [1] and in Theorem 7 of [13], all pseudo-symmetric numerical semigroups with multiplicity 3 are characterized. Let S be a numerical semigroup and let n be a positive integer. By Proposition 2.1.4, it is easy to verify that $n \in S$ if and only if $\frac{S}{n} = \mathbb{N}$. Using this observation, we conclude this chapter by offering the following result which uses fractions of numerical semigroups to characterize, in addition to $\langle 2, 3\rangle$, the symmetric numerical semigroups with multiplicity 3.

Proposition 3.0.22. *Let T be a numerical semigroup. The following are equivalent:*
(1) $\mathbb{N} = \frac{T}{3}$ *and T is symmetric;*
(2) $T = \langle 3, m\rangle$ *for some $m \in \mathbb{N} \backslash 3\mathbb{N}$;*
(3) *Either $T = \mathbb{N}$, $T = \langle 2, 3\rangle$, or T is symmetric with multiplicity 3.*

Proof. We prove (2) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) follows easily from Theorem 1.1.1 and the comment preceding Proposition 1.2.1. The implication (3) $\Rightarrow$ (1) holds by Proposition 2.1.4. It remains to show that (1) $\Rightarrow$ (2).

Suppose that $\mathbb{N} = \frac{T}{3}$ for some symmetric numerical semigroup T with Frobenius number $F(T)$. Define $S := 3\mathbb{N} \cup \{F(T)+1, \rightarrow\}$. Note that S is a subnumerical semigroup of T and $F(S) = F(T)$.

If $T = \mathbb{N}$, then $T = \langle 1, 3\rangle$ and the result holds. Thus, we may suppose $T \neq \mathbb{N}$. Since T is symmetric, $F(T)$ must be odd. Moreover, since $\mathbb{N} = \frac{T}{3}$, by Proposition 2.1.4 we have that $3\mathbb{N} \subseteq T$ so $F(T) \notin 3\mathbb{N}$. Therefore, $F(T) \equiv 1$ or 5 modulo 6. So, either $F(T) = 3(2k+1)+2$ or $F(T) = 3(2k)+1$ for some nonnegative integer k. We consider these two cases separately.

First, consider the case when $F(T) = 3(2k+1)+2$. We claim that $T \backslash S = \{3(0)+1, 3(1)+1, \ldots, 3(2k+1)+1 = F(T)-1\}$. Since T is symmetric, by Proposition 1.2.1, $|N(T)| = \frac{F(T)+1}{2} = 3k+3$. Furthermore, note that $|N(S)| = |\{3(0), 3(1), \ldots, 3(2k+1)\}| = 2k+2$. By Proposition 1.2.1, S is symmetric if and only if $k = -1$. Note that $k = -1$ if and only if $F(T) = -1$; and $F(T) = -1$ if and only if $T = \mathbb{N}$. Thus, we may suppose that S is not symmetric. In particular, $S \subset T$.

Let $x \in T \backslash S$. By the construction of S, note that $x < F(T)$ and $x \notin 3\mathbb{N}$. Furthermore, x is not congruent to $F(T)$ modulo 3 (otherwise there exists $i \in \mathbb{N}$ such that $F(T) = x+3i \in T$, a contradiction). Thus, $x \equiv 1 \pmod 3$. It follows that $T \backslash S \subseteq \{3(0)+1, 3(1)+1, \ldots, 3(2k+1)+1 = F(T)-1\}$. Since $F(T) = F(S)$ and T is symmetric, $|T \backslash S| = |N(T)| - |N(S)| = (3k+3)-(2k+2) = k+1$. Note that if $z \in T$ then $z + 3\mathbb{N} \subseteq T$. Hence, we must have that $T \backslash S = \{3(k+1)+1, 3(k+2)+1, \ldots, 3(2k+1)+1\}$.

Let $m := 3(k+1)+1$. By Theorem 1.1.1, $\langle 3, m\rangle$ is a numerical semigroup with Frobenius number $2m-3$. Since $\langle 3, m\rangle$ has embedding dimension 2, $\langle 3, m\rangle$ is symmetric. Therefore, by definition of symmetric, $\langle 3, m\rangle$ is maximal with Frobenius number $2m-3$. Note that $2m-3 = 6k+5 = 3(2k+1)+2 = F(T)$. Since $\langle 3, m\rangle \subseteq T$ and $\langle 3, m\rangle$ is maximal with Frobenius number $F(T)$, we must have that $\langle 3, m\rangle = T$. This completes the proof for the case when $F(T) = 3(2k+1)+2$.

Lastly, we consider the case $F(T) = 3(2k)+1$. Then $|N(S)| = |\{3(0), 3(1), \ldots, 3(2k)\}| = 2k+1$. By a similar argument as in the previous case, we can suppose S is not symmetric and $T \backslash S = \{3k+2, 3(k+1)+2, \ldots, 3(2k-1)+2\}$. Define $m := 3k+2$. By Theorem 1.1.1, the Frobenius number of $\langle 3, m\rangle$ is $2m-3$. Note that $2m-3 = 3(2k)+1 = F(T)$. Since $\langle 3, m\rangle$ is symmetric, hence maximal with Frobenius number $F(T)$, and $\langle 3, m\rangle \subseteq T$, we must have that $\langle 3, m\rangle = T$. $\square$

Remark 3.0.23. The preceding proof shows that for each $F \in \mathbb{N}$ of the form $3m-3-m$ where $\gcd(3, m) = 1$, there exists a unique symmetric numerical semigroup T with Frobenius number F which satisfies $\mathbb{N} = \frac{T}{3}$. In general, if p is an odd prime greater than 3, there may be more than one symmetric numerical semigroup A such that $\frac{A}{p} = \mathbb{N}$. For example, let $A := \langle 3, 5\rangle = \{0, 3, 5, 6, 8, \rightarrow\}$ and let $B := \langle 4, 5, 6\rangle = \{0, 4, 5, 6, 8, \rightarrow\}$. Note that $F(A) = F(B) = 7$. By condition (5) of Proposition 1.2.1, both A and B are symmetric. Moreover, $\frac{A}{5} = \mathbb{N} = \frac{B}{5}$, even though $A \neq B$.

Chapter 4

Fractions of Pseudo-symmetric Numerical Semigroups

We will now explore a pseudo-symmetric analog of Theorem 3.0.20. Let S be a numerical semigroup. It is interesting to note that, unlike the situation in Theorem 3.0.20, there may not exist a pseudo-symmetric numerical semigroup T such that $S = \frac{T}{2}$. In fact, the following is shown in Theorem 15 of [14].

Theorem 4.0.24 (Rosales)**.** *Let S be a numerical semigroup. There exists a pseudo-symmetric numerical semigroup T such that $S = \frac{T}{2}$ if and only if S is irreducible.*

For an example of a numerical semigroup which is not irreducible, see the comments following Proposition 1.2.2.

Combining Theorems 3.0.7 and 4.0.24 with Proposition 2.1.3, the following was proven in Theorem 13 of [14].

Theorem 4.0.25 (Rosales)**.** *Let S be a numerical semigroup. There exist infinitely many pseudo-symmetric numerical semigroups T such that $S = \frac{T}{4}$.*

Theorem 4.0.25 was later generalized in Theorem 6 of [21]. Our goal in this section is to offer a new proof of this generalization. More precisely, we will show that if S is a numerical semigroup and $k \geq 3$ is a positive integer, then S can be expressed as $S = \frac{T}{k}$ for infinitely many pseudo-symmetric numerical semigroups T. In light of Proposition 2.1.3 and Theorem 3.0.7, we have only to show that if S is a numerical semigroup and p is an odd prime number, then S can be expressed as $S = \frac{T}{p}$ for infinitely many pseudo-symmetric numerical semigroups T.

The proof will very closely mimic the proof from Chapter 3, where the construction from Lemma 3 of [12] was generalized to an arbitrary odd prime. As in the previous argument, when $p > 2$ the construction will fail to be pseudo-symmetric except in one special case. We shall have to adjoin additional elements as before in order to obtain a pseudo-symmetric numerical semigroup with all of the desired properties. Because of the similarity of the construction, occasionally we will withhold a full proof and instead direct the reader to those exact changes which will make a modification of an earlier proof successful.

Let S be a numerical semigroup with Frobenius number $F(S)$. Let p be an odd prime number. Let $A := A(S)$. For each $n \in \mathbb{N}$, define $m'_n := pF(S) + 2pn + 2p + 1$. Note that

$m'_n = m_n + p$ and so $m'_n \equiv 1 \pmod{p}$. Let $F'_n := pF(S) + m'_n + p - 2$; i.e., $F'_n = F_n + p$. Define $S'_n := pS \cup (pA + m'_n) \cup (pA + m'_n + 1) \cup \cdots \cup (pA + m'_n + p - 2)$.

The proof of the following result is identical to the proof of Proposition 3.0.9, with the exceptions that m'_n is substituted for m_n and F'_n is substituted for F_n in the original proof.

Proposition 4.0.26. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then S'_n contains every positive integer greater than F'_n.*

Just as the proof of the previous result followed closely the proof of Proposition 3.0.9, the following proposition may be proven by following the same argument as Proposition 3.0.10. To prove it, simply substitute m'_n, F'_n and their respective formulas for m_n, F_n and their respective formulas. We omit the proof.

Proposition 4.0.27. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then S'_n is a subnumerical semigroup of S, $F(S'_n) = F'_n$ and $S = \frac{S'_n}{p}$.*

The following is an analog of Proposition 3.0.11. We include the proof, which is similar to that of Proposition 3.0.11, though not identical.

Proposition 4.0.28. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $|N(S'_n)| = p|N(S)| + (p-1)|H(S)| + F(S) + 2n + 2$.*

Proof. Let $A := A(S)$ and $H := H(S)$. Recall that $S'_n := pS \cup (pA + m'_n) \cup (pA + m'_n + 1) \cup \cdots \cup (pA + m'_n + p - 2)$. Note that this union is disjoint because each element of pS is congruent to 0 modulo p and, since $m'_n \equiv 1 \pmod{p}$, each element of $pA + m'_n + i$ is congruent to $i + 1$ modulo p for each $i \in \{0, \ldots, p-2\}$. Thus, $|N(S'_n)| = |N(pS, S'_n)| + |N(pA + m'_n, S'_n)| + |N(pA + m'_n + 1, S'_n)| + \cdots + |N(pA + m'_n + p - 2, S'_n)|$. We consider each of these p disjoint sets separately.

By definition, $N(pS, S'_n) = \{px \in pS \mid px < F(S'_n)\}$. By Proposition 4.0.27, $F(S'_n) = F'_n = pF(S) + m'_n + p - 2$. Therefore, $N(pS, S'_n) = \{px \in pS \mid px < pF(S) + m'_n + p - 2\}$. Note that $\{px \in pS \mid px < pF(S) + m'_n + p - 2\}$ is the disjoint union $\{px \in pS \mid px \leq pF(S)\} \cup \{px \in pS \mid pF(S) < px < pF(S) + m'_n + p - 2\}$. Clearly, $\{px \in pS \mid px \leq pF(S)\} = p\{x \in S \mid x \leq F(S)\} = pN(S)$ and $|pN(S)| = |N(S)|$. By definition, $m'_n = pF(S) + 2pn + 2p + 1$. Hence, $pF(S) + m'_n + p - 2 = p(2F(S) + 2n + 3) - 1$ and so $\{px \in pS \mid pF(S) < px < pF(S) + m'_n + p - 2\} = \{px \in pS \mid pF(S) < px < p(2F(S) + 2n + 3) - 1\} = \{p(F(S) + 1), p(F(S) + 2), \ldots, p(2F(S) + 2n + 2)\}$. Therefore, $|N(pS, S'_n)| = |N(S)| + F(S) + 2n + 2$.

Now let $i \in \{0, \ldots, p-2\}$ and consider $N(pA + m'_n + i, S'_n)$. By definition, $N(pA + m'_n + i, S'_n) = \{x \in pA + m'_n + i \mid x < F(S'_n)\}$. Since $F(S'_n) = F'_n = pF(S) + m'_n + p - 2$, we have that $N(pA + m'_n + i, S'_n) = \{x \in pA + m'_n + i \mid x < pF(S) + m'_n + p - 2\}$. Note that by Proposition 3.0.8, A is the disjoint union $S \cup H$. Thus, $N(pA + m'_n + i, S'_n)$ is the disjoint union $\{px + m'_n + i \in pS + m'_n + i \mid px + m'_n + i < pF(S) + m'_n + p - 2\} \cup \{py + m'_n + i \in pH + m'_n + i \mid py + m'_n + i < pF(S) + m'_n + p - 2\}$. We consider these two disjoint sets separately.

