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Carleson-type inequalitites in harmonically weighted Dirichlet spaces

Gerardo Roman Chacon Perez
University of Tennessee - Knoxville, gchacon@utk.edu

To the Graduate Council:

I am submitting herewith a dissertation written by Gerardo Roman Chacon Perez entitled "Carleson-type inequalitites in harmonically weighted Dirichlet spaces." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Stefan Richter, Major Professor

We have read this dissertation and recommend its acceptance:

Carl Sundberg, Ken Stephenson, Andreas Nebenfuehr

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Carleson-type inequalitites in harmonically weighted Dirichlet spaces

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Gerardo Román Chacón Pérez
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Dedication

A Yinzú... por supuesto

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Abstract

Carleson measures for Harmonically Weighted Dirichlet Spaces are characterized. It is shown a version of a maximal inequality for these spaces. Also, Interpolating Sequences and Closed-Range Composition Operators are studied in this context.

Contents

Chapter 1

Introduction

1.1 Local Dirichlet spaces

In this section, we will define the $D(\mu)$ spaces and we will review some of the properties of these spaces that are going to be needed later on.

Let T be a bounded operator on a complex separable Hilbert space $\mathcal{H}$. It is said that (see [1]) T is a 2-*isometry* if it satisfies the operator equation:

$$T^{*^2}T^2 - 2T^*T + I = 0.$$

An operator T is *analytic* if $\cap_{n>0} T^n(\mathcal{H}) = \{0\}$. Analytic two isometries are studied in [1] and [27]. In [27], Richter shows that every analytic two-isometry such that dim kerT^* = 1 can be represented as muliplication by z on a Dirichlet-type space $D(\mu)$.

Dirichlet-type spaces were then introduced by Richter and have been studied ever since by several authors, see for example [4], [10], [11], [29], [31], [32], [40], [42] and [43].

Given a positive Borel measure μ defined on $\partial\mathbb{D}$ and let φ_μ be positive the harmonic function defined on the unit disc $\mathbb{D}$ by

$$\varphi_\mu(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} \frac{d\mu(t)}{2\pi}.$$

The Dirichlet type space $D(\mu)$ is defined as the space of all analytic functions on $\mathbb{D}$

such that

$$\int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z) < \infty$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. If $\mu = 0$ then define $D(\mu) = H^2$, the Hardy space on the unit disc. Notice that if $d\mu = dm$ is the arc-length Lebesgue measure on $\partial\mathbb{D}$, then the Dirichlet-type space $D(m)$ coincides with the classical Dirichlet space $\mathcal{D}$.

The following lemma shows that the space $D(\mu)$ is contained as a set in the space H^2.

Lemma 1.1.1 ([27]). *If $f \in D(\mu)$, then $f \in H^2$. In fact if $\mu \neq 0$ then there exists a constant $C > 0$ such that for any analytic function f on $\mathbb{D}$*

$$\|f\|_{H^2}^2 \leq C \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z) \right).$$

Consequently, every function $f \in D(\mu)$ has a nontangential limit almost everywhere in the bondary of the unit disc $\partial\mathbb{D}$ with respect to the Lebesgue arc-length measure. Another consequence of this is that we can now define a norm on the $D(\mu)$ space as

$$\|f\|_{D(\mu)}^2 := \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z).$$

Notice that if for each $z \in \mathbb{D}$ we define $\gamma_z : D(\mu) \to \mathbb{C}$ as $\gamma_z(f) := f(z)$, then γ_z defines a linear operator (called the evaluation functional at z) and since $D(\mu)$ is continuously contained in H^2 and it can be shown (see for example [15]) that evaluation functionals are bounded on H^2, then they are also bounded on $D(\mu)$. Moreover, since $D(\mu)$ spaces are Hilbert spaces with inner product given by

$$\langle f, g \rangle_{D(\mu)} := \langle f, g \rangle_{H^2} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} \varphi_\mu(z) dA(z),$$

then by Riesz's representation Theorem, we have that for each $z \in \mathbb{D}$ there exists a function $K_z^\mu \in D(\mu)$ such that for every $f \in D(\mu)$,

$$\gamma_z(f) = \langle f, K_z^\mu \rangle_{D(\mu)}.$$

The functions K_z^μ are called the *reproducing kernels* for the space $D(\mu)$. Reproducing kernels have received a lot of attention since sometimes one can characterize properties of the space in terms of properties of its reproducing kernels. In Chapter 3 we will exhibit some interesting properties and we will find an explicit expression for the reproducing kernels in one specific case of the measure $D(\mu)$; specifically, we will find a formula for the reproducing kernels of spaces of the form $D(a\delta_\lambda)$, where $a > 0$, $\lambda \in \partial\mathbb{D}$ and δ_λ denotes the Dirac measure at the point λ.

We mention two more properties of $D(\mu)$ spaces proved by Richter in [27]. First, the operator multiplication by z, $M_z : D(\mu) \to D(\mu)$ is bounded on $D(\mu)$. The proof of this result requires some estimates to the $D(\mu)$-norm of the truncation $\sum_{n=k}^{\infty} \hat{f}(n)z^n$ of an analytic function $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$. Another property proven in [27] is that the set of polynomials is dense in any $D(\mu)$ space.

In [28], Richter and Sundberg introduced the notion of *Local Dirichlet Integral*: For a function $f \in L^1(\partial\mathbb{D})$ and $\zeta \in \partial\mathbb{D}$, the local Dirichlet integral of f at ζ is given by

$$D_\zeta(f) = \int_0^{2\pi} \left| \frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta} \right|^2 \frac{dt}{2\pi}.$$

If $f(\zeta)$ does not exist, then we set $D_\zeta(f) = \infty$. It turns out that there is a nice and useful formula for expressing the norm of the Dirichlet-type spaces in terms of the local Dirichlet integral:

Proposition 1.1.2 ([28], Proposition 2.2)**.** *Let μ be a nonnegative finite Borel measure on $\partial\mathbb{D}$. If $f \in H^2$, then*

$$\int_{\partial\mathbb{D}} D_\zeta(f) d\mu(\zeta) = \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z) \tag{1.1}$$

Consequently, if a function f belongs to $D(\mu)$, then $D_\zeta(f)$ is finite for μ-almost every $\zeta \in \partial\mathbb{D}$ and the function g defined by

$$g(e^{it}) := \frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta}$$

is in H^2. Moreover, since for any function in H^2 we have that $(1 - |z|^2)|g(z)|^2 \to 0$ as $|z| \to 1$, then for $z \in \mathbb{D}$ we have

$$|f(z) - f(\zeta)| = |(z - \zeta)g(z)|^2 = \frac{|z - \zeta|^2}{(1 - |z|^2)}(1 - |z|^2)|g(z)|^2$$

and it follows that $f(z) \to f(\zeta)$ if $z \to \zeta$ in any oricyclic approach region:

$$O_\kappa(\zeta) := \{z \in \mathbb{D} : |z-\zeta|^2 < \kappa(1-|z|^2).$$

The following formula for the local Dirichlet integral is also proved in [28]. Then as a consequence of Equation 1.1, one obtains a formula for the norm of a function in $D(\mu)$.

Theorem 1.1.3. *Let $\zeta \in \partial\mathbb{D}$, let $f \in H^2$, and let $f = BSf_0$, that is, let*

$$f(z) = \prod_{j=1}^{\infty} \frac{\overline{\alpha_j}}{|\alpha_j|} \frac{\alpha_j - z}{1-\overline{\alpha_j}z} \exp\left(-\int \frac{e^{it}+z}{e^{it}-z} d\sigma(t)\right) \exp\left(\int \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})| \frac{dt}{2\pi}\right)$$

be the factorization of f into a Blaschke product, a singular inner function, and an outer function. Then

$$\begin{aligned} D_\zeta(f) &= \sum_{j=1}^{\infty} \frac{1-|\alpha_j|^2}{|\zeta-\alpha_j|^2}|f_0(\zeta)|^2 + \int_0^{2\pi} \frac{2}{|e^{it}-\zeta|^2} d\sigma(t)|f_0(\zeta)|^2 \\ &\quad + \int_0^{2\pi} \frac{e^{2u(e^{it})} - e^{2u(\zeta)} - 2e^{2u(\zeta)}(u(e^{it}) - u(\zeta))}{|e^{it}-\zeta|^2} \frac{dt}{2\pi}. \end{aligned} \tag{1.2}$$

Another important formula for the local Dirichlet integral that will be used later on is the following:

Proposition 1.1.4 ([28], Lemma 3.4)**.** *Let $\zeta \in \partial\mathbb{D}$, let φ be an inner function and $f \in H^2$. Then if $D_\zeta(f) < \infty$, then*

$$D_\zeta(\varphi f) = D_\zeta(\varphi)|f(\zeta)|^2 + D_\zeta(f).$$

Inequalities involving a maximal operator (either nontangential, radial or its modifications) received a lot of attention throughout the years. We mention here a question posed by Chartrand in [10] about the maximal operator on Dirichlet-type spaces. In Chapter 2, we will answer that question by showing that the nontangential maximal operator maps a function in $D(\mu)$ to a positive harmonic function with finite $D(\mu)$-integral. In order to do that, we will define the *harmonic Dirichlet-type spaces* $\mathcal{B}_\mu$ and we will use Wu's techniques from [44] to show that for any function $f \in D(\mu)$, it's norm $\|f\|_{D(\mu)}$ is comparable to the norm $\|\operatorname{Re} f\|_{\mathcal{B}_\mu}$. Then, we will use Marcinkiewicz interpolation to show the maximal inequality.

1.2 Carleson Measures and Interpolation Problems

In this Section, we will give a short introduction to interpolation problems and will review some results involving this problem in different spaces of analytic functions.

Let H^∞ be the algebra of bounded analytic functions in the open unit disc D. A sequence of points $(z_n) \subset \mathbb{D}$ is an *interpolating sequence* for H^∞ if for any sequence $(w_n) \in l^\infty$, there exists a bounded analytic function f on $\mathbb{D}$ such that $f(z_n) = w_n$ for all $n = 1, 2, \ldots$.

In 1958, L. Carleson [8] characterized the interpolating sequences for H^∞ in terms of a geometric condition.

Theorem 1.2.1. *Let (z_n) be a sequence of points in the unit disc $\mathbb{D}$, the following conditions are equivalent.*

(i) The sequence (z_n) is an interpolating sequence for H^∞.

(ii) There exists a constant $a > 0$ such that

$$\prod_{n\neq k} \frac{z_n - z_k}{1 - z_n\overline{z_k}} \geq \delta, \qquad k = 1, 2, \ldots$$

.

(iii) The sequence (z_n) is separated, that is, there exists a constant $c > 0$ such that

$$\left| \frac{z_n - z_k}{1 - z_n\overline{z_k}} \right| \geq c, \quad \text{for } n \neq k$$

and there exists a constant $C > 0$ such that for all Carleson squares $Q(z)$ the following inequality holds:

$$\sum_{z_n \in Q(z)} (1 - |z_n|) \leq C(1 - |z|)$$

For $z \in \mathbb{D}$, $z \neq 0$, a Carleson square is defined as the set

$$Q(z) := \{w \in \mathbb{D} : |\text{Arg}z - \text{Arg}w| \leq \pi(1 - |z|), 1 - |w| \leq 1 - |z|\}$$

and $Q(0) := \mathbb{D}$.

For a given finite positive Borel measure μ on the unit disc $\mathbb{D}$. We say that it is a *Carleson measure* for the space H^2 if there exists a constant $C > 0$ such that for every function $f \in H^2$ the following inequality holds:

$$\int |f|^2 d\mu \leq C\|f\|_{H^2}^2.$$

Carleson measures and interpolating sequences are of great importance in the proof of the Corona Theorem, by Carleson [9].

There is a well-known geometric characterization of Carleson measures for the Hardy space which is as follows (see for example [15]):

Theorem 1.2.2. *A positive finite Borel measure μ on the unit disc $\mathbb{D}$ is a Carleson measure if and only if there exists a constant $C > 0$ such that for every Carleson square $Q(z)$*

$$\mu(Q(z)) \leq C(1 - |z|).$$

Notice that the quantity $1 - |z|$ is comparable to the length of the arc determined on $\partial\mathbb{D}$ by $Q(z)$. Also notice that if we write $\mu_Z := \sum_{j=1}^{\infty}(1 - |z_n|)\delta_{z_n}$, then the second condition of (iii) can be written as $\mu_Z(Q(z)) \leq C(1 - |z|)$. That is, μ_Z is a Carleson measure for H^2.

Carleson measures are an important tool for studying the properties of operators on spaces of analytic functions. Thus it is important to characterize which measures satisfy what we call the *Carleson condition*. For example it is an important tool in the study of the multipliers of a space and the composition operators acting on a space of analytic functions (see for example [13]). We will say that a positive finite Borel measure ν is a Carleson measure for the Dirichlet space $D(\mu)$ if there exists a constant $C > 0$ such that for every function $f \in D(\mu)$ the following inequality holds:

$$\int |f|^2 d\nu \leq C\|f\|_{D(\mu)}^2.$$

For the case in which μ coincides with the Lebesgue arc-length measure, the space $D(\mu)$ is the classical Dirichlet space. In this case, Stegenga gave a characterization of Carleson measures in terms of the notion of capacity. In Chapter 2 we introduce a similar notion of capacity for the general $D(\mu)$ spaces and prove the corresponding characterization of Carleson measures. In order to to this, we will make use of the previously proven inequality about the nontangential maximal operator.

The concept of Carleson measures for Dirichlet type spaces was introduced and studied by Chartrand in [10] and [11] in order to characterize the multipliers of the $D(\mu)$ spaces. However, this concept does not coincide with the definition we present here. We consider our definition to be more natural since is the analogous definition to the one in the Hardy space and on the Dirichlet space. In Chapter 2 we show that the two definitions are not comparable. Also, the methods used lead to a characterization of Chartrand's definition of Carleson measures.

For the rest of this section we will introduce the notion of interpolation in other spaces of analytic functions. We will also present some known results on this subject.

