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Coefficients of modular forms and applications to partition theory

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To the Graduate Council:

I am submitting herewith a dissertation written by Michael Anthony Hanson entitled "Coefficients of modular forms and applications to partition theory." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Marie Jameson, Major Professor

We have read this dissertation and recommend its acceptance:

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Coefficients of modular forms and applications to partition theory

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To my father, Jon. The warp drive has been solved.

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Abstract

We begin with an overview of the theory of modular forms as well as some relevant sub-topics in order to discuss three results: the first result concerns positivity of self-conjugate t-core partitions under the assumption of the Generalized Riemann Hypothesis; the second result bounds certain types of congruences called "Ramanujan congruences" for an infinite class of eta-quotients - this has an immediate application to a certain restricted partition function whose congruences have been studied in the past; the third result strengthens a previous result that relates weakly holomorphic modular forms to newforms via p-adic limits.

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Chapter 1

Introduction

The theory of modular forms has a wide array of applications to number theory and combinatorics. There are two canonical examples in which modular forms are used to count certain objects: Jacobi's four square theorem, and the classification of Ramanujan's congruences for the integer partition function.

Jacobi's four square theorem provides an explicit formula for the number of ways a positive integer can be written as the sum of four integer squares. The theory of modular forms comes into play when we consider the generating function for sums of four squares:

$$\Theta(z) := \sum_{(w,x,y,z)\in\mathbb{Z}^4} q^{w^2+x^2+y^2+z^2} = \sum_{n\geq 0} r_4(n)q^n,$$

where $q := e^{2\pi iz}$, and where the right-most sum is obtained by gathering all of the like-powers of q in the previous sum. Notice that $r_4(n)$ counts the number of ways in which n can be written as the sum of four squares. We will see in Chapter 4 that $\Theta(z)$ is a modular form, the definition of which is given in Section 3.2 of Chapter 3. In fact, the particular vector space of modular forms that $\Theta(z)$ belongs to is two-dimensional, and one can show that $\Theta(z)$ is a constant multiple of one of the basis elements whose coefficients are divisor sums. In the end, we obtain Jacobi's beautiful identity

$$r_4(n) = 8 \sum_{\substack{0<d\mid n \\ 4\nmid d}} d,$$

which has the immediate consequence that every positive integer can be written as the sum of four squares.

Ramanujan's congruences concern the partition function $p(n)$, which counts the number of partitions of a positive integer n, i.e. the number of ways n can be written as a non-increasing sum of positive integers. For example, there are five partitions of the integer $n = 4$, namely $4 = 1 + 1 + 1 + 1$, $4 = 2 + 1 + 1$, $4 = 2 + 2$, $4 = 3 + 1$, and $4 = 4$, whence $p(4) = 5$. The behavior of $p(n)$ is quite mysterious. There is no known explicit closed formula for $p(n)$, although there are asymptotic formulas. One striking feature of $p(n)$ is its divisibility properties. Srinivasa Ramanujan discovered three congruences for $p(n)$ in the early twentieth century, the first two of which he proved in 1919:

$$\begin{cases} p(5n+4) \equiv 0 \pmod{5}, \\ p(7n+5) \equiv 0 \pmod{7}, \\ p(11n+6) \equiv 0 \pmod{11}. \end{cases} \tag{1.1}$$

After Ramanujan's death in 1920, mathematicians investigated the problem of completely classifying such congruences, i.e. congruences of the form $p(\ell n + a) \equiv 0 \pmod{\ell}$ for all $n \in \mathbb{Z}$, ℓ prime, and some fixed $a \in \mathbb{Z}$. These are aptly named *Ramanujan congruences*. Using the theory of modular forms S. Ahlgren and M. Boylan in 2003 [1] were able to show that the only Ramanujan congruences for $p(n)$ are the three in (1.1).

Both of these results have spawned a great body of research within the theory of modular forms to answer similar questions in combinatorics and number theory. On one hand, Jacobi's four square theorem involves the quadratic form $w^2 + x^2 + y^2 + z^2$. On the other hand, Ramanujan's congruences involve divisibility of the partition numbers $p(n)$. Can we answer similar questions for other types of quadratic forms? Are there other counting functions similar to $p(n)$ that have certain divisibility properties?

In this dissertation we will discuss both types of applications. Our first result relates a certain restricted partition function $sc_6(n)$ which counts *self-conjugate* 6*-core partitions* to the representation numbers of a certain ternary quadratic form. This helps us conditionally settle a conjecture of C.R.H. Hanusa and R. Nath on the positivity of $sc_6(n)$. More specifically, we show that, assuming the Generalized Riemann Hypothesis, $sc_6(n) > 0$ except for when $n \in \{2, 12, 13, 73\}$. This conditionally extends previous work on the positivity of $sc_t(n)$ for other values of t.

Our second result bounds the number of primes that result in Ramanujan congruences for a general class of eta-quotients, which are amalgamations of various scaled versions of Dedekind's eta-function. This result has many potential applications to interesting generating functions for restricted partition functions, such as $sc_t(n)$ discussed above. We specifically apply our theorem to another restricted partition function $\bar{t}(n)$, which counts the number of *overpartitions of* n *with restricted odd differences.*

Our final result concerns a p-adic relationship between cusp forms and weakly holomorphic modular forms for one-dimensional cusp spaces with trivial character. This extends work of S. Ahlgren and D. Samart, which in turn extends work of P. Guerzhoy, Z. Kent, and K. Ono on relating weakly holomorphic modular forms to CM newforms via p-adic limits.

Chapter 2

Some notation

The following notation is not comprehensive of this document. We only mention basic notation here, as well some notation that may not always be standard.

Let $\mathbb{N}$ denote the set $\{1, 2, 3, 4, \ldots\}$ of positive integers. When it is necessary, $\mathbb{N}_0$ will be used to denote the set $\mathbb{N} \cup \{0\}$. As is standard, the notations $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively. Denote by $\mathbb{H}$ the upper-half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Finally, for $p \in \mathbb{N}$ prime let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the p-adic integers and p-adic rational numbers, respectively, and let $\mathbb{F}_p$ denote the finite field with p elements.

Let $n_0 \in \mathbb{Z}$ and let $\{a_n\}$ be a sequence. We will use the notation $\sum_{n \geq n_0} a_n$ to denote the summation $\sum_{n=n_0}^{\infty} a_n$. The notation $\sum_{n \gg -\infty} a_n$ means that there is some $n_0 \in \mathbb{Z}$ such that $\{a_n\} = \{a_{n_0}, a_{n_0+1}, \ldots\}$, i.e. our sum is $\sum_{n \geq n_0} a_n$. The summation $\sum_{n \in \mathbb{Z}} a_n$ is equivalent to $\sum_{n=-\infty}^{\infty} a_n$. Products have analogous notation; for example $\prod_{n \geq n_0} a_n$ denotes the product $\prod_{n=n_0}^{\infty} a_n$.

Given a set S, we let $\#S$ denote its cardinality. If R is a ring, let $R^\times$ denote its group of units. If x is an indeterminate and R is a ring, let $R[\![x]\!]$ denote the ring of formal power series

$$R[\![x]\!] := \left\{ \sum_{n \geq 0} r_n x^n : r_n \in R \right\},$$

and let $R(\!(x)\!)$ denote the ring of formal Laurent series

$$R(\!(x)\!) := \left\{ \sum_{n \gg -\infty} r_n x^n : r_n \in R \right\}.$$

If $\alpha \in \mathbb{R}$ we let $\lfloor \alpha \rfloor$ denote the greatest integer less than α. Given $a_1, a_2, \ldots, a_r \in \mathbb{Z}$, the notation $(a_1, a_2, \ldots, a_r) = d$ is shorthand for $\gcd(a_1, a_2, \ldots, a_r) = d$.

Chapter 3

Modular forms

This chapter provides brief overviews of key concepts within the theory of modular forms that will help set up later chapters. We will not provide proofs of any propositions or theorems; however, more detailed information on these topics can be found in, for example, [21, 47].

3.1 Congruence subgroups

The group $\mathrm{SL}_2(\mathbb{Z})$ of determinant-one 2×2 integer-valued matrices acts on the upper-half plane $\mathbb{H} \subset \mathbb{C}$ by linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}, \quad z \in \mathbb{H}, \quad a, b, c, d \in \mathbb{Z}.$$

Any subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ inherits this action on $\mathbb{H}$. We are interested in special subgroups of $\mathrm{SL}_2(\mathbb{Z})$ called congruence subgroups, whose definition will be given shortly.

For $N \in \mathbb{N}$, define the homomorphism $\pi_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ that reduces the entries of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ modulo N. Then the kernel of π_N is precisely the *principal congruence subgroup*

$$\Gamma(N) := \ker \pi_N = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I \pmod{N}\},$$

where the congruence above is taken entry-wise. We can now see that $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and has finite index.

Definition 1. *A* ***congruence subgroup*** *is a subgroup* Γ *of* $SL_2(\mathbb{Z})$ *that contains* $\Gamma(N)$ *for some* $N \in \mathbb{N}$*. The* ***level*** *of* Γ *is the smallest such* N*.*

Since the principal congruence subgroups have finite index in $\mathrm{SL}_2(\mathbb{Z})$, so does any congruence subgroup.

We are particularly interested in the congruence subgroups

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : c \equiv 0 \pmod{N} \},$$
$$\Gamma_1(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \},$$

where we note that $\Gamma_1(N) \subset \Gamma_0(N)$. We observe that $\Gamma_0(N)$ is the preimage of π_N of the subgroup of upper-triangular matrices, and $\Gamma_1(N)$ is the preimage of π_N of the subgroup of unipotent matrices.

3.2 Modular forms

Let Γ be a congruence subgroup of level N. Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and a function $f : \mathbb{H} \to \mathbb{C}$, we define the **Petersson slash operator** of weight $k \in \mathbb{N}$ by

$$f \Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad z \in \mathbb{H}. \tag{3.1}$$

We will often not include z in this notation, or even the subscript k when the context is clear.

This action is transitive: given $\alpha, \beta \in \Gamma$ we have $f |_k \alpha |_k \beta = f |_k \alpha\beta$.

Definition 2. *Let* Γ *be a congruence subgroup of level* N*,* χ *a Dirichlet character modulo* N*, and* $k \in \mathbb{N}$*. A* ***modular form of weight*** k ***and Nebentypus*** χ ***for*** Γ *is a holomorphic function* $f : \mathbb{H} \to \mathbb{C}$ *satisfying the following properties.*

1. *(Automorphy) For any* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ *we have* $f |_k \gamma = \chi(d) f$*, i.e.*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z).$$

2. (Growth condition) For any $\alpha \in SL_2(\mathbb{Z})$, the function $f \mid_k \alpha$ is bounded for $Im(z) \to \infty$.

*If we strengthen the growth condition (2) to require $f \mid_k \alpha \to 0$ as $Im(z) \to \infty$, we call f a **cusp form of weight** k **for** Γ.*

Denote the $\mathbb{C}$-vector space of modular forms of weight k and Nebentypus χ for Γ by $M_k(\Gamma, \chi)$ and the corresponding subspace of cusp forms by $S_k(\Gamma, \chi)$. When χ is trivial we denote these spaces by $M_k(\Gamma)$ and $S_k(\Gamma)$. Also, if $f \in M_k(\Gamma, \chi)$ and it is clear from context, we may simply say that f is a weight k modular form for Γ without mention of the character χ.

Remark 1. *The term "Nebentypus" is standard in this definition. However, we may sometimes use the phrasing "modular form of weight k and character χ for Γ" instead.*

Remark 2. *If Γ is a congruence subgroup acting on $\mathbb{H}$, then we adjoin the projective rational line $\mathbb{Q} \cup \{\infty\}$ to $\mathbb{H}$ and then identify adjoined points of $\mathbb{Q} \cup \{\infty\}$ under Γ-equivalence. A Γ-equivalence class of points in $\mathbb{Q} \cup \{\infty\}$ is called a **cusp** of Γ.*

*It is known that each congruence subgroup has a finite number of cusps. If we represent each cusp of Γ by some element of $\mathbb{Q} \cup \{\infty\}$, then condition (2) above guarantees that modular forms are holomorphic at these limit points. In this case, we say that modular forms are **holomorphic at the cusps of** Γ.*

Observe that if $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ and f is a weight-k modular form for Γ, then f is periodic with period 1:

$$f(z+1) = f(z), \quad z \in \mathbb{H}$$

by the property (1) above. Thus f has a Fourier expansion of the form

$$f(z) = \sum_{n \geq n_0} a_f(n) q^n, \quad q := e^{2\pi i z}, \quad n_0 \geq 0.$$

When f is a cusp form, $n_0 > 0$. Fourier coefficients of modular forms often encode interesting arithmetic, geometric, or combinatorial data. For example, the **Eisenstein series of weight** k

$$E_k(z) := \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^k},$$

where $k \geq 4$ is even, is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ and has Fourier expansion of the form

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of $(k-1)$-powers of divisors of n, and where B_k is the k-th Bernoulli number.

Remark 3. *From this point on, the variable q expressed in Fourier expansions will always be the exponential $q = e^{2\pi i z}$.*

The space $M(\Gamma, \chi)$ of modular forms of any weight has a graded ring structure given by

$$M(\Gamma, \chi) = \bigoplus_k M_k(\Gamma, \chi).$$

An analogous decomposition holds for the space of all cusp forms $S(\Gamma, \chi)$. It is known that any of the components $M_k(\Gamma, \chi)$ is finite-dimensional, which follows from the *valence formula*. For example when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and χ is trivial, the valence formula for a modular form $f \in M_k(\Gamma)$ is given by

$$\mathrm{ord}_\infty(f) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \sum_{\substack{z \in \mathbb{F} \\ z \neq i, \rho}} \mathrm{ord}_z(f) = \frac{k}{12},$$

where $i^2 = -1$ is the imaginary unit, $\rho = e^{\pi i/3}$, and $\mathbb{F}$ is the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ (see [21, Chapter 2] for details). This formula can be used to show, for instance, that a nonzero form $f \in M_4(\mathrm{SL}_2(\mathbb{Z}))$ has order $\mathrm{ord}_\rho(f) = 1 = \mathrm{ord}_\rho(E_4)$ at ρ (where E_4 is the weight-4 Eisenstein series), whence $f/E_4 \in M_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}$, and so $M_4(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C} \cdot E_4$ is one-dimensional.

Consider the congruence subgroups $\Gamma_1(N)$ and $\Gamma_0(N)$ discussed above. The inclusion $\Gamma_1(N) \subset \Gamma_0(N)$ implies the inclusion $M_k(\Gamma_0(N)) \subset M_k(\Gamma_1(N))$ of spaces of modular forms. We can decompose the space $M_k(\Gamma_1(N))$ into smaller subspaces of modular forms on $\Gamma_0(N)$:

$$M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_0(N), \chi),$$

where the sum runs over all Dirichlet characters modulo N. An analogous grading holds for $S_k(\Gamma_1(N))$.

We may loosen condition (2) in Definition 2 by only requiring meromorphicity at the cusps of Γ. Such functions $f : \mathbb{H} \to \mathbb{C}$ still satisfy the automorphy condition (1).

Definition 3. *Functions $f : \mathbb{H} \to \mathbb{C}$ satisfying (1) in Definition 2 for some $k \in \mathbb{N}$ and character χ modulo N and that are meromorphic at the cusps of Γ are called* ***weakly holomorphic modular forms****. The space of weakly holomorphic modular forms of weight k and Nebentypus χ for Γ is denoted $M_k^!(\Gamma, \chi)$. Again, we drop the character χ from this notation when χ is trivial.*

Remark 4. *The spaces $M_k^!(\Gamma, \chi)$ are typically not finite-dimensional, although they always contain the finite-dimensional modular forms spaces $M_k(\Gamma, \chi)$.*

Let Γ be either $\Gamma_0(N)$ or $\Gamma_1(N)$. Then weakly holomorphic modular forms for Γ have Fourier expansions of the form

$$\sum_{n \gg -\infty} a(n) q^n,$$

i.e. they are Laurent series in $q = e^{2\pi i z}$, $z \in \mathbb{H}$. A classical example of a weakly holomorphic modular form is the j-invariant for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$j(z) := 1728 \frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

which parameterizes elliptic curves over $\mathbb{C}$.

3.3 Modular forms of half-integral weight

Modular forms of half-integral weight $k \in \frac{1}{2}\mathbb{Z}$ are extremely important in the theory. Often times generating functions for combinatorial objects (such as the partition function) happen to be half-integral weight modular forms. A canonical example is Dedekind's eta-function

$$\eta(z) := q^{1/24} \prod_{n \geq 1} (1 - q^n), \quad q = e^{2\pi i z},$$

a modular form of weight 1/2. It turns out that the reciprocal of $\eta(z)$ is essentially the generating function for the partition numbers $p(n)$. A *partition* of a nonnegative integer n is a non-increasing sequence of positive integers (called parts) that sum to n. For example, the integer 4 has five partitions, namely $(1,1,1,1),(2,1,1),(2,2),(3,1),(4)$, and so $p(4)=5$.