Let $k := |N(S)|$. Then $S = \{0 = s_0, s_1, \ldots, s_{k-1}, s_k, \rightarrow\}$. Note that since $s_k = F(S) + 1$ and $0 \leq i < p$, $ps_k + m'_n + i = pF(S) + p + m'_n + i \geq pF(S) + p + m'_n > pF(S) + m'_n + i$. Therefore, $\{px + m'_n + i \in pS + m'_n + i \mid px + m'_n + i < pF(S) + m'_n + p - 2\} = \{ps_0 + m'_n + i, ps_1 + m'_n + i, \ldots, ps_{k-1} + m'_n + i\}$, which has cardinality $N(S)$. Now consider the set $\{py+m'_n+i \in pH+m'_n+i \mid py+m'_n+i < pF(S)+m'_n+p-2\}$. Note that $F(S) \notin H$. Thus, if $y \in H$, then $y < F(S)$. Hence, $py+m'_n+i < pF(S)+m'_n+p-2$ for all $y \in H$ and $\{py+m'_n+i \in pH+m'_n+i \mid py+m'_n+i < pF(S)+m'_n+p-2\}$ has cardinality $|H|$. Therefore, $|N(pA+m'_n+i, S'_n)| = |N(S)|+|H|$ for each $i \in \{0,\dots,p-2\}$.

By combining the above observations, we have that $|N(S'_n)| = |N(pS,S'_n)| + |N(pA+m'_n, S'_n)| + |N(pA+m'_n+1, S'_n)| + \cdots + |N(pA+m'_n+p-2, S'_n)| = |N(S)|+F(S)+2n+2+(p-1)(|N(S)|+|H|) = p|N(S)|+(p-1)|H(S)|+F(S)+2n+2$. ☐

Combining Propositions 3.0.12 and 4.0.28, as in the proof of Proposition 3.0.13, gives the following.

Proposition 4.0.29. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $|N(S'_n)| = pF(S)+2n+p+1-(p-2)|N(S)|$.*

Proof. Since S is a numerical semigroup, $|\mathbb{N}\backslash S| = F(S)+1-|N(S)|$. By Proposition 3.0.12, $|H(S)| = 2|\mathbb{N}\backslash S| - F(S) - 1$. Combining these results with Proposition 4.0.28 gives $|N(S'_n)| = p|N(S)|+(p-1)|H(S)|+F(S)+2n+2 = p|N(S)|+(p-1)(2|\mathbb{N}\backslash S|-F(S)-1)+F(S)+2n+2 = p|N(S)|+(p-1)(2(F(S)+1-|N(S)|)-F(S)-1)+F(S)+2n+2 = pF(S)+2n+p+1-(p-2)|N(S)|$. ☐

Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Recall Proposition 1.2.2, which states that a numerical semigroup T with even Frobenius number $F(T)$ is pseudo-symmetric if and only if $|N(T)| = \frac{F(T)}{2}$. Recall also that $F(S'_n) = F'_n = pF(S)+m'_n+p-2$ and $m'_n := pF(S)+2pn+2p+1$. Since p is odd, it is easy to verify that $F(S'_n)$ is even and that $\frac{F(S'_n)}{2}$ is an integer.

We now define $k'_n := \frac{F(S'_n)}{2} - |N(S'_n)|$. In light of Proposition 1.2.2, k'_n may be viewed as the number of elements that would have to be "adjoined" to S'_n in order to obtain a pseudo-symmetric numerical semigroup with Frobenius number F'_n.

As in the previous chapter, we first obtain alternative formulas for k'_n.

Proposition 4.0.30. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $k'_n = \frac{p-3}{2}+(p-2)(n+|N(S)|)$.*

Proof. By Proposition 4.0.27, $k'_n = \frac{F(S'_n)}{2} - |N(S'_n)| = \frac{F'_n}{2} - |N(S'_n)| = \frac{pF(S)+m'_n+p-2}{2} - |N(S'_n)|$. Recall that $m'_n = pF(S)+2pn+2p+1$. Therefore, $k'_n = \frac{2pF(S)+2pn+2p+p-1}{2} - |N(S'_n)| = pF(S)+pn+p+\frac{p-1}{2} - |N(S'_n)|$. By Proposition 4.0.29, $k'_n = pF(S)+pn+p+\frac{p-1}{2} - (pF(S)+2n+p+1-(p-2)|N(S)|) = \frac{p-3}{2}+(p-2)(n+|N(S)|)$. ☐

As promised, we can now demonstrate that when $p > 2$, the construction from Lemma 3 of [12] fails to be pseudo-symmetric except in a very special case.

Proposition 4.0.31. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. The following are equivalent:*

(1) *S'_n is pseudo-symmetric;*
(2) *$k'_n = 0$;*
(3) *$p = 3$, $n = 0$ and $S = \mathbb{N}$.*

Proof. We prove (1)⇒(2)⇒(3)⇒(1).

Note that (1)⇒(2) by Proposition 1.2.2.

We show that (2)⇒(3). Suppose $k'_n = 0$. By Proposition 4.0.30, $k'_n = \frac{p-3}{2} + (p-2)(n + |N(S)|)$. Since $\frac{p-3}{2} \in \mathbb{N}$, note that $(p-2)(n+|N(S)|) \in \mathbb{N}$ and $p > 2$, we must have that $p = 3$ and $n + |N(S)| = 0$. Therefore, both n and $|N(S)|$ must be 0. Furthermore, since $|N(S)| = 0$, we must have that $S = \mathbb{N}$.

Finally, we show that (3)⇒(1). If $p = 3$, $n = 0$ and $S = \mathbb{N}$, then $k'_n = 0$ by Proposition 4.0.30. Therefore, by definition of k'_n, we have that $\frac{F(S'_n)}{2} = |N(S'_n)|$. Hence, S'_n is pseudo-symmetric by Proposition 1.2.2. □

Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. In light of Proposition 4.0.31, from now on we will always assume that S'_n is not pseudo-symmetric. The goal now is to adjoin exactly k'_n elements of $S \backslash S'_n$ to S'_n so that the resulting set, which will be denoted T'_n, is a pseudo-symmetric numerical semigroup with Frobenius number $F(T'_n)$ which satisfies $S = \frac{T'_n}{p}$.

Let $v := |N(S)|$. As before, it will be convenient to index the elements $\{s_i\}_{i\in\mathbb{N}}$ of S so that $s_i < s_j$ whenever $i < j$. Then $S = \{0 = s_0, s_1, \ldots, s_{v-1}, s_v, \rightarrow\}$. Note that $F(S) = s_v - 1$ and, for each integer $i \geq 0$, $s_{v+i} = s_v + i$.

We now define the k'_n elements of $S\backslash S'_n$ which can be adjoined to S'_n in order to obtain the desired result. For each integer $i \in \{0, 1, \ldots, n+v\}$ and for each integer $j \in \{0, 1, \ldots, p-3\}$, define $d'_{i,j} := pF(S) + m'_n + j - ps_i$. Let $D'(n) := \{d'_{i,j} \mid 0 \leq i < n+v, 0 \leq j \leq p-3\} \cup \{d'_{n+v,j} \mid \frac{p-1}{2} \leq j \leq p-3\}$.

We will now show that if we adjoin the set $D'(n)$ to S'_n, we obtain a pseudo-symmetric numerical semigroup with the desired properties. For clarity, we will again present the proof in separate propositions. Our first goal is to show that $D'(n) \subseteq S$.

Proposition 4.0.32. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $d'_{i,j} > pF(S)$ for each $d'_{i,j} \in D'(n)$. In particular, $D'(n) \subseteq S$.*

Proof. Let $D := D'(n)$. We claim that $d'_{n+v,\frac{p-1}{2}} = \min(D)$. Note that $d'_{i,j}$ decreases as i increases and j decreases. Thus, it is easy to see that $d'_{n+v,\frac{p-1}{2}} = \min\{d'_{n+v,j} \mid \frac{p-1}{2} \leq j \leq p-3\}$ and that $d'_{n+v-1,0} = \min\{d'_{i,j} \mid 0 \leq i < n+v, 0 \leq j \leq p-3\}$.

To prove the claim, it now suffices to show that $d'_{n+v,\frac{p-1}{2}} < d'_{n+v-1,0}$. First, note that $s_v = F(S) + 1$. Hence, $s_{n+v} = F(S) + 1 + n$ for each $n \in \mathbb{N}$. If $n \geq 1$, it follows that $s_{n+v-1} = F(S) + 1 + n - 1 = s_{n+v} - 1$. If $n = 0$, notice that $s_{n+v-1} = s_{v-1} \leq F(S) - 1 \leq F(S) = s_v - 1 = s_{n+v} - 1$. In particular, $s_{n+v-1} \leq s_{n+v} - 1$ for all $n \in \mathbb{N}$. Therefore, by definition, $d'_{n+v,\frac{p-1}{2}} = pF(S) + m'_n + \frac{p-1}{2} - ps_{n+v} \leq pF(S) + m'_n + \frac{p-1}{2} - p(s_{n+v-1} + 1) < pF(S) + m'_n - ps_{n+v-1} = d'_{n+v-1,0}$. This completes the proof of the claim that $d'_{n+v,\frac{p-1}{2}} = \min(D)$.

To finish the proof of the proposition, it now suffices to show that $d'_{n+v,\frac{p-1}{2}} > pF(S)$. Recall that $m'_n = pF(S) + 2pn + 2p + 1$. Therefore, $d'_{n+v,\frac{p-1}{2}} = pF(S) + m'_n + \frac{p-1}{2} - ps_{n+v} = pF(S) + m'_n + \frac{p-1}{2} - p(F(S) + 1 + n) = 2pF(S) + 2pn + 2p + 1 + \frac{p-1}{2} - p(F(S) + 1 + n) = pF(S) + pn + p + 1 + \frac{p-1}{2} > pF(S)$. □

Next, we show that $D'(n) \cap S'_n = \emptyset$.

Proposition 4.0.33. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then $D'(n) \cap S'_n = \emptyset$.*

Proof. Suppose the assertion is not true and there exists $d'_{i,j} \in D'(n) \cap S'_n = \emptyset$. Let $A := A(S)$ and let $H := H(S)$. By definition, $d'_{i,j} = pF(S) + m'_n + j - ps_i = p(F(S) - s_i) + m'_n + j$ with $0 \leq j \leq p-3$. Therefore, by the construction of S'_n, we must have that $d'_{i,j} \in pA + m'_n + j$. By Proposition 3.0.8, $d'_{i,j} \in pS + m'_n + j$ or $d'_{i,j} \in pH + m'_n + j$. We consider these two cases separately.

If $d'_{i,j} \in pS + m'_n + j$, then $p(F(S) - s_i) + m'_n + j \in pS + m'_n + j$. Thus, $F(S) - s_i \in S$. It follows that $F(S) - s_i = s$ for some $s \in S$. Hence, $F(S) = s + s_i \in S$, a contradiction. Therefore, we may assume that $d'_{i,j} \in pH + m'_n + j$ and $F(S) - s_i = h$ for some $h \in H$.

Since $h \in H$, $F(S) - h \notin S$. However, note that $F(S) - h = F(S) - (F(S) - s_i) = s_i \in S$, contradicting $h \in H$. Therefore, $d'_{i,j} \notin S'_n$ and $D'(n) \cap S'_n = \emptyset$. □

Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Define $T'_n := S'_n \cup D'(n)$. We will now show that T'_n is a pseudo-symmetric numerical semigroup such that $F(T'_n) = F'_n$ and $S = \frac{T'_n}{p}$. As with the construction in Chapter 3, we present the proof in two separate propositions.

Proposition 4.0.34. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then T'_n is a subnumerical semigroup of S with Frobenius number F'_n.*

Proof. We first show that T'_n is a subnumerical semigroup of S. Let $A := A(S)$, let $D := D'(n)$ and let $v := |N(S)|$. Then $S = \{0 = s_0, s_1, \ldots, s_{v-1}, s_v, \rightarrow\}$. Since $S'_n \subseteq S$ by Proposition 4.0.27 and $D \subseteq S$ by Proposition 4.0.32, we have only to show that T'_n is a numerical semigroup. Clearly $0 \in pS \subseteq T'_n$. Since S'_n is a numerical semigroup, $\mathbb{N} \backslash S'_n$ is finite. Since $S'_n \subseteq T'_n$, we have that $\mathbb{N} \backslash T'_n$ is finite.

It remains to be shown that T'_n is closed under addition. Since S'_n is a numerical semigroup contained in T'_n, it is clear that $S'_n + S'_n \subseteq T'_n$. Therefore, we must show that $D + D \subseteq T'_n$ and that $D + S'_n \subseteq T'_n$.

We show that $D + D \subseteq T'_n$. Recall from the proof of Proposition 4.0.32 that $d'_{n+v,\frac{p-1}{2}} = \min(D)$. Thus, since $F(S'_n) = F'_n$ by Proposition 4.0.27, it suffices to show that $2d'_{n+v,\frac{p-1}{2}} > F'_n$, for then $D + D \subseteq S'_n \subseteq T'_n$. Note that since $s_{n+v} = F(S) + 1 + n$, we have that $2d'_{n+v,\frac{p-1}{2}} = 2pF(S) + 2m'_n + p - 1 - 2ps_{n+v} = 2pF(S) + 2m'_n + p - 1 - 2pF(S) - 2p - 2pn = 2m'_n + p - 1 - 2p - 2pn$. By the definition of m'_n, we have that $2m'_n + p - 1 - 2p - 2pn = pF(S) + 2pn + 2p + 1 + m'_n + p - 1 - 2p - 2pn = pF(S) + m'_n + p > pF(S) + m'_n + p - 2 = F'_n$.