First, we introduce the notion of interpolating sequence for the case of a Hilbert space $\mathcal{H}$ of analytic functions on the unit disc $\mathbb{D}$ in which the point evaluations are continuous (and consequently there are reproducing kernels $k_z^{\mathcal{H}}$ in the space). A sequence (z_n) is called *interpolating for* $\mathcal{H}$ if for any sequence $(w_n) \subset \mathbb{C}$ such that the sequence $(w_n/\|k_{z_n}^{\mathcal{H}}\|)$ belongs to l^2, there exists a function $f \in \mathcal{H}$ such that $f(z_n) = w_n$ for every $n = 1, 2, \ldots$

Similarly, a sequence of points $(z_n) \subset \mathbb{D}$ is called *interpolating for the Hardy space* H^p if for any sequence (w_n) such that $(w_n/\|T_{z_n}\|) \in l^p$, the interpolation problem $f(z_n) = w_n \; n = 1, 2, \ldots$ is solvable with a function $f \in H^p$. Here, $\|T_{z_n}\|$ denotes the norm of the point evaluation functional at the point z, that is, $T_z : \mathcal{B} \to \mathbb{C}$ is given by $T_z(f) := f(z)$

In 1961, H. Shapiro and A. Shields [41] showed that a sequence of points is interpolating for the space H^p with $1 \le p < \infty$ if and only if the sequence is interpolating for H^∞. The result also holds for the case $0 < p < 1$ [19]. For the case of Bergman spaces A^p, interpolating sequences are defined in the obvious analogous way. In this case, interpolating sequences were characterized by K. Seip [38] in 1993 using a density condition.

Interpolating sequences in the Dirichlet space were described by D. Marshall - C. Sundberg [24] and C. Bishop [5] simultaneously in the 1994 using different techniques. An important observation proved in [24] is that the interpolating sequences for the Dirichlet space $\mathcal{D}$ and the interpolating sequences for its space of multipliers $\mathcal{M}(\mathcal{D})$ are the same (just as in the case of H^p spaces). Marshall and Sundberg also give a new proof of the characterization of interpolating sequences for H^p using Hilbert space techniques.

In 2002, B. Boe [6] characterized the interpolating sequences for the Besov spaces

B_p in terms of a separation condition and a Carleson measure condition. He also finds in [7] a necessary and sufficient condition for a sequence to be interpolating in certain Hilbert spaces satisfying the Nevanlinna-Pick property and another technical condition about the Grammian matrix associated with the sequence.

Marshall and Sundberg showed in [24] that if a Hilbert space has the Nevanlinna-Pick property, then the interpolating sequences for H and for $\mathcal{M}(H)$ are the same. In [43] Shimorin showed that $D(\mu)$ spaces have the Nevannlina-Pick property. Spaces with this property have received a lot of attention lately and several open problems concerning them have been resolved for the case of spaces with radially symmetric weights (see for example [2] and the references therein for solutions of the interpolation problem in this setting).

As can be seen from the previous paragraphs, there is a lot of work in the subject of characterizing interpolating sequences in spaces of analytic functions. A good exposition of recent work and open questions can be found in [39] and in the more recent [37]. In Chapter 3 we will use some results due to Serra [40] and some ideas from Sarason [31] and we will characterize interpolating sequences for $D(\mu)$ spaces in the case in which μ is a finitely atomic measure. As far as we know, these are the first examples of spaces with the Nevannlinna-Pick property with weights that are not radially symmetric and where still the analogous results of interpolation remain true. In order to do that, we start by studying Dirichlet type spaces of the form $D(a\delta_\lambda)$ and we use that these space can be seen as *De Branges Rovnyak* spaces (see [31]). We then use the explicit form of the reproducing kernels to characterize interpolating sequences in terms of a separation condition and a Carleson measure condition. Then, we study the more general case of μ being a finitely atomic measure. In this case, we combine a result from McCullough and Trent [25] with a result from Richter and Sundberg [29] and Aleman [4] to obtain that the quotient of the reproducing kernel of this space and the reproducing kernel of the space $D(a\delta_\lambda)$ is positive definite. Then we reduce the problem to the previously proven result.

1.3 Composition operators on $D(\mu)$ spaces

In this Section we will give a short introduction to composition operators. Let φ denote a nonconstant holomorphic self-map of $\mathbb{D}$. The corresponding composition operator will be denoted by C_φ and it is formally defined as

$$C_\varphi(f)(z) := f(\varphi(z))$$

for any function f defined in $\mathbb{D}$. The subordination principle of J. E. Littlewood guarantees that C_φ acts boundedly on the Hardy space H^2 (see [13] or [35]).

Composition operators have received a lot of attention since they relate the function-theoretical properties of the symbol φ with the operator-theoretical properties of the operator C_φ. Questions such as boundedness and compactness have been studied in many spaces of analytic functions and characterized in terms of the properties of the symbol. Also some further questions about composition operators have been studied. For example, When does a composition operator has closed range? or what is the spectrum of a composition operator? or when is a composition operator a Fredholm operator?. Also the topological structure of the sets of composition operators on the space of linear operators has been investigated. For a nice recompilation of properties of composition operators acting on various spaces of analytic functions see [13] or [35].

The Nevanlinna counting function for φ is the function N_φ defined in $\mathbb{D}$ by

$$N_\varphi(w) := \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|}; \tag{1.3}$$

where the sum on the right side here is to be interpreted as taking account of multiplicities, and it is to be interpreted as 0 if w is not in the image set $\varphi(\mathbb{D})$. If $\varphi(w) \neq 0$, then it is a consequence of the Blaschke condition that the sum on the right is finite.

The following Theorem due to Shapiro [34] characterizes compactness of composition operators acting on the Hardy space H^2 in terms of the Nevanlinna counting function:

Theorem 1.3.1. *The operator C_φ acts compactly on H^2 if and only if*

$$N_\varphi(w) = o\left(\log\frac{1}{|w|}\right) \quad as\ |w| \to 1.$$

Shapiro's proof of this result relies on the following change of variables formula and on the Littelwood-Paley identity that defines a norm that is equivalent to the H^2-norm.

Theorem 1.3.2. *Let u and v be nonnegative measurable functions on $\mathbb{D}$. Then*

$$\int_{\mathbb{D}} u(\varphi(z))|\varphi'(z)|^2 v(z)dA(z) = \int_{\varphi(\mathbb{D})} u(w)\left(\sum_{z\in\varphi^{-1}(w)} v(z)\right) dA(w)$$

Using these ideas and the formula for the local Dirichlet integral found by Richter and Sundberg in [28], Sarason and Silva [33] characterized boundedness and compactness of composition operators for the Dirichlet type spaces $D(\mu)$. Their characterization is given in terms of the counting function

$$R^{\mu}_{\varphi}(w) := \sum_{z\in\varphi^{-1}(w)} P_{\mu}(z)$$

where $P_{\mu}(z)$ denotes the Poisson integral of μ. This characterization becomes particulary simple for the case in which $\mu = \delta_{\zeta}$ for $\zeta \in \partial\mathbb{D}$.

In Chapter 4 we will characterize those composition operators which have closed range in the space $D(\delta_1)$, In the proof we will use a characterization of dominant sets for $D(\delta_1)$ that we prove by following ideas of Luecking from [21] and [21].

Composition operators with closed range have been studied in several settings. Nina Zorboska [46] characterized the composition operators with closed range in the Hardy space $H^2(\mathbb{D})$, she uses Luecking's ideas about dominant sets and the Littlewood-Paley identity. Furthermore, composition operators with closed range have been investigated on weighted Bergman spaces and on the Bloch space in one and several variables (see [3], [16] and [26]).

In the Dirichlet space, composition operators with closed range have also been studied, and people have tried to use the same ideas. That is, one tries to convert the norm estimates into reverse Carleson measure inequalities using the change of variables formula. The measure corresponding to the composition operator C_{φ} in the Dirichlet space is $d\nu_{\varphi} = n_{\varphi}dA$, where $n_{\varphi}(w)$ is another "counting function" that measures the cardinality of the set $\{z \in \mathbb{D} : \varphi(z) = w\}$. Consequently the characterization of the composition operators with closed range corresponds to a characterization of a measure ν_{φ} which satisfies a reverse Carleson inequality. For the case of the Hardy space and some weighted Bergman spaces this corresponds to finding a characterization of the so-called dominant sets. In the Dirichlet space Luecking and others had conjectured that a sufficient condition for the function φ to induce a composition operator having closed range was that there exists a constant $\delta > 0$ such that for every $a \in \mathbb{D}$ and $0 < \eta < 1$, $\int_{\Delta(a,\eta)} n_\varphi dA \geq \delta|\Delta(a,\eta)|$, where $\Delta(a,\eta)$ denotes a pseudohyperbolic disc with center at a and radius η, and $|\Delta(a,\eta)|$ denotes its area. However, in 1999 Luecking [23] showed an example of a function φ in the Dirichlet space that induces a bounded composition operator in the Dirichlet space, and satisfies that condition but the induced composition operator has no closed range.

Chapter 2

Carleson measures on Dirichlet-type spaces

2.1 The non-tangential maximal function on $D(\mu)$

In this Section, we will show an inequality for the local Dirichlet integral of the nontangential maximal function of a function in $D(\mu)$. First, we will reduce the problem to one of harmonic functions by using techniques that can be found in [44] in which the case of the Dirichlet spaces D^p_α is considered.

Recall that if μ is a finite, positive Borel measure on $\partial\mathbb{D}$. Then the norm of the Dirichlet-type space $D(\mu)$ is given by $\|f\|^2_{D(\mu)} := \|f\|_{H^2} + \int D_\zeta(f)d\mu(\zeta) < \infty$, where by a change of variables we can write the local Dirichlet integral as:

$$D_\zeta(f) = \int_0^{2\pi} \left| \frac{f(e^{it}\zeta) - f(\zeta)}{e^{it} - 1} \right|^2 \frac{dt}{2\pi}.$$

Definition 2.1.1. The harmonic Dirichlet space $\mathcal{B}_\mu$ consists of all real functions $f \in L^2(\partial\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{B}_\mu} := \|f\|^2_{L^2(\partial\mathbb{D})} + \int_{\partial\mathbb{D}} D_\zeta(f)d\mu(\zeta) < \infty.$$

We will show that for every $f \in D(\mu)$ we have that $\|f\|_{D(\mu)} \sim \|\mathrm{Re} f\|_{\mathcal{B}_\mu}$ and consequently it will be enough to prove a maximal inequality for harmonic functions. For that, we will use the following proposition.

Proposition 2.1.2 ([10], Prop. 2.9)**.** *Let f be a harmonic function on $\mathbb{D}$ of the form $f = f_+ + f_-$, where $f_+, f_- \in D(\mu)$, and $f_-(0) = 0$. Then*

$$\int_{\partial\mathbb{D}} D_\zeta(f)d\mu(\zeta) = \int_{\mathbb{D}} |\nabla(f)|^2 P_\mu dA$$

Now, with this in hand, we have that if $f \in D(\mu)$, then there exists harmonic functions h_1 and h_2 such that $f = h_1 + ih_2$ and h_1 satisfies the conditions of the previous proposition. Moreover, since h_1 and h_2 are harmonic conjugates, then (see [15], Theorem 4.1) $\|h_2\|_{L^2(\partial\mathbb{D})} \lesssim \|h_1\|_{L^2(\partial\mathbb{D})}$. Consequently,

$$\begin{aligned} \|f\|^2_{D(\mu)} &= \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'|^2 P_\mu dA \\ &\lesssim \|h_1\|^2_{L^2(\mu)} + \|h_2\|^2_{L^2(\mu)} + \int_{\mathbb{D}} |\nabla h_1|^2 P_\mu dA \\ &\lesssim \|h_1\|^2_{L^2(\mu)} + \int_{\mathbb{D}} |\nabla h_1|^2 P_\mu dA \\ &= \|h_1\|^2_{\mathcal{B}_\mu}. \end{aligned}$$

On the other hand, it is clear that $\|h\|_{\mathcal{B}_\mu} \lesssim \|f\|_{D(\mu)}$. Consequently we have that

$$\|f\|_{D(\mu)} \sim \|\mathrm{Re} f\|_{\mathcal{B}_\mu}. \tag{2.1}$$

Now, we will use a truncation method to show the maximal inequality for functions in $\mathcal{B}_\mu$ and consecuently in $D(\mu)$.

Let φ be a nondecreasing function in $C_0^\infty(\mathbb{R})$ which satisfies

$$\varphi(t) = \begin{cases} 0, & \text{if } t \leq 1/2; \\ 1, & \text{if } t \geq 1. \end{cases}$$

and consider the smooth truncation $\{F_j\}_{-\infty}^{\infty}$:

$$F_j(f) := 2^j \varphi\left(\frac{|f|}{2^j}\right), \qquad j = 0, \pm 1, \pm 2, \ldots$$

Then for each j we have that

$$\begin{aligned} \|F_j(f)\|^2_{L^2(\partial\mathbb{D})} &= 2^{2j}\int_{\partial\mathbb{D}} \left|\varphi\left(\frac{|f(e^{it})|}{2^j}\right)\right|^2 \frac{dt}{2\pi} \\ &= 2^{2j}\int_{\{|f|>2^{j-1}\}} \left|\varphi\left(\frac{|f(e^{it})|}{2^j}\right)\right|^2 \frac{dt}{2\pi} \\ &\leq 2^{2j}|\{|f|>2^{j-1}\}|, \end{aligned}$$

where for a set $A \subset \partial\mathbb{D}$, $|A|$ denotes the normalized Lebesgue measure of A on $\partial\mathbb{D}$. Consequently,

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \|F_j(f)\|^2_{L^2(\partial\mathbb{D})} &\leq \sum_{j=-\infty}^{\infty} 2^{2j}|\{|f|>2^{j-1}\}| \\ &\lesssim \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} 2^{j-1}|\{|f|>2^{j-1}\}|dt \\ &\lesssim \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} |\{|f|>t\}|tdt \\ &= \int_0^\infty |\{|f|>t\}|tdt \\ &= \|f\|^2_{L^2(\partial\mathbb{D})} \end{aligned}$$

Lemma 2.1.3. *There exists a constant $C>0$ such that*

$$\sum_{j=-\infty}^{\infty} \|F_j(f)\|^2_{\mathcal{B}_\mu} \leq C\|f\|^2_{\mathcal{B}_\mu}$$

Proof. The corresponding inequality for the $L^2(\partial\mathbb{D})$-norm is shown above. Thus, it is enough to show that there exist a constant $C>0$ such that for any $t, s \in \mathbb{R}$,

$$\sum_{l=-\infty}^{\infty} \left|\frac{F_l(f(e^{it})) - F_l(f(e^{is}))}{e^{it}-e^{is}}\right|^2 \leq \left|\frac{f(e^{it})-f(e^{is})}{e^{it}-e^{is}}\right|^2. \tag{2.2}$$

The proof of this is done in [44], we include it here for the sake of completeness. Let j and k be integers such that $2^{j-1} \leq |f(e^{it})| < 2^j$ and $2^{k-1} \leq |f(e^{it})| < 2^k$. Without loss of generality we may assume that $j \geq k$, then for the case $j=k$ we use the Mean Value Theorem for the function φ to obtain that there exists a constant $c \in (0,1)$ such that

$$\begin{aligned}\sum_{l=-\infty}^{\infty} |F_l(f(e^{it})) - F_l(f(e^{is}))|^2 &= |F_j(f(e^{it})) - F_j(f(e^{is}))|^2 \\ &= |\varphi'(c)|^2 ||f(e^{it})| - |f(e^{is})||^2 \\ &\lesssim |f(e^{it}) - f(e^{is})|^2\end{aligned}$$

and inequality (2.2) holds.