The function $p(n)$ is rather mysterious and grows quite quickly as n increases. Its generating function is intimately tied to $\eta(z)$ in the following identity:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{q^{1/24}}{\eta(z)} = \prod_{n\geq 1}(1-q^n)^{-1}.$$

Thus, properties of $p(n)$ can be extracted from the half-integral weight modular form $\eta(z)$. We now build the definition of half-weight modular forms.

Define $\theta : \mathbb{H} \to \mathbb{C}$ by $\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2}$ (this is an example of what is known as a *theta function*). It is a classical fact that $\theta^2 \in M_1(\Gamma_1(4), \chi_4)$, where χ_4 is the primitive Dirichlet character modulo 4. We thus have that $\theta^2(\gamma z) = \chi_4(d)(cz+d)\theta^2(z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4)$. Define

$$j(\gamma, z) := \frac{\theta(\gamma z)}{\theta(z)}.$$

Let $GL_2^+(\mathbb{R})$ be the invertible 2×2 real matrices with positive determinant. Define the set

$$\mathscr{G} := \{(\gamma,\phi) : \gamma \in GL_2^+(\mathbb{R}), \phi : \mathbb{H} \to \mathbb{C} \text{ holomorphic}, \phi^2(z) = \zeta \det(\gamma)^{-1/2}(cz+d)\},$$

where ζ lies on the unit circle in $\mathbb{C}$ and depends on γ and ϕ (but not on $z \in \mathbb{H}$). The natural map $\pi : \mathscr{G} \to GL_2^+(\mathbb{R})$ is surjective since the map $z \mapsto cz+d$ has a holomorphic square root. We make $\mathscr{G}$ into a group by defining

$$(\gamma_1,\phi_1)(\gamma_2,\phi_2) := (\gamma_1\gamma_2, \phi_1(\gamma_2 z)\phi_2(z)).$$

Now define for $k \in \mathbb{N}$ odd and $f : \mathbb{H} \to \mathbb{C}$ the weight-$k/2$ slash operator

$$f\mid_{k/2}(\gamma,\phi) := \phi(z)^{-k} f(\gamma z). \tag{3.2}$$

Let $\mathscr{G}_0(4) := \pi^{-1}(\Gamma_0(4))$. The induced map $\mathscr{G}_0(4) \to \Gamma_0(4)$ has a splitting given by $\gamma \mapsto \gamma^* := (\gamma, j(\gamma, z))$. Thus if Γ is a congruence subgroup of level $N \in 4\mathbb{N}$ then we define $\gamma^* := (\gamma, j(\gamma, z))$ to be the image of $\gamma \in \Gamma$ in $\mathscr{G}_0(4)$. Thus we may discuss holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying $f \mid_{k/2} \gamma^* = f$ for $\gamma \in \Gamma$.

Moreover, let $s \in \mathbb{Q} \cup \{\infty\}$ be a representative of a cusp of Γ, i.e. choose $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ with $\alpha \cdot \infty = s$. We can lift α to $\tilde{\alpha} \in \mathscr{G}$. Then $f \mid_{k/2} \tilde{\alpha}$ will be invariant under $z \mapsto z + h$ for some $h > 0$, whence $f \mid_{k/2} \tilde{\alpha}$ has a Fourier expansion. We have the following definition.

Definition 4. *Let Γ be a congruence subgroup of level $N \in 4\mathbb{N}$ and let $k \in \mathbb{N}$ be odd. A* ***modular form of weight*** $k/2$ ***for*** Γ *is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following.*

1. *For $\gamma \in \Gamma$ we have $f \mid_{k/2} \gamma^* = f$, i.e. $f(\gamma z) = j(\gamma, z)^{k/2} f(z)$.*

2. *For $\alpha \in SL_2(\mathbb{Z})$ and $\tilde{\alpha}$ as above, $f \mid_{k/2} \tilde{\alpha}$ has a Fourier expansion with no terms of negative degree.*

We denote the space of modular forms (respectively cusp forms) of weight $k/2$ for Γ by $M_{k/2}(\Gamma)$ (respectively $S_{k/2}(\Gamma)$). If χ is an even Dirichlet character modulo N then we let $M_{k/2}(\Gamma_0(N), \chi)$ (respectively $S_{k/2}(\Gamma_0(N), \chi)$) be the space of $f \in M_{k/2}(\Gamma_0(N))$ with $f \mid_{k/2} \gamma^* = \chi(d) f$ (as usual d is the lower right entry of γ). We obtain a similar grading as before:

$$M_{k/2}(\Gamma_1(N)) = \bigoplus_{\chi} M_{k/2}(\Gamma_0(N), \chi),$$

with an analogous grading for $S_{k/2}(\Gamma_1(N))$.

We saw that Dedekind's eta-function $\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is an example of a modular form of half-integer weight. As another example, let ψ be an even, primitive Dirichlet character with conductor m. Then the theta function

$$\theta_\psi := \sum_{n \in \mathbb{Z}} \psi(n) q^{n^2} \in M_{1/2}(\Gamma_1(4m^2), \psi)$$

is a half-integral weight modular form.

3.4 Eta-quotients

Recall Dedekind's eta-function

$$\eta(z) := q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

Then $\eta \in M_{1/2}(\mathrm{SL}_2(\mathbb{Z}), \epsilon)$ for some character ϵ.

Remark 5. *The character ϵ has an explicit formula which we do not give here. However, the scaling $\eta(24z)$ belongs to the space $M_{1/2}(\Gamma_0(576), (\frac{12}{\cdot}))$.*

We have seen that the partition function $p(n)$ has generating function essentially equal to the reciprocal of $\eta(z)$. In fact, many partition-type functions are built out of scaled versions of $\eta(z)$. For example, consider the number $p_r(n)$ of r*-regular* partitions of n for $r \in \mathbb{N}$, i.e. partitions of n such that no part is divisible by r. Then the generating function for $p_r(n)$ is

$$\sum_{n \geq 0} p_r(n) q^n = \prod_{n \geq 1} \frac{(1 - q^{rn})}{1 - q^n} = q^{(1-r)/24} \frac{\eta(rz)}{\eta(z)}.$$

This is an example of what is known as an eta-quotient, whose definition we give below.

Definition 5. *Let $N \in \mathbb{N}$. An* ***eta-quotient*** *is any function of the form*

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}, \tag{3.3}$$

where $r_\delta \in \mathbb{Z}$.

Remark 6. *It may be the case that all of the r_δ are non-negative, in which case $f(z)$ above is not a quotient. In this case, $f(z)$ is typically called an* ***eta-product****. Nonetheless, we will continue to refer to functions of the form (3.3) as eta-quotients.*

We have the following theorems, all of which can be found in [47].

Theorem 1. *Let $f(z)$ be an eta-quotient as in Definition 5, and let $k := \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z}$. Assume that*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then $f \mid_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi(d) f(z)$ *for every* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$*. Here, the character* χ *is defined by* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$ *where* $s := \prod_{\delta|N} \delta^{r_\delta}$*.*

3.5 Hecke operators

Hecke operators are operators that preserve spaces of modular forms of a given weight. They are indexed by integers $n \in \mathbb{N}$, and when n is prime to the level N of the space of modular forms, it turns out that the Hecke operators are self-adjoint with respect to the Petersson inner product (given in Section 3.6 below). The spectral theorem then implies that there is a basis of modular forms that are eigenfunctions for these Hecke operators. This allows one to compute bases for specific spaces of modular forms. This material can be found in [21].

For simplicity we only consider the congruence subgroup $\Gamma_1(N)$.

Definition 6. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$*. The* ***$d$-th Diamond operator*** $\langle d \rangle : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$ *is given by*

$$\langle d \rangle f := f \mid_k \gamma.$$

Note that $\langle d \rangle f$ *is completely determined by* $d \pmod{N}$*.*

For $p \in \mathbb{N}$ *prime let* $\alpha_p := \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$*. The* ***$p$-th Hecke operator*** $T_k(p) : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$ *of weight* k *is given by*

$$f \mid T_k(p) := \sum_j f \mid_k \beta_j,$$

where $\{\beta_j\}$ *are orbit representatives, i.e.* $\Gamma_1(N)\alpha_p\Gamma_1(N) = \bigcup_j \Gamma_1(N)\beta_j$ *is a disjoint union. We sometimes write* $T(p)$ *instead of* $T_k(p)$ *when the context is clear.*

We set $T_k(1)$ *to be the identity operator. For* $r \geq 2$ *define*

$$T_k(p^r) := T_k(p)T_k(p^{r-1}) - p^{k-1}\langle p \rangle T_k(p^{r-2}).$$

Proposition 1. *Let $p, q \in \mathbb{N}$ be prime and $r, s \in \mathbb{N}$. Then*

1. $T_k(p^r)T_k(q^s) = T_k(q^s)T_k(p^r)$.

This allows us to define $T_k(n)$ for general n.

Definition 7. *For $n \in \mathbb{N}$ with prime factorization $n = \prod_i p_i^{e_i}$, define the **n-th Hecke operator** by*

$$T_k(n) := \prod_i T_k(p_i^{e_i}).$$

Corollary 1. *If $(n, m) = 1$ then $T_k(n)T_k(m) = T_k(m)T_k(n)$.*

Proposition 2. *Let $f \in M_k(\Gamma_1(N))$ have Fourier expansion $f(z) = \sum_{n\geq 0} a_n(f)q^n$. We have the following.*

1. *Let χ_0 be the trivial character modulo N. Then $f \mid T_k(p)$ has Fourier expansion*

$$(f \mid T_k(p))(z) = \sum_{n\geq 0} \left(a_{np}(f) + \chi_0(p)p^{k-1}a_{n/p}(\langle p\rangle f)\right) q^n,$$

where $a_{n/p}(\langle p\rangle f) = 0$ when $n/p \notin \mathbb{N}$.

2. *Let χ be a character modulo N. Then $T_k(p) : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \chi)$ and $f \mid T_k(p)$ has Fourier expansion*

$$(f \mid T_k(p))(z) = \sum_{n\geq 0} \left(a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f)\right) q^n.$$

Definition 8. *Let Γ be a congruence subgroup. Define a topology on the extended upper-half plane $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ whose basis consists of the usual topology on $\mathbb{H}$ along with the open balls $B_t := \{z \in \mathbb{H} : Im(z) > t\} \cup \{\infty\}$ for $t > 0$, as well as the images of B_t under the action of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $(a, c) = 1$. The quotient space $X(\Gamma) := \Gamma \backslash \mathbb{H}^*$, endowed with a complex structure, is a compact Riemann surface called the **modular curve** corresponding to Γ. We refer the reader to [21, Chapter 1] for more details on modular curves*

The following definition gives us an inner product on $S_k(\Gamma_1(N))$.

Definition 9. *Let Γ be a congruence subgroup, and let $X(\Gamma)$ be the corresponding modular curve. Let $f, g \in S_k(\Gamma, \chi)$. The* ***Petersson inner product*** *of f, g is given by*

$$\langle f, g\rangle_\Gamma := \frac{1}{V_\Gamma}\int_{X(\Gamma)} f(z)\overline{g(z)}(Im(z))^k d\mu(z),$$

where $V_\Gamma := \int_{X(\Gamma)} d\mu(z)$ is the volume of the modular curve $X(\Gamma)$ and $d\mu(z) = dxdy/y^2$ (for $z = x + iy \in \mathbb{H}$).

Proposition 3. *The Hecke operators $T_k(n)$ for n prime to N are self-adjoint with respect to the Petersson inner product. In particular, there is an orthogonal basis of eigenfunctions for all $T_k(n)$ in $S_k(\Gamma_1(N))$.*

Definition 10. *An eigenfunction as in Proposition 3 is called a* ***Hecke eigenform****, or just an* ***eigenform****. Let f be a cusp form with Fourier expansion $f(z) = \sum_{n\geq 1} a_n(f)q^n$. We say that f is* ***normalized*** *when $a_1(f) = 1$.*

Proposition 4. *Let $f \in S_k(\Gamma_1(N))$ be a normalized eigenform for all $T_k(n)$ with $(n, N) = 1$, where f has Fourier expansion $f(z) = \sum_{n\geq 1} a_n(f)q^n$. Then $a_1(f \mid T_k(n)) = a_n(f)$. In particular, $(f \mid T_k(n))(z) = a_n(f)f(z)$ when $(n, N) = 1$.*

There is an analogous notion of Hecke operators on spaces of half-integral weight modular forms, denoted $T_k(p^2)$ when $k \in \mathbb{Z} + \frac{1}{2}$. There is an analogue of Proposition 2 for these operators, and also a notion of newform (given in 3.6 below) of half-integral weight. We do not discuss the details here, but they can be found, for example, in [47, Chapter 3].

3.6 Newforms and complex multiplication

Here we discuss special types of cusp forms called newforms. These objects are most relevant in Chapter 6, where newforms are p-adically related to weakly holomorphic modular forms.

Definition 11. *Let $N \in \mathbb{N}$, and for each $d \mid N$ define*

$$i_d : (S_k(\Gamma_1(N/d)))^2 \to S_k(\Gamma_1(N)),$$

by

$$(f,g) \mapsto f + d^{k-1}g(dz).$$

The subspace of ***oldforms*** *for* $\Gamma_1(N)$ *is*

$$S_k(\Gamma_1(N))^{old} := \sum_{d|N} i_d\left((S_k(\Gamma_1(N/d)))^2\right).$$

The subspace of ***newforms*** $S_k(\Gamma_1(N))^{new}$ *for* $\Gamma_1(N)$ *is the orthogonal complement of* $S_k(\Gamma_1(N))^{old}$ *with respect to the Petersson inner product.*

Proposition 5. *The spaces of oldforms and newforms are each stable under the Hecke operators. Thus they each have orthogonal bases of eigenforms for the Hecke operators* $T_k(n)$ *where* $(n,N)=1$*.*

Definition 12. *A* ***newform*** *is a normalized eigenform in* $S_k(\Gamma_1(N))^{new}$*.*

Proposition 6. *The set of newforms in* $S_k(\Gamma_1(N))^{new}$ *is an orthogonal basis of the space. Each newform lies in some* $S_k(\Gamma_0(N),\chi)$ *for character* χ *modulo* N*.*

The last topic in this section is the notion of a modular form with complex multiplication. Essentially this means that there is a quadratic imaginary field extension K such that the p-th Fourier coefficients are zero whenever p is an inert prime in K. We will build the precise definition of a modular form with complex multiplication below. For a more detailed overview on this topic, see, for example, [46, Section 1.2.2] or [11, Section 5].

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field extension with $\mathcal{O}_K$ its ring of integers. Let $\Lambda \subset \mathcal{O}_K$ be a nontrivial ideal, and let $I(\lambda)$ be its group of fractional ideals that are prime to Λ. A *Hecke Grössencharacter* with modulus Λ is a homomorphism $\phi : I(\Lambda) \to \mathbb{C}^\times$ such that if $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\Lambda}$ then $\phi(\alpha\mathcal{O}_K) = \alpha^{k-1}$.

Let ω_ϕ be the Dirichlet character $\omega_\phi(n) := \phi((n))/n^{k-1}$ for $(n,\Lambda)=1$. Define the q-expansion

$$\Psi(z) := \sum_{\substack{\mathfrak{a} \text{ integral ideal} \\ (\mathfrak{a},\Lambda)=1}} \phi(\mathfrak{a})q^{N(\mathfrak{a})} =: \sum_{n\geq 1} a(n)q^n, \tag{3.4}$$

where $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$.

Theorem 2. *We have that* $\Psi(z) \in S_k\left(\Gamma_0(D \cdot N(\Lambda)), \left(\frac{-D}{\cdot}\right)\omega_\phi\right)$. *Further,* Ψ *is a newform as in Definition 12.*

Definition 13. *Such a cusp form* $\Psi(z)$ *as in (3.4) is called a* ***modular form with complex multiplication****, or more simply a* ***CM newform****.*

For example, the modular form

$$\eta^2(4z)\eta^2(8z) = q - 2q^5 - 3q^9 + \cdots \in S_2(\Gamma_0(32))$$

is a CM newform of weight 2. We will revisit this particular modular form in Chapter 6.

3.7 L-functions

The topics in this chapter will be directly applicable to Chapter 4.