We now show that $D + S'_n \subseteq T'_n$. To do this, it suffices to show that $D + pS \subseteq T'_n$ and that $D + pA + m'_n + b \subseteq T'_n$ for each $b \in \{0, \ldots p-2\}$.

We show that $D + pS \subseteq T'_n$. Let $d'_{i,j} \in D$ and let $ps \in pS$. Note that $0 \leq j \leq p-3$. If $s > s_i$, then $d'_{i,j} + ps = pF(S) + m'_n + j - ps_i + ps = p(F(S) + s - s_i) + m'_n + j \in pS + m'_n + j \subseteq pA + m'_n + j \subseteq S'_n \subseteq T'_n$. If $s = s_i$, then since $s_0 = 0$, we have that $d'_{i,j} + ps = d'_{i,j} + ps_i = pF(S) + m'_n + j - ps_i + ps_i = pF(S) + m'_n + j - ps_0 = d'_{0,j} \in D \subseteq T'_n$. We must show that $d'_{i,j} + ps \in T'_n$ when $s < s_i$.

Suppose $s < s_i$. Then $s = s_r$ for some $r \in \{0, \ldots, i-1\}$ and $s_i - s_r \in \mathbb{N}$. If $s_i - s_r \in S$, then $s_i - s_r = s_u$ for some $u \in \{0, 1, \ldots, v+n-1\}$ and it is clear that $d'_{i,j} + ps_r = pF(S) + m'_n + j - ps_u = d'_{u,j} \in D \subseteq T'_n$. Therefore, we may suppose that $s_i - s_r \in \mathbb{N} \backslash S$.

Then $d'_{i,j} + ps_r = p(F(S) - (s_i - s_r)) + m'_n + j$. If $s_i - s_r \notin H(S)$, then $F(S) - (s_i - s_r) \in S$ and, since $j \in \{0, \dots, p-3\}$, we have that $d'_{i,j} + ps_r \in pS + m'_n + j \subseteq pA + m'_n + j \subseteq S'_n \subseteq T'_n$. Hence, we may suppose that $s_i - s_r \in H(S)$. Then $F(S) - (s_i - s_r) \notin S$ and $F(S) - (F(S) - (s_i - s_r)) = s_i - s_r \in \mathbb{N} \backslash S$. Thus, by definition, $F(S) - (s_i - s_r) \in H(S)$. Therefore, $d'_{i,j} + ps_r = p(F(S) - (s_i - s_r)) + m'_n + j \in pH(S) + m'_n + j \subseteq pA + m'_n + j \subseteq S'_n \subseteq T'_n$. This completes the proof that $D + S'_n \subseteq T'_n$.

We now have only to show that $D + pA + m'_n + b \in T'_n$ for each $b \in \{0, \dots p-2\}$. Let $d'_{i,j} \in D$, let $a \in A$ and let $b \in \{0, \dots p-2\}$. We will prove the inclusion by showing that $d'_{i,j} + pa + m'_n + b > F(S'_n)$. By definition of $d'_{i,j}$, we have that $d'_{i,j} + pa + m'_n + b = pF(S) + 2m'_n + j - ps_i + pa + b$. Since $i \in \{0, 1, \dots, n+v\}$, we have that $s_i \leq s_{n+v} = F(S) + 1 + n$. Thus, $d'_{i,j} + pa + m'_n + b \geq pF(S) + 2m'_n + j - p(F(S) + 1 + n) + pa + b$. By definition of m'_n, note that $pF(S) + 2m'_n + j - p(F(S) + 1 + n) + pa + b = pF(S) + m'_n + pn + p + 1 + j + pa + b \geq pF(S) + m'_n + p + 1 + j + b > pF(S) + m'_n + p - 2 = F'_n = F(S'_n)$. This completes the proof that T'_n is closed under addition and is therefore a numerical semigroup.

To see that $F(T'_n) = F'_n$, note that since $S'_n \subseteq T'_n$, it suffices to show that $F'_n \notin T'_n$. Suppose this assertion is not true and $F'_n \in T'_n$. It is clear by Proposition 4.0.27 that if $F'_n \in T'_n$, we must have that $F'_n \in D$. Thus, $F'_n = d'_{i,j}$ for some i, j where $j \leq p-3$. It follows that $F'_n = d'_{i,j} = pF(S) + m'_n + j - ps_i \leq pF(S) + m'_n + j < pF(S) + m'_n + p - 2 = F'_n$, which is a contradiction. Therefore, $F(T'_n) = F'_n$. □

Proposition 4.0.35. *Let S be a numerical semigroup, let p be an odd prime number and let $n \in \mathbb{N}$. Then T'_n is pseudo-symmetric and $S = \frac{T'_n}{p}$.*

Proof. We show that $S = \frac{T'_n}{p}$. By Proposition 2.1.4, it suffices to show that if $x \in \mathbb{N}$, then $x \in S$ if and only if $px \in T'_n$.

Let $D := D'(S)$. If $x \in S$, then $px \in pS \subseteq S'_n \subseteq T'_n$. Conversely, let $px \in T'_n$. By definition, $T'_n = S'_n \cup D$. It is easy to check that $D \cap p\mathbb{N} = \emptyset$. Thus, $px \in S'_n$. By Proposition 4.0.27, $S = \frac{S'_n}{p}$. Hence, by Proposition 2.1.4, $x \in S$. This shows that $S = \frac{T'_n}{p}$.

To show that T'_n is pseudo-symmetric, by Proposition 1.2.2 it suffices to show that $|N(T'_n)| = \frac{F(T'_n)}{2}$. Since $T'_n = S'_n \cup D$ and $D \cap S'_n = \emptyset$, note that $|N(T'_n)| = |N(S'_n, T'_n)| + |N(D, T'_n)|$. By Propositions 4.0.34 and 4.0.27, $|N(S'_n, T'_n)| = |N(S'_n)|$. Moreover, since $D \cap S'_n = \emptyset$, we have $d'_{i,j} < F(S'_n) = F'_n = F(T'_n)$ for each $d'_{i,j} \in D$. Therefore, $|N(D, T'_n)| = |D|$. By Proposition 4.0.30, $|D| = k'_n$. By definition, $k'_n = \frac{F(S'_n)}{2} - |N(S'_n)|$. Thus, $|N(T'_n)| = |N(S'_n)| + |D| = |N(S'_n)| + \frac{F(S'_n)}{2} - |N(S'_n)| = \frac{F(S'_n)}{2} = \frac{F(T'_n)}{2}$, completing the proof. □

We now prove the main result of this section, which generalizes Theorem 4.0.25. As mentioned at the beginning of this chapter, the following may also be found as Theorem 6 of [21]. The proof is exactly like the proof of Theorem 3.0.20 and is omitted.

Theorem 4.0.36. *Let S be a numerical semigroup and let $k \geq 3$ be a positive integer. Then $S = \frac{T}{k}$ for infinitely many pseudo-symmetric numerical semigroups T.*

We may now state a Kunz analog of Proposition 3.0.21. The proof will be omitted. It is the same as that of Proposition 3.0.21, with the exception that Theorem 1.2.4 is cited instead of Theorem 1.2.3.

Proposition 4.0.37. *Let S be a numerical semigroup. Let p be an odd prime number and let K be a field of characteristic p. Then there exist infinitely many Kunz subrings R of $K[[S]]$ such that $K[[S]] = \{f \in K[[x]] \mid f^p \in R\}$.*

We conclude this chapter by offering an alternate (elementary) proof of Theorem I.4.4 of [1] and Theorem 7 of [13], both of which characterize all pseudo-symmetric numerical semigroups with multiplicity 3. Recall that every numerical semigroup generated by two relatively prime positive integers is symmetric. Note that the lack of a pseudo-symmetric analog of this result makes the following proof considerably more difficult than that of Proposition 3.0.22.

Proposition 4.0.38. *T is a pseudo-symmetric numerical semigroup such that $\mathbb{N} = \frac{T}{3}$ if and only if $T = \langle 3, m, F(\langle 3, m\rangle)\rangle$ for some $m \in \mathbb{N}\backslash 3\mathbb{N}$ such that $m \geq 4$.*

Proof. ($\Leftarrow$) Let $m \in \mathbb{N}\backslash 3\mathbb{N}$ and let $T := \langle 3, m, F(\langle 3, m\rangle)\rangle$. Since $m \notin 3\mathbb{N}$, note that either $m = 3k+1$ or $m = 3k+2$, for some positive integer k. We claim that $F(T) = F(\langle 3, m\rangle) - 3$. To prove this, we first show that $F(\langle 3, m\rangle) - 3 \notin T$. Suppose this assertion is not true and $F(\langle 3, m\rangle) - 3 \in T$. Since $F(\langle 3, m\rangle) > F(\langle 3, m\rangle) - 3 \in T$, we must have that $F(\langle 3, m\rangle) - 3 \in \langle 3, m\rangle$, which leads to a contradiction since $3 \in \langle 3, m\rangle$.

We next show that every positive integer greater than $F(\langle 3, m\rangle) - 3$ lies in T. By Theorem 1.1.1, $F(\langle 3, m\rangle) - 3 = 2m - 6$. Thus, to complete the proof of the claim, it suffices to show that $\{2m-5, 2m-4, 2m-3\} \subseteq T$. Clearly $2m-3 = F(\langle 3, m\rangle) \in T$ by Theorem 1.1.1. If $m = 3k+1$, then $2m-5 = 3(2k-1) \in 3\mathbb{N} \subseteq T$ and $2m-4 = m+3(k-1) \in m+3\mathbb{N} \subseteq T$. If $m = 3k+2$, then $2m-5 = m+3(k-1) \in m+3\mathbb{N} \subseteq T$ and $2m-4 = 6k \in 3\mathbb{N} \subseteq T$. This completes the proof that $F(T) = F(\langle 3, m\rangle) - 3$.

By Proposition 1.2.2, to prove that T is pseudo-symmetric, it is enough to show that $|N(T)| = \frac{F(T)}{2}$. Recall that $N(T) = T \cap \{0, 1, \ldots, F(T)\}$. Note that $N(T)$ may be viewed as a disjoint union $\cup_{i=0}^{2}\{x \in T \mid x < F(T), x \equiv i (\text{mod } 3)\}$. Since 3 is prime and $m \notin 3\mathbb{N}$, m is not congruent to $2m$ modulo 3. Moreover, since $3 \in T$, the smallest element of T congruent to $F(T)$ modulo 3 must be $F(T)+3 = 2m-3$. Also, $|N(T)| = |\{x \in 3\mathbb{N} \mid x < F(T)\}| + |\{x \in m+3\mathbb{N} \mid x < F(T)\}|$. If $m = 3k+1$, then $F(T) = 2m-6 = 6k-4 = 3(2k-2)+2$; thus, $|N(T)| = |\{0, 3, \ldots, 3(2k-2)\}| + |\{3k+1, 3(k+1)+1, \ldots, 3(2k-2)+1\}| = 2k-1+k-1 = 3k-2 = \frac{F(T)}{2}$. If $m = 3k+2$, then $F(T) = 2m-6 = 6k-2 = 3(2k-1)+1$; thus, $|N(T)| = |\{0, 3, \ldots, 3(2k-1)\}| + |\{3k+2, 3(k+1)+2, \ldots, 3(2k-2)+2 = F(T)-2\}| = 2k+k-1 = 3k-1 = \frac{F(T)}{2}$. Thus, T is pseudo-symmetric; and since $3 \in T$, we also have that $\frac{T}{3} = \mathbb{N}$.

($\Rightarrow$) Let T be a pseudo-symmetric numerical semigroup and suppose that $\mathbb{N} = \frac{T}{3}$. Since T is pseudo-symmetric, $F(T)$ is even. Moreover, since $\mathbb{N} = \frac{T}{3}$, we must have that $3\mathbb{N} \subseteq T$. Hence, $F(T) \notin 3\mathbb{N}$. It follows that $F(T)$ is an even integer congruent to 1 or 2 modulo 3 and therefore is of the form either $3(2k+1)+1$ or $3(2k)+2$, for some $k \in \mathbb{N}$. We consider each case separately.

First, consider the case when $F(T) = 3(2k+1)+1$, with $k \in \mathbb{N}$. If $k = 0$, then $T = \{0, 3, 5, \rightarrow\} = \langle 3, 5, 7\rangle$ and since $7 = F(\langle 3, 5\rangle)$ by Theorem 1.1.1, the result holds. Therefore, we may suppose that $k \geq 1$. Let $S := 3\mathbb{N} \cup \{F(T)+1, \rightarrow\}$. Note that S is a subnumerical semigroup of T and $F(S) = F(T)$. Moreover, $|N(S)| = |\{0, 3(1), 3(2), \ldots, 3(2k+1)\}| = 2k+2$. Since T is pseudo-symmetric, Proposition 1.2.2 gives $|N(T)| = \frac{F(T)}{2} = \frac{3(2k+1)+1}{2} = 3k+2$. Thus, $T \backslash S \neq \emptyset$.