Now, if $j \geq k+1$, then we use the Mean Value Theorem twice to find two constants c and d in $(0,1)$ such that

$$\begin{aligned}\sum_{l=-\infty}^{\infty} |F_l(f(e^{it})) - F_l(f(e^{is}))|^2 &= |2^k - F_k(f(e^{is}))|^2 + |F_j(f(e^{it})|^2 \\ &= 2^{2k}|\varphi(1) - \varphi(2^{-k}|f(e^{is}|))|^2 \\ &\quad +2^{2j}|\varphi(2^{-j}|f(e^{it})|) - \varphi(1/2)|^2 \\ &= |\varphi'(c)|^2(2^k - |f(e^{is})|)^2 \\ &\quad +|\varphi'(d)|^2(|f(e^{it})| - 2^{j-1})^2 \\ &\lesssim (|f(e^{it}) - f(e^{is})|)^2 \lesssim |f(e^{it}) - f(e^{is})|^2\end{aligned}$$

and inequality (2.2) also holds. □

Definition 2.1.4. For any open set $O \subset \partial\mathbb{D}$ define the $\mathcal{B}_\mu$-capacity of O by

$$\text{cap}_{\mathcal{B}_\mu}(O) := \inf\{\|f\|^2_{\mathcal{B}_\mu} : f \geq 1 \text{ on } O\}$$

Lemma 2.1.5. *For all $f \in \mathcal{B}_\mu$ we have the estimate:*

$$\int_0^\infty \text{cap}_{\mathcal{B}_\mu}(\{|f| > t\})tdt \lesssim \|f\|^2_{\mathcal{B}_\mu}$$

Proof. Since

$$\begin{aligned}\int_0^\infty \operatorname{cap}_{\mathcal{B}_\mu}(\{|f|>t\})tdt &= \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}} \operatorname{cap}_{\mathcal{B}_\mu}(\{|f|>t\})tdt \\ &\le \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}} 2^j \operatorname{cap}_{\mathcal{B}_\mu}(\{|f|>2^j\})dt \\ &= \sum_{j=-\infty}^{\infty} 2^{2j} \operatorname{cap}_{\mathcal{B}_\mu}(\{|f|>2^j\})\end{aligned}$$

and $2^{-k}F_k(f)\ge 1$ on the set $\{|f|>2^k\}$, then using Lemma 2.1.3, we have that

$$\sum_{-\infty}^{\infty} 2^{2k} \operatorname{cap}_{\mathcal{B}_\mu}(\{|f|>2^k\}) \le \sum_{-\infty}^{\infty} \|F_k(f)\|^2_{\mathcal{B}_\mu} \lesssim \|f\|^2_{\mathcal{B}_\mu}.$$

□

Definition 2.1.6. Given a function f on $\mathbb{D}$, the nontangential maximal function of f is the function on $\partial\mathbb{D}$ defined by

$$N(f)(e^{i\theta}) := \sup_{z\in\Gamma(e^{i\theta})} |f(z)|$$

where $\Gamma(e^{i\theta})$ denotes the convex hull of the disk $\{|z|<1/2\}$ and the point $e^{i\theta}$.

We will show that the operator N satisfies:

$$D_\zeta(Nf) \lesssim D_\zeta(f).$$

For a function $g\in L^1(\partial\mathbb{D})$ define the following function as

$$Mg(e^{ix}) := \sup_{1\in I}\frac{1}{|I|}\int_I \frac{|e^{it}-1||g(e^{i(x+t)})-g(e^{it})|}{|e^{ix}-1|}\frac{dt}{2\pi},$$

where the supremum is taken over all the open intervals $I\subset\partial\mathbb{D}$ centered at 1. It is well known (see for example [30]) that for every $e^{ix}\in\partial\mathbb{D}$,

$$\sup_{z\in\Gamma(1)}|g(ze^{ix})-g(z)| \lesssim \sup_{1\in I}\frac{1}{|I|}\int_I |g(e^{i(x+t)})-g(e^{it})|\frac{dt}{2\pi}. \tag{2.3}$$

We will also need the following lemmas.

Lemma 2.1.7. *Let $g \in L^1(\partial\mathbb{D})$, then for every $\lambda > 0$*

$$\left|\left\{e^{ix} \in \partial\mathbb{D} : Mg(e^{ix}) > \lambda\right\}\right| \lesssim \frac{\|g\|_{L^1(\partial\mathbb{D})}}{\lambda},$$

i.e. M maps $L^1(\partial\mathbb{D})$ to weak-$L^1(\partial\mathbb{D})$.

Proof. Notice that

$$\begin{aligned} Mg(e^{ix}) &\leq \sup_{1 \in I} \int_I \frac{|g(e^{i(x+t)}) - g(e^{it})|}{|e^{ix} - 1|} \frac{dt}{2\pi} \\ &\lesssim \frac{1}{|e^{ix} - 1|} \|g\|_{L^1(\partial\mathbb{D})}. \end{aligned}$$

Consequently,

$$\left\{e^{ix} \in \partial\mathbb{D} : Mg(e^{ix}) > \lambda\right\} \subset \left\{e^{ix} \in \partial\mathbb{D} : \frac{1}{|e^{ix} - 1|} \|g\|_{L^1(\partial\mathbb{D})} > \lambda\right\}$$

and the result follows. □

By equation (2.3) we have that

$$\sup_{z \in \Gamma(1)} |z - 1| \frac{|g(ze^{ix}) - g(z)|}{|e^{ix} - 1|} \lesssim Mg(e^{ix})$$

which, by the previous lemma implies that the operator defined as

$$\widetilde{M}g(e^{ix}) := \sup_{z \in \Gamma(1)} |z - 1| \frac{|g(ze^{ix}) - g(z)|}{|e^{ix} - 1|}$$

maps $L^1(\partial\mathbb{D})$ to weak-$L^1(\partial\mathbb{D})$.

Lemma 2.1.8. *The sublinear operator $\widetilde{M}$ maps $L^\infty(\partial\mathbb{D})$ to $L^\infty(\partial\mathbb{D})$.*

Proof. Suppose $g \in L^\infty(\partial\mathbb{D})$, then

$$
\begin{aligned}
\widetilde{M}g(e^{ix}) &= \sup_{z\in\Gamma(1)} |z-1| \frac{|g(ze^{ix}) - g(z)|}{|e^{ix}-1|} \\
&\sim \sup_{0\le r<1} \frac{(1-r)}{|e^{ix}-1|} \left| \int_0^{2\pi} \left(\frac{1-r^2}{|e^{i(t-x)}-r|^2} - \frac{1-r^2}{|e^{it}-r|^2} \right) g(e^{it}) \frac{dt}{2\pi} \right| \\
&\le \sup_{0\le r<1} \frac{(1-r)(1-r^2)}{|e^{ix}-1|} \int_0^{2\pi} \left(\frac{|2\,\mathrm{Re}\, re^{-it}(1-e^{ix})|}{|e^{i(t-x)}-r|^2 |e^{it}-r|^2} \right) |g(e^{it})| \frac{dt}{2\pi} \\
&\lesssim \sup_{0\le r<1} \frac{(1-r)(1-r^2)}{|e^{ix}-1|} \int_0^{2\pi} \left(\frac{|2\,\mathrm{Re}\,(re^{-it}-1)(1-e^{ix})|}{|e^{i(t-x)}-r|^2 |e^{it}-r|^2} \right) |g(e^{it})| \frac{dt}{2\pi} \\
&\quad + \sup_{0\le r<1} \frac{(1-r)(1-r^2)}{|e^{ix}-1|} \int_0^{2\pi} \left(\frac{|2\,\mathrm{Re}\,(1-e^{ix})|}{|e^{i(t-x)}-r|^2 |e^{it}-r|^2} \right) |g(e^{it})| \frac{dt}{2\pi} \\
&\lesssim \sup_{0\le r<1} \int_0^{2\pi} \frac{(1-r^2)^2}{|e^{i(t-x)}-r|^2 |e^{it}-r|} \frac{dt}{2\pi} \|g\|_{L^\infty(\partial\mathbb{D})} \\
&\quad + \sup_{0\le r<1} |e^{ix}-1| \int_0^{2\pi} \frac{(1-r^2)^2}{|e^{i(t-x)}-r|^2 |e^{it}-r|^2} \frac{dt}{2\pi} \|g\|_{L^\infty(\partial\mathbb{D})} \\
&\lesssim \sup_{0\le r<1} \int_0^{2\pi} \frac{(1-r^2)}{|e^{i(t-x)}-r|^2} \frac{dt}{2\pi} \|g\|_{L^\infty(\partial\mathbb{D})} \\
&\quad + \sup_{0\le r<1} |e^{ix}-1| \sum_{n=-\infty}^{\infty} r^{|n|} \int_0^{2\pi} e^{int} \frac{1-r^2}{|e^{i(t-x)}-r|^2} \frac{dt}{2\pi} \|g\|_{L^\infty(\partial\mathbb{D})} \\
&= \|g\|_{L^\infty(\partial\mathbb{D})} + |e^{ix}-1| \|g\|_{L^\infty(\partial\mathbb{D})} \sup_{0\le r<1} \sum_{n=-\infty}^{\infty} r^{2|n|} e^{inx} \\
&= \|g\|_{L^\infty(\partial\mathbb{D})} + |e^{ix}-1| \|g\|_{L^\infty(\partial\mathbb{D})} \sup_{0\le r<1} \frac{1-r^2}{|e^{ix}-r^2|^2} \\
&\lesssim \|g\|_{L^\infty(\partial\mathbb{D})}
\end{aligned}
$$

□

Now we can use Marcinkiewicz Interpolation Theorem (see for example [45]) to conclude that the operator $\widetilde{M}$ maps $L^p(\partial\mathbb{D})$ boundedly to itself for any $1 < p \le \infty$. Therefore, if a function $f \in H^1(\mathbb{D})$ is such that

$$D_1^p(f) := \int_0^{2\pi} \left| \frac{f(e^{it}) - f(1)}{e^{it}-1} \right|^p \frac{dt}{2\pi} < \infty$$

then the function $g(e^{it}) := \dfrac{f(e^{it}) - f(1)}{e^{it} - 1}$ belongs to $L^p(\partial\mathbb{D})$ and

$$\begin{aligned} f(ze^{ix}) - f(z) &= (ze^{ix} - 1)g(ze^{ix}) - (z-1)g(z) \\ &= z(e^{ix} - 1)g(ze^{ix}) + (z-1)(g(ze^{ix}) - g(z)) \end{aligned}$$

and consequently,

$$\frac{|f(ze^{ix}) - f(z)|}{|e^{ix} - 1|} \leq |z||g(ze^{ix})| + |z-1|\frac{|g(ze^{ix}) - g(z)|}{|e^{ix} - 1|}.$$

Hence,

$$T(f)(e^{ix}) \leq N(g)(e^{ix}) + \sup_{z\in\Gamma(1)} |z-1|\frac{|g(ze^{ix}) - g(z)|}{|e^{ix} - 1|}. \tag{2.4}$$

where T is defined as the sublinear operator

$$Tf(e^{ix}) := \frac{\sup_{z\in\Gamma(1)} |f(ze^{ix}) - f(z)|}{|e^{ix} - 1|}.$$

Thus,

$$\begin{aligned} (D_1^p(Nf))^{1/p} &\leq \left(\int_0^{2\pi} (Tf(e^{it})^p \frac{dt}{2\pi}\right)^{1/p} \\ &\leq \|Ng\|_{L^p(\partial\mathbb{D})} + \|\widetilde{M}g\|_{L^p(\partial\mathbb{D})} \\ &\lesssim \|g\|_{L^p(\partial\mathbb{D})} \end{aligned}$$

where we have used the fact that the operator N maps $L^p(\partial\mathbb{D})$ boundedly to itself. Therefore for any $1 < p \leq \infty$,

$$D_1^p(Nf) \lesssim D_1^p(f)$$

and notice that if $\zeta \in \partial\mathbb{D}$, for $f \in D(\mu)$ we define $g(z) := f(z\zeta)$, then $D_\zeta(f) = D_1(g)$ and $D_\zeta(Nf) = D_1(Ng)$. Therefore, we have the more general equation:

$$D_\zeta(Nf) \lesssim D_\zeta(f) \tag{2.5}$$

where the constant involved does not depend on ζ and consequently we have the following theorem

Theorem 2.1.9. *Let μ be a finite, positive Borel measure on $\partial\mathbb{D}$, then there exists a constant $C > 0$ such that for every $f \in D(\mu)$*

$$\|Nf\|_{D(\mu)} \le C\|f\|_{D(\mu)}$$

Proof. By equation (2.5) we have that $\int_{\partial\mathbb{D}} D_\zeta(Nf)d\mu(\zeta) \lesssim \int_{\partial\mathbb{D}} D_\zeta(f)d\mu(\zeta)$ and using again the fact that $\|Nf\|_{H^2} \lesssim \|f\|_{H^2}$ we have the result. □

This theorem answers a question asked by Chartrand [11] and generalizes lemma 3.12 of [11] where the result is proven for the case in which the measure μ is absolutely continuous with respect to the Lebesgue measure and satisfies the Muckenhoupt's condition.