Definition 14. *Given a Dirichlet character* χ *modulo* N *has an associated* ***Dirichlet L-function*** *given by*

$$L(\chi, s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s}, \quad Re(s) > 1.$$

Dirichlet L-functions converge for $\mathrm{Re}(s) > 0$, and they converge absolutely for $\mathrm{Re}(s) > 1$. These L-functions extend meromorphically to all of $\mathbb{C}$; this extension is entire except when χ is the trivial character χ_0 modulo N. In this case, $L(\chi_0, s)$ is essentially the Riemann zeta function and has a simple pole at $s = 1$.

Definition 15. *Given a modular form* $f \in M_k(\Gamma, \chi)$ *with Fourier expansion* $f(z) = \sum_{n\geq 0} a(n)q^n$, *we can associate to it a* ***modular L-function*** $L(f, s)$ *by formally writing*

$$L(f, s) := \sum_{n \geq 1} \frac{a(n)}{n^s}, \quad s \in \mathbb{C}.$$

Although the L-function $L(f, s)$ in Definition 15 is written formally, we can still discuss its convergence by estimating the Fourier coefficients $a(n)$ of f. More specifically, when the

$a(n)$ satisfy polynomial growth $|a(n)| = O(n^\sigma)$ as $n \to \infty$, the corresponding L-function $L(f, s)$ converges uniformly in the half-plane $\mathrm{Re}(s) > \sigma$.

We end by stating arguably the most famous conjecture within the theory.

Conjecture 3 (Generalized Riemann Hypothesis (GRH))**.** *Let $L(s)$ be the meromorphic continuation of a modular or Dirichlet L-function to the complex plane $\mathbb{C}$. If $L(s) = 0$ and s is not a negative real number, then $Re(s) = 1/2$.*

Chapter 4

Self-conjugate 6-cores and quadratic forms

The results of this chapter can be found in [32].

4.1 Introduction and statement of results

Recall from Section 3.3 of Chapter 3 that a *partition* of a nonnegative integer n is a non-increasing sequence of positive integers (called parts) that sum to n. Let $p(n)$ denote the number of partitions of n. Each partition of n can be represented by a *Ferrers diagram*, in which the number of cells in the ith row of the diagram is the ith part of the partition. The *hook length* of a cell in the Ferrers diagram is the number of squares below or to right of the cell (including itself). For example, (4,2,1,1) is a partition of 8 which has the following Ferrers diagram, where each cell is labeled with its hook length.

7	4	2	1
4	1		
2			
1			

For $t \in \mathbb{N}$, a *t-core partition* is a partition where no hook length in the Ferrers diagram is a multiple of t. Denote the number of t-core partitions of n by $c_t(n)$. Partition hook lengths and t cores are objects of fundamental importance which appear in several areas of mathematics; for example, they have connections to the representation theory of S_n and A_n, congruences for $p(n)$, class numbers, and more (see, for example, [26, 27, 50, 53]).

Here, we are interested in *self-conjugate t-core partitions*, which are t-core partitions whose Ferrers diagram remains the same after switching its columns and rows. We denote the number of self-conjugate t-core partitions by $sc_t(n)$. Here also, we find deep connections between $sc_t(n)$ and other mathematical objects; for example, K. Ono and W. Raji [48] proved that in many cases, $sc_7(n)$ is equal to a Hurwitz class number. Work of K. Bringmann, B. Kane, J. Males, and others also made connections between self-conjugate t-cores and t-cores, Hurwitz class numbers, and sums of squares (see, for example, [8, 18, 44]). Much of this work relies on connecting self-conjugate t-core partitions to the theory of modular forms.

One of the first questions that arise in this study is the following: when is $c_t(n) > 0$, and when is $sc_t(n) > 0$? For $c_t(n)$, this came in the form of the *t-core positivity conjecture,* which asserts that $c_t(n) > 0$ for every integer $t \geq 4$. This was proved by A. Granville and K. Ono [27]. For $sc_t(n)$, work of Baldwin et al. [4] shows that for $n \neq 2$ and $t = 8$ or $t \geq 10$, we have that $sc_t(n) > 0$. However, they note that $sc_6(n)$ is not always positive, since (for example) $sc_6(13) = 0$. After computing many values of $sc_6(n)$, Hanusa and Nath [34] made a precise conjecture regarding the positivity of $sc_6(n)$.

Conjecture 4 (Conjecture 3.5 of [34]). *Let n be a positive integer. Then $sc_6(n) > 0$ except when $n \in \{2, 12, 13, 73\}$.*

A key step in this direction was made by L. Alpoge [3], who used the generating function for $sc_6(n)$

$$\sum_{n\geq 0} sc_6(n)q^n = \prod_{n\geq 1} \frac{(1-q^{2n})^2(1-q^{12n})^3}{(1-q^n)(1-q^{4n})}$$

to make the following connection between $sc_6(n)$ and representation numbers of a certain ternary quadratic form.

Theorem 5. *For all $n \geq 0$,*

$$sc_6(n) = \frac{1}{12}\#\{(x,y,z) \in \mathbb{Z}^3 : 24n + 35 = 3x^2 + 32y^2 + 32yz + 32z^2\}.$$

Remark 7. *The statement of this theorem has been adjusted to correct an error in Alpoge's calculations.*

This theorem is crucial because it reduces Hanusa-Nath's positivity conjecture to the question of which nonnegative integers of the form $24n+35$ are represented by the quadratic form

$$Q := 3x^2 + 32y^2 + 32yz + 32z^2.$$

Alpoge then applies deep results of Duke and Schulze-Pillot [24] to this quadratic form to prove that $sc_6(n) > 0$ for sufficiently large n, but this result is ineffective. There is at present no unconditional way to resolve Conjecture 4; counting the representations of a sufficiently large integer n by Q is approximated by an expression involving a class number of an imaginary quadratic field and so is intimately related to the value of a Dirichlet L-function, which can be ineffectively bounded from below by Siegel's theorem.

In this work, we assume the Generalized Riemann Hypothesis (GRH) (see Conjecture 3) in order to prove the following statement about this quadratic form Q. In order to state the theorem, we let $r_Q(n)$ denote the number of representations of n by Q, i.e., $r_Q(n) := \#\{\mathbf{x} \in \mathbb{Z}^3 : n = Q(\mathbf{x})\}$.

Theorem 6. *Assume the GRH for all Dirichlet L-functions and all modular L-functions and let n be a positive integer. Then* $r_Q(24n+35) > 0$ *except when* $n \in \{2, 12, 13, 73\}$.

This conditionally settles Hanusa-Nath's positivity conjecture.

Corollary 2. *Assume the GRH for all Dirichlet L-functions and all modular L-functions. Then Conjecture 4 is true.*

Remark 8. *It is natural to investigate where* $r_Q(N) > 0$ *when* N *belongs to other congruence classes modulo 24. Because the numbers* $N = 24n + 35$ *lie in the progression* $\{24m + 11 : m \geq 0\}$*, Theorem 6 implies that* $r_Q(24n + 11) > 0$ *except when* $n \in \{3, 13, 14, 74\}$*. The techniques involved in this chapter utilize work of Chandee [12] to bound the number of* N *for which* $r_Q(N) = 0$*, and most of the results rely on* N *being square-free, with the exception of Theorem 9. These techniques can also be applied to square-free numbers* $N = 24n + 3$*.*

However, Chandee's bound is too large for this progression, and determining which n give $r_Q(24n+3)=0$ is not computationally feasible.

For the square-free classes $N \equiv 1,2,4,6,7,9,13,14,15,16,18,21 \pmod{24}$, both $a=0$ and $d=0$ in Equation (4.7), meaning that $r_Q(N)=0$ for all such N. For the square-free classes $N \equiv 0,8,12,20 \pmod{24}$ we have $d=0$, so the techniques in this work do not apply. However, there are $N \equiv 0,8,12,20 \pmod{24}$ for which $r_Q(N)>0$. We are not certain what can be said about the remaining square-free classes $N \equiv 5,10,17,22,23 \pmod{24}$.

In Section 4.2 we provide key results from the theory of modular forms and quadratic forms that will be used in later sections. We prove Theorem 5 in Section 4.3, and Theorem 6 will be proved in Section 4.5.

4.2 Background

Here we provide a brief overview of some key concepts in the theory of quadratic forms and modular forms that we use to prove Theorem 6. See, for example, [36].

Let $Q = Q(x_1, x_2, x_3)$ be a positive definite integral ternary quadratic form. That is, Q is a homogeneous degree-2 polynomial in three variables with coefficients in $\mathbb{Z}$ that can be expressed as

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t A \mathbf{x},$$

where A is a positive definite symmetric matrix with integer entries (which are even on the diagonal). We wish to understand the behavior of the function $r_Q(n) := \#\{\mathbf{x} \in \mathbb{Z}^3 : n = Q(\mathbf{x})\}$.

It is known that the *theta function* associated to a ternary quadratic form Q,

$$\theta_Q(z) := \sum_{\mathbf{x}\in\mathbb{Z}^3} q^{Q(\mathbf{x})} = \sum_{n\geq 0} r_Q(n) q^n,$$

is a modular form of weight $3/2$, level $2N$, and character $\left(\frac{2\det(A)}{\cdot}\right)$, where N is the least integer for which NA^{-1} has integer entries (although N may not be the minimal level).

In order to prove Theorem 6, we must understand which integers are represented by Q, so we must understand which Fourier coefficients of θ_Q are nonzero. To do this, we follow the approach introduced by K. Ono and K. Soundararajan [49] (see also [41, 51]). First we decompose the theta function as

$$\theta_Q(z) = E(z) + C(z)$$

where $E(z)$ is an Eisenstein series and $C(z)$ is a cusp form. Note that this decomposition can be computed quickly since the Eisenstein series is equal to a weighted sum of theta functions of the forms in the *genus* $\mathcal{G}(Q)$ of Q by

$$E(z) = \frac{\sum_{Q' \in \mathcal{G}(Q)} (1/|\mathrm{Aut}(Q')|)\theta_{Q'}}{\sum_{Q' \in \mathcal{G}(Q)} (1/|\mathrm{Aut}(Q')|)}.$$

Here, the genus of Q is the set of ternary forms Q' that are equivalent to Q over the local rings $\mathbb{Z}_p$ for each prime p, as well as over $\mathbb{R}$. Next, we work to understand the coefficients of $E(z)$ and $C(z)$.

Letting $a_E(n)$ denote the Fourier coefficients of $E(z)$, it is known that if $n \geq 1$ is squarefree then

$$a_E(n) = \frac{24h(-nM)}{Mw(-nM)} \prod_{p|2N} \beta_p(n) \cdot \frac{1 - \chi(p)\left(\frac{n}{p}\right)p^{-1}}{1 - 1/p^2}, \tag{4.1}$$

where M is a rational number depending on $n \pmod{8N^2}$ with the property that nM is a fundamental discriminant, $h(-nM)$ is the class number of the ring of integers in $\mathbb{Q}(\sqrt{-nM})$, $w(-nM)$ is half of the number of roots of unity in $\mathbb{Q}(\sqrt{-nM})$, and the $\beta_p(n)$ are certain local densities depending on the image of n in the set

$$\prod_{p|\Delta} \mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2.$$

Thus, for all n in a fixed square class, we may write $a_E(n) = ah(-bn)$, where the constants a and b depend only on the square class under consideration.

In order to study the coefficients of the cusp form $C(z)$, we will first apply the Shimura correspondence in order to obtain an integer weight modular form (see e.g. [51, Section 6]).

Theorem 7. *Suppose that* $f(z) = \sum_{n\geq 1} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$ *is a half-integral weight cusp form with* $\lambda \geq 1$*. Let* t *be a positive square-free integer and set*

$$\mathcal{S}_t(f(z)) := \sum_{n\geq 1} \left(\sum_{d|n} \chi(d) \left(\frac{(-1)^\lambda t}{d} \right) d^{\lambda-1} a(tn^2/d^2) \right) q^n.$$

Then $\mathcal{S}_t(f(z)) \in M_{2\lambda}(\Gamma_0(2N), \chi^2)$*. It is a cusp form if* $\lambda > 1$*, and if* $\lambda = 1$ *it is a cusp form if* $f(z)$ *is orthogonal to all cusp forms* $\sum_{n\geq 1} \psi(n) n q^{n^2}$*, where* ψ *is an odd Dirichlet character.*

Moreover, one can show that if p is a prime not dividing $4tN$ then $\mathcal{S}_t(f(z) \mid T(p^2)) = \mathcal{S}_t(f(z)) \mid T(p)$. Thus, if $C(z)$ is an eigenform and $2t \mid N$, this guarantees that $F(z) := \mathcal{S}_t(C(z))$ is also an eigenform with the same eigenvalues. Finally, a deep theorem of Waldspurger [58] allows us to write the Fourier coefficients of $C(z)$ in terms of the central critical L-values of twists of $F(z)$. By a *twist* of a form $F = \sum_{n\geq 0} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ by a Dirichlet character ψ modulo N, we mean the modular form $F \otimes \psi := \sum_{n\geq 0} \psi(n) a(n) q^n$.

Theorem 8 (Waldspurger)**.** *Suppose* $f(z) = \sum_{n\geq 1} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$ *is a Hecke eigenform for all Hecke operators* $T(p^2)$ *for primes* $p \nmid N$*. Let* $F(z) := \mathcal{S}_t(f(z))$ *be the Shimura lift of* $f(z)$ *such that* $F(z) \in S_{2\lambda}^{new}(\Gamma_0(2N), \chi^2)$*. If* $n_1, n_2 \in \mathbb{N}$ *are square-free with* $n_1/n_2 \in \left(\mathbb{Q}_p^\times\right)^2$ *for all* $p \mid N$*, then*

$$a(n_1)^2 L(F \otimes \chi^{-1}\chi_{n_2(-1)^\lambda}, \lambda) \chi(n_2/n_1) n_2^{\lambda-1/2} = a(n_2)^2 L(F \otimes \chi^{-1}\chi_{n_1(-1)^\lambda}, \lambda) n_1^{\lambda-1/2}.$$

Thus, for all square-free n in a fixed square class, we may write the Fourier coefficients of $C(z)$ as $a_C(n) = \pm d n^{1/4} L(F \otimes \chi^{-1}\chi_{-n}, 1)^{1/2}$, where χ is the Nebentypus character of $C(z)$ and $\chi_{-n}(\cdot) = \left(\frac{-n}{\cdot}\right)$.

Putting this together, for all square-free n in a fixed square class, we have that

$$r_Q(n) = ah(-bn) \pm d n^{1/4} L(F \otimes \chi^{-1}\chi_{-n}, 1)^{1/2}$$

for some constants a, b, d (which depend on the square class). Since Dirichlet's class number formula gives

$$h(-bn) = \frac{w\sqrt{bn}}{2\pi} L(\chi_{-bn}, 1), \quad w := \begin{cases} 2, & -bn < -4, \\ 4, & -bn = -4, \\ 6, & -bn = -3, \end{cases} \tag{4.2}$$

we know that if n is not represented by Q, it follows that

$$\frac{L(F \otimes \chi^{-1}\chi_{-n}, 1)^{1/2}}{L(\chi_{-bn}, 1)} \geq \frac{a\sqrt{b}}{d\pi} n^{1/4}.$$

On the other hand, results of Chandee [12] can give us upper bounds for this expression. This allows us to restrict the possible values of square-free n that are not represented by Q to a finite set; a computer can then check these cases individually.

Integers n that are not square-free must be considered using a different approach. For the quadratic form of interest here, Theorem 9 shows that all such integers are represented.

4.3 Proof of Theorem 5

For completeness, we now give Alpoge's proof (see Theorem 6 of [3]) but correct a minor error in his calculations.