Let $m := \min(T \backslash S)$. Since $3 \in T$, the smallest element of T congruent to $F(T)$ modulo 3 must be $F(T) + 3$. Therefore, since $F(T) \equiv 1 \pmod 3$, every element of $T \backslash S$ must be congruent to 2 modulo 3. Note that $m + 3\mathbb{N} \subseteq T$ and that $N(S)$ contains no elements congruent to 2 modulo 3. Hence, it follows that $T \backslash S = \{m, m+3, \ldots, F(T)-2 = 3(2k)+2\}$. Since $|T \backslash S| = (3k+2) - (2k+2) = k$, we must have that $m = 3(k+1)+2$. Note that by Theorem 1.1.1, a simple calculation shows that $F(\langle 3, m \rangle) = 2m - 3 = 2(3(k+1)+2) - 3 = 6k + 7 = 3(2k+1) + 4 = F(T) + 3$. Thus, $\langle 3, m, F(\langle 3, m \rangle) \rangle \subseteq T$.

To see that $\langle 3, m, F(\langle 3, m \rangle) \rangle = T$, note that the fact that $m \equiv 2 \pmod 3$ implies that if $x \in T$, then x must be congruent to $0, m$, or $F(\langle 3, m \rangle)$ modulo 3. Hence, since m and $F(\langle 3, m \rangle)$ are the smallest elements of T in their respective residue classes modulo 3, we must have that either $x \in 3\mathbb{N}$, $x \in m + 3\mathbb{N}$, or $x \in F(\langle 3, m \rangle) + 3\mathbb{N}$. Therefore, $\langle 3, m, F(\langle 3, m \rangle) \rangle = T$.

Next, consider the case $F(T) = 3(2k) + 2$ for some $k \in \mathbb{N}$. Note that if $k = 0$, then $F(T) = 2$. By Theorem 1.1.1, $T = \{0, 3, \rightarrow\} = \langle 3, 4, 5 \rangle = \langle 3, 4, F(\langle 3, 4 \rangle) \rangle$ and the proof is complete. Therefore, we may suppose that $k \geq 1$. Let $S := 3\mathbb{N} \cup \{F(T) + 1, \rightarrow\}$. Note that S is a subnumerical semigroup of T and $F(S) = F(T)$. Moreover, $|N(S)| = |\{0, 3(1), 3(2), \ldots, 3(2k)\}| = 2k + 1$. Since T is pseudo-symmetric, by Proposition 1.2.2 $|N(T)| = \frac{F(T)}{2} = \frac{3(2k)+2}{2} = 3k + 1$. Thus, $T \backslash S \neq \emptyset$.

Let $m := \min(T \backslash S)$. Since $3 \in T$, the smallest element of T congruent to $F(T)$ modulo 3 must be $F(T) + 3$. Therefore, since $F(T) \equiv 2 \pmod 3$, every element of $T \backslash S$ must be congruent to 1 modulo 3. Note that $m + 3\mathbb{N} \subseteq T$ and that $N(S)$ contains no elements congruent to 1 modulo 3. Hence, it follows that $T \backslash S = \{m, m+3, \ldots, F(T)-1 = 3(2k)+1\}$. Since $|T \backslash S| = k$, we must have that $m = 3(k+1)+1$. Note that by Theorem 1.1.1, a simple calculation shows that $F(\langle 3, m \rangle) = 2m - 3 = 2(3(k+1)+1) - 3 = 6k + 5 = 3(2k) + 5 = F(T) + 3$. Thus, $\langle 3, m, F(\langle 3, m \rangle) \rangle \subseteq T$.

To see that $\langle 3, m, F(\langle 3, m \rangle) \rangle = T$, note that if $x \in T$, then x must be congruent to $0, m$, or $F(\langle 3, m \rangle)$ modulo 3. Hence, since m and $F(\langle 3, m \rangle)$ are the smallest elements of T in their respective residue classes modulo 3, we must have that either $x \in 3\mathbb{N}$, $x \in m + 3\mathbb{N}$, or $x \in F(\langle 3, m \rangle) + 3\mathbb{N}$. Therefore, $\langle 3, m, F(\langle 3, m \rangle) \rangle = T$. □

Remark 4.0.39. The preceding proof shows that if F is an even positive integer which is not divisible by 3, there exists a unique pseudo-symmetric numerical semigroup T with Frobenius number F which satisfies $\mathbb{N} = \frac{T}{3}$. In general, if p is a prime greater than 3 and A and B are pseudo-symmetric numerical semigroups such that $F(A) = F(B)$ and $\frac{A}{p} = \mathbb{N} = \frac{B}{p}$, then A and B need not be the same. For example, let $A := \langle 5, 6, 7, 9 \rangle = \{0, 5, 6, 7, 9, \rightarrow\}$ and let $B := \langle 3, 7, 11 \rangle = \{0, 3, 6, 7, 9, \rightarrow\}$. Note that $F(A) = F(B) = 8$. Moreover, both A and B are pseudo-symmetric by Proposition 1.2.2. Moreover, note that $\frac{A}{7} = \mathbb{N} = \frac{B}{7}$ even though $A \neq B$.

Chapter 5

Fractions of MED Numerical Semigroups

5.1 A New Characterization of Maximal Embedding Dimension

In this section we will examine the numerical semigroups that are fractions of a numerical semigroup of maximal embedding dimension. We will demonstrate behavior that is qualitatively different from that of fractions of symmetric or pseudo-symmetric numerical semigroups given in [15], [17], [14] and [21]. Given a non-MED numerical semigroup S and a positive integer $k \geq 2$, we shall find in Proposition 5.1.3 a restriction on the multiplicity of any MED numerical semigroup T which satisfies $S = \frac{T}{k}$. Because of this restriction, we will see in Example 5.1.8 that (unlike the symmetric or pseudo-symmetric cases) it is not true in general that $S = \frac{T}{k}$ for some MED numerical semigroup T. This restriction will allow us to obtain a new characterization of MED numerical semigroups in Theorem 5.1.7.

The following two simple results will be needed in the sequel.

Proposition 5.1.1. *If S is a numerical semigroup and $m := \mu(S)$, there cannot exist m gaps $g_1 < \cdots < g_m$ of S such that $g_i - g_j \notin S$ whenever $i > j$.*

Proof. Consider gaps $g_1 < \cdots < g_m$ of S. Then each $g_i \notin m\mathbb{N}$ since $m\mathbb{N} \subseteq S$. By the Pigeonhole Principle, since no g_i is congruent to 0 modulo m, there must exist $i > j$ such that $g_i \equiv g_j \pmod{m}$. Hence, $g_i - g_j \in m\mathbb{N} \subseteq S$. □

Proposition 5.1.2. *Let S be a numerical semigroup and let $s_1 < \cdots < s_e$ be the elements of the minimal generating set of S. Then $s_i \notin S^* + S^*$ for each i.*

Proof. Suppose the assertion is not true. Then there exist $a, b \in S^*$ such that $a + b = s_i$ for some element s_i of the minimal generating set of S. Since $a, b < s_i$, we have that $a = \sum_{j=1}^{i-1} n_j s_j$ and $b = \sum_{j=1}^{i-1} m_j s_j$ where each $n_j, m_j \in \mathbb{N}$. Thus, $s_i = a + b = \sum_{j=1}^{i-1}(n_j + m_j)s_j$ can be generated by the elements $s_1, \ldots, s_{i-1}$. This implies that $S = \langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_e\rangle$, contradicting the minimality of $\{s_1, \ldots, s_e\}$. □

Let S be a numerical semigroup, let $2 \leq k \in \mathbb{N}$ and let $n \in S$ such that $\gcd(k, n) = 1$. Define $E_{kS,n} := \langle kS, n\rangle$. It is easy to see that $E_{kS,n}$ is a subnumerical semigroup of S which satisfies $S = \frac{E_{kS,n}}{k}$. Also, $E_{kS,n} = kS \cup (kS+n) \cup (kS+2n) \cup \cdots \cup (kS+(k-1)n)$. Additional properties of $E_{kS,n}$ may be found in Appendix A.

Note that any subnumerical semigroup T of S which contains n and satisfies $S = \frac{T}{k}$ must contain $E_{kS,n}$ (this follows easily from condition 4 of Proposition 2.1.4). This required containment will be used in the proof of the following useful restriction.

Proposition 5.1.3. *Let S be a numerical semigroup and let $2 \leq k \in \mathbb{N}$. If S is not MED and $S = \frac{T}{k}$ where T is an MED numerical semigroup, then $\mu(T) < k\mu(S)$.*

Proof. Suppose that S is not MED, $m := \mu(S)$, $e := e(S)$ and $S = \frac{T}{k}$ for some MED numerical semigroup T. Let $m = s_1 < \cdots < s_e$ be the minimal system of generators for S and let $t := \mu(T)$. Since $S = \frac{T}{k}$, we have $kS \subseteq T$ by Proposition 2.1.4 and so $t \leq k\mu(S) = km$.

Suppose that the assertion fails and $t = km$. Let $n := \min\{x \in T \mid \gcd(x, k) = 1\}$. Note that $n > km$. Since $n \in T$ and $kS \subseteq T$, note that $E := E_{kS,n} = \langle ks_1, \ldots, ks_e, n\rangle$ is a subnumerical semigroup of T and $e(E) \leq e + 1$. Moreover, since S is not MED and $k > 1$, note that $e + 1 \leq m < km = \mu(E)$. Thus, $E \subseteq T$ and E is not MED. In particular, $E \subset T$.

Note that $e(T) = km$ and $e(E) \leq m$. Moreover, any elements of the minimal generating set of T that belong to E must also belong to the minimal generating set of E. Thus, the set $T \backslash E$ contains at least $km - m = (k-1)m$ elements of the minimal generating set of T. Since E and T satisfy $\frac{E}{k} = S = \frac{T}{k}$, by Proposition 2.1.4 none of these elements in both $T \backslash E$ and the minimal generating set of T are elements of $k\mathbb{N}$. We claim that no $m+1$ of them can lie in the same residue class modulo k.

Suppose the above claim fails. Then there exist $m+1$ elements of the minimal generating set of T which are of the form $a < a + kg_1 < \cdots < a + kg_m$ where each g_i is a positive integer. Moreover, by Proposition 5.1.2, each $g_i \notin S$ and $a + kg_i \notin a + kg_j + kS$ whenever $i > j$. Thus, $g_i - g_j \notin S$ whenever $i > j$, violating Proposition 5.1.1. This proves the above claim.

By the claim, since $T \backslash E$ contains at least $(k-1)m$ elements of the minimal generating set of T, each nonzero residue class modulo k must contain exactly m of the elements of the minimal generating set of T. Hence, exactly m of these elements are congruent to n modulo k. Since n must, by definition, be the smallest element of T in its residue class, these m elements must all have the form $n + kh_1 < \cdots < n + kh_m$ where each h_i is a positive integer. Moreover, by Proposition 5.1.2, each $h_i \notin S$ and $n + kh_i \notin n + kh_j + kS$ whenever $i > j$. Therefore, $h_i - h_j \notin S$ whenever $i > j$, again violating Proposition 5.1.1, and the proof is complete. □

The restriction imposed by Proposition 5.1.3 immediately gives us the following three corollaries.

Corollary 5.1.4. *Let S be a numerical semigroup and let $2 \leq k \in \mathbb{N}$. If S is not MED and $S = \frac{T}{k}$ where T is an MED numerical semigroup, then $\mu(T) \in S \backslash k\mathbb{N}$.*

Proof. Of course, $\mu(T) \in T \subseteq S$. Let $m := \mu(S)$. If the result fails, $\mu(T) = kn$ for some integer $n > 0$. Then $km > kn$ by Proposition 5.1.3. Since $S = \frac{T}{k}$, by Proposition 2.1.4 we have $m > n \in S \backslash \{0\}$, which is a contradiction. □

Corollary 5.1.5. *Let S be a numerical semigroup. If S is not MED and $S = \frac{T}{2}$ where T is an MED numerical semigroup, then $\mu(T)$ is an odd integer which is a member of the minimal generating set of S.*

Proof. We prove that $\mu(T)$ is odd. Suppose the assertion is not true and $\mu(T)$ is even. Then $\mu(T) = 2x$ for some positive integer x. By Proposition 5.1.3, $2x = \mu(T) < 2\mu(S)$. Thus, $0 < x < \mu(S)$. Since $S = \frac{T}{2}$, by Proposition 2.1.4 we have that $x \in S$, which contradicts the assertion that $\mu(S)$ is the smallest nonzero element of S. This proves that $\mu(T)$ is odd.