2.2 Carleson measures on $D(\mu)$ spaces

In this Section, we will characterize Carleson measures for the $D(\mu)$ spaces. In order to do that, we will rely on results from the previous section. Specifically, notice that from equation (2.1) we can conclude that a positive measure ν on $\mathbb{D}$ is a $D(\mu)$-Carleson measure if and only if there exists a constant $C > 0$ such that for every $h \in \mathcal{B}_\mu$ we have that

$$\int_{\mathbb{D}} |h(z)|^2 d\nu(z) \le C\|h\|^2_{\mathcal{B}_\mu}.$$

Theorem 2.2.1. *Let μ be a finite, positive Borel measure on $\partial\mathbb{D}$. Then a positive Borel measure ν is a $D(\mu)$-Carleson measure if and only if there exists a constant $C > 0$ such that for any open set $O \subset \partial\mathbb{D}$*

$$\nu(T(O)) \le C \operatorname{cap}_{\mathcal{B}_\mu}(O)$$

where $T(O) := \{z \in \mathbb{D} : \{e^{i\theta} : |e^{i\theta} - z/|z|| < 1 - |z|\} \subset O\}$.

Proof. Suppose ν is a $D(\mu)$-Carleson measure. By definition, there exists a function $h \in \mathcal{B}_\mu$ such that $h \ge 1$ on O and $\|h\|^2_{\mathcal{B}_\mu} \le 2 \operatorname{cap}_{\mathcal{B}_\mu}(O)$. Since $\||h|\|_{\mathcal{B}_\mu} \le \|h\|_{\mathcal{B}_\mu}$ we can assume that $h \geq 0$ on $\partial\mathbb{D}$. Let $O = \cup_j I_j$, where $\{I_j\}$ are disjoint arcs on $\partial\mathbb{D}$. Note that $T(O) = \cup_j T(I_j)$. Now, since for any $z \in T(I_j)$ we have that $h(z) \geq \frac{1}{4\pi}$, then

$$\nu(T(O)) \leq (4\pi)^2 \int_{T(O)} |h|^2 d\nu \leq (4\pi)^2 \int_{\mathbb{D}} |h|^2 d\nu \leq C\|h\|^2_{\mathcal{B}_\mu} \leq C \operatorname{cap}_{\mathcal{B}_\mu}(O).$$

Conversely, since $\nu(\{z \in \mathbb{D} : |f(z)| > t\}) \leq \nu(T(\{N(f) > t\}))$, then by the hypothesis and the previous lemmas,

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 d\nu &= \int_0^\infty \nu(\{z \in \mathbb{D} : |f(z)| > t\}) t dt \\ &\lesssim \int_0^\infty \operatorname{cap}_{\mathcal{B}_\mu}(T(\{N(f) > t\})) t dt \\ &\lesssim \|N(f)\|_{\mathcal{B}_\mu} \\ &\lesssim \|f\|_{\mathcal{B}_\mu} \end{aligned}$$

□

In [10], Chartrand defined a Carleson type measure that is different from ours. We will refer to that condition as condition (Ch).

Definition 2.2.2 ([10])**.** A finite, positive Borel measure ν is said to satisfy condition (Ch) for $D(\mu)$ if there exists a constant $C > 0$ such that for every $f \in D(\mu)$

$$\int P(|f|^2\mu) d\nu \leq C\|f\|^2_{D(\mu)}$$

where $P(|f|^2\mu)$ denotes the Poisson extension of the measure $|f|^2 d\mu$ to the unit disc, i.e.

$$P(|f|^2\mu)(z) := \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} |f(\zeta)|^2 d\mu(\zeta)$$

In [11] Chartrand characterizes the measures ν that satisfy condition (Ch) for measures μ that are either a finite sum of atoms or absolutely continuous with respect to Lebesgue measure and satisfying the Muckenhoupt's condition. We will show Chartrand's definition of Carleson measures (condition (Ch)) and the definition presented in these notes are different by exhibiting two examples. In order to do that, we will need a result from [11].

Proposition 2.2.3. *Let $\mu = \sum a_k \delta_{\zeta_k}$, a finite sum of atoms on $\partial\mathbb{D}$. Let ν be a finite, positive, Borel measure on $\mathbb{D}$. Then ν satisfies condition* (Ch) *if and only if $S_\nu(\zeta_k) < \infty$ for each k, where $S_\nu(\zeta) := \int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|} d\nu(z)$.*

Example 2.2.4. Suppose $\mu = \delta_1$, we will show that ν is a $D(\delta_1)$-Carleson measure if, and only if $|z-1|^2 d\nu$ is a Carleson measure for the Hardy space H^2.

Suppose ν is a $D(\delta_1)$-Carleson measure and let $g \in H^2$. Define $f(z) := (z-1)g(z)$, then $f \in D(\delta_1)$ and

$$\begin{aligned} \int |z-1|^2 |g(z)|^2 d\nu(z) &= \int |f|^2 d\nu \\ &\lesssim \|f\|_{H^2}^2 + \|g\|_{H^2}^2 \\ &\lesssim \|g\|_{H^2}^2. \end{aligned}$$

Hence, $|z-1|^2 d\nu(z)$ is a Carleson measure for the Hardy space.

Conversely, suppose $|z-1|^2 d\nu(z)$ is a H^2-Carleson measure and let $f \in D(\delta_1)$, then $g \in H^2$, where $g(z) := \dfrac{f(z) - f(1)}{z-1}$ and

$$\begin{aligned} \int |f|^2 d\nu &= \int |f(1) + (z-1)g(z)|^2 d\nu \\ &\lesssim |f(1)|^2 \nu(\mathbb{D}) + \int |z-1|^2 |g(z)|^2 d\nu(z) \\ &\lesssim \|f\|_{D(\delta_1)}^2 + \|g\|_{H^2}^2 \\ &\lesssim \|f\|_{D(\delta_1)}^2. \end{aligned}$$

Now, for $i \in \mathbb{Z}^+$ consider the sequences $r_i := 1 - \dfrac{1}{i}$ and $a_i := \dfrac{1}{i^2}$. Take $\nu = \sum_{i=1}^\infty a_i \delta_{r_i}$. Note that $\sum a_i < \infty$ and consequently ν is a finite measure. Moreover, if $I \subset \partial\mathbb{D}$ is an interval such that $1 \in I$, then

$$\begin{aligned} \int_{S(I)} |z-1|^2 d\nu(z) &= \sum_{r_i > 1-|I|} a_i |r_i - 1|^2 \\ &\leq |I|^2 \sum_{i > 1/|I|}^{\infty} \frac{1}{i^2} \\ &\lesssim |I|. \end{aligned}$$

Therefore $|z-1|^2 d\nu(z)$ is a H^2-Carleson measure and hence ν is a $D(\delta_1)$-Carleson measure. However, by proposition (2.2.3) ν satisfies condition (Ch) if, and only if $\int \frac{1-|z|^2}{|1-z|^2} d\nu(z) < \infty$. But,

$$\begin{aligned}\int \frac{1-|z|^2}{|1-z|^2} d\nu(z) &= \sum_{i=1}^{\infty} \frac{1-r_i^2}{(1-r_i)^2} a_i \\ &= \sum_{i=1}^{\infty} \frac{1+r_i}{(1-r_i)} a_i \\ &\gtrsim \sum_{i=1}^{\infty} \frac{1}{i} = \infty,\end{aligned}$$

So, ν is a $D(\delta_1)$-Carleson measure but it does not satisfy condition (Ch).

On the other hand, define the sequences $s_i := \dfrac{1}{3^i} - 1$ and $b_i := \dfrac{1}{2^i}$. Then the measure $\sigma = \sum_{i=1}^{\infty} b_i \delta_{s_i}$ is finite.

Now, consider for each nonnegative integer k, the interval $I_k \subset \partial\mathbb{D}$ centered at -1 and with length $|I_k| = \dfrac{1}{3^k}$, then

$$\begin{aligned}\int_{S(I_k)} |1-z|^2 d\sigma(z) &= \sum_{r_i < |I_k| - 1} (1-s_i)^2 b_i \\ &\gtrsim \sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^k}.\end{aligned}$$

Thus, $\dfrac{\int_{S(I_k)} |1-z|^2 d\sigma(z)}{|I_k|} \geq \left(\dfrac{3}{2}\right)^k \to \infty$ when k tends to infinity. Hence σ is not a $D(\delta_1)$-Carleson measure. However, σ satisfies condition (Ch):

$$\int \frac{1-|z|^2}{|1-z|^2} d\sigma(z) \lesssim \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Chapter 3

Interpolating Sequences for $D(\mu)$

3.1 Interpolating Sequences for $D(a\delta_\lambda)$

In this Section we will observe some properties of interpolating sequences for Dirichlet type spaces when the measure μ is a point mass ($\mu = a\delta_\lambda$, $a > 0$, $\lambda \in \partial\mathbb{D}$). First we will recall some definitions.

Definition 3.1.1. Let H be a reproducing kernel Hilbert space formed by analytic functions defined on the unit disc $\mathbb{D}$.

- A sequence $Z = (z_j)$ of distinct points in $\mathbb{D}$ is said to be an interpolating sequence for H if the interpolation problem $f(z_j) = a_j$ has a solution $f \in H$ whenever $(a_j/\|K_{z_j}\|_H) \in l^2$ where K_{z_j} denotes the reproducing kernel of the space H at the point z_j.

- A sequence $Z = (z_j)$ of distinct points in $\mathbb{D}$ is said to be an interpolating sequence for $\mathcal{M}(H)$ (the space of multipliers of H) if the interpolation problem $f(z_j) = a_j$ has a solution $f \in \mathcal{M}(H)$ whenever $(a_j) \in l^\infty$.

- A sequence $Z = (z_j)$ is said to be H-separated if

$$\sup_{j\neq l} \frac{|k_{z_j}(z_l)|^2}{k_{z_j}(z_j)k_{z_l}(z_l)} < 1$$

Proposition 3.1.2 (See [39]). *Let H be a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$, and let $Z = (z_j) \subset \mathbb{D}$ be a sequence of distinct points. Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d)* *(a) Z is an interpolating sequence for $\mathcal{M}(H)$.*

(b) $\|\sum_j b_j K_{z_j}\|_H \lesssim \|\sum_j a_j K_{z_j}\|_H$ whenever $|b_j| \le |a_j|$ for every j.

(c) $\|\sum_j a_j K_{z_j}/\|K_{z_j}\|_H\|_H \sim \|(a_j)\|_{l^2}$.

(d) Z is H-separated and $\sum_j \|K_{z_j}\|^{-2}\delta_{z_j}$ is a Carleson measure for H.

For the cases $H = H^2$ or $H = D$, the Dirichlet space, these conditions are equivalent (see for example [39]). There is a conjecture that the four conditions are equivalent in every space H with complete Nevanlinna-Pick kernel. In these spaces, Marshall and Sunberg showed in [24] that a sequence Z is interpolating for $\mathcal{M}(H)$ if and only if it is interpolating for H. In [43], Shimorin shows that the spaces $D(\mu)$ have a complete Nevanlinna-Pick kernel. Now, we will show that the four conditions are equivalent in the case in which $\mu = a\delta_\lambda$, for $\lambda \in \partial\mathbb{D}$ and $a > 0$. We will use the following result by Serra. Later we will see that the four conditions are also equivalent in the space $D(\mu)$ in the case in which the measure μ in finitely atomic.

Proposition 3.1.3 ([40]). *A sequence $(z_j) \subseteq \mathbb{D}$ is an interpolating sequence for $M(D(\delta_1)$ if and only if (z_j) satisfies the following two conditions:*

(i) (z_j) is uniformly separated, i.e.

$$\prod_{j\neq l}\left|\frac{z_j - z_l}{1-\overline{z_j}z_l}\right| \ge \delta, \qquad l = 1, 2, \ldots$$

for some constant δ independent of l.

(ii) The sequence

$$(P_1(z_j)) := \left(\frac{1-|z_j|^2}{|1-z_j|^2}\right)$$

belongs to l^1.

We will show that the space $D(a\delta_\lambda)$ can be written as a de Branges-Rovnyak space. We will use a reasoning analogous to that in [31] in which the result is proven for the case $a = 1$.

Consider the function

$$b_\lambda(z) := \frac{(1-w_a)\overline{\lambda}z}{1-w_a\overline{\lambda}z}$$

where w_a is the solution of the equation $(x-1)^2 = ax$ that belongs to the unit disc. That is, let

$$w_a := \frac{2+a-((2+a)^2-4)^{1/2}}{2}.$$

We will show that the reproducing kernel of the space $D(a\delta_\lambda)$ can be written as

$$K^\lambda(z,w) = \frac{1-b_\lambda(z)\overline{b_\lambda(w)}}{1-z\overline{w}},$$

i.e. $D(a\delta_\lambda)$ is the de Branges-Rovnyak space $H(b_\lambda)$.

First, we will use some facts about de Branges-Rovnyak spaces that can be found in [31] and in [14]. If a function γ_1 is such that $1-|\gamma_1|^2$ is log-integrable on $\partial\mathbb{D}$ and $\|\gamma_1\|_\infty = 1$ then there is a unique outer function γ_2 such that $\gamma_2(0) > 0$ and $|\gamma_1|^2 + |\gamma_2|^2 = 1$ almost everywhere on $\partial\mathbb{D}$. Then for $f \in H^2$, f belongs to the de Branges-Rovnyak space $H(\gamma_1)$ if and only if there is a unique function $f^+ \in H^2$ such that $T_{\overline{\gamma_1}}f = T_{\overline{\gamma_2}}f^+$, where $T_{\overline{\gamma_j}}$ is the Toeplitz operator with symbol γ_j defined on H^2. In this case,

$$\|f\|^2_{\gamma_1} = \|f\|^2_{H^2} + \|f^+\|^2_{H^2}.$$

In our case, let $\gamma_1 = b_\lambda$ and notice that for $|z| = 1$

$$\begin{aligned}
1-|b_\lambda(z)|^2 &= 1-\left|\frac{(1-w_a)\overline{\lambda}z}{1-w_a\overline{\lambda}z}\right|^2 \\
&= 1-\frac{(1-w_a)^2}{|1-w_a\overline{\lambda}z|^2} = 1-\frac{aw_a}{|1-w_a\overline{\lambda}z|^2} \\
&= \frac{1-2w_a\operatorname{Re}(\overline{\lambda}z)+w_a^2-aw_a}{|1-w_a\overline{\lambda}z|^2} \\
&= \frac{w_a 2\operatorname{Re}(1-\overline{\lambda}z)}{|1-w_a\overline{\lambda}z|^2} \\
&= \frac{a^{-1}(1-w_a)^2|1-\overline{\lambda}z|^2}{|1-w_a\overline{\lambda}z|^2}.
\end{aligned}$$

Hence we can define

$$\gamma_2(z) := \frac{a^{-1/2}(1-w_a)(1-\overline{\lambda}z)}{(1-w_a\overline{\lambda}z)}.$$

Now, for $f,g \in H^2$ $T_{\overline{\gamma_1}}f = T_{\overline{\gamma_2}}g$ if and only if the function $\overline{\gamma_1}f - \overline{\gamma_2}g$ is orthogonal to H^2, but for $|z| = 1$

$$\begin{aligned}\overline{\gamma_1} f - \overline{\gamma_2} g &= \frac{(1-w_a)\lambda\overline{z}}{1-w_a\lambda\overline{z}} f(z) - \frac{a^{-1/2}(1-w_a)(1-\lambda\overline{z})}{1-w_a\lambda\overline{z}} g(z) \\ &= \frac{1-w_a}{1-w_a\lambda\overline{z}}\left[\lambda\overline{z} f(z) - a^{-1/2}(1-\lambda\overline{z}) g(z)\right] \\ &= \frac{1-w_a}{1-w_a\lambda\overline{z}}\lambda\overline{z}[f(z) - (z-\lambda)\overline{\lambda} a^{-1/2} g(z)],\end{aligned}$$

so $T_{\overline{\gamma_1}} f = T_{\overline{\gamma_2}} g$ if and only if there exists a constant c such that

$$f(z) = c + a^{-1/2}\overline{\lambda}(z-\lambda) g(z).$$

Here we have used the fact that the function $\dfrac{1-w_a}{1-w_z\bar{\lambda}z}$ is cyclic in H^2.