Proof. By work of C.R.H Hanusa and R. Nath [34, equation (2)], the generating function for $sc_6(n)$ is

$$\sum_{n\geq 0} sc_6(n) q^{24n+35} = \left(\frac{\eta(48z)^2}{\eta(24z)}\right)\left(\frac{\eta(288z)^3}{\eta(96z)}\right),$$

where $\eta(z) := q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function. It is known that the first factor is

$$\frac{\eta(48z)^2}{\eta(24z)} = \sum_{n\geq 0} q^{3(2n+1)^2} = \frac{1}{2}\sum_{n\in\mathbb{Z}} q^{3(2n+1)^2}$$

and the second is ([34])

$$\frac{\eta(288z)^3}{\eta(96z)} = \sum_{n\geq 0} c_3(n) q^{32(3n+1)}$$

where $c_3(n)$ is the number of 3-cores of n. Work of G. Han and K. Ono [30, Lemma 2.5] tells us that

$$c_3(n) = \frac{1}{6}\#\{(x,y) \in \mathbb{Z}^2 : 3n+1 = x^2+xy+y^2\}$$

and thus it follows that

$$sc_6(n) = \frac{1}{12}\#\{(x,y,z) \in \mathbb{Z}^3 : 24n+35 = 3x^2+32y^2+32yz+32z^2\}$$

(noting that if $24n+35 = 3x^2+32y^2+32yz+32z^2$ then x must be odd) as desired. $\square$

4.4 Initial calculations

In this section, we will set some notation and make some initial calculations that will be helpful in proving Theorem 6. Let $Q := 3x^2+32y^2+32yz+32z^2$, which has associated matrix

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 64 & 32 \\ 0 & 32 & 64 \end{pmatrix}.$$

Using Sage or Magma, we find that the theta function corresponding to Q is

$$\theta_Q(z) = \sum_{\mathbf{x}\in\mathbb{Z}^3} q^{Q(\mathbf{x})} = \sum_{n\geq 0} r_Q(n)q^n = 1+2q^3+2q^{12}+2q^{27}+6q^{32}+12q^{35}+O(q^{40}) \in M_{3/2}(\Gamma_0(96)).$$

It is convenient that the genus of Q has size 2, and the other form is $Q' := 11x^2+10xy+11y^2+6xz-6yz+27z^2$. Thus one may compute that

$$E(z) = \sum_{n\geq 0} a_E(n)q^n = \frac{1}{4}\theta_Q(z) + \frac{3}{4}\theta_{Q'}(z) = 1 + \frac{1}{2}q^3 + 3q^{11} + 2q^{12} + \frac{7}{2}q^{27} + 6q^{32} + 6q^{35} + O(q^{40}),$$

$$C(z) = \sum_{n\geq 0} a_C(n)q^n = \frac{3}{4}\left(\theta_Q(z) - \theta_{Q'}(z)\right) = \frac{3}{2}q^3 - 3q^{11} - \frac{3}{2}q^{27} + 6q^{35} + O(q^{40}).$$

Applying the Shimura correspondence to $C(z)$ when $t = 3$ gives a constant multiple of the newform

$$F(z) = \sum_{n\geq 0} A(n)q^n = q - q^3 - 2q^5 + q^9 + 4q^{11} - 2q^{13} + 2q^{15} + 2q^{17} - 4q^{19} + O(q^{20}) \in S_2(\Gamma_0(24)),$$

which is the cusp form associated to the elliptic curve $E : y^2 = x^3 - x^2 + x$.

With Theorem 8 in mind, we will eventually consider certain twists $F \otimes \chi_{-(24n+35)}$ of F. These twists translate over to the associated elliptic curve E by defining the twisted curve

$$E \otimes \chi_{-N} : y^2 = x^3 + Nx^2 + N^2x,$$

where $N = 24n + 35$.

The following proposition will be useful in Section 4.5.

Proposition 7. *If $N = 24n + 35$ is square-free, then the conductor of $E \otimes \chi_{-N}$ is $24N^2$.*

Proof. We use the Weierstrass equation $y^2 = x^3 + Nx^2 + N^2x$ for the twisted curve $E \otimes \chi_{-N}$, which has discriminant $\Delta := -48N^6$ (see, for example, [55]). The conductor q of $E \otimes \chi_{-N}$ is given by

$$q = \prod_{p|\Delta} p^{f_p},$$

where the exponents f_p can be computed using Tate's algorithm (see [54, 16]). □

4.5 Proof of Theorem 2

In order to prove Theorem 6 we must first restrict our attention to values of $24n + 35$ that are square-free so that we may apply Theorem 8 and equation (4.1). In order to do this, we proceed as in Sections 2 and 3 of [49].

Let $R_Q(N)$ and $R_{Q'}(N)$ be the number of *primitive* representations of N by Q and Q', respectively. There are 12 automorphs of Q (i.e., matrices B of determinant 1 such that

$B^T AB = A$), namely

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, there are 4 automorphs of Q', namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We say that two representations of N are *essentially distinct* if there is no automorph that takes one to the other, and we let $G(N)$ be the number of essentially distinct primitive representations of N by the genus of Q. When N is square-free and coprime to 6 one can check that

$$G(N) = \frac{1}{12} R_Q(N) + \frac{1}{4} R_{Q'}(N). \tag{4.3}$$

Also, Theorem 86 in [39] gives us that

$$G(N) = \frac{1}{2} h(-4N), \tag{4.4}$$

with $-4N$ representing the discriminant rather than the determinant of the corresponding binary forms from [39].

Theorem 9. *If $N = 24n + 35$ is not square-free, then $r_Q(N) > 0$.*

Proof. First note that if $N = 24n + 35$ is not square-free, then it suffices to find $d^2 \mid N$ such that $r_Q(N/d^2) > 0$ (since if $Q(x_1, x_2, x_3) = N/d^2$ then $Q(dx_1, dx_2, dx_3) = N$). Thus, it suffices to prove the following statement: if $N = 24n + 35$ is square-free and $r_Q(N) = 0$, then $r_Q(Np^2) > 0$ for any prime $p \geq 5$.

As in Section 4.4, consider $C(z) = \frac{3}{4}(\theta_Q(z) - \theta_{Q'}(z)) = \sum_{n\geq 1} a_C(n)q^n \in S_{3/2}(\Gamma_0(96))$. Note that $C(z)$ is a Hecke eigenform and its Shimura lift is a multiple of the newform $F(z) = \sum_{n\geq 1} A(n)q^n \in S_2(\Gamma_0(24))$. Thus for $p \geq 5$ it follows that $A(p)$ is the Hecke eigenvalue when $T(p^2)$ is applied to $C(z)$, and so

$$A(p)a_C(n) = a_C(p^2n) + \left(\frac{-n}{p}\right)a_C(n) + pa_C(n/p^2).$$

Since $a_C(n) = \frac{3}{4}(r_Q(n) - r_{Q'}(n))$, it follows that for square-free n we have

$$r_Q(np^2) - r_{Q'}(np^2) = \left(A(p) - \left(\frac{-n}{p}\right)\right)(r_Q(n) - r_{Q'}(n)).$$

Now let $p \geq 5$ be prime and $N = 24n + 35$ be a square-free integer such that $r_Q(N) = 0$. Suppose for contradiction that $r_Q(Np^2) = 0$, so

$$\frac{r_{Q'}(Np^2)}{r_{Q'}(N)} = A(p) - \left(\frac{-N}{p}\right) \leq A(p) + 1. \tag{4.5}$$

Since N is square-free, we have that

$$r_{Q'}(Np^2) = R_{Q'}(Np^2) + R_{Q'}(N) = R_{Q'}(Np^2) + r_{Q'}(N)$$

and (using equation (4.3))

$$4G(N) = \frac{1}{3}R_Q(N) + R_{Q'}(N) = \frac{1}{3}r_Q(N) + r_{Q'}(N) = r_{Q'}(N).$$

Also, since $Np^2 \neq 0$, every primitive essentially distinct representation of Np^2 by Q' has at least 2 different automorphs, whence $2G(Np^2) \leq R_{Q'}(Np^2)$. Therefore,

$$\frac{r_{Q'}(Np^2)}{r_{Q'}(N)} = 1 + \frac{R_{Q'}(Np^2)}{r_{Q'}(N)} \geq 1 + \frac{2G(Np^2)}{4G(N)} = 1 + \frac{G(Np^2)}{2G(N)}. \tag{4.6}$$

Using equation (4.4) along with Corollary 7.28 from [14], we get

$$\frac{G(Np^2)}{G(N)} = \frac{h(-4Np^2)}{h(-4N)} = p - \left(\frac{-4N}{p}\right) \geq p - 1.$$

Substituting this into (4.6) yields

$$\frac{r_{Q'}(Np^2)}{r_{Q'}(N)} \geq \frac{p+1}{2}.$$

This coupled with (4.5) tells us that $A(p) + 1 \geq (p+1)/2$, whence $A(p) \geq (p-1)/2$. This contradicts Hasse's bound $|A(p)| \leq 2\sqrt{p}$ for $p > 17$. Finally, there are no primes $5 \leq p \leq 17$ that satisfy $A(p) \geq (p-1)/2$. This completes the proof. □

Proof of Theorem 6. Let $N = 24n + 35$ be square-free. It suffices to show that $r_Q(N) > 0$ except when $n \in \{2, 12, 13, 73\}$.

By considering the decomposition $\theta_Q = E(z) + C(z)$ and applying both equation (4.1) and Theorem 8, we can find constants a, b, and d such that

$$r_Q(N) = ah(-bN) \pm dN^{1/4} L(E \otimes \chi_{-N}, 1)^{1/2}. \tag{4.7}$$

In fact, we have $a = 3$, $b = 1$, and $d = 1.63384...$ Dirichlet's class number formula (equation (4.2)) gives us

$$h(-N) = \frac{1}{\pi}\sqrt{N} L(\chi_{-N}, 1),$$

so if N is not represented by Q then it must be that

$$\frac{L(E \otimes \chi_{-N}, 1)^{1/2}}{L(\chi_{-N}, 1)} = \frac{a\sqrt{b}}{d\pi} N^{1/4} \geq 0.5844 N^{1/4}. \tag{4.8}$$

On the other hand, Proposition 7 allows us to utilize work of Chandee (see Section 4 of [12]) to obtain the upper bound

$$\frac{L(E \otimes \chi_{-N}, 1)^{1/2}}{L(\chi_{-N}, 1)} \leq 2.5889 N^{0.14157}. \tag{4.9}$$

Equations (4.8) and (4.9) together tell us that $n \leq 916347.7794$. That is, if $r_Q(N) = 0$ then $n \leq 916347.7794$. Using Sage or Magma, we find that the only such $N = 24n + 35$ correspond to $n \in \{2, 12, 13, 73\}$. This completes the proof. □

Chapter 5

Ramanujan congruences for overpartitions with restricted odd differences

The contents of this chapter can be found in [33].

5.1 Introduction

Perhaps one of the most famous results of Ramanujan was his collection of congruences for the partition function $p(n)$. It is common notation to write a partition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ of n in the concatenated form $\lambda_1\lambda_2 \ldots \lambda_r$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Ramanujan's famous result on $p(n)$ is the collection of congruences

$$\begin{cases} p(5n+4) \equiv 0 \pmod{5}, \\ p(7n+5) \equiv 0 \pmod{7}, \\ p(11n+6) \equiv 0 \pmod{11}. \end{cases} \tag{5.1}$$

These are proven starting with the fact that the generating function for $p(n)$ takes the form $\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1}(1-q^n)^{-1}$ with $q := e^{2\pi iz}$, which is essentially the inverted Dedekind's eta-function $\eta(z) := q^{1/24}\prod_{n\geq 1}(1-q^n)$ (see Section 3.3 of Chapter 3). Ramanujan's congruences for $p(n)$ all take the form $p(\ell n + a) \equiv 0 \pmod{\ell}$ for some prime ℓ, hence all such congruences are called *Ramanujan congruences*. It is natural to ask whether there are other primes ℓ for which $p(n)$ has Ramanujan congruences. But as it turns out, work of Ahlgren and Boylan [1] show that the Ramanujan congruences (5.1) are the only ones for $p(n)$.

Ramanujan's work has inspired many others to consider such congruences for other modified partition functions. In this work, we consider the modified partition function $\overline{t}(n)$ that counts "overpartitions with restricted odd parts," as originally described in [6]. An *overpartition* of n is a partition of n in which the final occurrence of a number may be overlined. We let $\overline{p}(n)$ be the number of overpartitions of n. Thus, for example, $\overline{p}(4) = 14$, with the 14 partitions given as

$$4, \quad \overline{4}, \quad 3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1}, \quad 2+2, \quad 2+\overline{2}, \quad 2+1+1,$$
$$\overline{2}+1+1, \quad 2+1+\overline{1}, \quad \overline{2}+1+\overline{1}, \quad 1+1+1+1, \quad 1+1+1+\overline{1}.$$

Our function of interest $\overline{t}(n)$ counts the number of overpartitions of n with the following restrictions.

(i) The difference between two successive parts may be odd only if the larger part is overlined.

(ii) If the smallest part is odd, then it is overlined.

For example, using $n = 4$ again we have $\overline{t}(4) = 8$, with the 8 such partitions given as

$$4, \quad \overline{4}, \quad 3+\overline{1}, \quad \overline{3}+\overline{1}, \quad 2+2, \quad 2+\overline{2}, \quad \overline{2}+1+\overline{1}, \quad 1+1+1+\overline{1}.$$

The authors of [6] show that the generating function for $\overline{t}(n)$ is given by

$$\sum_{n\geq 0} \overline{t}(n)q^n = \frac{\eta(3z)}{\eta(2z)\eta(z)}.$$

This is a weakly holomorphic modular form of weight $-1/2$ for the congruence subgroup $\Gamma_1(144)$. Many congruences have been proven for $\bar{t}(n)$. In particular, congruences modulo 2, 3, and 5 have been found:

- Theorem 1.1 of [35]: For $n \geq 0$ we have

$$\bar{t}(n) \equiv \begin{cases} (-1)^{k+1} \pmod 3 & n = k^2 \text{ some } k, \\ 0 \pmod 3 & \text{else.} \end{cases}$$

- Theorem 1.2 of [35]: For $n \geq 1$ we have

$$\bar{t}(2n) \equiv \begin{cases} 1 \pmod 2 & n = (3k+1)^2 \text{ some } k, \\ 0 \pmod 2 & \text{else.} \end{cases}$$

- Theorem 1.1 of [42]: For all $\alpha, n \geq 0$ we have

$$\bar{t}(9^\alpha(45n+30)) \equiv 0 \pmod 5.$$

More congruences can be found in [13, 35, 42, 45]. We provide two more congruences modulo 5, proven in Section 5.7 using the theory of modular forms.

Proposition 8. *The following congruences hold for all $n \geq 0$:*

$$\bar{t}(80n+40) \equiv 0 \pmod 5, \tag{5.2}$$

$$\bar{t}(80n+60) \equiv 0 \pmod 5. \tag{5.3}$$

Remark 9. *We note that although this result is proven using standard techniques from the theory of modular forms, these congruences do not appear in the current literature. We bring this to the attention of combinatorialists who may like to prove this using only manipulation of generating functions.*

However, one might question if any Ramanujan congruences hold for $\bar{t}(n)$ (i.e. congruences of the form $\bar{t}(\ell n + a) \equiv 0 \pmod \ell$ for a prime ℓ). The results of [35] above show that the only Ramanujan congruence mod 2 or 3 is $\bar{t}(3n+2) \equiv 0 \pmod 3$, and one can check that there are no Ramanujan congruences mod 5. In fact, we have the following.

Theorem 10. *The only Ramanujan congruence for $\bar{t}(n)$ is*

$$\bar{t}(3n+2) \equiv 0 \pmod 3.$$

In order to prove Theorem 10, it suffices to show that there are no Ramanujan congruences for primes $\ell > 5$. This is an immediate consequence of Theorem 12 below, which is a generalization of Sinick's Theorem 1.2 in [56]:

Theorem 11 (Theorem 1.2 of [56])**.** *Let $S = (a_1, a_2, \ldots, a_j)$ be a sequence of positive integers with j even and define $c(n)$ by*

$$\prod_{n=\geq 1} \prod_{i=1}^{j} \frac{1}{1-q^{a_i n}} = \sum_{n\geq 0} c(n) q^n.$$

Let $N = lcm(a_1, a_2, \ldots, a_j)$. If $c(n)$ obeys a Ramanujan congruence modulo ℓ, then $\ell \mid N$ or $\ell \leq \max(5, j+4)$.

We extend this theorem to bound Ramanujan congruences for more general eta-quotients in the following theorem.

Theorem 12. *Let $\lambda = \lambda_1\lambda_2\ldots\lambda_r$ and $\mu = \mu_1\mu_2\ldots\mu_s$ be partitions of $u \in \mathbb{N}$ and $v \in \mathbb{N}$, respectively, where we assume without loss of generality that $\lambda_i \neq \mu_j$ for all i, j. Define*

$$h(z) := \prod_{n\geq 1} \frac{(1-q^{\lambda_1 n})(1-q^{\lambda_2 n})\cdots(1-q^{\lambda_r n})}{(1-q^{\mu_1 n})(1-q^{\mu_2 n})\cdots(1-q^{\mu_s n})} =: \sum_{n\geq 0} c(n) q^n.$$

Let $N := lcm(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s)$, and let γ be the number of occurrences of the smallest element of $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s\}$. Let ℓ be prime such that $\ell > \max(5, |r-s|+4)$ and $\ell \nmid \gamma N$. Then the following statements hold.

(i) If $r-s$ is even, then $c(n)$ does not obey a Ramanujan congruence modulo ℓ.

(ii) If $r-s \neq 1, 3$ is odd and $u \equiv v \pmod{\ell}$, then $c(n)$ does not obey a Ramanujan congruence modulo ℓ.