We now prove that $\mu(T)$ is a member of the minimal generating set of S. Suppose the assertion is not true and $\mu(T)$ is not a member of the minimal generating set of S. Then there exist $a, b \in S^*$ such that $\mu(T) = a+b$. However, by Proposition 5.1.3, $\mu(T) < 2\mu(S) \leq a+b$, which is a contradiction. □

Corollary 5.1.6. *Let S be a numerical semigroup and let p be a prime number. If S is not MED, then there are at most finitely many MED numerical semigroups T such that $S = \frac{T}{p}$.*

Proof. Let $m := \mu(S)$. By Proposition 5.1.3, any MED numerical semigroup T satisfying $S = \frac{T}{p}$ has multiplicity $t < pm$. By Corollary 5.1.4, $t \in S \backslash p\mathbb{N}$. Hence, $\gcd(t, p) = 1$ and T contains the numerical semigroup $E := E_{pS,t} := \langle pS, t\rangle$. The result is now immediate because there are only finitely many possibilities for t and each numerical semigroup E has only finitely many overnumerical semigroups. □

We shall see in Example 5.1.9 that the above corollary does not always hold if p is not prime.

Recall Proposition 1.2.5, which says that a numerical semigroup S is MED if and only if $S^* - \mu(S)$ is a numerical semigroup. Using this result, we obtain the following characterization of MED numerical semigroups.

Theorem 5.1.7. *Let S be a numerical semigroup. The following are equivalent:*

(1) *There exists a prime number p and infinitely many MED numerical semigroups T such that $S = \frac{T}{p}$;*
(2) *For each prime number p, there exist infinitely many MED numerical semigroups T such that $S = \frac{T}{p}$;*
(3) *For each positive integer $k \geq 2$, there exist infinitely many MED numerical semigroups T such that $S = \frac{T}{k}$;*
(4) *S is an MED numerical semigroup.*

Proof. Clearly (3) ⇒ (2) ⇒ (1); and (1) ⇒ (4) by Corollary 5.1.6. We will show that (4) ⇒ (3).

Let S be MED with multiplicity m. Let $2 \leq k \in \mathbb{N}$. Choose $n \in S$ such that $\gcd(n, k) = 1$ and so that $n > \max\{km, F(S) + (k-2)m\}$. Let $T := T_n$ where

$$T_n := kS \cup (k\mathbb{N}+n) \cup (k\mathbb{N}+2n-km) \cup (k\mathbb{N}+3n-2km) \cup \cdots \cup (k\mathbb{N}+(k-1)n-(k-2)km)$$

We claim that T is an MED numerical semigroup satisfying $S = \frac{T}{k}$.

First, we prove that T is a numerical semigroup. Clearly $0 \in kS \subseteq T$. We show that T is closed under addition. It is clear that $kS+kS \subseteq T$ and that $kS+k\mathbb{N}+in-(i-1)km \subseteq T$ for each $i \in \{1, \ldots, k-1\}$. We must show that $k\mathbb{N}+in+(i-1)km+k\mathbb{N}+jn+(j-1)km \subseteq T$ for each $i, j \in \{1, \ldots, k-1\}$. Let $i, j \in \{1, \ldots, k-1\}$ and let $a, b \in \mathbb{N}$. If $i+j < k$ then

$(ka+in-(i-1)km)+(kb+jn-(j-1)km) = k(a+b+m)+(i+j)n-(i+j-1)km \in T$. If $i+j = k+r$ for some $0 \leq r \leq k-2$, then $(ka+in-(i-1)km)+(kb+jn-(j-1)km) = k(a+b+m+n-km)+rn-(r-1)km$, which lies in $k\mathbb{N}+rn-(r-1)km$ if $r > 0$ (since $n > km$) and in kS if $r = 0$ (since $n > F(S)+(k-2)m$). This shows that T is closed under addition. To see that $\mathbb{N}\backslash T$ is finite, note that since $\gcd(n,k) = 1$, S is a numerical semigroup, and T is closed under addition, $E_{kS,n} = \langle kS, n\rangle$ is a numerical semigroup contained in T. Hence, $\mathbb{N}\backslash T$ is finite. This completes the proof that T is a numerical semigroup.

Next, we show that T is MED. Since $n > km$, we have $\mu(T) = km$. Therefore, by Proposition 1.2.5, it suffices to show that $T^* - km$ is a numerical semigroup. Note that

$$T^* - km = (kS^* - km) \cup (k\mathbb{N} + n - km) \cup (k\mathbb{N} + 2(n-km)) \cup \cdots \cup (k\mathbb{N} + (k-1)(n-km)).$$

Since $m = \mu(S)$, $0 \in kS^* - km \subseteq T^* - km$.

We show that $T^* - km$ is closed under addition. Clearly $kS^* - km \subseteq k\mathbb{N}$; so $kS^* - km + k\mathbb{N} + i(n-km) \subseteq T^* - km$ for each $i \in \{1, \ldots, k-1\}$. Let $i, j \in \{1, \ldots, k-1\}$. If $i+j < k$ then $k\mathbb{N} + i(n-km) + k\mathbb{N} + j(n-km) \subseteq k\mathbb{N} + (i+j)(n-km) \subseteq T^* - km$. If $i+j = k+r$ for some $0 \leq r \leq k-2$, then $k\mathbb{N} + i(n-km) + k\mathbb{N} + j(n-km) \subseteq k\mathbb{N} + r(n-km) + k(n-km) \subseteq T^* - km$ (since $n > km$). Since S is MED and $m = \mu(S)$, we have that $S^* - m$ is a numerical semigroup. Thus, $k(S^* - m) = kS^* - km$ is closed under addition and $(kS^* - km) + (kS^* - km) \subseteq T^* - km$. Therefore, $T^* - km$ is closed under addition.

We show that the complement of $T^* - km$ in $\mathbb{N}$ is finite. Note that $n - km \in T^* - km$ and $ks - km = k(s-m) \in T^* - km$ for all $s \in S^*$. Since $\gcd(n,k) = 1$ by hypothesis, it follows that $\gcd(n-km, k) = 1$.

Since $\{F(S)+1, \rightarrow\} \subseteq S^*$, there exists some large prime $p \in S^*$ such that $p > n-km$. Let $s := p+m$. Note $s \in S^*$. Then $ks-km \in T^*-km$. Note that $ks-km = k(s-m) = kp$. Since kp and $n-km$ are relatively prime and T^*-km is closed under addition, $\langle ks-km, n-km\rangle$ is a numerical semigroup contained in $T^* - km$. Therefore, the complement of $T^* - km$ in $\mathbb{N}$ is finite. This completes the proof that $T^* - km$ is a numerical semigroup. Hence, T is MED.

Finally, we show that $S = \frac{T}{k}$. Clearly $kS \subseteq T$. Suppose $kx \in T$ for some integer x. Then since $k\mathbb{N} \cap T = kS$ by construction, $kx \in kS$ and so $x \in S$. It follows that $S = \frac{T}{k}$ by condition 4 of Proposition 2.1.4. The above construction leads to infinitely many suitable T because $n = \min(T\backslash k\mathbb{N})$ and there exist infinitely many suitable values of n. □

Example 5.1.8. Let $2 \leq k \in \mathbb{N}$. Using Proposition 5.1.3, we next describe an infinite family of numerical semigroups S that cannot be expressed in the form $\frac{T}{k}$ for any MED numerical semigroup T.

Let $2 \leq n \in \mathbb{N}$. Define $S := S_n := \langle nk, nk^2+1\rangle$. Note that $(-k)nk + nk^2 + 1 = 1$, so nk and nk^2+1 are relatively prime and S is a numerical semigroup. Furthermore, $\mu(S) = nk \geq 4$ so S is not MED (by the definition of MED). Suppose $S = \frac{T}{k}$ for some MED numerical semigroup T. By Proposition 5.1.3, we have that $\mu(T) \in S \cap \{1, \ldots, nk^2-1\} = \{nk, 2nk, \ldots, (k-1)nk\}$. Then we must have $\mu(T) = ink$, where $i \in \{1, \ldots, k-1\}$. Since $S = \frac{T}{k}$, by condition 4 of Proposition 2.1.4 we must have that $in \in S$. Note that $0 < in < kn = \mu(S)$, a contradiction. Finally, note that $2 \leq a < b$ in $\mathbb{N}$ gives $S_a \neq S_b$.

Theorem 5.1.7, in conjunction with Proposition 2.1.3, can now be used to produce an example which shows that the conclusion of Corollary 5.1.6 does not hold in general when the denominator is not a prime number.

Example 5.1.9. Let $S := \langle 5,6,7\rangle = \{0,5,6,7,10,\rightarrow\}$ and let $T := \langle 5,12,14,21,23\rangle = \{0,5,10,12,14,15,17,19,\rightarrow\}$. It is clear that S is not MED, T is MED, and it is easy to verify that $S = \frac{T}{2}$. By Theorem 5.1.7, if $k \geq 2$ is an integer, then we may express T as $T = \frac{T'}{k}$ for infinitely many MED numerical semigroups T'. Therefore, by Proposition 2.1.3, $S = \frac{T}{2} = \frac{T'/k}{2} = \frac{T'}{2k}$ for infinitely many MED numerical semigroups T'.

5.2 Realizing a Numerical Semigroup as a Fraction of an MED Numerical Semigroup

We will show in Theorem 5.2.8 that given any numerical semigroup S, there exists a prime number q such that for all prime numbers $p \geq q$, S satisfies $S = \frac{T}{p}$ for some MED numerical semigroup T. In order to do this, we first find (in Proposition 5.2.3) a condition, given a prime number p, for a numerical semigroup S that is not MED to satisfy $S = \frac{T}{p}$ for some MED numerical semigroup T. The main result of this section, Theorem 5.2.11, generalizes Theorem 5.2.8 by replacing the primes $p \geq q$ with the possibly composite integers $n \geq N$. In light of Proposition 5.1.3, the requirement that S contain an element $n \in S\backslash p\mathbb{N}$ such that $n < p\mu(S)$ will be quite common.

The following is a direct consequence of Proposition 1.2.5.

Proposition 5.2.1. *Let $S \subseteq \mathbb{Z}$. Then S is a numerical semigroup and $n \in S^*$ if and only if $(S+n)\cup\{0\}$ is an MED numerical semigroup with multiplicity n.*

Proof. ($\Rightarrow$) Let S be a numerical semigroup and let $n \in S^*$. It is clear that $(S+n)\cup\{0\} \subseteq \mathbb{N}$ is closed under addition and contains 0. Note that $(S+n)\cup\{0\}$ has finite complement in $\mathbb{N}$ since every integer greater than $F(S)+n$ lies in $(S+n)\cup\{0\}$. Thus, $(S+n)\cup\{0\}$ is a numerical semigroup. Moreover, since $0 \in S$, $(S+n)\cup\{0\}$ has multiplicity n. Finally, note that $((S+n)\cup\{0\})^* - n = (S+n) - n = S$, which is a numerical semigroup. Thus, by Proposition 1.2.5, $(S+n)\cup\{0\}$ is of maximal embedding dimension.

($\Leftarrow$) Suppose that $(S+n)\cup\{0\}$ is an MED numerical semigroup with multiplicity n. Since, by definition, n is the smallest nonzero element of $(S+n)\cup\{0\}$, we have $S \subseteq \mathbb{N}$. By Proposition 1.2.5, note that $((S+n)\cup\{0\})^* - n$ is a numerical semigroup. Moreover, $((S+n)\cup\{0\})^* - n = (S+n) - n = S$, so S is a numerical semigroup. Since $(S+n)\cup\{0\}$ is a numerical semigroup and $n \in (S+n)\cup\{0\}$, we also have $n+n \in (S+n)\cup\{0\}$. Thus, $n+n \in S+n$ and $n \in S$. Since n is nonzero, $n \in S^*$. □

Let S be a numerical semigroup with multiplicity m and let p be a prime number. Suppose that $n \in S\backslash p\mathbb{N}$ with $n < pm$. Recall that $E := E_{pS,n} = \langle pS, n\rangle$ is a numerical semigroup. Define the numerical semigroup $W_{pS,n} := (\langle E^* - n\rangle + n)\cup\{0\}$. Using this construction and Proposition 5.2.1, we obtain the following result.

Proposition 5.2.2. *Let S be a numerical semigroup, let $m = \mu(S)$, let p be a prime number and let $n \in S\backslash p\mathbb{N}$ such that $n < pm$. Then $W_{pS,n}$ is an MED numerical semigroup with $\mu(W_{pS,n}) = n$. Moreover, any MED numerical semigroup with multiplicity n containing pS also contains $W_{pS,n}$.*

Proof. Let $E := E_{pS,n}$. Since $n < pm$ and $pm \in E^*$, note that $\langle E^* - n\rangle$ is a numerical semigroup. Since $n \in E^*$ and E^* is closed under addition, $2n \in E^*$. Thus, $n = 2n - n \in E^* - n \subseteq \langle E^* - n\rangle$ and, by Proposition 5.2.1, $W_{pS,n}$ is an MED numerical semigroup with multiplicity n.