Finally, since a function f belongs to $D(a\delta_\lambda)$ if and only if there exists a function $h \in H^2$ such that $f(z) = f(\lambda) + (z-\lambda)h(z)$ and $D_\lambda(f) = \|h\|^2_{H^2}$, then $f \in D(a\delta_\lambda)$ if and only if the function $g(z) := \bar{\lambda} a^{-1/2} h(z)$ belongs to H^2 and $\|g\|^2_{H^2} = a\|h\|^2_{H^2} = aD_\lambda(f)$. Thus, the spaces $H(b_\lambda)$ and $D(a\delta_\lambda)$ coincide and

$$\|f\|^2_{D(a\delta_\lambda)} = \|f\|^2_{H^2} + aD_\lambda(f) = \|f\|^2_{H^2} + \|g\|^2_{H^2} = \|f\|^2_{H(b_\lambda)}.$$

We will show that condition (d) of Proposition 3.1.2 implies conditions (i) and (ii) of Proposition 3.1.3. This will imply that conditions (a) through (d) are equivalent for the space $D(\delta_\lambda)$. Actually, for simplifying the notation we will assume $\lambda = 1$ but the general result can be proven in a similar way, we will also denote $b = b_1$. We will use a few lemmas.

Lemma 3.1.4. *If the sequence (z_j) is $D(\delta_1)$-separated, then (z_j) is uniformly discrete, i.e. there exist a constant $\delta > 0$ such that*

$$\left|\frac{z_j - z_l}{1-\overline{z_j} z_l}\right| \geq \delta \qquad \forall j \neq l.$$

Proof. If (z_j) is $D(\delta_1)$-separated, then there exists a constant $0 < c < 1$ such that

$$\frac{|K_{z_j}(z_l)|^2}{K_{z_j}(z_j) K_{z_l}(z_l)} \leq c$$

and since $K_{z_j}(z_l) = \dfrac{1-\overline{b(z_j)}b(z_l)}{1-\overline{z_j}z_l}$ we have that

$$\frac{(1-|z_j|^2)(1-|z_l|^2)}{|1-\overline{z_j}z_l|^2} \leq c\frac{(1-|b(z_j)|^2)(1-|b(z_l)|^2)}{|1-\overline{b(z_j)}b(z_l)|^2},$$

but since

$$\frac{(1-|b(z_j)|^2)(1-|b(z_l)|^2)}{|1-\overline{b(z_j)}b(z_l)|^2} = 1-\left|\frac{b(z_j)-b(z_l)}{1-\overline{b(z_j)}b(z_l)}\right|^2 \leq 1,$$

we have

$$1-\left|\frac{z_j-z_l}{1-\overline{z_j}z_j}\right|^2 \leq c$$

and consequently (z_j) is uniformly discrete. □

Lemma 3.1.5. *If a sequence $(z_j) \subset \mathbb{D}$ is such that $\dfrac{|1-z_j|^2}{1-|z_j|^2} \to 0$, then the sequence $(\|K_{z_j}\|_{D(a\delta_1)})$ converges to $\|K_1\|_{D(a\delta_1)}$.*

Proof. First note that K_1 is well defined since every function in $D(a\delta_1)$ has a non-tangential limit at 1, so the evaluation functional $f \mapsto f(1)$ is well defined on $D(a\delta_1)$; its kernel is $(z-1)H^2$ which is a closed subspace of $D(a\delta_1)$, hence the functional is bounded (see [32]).

Also, note that if $\dfrac{|1-z_j|^2}{1-|z_j|^2} \to 0$, then $z_j \to 1$ and that $b(z) := b_1(z)$ converges to 1 as z converges to 1 because

$$|1-b(z)| = \left|\frac{1-z}{1-w_a z}\right| \to 0 \qquad \text{as } z \to 1.$$

Consequently for every $w \in \mathbb{D}$

$$K_w(z) = \frac{1-\overline{b(w)}b(z)}{1-\overline{w}z} \to \frac{1-\overline{b(w)}}{1-\overline{w}} \qquad \text{as } z \to 1$$

so

$$K_1(w) = \frac{1-b(w)}{1-w} = \frac{1}{1-w_a w},$$

hence

$$\|K_1\|^2_{D(a\delta_1)} = \frac{1}{1-w_a}.$$

Now,

$$\begin{aligned}\|K_{z_j}\|^2_{D(a\delta_1)} &= \frac{1-|b(z_j)|^2}{1-|z_j|^2} \\ &= \frac{1}{|1-w_a z_j|^2} + \frac{2w_a(|z_j|^2 - \operatorname{Re} z_j)}{(1-|z_j|^2)|1-w_a z_j|^2}\end{aligned}$$

and notice that

$$\left|\frac{|z_j|^2 - \operatorname{Re} z_j}{1-|z_j|^2} + \frac{1}{2}\right| = \frac{|1-z_j|^2}{2(1-|z_j|^2)} \to 0.$$

Hence $\dfrac{|z_j|^2 - \operatorname{Re} z_j}{1-|z_j|^2} \to \dfrac{-1}{2}$, and consequently $\|K_{z_j}\|^2_{D(a\delta_1)} \to \dfrac{1}{1-w_a}$. □

Lemma 3.1.6. *Suppose a sequence $(z_j) \subset \mathbb{D}$ is such that $\sum_j \|K_{z_j}\|^{-2}_{D(a\delta_1)} \delta_{z_j}$ is a $D(a\delta_1)$-Carleson measure, then the sequence $\left(\dfrac{|1-z_j|^2}{1-|z_j|^2}\right)$ is bounded away from 0.*

Proof. If there were a subsequence (z_{j_n}) such that the sequence $\left(\dfrac{|1-z_{j_n}|^2}{1-|z_{j_n}|^2}\right)$ converges to zero, then by the previous lemma, we have that $\|K_{z_{j_n}}\|_{D(a\delta_1)}$ converges to $\|K_1\|_{D(a\delta_1)}$. However, by hypothesis, there exists a constant $C > 0$ such that for every $f \in D(a\delta_1)$

$$\sum_j \|K_{z_j}\|^{-2}_{D(a\delta_1)} |f(z_j)|^2 \leq C\|f\|^2_{D(a\delta_1)}.$$

In particular, taking $f \equiv 1$ we have that $\|K_{z_{j_n}}\|^{-2}_{D(a\delta_1)} \to 0$ which is a contradiction. □

Lemma 3.1.7. *Let $0 < \varepsilon < 1$, and define the set*

$$A_\varepsilon := \left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} \geq \varepsilon \right\},$$

then for every $z \in A_\varepsilon$,

$$\frac{w_a}{|1-w_a z|^2} \leq \frac{1-|b(z)|^2}{|1-z|^2} \leq \frac{\varepsilon w_a + 1 - w_a}{\varepsilon |1-w_a z|^2}. \tag{3.1}$$

Proof.

$$
\begin{aligned}
&\frac{1-|b(z)|^2}{|1-z|^2} = \frac{|1-w_a z|^2 - (1-w_a)^2|z|^2}{|1-w_a z|^2|1-z|^2} \\
= \;& \frac{1-|z|^2 - 2w_a \operatorname{Re} z + 2w_a|z|^2}{|1-w_a z|^2|1-z|^2} \\
= \;& \frac{1-|z|^2}{|1-z|^2}\frac{1}{|1-w_a z|^2} + \frac{w_a}{|1-w_a z|^2}\left(\frac{2|z|^2 - 2\operatorname{Re} z}{|1-z|^2}\right) \\
= \;& \frac{1-|z|^2}{|1-z|^2}\frac{1}{|1-w_a z|^2} + \frac{w_a}{|1-w_a z|^2}\left(\frac{2|z|^2 - 2\operatorname{Re} z + 1 - |z|^2}{|1-z|^2} - \frac{1-|z|^2}{|1-z|^2}\right) \\
= \;& \frac{1-|z|^2}{|1-z|^2}\frac{1}{|1-w_a z|^2} + \frac{w_a}{|1-w_a z|^2}\left(1 - \frac{1-|z|^2}{|1-z|^2}\right) \\
= \;& \frac{1}{|1-w_a z|^2}\left(w_a + \frac{1-|z|^2}{|1-z|^2}(1-w_a)\right)
\end{aligned}
$$

and since $z \in A_\varepsilon$ inequality (3.1) follows. □

Lemma 3.1.8. *Suppose that a given sequence $(z_j) \subset \mathbb{D}$ is such that the measure $\sum \|K_{z_j}\|^{-2}_{D(a\delta_1)}\delta_{z_j}$ is a $D(a\delta_1)$-Carleson measure, then there exists $0 < \varepsilon < 1$ such that $(z_j) \subset A_\varepsilon$.*

Proof. Suppose that for every $0 < \varepsilon < 1$, $(z_j) \not\subset A_\varepsilon$, then there exists a subsequence $(z_{j_n}) \subset (z_j)$ such that for every $n \in \mathbb{N}$,

$$\frac{|1-z_{j_n}|^2}{1-|z_{j_n}|^2} < \frac{1}{n}$$

so $\dfrac{|1-z_{j_n}|^2}{1-|z_{j_n}|^2} \to 0$ and since $\sum \|K_{z_{j_n}}\|^{-2}_{D(a\delta_1)}\delta_{z_{j_n}}$ is also a $D(a\delta_1)$-Carleson measure, this contradicts Lemma 3.1.6. □

Now we are ready to prove the main result of this section.

Theorem 3.1.9. *Suppose that a sequence $(z_j) \subset \mathbb{D}$ is $D(a\delta_1)$-separated and that $\sum \|K_{z_j}\|^{-2}_{D(a\delta_1)}\delta_{z_j}$ is a $D(a\delta_1)$-Carleson measure. Then (z_j) is uniformly separated and the sequence $\left(\dfrac{1-|z_j|^2}{|1-z_j|^2}\right)$ belongs to l^1.*

Proof. We showed in Example 2.2.4 that a measure ν is a $D(\delta_1)$-Carleson measure if and only if $|z-1|^2 d\nu$ is a H^2-Carleson measure. Now, since the norms of $D(\delta_1)$

and $D(a\delta_1)$ are equivalent, then the same holds for the case of a $D(a\delta_1)$-Carleson measure. Hence if $\sum \|K_{z_j}\|_{D(a\delta_1)}^{-2}\delta_{z_j}$ is a $D(a\delta_1)$-Carleson measure, then there exists a constant $C > 0$ such that for every $f \in H^2$

$$\sum_j \frac{|1 - z_j|^2}{\|K_{z_j}\|_{D(a\delta_1)}^2}|f(z_j)|^2 \leq C\|f\|_{H^2}^2. \tag{3.2}$$

Now, note that by Lemma 3.1.8 there exists $0 < \varepsilon < 1$ such that $(z_j) \subset A_\varepsilon$ and consequently by Lemma 3.1.7 we have that

$$\frac{1 - |b(z_j)|^2}{(1 - |z_j|^2)|1 - z_j|^2} \sim \frac{1}{(1 - |z_j|^2)|1 - w_a z_j|^2} \sim \frac{1}{1 - |z_j|^2}.$$

Thus

$$\frac{|1 - z_j|^2}{\|K_{z_j}\|_{D(a\delta_1)}^2} \sim \frac{1}{\|K_{z_j}^{H^2}\|_{H^2}^2}, \tag{3.3}$$

where $K_{z_j}^{H^2}$ denotes the reproducing kernel for the space H^2 at z_j. Then Equation (3.2) can be written as:

$$\sum_j \frac{|f(z_j)|^2}{\|K_{z_j}^{H^2}\|_{H^2}^2} \lesssim \|f\|_{H^2}^2.$$

Thus $\sum_j \|K_{z_j}^{H^2}\|_{H^2}^{-2}\delta_{z_j}$ is a H^2-Carleson measure. But it is known (see [39]) that if a sequence (z_n) satisfies this condition and is uniformly discrete (which is because of Lemma 3.1.4), then it is uniformly separated; this proves the first part of the theorem.

Finally, for the second part note that by Equation (3.3) we have that

$$\frac{1}{\|K_{z_j}\|_{D(a\delta_1)}^2} \sim \frac{1 - |z_j|^2}{|1 - z_j|^2}$$

and consequently

$$\sum \frac{1 - |z_j|^2}{|1 - z_j|^2} \sim \sum \frac{1}{\|K_{z_j}\|_{D(a\delta_1)}^2} \leq C,$$

because $\sum \|K_{z_j}\|_{D(a\delta_1)}^{-2}\delta_{z_j}$ is a $D(a\delta_1)$-Carleson measure. This proves the result. □

3.2 Interpolating sequences for $D(\sum_{k=1}^{n} \mu_k \delta_{\zeta_k})$

In this section we will show that for the case of $\mu = \sum_{k=1}^{n} \mu_k \delta_{\zeta_k}$, $\mu_k > 0$ for every $k = 1, \ldots, n$, conditions (a) through (d) of Proposition 3.1.2 are equivalent. For this, we will rely upon the corresponding result for one point mass (Theorem 3.1.9) and some preliminary results.