Remark 10. *The case $r - s \neq 1, 3$ is odd and $u \not\equiv v \pmod{\ell}$ is not included in Theorem 12. We leave open the problem of whether one can establish an upper bound for primes ℓ that could admit a Ramanujan congruence in this case.*

In Section 5.2, we provide a brief introduction to modular forms modulo ℓ, and we give a few results within the theory that will be used in later sections. Section 5.3 provides motivation and an outline for the proof of Theorem 12. Section 5.4 proves the most interesting and essential result for the proof of Theorem 12. Section 5.5 disposes of some necessary calculations needed for Section 5.6 which completes the proofs of Theorems 12 and 10, respectively. Section 5.7 proves Proposition 8. Section 5.8 provides a helpful example pertaining to the discussion following Proposition 10.

5.2 Modular forms modulo ℓ

In this section, we introduce the notion of "modular forms mod ℓ" as well as some preliminary results that will be used in later sections. We refer the reader to [47] for a more detailed account of this material.

Define $\theta := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$, where $q := e^{2\pi i z}$, so that on Fourier series we have

$$\theta\left(\sum_{n\geq 0} a(n)q^n\right) = \sum_{n\geq 0} na(n)q^n.$$

Also define the m-th U-operator on Fourier series as

$$\sum_{n\geq 0} a(n)q^n \mid U(m) := \sum_{n\geq 0} a(mn)q^n.$$

Eisenstein series are canonical examples of modular forms for $\mathrm{SL}_2(\mathbb{Z})$, and they play an important role in Lemma 1 below. For even $k > 2$, recall from Chapter 3 that the weight-k *Eisenstein series* for $\mathrm{SL}_2(\mathbb{Z})$ is

$$E_k(z) := 1 - \frac{2k}{B_k}\sum_{n\geq 1} \sigma_{k-1}(n)q^n,$$

where B_k is the k-th Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. When $k = 2$, $E_2(z) := 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$ is not a modular form, but rather a "quasi-modular form." It has the transformation law

$$E_2(z) \mid M = E_2(z) - \frac{6ic}{\pi(cz+d)}, \tag{5.4}$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We also have the *Delta function*

$$\Delta(z) = \eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} \in M_{12}(\mathrm{SL}_2(\mathbb{Z})),$$

which vanishes at the cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$.

Given a modular form $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[\![q]\!]$, one can reduce the Fourier coefficients of f modulo a prime ℓ, giving an element $\tilde{f}$ of $\mathbb{F}_\ell[\![q]\!]$. We call $\tilde{f}$ a *modular form modulo* ℓ for $\Gamma_1(N)$. The *filtration* of f is defined as

$$w_\ell(f) := \min\{k' : \tilde{f} \in \widetilde{M}_{k'}(\Gamma_1(N))\},$$

where

$$\widetilde{M}_{k'}(\Gamma_1(N)) := \{\tilde{g} : g \in M_{k'}(\Gamma_1(N))\}.$$

We will also refer to preimages of $\tilde{f}$ under the reduction map as "modular forms modulo ℓ".

By using (5.4), one can easily generalize Lemma 3 of [57] for $N > 1$ to conclude that if $f \in M_k(\Gamma_1(N))$ then $12\theta f - kE_2 f \in M_{k+2}(\Gamma_1(N))$. Theorem 2(i) from [57] implies that $E_{\ell-1} \equiv 1 \pmod{\ell}$ and $E_{\ell+1} \equiv E_2 \pmod{\ell}$. These facts come together to prove Lemma 1.

Lemma 1 (Lemma 2.1 of [56])**.** *If $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[\![q]\!]$, then defining R to be*

$$R := \left(\theta f - \frac{k}{12}E_2 f\right)E_{\ell-1} + \frac{k}{12}E_{\ell+1}f, \tag{5.5}$$

R is a modular form of weight $k + \ell + 1$ such that $R \equiv \theta f \pmod{\ell}$. In particular, θf is a modular form $\pmod{\ell}$ for $\Gamma_1(N)$. It follows that if $\tilde{f} \not\equiv 0 \pmod{\ell}$, then $w_\ell(\theta f) \leq w_\ell(f) + \ell + 1$.

We will also need the following facts about filtrations.

Lemma 2. *Let $N \geq 4$, let $f \in M_{k_1}(\Gamma_1(N)) \cap \mathbb{Z}[\![q]\!]$ and $g \in M_{k_2}(\Gamma_1(N)) \cap \mathbb{Z}[\![q]\!]$, and let $\ell \geq 5$ be prime. Then:*

(i) We have $w_\ell(\theta f) = w_\ell(f) + \ell + 1$ if and only if $w_\ell(f) \not\equiv 0 \pmod{\ell}$.

(ii) If $\tilde{f} \equiv \tilde{g} \not\equiv 0 \pmod{\ell}$, then $k_1 \equiv k_2 \pmod{\ell - 1}$.

(iii) If $\ell \nmid N$ then for $i \geq 0$ we have $w_\ell(f^i) = i \cdot w_\ell(f)$.

The proofs of (ii) and the reverse implication of (i) are given directly in Section 4 of [28]. The remaining facts are quick consequences of the results given in the same section.

The following elementary fact will be useful in Section 5.6: if $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[\![q]\!]$ and ℓ is a prime, then $(f \mid U_\ell)^\ell \equiv f - \theta^{\ell-1} f \pmod{\ell}$. It follows that

$$f \mid U_\ell \equiv 0 \pmod{\ell} \iff \theta^{\ell-1} f \equiv f \pmod{\ell}.$$

5.3 Keys to the proof of Theorem 12

The statement of Theorem 12 concerns the existence of Ramanujan congruences for a particular type of eta-quotient. We hope to apply the following proposition originally due to I. Kiming and J. Olsson [40] and then corrected by J. Sinick [56, Proposition 3.2].

Proposition 9. *Let $\ell \geq 5$ be prime and $N \geq 4$, $\ell \nmid N$. Suppose that $f(z) \in M_k(\Gamma_1(N))$ has ℓ-integral Fourier coefficients, $w_\ell(f(z)) \not\equiv 0 \pmod{\ell}$, and $\theta(f(z)) \not\equiv 0 \pmod{\ell}$. Suppose further that $w_\ell(\theta^m f(z)) \geq w_\ell(f(z))$ for any integer $m \geq 0$. Then if the Fourier coefficients $d(n)$ of $f(z)$ satisfy $d(\ell n + b) \equiv 0 \pmod{\ell}$, one of the following is true: $b = 0$, $w_\ell(f(z)) \equiv (\ell+1)/2 \pmod{\ell}$, or $w_\ell(f(z)) \equiv (\ell+3)/2 \pmod{\ell}$.*

There is a glaring problem with naively applying this proposition to the eta-quotient in Theorem 12: the fact that the eta-quotient is not necessarily an integer weight holomorphic modular form. We will fix this by defining an integer weight modular form for each prime ℓ for which a Ramanujan congruence exists mod ℓ if and only if a Ramanujan congruence exists mod ℓ for the given eta-quotient. This definition is given below.

Let N, λ, and μ be as in Theorem 12. Define

$$F_\ell(z) := \Delta(z)^{\ell^t} \left(\frac{\Delta(\mu_1 z)\Delta(\mu_2 z)\cdots\Delta(\mu_s z)}{\Delta(\lambda_1 z)\Delta(\lambda_2 z)\cdots\Delta(\lambda_r z)} \right)^{\delta_\ell} =: \sum_{n\geq 0} D(n)q^n, \tag{5.6}$$

where $\delta_\ell := \frac{\ell^2-1}{24}$, and where $t \geq 2$ is the smallest integer such that F_ℓ is holomorphic at the cusps of $\Gamma_1(N)$.

Remark 11. *Note that $F_\ell(z)$ is a modular form of weight $\frac{(\ell^2-1)(s-r)}{2} + 12\ell^t$ for $\Gamma_1(N)$. In what follows, we sometimes need $N \geq 4$. We can substitute $4N$ for N without loss of generality when $N < 4$.*

Remark 12. *We also note that $F_\ell(z)$ serves the same purpose as the modular form in equation (1.3) of [56]. Our form $F_\ell(z)$ differs from Sinick's in that it requires an appropriate power of $\Delta(z)$ to account for any poles coming from the numerator of $h(z)$ in Theorem 12.*

The fact that the Ramanujan congruences for $F_\ell(z)$ are in correspondence with those of $h(z) = \sum_{n\geq 0} c(n)q^n$ is the following lemma.

Lemma 3. *Let u, v be as in Theorem 12. With notation as above, we have that $D(\ell n+b) \equiv 0 \pmod{\ell}$ if and only if $c(\ell n + a) \equiv 0 \pmod{\ell}$, where b is defined by $24a \equiv 24b + (u-v) \pmod{\ell}$.*

The proof of this will be given later in Section 5.5. Notice that if $u \equiv v \mod \ell$, then $a \equiv b \pmod{\ell}$ (recall $\ell > 5$). Now that we have defined an appropriate modular form, we need to proceed by checking that F_ℓ satisfies the other assumptions of Proposition 9. The fact that $\theta F_\ell \not\equiv 0 \pmod{\ell}$ is a simple calculation (see Proposition 11) which will be done in Section 5.5. We note that for the corresponding modular form in Sinick's case, this is an immediate verification. On a technical note, the necessity of γ in the statement of Theorem 12 comes from the proof of Proposition 11. The most difficult assumptions to verify in Proposition 9 are precisely those that deal with the filtrations. This will be accomplished in Section 5.4. By these calculations, we will see that the two congruences $w_\ell(F_\ell(z)) \equiv (\ell+1)/2 \pmod{\ell}$ and $w_\ell(F_\ell(z)) \equiv (\ell+3)/2 \pmod{\ell}$ are impossible. Lastly, in order to get the full strength of Theorem 12, we must dispose of the possibility that $b = 0$ in the statement of Proposition 9. This is a technical point which is resolved in Section 5.6.

5.4 Calculating the filtrations

In this section, we show that $w_\ell(\theta^m F_\ell) \geq w_\ell(F_\ell) = \frac{(\ell^2-1)(s-r)}{2} + 12\ell^t$ by proving the more general Proposition 10.

Proposition 10. *Let $F \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[\![q]\!]$, $\ell \geq 5$ prime, $\ell \nmid N$, $\theta F \not\equiv 0 \pmod{\ell}$, and suppose that F does not vanish on $\mathbb{H}$. Then $w_\ell(F) = k$ and $w_\ell(\theta^m F) \geq w_\ell(F)$ for any integer $m \geq 0$.*

Following the discussion below Lemma 4.1 of [56], we enumerate the cosets of $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ by $\{i\}_{1 \leq i \leq 2d_N}$. Let M_i be a representative of the i-th coset. Let α_i be the cusp that M_i sends to ∞. Denote the minimal period of $F \mid M_i$ by t_i. Then $F \mid M_i$ has a Fourier expansion in powers of $q_{t_i} := e^{2\pi i z/t_i}$, and the order of vanishing of F at α_i is the index of the first non-vanishing Fourier coefficient of F in powers of q_{t_i}, denoted $\mathrm{ord}_{\alpha_i}(F)$. Though the coefficients of these q_{t_i}-Fourier expansions of F need not be integral, [20, §12.3] tells us that they lie in $\mathbb{Z}[\zeta_N]$ with ζ_N a primitive N-th root of unity.

Instead of considering modular forms modulo ℓ, one may choose an algebraic number field L and look at forms $g \in M_k(\Gamma_1(N)) \cap L[\![q]\!]$. We can then reduce g modulo v for any prime $v \in \mathcal{O}_L$ such that the v-adic valuation of g is 0. This allows us to define the notion of "modular form modulo v", and we can define the filtration w_v for nonvanishing forms $\pmod{v}$ in the obvious way. Hence we can define the v-adic valuation of the corresponding power series to be the minimum of the v-adic valuations of the coefficients. Defining $\widetilde{\mathrm{ord}}_{\alpha_i}(f)$ to be the order of vanishing of $f \pmod{v}$ at the cusp α_i (this is well-defined; see for example Remark 2.4 of [19]), we have Lemma 4.2 of [56], stated below.

Lemma 4. *Let $m \geq 1$ be an integer and let v be a prime in $\mathbb{Z}[\zeta_N]$ such that $v \nmid 2, 3, N$. Let $f(z)$ be a modular form for $\Gamma_1(N)$ such that $f(z) \mid M_i$ has coefficients in $\mathbb{Q}(\zeta_N)$ and v-adic valuation 0. Let α_i be a cusp of $\Gamma_1(N)$. Then*

$$\widetilde{ord}_{\alpha_i}(\theta^m f) \geq \widetilde{ord}_{\alpha_i}(f).$$

Remark 13. *Sinick's proof states, "Since $v \nmid N$ and $f(z) \mid M_i$ has v-adic valuation 0, the Fourier expansion of $\theta(f \mid M_i)$ has v-adic valuation 0." However, this statement is false* *by taking a prime above 5 in $\mathbb{Z}[\zeta_7]$ and considering $E_4(z)$ as a modular form for $\Gamma_1(7)$ for example. One needs the additional assumption that $\theta f \not\equiv 0 \pmod{v}$.*

Proof of Proposition 10. We let M_i, α_i, t_i, and $v \in \mathbb{Z}[\zeta_N]$ be as above, v being a prime above ℓ. Since $\theta F \not\equiv 0 \pmod{\ell}$ by assumption, $F \not\equiv 0 \pmod{\ell}$. In particular, $F \not\equiv 0 \pmod{v}$, and so Theorem 12.3.4 and Remark 12.3.5 of [20] assert that $F \mid M_i \not\equiv 0 \pmod{v}$. Define

$$G(z) := \prod_{i=1}^{2d_N} (F \mid M_i),$$

a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2d_N k$. As F is zero-free on $\mathbb{H}$, so is G, and so by the valence formula we have that G is a non-zero constant multiple of $\Delta(z)^e$ with $e := \frac{2d_N k}{12} = \frac{d_N k}{6}$. It follows that $w_v(G) = 12e$ since if there existed a modular form of smaller weight that is congruent to G, Sturm's Theorem [47, Theorem 2.58] would imply that it would have to vanish mod v. Thus $w_v(F) = k$, and since $F \in \mathbb{Z}[\![q]\!]$, we in fact have that $w_\ell(F) = k$, as desired.

Now we show that $w_\ell(\theta^m F) \geq w_\ell(F)$. Notice that $\widetilde{\mathrm{ord}}_\infty(G) = e$, whence $\widetilde{\mathrm{ord}}_\infty(F \mid M_i) = \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(F)}{t_i}$. Thus

$$\sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(F)}{t_i} = \widetilde{\mathrm{ord}}_\infty(G) = e.$$

Define

$$H := \prod_{i=1}^{2d_N} (\theta^m F) \mid M_i.$$

Notice that H is a modular form modulo v for $\mathrm{SL}_2(\mathbb{Z})$ by the modified version of Lemma 1, where we replace ℓ with v. Then

$$\widetilde{\mathrm{ord}}_\infty(H) = \sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(\theta^m F)}{t_i} \geq \sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(F)}{t_i} = e,$$

the inequality being a consequence of Lemma 4. Since $\theta^m F$ is a modular form modulo v that does not vanish, Theorem 12.3.4 and Remark 12.3.5 of [20] assert that $(\theta^m F) \mid M_i \not\equiv 0 \pmod{v}$. This shows that $H \not\equiv 0 \pmod{v}$. Sturm's Theorem [47, Theorem 2.58] now tells us that $w_v(H) \geq 12e$, and so $w_v(\theta^m F) = w_\ell(\theta^m F) \geq k = w_\ell(F)$, the first equality coming from the fact that F has integral Fourier coefficients. $\square$

5.5 Necessary calculations

In this section, we produce two calculations that are necessary for the application of Proposition 9.

Proposition 11. *We have that* $\theta F_\ell \not\equiv 0 \pmod{\ell}$.