Suppose that T is an MED numerical semigroup with multiplicity n and $pS \subseteq T$. Then $E \subseteq T$ so $E^* - n \subseteq T^* - n$. Since T is MED, $T^* - n$ is a numerical semigroup and $\langle E^* - n\rangle \subseteq T^* - n$. Thus, $\langle E^* - n\rangle + n \subseteq T^*$ and the result follows. □

Proposition 5.2.3. *Let S be a numerical semigroup, let $m := \mu(S)$, let p be a prime number, let $n \in S\backslash p\mathbb{N}$ such that $n < pm$ and let $W := W_{pS,n}$. If S is not an MED numerical semigroup, then $S = \frac{T}{p}$ for some MED numerical semigroup T with multiplicity n if and only if $W \subseteq S\backslash p(\mathbb{N}\backslash S)$.*

Proof. ($\Rightarrow$) Since $S = \frac{T}{p}$, it is clear that $pS \subseteq T$ by Proposition 2.1.4. By Proposition 5.2.2, $W \subseteq T \subseteq S$. Thus, $W \cap p(\mathbb{N}\backslash S) \subseteq T \cap p(\mathbb{N}\backslash S)$. As $S = \frac{T}{p}$, it follows by Proposition 2.1.4 that $T \cap p(\mathbb{N}\backslash S) = \emptyset$. Therefore, $W \cap p(\mathbb{N}\backslash S) = \emptyset$ and so $W \subseteq S\backslash p(\mathbb{N}\backslash S)$.

($\Leftarrow$) By Proposition 5.2.2, W is an MED numerical semigroup with multiplicity n. We show that $S = \frac{W}{p}$, which will complete the proof. Clearly $pS \subseteq E_{pS,n} \subseteq W$. If $px \in W$ for some positive integer x, then since $W \cap p(\mathbb{N}\backslash S) = \emptyset$, we must have $x \in S$. Thus, by Proposition 2.1.4, $S = \frac{W}{p}$. □

We now present two simple examples showing distinct ways in which the condition $W \subseteq S\backslash p(\mathbb{N}\backslash S)$ in Proposition 5.2.3 may fail. Thus, in view of Proposition 5.1.3, we see that Examples 5.2.4 and 5.2.5 give specific instances of data (S, p) for which no numerical semigroup T of maximal embedding dimension can satisfy $S = \frac{T}{p}$.

Example 5.2.4. We show that W may not satisfy $W \subseteq S$. Let $p = 3$, let $S = \langle 5, 7\rangle = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24, \rightarrow\}$ and let $n = 7$. Note that $F(S) = 23 \notin S$ and that $23 = 2(5p - 7) + 7 \in W$.

Example 5.2.5. We show that W may not satisfy $W \cap p(\mathbb{N}\backslash S) = \emptyset$. Let $p = 3$, let $S = \langle 5, 7, 11\rangle = \{0, 5, 7, 10, 11, 12, 14, \rightarrow\}$ and let $n = 7$. Note that $F(S) = 13 \in \mathbb{N}\backslash S$. So $39 = 3F(S) \in p(\mathbb{N}\backslash S)$, although $39 = 4(5p - 7) + 7 \in W$.

The existence of n in Proposition 5.2.3 is not automatic. For instance, in Example 5.1.8, if $p = k$, then for each positive integer m, $S_m = \langle mp, mp^2 + 1\rangle$ and there exist no elements n of S_m such that $n \in S_m\backslash p\mathbb{N}$ and $n < p\mu(S_m)$. However, the next remark will show that n exists in all the cases of interest.

Remark 5.2.6. Given a numerical semigroup S that is not of maximal embedding dimension and a prime number p, there may not exist $n \in S\backslash p\mathbb{N}$ such that $n < p\mu(S)$. Perhaps, the simplest example of this is given by taking $p := 2$ and $S := \langle 4, 9\rangle = \{0, 4, 8, 9, 12, 13, 16, 17, 18, 20, 21, 22, 24, \rightarrow\}$. The utility of the criterion in Proposition 5.2.3 resides in the following fact. If S is any numerical semigroup other than $\mathbb{N}$ and $m := \mu(S)$, then for all sufficiently large prime numbers p, there exists $n \in S\backslash p\mathbb{N}$ such that $n < pm$.

For a proof, note that $F(S) > 1$ since $S \neq \langle 2, 3\rangle$ (because S is not of maximal embedding dimension). Similarly, $F(S) \neq 2$ since $S \neq \langle 3, 4, 5\rangle$. Now, if p is any prime number such that $p > \frac{2F(S)}{m}$, note that $\{F(S) + i \mid 1 \leq i \leq F(S) - 1\}$ is a set of $F(S) - 1 \geq 2$ consecutive elements of S, none of which is divisible by p, and each of which is less than $2F(S) \leq pm$. Thus, any of these $F(S)-1$ elements is a suitable n, to complete the proof.

If S is a numerical semigroup with multiplicity m, p is a prime number and $n \in S \backslash p\mathbb{N}$ with $n < pm$, then it is easy to see that $E_{pS,n}$ is a subnumerical semigroup of $W_{pS,n}$. When p is odd, the following result gives us a useful lower bound on the elements of $W_{pS,n} \backslash E_{pS,n}$ that are multiples of p.

Proposition 5.2.7. *Let S be a numerical semigroup with multiplicity m, let p be an odd prime number and let $n \in S \backslash p\mathbb{N}$ such that $n < pm$ and $\gcd(m,n) = 1$. If $x \in (W_{pS,n} \backslash E_{pS,n}) \cap p\mathbb{N}$, then $x \geq (p+1)(pm-n)+n$.*

Proof. Let $S = \langle s_1, \ldots, s_e \rangle$ where $m = s_1$, let $E := E_{pS,n}$, let $W := W_{pS,n}$ and let $x \in (W \backslash E) \cap p\mathbb{N}$. Since $x \neq 0$, we have $x - n \in \langle E^* - n \rangle \backslash E$. We show that $x - n \geq (p+1)(pm-n)$, which will complete the proof.

We claim that $\langle E^* - n \rangle = \langle n, ps_1 - n, \ldots, ps_e - n \rangle$. Clearly $n = 2n - n \in E^* - n \subseteq \langle E^* - n \rangle$ and, for each $i \in \{1, \ldots, e\}$, $ps_i - n \in E^* - n \subseteq \langle E^* - n \rangle$. Thus, $\langle n, ps_1 - n, \ldots, ps_e - n \rangle \subseteq \langle E^* - n \rangle$. To show the reverse inclusion, it's enough to show that $E^* - n \subseteq \langle n, ps_1 - n, \ldots, ps_e - n \rangle$. Let $x \in E^* - n$. Then $x + n \in E^*$. Note that $E = \langle n, ps_1, \ldots, ps_e \rangle$. Thus, $x + n = kn + \sum_{i=1}^{e} k_i p s_i$ for some $k, k_1, \ldots, k_e \in \mathbb{N}$ such that $k > 0$ or $k_i > 0$ for some $i \in \{1, \ldots, e\}$. It follows that $x = kn - n + \sum_{i=1}^{e} k_i n + \sum_{i=1}^{e} (k_i p s_i - k_i n) = (k - 1 + \sum_{i=1}^{e} k_i) n + \sum_{i=1}^{e} k_i (ps_i - n) \in \langle n, ps_1 - n, \ldots, ps_e - n \rangle$, completing the proof of the claim.

Next, we claim that $x - n$ is not an element of the minimal generating set of $\langle E^* - n \rangle$. Since every system of generators for a numerical semigroup contains the minimal generating set, by the previous claim it suffices to show that $x - n \notin \{n, ps_1 - n, \ldots, ps_e - n\}$. If $x - n = n$, then $2n = x \in p\mathbb{N}$, a contradiction since $p \nmid 2$ and $p \nmid n$. Furthermore, if $x - n = ps_i - n$ for some $i \in \{1, \ldots, e\}$, then $x = ps_i \in E$, a contradiction. This proves the claim that $x - n$ is not an element of the minimal generating set of $\langle E^* - n \rangle$.

In any event, we have $x - n = kn + \sum_{i=1}^{e} k_i (ps_i - n)$ for some $k, k_1, \ldots, k_e \in \mathbb{N}$. Note that $x - n \geq (pm - n) \sum_{i=1}^{e} k_i$. To complete the proof, it will suffice to show that $\sum_{i=1}^{e} k_i \geq p + 1$. Note that $x - n = (k - \sum_{i=1}^{e} k_i) n + p \sum_{i=1}^{e} k_i s_i$. If $k \geq \sum_{i=1}^{e} k_i$, then $x \in \langle pS, n \rangle = E$, a contradiction. Hence, $k < \sum_{i=1}^{e} k_i$. Moreover, since $x \in p\mathbb{N}$ and n is invertible modulo p, we have that $k - \sum_{i=1}^{e} k_i \equiv -1 \pmod{p}$. Thus, $k - \sum_{i=1}^{e} k_i \in \{-1, -p-1, -2p-1, \ldots\}$. If $k - \sum_{i=1}^{e} k_i = -1$, then $x - n = -n + p \sum_{i=1}^{e} k_i s_i$, whence $x \in pS \subseteq E$, a contradiction. Therefore, $k - \sum_{i=1}^{e} k_i = -rp - 1$ for some positive integer r, and it follows that $\sum_{i=1}^{e} k_i = k + rp + 1 \geq p + 1$, as required. □

Recall Theorem 1.1.1, which says that if m and n are positive integers such that $\gcd(m,n) = 1$, then $\langle m, n \rangle$ is a numerical semigroup with Frobenius number $mn - m - n$. Using this result and Proposition 5.2.7, we may now prove one of the main results of this chapter.

Theorem 5.2.8. *Let S be a numerical semigroup. There exists a prime number q such that for each prime number $p \geq q$, there exists an MED numerical semigroup T such that $S = \frac{T}{p}$.*

Proof. By Theorem 5.1.7, we may assume that S is not an MED numerical semigroup. Let $m := \mu(S)$, let $E := E_{pS,n}$ and suppose $S = \langle m = s_1, \ldots, s_e \rangle$. Choose $n \in S$ such that $\gcd(m, n) = 1$. By the Archimedean property of $\mathbb{N}$, we can find an odd prime number p such that $p > n$ and $pm - n > F(S)$. We will show that the least such p is a satisfactory value for q.

Note that $ps_i - n \in S$ for all $i \in \{1, \ldots, e\}$ (since, without loss of generality, each such $s_i > 0$). Thus, $\langle E^* - n \rangle = \langle n, ps_1 - n, \ldots, ps_e - n \rangle \subseteq S$, and so $W := W_{pS,n} \subseteq S$. Also, if $x \in (W \backslash E) \cap p\mathbb{N}$, then by Proposition 5.2.7, $x \geq (p+1)(pm - n) + n = p^2 m - pn + pm > pnm - pn - pm = pF(\langle n, m \rangle) \geq pF(S)$. Moreover, it is easy to check that $E \cap p(\mathbb{N} \backslash S) = \emptyset$. Indeed, if $ps + kn = pg$ for some $s \in S$, $k \in \mathbb{N}$, and $g \in \mathbb{N} \backslash S$, then we must have that $k = pr$ for some $r \in \mathbb{N}$, whence $g = s + rn \in S$, a contradiction. Therefore, $pF(S) \notin W$. It then follows that $W \cap p(\mathbb{N} \backslash S) = \emptyset$ and so, by Proposition 5.2.3, $S = \frac{T}{p}$ for some MED numerical semigroup T. □

We will now expand upon the last result. In Theorem 5.2.11, we will show that, given any numerical semigroup S, there exists a positive integer N such that for all $k > N$, there exists an MED numerical semigroup T which satisfies $S = \frac{T}{k}$. Recall that given a numerical semigroup S and prime number p, there may not exist an MED numerical semigroup T such that $S = \frac{T}{p}$. Hence, before establishing Theorem 5.2.11, we first show, given any numerical semigroup S and prime number p, that there does exist a positive integer $k := k_{p,S}$ such that $S = \frac{T}{p^k}$ for some MED numerical semigroup T.

Given a numerical semigroup S and an element $n \in S$, the **Apéry set of n in S** is defined as $Ap(S, n) := \{s \in S \mid s - n \notin S\}$. In other words, an element $s \in Ap(S, n)$ if and only if s is the smallest element of S congruent to s modulo n. It is well known that $Ap(S, n)$ has cardinality n and, in fact, is a complete residue system modulo n (see [9]). The following preliminary result will be needed in the construction that follows.