First, we will need a general result about complete Nevannlina-Pick reproducing kernels. Recall that a reproducing kernel k on the unit disc is a *complete Nevannlina-Pick kernel (complete NP kernel)* if $k_0(z) = 1$ for all $z \in \mathbb{D}$ and if there exists a sequence of analytic functions $\{b_n\}_{n\geq 1}$ on $\mathbb{D}$ such that

$$1 - \frac{1}{k_\lambda(z)} = \sum_{n\geq 1} b_n(z)\overline{b_n(\lambda)}, \qquad \text{for all } \lambda, z \in \mathbb{D}.$$

This condition is equivalent to the assumption that $1 - 1/k$ is positive definite. We mentioned before that Shimorin in [43] showed that the $D(\mu)$ spaces have a complete NP kernel. The first result we will need is due to McCullough and Trent [25]. We will say that a subspace $\mathcal{M}$ of a Hilbert space H is a *multiplier invariant subspace* if $\varphi\mathcal{M} \subset \mathcal{M}$ for every $\varphi \in M(H)$, the space of multipliers of H.

Theorem 3.2.1 ([25]). *Let k be a complete NP kernel and let $\mathcal{M}$ be a multiplier invariant subspace. Then there exists a sequence of multipliers $\{\varphi_n\} \subset \mathcal{M}$ such that*

$$P_{\mathcal{M}} = \sum_{n\geq 1} M_{\varphi_n} M^*_{\varphi_n} \,(SOT)$$

where $P_{\mathcal{M}}$ denotes the projection onto $\mathcal{M}$ and M_{φ_n} denotes the multiplication operator: $f \mapsto \varphi_n f$.

In particular, notice that if we take the function k_z, $z \in \mathbb{D}$, we have that

$$P_{\mathcal{M}} k_z = \sum_{n\geq 1} M_{\varphi_n} M^*_{\varphi_n} k_z.$$

Since $M^*_{\varphi_n} k_z = \overline{\varphi_n(z)} k_z$, then we have that for every $w \in \mathbb{D}$,

$$P_{\mathcal{M}} k_z(w) = \sum_{n\geq 1} \varphi_n(w)\overline{\varphi_n(z)} k_z(w),$$

or equivalently,

$$\frac{P_{\mathcal{M}} k_z(w)}{k_z(w)} = \sum_{n\geq 1} \varphi_n(w)\overline{\varphi_n(z)}, \tag{3.4}$$

i.e. $\frac{P_{\mathcal{M}} k_z(w)}{k_z(w)}$ is positive definite.

We will also need the following result which is due to Richter and Sundberg [27, 29] and Aleman [4].

Theorem 3.2.2. *Let $\mathcal{M}$ be a multiplier invariant subspace of $D(\mu)$, then $dim\mathcal{M} \ominus z\mathcal{M} = 1$ and if $f \in \mathcal{M} \ominus z\mathcal{M}$, $\|f\|_{D(\mu)} = 1$, then*

(i) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

(ii) $\|fg\|_{D(\mu)} = \|g\|_{D(\mu_f)}$, *for every* $g \in D(\mu_f)$, *where* $d\mu_f = |f|^2 d\mu$.

(iii) *For every* $g \in \mathcal{M}$, *there exists* $h \in D(\mu_f)$ *such that* $g = fh$,

Lemma 3.2.3. *Let $\{\zeta_1, \ldots, \zeta_n\} \subset \partial\mathbb{D}$, and $\mu_1, \ldots, \mu_n > 0$. If $\mu := \sum_{k=1}^n \mu_k \delta_{\zeta_k}$, and if K_z^μ denotes the reproducing kernel of the space $D(\mu)$ at z. Then for every $j = 1, \ldots n$, there exists a positive constant a_j such that the reproducing kernel K_z^j of the space $D(a_j\delta_{\zeta_j})$ satisfies that $\frac{K^j}{K^\mu}$ is positive definite.*

Proof. First, notice that the kernel K^μ is never zero (see [42]) and consequently the quotient is well defined. Let $j \in \{1, \ldots, n\}$ be fixed and define

$$\mathcal{M}_j := \{f \in D(\mu) : f(\zeta_k) = 0 \; \forall k \neq j\},$$

then $\mathcal{M}_j$ is a multiplier invariant subspace of $D(\mu)$. Let $\phi_j \in \mathcal{M}_j \ominus z\mathcal{M}_j$, $\|\phi_j\|_{D(\mu)} = 1$, then by Theorem 3.2.2 we have that the multiplication operator $M_{\phi_j} : D(\mu_{\phi_j}) \to \mathcal{M}_j$ is an onto isometry (and consequently a unitary operator). Here,

$$d\mu_{\phi_j} = |\phi_j|^2 d\mu = \sum_{k=1}^n \mu_k |\phi_j(\zeta_k)|^2 d\delta_{\zeta_j} = \mu_j |\phi_j(\zeta_j)|^2 d\delta_{\zeta_j}.$$

Define $a_j := \mu_j |\phi_j(\zeta_j)|^2$, then the reproducing kernel for the space $\mathcal{M}_j$ is given by

$$K_z^{\mathcal{M}_j}(w) = \overline{\phi_j(z)} \phi_j(w) K_z^j(w).$$

On the other hand, we also know that $K_z^{\mathcal{M}_j} = P_{\mathcal{M}_j} K_z^\mu$, hence

$$\begin{aligned}\frac{K_z^j(w)}{K_z^\mu(w)} &= \frac{1}{\overline{\phi_j(z)}\phi_j(w)} \frac{K_z^{\mathcal{M}_j}(w)}{K_z^\mu(w)} \\ &= \frac{1}{\overline{\phi_j(z)}\phi_j(w)} \frac{P_{\mathcal{M}_j} K_z^\mu(w)}{K_z^\mu(w)}\end{aligned}$$

and since each one of the factors is positive definite, then the result follows. □

From now on, we will use the same notation as in the hypothesis of the previous lemma. A consequence of the lemma is the following: take $z = 0$, then $K_z^j(w) = K_z^\mu(w) = 1$ for every $w \in \mathbb{D}$ and by the positive definiteness of $\frac{K^j}{K^\mu}$ we have that

$$1 \leq \frac{\|K_w^j\|^2_{D(a_j \delta_{\zeta_j})}}{\|K_w^\mu\|^2_{D(\mu)}}. \tag{3.5}$$

Another consequence of Lemma 3.2.3 is the following.

Lemma 3.2.4. *If a sequence $(z_j) \subset \mathbb{D}$ is $D(\mu)$-separated, then it is $D(a_k \delta_{\zeta_k})$-separated for every $k = 1, \ldots, n$.*

Proof. By Lemma 3.2.3 we have that $\frac{K^k}{K^\mu}$ is positive definite, consequently given $z, w \in \mathbb{D}$ we have that

$$\left| \frac{K_z^k(w)}{K_z^\mu(w)} \right|^2 \leq \frac{\|K_z^k\|^2_{D(a_k \delta_{\zeta_k})} \|K_w^k\|^2_{D(a_k \delta_{\zeta_k})}}{\|K_z^\mu\|^2_{D(\mu)} \|K_w^\mu\|^2_{D(\mu)}}$$

and since the assumption implies that for some $C > 0$,

$$\frac{|K_z^\mu(w)|^2}{\|K_z^\mu\|^2_{D(\mu)} \|K_w^\mu\|^2_{D(\mu)}} \leq C < 1$$

the result follows. □

Notice that we could have proved Lemma 3.1.4 in a similar fashion: If we consider $\mathcal{M} := \{f \in D(\delta_1) : f(1) = 0\}$ and use Theorem 3.2.2 to identify $\mathcal{M}$ with the Hardy space H^2, then by use of Lemma 3.2.3 we obtain the result.

Lemma 3.2.5. *Let $(z_j) \subset \mathbb{D}$ be a sequence such that for some $m \in \{1, \ldots n\}$, $\dfrac{|\zeta_m - z_j|^2}{1 - |z_j|^2} \to 0$, then $\|K_{z_j}^\mu\|_{D(\mu)} \to \|K_{\zeta_m}^\mu\|_{D(\mu)}$.*

Proof. By Lemma 3.1.5 we have that $\|K^m_{z_j}\|_{D(a_m\delta_{\zeta_m})} \to \|K^m_{\zeta_m}\|_{D(a_m\delta_{\zeta_m})}$ and by the reproducing property we have that $\|K^m_{z_j} - K^m_{\zeta_m}\|_{D(a_m\delta_{\zeta_m})} \to 0$. Now, consider the inclusion operator $J : D(\mu) \to D(a_m\delta_m)$, then J is bounded and so is J^*. Notice that $J^*K^m_z = K^\mu_z$ and by the continuity of J^* we have that $\|K^\mu_{z_j} - K^\mu_{\zeta_m}\|_{D(\mu)} \to 0$ □

Lemma 3.2.6. *Suppose a sequence $(z_j) \subset \mathbb{D}$ is such that $\sum_j \|K^\mu_{z_j}\|^{-2}_{D(\mu)}\delta_{z_j}$ is a $D(\mu)$-Carleson measure, then the sequence $\left(\frac{|\zeta_m - z_j|^2}{1-|z_j|^2}\right)$, is bounded away from 0 for every $m \in \{1, \dots, n\}$.*

Proof. The proof follows using the previous lemma and a reasoning analogous to that on the proof on Lemma 3.1.6. □

For each $m \in \{1, \dots, n\}$ and $0 < \epsilon < 1$, define the sets

$$A^m_\varepsilon := \left\{ z \in \mathbb{D} : \frac{|\zeta_m - z|^2}{1-|z|^2} \geq \varepsilon \right\}$$

Lemma 3.2.7. *Suppose the sequence $(z_j) \subset \mathbb{D}$ is such that the measure $\sum \|K_{z_j}\|^{-2}_{D(\mu)}\delta_{z_j}$ is a $D(\mu)$-Carleson measure, then there exist $0 < \varepsilon_1, \dots, \varepsilon_n < 1$ such that $(z_j) \subset A^1_{\varepsilon_1} \cap \dots \cap A^n_{\varepsilon_n}$.*

Proof. The proof is similar to that of Lemma 3.1.8. □

Theorem 3.2.8. *Suppose that a sequence $(z_j) \subset \mathbb{D}$ is $D(\mu)$-separated and that $\sum \|K^\mu_{z_j}\|^{-2}_{D(\mu)}\delta_{z_j}$ is a $D(\mu)$-Carleson measure. Then (z_j) is an interpolating sequence for the space of multipliers $M(D(\mu))$.*

Proof. Fix $m \in \{1, \dots, n\}$ and notice that by the previous lemma, there exists $0 < \varepsilon_m < 1$ such that $(z_j) \subset A^m_{\varepsilon_m}$ so we can use identity (3.3) to conclude that

$$\frac{1}{\|K^m_{z_j}\|^2_{D(a_m\delta_{\zeta_m})}} \sim \frac{1-|z_j|^2}{|\zeta_m - z_j|^2}$$

but since by Equation (3.5) for every $z \in \mathbb{D}$ $\|K^\mu_z\|_{D(\mu)} \leq \|K^m_z\|_{D(a_m\delta_{\zeta_m})}$, then we have that there exists a constant $C > 0$ such that

$$\sum_j \frac{1-|z_j|^2}{|\zeta_m - z_j|^2} \lesssim \sum_j \frac{1}{\|K^\mu_{z_j}\|^2_{D(\mu)}} \leq C.$$

Thus by Theorem 3.1.9 (z_j) is an interpolating sequence for $M(a_m D(\delta_{\zeta_m}))$. Now we use another result of Serra ([40]) that says that if (z_n) is interpolating for each $M(D(a_m \delta_{\zeta_m}))$, then it is interpolating for $M(D(\mu))$. $\square$

Chapter 4

Composition Operators with closed range on $D(\delta_1)$

Composition operators for the Dirichlet-type spaces where introduced by Silva in his Doctoral dissertation [36], in which he characterized boundedness and compactness of composition operators on the Dirichlet type spaces $D(\delta_1)$. Four years later, Sarason and Silva [33] characterized boundedness and compactness on general Dirichlet type spaces $D(\mu)$.

The characterization they present is given in terms of a counting function that resembles the Nevannlina counting function (which is used by Shapiro to characterize compact composition operators on the Hardy space). When analyzing the condition for boundedness, Sarason and Silva discovered that the only possible cases for C_φ to be bounded in $D(\delta_1)$ is that $\varphi(1) \in \mathbb{D}$ or $\varphi(1) = 1$. Note that if C_φ is bounded then $\varphi \in D(\delta_1)$, hence $\varphi(1)$ exists as a nontangential limit of φ. They completely characterized the analytic functions mapping 1 to 1 such that C_φ is bounded in terms on its angular derivative at 1, and they also study the case $|\varphi(1)| < 1$ obtaining a partial answer.

In this Chapter we study composition operators on $D(\delta_1)$ with a closed range and characterize them in terms of certain properties of the counting function associated to φ that was used by Sarason and Silva. This counting function is the counterpart of the Nevannlina counting function for functions in the Hardy space. All the results in this chapter can be easily generalized for spaces $D(\delta_\zeta)$ with $\zeta \in \partial\mathbb{D}$.

Definition 4.0.9. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function, we define the composition operator $C_\varphi : D(\delta_1) \to D(\delta_1)$ with symbol φ as

$$C_\varphi(f) := f \circ \varphi$$

Theorem 4.0.10 ([33])**.** *C_φ is bounded on $D(\delta_1)$ if and only if there exists a constant $C > 0$ such that $R_\varphi(w) \leq CP_1(w)$ for every $w \in \mathbb{D}$. Here, R_φ is the counting function:*

$$R_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \frac{1-|z|^2}{|1-z|^2}$$

where the summation on the left hand side takes into account the multiplicities, and P_1 is the Poisson kernel:

$$P_1(w) = \frac{1-|w|^2}{|1-w|^2}.$$

For the case in which $\varphi(1) = 1$, then C_φ is bounded if and only if φ has an angular derivative at 1. If $\limsup_{|z|\to 1} |\varphi(z)| < 1$, then C_φ is bounded

Notice that in the following equation, R_φ comes naturally from the Change of Variables Theorem:

$$\begin{aligned} \|C_\varphi(f)\|^2_{D(\delta_1)} &= \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 P_1(z) dA(z) \\ &= \int_{\varphi(\mathbb{D})} |f'(w)|^2 R_\varphi(w) dA(w) \end{aligned}$$

4.1 Dominant sets for $D(\delta_1)$

In this section we characterize dominant sets for $D(\delta_1)$. The characterization is similar to one given by Luecking in [21] and [22] where he studies the problem for weighted Bergman spaces. Intuitively, it says that a set is dominant if it is sufficiently spread out in the unit disc.