Proof. We note that by definition of θ it is sufficient to compute the Fourier expansion modulo ℓ. We have

$$\begin{aligned} F_\ell(z) &= \Delta(z)^{\ell t}\left(\frac{\Delta(\mu_1 z)\Delta(\mu_2 z)\cdots\Delta(\mu_s z)}{\Delta(\lambda_1 z)\Delta(\lambda_2 z)\cdots\Delta(\lambda_r z)}\right)^{\delta_\ell} \\ &= q^{\ell t+(v-u)\delta_\ell}\prod_{n=1}^{\infty}\left[(1-q^n)^{\ell t}\left(\prod_i (1-q^{\mu_i n})^{\ell^2-1}\right)\left(\prod_j \frac{1}{(1-q^{\lambda_j n})^{\ell^2-1}}\right)\right]. \end{aligned}$$

Using a geometric series expansion, we have

$$\begin{aligned} F_\ell(z) &= q^{\ell t+(v-u)\delta_\ell}\prod_{n=1}^{\infty}\left[(1-q^n)^{\ell t}\left(\prod_i (1-q^{\mu_i n})^{\ell^2-1}\right)\left(\prod_j \left(\sum_{k=0}^{\infty} q^{\lambda_j n k}\right)^{\ell^2-1}\right)\right] \\ &= q^{\ell t+(v-u)\delta_\ell}\prod_{n=1}^{\infty}\left[(1-q^n)^{\ell t}\left(\prod_i (1-q^{\mu_i n})^{\ell^2-1}\right)\left(\prod_j (1+q^{\lambda_j n}+\ldots)^{\ell^2-1}\right)\right] \\ &= q^{\ell t+(v-u)\delta_\ell}\prod_{n=1}^{\infty}\left[(1-q^n)^{\ell t}\left(\prod_i (1-(\ell^2-1)q^{\mu_i n}+\ldots)\right)\left(\prod_j (1+(\ell^2-1)q^{\lambda_j n}+\ldots)\right)\right] \\ &\equiv q^{\ell t+(v-u)\delta_\ell}\prod_{n=1}^{\infty}\left[(1-q^n)^{\ell t}\left(\prod_i (1+q^{\mu_i n}+\ldots)\right)\left(\prod_j (1-q^{\lambda_j n}+\ldots)\right)\right] \pmod{\ell} \\ &\equiv q^{\ell t+(v-u)\delta_\ell}\prod_{n=1}^{\infty}\left[(1-q^{\ell^t n})(1+\alpha q^{\mu_s n}+\ldots)(1-\beta q^{\lambda_r n}+\ldots)\right] \pmod{\ell}, \end{aligned}$$

where α is the number of occurrences of μ_s in μ and β is the number of occurrences of λ_r in λ. Hence,

$$\begin{aligned}F_\ell(z) &\equiv q^{\ell^t+(v-u)\delta_\ell}(1 \pm \cdots \pm \gamma q^m \pm \cdots)\\ &\equiv q^{\ell^t+(v-u)\delta_\ell} \pm \cdots \pm \gamma q^{m+\ell^t+(v-u)\delta_\ell} \pm \cdots \pmod{\ell},\end{aligned}$$

where $m = \min(\mu_s, \lambda_r)$. So, by applying the theta operator,

$$\theta F_\ell(z) \equiv (\ell^t + (v-u)\delta_\ell)q^{\ell^t+(v-u)\delta_\ell} \pm \cdots \pm \gamma(m + \ell^t + (v-u)\delta_\ell)q^{m+\ell^t+(v-u)\delta_\ell} \pm \cdots \pmod{\ell}.$$

One of these coefficients does not vanish mod ℓ because either $(v-u)\delta_\ell$ is divisible by ℓ or not, and from the fact that $\ell \nmid \gamma N$, which implies $\ell \nmid \gamma m$, the result follows. □

We also provide the proof of the correspondence of Ramanujan congruences of the eta-quotient to an integer weight modular form.

Proof of Lemma 3. We have

$$\begin{aligned}\sum_{n\geq 0} c(n)q^n &= \prod_{n\geq 1} \frac{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}\\
&= \prod_{n\geq 1} \left(\frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{-\ell^2} \left(\frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{\ell^2-1}\\
&= \prod_{n\geq 1} \left(\frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{-\ell^2} \frac{q^{\delta_\ell(\lambda_1+\cdots+\lambda_r)}}{q^{\delta_\ell(\mu_1+\cdots+\mu_s)}}\\
&\quad \times \prod_{n\geq 1} \left(\frac{q^{\frac{1}{24}(\mu_1+\cdots+\mu_s)}(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{q^{\frac{1}{24}(\lambda_1+\cdots+\lambda_r)}(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{\ell^2-1}\\
&= \left(\prod_{n\geq 1} \frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{-\ell^2} q^{\delta_\ell(u-v)} \left(\frac{\Delta(\mu_1 z)\cdots\Delta(\mu_s z)}{\Delta(\lambda_1 z)\cdots\Delta(\lambda_r z)}\right)^{\delta_\ell}.\end{aligned}$$

Hence,

$$q^{\delta_\ell(u-v)} \left(\frac{\Delta(\mu_1 z)\cdots\Delta(\mu_s z)}{\Delta(\lambda_1 z)\cdots\Delta(\lambda_r z)}\right)^{\delta_\ell} = \left(\prod_{n\geq 1} \frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{\ell^2} \sum_{n\geq 0} c(n)q^n.$$

Multiply both sides by $\Delta(z)^{\ell t}$ to get

$$q^{\delta_\ell(u-v)}F_\ell(z) = \Delta(z)^{\ell t}\left(\prod_{n\geq 1}\frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{\ell^2}\sum_{n\geq 0}c(n)q^n. \tag{5.7}$$

Now we apply $U(\ell)$ to equation (5.7), reduce modulo ℓ, and multiply both sides by q^{-a} to get

$$\sum_{n\geq 0} D\left(\ell n + \delta_\ell(u-v)+a\right)q^n$$
$$\equiv q^{-a}\Delta(z)^{\ell t-1}\left(\prod_{n\geq 1}\frac{(1-q^{\mu_1 n})\cdots(1-q^{\mu_s n})}{(1-q^{\lambda_1 n})\cdots(1-q^{\lambda_r n})}\right)^{\ell}\sum_{n\geq 0}c(\ell n)q^n \pmod{\ell}.$$

Applying Proposition (3) of [46] to the right-hand side, we get

$$\sum_{n\geq 0} D\left(\ell n+\delta_\ell(u-v)+a\right)q^n \equiv 0 \pmod{\ell} \iff \sum_{n\geq 0} c(\ell(n-\ell^{t-1})+a)q^n \equiv 0 \pmod{\ell}.$$

Since $c(n) = 0$ for $n < 0$ this proves the lemma. □

5.6 Proof of Theorem 12

By using the results in Sections 5.4 and 5.5, we may apply Proposition 9 to F_ℓ. Notice that the two congruences $w_\ell(F_\ell(z)) \equiv (\ell+1)/2 \pmod{\ell}$ and $w_\ell(F_\ell(z)) \equiv (\ell+3)/2 \pmod{\ell}$ are impossible given that $\ell > \max(5, |r-s|+4)$ and also the fact that $r - s \neq 1, 3$. This shows that if F_ℓ has a Ramanujan congruence, then $b = 0$. Thus, checking that h has no Ramanujan congruences modulo ℓ is equivalent to checking that $\sum_{n\geq 0} D(\ell n)q^n \not\equiv 0 \pmod{\ell}$. In the case that $u \equiv v \pmod{\ell}$, we have that $F_\ell = q^{\ell M}(1+\cdots)$ for some M and therefore $\sum_{n\geq 0} D(\ell n)q^n \not\equiv 0 \pmod{\ell}$.

The case $u \neq v$ and $s - r \in 2\mathbb{N}_0$ is more subtle, but luckily the proof of Theorem 1.2 in [56] works just as well in our situation. For the remainder of Section 5.6, we assume that $u \not\equiv v \pmod{\ell}$ and $s - r \in 2\mathbb{N}_0$. We start by stating Proposition 5.1 of [56].

Proposition 12 (Proposition 5.1 of [56]). *Let $\ell \geq 5$ be prime and $N \geq 4$, $\ell \nmid N$. Suppose that $f(z) \in M_k(\Gamma_1(N))$ has ℓ-integral Fourier coefficients, $w_\ell(f(z)) \not\equiv 0 \pmod{\ell}$, and $\theta f \not\equiv 0 \pmod{\ell}$. Suppose further that $w_\ell(\theta^m f(z)) \geq w_\ell(f(z))$. Let $i_1 < i_2 < \cdots < i_c$ be those $i \in \{0, 1, \ldots, \ell-1\}$ for which $w_\ell(\theta^i f) \equiv 0 \pmod{\ell}$. Write $w_\ell(\theta^{i_j+1} f) = w_\ell(\theta^{i_j} f) + (\ell+1) - s_j(\ell-1)$. Write $k = w_\ell(f)$ and let $k_0 \in \{1, \ldots, \ell-1\}$ be such that $k \equiv -k_0 \pmod{\ell}$. Then one of the four cases below holds:*

(I) $k \equiv 1 \pmod{\ell}$, $c = 1$, $i_1 = \ell - 1$, *and* $s_1 = \ell + 1$.

(II) $k \equiv 2 \pmod{\ell}$, $c = 1$, $i_1 = \ell - 2$, *and* $s_1 = \ell + 1$.

(III) $k \not\equiv 1 \pmod{\ell}$, $c = 2$, $(i_1, i_2) = (k_0, \ell - 1)$, *and* $(s_1, s_2) = (k_0 + 1, \ell - k_0)$.

(IV) $k \not\equiv 1 \pmod{\ell}$, $c = 2$, $(i_1, i_2) = (k_0, \ell - 2)$, *and* $(s_1, s_2) = (k_0 + 2, \ell - k_0 - 1)$.

We have $w_\ell(f) = w_\ell(\theta^{\ell-1} f)$ if and only if case (II) or case (IV) holds.

We proceed by contradiction by assuming $D(\ell n) \equiv 0 \pmod{\ell}$ for all n which implies that $\theta^{\ell-1} F_\ell \equiv F_\ell \pmod{\ell}$. So by the last statement in Proposition 12, we are in cases (II) or (IV). But in case (II) we have

$$w_\ell(F_\ell) = \frac{(s-r)(\ell^2-1)}{2} + 12\ell^t \equiv 2 \pmod{\ell} \iff r - s \equiv 4 \pmod{\ell},$$

which contradicts that $\ell > s - r + 4$. So we are in case (IV). Here, we have

$$k_0 \equiv -k \equiv \frac{s-r}{2} \pmod{\ell}.$$

Using Lemma 2, the identity $w_\ell(\theta^{i_j+1} f) = w_\ell(\theta^{i_j} f) + (\ell+1) - s_j(\ell-1)$ with $s_j = s_1 = k_0 + 2$ becomes

$$\begin{aligned} w_\ell(\theta^{k_0+1} F_\ell) &= w_\ell(F_\ell) + (\ell+1)(k_0+1) - (k_0+2)(\ell-1) \\ &= w_\ell(F_\ell) + 2k_0 + 3 - \ell. \end{aligned} \tag{5.8}$$

Now, $2k_0 \equiv s - r \pmod{\ell}$. Since $s - r$ is even and $\ell > s - r \in \mathbb{N}_0$, we must have that

$$k_0 = \frac{s-r}{2}.$$

Thus, (13) becomes

$$w_\ell(F_\ell) + s - r + 3 - \ell.$$

But $\ell > s - r + 3$, so this implies that

$$w_\ell(\theta^{k_0+1} F_\ell) = w_\ell(F_\ell) + s - r + 3 - \ell < w_\ell(F_\ell),$$

contradicting Proposition 10. This concludes the proof of Theorem 12.

5.7 Proof of Proposition 8

In this section we use modular forms of half-integral weight. See §3.3 of Chapter 3 for background. We follow in the same theme as Ono [46]. We first define the eta-quotient

$$f(z) := \frac{\eta(3z)}{\eta(2z)\eta(z)}\ \eta^{12}(80z) = \sum_{m \geq 0} b(m) q^m.$$

Using the appropriate theorems for eta-quotients (see §3.4 of Chapter 3), one can see that $f \in M^!_{\frac{11}{2}}(\Gamma_1(1440))$. To make this holomorphic at the cusps, we introduce the factor $\eta^5(z)/\eta(5z)$. We make the following definition:

$$F(z) := f(z)\ \frac{\eta^5(z)}{\eta(5z)} = \sum_{m=0}^{\infty} c(m) q^m \in S_{\frac{15}{2}}(\Gamma_1(1440)).$$

Notice $\eta^5(z)/\eta(5z) \equiv 1 \pmod 5$. So, $F(z) \equiv f(z) \pmod 5$. Also, if $a(n)$ for $n \geq 0$ are the Fourier coefficients of $\eta^{12}(80z)/q^{40}$ then we have $\eta^{12}(80z)/q^{40} = 1 + \sum_{m>0} a(80m) q^{80m}$. Thus, if we can show that

$$b(80n) \equiv 0 \pmod 5 \text{ for all } n \geq 1, \tag{5.9}$$

then (5.2) holds. Since $80 \mid 1440$, we have

$$F(z) \mid U(80) = \sum_{n=0}^{\infty} c(80n)q^n \in S_{\frac{15}{2}}(\Gamma_1(1440)).$$

Recall that $b(80n) \equiv c(80n) \pmod 5$. By Sturm's Criterion, we need to verify that $c(80n) \equiv 0 \pmod 5$ for $0 \leq n \leq \frac{15}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(1440)] + 1 = 2161$. We do this by using SAGE and computing the series $\sum_{m\geq 0} b(m)q^m$ up to $80 \cdot 2161 = 172,880$ terms and then focusing on the coefficients whose index is divisible by 80. This proves the first congruence.

The second congruence is proved in a nearly identical fashion. First, we define the eta-quotient

$$g(z) := \frac{\eta(3z)}{\eta(2z)\eta(z)}\eta^6(80z) = \sum_{m\geq 0} \beta(m)q^m \in M^!_{\frac{5}{2}}(\Gamma_1(2880)).$$

Define

$$G(z) := g(z)\frac{\eta^5(z)}{\eta(5z)} \in S_{\frac{9}{2}}(\Gamma_1(2880)).$$

We notice that $G(z) \equiv g(z) \pmod 5$. Also, letting $\alpha(n)$ for $n \geq 0$ be the Fourier coefficients of $\eta^6(80z)/q^{20}$, we have $\eta^6(80z)/q^{20} = q + \sum_{m>0} \alpha(80m)q^{80m}$. Thus, if we can show that

$$\beta(80n) \equiv 0 \pmod 5 \text{ for all } n \geq 1, \tag{5.10}$$

then (5.3) holds. Since $80 \mid 2880$, then we have

$$G(z) \mid U(80) = \sum_{n=0}^{\infty} \gamma(80n)q^n \in S_{\frac{9}{2}}(\Gamma_1(2880)).$$

Hence, $\beta(80n) \equiv \gamma(80n) \pmod 5$. By Sturm's Criterion, we need to verify that $\gamma(80n) \equiv 0 \pmod 5$ for $0 \leq n \leq 9/24[\mathrm{SL}_2(\mathbb{Z} : \Gamma_0(2880)] + 1 = 2593$. We do this by using SAGE and computing the series $\sum_{m\geq 0} \beta(m)q^m$ up to $80 \cdot 2593 = 207440$ terms and then focus on the coefficients whose index is divisible by 80. This completes the proof.

5.8 Example of Proposition 2

In this section, we give an explicit example of Proposition 10 by following the steps outlined in the proof. Let $\ell = 7$ and

$$h(z) = \frac{\eta(3z)}{\eta(2z)\eta(z)}.$$

The corresponding F_7 as defined in (5.6) is

$$F_7(z) = \Delta^{49}(z)\left(\frac{\eta(z)\eta(2z)}{\eta(3z)}\right)^{48},$$

which is a holomorphic modular form for $\Gamma_1(6)$ of weight 612. Note that $[\Gamma_1(6) : \mathrm{SL}_2(\mathbb{Z})] = 24$. We need to find a set of coset representatives for $\Gamma_1(6)$ in $\mathrm{SL}_2(\mathbb{Z})$. The following lemma will make this easier.

Lemma 5. *Let $\gamma, \gamma' \in SL_2(\mathbb{Z})$, and for $(x, y) \in (\mathbb{Z}/N\mathbb{Z})^2$ let $|(x, y)|$ denote the order of (x, y) in $(\mathbb{Z}/N\mathbb{Z})^2$. We have*

$$\Gamma_1(N)\gamma = \Gamma_1(N)\gamma' \iff |(c, d)| = |(c', d')| = N \text{ and } (c, d) \neq (c', d') \text{ in } (\mathbb{Z}/N\mathbb{Z})^2,$$

where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\gamma' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$.