Lemma 5.2.1. *Let S be a numerical semigroup, let p be a prime number, let $n \in S$ such that $n \notin p\mathbb{N}$ and let $E := E_{pS,n}$. Then $Ap(E, n) = pAp(S, n)$.*

Proof. Since both Apéry sets have cardinality n, if suffices to show that if $x \in Ap(S, n)$, then $px \in Ap(E, n)$. Since $x \in S$ and $E = \langle pS, n \rangle$, clearly $px \in E$. Note that $E = pS \cup (pS + n) \cup (pS + 2n) \cup \cdots \cup (pS + (p-1)n)$. Suppose the assertion fails. Then $px \notin Ap(E, n)$; that is, $px - n \in E$. It must be the case that $px - n \in pS + (p-1)n$. (The point is that $px - n \notin pS + jn$ if $0 \leq j \leq p - 2$ in $\mathbb{N}$, for otherwise $(j + 1)n \in p\mathbb{N}$ with $p \nmid (j + 1)$ and $p \nmid n$, contradicting that p is prime.) Therefore, $px - n = p(s + n) - n$ for some $s \in S$, and so $x = s + n$. Therefore, $x - n = s \in S$, contradicting the hypothesis that $x \in Ap(S, n)$. □

The following construction will be used repeatedly to establish the main result of this section.

Proposition 5.2.9. *Let S be a numerical semigroup with multiplicity m and minimal generating set $\{m = s_1, \ldots, s_e\}$ and let p be a prime number. Suppose S is not an MED numerical semigroup, $m \notin p\mathbb{N}$ and $s_i \in p\mathbb{N}$ for each $i \in \{2, \ldots, e\}$. Then there exists a numerical semigroup T such that $\mu(T) = m$, $e(T) = e(S) + 1$ and $S = \frac{T}{p}$.*

Proof. Since S is not MED, clearly $m > e$. As $Ap(S,m)$ has cardinality m, there must exist a positive integer $a := \max(Ap(S,m)\backslash\{0, s_2, \dots, s_e\})$. Since $0 \neq a \in S$ and $m \notin Ap(S,m)$, we have $m < a$ and so $m \leq pm - m < pa - m$. Let $E := E_{pS,m}$ and let $T := \langle m, ps_2, \dots, ps_e, pa - m\rangle$. Note that T is a numerical semigroup since E is a numerical semigroup and $T = \langle E, pa - m\rangle$. Clearly $m < ps_i$ for each $i \in \{2, \dots, e\}$. As $m < pa - m$, we have $\mu(T) = m$.

To show that $e(T) = e(S) + 1$, it suffices to show that $\{m, ps_2, \dots, ps_e, pa - m\}$ is the minimal generating set of T. Because $m = \mu(T)$, it is clear that $m \notin \langle ps_2, \dots, ps_e, pa - m\rangle$. By Proposition 5.2.1, $pa \in Ap(E,m)$, and so $pa - m \notin \langle m, ps_2, \dots, ps_e\rangle$. Hence, we have only to show, for each $i \in \{2, \dots, e\}$, that $ps_i \notin \langle m, ps_2, \dots, ps_{i-1}, ps_{i+1}, \dots, ps_e, pa - m\rangle$.

Suppose, on the contrary, that $ps_i \in \langle m, ps_2, \dots, ps_{i-1}, ps_{i+1}, \dots, ps_e, pa - m\rangle$ for some $i \in \{2, \dots, e\}$. Then there exists $ps \in \langle ps_2, \dots, ps_{i-1}, ps_{i+1}, \dots, ps_e\rangle$, with $s \in S$, such that $ps_i = km + ps + n(pa - m)$ for some $k, n \in \mathbb{N}$. We show that each of the three possibilities $n < k$, $n > k$ and $n = k$ lead to a contradiction.

If $n < k$, then $ps_i = (k - n)m + p(s + na)$. Since p is prime and $m \notin p\mathbb{N}$, $k - n = pc$ for some $0 < c \in \mathbb{N}$. Thus, $ps_i = p(cm + s + na)$, and so $s_i = cm + s + na$. Because $c > 0$ and s_i is an element of the minimal generating set of S, Proposition 5.1.2 yields that we must have $s_i = m = s_1$, a contradiction.

If $n > k$, then $ps_i = -(n-k)m + p(s+na)$. Since p is prime and $m \notin p\mathbb{N}$, $n - k = pd$ for some $0 < d \in \mathbb{N}$. Thus, $ps_i = p(s+na-dm) = p(s+(pd+k)a-dm) = p(s+d(pa-m)+ka)$, and so $s_i = s+d(pa-m)+ka$. We claim that $pa-m \in S^*$. Since $m < a < pa$, it's clear that $pa - m \neq 0$. As $pa \in S$, the condition $pa - m \notin S$ would lead to $pa \in Ap(S,m)$. Moreover, because $pa > a$ and $a = \max(Ap(S,m)\backslash\{0, s_2, \dots, s_e\})$, we would then have $pa = s_j$ for some $j \in \{2, \dots, e\}$. Note that this contradicts Proposition 5.1.2 because each such s_j is an element of the minimal generating set of S and $a \in S$. This proves the above claim that $pa - m \in S^*$. Because $d > 0$ and s_i is an element of the minimal generating set of S, Proposition 5.1.2 shows that $s_i = pa - m$. Note that this leads to a contradiction because p is prime, $s_i \in p\mathbb{N}$ and $m \notin p\mathbb{N}$.

If $n = k$, then $ps_i = p(s + na)$. Thus, $s \in \langle s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_e\rangle$ satisfies $s_i = s + na$. Since s_i is an element of the minimal generating set of S, Proposition 5.1.2 yields that either $s = 0$ or $na = 0$. If $s = 0$, then Proposition 5.1.2 yields that $s_i = na = a$, contradicting the definition of a. If $na = 0$, then $k = n = 0$, whence $s_i = s$, contradicting minimality of the given generating set of S. Therefore, the case analysis reveals that $e(T) = e + 1 = e(S) + 1$.

Finally, we show that $S = \frac{T}{p}$. Clearly $pS \subseteq T$ by Proposition 2.1.4. It remains only to show that $T \cap p\mathbb{N} \subseteq pS$. Suppose $x \in \mathbb{N}$ such that $px \in T$. Then $px = km + ps + n(pa - m)$ for some $k, n \in \mathbb{N}$ and some $s \in S$. If $n = k$, then $px = p(s + na) \in pS$. Suppose next that $n < k$. Then $px = (k - n)m + p(s + na)$. Since p is prime and $m \notin p\mathbb{N}$, $k - n = pc$ for some $0 \leq c \in \mathbb{N}$, and so $px = p(cm + s + na) \in pS$. In the final case, suppose that $n > k$. Then $px = -(n - k)m + p(s + na)$. Since p is prime and $m \notin p\mathbb{N}$, $n - k = pd$ for some $0 \leq d \in \mathbb{N}$, and so $px = p(s + na - dm) = p(s + (pd + k)a - dm) = p(s + d(pa - m) + ka) \in pS$, the last step holding because we proved above that $pa - m \in S$. □

We now show that we may always obtain conditions that allow us to apply the construction in Proposition 5.2.9. Once those conditions are met, we simply repeat the construction as often as needed, increasing the embedding dimension by 1 with each iteration, until a numerical semigroup of maximal embedding dimension is obtained.

Theorem 5.2.10. *Let S be a numerical semigroup and let p be a prime number. Then there exists a positive integer k such that $S = \frac{T}{p^k}$ for some MED numerical semigroup T.*

Proof. By Theorem 5.1.7, we may assume that S is not an MED numerical semigroup. Let $\{s_1, \ldots, s_e\}$ be the minimal generating set of S with $s_1 = \mu(S)$. Then we can choose $j \in \{1, \ldots, e\}$ such that $s_j \notin p\mathbb{N}$. For some positive integer k_0, note that $s_j < p^{k_0} s_i$ for each $i \in \{1, \ldots, e\}$ such that $i \neq j$. Let $t_1 := s_j$. Let $t_2 := \min(\{p^{k_0}s_2, \ldots, p^{k_0}s_e\} \backslash \{p^{k_0}s_j\})$; and for each $i \in \{2, \ldots, e\}$, let $t_i := \min(\{p^{k_0}s_2, \ldots, p^{k_0}s_e\} \backslash \{p^{k_0}s_j, t_2, \ldots, t_{i-1}\})$. Then $T_0 := \langle t_1, t_2, \ldots, t_e \rangle = \langle t_1, p^{k_0}S \rangle$ is a numerical semigroup which satisfies $S = \frac{T_0}{p^{k_0}}$. Since $e(T_0) \leq e < t_1 = \mu(T_0)$, T_0 is not MED. Moreover, note that $\mu(T_0) = t_1 \notin p\mathbb{N}$ and $t_i \in p\mathbb{N}$ for all $i \in \{2, \ldots, e\}$. Therefore, by Propositions 5.2.9 and 2.1.3, there exists a numerical semigroup T_1 such that $\mu(T_1) = t_1$, $e(T_1) = e(T_0) + 1$, and $S = \frac{T_1/p}{p^{k_0}} = \frac{T_1}{p^{k_0+1}}$.

Let $m := t_1$ and iterate the above argument. For each $i \geq 1$, if T_i is not MED, there exists a numerical semigroup T_{i+1} such that $\mu(T_{i+1}) = m$, $e(T_{i+1}) = e(T_i)+1$, and $S = \frac{T_i}{p^{k_0+i}}$. Thus, $T := T_{m-e}$ has multiplicity m and embedding dimension $e + (m - e) = m$, and is therefore MED. Furthermore, if $k := k_0 + m - e$, then $S = \frac{T}{p^k}$. □

Let S be a numerical semigroup. Consider the question of whether $S = \frac{T}{n}$ for some MED numerical semigroup T and some positive integer n. If n is prime, then Theorem 5.2.8 tells us, given S, we need only worry about a finite number of values of n. If n is not prime, Proposition 2.1.3 tells us that once we deal with the "small primes" via Theorem 5.2.10, we should be able to find a natural number N beyond which every integer n satisfies $S = \frac{T}{n}$ for some MED numerical semigroup T. The next proof makes this intuition rigorous.

Theorem 5.2.11. *Let S be numerical semigroup. Then there exists $N \in \mathbb{N}$ such that for each positive integer $n \geq N$, there exists an MED numerical semigroup T such that $S = \frac{T}{n}$.*

Proof. By Theorem 5.1.7, we may assume that S is not an MED numerical semigroup. By Theorem 5.2.8, there exists a prime number q such that, for all prime numbers $p \geq q$, there exists an MED numerical semigroup T which satisfies $S = \frac{T}{p}$. Let $p_1 < \cdots < p_r$ be a list of all the prime numbers less than or equal to q. By Theorem 5.2.10, for each $i \in \{1, \ldots, r\}$, there exists $k_i \in \mathbb{N}$ such that $S = \frac{T_i}{p_i^{k_i}}$ for some MED numerical semigroup T_i. Define $N := p_1^{k_1} \cdots p_r^{k_r}$, and suppose $n \geq N$.

If there exists a prime number $w > p_r = q$ such that $w|n$, write $n = wc$ for some $0 < c \in \mathbb{N}$. Then $S = \frac{T'}{w}$ for some MED numerical semigroup T' (since $w > p_r$); and $T' = \frac{T}{c}$ for some MED numerical semigroup T by Theorem 5.1.7. Hence, by Proposition 2.1.3, $S = \frac{T'}{q} = \frac{T/c}{q} = \frac{T}{cq}$ and the result holds. In the remaining case, the only prime numbers that divide n belong to $\{p_1, \ldots, p_r\}$. As $n \geq N$, there exists an index i such that $p_i^{k_i}|n$. Then Theorem 5.1.7 gives an MED numerical semigroup T'' such that $T_i = \frac{T''}{n/p_i^{k_i}}$; and Proposition 2.1.3 gives $S = \frac{T_i}{p_i^{k_i}} = \frac{T''/(n/p_i^{k_i})}{p_i^{k_i}} = \frac{T''}{n}$. □

It follows from Example 5.1.8 that Theorem 5.2.11 is best possible. In other words, for any integer $n \geq 2$, there exists a numerical semigroup S such that no MED numerical semigroup T can satisfy $S = \frac{T}{n}$.

We close by showing that Proposition 5.2.9 leads to a family of examples that generalizes Example 5.1.9.

Example 5.2.12. If $n \geq 2$ is a composite integer, there exists a numerical semigroup S that is not MED such that $S = \frac{T}{n}$ for infinitely many numerical semigroups T of maximal embedding dimension. As in the proof of Example 5.1.9, it is enough to prove this when n is a prime number, say p. One construction that suffices is the following.

Since $p+1$ and $p(p+2)$ are relatively prime, $S := \langle p+1, p(p+2), \ldots, p(p+p) \rangle$ is a numerical semigroup. Note that $\mu(S) = p+1$.

We claim that $e(S) = p$. To prove this, it suffices to show that the minimal generating set of S is $\{p+1, p(p+2), \ldots, p(p+p)\}$. Note that $p(p+2)+p(p+2) > p(p+p) \geq p(p+i)$ for all $i \in \{2, \ldots, p\}$. Therefore, if there exists $j \in \{2, \ldots, p\}$ such that $p(p+j) \in \langle p+1, p(p+2), \ldots, p(p+j-1) \rangle$, we must have that $p(p+j) = a(p+1) + bp(p+i)$ for some positive integer a, some $b \in \{0, 1\}$ and some $i \in \{2, \ldots, j-1\}$. We analyze separately the cases where $b = 0$ and $b = 1$.