Definition 4.1.1. A Borel set $G \subset \mathbb{D}$ is said to be a dominant set for $D(\delta_1)$ if there exists a constant $C > 0$ such that for every $f \in D(\delta_1)$ the following inequality holds

$$\int_G |f'(z)|^2 P_1(z) dA(z) \geq C \int_{\mathbb{D}} |f'(z)|^2 P_1(z) dA(z) \tag{4.1}$$

Theorem 4.1.2. *Let $G \subset \mathbb{D}$ be a Borel set. Then G is a dominant set for $D(\delta_1)$, if and only if there exists $0 < \eta < 1$ and $K > 0$ such that for every $a \in \mathbb{D}$*

$$|G \cap \Delta(a,\eta)| \geq K|\Delta(a,\eta)| \tag{4.2}$$

where $\Delta(a,\eta)$ denotes the pseudohyperbolic disc with pseudohyperbolic center at a and pseudohyperbolic radious η, i.e. $\Delta(a,\eta) = \{z \in \mathbb{D} : \rho(z,a) < \eta\}$, $\rho(z,a) = \left|\frac{z-a}{1-\bar{z}a}\right|$ and $|\Delta(a,\eta)|$ denotes its Lebesgue area measure.

Proof. Suppose first that Equation (4.1) holds for every $f \in D(\delta_1)$. Let $a \in \mathbb{D}$ and consider the function

$$f_a(z) := \frac{|1-\bar{a}|}{(1-|a|^2)^{1/2}} \frac{1-\bar{a}}{1-a}\varphi_a(z)$$

where $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ is an automorphism of the unit disc. Then since (see [28], Proposition 3.5) φ_a belongs to $D(\delta_1)$ and $D_1(\varphi_a) = |\varphi_a'(1)|$, we have that f_a also belongs to $D(\delta_1)$ and $D_1(f_a) = \dfrac{|1-\bar{a}|^2}{(1-|a|^2)}|\varphi_a'(1)| = 1$ (here, $\varphi_a'(1)$ denotes the angular derivative of φ_a at 1). Hence by hypothesis,

$$\int_G |f_a'(z)|^2 P_1(z)dA(z) \geq CD_1(f_a) = C. \tag{4.3}$$

On the other hand, since $\displaystyle\int_{\mathbb{D}} P_1(z)dA(z) = 1$, then there exists $0 < \eta < 1$ such that

$$\int_{\Delta(0,\eta)} P_1(z)dA(z) > 1 - C/2.$$

Now, let $g_a(z) := \dfrac{1-\bar{a}}{1-a}\varphi_a$, then by a change of variables we obtain

$$\int_{\Delta(a,\eta)} P_1(g_a(z))|g_a'(z)|^2 dA(z) \geq 1 - C/2.$$

Since g_a is an automorphism of $\mathbb{D}$ that fixes 1, it has an angular derivative at 1, and for every $z \in \mathbb{D}$ we have that

$$\frac{1-|z|^2}{|1-z|^2} = |g_a'(1)|\frac{1-|g_a(z)|^2}{|1-g_a(z)|^2}.$$

Consequently,

$$|\varphi_a'(1)|^{-1}\int_{\Delta(a,\eta)} P_1(z)|\varphi_a'(z)|^2 dA(z) \geq 1 - C/2$$

Hence,

$$\int_{\Delta(a,\eta)} |f_a'(z)|^2 P_1(z) dA(z) \geq 1 - C/2. \tag{4.4}$$

Putting Equations (4.3) and (4.4) together we get

$$\begin{aligned}\int_{G\cap\Delta(a,\eta)} |f_a'(z)|^2 P_1(z) dA(z) &\geq \int_G |f_a'(z)|^2 P_1(z) dA(z) - \int_{\mathbb{D}\setminus\Delta(a,\eta)} |f_a'(z)|^2 P_1(z) dA(z)\\ &\geq C - C/2 = C/2.\end{aligned}$$

For $z \in \Delta(a,\eta)$, $\dfrac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \sim (1-|a|)^{-2}$ hence

$$\begin{aligned}\frac{1}{(1-|a|)^2}\int_{G\cap\Delta(a,\eta)} \frac{1-|z|^2}{|1-z|^2} dA(z) &\gtrsim \int_{G\cap\Delta(a,\eta)} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\frac{1-|z|^2}{|1-z|^2} dA(z)\\ &= \frac{1-|a|^2}{|1-\bar{a}|^2}\int_{G\cap\Delta(a,\eta)} |f_a'(z)|^2 P_1(z) dA(z)\\ &\geq \frac{C}{2}\frac{1-|a|^2}{|1-\bar{a}|^2}.\end{aligned}$$

It can be shown (see for example [18]) that for $z \in \Delta(a,\eta)$ we have $P_1(z) \sim P_1(a)$ with the constants depending only on η. Hence from the previous equation we get

$$\begin{aligned}\frac{1}{(1-|a|)^2}P_1(a)\int_{G\cap\Delta(a,\eta)} dA(z) &\gtrsim \frac{1}{(1-|a|)^2}\int_{G\cap\Delta(a,\eta)} \frac{1-|z|^2}{|1-z|^2} dA(z)\\ &\gtrsim CP_1(a).\end{aligned}$$

Thus condition (4.2) holds.

For the converse, suppose Equation (4.2) holds and let $0 < \beta < 1/4$ and $d\mu(z) := \chi_G(z)P_1(z)dA(z)$. We will follow Luecking's ideas from [22]. It can be shown that there exists a constant C_1 such that for $0 < |z| < \beta$

$$\left|\frac{f'(z) - f'(0)}{z}\right|^2 \leq C\int_{\Delta(0,1/2)} |f'(\zeta)|^2 dA(\zeta)$$

for all f analytic on $\mathbb{D}$.

Let

$$\chi_\beta(z,w) := \begin{cases} 1, & \text{if } \rho(z,w) < \beta \\ 0, & \text{otherwise} \end{cases}$$

where $\rho(z,w) := |\varphi_z(w)|$, then

$$\chi_\beta(z,0)|f'(z) - f'(0)|^2 \le C\beta^2 \chi_\beta(z,0) \int_{\Delta(0,1/2)} |f'(\zeta)|^2 dA(\zeta).$$

Let $w = \varphi_a(z)$ and apply this for the function $f \circ \varphi_a$, then changing variables in the integral we get that for another possibly different constant C_1,

$$\chi_\beta(w,a)|f'(w) - f'(a)|^2 \le C_1\beta^2 \chi_\beta(w,a) \int_{\Delta(a,1/2)} |f'(\zeta)|^2 \frac{(1-|a|^2)^2}{|1-\bar{a}\zeta|^4} dA(\zeta).$$

We now use again the fact that for $\zeta \in \Delta(a,1/2)$, $\dfrac{(1-|a|^2)^2}{|1-\bar{a}\zeta|^4} \sim (1-|\zeta|)^{-2}$ and Fubini's Theorem to obtain

$$\begin{aligned} &\int_{\mathbb{D}} \frac{\chi_\beta(w,a)}{|\Delta(a,\beta)|} |f'(w) - f'(a)|^2 dA(w) \\ &\le \frac{\beta^2 C_1}{|\Delta(a,\beta)|} \int \chi_\beta(w,a) \int_{\Delta(a,1/2)} |f'(\zeta)|^2 \frac{(1-|a|^2)^2}{|1-\bar{a}\zeta|^4} dA(\zeta) dA(w) \\ &\le \beta^2 C_2 \int_{\mathbb{D}} |f'(\zeta)|^2 \chi_{\Delta(a,1/2)}(\zeta) \frac{1}{(1-|\zeta|)^2} dA(\zeta). \end{aligned}$$

Consequently

$$\begin{aligned} &\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\chi_\beta(w,a)}{|\Delta(a,\beta)|} |f'(w) - f'(a)|^2 dA(w) d\mu(a) \\ &\le \int_{\mathbb{D}} \beta^2 C_2 \int_{\mathbb{D}} |f'(\zeta)|^2 \chi_{\Delta(a,1/2)}(\zeta) \frac{1}{(1-|\zeta|)^2} dA(\zeta) d\mu(a) \\ &= \beta^2 C_2 \int_{\mathbb{D}} |f'(\zeta)|^2 \frac{\mu(\Delta(\zeta,1/2))}{(1-|\zeta|)^2} dA(\zeta) \\ &\le \beta^2 C_2 \int_{\mathbb{D}} |f'(\zeta)|^2 \frac{1}{(1-|\zeta|)^2} \int_{\Delta(\zeta,1/2)} P_1(s) dA(s) dA(\zeta). \end{aligned}$$

As P_1 is a harmonic function, we have $\int_{\Delta(\zeta,1/2)} P_1(s) dA(s) = R^2 P_1(b)$ where R and b are respectively the euclidean radius and the euclidean center of the disc $\Delta(\zeta,1/2)$.

But again since $\rho(\zeta, b) \le 1$, then there exists a constant $C_3 > 0$ (not depending on ζ) such that $P_1(b) \le C_3 \rho(\zeta, b) P_1(\zeta) \le C_3 P_1(\zeta)$. Moreover, $R^2 \lesssim (1 - |\zeta|)^2$. Therefore, there exists a constant $C_4 > 0$ such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\chi_\beta(w,a)}{|\Delta(a,\beta)|} |f'(w) - f'(a)|^2 dA(w) d\mu(a) \le C_4 \beta^2 \int_{\mathbb{D}} |f'(\zeta)|^2 P_1(\zeta) dA(\zeta). \qquad (4.5)$$

Now, by use of the hypothesis we have that for every $w \in \mathbb{D}$,

$$\int_{\mathbb{D}} \frac{\chi_\beta(a,w)}{|\Delta(a,\beta)|} d\mu(a) = \int_{\Delta(w,\beta)} \frac{d\mu(a)}{|\Delta(a,\beta)|} \gtrsim \frac{\mu(\Delta(w,\beta))}{|\Delta(w,\beta)|} \ge K P_1(w).$$

Thus by Fubini's Theorem

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\chi_\beta(a,w)}{|\Delta(a,\beta)|} |f'(w)|^2 dA(w) d\mu(a) \ge K \int_{\mathbb{D}} |f'(w)|^2 P_1(w) dA(w)$$

and on the other hand,

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\chi_\beta(a,w)}{|\Delta(a,\beta)|} |f'(a)|^2 dA(w) d\mu(a) = \int_{\mathbb{D}} |f'(a)|^2 d\mu(a).$$

This, together with the previous equation and Equation 4.5 gives

$$\begin{aligned} \left(K \int_{\mathbb{D}} |f'(w)|^2 P_1(w) dA(w)\right)^{1/2} &- \left(\int_{\mathbb{D}} |f'(a)|^2 d\mu(a)\right)^{1/2} \\ &\le \left(C_4 \beta^2 \int_{\mathbb{D}} |f'(\zeta)|^2 P_1(\zeta) dA(\zeta)\right)^{1/2}. \end{aligned}$$

If we now choose β small enough so that $K - C_4\beta^2 > 0$ then we see that

$$D_1(f) \lesssim \int_{G_c} |f'(a)|^2 d\mu(a)$$

which finishes the proof of the theorem. □

4.2 Reverse Carleson Inequality for $D(\delta_1)$

In this section we will characterize composition operators with closed range on $D(\delta_1)$. First notice that it is a consequence of the Closed Gaph Theorem (see for example

[13]) that a composition operator acting on a Hilbert space of analytic functions has closed range if and only if it is bounded below. We will show that this condition is equivalent to having a reverse Carleson inequality for a measure related to the symbol of the composition operator. From now on, we will assume φ is not constant.

Theorem 4.2.1. *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ belongs to $D(\delta_1)$, and that the function*

$$\tau_\varphi(w) := \frac{R_\varphi(w)}{P_1(w)}$$

is bounded. Then the following propositions are equivalent:

(i) There exists a constant $K > 0$ such that for every $f \in D(\delta_1)$

$$\int_{\mathbb{D}} |f'(w)|^2 R_\varphi(w) dA(w) \geq K \int_{\mathbb{D}} |f'(w)|^2 P_1(w) dA(w)$$

(ii) There exists constants $c > 0$, $K' > 0$ and $0 < \eta < 1$ such that

$$|G_c \cap \Delta(a, \eta)| \geq K' |\Delta(a, \eta)|$$

where $G_c := \{w \in \mathbb{D} : \tau_\varphi(w) > c\}$

Proof. First suppose that (ii) holds, then by the previous theorem we have that there exist $K > 0$ such that

$$\begin{aligned} \int_{\mathbb{D}} |f'(w)|^2 R_\varphi(w) dA(w) &= \int_{\mathbb{D}} |f'(w)|^2 \tau_\varphi(w) P_1(w) dA(w) \\ &\geq \int_{G_c} |f'(w)|^2 \tau_\varphi(w) P_1(w) dA(w) \\ &\geq c \int_{G_c} |f'(w)|^2 P_1(w) dA(w) \\ &\geq K \int_{\mathbb{D}} |f'(w)|^2 P_1(w) dA(w) \end{aligned}$$

For the converse, suppose that (i) holds and choose $0 < c < K/2$; then

$$\begin{aligned} K\int_{\mathbb{D}}|f'(w)|^2P_1(w)dA(w) &\leq \int_{\mathbb{D}}|f'(w)|^2\tau_\varphi(w)P_1(w)dA(w) \\ \leq \int_{G_c}|f'(w)|^2\tau_\varphi(w)P_1(w)dA(w) &+ \int_{\mathbb{D}\setminus G_c}|f'(w)|^2\tau_\varphi(w)P_1(w)dA(w) \\ \leq \|\tau_\varphi\|_\infty\int_{G_c}|f'(w)|^2P_1(w)dA(w) &+ c\int_{\mathbb{D}}|f'(w)|^2P_1(w)dA(w) \end{aligned}$$

and consequently,

$$\int_{\mathbb{D}}|f'(w)|^2P_1(w)dA(w) \leq \frac{\|\tau_\varphi\|_\infty}{K-c}\int_{G_c}|f'(w)|^2P_1(w)dA(w)$$

and the result follows from the previous theorem. □

Before proving the main Theorem of this section, we will consider an example that will give the flavor of the Theorem. We will show that for a composition operator to have closed range in $D(\delta_1)$ is equivalent to the set G_c to be "big" in $\mathbb{D}$ in a sense that will be made precise soon. We will show that every inner function that induce a bounded composition operator on $D(\delta_1)$, induces a composition operator with closed range. The corresponding results for inner functions for the case of Hardy spaces and weighted Bergman spaces have been studied in [3], and [46].