Remark 14. *One needs to be slightly careful when applying the above lemma. For instance, when $N = 6$, $|(5, 5)| = 6$ but since c and d in this case are not relatively prime, then one cannot construct a matrix $\begin{bmatrix} a & b \\ 5 & 5 \end{bmatrix} \in SL_2(\mathbb{Z})$. However, $(5, 5) \equiv (5, -1) \pmod 6$ and this problem is resolved. The fact that this can always be resolved is Lemma 3.8.4 of [22].*

The above lemma states that all we must find is the pairs (c, d) in $(\mathbb{Z}/6\mathbb{Z})^2$ with order 6. Furthermore, by consulting the formulas for computing q-expansions of eta-quotients in [52] and using the fact that we are raising the eta-quotient to the 48th power, we notice that the q-expansions are only dependent on these two matrix entries. There are 24 such elements of $(\mathbb{Z}/6\mathbb{Z})^2$; however, not all of these elements produce unique q-expansions. This is because (c, d) and $(-c, -d)$ give rise to the same expansion and are not equivalent because

$3 \nmid \gcd(c,d) = 1$. Thus, there are twelve expansions to compute. For example,

$$F_7(z) \mid \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \Delta^{51}(z)\frac{1}{2^{24}}q\left[\prod_{n=1}^{\infty}(1-(-q^{1/2})^n)\right]^{48} 3^{24}\zeta_3 q^{-2/3}\left[\prod_{n=1}^{\infty}(1-(\zeta_3 q^{1/3})^n)\right]^{-48}.$$

By multiplying all of these expansions together, we obtain a formula for G as in the proof of Proposition 10,

$$G(z) = \frac{3^{432}}{2^{384}}q^4\frac{\Delta^{1224}(z)\Delta^{16}(2z)}{\Delta^{12}(3z)}\prod_{n=1}^{\infty}\frac{(1-q^{n/2})^{384}(1-(-q^{1/2})^n)^{384}}{(1-q^{n/3})^{288}(1-(\zeta_3 q^{1/3})^n)^{288}(1-(\zeta_3^2 q^{1/3})^n)^{288}},$$

which (as expected) is equal to $C\Delta^{1224}(z)$ where $C = \frac{3^{432}}{2^{384}}$. We will also show that H (as in the proof of Proposition 10 with $m = 1$) does not vanish mod 7. To accomplish this, we use the fact $\theta(F_7 \mid M) \equiv (\theta F_7) \mid M \pmod 7$ which comes from the proof of Lemma 4.2 in [56]. Using SAGE, we obtain

$$H \equiv 2q^{1252} + 5q^{1253} + q^{1254} + 2q^{1255} + 2q^{1256} + \cdots \pmod 7$$

which is clearly not equivalent to 0. We now see that $\widetilde{\mathrm{ord}}_\infty(H) = 1252 \geq 1224$.

Chapter 6

Cusp forms as p-adic limits

The results in this chapter can be found in [31].

6.1 Introduction and statement of results

In [2], S. Ahlgren and D. Samart considered the cusp form with complex multiplication (see Section 3.6 of Chapter 3)

$$g(z) = \eta^2(4z)\eta^2(8z) = q - 2q^5 - 3q^9 + \cdots \in S_2(\Gamma_0(32))$$

as well as the weakly holomorphic modular form

$$F(z) = -g(z)\frac{\eta^6(16z)}{\eta^2(8z)\eta^4(32z)} = \sum_{n \geq -1} C(n)q^n = -\frac{1}{q} + 2q^3 + q^7 + \cdots \in M_2^{\infty}(\Gamma_0(32)),$$

whose only pole lies at the cusp ∞. Here, $\eta(z)$ is Dedekind's eta-function $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. Applying a theorem of P. Guerzhoy, Z. Kent, and K. Ono [29], one has that if $p \equiv 3 \pmod 4$ is a prime for which $p \nmid C(p)$, then as a p-adic limit, we have that

$$\lim_{m\to\infty} \frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} = g.$$

However, Ahlgren and Samart were able to provide a strengthened result in this case, proving that for all primes $p \equiv 3 \pmod 4$ and integers $m \geq 0$, we have

$$v_p(C(p^{2m+1})) = m$$
$$v_p\left(\frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} - g\right) \geq m+1,$$

where $v_p(\cdot)$ denotes the p-adic valuation on $\mathbb{Z}[\![q]\!]$.

Furthermore, Ahlgren and Samart obtained analogous improved results for two other examples involving normalized cusp forms lying in $S_4(\Gamma_0(9))$ and $S_3(\Gamma_0(16), \chi)$, where χ is the non-trivial Dirichlet character modulo 4. In each of the three examples given in [2], the relevant space of cusp forms is one-dimensional and the unique normalized cusp form g in that space has complex multiplication.

Ahlgren and Samart mention that their approach would give similar results for a number of other spaces of modular forms. We resolve this claim in Theorem 13 for all such one-dimensional spaces whose character is trivial.

Theorem 13. *Suppose that $S_k(\Gamma_0(N))$ is one-dimensional and that the unique normalized cusp form*

$$g = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N))$$

has complex multiplication. There exists

$$F = \sum_{n=-1}^{\infty} C(n)q^n \in S_k^{\infty}(\Gamma_0(N))$$

such that for every odd prime p that is inert in the field of complex multiplication for g and every integer $m \geq 0$ we have that

$$v_p(C(p^{2m+1})) = (k-1)m \tag{6.1}$$
$$v_p\left(\frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} - g\right) \geq (k-1)(m+1). \tag{6.2}$$

Remark 15. *Here, we assume that p is an odd prime since this theorem depends on Proposition 17, which may not hold when $p = 2$. However, Proposition 17 and Theorem 13 are both true for the space $S_2(\Gamma_0(27))$. A modified version of Theorem 13 also holds for $S_4(\Gamma_0(9))$, as described in [2, Theorem 4.1].*

Remark 16. *Work of R. Dicks [23] also establishes analogous results for weight 2 newforms with complex multiplication that can be expressed as eta-quotients.*

In order to prove this theorem, for each space $S_k(\Gamma_0(N))$ described above we will construct (see Proposition 15) two families of modular forms with integer coefficients of the form

$$\phi_n = q^{-n} + \sum_m A_n(m)q^m \in M^{\infty}_{2-k}(\Gamma_0(N)) \tag{6.3}$$

$$F_m = -q^{-m} + \sum_n C_m(n)q^n \in S^{\infty}_k(\Gamma_0(N)), \tag{6.4}$$

where the indices n, m are defined for appropriate sets of integers. In particular, if $k = 2$ we consider integers $n \geq 2$ and $m \in \{-1\} \cup \mathbb{N}$, and if $k > 2$ then we consider integers $n \geq 2$ and $m \geq -1$.

These families of modular forms also satisfy a beautiful property known as Zagier duality. For example, for the one-dimensional space $S_2(\Gamma_0(27))$ one can construct the Fourier expansions of a few ϕ_n (where, for convenience, we restrict only to the first few n that are congruent to $2 \pmod 3$).

$$\begin{aligned}
\phi_2 &= q^{-2} + q + 2q^4 - q^7 + q^{10} - q^{13} + O(q^{15}) \\
\phi_5 &= q^{-5} + q + 2q^4 + 7q^7 + 8q^{10} - 10q^{13} + O(q^{15}) \\
\phi_8 &= q^{-8} + 6q + 5q^4 + 14q^7 - 8q^{10} + 30q^{13} + O(q^{15}) \\
\phi_{11} &= q^{-11} - 6q - 8q^4 + 26q^7 + 44q^{10} + 95q^{13} + O(q^{15}) \\
\phi_{14} &= q^{-14} - 7q + 21q^4 - 27q^7 + 21q^{10} + 49q^{13} + O(q^{15})
\end{aligned}$$

One can compare these expansions to those of the first few F_m (where, again for convenience, m is congruent to $1 \pmod 3$) to find that the coefficients that appear are exactly the same (although they are arranged differently).

$$F_1 = -q^{-1} + q^2 + q^5 + 6q^8 - 6q^{11} - 7q^{14} + O(q^{15})$$
$$F_4 = -q^{-4} + 2q^2 + 2q^5 + 5q^8 - 8q^{11} + 21q^{14} + O(q^{15})$$
$$F_7 = -q^{-7} - q^2 + 7q^5 + 14q^8 + 26q^{11} - 27q^{14} + O(q^{15})$$
$$F_{10} = -q^{-10} + q^2 + 8q^5 - 8q^8 + 44q^{11} + 21q^{14} + O(q^{15})$$
$$F_{13} = -q^{-13} - q^2 - 10q^5 + 30q^8 + 95q^{11} + 49q^{14} + O(q^{15})$$

In fact, the coefficients that appear in the q-expansions of these two families are always the same. In particular, we have the following result.

Theorem 14. *For all integers n, m as described above, we have that*

$$C_m(n) = A_n(m).$$

After giving some background in Section 6.2, we show in Section 6.3 that there are only finitely many one-dimensional cusp form spaces with trivial Nebentypus (Proposition 14). In Section 6.4 we build and study the ϕ_n and F_m necessary for proving Theorem 13, and we also prove Theorem 14. These families then provide congruences for the coefficients $C(n)$ of F in Section 6.5. Finally, we prove Theorem 13 in Section 6.6.

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6.2 Background

Recall from Chapter 3 the space $M_k^!(\Gamma_0(N))$ of weakly holomorphic modular forms for $\Gamma_0(N)$. We are particularly interested in its subspace

$$M_k^\infty(\Gamma_0(N)) := \{f \in M_k^!(\Gamma_0(N)) : f \text{ is holomorphic at every cusp except possibly } \infty\},$$

and its cuspidal subspace

$$S_k^{\infty}(\Gamma_0(N)) := \{f \in M_k^!(\Gamma_0(N)) : f \text{ vanishes at every cusp except possibly } \infty\}.$$

Recall the differential operator $\theta := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$ from Chapter 5. It is well-known that

$$\theta^{k-1} : M_{2-k}^{\infty}(\Gamma_0(N), \chi) \to S_k^{\infty}(\Gamma_0(N), \chi).$$

Recall from Chapter 5 the m-th U-operator

$$\sum a(n)q^n \mid U(m) := \sum a(mn)q^n.$$

We also have the V-operator

$$\sum a(n)q^n \mid V(m) := \sum a(n)q^{mn}.$$

Letting $T_k(m)$ be the usual Hecke operator on $M_k^!(\Gamma_0(N))$, we have (see, for example, [21])

$$\sum a(n)q^n \mid T_k(m) = \sum_n \left(\sum_{d|(m,n)} \chi(d)d^{k-1}a(mn/d^2) \right) q^n.$$

In particular, for $f = \sum a(n)q^n$ we have

$$f \mid T_k(p^n) = \sum_{j=0}^{n} \chi(p^j)p^{(k-1)j} f \mid U(p^{n-j}) \mid V(p^j).$$

It is well-known that if $(m, N) = 1$ then

$$T_k(m) : M_k^{\infty}(\Gamma_0(N), \chi) \to M_k^{\infty}(\Gamma_0(N), \chi),$$

and in fact $T_k(m)$ also preserves the subspace $S_k^{\infty}(\Gamma_0(N), \chi)$.

We restrict our attention to the spaces $S_k(\Gamma_0(N))$ that are one-dimensional and spanned by a normalized cusp form g having complex multiplication by a quadratic field K with fundamental discriminant $D < 0$. This essentially means that the coefficients of g are supported on those $n \in \mathbb{Z}$ for which $\left(\frac{D}{n}\right) = 1$. See Section 3.6 of Chapter 3 for a more detailed account of CM forms.

Recall the definition of *filtration* in Section 5.2 of Chapter 5. The following fact can be found in Section 1 of [38].

Proposition 13. *If $f \in M_k(\Gamma_0(N))$ has p-integral rational coefficients and $w_p(f) \neq -\infty$, then $w_p(f) \equiv k \pmod{p-1}$. Moreover, $w_p(f \mid V(p)) = pw_p(f)$.*

6.3 Reducing to a finite list of (k, N)

First, we will describe the set of all pairs (k, N) where $S_k(\Gamma_0(N))$ is one-dimensional and spanned by a cusp form with complex multiplication. First, we have that (as in, for example, [21, Theorem 3.5.1])

$$\dim(S_k(\Gamma_0(N))) = \begin{cases} (k-1)(g_N - 1) + \lfloor \frac{k}{4} \rfloor \varepsilon_2 + \lfloor \frac{k}{3} \rfloor \varepsilon_3 + (\frac{k}{2} - 1)\varepsilon_\infty & \text{if } k \geq 4 \\ g_N & \text{if } k = 2 \\ 0 & \text{if } k \leq 0 \end{cases},$$

where g_N is the genus of the modular curve $X(\Gamma_0(N))$, ε_2 is the number of elliptic points with period 2, ε_3 is the number of elliptic points with period 3, and ε_∞ is the number of cusps. So, to find the set of all one-dimensional spaces $S_k(\Gamma_0(N))$, we need only consider the levels N for which $g_N \leq 1$.

The number of congruence subgroups of a fixed genus is known to be finite, and work of D.A. Cox and W.R. Parry gives explicit bounds on the level (see [15, 17]), proving that

$$N \leq \begin{cases} 168 & \text{if } g_N = 0 \\ 12g_N + \frac{1}{2}(13\sqrt{48g_N + 121}) + 145) & \text{if } g_N \geq 1 \end{cases}.$$

Thus, we need only consider (k, N) where $N \leq 12 + \frac{1}{2}(13\sqrt{48 + 121}) + 145) = 241.5$. Finally, after a short Sage calculation, it follows that the only pairs (k, N) such that $S_k(\Gamma_0(N))$ is one-dimensional and spanned by a cusp form with complex multiplication are

$$(2, 27), (2, 32), (2, 36), (2, 49), \text{and } (4, 9).$$

This proves the following proposition.

Proposition 14. *The only one-dimensional spaces $S_k(\Gamma_0(N))$ that are spanned by a cusp form with complex multiplication are*

$$S_2(\Gamma_0(27)), S_2(\Gamma_0(32)), S_2(\Gamma_0(36)), S_2(\Gamma_0(49)) \textit{and } S_4(\Gamma_0(9)).$$

Remark 17. *In particular, there are only five normalized cusp forms $g \in S_k(\Gamma_0(N))$ that have complex multiplication and $\dim(S_k(\Gamma_0(N))) = 1$. Thus, in order to prove Theorem 13, it suffices to consider these five cases. These five cusp forms are listed in Table 1, together with their LMFDB labels and field $K = \mathbb{Q}(\sqrt{D})$ of complex multiplication (where $D < 0$ is the fundamental discriminant of K) [43].*

Remark 18. *Note that S. Ahlgren and D. Samart [2] proved Theorem 13 for $g \in S_2(\Gamma_0(32))$ and $g \in S_4(\Gamma_0(9))$.*

6.4 Defining ϕ_n and F_m

First, we recall the following well-known fact (see, for example, the proof of Corollary 2.4 of [25] or Lemma 2.1 of [37]).

Lemma 6. *If $f \in S_2^{\infty}(\Gamma_0(N))$ then the constant term of f must vanish.*

For each pair (k, N) identified in Proposition 14, we can now define the two families of modular forms described in equations (6.3) and (6.4).

Proposition 15. *For each space $S_k(\Gamma_0(N))$ identified in Proposition 14, we have the following.*

(a) For all integers $n \geq 2$ there exists $\phi_n \in M_{2-k}^{\infty}(\Gamma_0(N)) \cap \mathbb{Z}(\!(q)\!)$ of the form

$$\phi_n = q^{-n} + A_n(-1)q^{-1} + \sum_{m=1}^{\infty} A_n(m)q^m$$

if $k = 2$ and of the form

$$\phi_n = q^{-n} + \sum_{m=-1}^{\infty} A_n(m)q^m$$

if $k = 4$.

(b) For $m = -1$ and for all integers $m \geq 1$ there exists a unique $F_m \in S_k^{\infty}(\Gamma_0(N)) \cap \mathbb{Z}(\!(q)\!)$ of the form

$$F_m = -q^{-m} + \sum_{n=2}^{\infty} C_m(n)q^n.$$

If $k = 4$, then a unique F_m of the given form exists for all integers $m \geq -1$.

(c) Define $F := F_1 = \sum_{n=-1}^{\infty} C(n)q^n$. If p is prime such that $p \nmid N$, then we have for each $n \geq 0$ that

$$F \mid T_k(p^n) = p^{(k-1)n} F_{p^n} + C(p^n)g.$$

Proof. (a) For $(k, N) = (2, 32)$, note that the modular functions $\phi_n \in M_0^{\infty}(\Gamma_0(32)) \cap \mathbb{Z}(\!(q)\!)$ are constructed by A. El-Guindy and K. Ono in [25, Lemma 2.3]. We will construct the ϕ_n similarly for $(k, N) \in \{(2, 27), (2, 36), (2, 49)\}$. First, we explicitly define ϕ_2 and ϕ_3 in those cases as in Table 2.

Note that each eta-quotient defined here is an element of $M_0^{\infty}(\Gamma_0(N))$ (see, for example, [46, Theorems 1.64 and 1.65]). Then, the modular functions $\phi_4, \phi_5, \ldots$ can be defined inductively as the appropriate polynomials in ϕ_2, ϕ_3 to obtain the desired principal parts.