We first show that $b = 0$ leads to a contradiction. Suppose $b = 0$. Then $p(p+j) = a(p+1)$. Since p is prime, we must have that $p|a$, and so $a = pc$ for some $1 \leq c \in \mathbb{N}$. Thus $p(p+j) = pc(p+1)$, and so $p+j = cp+c$. Then $j - c = p(c-1) \geq 0$. Note that the prime number p cannot divide any positive integer less than p, and so $j - c = 0$. Hence, $p+j = cp+c = jp+j > p+j$, the desired contradiction.

Since $b = 1$, we have $p(p+j) = a(p+1)+p(p+i)$. Once again, since p is prime, we have $a = pc$ for some $1 \leq c \in \mathbb{N}$. Thus $p(p+j) = pc(p+1)+p(p+i)$, and so $p+j = c(p+1)+p+i$. Hence, $j = cp+c+i \geq p+1+i > p \geq j$, a contradiction.

The above case analysis proves the claim that $e(S) = p$. Hence, since S satisfies the conditions of Proposition 5.2.9, there exists a numerical semigroup T such that $S = \frac{T}{p}, \mu(T) = \mu(S) = p+1$, and $e(T) = e(S)+1 = p+1$. In particular, T is of maximal embedding dimension, as desired.

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Appendices

Appendix A

Miscellaneous Results

We include some miscellaneous results which were not needed in the dissertation. Let S be a numerical semigroup, let p be a prime number and let $n \in S \backslash p\mathbb{N}$. The first three results provide additional information about the numerical semigroup $E_{pS,n} := \langle pS, n\rangle$ using in the dissertation.

Proposition A.0.13. *Let p be a prime number and let S be a numerical semigroup with minimal generating set $\{s_1, \ldots, s_e\}$. Then $\{ps_1, \ldots, ps_e\}$ is the minimal generating set for the monoid pS.*

Proof. First, we show that $pS = \langle ps_1, \ldots, ps_e\rangle$. Clearly $\langle ps_1, \ldots, ps_e\rangle \subseteq pS$ since $ps_i \in pS$ for each $i \in \{1, \ldots, e\}$. Let $x \in pS$. Then $x = ps$ for some $s \in S$. Since $S = \langle s_1, \ldots, s_e\rangle$, $s = a_1s_1 + \cdots + a_es_e$ for some $a_1, \ldots, a_e \in \mathbb{N}$. Hence, $x = ps = p(a_1s_1 + \cdots + a_es_e) = a_1ps_1 + \cdots + a_eps_e \in \langle ps_1, \ldots, ps_e\rangle$. This proves that $pS = \langle ps_1, \ldots, ps_e\rangle$.

Next, we show minimality of the above generating set. If some $ps_i \in \langle ps_1, \ldots, \widehat{ps_i}, \ldots, ps_e\rangle$, where "$\widehat{\ }$" denotes omission, division by p shows that $s_i \in \langle s_1, \ldots, \hat{s}_i, \ldots, s_e\rangle$, contrary to the minimality of $\{s_1, \ldots, s_e\}$. □

Proposition A.0.14. *Let S be a numerical semigroup, let p be a prime number and let $n \in S \backslash p\mathbb{N}$. Then $E_{pS,n} = \langle pS, n\rangle$ is a subnumerical semigroup of S, $F(E_{pS,n}) = pF(S)+(p-1)n$ and $S = \frac{E_{pS,n}}{p}$.*

Proof. We first show that $E := E_{pS,n} = \langle pS, n\rangle$ is a subnumerical semigroup of S. Since $\{n\}, pS \subseteq S$, it suffices to show that $\mathbb{N}\backslash E$ is finite. Since S is a numerical semigroup, $\mathbb{N}\backslash S$ is finite and, therefore, there exists $s \in S$ such that $\gcd(n, s) = 1$. Since $\gcd(n, p) = 1$ by hypothesis, it follows that $\gcd(n, ps) = 1$ and $\langle n, ps\rangle$ is a numerical semigroup. Since $\langle n, ps\rangle \subseteq E$, $\mathbb{N}\backslash E$ is finite.

We now show that $S = \frac{E}{p}$. Clearly $pS \subseteq E$, and so $S \subseteq \frac{E}{p}$. By Proposition 2.1.4, it suffices to show that $x \in S$ whenever $x \in \mathbb{N}$ and $px \in E$. Let $x \in \mathbb{N}$ with $px \in E$. Then $px = an + bps$ for some $a, b \in \mathbb{N}$ and some $s \in S$. Since p is prime and $n \notin p\mathbb{N}$, we must have that $a = pc$ for some $c \in \mathbb{N}$. Hence, $px = p(cn + bs) \in pS$ and so $x = cn + bs \in S$.

We next show that $F(E) = pF(S) + (p-1)n$. First, we claim that every integer greater than $pF(S) + (p-1)n$ belongs to E. Let $x = pF(S) + (p-1)n + a$ where $0 < a \in \mathbb{N}$. If $a = pb$ for some $0 < b \in \mathbb{N}$, then $x = p(F(S)+b)+(p-1)n \in pS+(p-1)n \in E$. Therefore, we may suppose that $a = pb + c$ where $b \in \mathbb{N}$ and $c \in \{1, \ldots, p-1\}$. Note that since

$n \notin p\mathbb{N}$, $n = qp + r$ for some $q \in \mathbb{N}$ and $r \in \{1, \ldots, p-1\}$. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, there exists $d \in \{1, \ldots, p-1\}$ such that $dr \equiv p - c \pmod{p}$. Let $dr = pf + p - c$ where $f \in \mathbb{N}$. Thus, $x = pF(S) + (p-1)n + a = pF(S) + (p-1-d)n + dn + pb + c = pF(S) + (p-1-d)n + d(qp+r) + pb + c = pF(S) + (p-1-d)n + dqp + dr + pb + c = pF(S) + (p-1-d)n + dqp + pf + p - c + pb + c = pF(S) + (p-1-d)n + dqp + pf + p + pb = p(F(S) + dq + f + 1 + b) + (p-1-d)n \in E$, proving the claim.

To complete the proof that $F(E) = pF(S) + (p-1)n$, it is now enough to show that $pF(S) + (p-1)n \notin E$. Suppose the assertion is not true and $pF(S) + (p-1)n \in E$. Then $pF(S) + (p-1)n = ps + kn$ for some $s \in S$ and $k \in \mathbb{N}$. As $n \notin p\mathbb{N}$ and $F(S) \notin S$, we must have $k > 0$ and $s \neq F(S)$. We finish the proof by showing that the cases $s > F(S)$ and $s < F(S)$ each lead to a contradiction.

If $s > F(S)$, then $k < p - 1$. Thus, $(p-1-k)n = p(s - F(S)) \in p\mathbb{N}$, which is a contradiction since $p \nmid n$ and $p \nmid (p-1-k)$. Hence, we may suppose that $s < F(S)$. Then $k > p - 1$ and $(k - p + 1)n = p(F(S) - s) \in p\mathbb{N}$. Therefore, since $n \notin p\mathbb{N}$, we must have that $k - p + 1 = pr$ for some $0 < r \in \mathbb{N}$. It now follows that $pF(S) + (p-1)n = ps + kn = ps + (pr + p - 1)n = p(s + rn) + (p-1)n$ and so $F(S) = s + rn \in S$, a contradiction. □

Proposition A.0.15. *Let p be a prime number, let S be a numerical semigroup with minimal generating set $\{s_1, \ldots, s_e\}$, let $n \in S \backslash p\mathbb{N}$ and let $E_{pS,n} = \langle pS, n \rangle$. Then:*

(1) *If $n = s_i$ for some $i \in \{1, \ldots, e\}$, then $\{n, ps_1, \ldots, ps_{i-1}, ps_{i+1}, \ldots, ps_e\}$ is the minimal generating set for $E_{pS,n}$.*

(2) *If $n \neq s_i$ for each $i \in \{1, \ldots, e\}$, then $\{n, ps_1, \ldots, ps_e\}$ is the minimal generating set for $E_{pS,n}$.*

Proof. (1) Suppose that $n = s_j$ where $j \in \{1, \ldots, e\}$. Let $E := E_{pS,n}$ and let $A := \{n, ps_1, \ldots, ps_{j-1}, ps_{j+1}, \ldots, ps_e\}$. It's clear that $E = \langle A \rangle$. We must prove the minimality of A. Suppose that A is not the minimal generating set of E. Then there exists a subset $\{q_1, \ldots, q_r\}$ of E, with $r < e$, which is the minimal generating set of E. Since $n = \min(E \backslash p\mathbb{N})$, we must have that $n = q_i$ for some $i \in \{1, \ldots, r\}$. Without loss of generality, let $n = q_1$. Note that since $E = \langle pS, n \rangle$, every element of E is generated by n and pS. In particular, every element of $E \backslash p\mathbb{N}$ is generated by n and pS. It follows that n is the only element of $E \backslash p\mathbb{N}$ which can belong to the minimal generating set of E. Therefore, for each $i \in \{2, \ldots, r\}$, there exists $a_i \in \mathbb{N}$ such that $q_i = pa_i$ and so $E = \langle n, pa_2, \ldots, pa_r \rangle$.

It is clear that $\langle pn, pa_2, \ldots, pa_r \rangle \subseteq pS$. Moreover, if $s \in S$ then $ps \in E = \langle n, pa_2, \ldots, pa_r \rangle$. Hence, $ps = k_1 n + k_2 pa_2 + \cdots + k_r pa_r$ for some $k_1, \ldots, k_r \in \mathbb{N}$ and so $ps \in \langle pn, pa_2, \ldots, pa_r \rangle$ since $p \nmid n$. Therefore, $pS = \langle pn, pa_2, \ldots, pa_r \rangle$. However, since $r < e$, this contradicts Proposition A.0.13. Thus, A is the minimal generating set of E.

(2) Suppose that $n \neq s_i$ for each $i \in \{1, \ldots, e\}$. Let $E := E_{n,pS}$ and let $A := \{n, ps_1, \ldots, ps_e\}$. It's clear that $E = \langle A \rangle$. We must prove the minimality of A. Suppose that A is not the minimal generating set of E. Then there exists a subset $\{q_1, \ldots, q_r\}$ of E, with $r < e + 1$, which is the minimal generating set of E.

Since $n = \min(E \backslash p\mathbb{N})$, we must have that $n = q_i$ for some $i \in \{1, \ldots, r\}$. Without loss of generality, let $n = q_1$. Note that since $E = \langle pS, n \rangle$, every element of E is generated by n and pS. In particular, every element of $E \backslash p\mathbb{N}$ is generated by n and pS. It follows that n is the only element of $E \backslash p\mathbb{N}$ which can belong to the minimal generating set of E. Therefore, for each $i \in \{2, \ldots, r\}$, there exists $a_i \in \mathbb{N}$ such that $q_i = pa_i$ and so $E = \langle n, pa_2, \ldots, pa_r \rangle$.

It is clear that $\langle pn, pa_2, \ldots, pa_r\rangle \subseteq pS$. Moreover, if $s \in S$ then $ps \in E = \langle n, pa_2, \ldots, pa_r\rangle$. Hence, $ps = k_1n + k_2pa_2 + \cdots + k_rpa_r$ for some $k_1, \ldots, k_r \in \mathbb{N}$ and so $ps \in \langle pn, pa_2, \ldots, pa_r\rangle$. Thus, $pS = \langle pn, pa_2, \ldots, pa_r\rangle$. Since $r \leq e$ and $\{s_1, \ldots, s_e\}$ is the minimal generating set for S, by Proposition A.0.13 we must have that $r = e$ and $\{n, a_2, \ldots, a_e\} = \{s_1, \ldots, s_e\}$. Therefore, $n = s_i$ for some $i \in \{1, \ldots, e\}$, which is a contradiction. $\square$

Vita

Harold J. Smith earned his B.S. in Chemical Engineering from North Carolina State University in Raleigh, NC in 1993. After graduation, he spent 11 years working in various engineering roles in the adhesive, coating and converting industries. As a Technical Service Engineer with National Starch and Chemical Company in Bridgewater, NJ, he worked with solventborne, emulsion, hotmelt and radiation curable technologies and served customers in industrial, graphics and paper label markets. Later, he worked as a Senior Process Engineer with Rexam Image Products in Matthews, NC and Pechiney Plastic Packaging in Asheville, NC. As a process engineer, he worked primarily with various coating, printing and finishing technologies including reverse roll and slot die coating and both flexographic and rotogravure printing.

In the Spring of 2004, while taking classes part-time at the University of North Carolina at Asheville, he took his first courses in higher Mathematics. In 2005, he left industry and began graduate studies at the University of Tennessee. He was awarded a Ph.D. in the Spring of 2010 under the direction of Dr. David E. Dobbs.