Example 4.2.2. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an inner function. Then, by a theorem of O. Frostman, (see [17], Theorem 6.4), the function $\varphi_w \circ \varphi$ defines a Blaschke product for every $w \in \mathbb{D}$ except for a set of logarithmic capacity zero S. Thus, let $w \in \mathbb{D} \setminus S$ and notice that $\{z \in \mathbb{D} : \varphi(z) = w\} = \{z \in \mathbb{D} : \varphi_w(\varphi(z)) = 0\}$; consequently

$$R_\varphi(w) = R_{\varphi_w\circ\varphi}(0) = \sum_{j=1}^{\infty}\frac{1-|z_j|^2}{|1-z_j|^2}$$

where the sequence (z_j) are the zeroes of the Blaschke product $\varphi_w \circ \varphi$.

Now, we use the formula for the local Dirichlet integral given in [28] to conclude that

$$R_\varphi(w) = D_1(\varphi_w \circ \varphi)$$

which, together with the fact (also from [28]) that $P_1(w) = |\varphi_w'(1)| = D_1(\varphi_w)$ and that the chain rule holds for angular derivatives, gives that if $\varphi(1) = 1$ (i.e. if C_φ is bounded on $D(\delta_1)$), then

$$R_\varphi(w) = D_1(\varphi_w \circ \varphi) = |(\varphi_w \circ \varphi)'(1)| = |\varphi_w'(1)||\varphi'(1)| = |\varphi'(1)|P_1(w).$$

Therefore the set $\{w \in \mathbb{D} : \tau_\varphi(w) > |\varphi'(1)|/2\} = \mathbb{D} \setminus S$, and we will see that since S has Lebesgue measure zero, this implies that C_φ is bounded below. Thus, an inner function $\varphi : \mathbb{D} \to \mathbb{D}$ induces a composition operator with closed range in $D(\delta_1)$ if and only if its angular derivative exists at 1.

For the main Theorem of this section, we will need the following lemma that gives an equivalent norm for the space $D(\delta_1)$. This will make the proof of the main result easier.

Lemma 4.2.3. *There exist constants $c > 0$ and $C > 0$ such that for every $f \in D(\delta_1)$*

$$c\|f\|^2_{D(\delta_1)} \leq |f(0)|^2 + D_1(f) \leq C\|f\|^2_{D(\delta_1)}$$

Proof. It is shown in [27] that there exists $K > 0$ such that for every $f \in D(\delta_1)$ $\|f\|^2_{H^2} \leq K(|f(0)|^2 + D_1(f))$. Hence, $\|f\|^2_{H^2} + D_1(f) \leq (K+1)(|f(0)|^2 + D_1(f))$. On the other hand, since $|f(0)|^2 \leq \|f\|^2_{H^2}$, then $|f(0)|^2 + D_1(f) \leq \|f\|^2_{H^2} + D_1(f)$ and this finishes the proof. □

Therefore we have that the norm $|f(0)|^2 + D_1(f)$ is equivalent to $\|f\|_{D(\delta_1)}$. We are ready to prove the main result of this section.

Theorem 4.2.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be such that the composition operator $C_\varphi : D(\delta_1) \to D(\delta_1)$ is bounded. Then C_φ has closed range if and only if there are positive constants $c > 0$, $K > 0$ and $0 < \eta < 1$ such that if $G_c := \{w \in \mathbb{D} : \tau_\varphi(w) > c\}$, then*

$$|G_c \cap \Delta(a, \eta)| \geq K|\Delta(a, \eta)| \tag{4.6}$$

Proof. We first prove the theorem under the additional hypothesis that $\varphi(0) = 0$. Let $D^0(\delta_1) := \{f \in D(\delta_1) : f(0) = 0\}$. Then $D^0(\delta_1)$ is invariant under C_φ. Suppose that there are positive constants $c > 0$, $K > 0$ and $0 < \eta < 1$ such that

$$|G_c \cap \Delta(a, \eta)| \geq K|\Delta(a, \eta)|.$$

Then by Theorem (4.2.1) and the Change of Variables Theorem (Theorem 1.3.2), we have that for $f \in D^0(\delta_1)$

$$\begin{aligned} \|C_\varphi f\|^2_{D^0(\delta_1)} &= \int_{\mathbb{D}} |f'(w)|^2 R_\varphi(w) dA(w) \\ &\gtrsim \int_{\mathbb{D}} |f'(w)|^2 P_1(w) dA(w) \\ &= \|f\|_{D^0(\delta_1)}. \end{aligned}$$

Hence C_φ is bounded below on $D^0(\delta_1)$. But if $f \in D(\delta_1)$ then $f = g + f(0)$ for some $g \in D^0(\delta_1)$, $\|f\|^2_{D(\delta_1)} \sim |f(0)|^2 + D_1(g)$ and $C_\varphi(f) = g \circ \varphi + f(0)$ and consequently

$$\|C_\varphi(f)\|^2_{D(\delta_1)} \sim |f(0)|^2 + D_1(g \circ \varphi) \gtrsim |f(0)|^2 + \|g\|^2_{D^0(\delta_1)} \sim \|f\|_{D(\delta_1)}.$$

For the converse, if C_φ is bounded below, then again by the Change of Variables Theorem we have that condition (i) of Theorem 4.2.1 holds and consequently there exist constants $c > 0$, $K > 0$ and $0 < \eta < 1$ such that for every $a \in \mathbb{D}$,

$$|G_c \cap \Delta(a, \eta)| \geq K|\Delta(a, \eta)|$$

which proves the theorem for the case $\varphi(0) = 0$.

For the general case, suppose $\varphi(0) = u$ and let $\psi := g_u \circ \varphi$, where $g_u(z) = \dfrac{1-\bar{u}}{1-u}\varphi_u(z)$. Then notice that since the operator C_{g_u} is invertible then we have that $\{C_\varphi(f) : f \in D(\delta_1)\} = \{C_\psi(f) : f \in D(\delta_1)\}$ and therefore C_φ has closed range if and only if C_ψ has closed range; which happens if and only if there exists $c_1 > 0$, $K_1 > 0$ and $0 < \eta_1 < 1$ such that for every $a \in \mathbb{D}$,

$$|G^\psi_{c_1} \cap \Delta(a, \eta_1)| \geq K_1|\Delta(a, \eta_1)|$$

where $G^\psi_{c_1} := \{w \in \mathbb{D} : \tau_\psi(w) > c_1\}$.

Now, if $w \in G^\psi_{c_1}$, then

$$\sum_{z \in \varphi^{-1}(g_u^{-1}(w))} P_1(z) \geq c_1 P_1(w)$$

and if $s := g_u^{-1}(w)$,

$$\sum_{z\in\varphi^{-1}(s)} P_1(z) \geq c_1 P_1(g_u(s)) = |\varphi_u'(1)|^{-1} c_1 P_1(s).$$

Let $c_2 := c_1|\varphi_u'(1)|^{-1}$, then $g_u^{-1}(w) \in G_{c_2}^{\varphi}$ and consequently $g_u^{-1}(G_{c_1}^{\psi}) \subset G_{c_2}^{\varphi}$, where $g_u^{-1}(z) = \varphi_u(u\gamma)$, and $\gamma = \dfrac{(1-u)}{1-\bar{u}}$. Therefore if $a \in \mathbb{D}$, then let $b \in \mathbb{D}$ such that $g_u^{-1}(\Delta(b,\eta_1)) = \Delta(a,\eta_1)$. Then

$$\begin{aligned}
\int_{G_{c_2}^{\varphi}\cap\Delta(a,\eta_1)} P_1(w)dA(w) &\geq \int_{g_u^{-1}(G_{c_1}^{\psi})\cap g_u^{-1}(\Delta(b,\eta_1))} P_1(w)dA(w) \\
&\geq \int_{\varphi_u(\gamma(G_{c_1}^{\psi}\cap\Delta(b,\eta_1)))} P_1(w)dA(w) \\
&= \int_{G_{c_1}^{\psi}\cap\Delta(b,\eta_1)} P_1(\varphi_u(\gamma w))|\varphi_u'(\gamma w)|^2 dA(w) \\
&= |\varphi_u'(\gamma)|^{-1} \int_{G_{c_1}^{\psi}\cap\Delta(b,\eta_1)} P_1(w)|\varphi_u'(\gamma w)|^2 dA(w) \\
&\gtrsim \int_{G_{c_1}^{\psi}\cap\Delta(b,\eta_1)} P_1(w)dA(w) \\
&\gtrsim |\Delta(b,\eta_1)|P_1(b) \sim |\Delta(a,\eta_1)|P_1(a)
\end{aligned}$$

where the constants involved depend only on the constants for the corresponding result for ψ or on u. Hence,

$$|G_{c_2}^{\varphi} \cap \Delta(a,\eta_1)| \gtrsim |\Delta(a,\eta_1)|$$

and consequently C_φ has closed range in $D(\delta_1)$ and this finishes the proof. □

In [46], Zorboska uses Luecking's ideas about the characterization of dominant sets for the Bergman space A^2 [20, 22] and the Change of Variables Theorem to find a characterization for the composition operators which have closed range on the Hardy space H^2 and on some weighted Bergman spaces A_α^2. Both characterizations are very similar to the result we just proved substituting the right version of counting functions for each space. The following is Zorboska's Theorem.

Theorem 4.2.5 ([46])**.** *A composition operator C_ϕ on H^2 has closed range if, and only if there exist positive constants $c > 0$, $0 < \eta < 1$ and $K > 0$ such that the set*

$F_c := \{z \in \mathbb{D} : \upsilon_\phi(z) > c\}$ *satisfies that for every* $a \in \mathbb{D}$,

$$|F_c \cap \Delta(a, \eta)| \geq K|\Delta(a, \eta)|$$

where

$$\upsilon_\phi(z) = \frac{N_\phi(z)}{\log(1/|z|)}$$

and recall that N_ϕ *denotes the Nevanlinna counting function defined in Equation (1.3).*

Zorboska uses this to show that in every inner function induces a composition operator with closed range in H^2. This does not hold for weighted Bergman spaces and the problem is studied by Akeroyd and Gathage in [3].

For $\alpha > -1$, define a measure on $\mathbb{D}$ as

$$dm_\alpha(z) = \frac{1}{\Gamma(\alpha+1)} \left(\log \frac{1}{|z|}\right)^\alpha dA(z).$$

The weighted Bergman space A^2_α is defined to be the set of all analytic functions on $\mathbb{D}$ such that

$$\|f\|^2_\alpha := \int_{\mathbb{D}} |f(z)|^2 dm_\alpha(z).$$

For A^2_α we define the functions corresponding to N_ϕ, and υ_ϕ as follows:

$$N_{\phi,\alpha}(w) := \sum_{z \in \phi^{-1}(w)} (\log(1/|z|))^\alpha$$

and

$$\upsilon_{\phi,\alpha}(w) := \frac{N_{\phi,\alpha}(w)}{(\log(1/|w|))^\alpha}.$$

Then we have the following characterization of the composition operators having closed range in A^2_α.

Theorem 4.2.6 ([46])**.** *A composition operator* C_ϕ *on* A^2_α *has closed range if, and only if there exist positive constants* $c > 0$, $0 < \eta < 1$ *and* $K > 0$ *such that the set* $E_c := \{z \in \mathbb{D} : \upsilon_{\phi,\alpha}(z) > c\}$ *satisfies that for every* $a \in \mathbb{D}$,

$$|E_c \cap \Delta(a, \eta)| \geq K|\Delta(a, \eta)|$$

In his paper [20], Luecking shows that the geometric condition (4.6) over the set G_c is equivalent to some other geometric conditions taking the intersection with euclidean discs of the form $D_\eta(a) = \{z \in \mathbb{D} : |z - a| < \eta(1 - |a|)\}$ or with euclidean discs centered in the boundary of the unit disc instead of pseudohyperbolic discs. An advantage of using euclidean disc centered in $\partial\mathbb{D}$ is that in this case condition (4.6) does not depend on the radius of the disc.

Nina Zorboska uses this fact to show in [46] some interesting examples of functions inducing bounded composition operators in H^2 or in A^2_α not having closed range. Since some of the examples depend just on the range of the function and the geometric condition over the set to be dominant in H^2 or in A^2_α is the same as (4.6), then the same examples work here. So, if for example the range of a function φ on $\mathbb{D}$ misses a neighborhood of a point in $\partial\mathbb{D}$, then C_φ does not have closed range in $D(\delta_1)$. Similarly, if the range of a function has a hole that includes a disc internally tangent to the unit disc, then the corresponding composition operator does not have closed range in $D(\delta_1)$. This is just because in the first case, it is possible to find an euclidean disc D centered at the given point in $\partial\mathbb{D}$ and inside the given neighborhood so that $|G_c \cap D| = 0$. In the second case, it is better to use an euclidean disc of the form $D_\eta(a)$ with a closed enough to $\partial\mathbb{D}$ so that for any choice of $0 < \eta < 1$, $D_\eta(a)$ is completely included in the hole of the range of the function and hence $|G_c \cap D_\eta(a)| = 0$.

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Vita

G. R. Chacón was born in San Cristóbal, Venezuela on November 3, 1979. He graduated from the University of los Andes in 2002 when he was awarded the degree of Licenciado in Mathematics. After that, in 2004 he was awarded the degree of Magister Scientiae in Mathematics also by the University of los Andes.

In 2002 he won a position as a Professor at the University of los Andes in the Facultad de Humanidades y Educación. This University awarded him a grant to pursue the Doctorate of Philosophy in Mathematics in the University of Tennessee in the year 2005. He attended the University of Tennessee until the year 2009.

G. R. Chacón is currently working as a professor in the University of los Andes in Venezuela and his research interests are in the area of *Hilbert spaces of Analytic Functions* and particulary in the theory of *local Dirichlet spaces*.