For example, when $(k, N) = (2, 49)$, we have

$$\phi_4 := \phi_2^2 + 2\phi_3 - \phi_2 = q^{-4} + q^{-1} + q^3 - q^5 + q^6 + \cdots$$

$$\phi_5 := \phi_3\phi_2 + \phi_2^2 + 2\phi_3 - \phi_2 - 3 = q^{-5} - q + 2q^2 - q^4 + q^8 + 2q^9 + \cdots$$

$$\phi_6 := \phi_2^3 + 3\phi_3\phi_2 + \phi_3 - 3 = q^{-6} + 2q + q^2 + q^4 + 3q^8 + \cdots,$$

etc. This completes the proof of (a) when $k = 2$.

If $(k, N) = (4, 9)$, note that we cannot define $\phi_n \in M^{\infty}_{-2}(\Gamma_0(9)) \cap \mathbb{Z}((q))$ in quite the same way because they are not modular functions. Instead, we follow the proof of Lemma 4.3 of [2] (with the appropriate modifications to consider all positive integers $n \geq 2$). Set

$$\phi_2 := \frac{\eta^2(3z)}{\eta^6(9z)} = q^{-2} - 2q - q^4 - 8q^7 + \cdots \in M^{\infty}_{-2}(\Gamma_0(9)) \cap \mathbb{Z}((q))$$
$$L := \frac{\eta^3(z)}{\eta^3(9z)} + 3 = q^{-1} + 5q^2 - 7q^5 + 3q^8 + \cdots \in M^{\infty}_{0}(\Gamma_0(9)) \cap \mathbb{Z}((q)).$$

Then, the modular forms $\phi_3, \phi_4, \ldots$ can be defined inductively as the appropriate integer linear combinations of $\phi_2, \phi_2 L, \phi_2 L^2, \ldots$ to obtain the desired principal parts. For example, we have that

$$\phi_3 := \phi_2 L = q^{-3} + 3 - 18q^3 + 20q^6 + 45q^9 + \cdots$$
$$\phi_4 := \phi_2 L^2 = q^{-4} + 8q^{-1} - 10q^2 - 88q^5 + 295q^8 + \cdots$$
$$\phi_5 := \phi_2 L^3 - 13\phi_2 = q^{-5} + 49q - 178q^4 - 140q^7 + \cdots ,$$

etc.

(b) We first build the $F_m \in S^{\infty}_{2}(\Gamma_0(N)) \cap \mathbb{Z}((q))$ for all of the weight 2 spaces. In each such case, set $F_{-1} := -g$. Then, to inductively construct the F_m for $m \geq 1$ we consider $F_{-1}\phi_{m+1} \in S^{\infty}_{2}(\Gamma_0(N)) \cap \mathbb{Z}((q))$. Note that by Lemma 6 the constant coefficient of $F_{-1}\phi_{m+1}$ is zero. Inductively, we may add the appropriate linear combinations of the previous F_r, for $r < m$, to get rid of the undesired terms in the principal part of $F_{-1}\phi_{m+1}$. For example, when $(k, N) = (2, 49)$ we have

$$F_1 := F_{-1}\phi_2 + F_{-1} = -q^{-1} - q^3 - q^5 + 2q^6 + \cdots$$
$$F_2 := F_{-1}\phi_3 - F_1 + F_{-1} = -q^{-2} + q^3 + 2q^5 + q^6 + \cdots$$
$$F_3 := F_{-1}\phi_4 - F_2 - F_{-1} = -q^{-3} + q^2 + q^4 - q^8 - 4q^9 + \cdots$$
$$F_4 := F_{-1}\phi_5 - F_3 + F_1 = -q^{-4} + q^3 - q^5 + q^6 + \cdots ,$$

etc.

When $(k, N) = (4, 9)$, we again must adjust our argument slightly since the modular forms ϕ_m have weight -2. First, set $F_{-1} := -g$. Then, the F_m for $m \geq 0$ can be constructed inductively by adding the appropriate linear combinations of the previous F_r, for $r < m$, to $F_{-1}L^{m+1}$.

Uniqueness is proved for all five spaces simultaneously: if $G_m = q^{-m} + O(q^2) \in S_k^{\infty}(\Gamma_0(N))$ then $F_m - G_m = O(q^2) \in S_k(\Gamma_0(N))$. Since the CM form $g = q + O(q^2)$ spans the one-dimensional space $S_k(\Gamma_0(N))$, we must have $F_m = G_m$.

(c) First observe that

$$F \mid T_k(p^n) = -p^{n(k-1)}q^{-p^n} + C(p^n)g + O(q^2).$$

Hence

$$F \mid T_k(p^n) - p^{n(k-1)}F_{p^n} = C(p^n)g + O(q^2) = C(p^n)g,$$

the last equality following from $\dim S_k(\Gamma_0(N)) = 1$. This gives the desired identity.

□

We now prove Theorem 14.

Proof of Theorem 14. Note that for m, n as given above, we have that $F_m\phi_n \in S_2^{\infty}(\Gamma_0(N))$. By Lemma 6, it follows that $F_m\phi_n$ has constant term

$$0 = C_m(n) - A_n(m),$$

as desired. □

Remark 19. *Note that Theorem 14 guarantees that the ϕ_n are uniquely defined, since the F_m are unique.*

Remark 20. *By Theorem 14 we have that*

$$A_n(-1) = C_{-1}(n) = a(n).$$

Thus, if p is a prime that is inert in the field of complex multiplication then $A_p(-1) = 0$.

6.5 Congruence results for $C(p^{2m+1})$

Proposition 16. *For each of the spaces $S_k(\Gamma_0(N))$ listed in Proposition 14, if p is an inert prime in the field of complex multiplication for the corresponding newform g and $m \geq 0$ is an integer, we have*

$$C(p^{2m+1}) \equiv (-1)^m p^{(k-1)m} C(p) \pmod{p^{(k-1)(m+1)}}.$$

Proof. By Remark 20, if p is an inert prime in the CM field then ϕ_p has the form $\phi_p = q^{-p} + \sum_{n\geq 1} A_p(n)q^n$. Theorem 14 implies that $A_p(1) = C(p)$. We thus have

$$\theta^{k-1}(\phi_p) = -p^{k-1}q^{-p} + C(p)q + O(q^2) \in S_k^\infty(\Gamma_0(N)).$$

On the other hand, part (c) of Proposition 15 gives

$$F \mid T_k(p) = -p^{k-1}q^{-p} + C(p)q + O(q^2) \in S_k^\infty(\Gamma_0(N)),$$

and so

$$F \mid T_k(p) = \theta^{k-1}(\phi_p).$$

That is,

$$F \mid U(p) = \theta^{k-1}(\phi_p) - p^{k-1} F \mid V(p).$$

Applying $U(p^2)$ to both sides and arguing inductively, we obtain for each $m \geq 0$

$$F \mid U(p^{2m+1}) = \sum_{j=0}^{m} (-1)^{m-j} p^{(k-1)(m-j)} \theta^{k-1}(\phi_p) \mid U(p^{2j}) + (-1)^{m+1} p^{(k-1)(m+1)} F \mid V(p).$$

For any $j \geq 0$ we have $\theta^{k-1}(\phi_p) \mid U(p^{2j}) \equiv 0 \pmod{p^{(k-1)2j}}$. Hence for $m \geq 0$,

$$F \mid U(p^{2m+1}) \equiv (-1)^m p^{(k-1)m} \theta^{k-1}(\phi_p) \pmod{p^{(k-1)(m+1)}}.$$

The result follows by comparing coefficients of q above. $\square$

Proposition 17. *For each space $S_k(\Gamma_0(N))$ listed in Proposition 14, if p is an odd, inert prime in the field of complex multiplication for the corresponding newform g, then $p \nmid C(p)$.*

Proof. When $N \in \{9, 32\}$ we refer the reader to [2, Lemma 3.3, Lemma 4.4]. For $N \in \{27, 36, 49\}$ we argue analogously. For each such N, suppose $p \mid C(p)$. Proposition 15(c) asserts that

$$\theta(\phi_p) = F \mid T_2(p) = pF_p + C(p)g \equiv 0 \pmod{p}.$$

Thus we have

$$\phi_p \equiv q^{-p} + \sum_{n=1}^{\infty} A_p(np)q^{np} \pmod{p}.$$

We now consider each case separately.

First let $N = 27$. By using Sage (for example) to inspect a basis for $M_2(\Gamma_0(27))$, one can see that there exists an element

$$f(z) := \frac{\eta^6(27z)}{\eta^2(9z)} = q^6 + 2q^{15} + O(q^{24}) \in M_2(\Gamma_0(27)).$$

Then $f^p \in M_{2p}(\Gamma_0(27))$ has the form

$$f^p \equiv \sum_{n=6}^{\infty} B_p(np)q^{np} \equiv q^{6p} + \cdots \pmod{p}.$$

As $\phi_p \in M_0^{\infty}(\Gamma_0(27))$, we find that $h_p := \phi_p f^p \in M_{2p}(\Gamma_0(27))$ has the form

$$h_p \equiv \sum_{n=5}^{\infty} D_p(np)q^{np} \equiv q^{5p} + \cdots \pmod{p}.$$

Hence

$$h_p \equiv h_p \mid U(p) \mid V(p) \pmod{p}.$$

By Proposition 13, we get

$$w_p(h_p) = pw_p(h_p \mid U(p)).$$

As $w_p(h_p) \equiv 2p \pmod{p-1}$ by Proposition 13 and $p \mid w_p(h_p)$, we have that $w_p(h_p) = 2p$, so that $w_p(h_p \mid U(p)) = 2$. So there exists $h_0 \in M_2^{(p)}(\Gamma_0(27))$ such that

$$h_0 \equiv h_p \mid U(p) = q^5 + O(q^6) \pmod{p}.$$

But by examining a basis for the six-dimensional space $M_2(\Gamma_0(27))$, we find that there is no such h_0 of this form. This completes the proof of the claim for $N = 27$.

Now let $N = 36$. Following a similar argument as above, define

$$f(z) = q^{12} - 2q^{18} + 3q^{24} + O(q^{30}) \in M_2(\Gamma_0(36)).$$

Then $f^p \in M_{2p}(\Gamma_0(36))$ has the form

$$f^p \equiv \sum_{n=12}^{\infty} B_p(np)q^{np} \equiv q^{12p} + \cdots \pmod{p}.$$

Since $\phi_p \in M_0^{\infty}(\Gamma_0(36))$, we have that $h_p := \phi_p f^p \in M_{2p}(\Gamma_0(36))$ and has the form

$$h_p \equiv \sum_{n=11}^{\infty} D_p(np)q^{np} \equiv q^{11p} + \cdots \pmod{p}.$$

Then $h_p \mid U(p) \mid V(p) \equiv h_p \pmod{p}$. By Proposition 13, we get

$$w_p(h_p) = p w_p(h_p \mid U(p)).$$

Since $w_p(h_p) \equiv 2p \pmod{p-1}$ and $p \mid w_p(h_p)$, we have $w_p(h_p) = 2p$. Hence $w_p(h_p \mid U(p)) = 2$. So there exists $h_0 \in M_2^{(p)}(\Gamma_0(36))$ such that

$$h_0 \equiv h_p \mid U(p) = q^{11} + O(q^{12}) \pmod{p}.$$

But examining a basis of $M_2(\Gamma_0(36))$, we find that no such form h_0 can exist. This completes the proof of the claim for $N = 36$.

Let $N = 49$. If $p = 3$ then $3 \nmid C(3) = -1$ (see the last paragraph on filtrations in §6.2; namely the condition $p \nmid 6N$). Let $p > 3$ be inert. There exists an element

$$f(z) = q^{29} + q^{30} - q^{32} + O(q^{36}) \in M_8(\Gamma_0(49))$$

so that

$$f^p \equiv \sum_{n=29}^{\infty} B_p(np)q^{np} \equiv q^{29p} + \cdots \pmod{p}.$$

Then $h_p := \phi_p f^p \in M_{8p}(\Gamma_0(49))$ has the form

$$h_p \equiv \sum_{n=28}^{\infty} D_p(pn)q^{pn} \equiv q^{28p} + \cdots \pmod{p}.$$

and so $h_p \equiv h_p \mid U(p) \mid V(p) \pmod{p}$. As before, Proposition 13 shows us that

$$w_p(h_p) = pw_p(h_p \mid U(p)).$$

Since $w_p(h_p) \equiv 8p \pmod{p-1}$ and $p \mid w_p(h_p)$, we have $w_p(h_p) = 8p - ap(p-1)$ for some integer $a \geq 0$. As $p > 9$, we must have $a = 0$. Hence $w_p(h_p) = 8p$, so that $w_p(h_p \mid U(p)) = 8$. Thus there exists $h_0 \in M_8^{(p)}(\Gamma_0(49))$ such that

$$h_0 \equiv h_p \mid U(p) = q^{28} + O(q^{29}) \pmod{p}.$$

But by examining a basis for the 36-dimensional space $M_8(\Gamma_0(49))$, we find that no such h_0 exists. This completes the proof. □

6.6 Proof of Theorem 13

Equation (6.1) of Theorem 13 follows from Propositions 16 and 17. To prove equation (6.2), note that from part (c) of Proposition 15 it follows that

$$\frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} - g = \frac{1}{C(p^{2m+1})}\left(p^{(k-1)(2m+1)}F_{p^{2m+1}} - \sum_{j=1}^{2m+1} p^{(k-1)j}F \mid U(p^{2m+1-j}) \mid V(p^j)\right). \tag{6.5}$$

Observe

$$F \mid T(p^{2m}) = \sum_{j=1}^{2m+1} p^{(k-1)(j-1)}F \mid U(p^{2m+1-j}) \mid V(p^{j-1})$$

and on the other hand

$$F \mid T(p^{2m}) = p^{(k-1)2m}F_{p^{2m}} + C(p^{2m})g = p^{(k-1)2m}F_{p^{2m}}$$

by part (c) of Proposition 15 together with the fact that $C(p^{2m}) = 0$ since p is inert. Therefore, (6.5) becomes

$$\frac{F \mid U(p^{2m+1})}{C(p^{2m+1})} - g = \frac{1}{C(p^{2m+1})}\left(p^{(k-1)(2m+1)}F_{p^{2m+1}} - p^{(k-1)(2m+1)}F_{p^{2m}} \mid V(p)\right). \tag{6.6}$$

By Proposition 16, this completes the proof.

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Appendix

A List of tables

Table 1: Cusp forms g with CM that span $S_k(\Gamma_0(N))$

(k,N)	$g=\sum a(n)q^n \in S_k(\Gamma_0(N))$	LMFDB	$K=\mathbb{Q}(\sqrt{D})$
$(2,27)$	$g=\eta(3z)^2\eta(9z)^2=q-2q^4-q^7+\cdots$	27.2.a.a	$\mathbb{Q}(\sqrt{-3})$
$(2,32)$	$g=\eta(4z)^2\eta(8z)^2=q-2q^5-3q^9+\cdots$	32.2.a.a	$\mathbb{Q}(\sqrt{-4})$
$(2,36)$	$g=\eta(6z)^4=q-4q^7+\cdots$	36.2.a.a	$\mathbb{Q}(\sqrt{-3})$
$(2,49)$	$g=q+q^2-q^4-3q^8-3q^9+\cdots$	49.2.a.a	$\mathbb{Q}(\sqrt{-7})$
$(4,9)$	$g=\eta(3z)^8=q-8q^4+20q^7+\cdots$	9.4.a.a	$\mathbb{Q}(\sqrt{-3})$

Table 2: Defining $\phi_2,\phi_3\in M^{\infty}_{2-k}(\Gamma_0(N))$ when $k=2$

(k,N)	$\phi_2,\phi_3\in M^{\infty}_{2-k}(\Gamma_0(N))$
$(2,27)$	$\phi_2:=\dfrac{\eta^4(9z)}{\eta(3z)\eta^3(27z)}=q^{-2}+q+2q^4-q^7+\cdots$ $\phi_3:=\dfrac{\eta^3(3z)}{\eta^3(27z)}+3=q^{-3}+5q^6+\cdots$
$(2,32)$	$\phi_2:=\dfrac{\eta^6(16z)}{\eta^2(8z)\eta^4(32z)}=q^{-2}+2q^6+\cdots$ $\phi_3:=\dfrac{\eta^4(8z)\eta^2(16z)}{\eta^2(4z)\eta^4(32z)}=q^{-3}+2q+q^5+2q^9+\cdots$
$(2,36)$	$\phi_2:=\dfrac{\eta(12z)\eta^3(18z)}{\eta(6z)\eta^3(36z)}=q^{-2}+q^4+\cdots$ $\phi_3:=\dfrac{\eta^3(9z)\eta(12z)}{\eta(3z)\eta^3(36z)}-1=q^{-3}+2q^3+q^9+\cdots$
$(2,49)$	$\phi_2:=\dfrac{\eta(z)}{\eta(49z)}+1=q^{-2}-q^{-1}+q^3+q^5+\cdots$ $\phi_3:=\phi_2 \mid T_0(2)-\frac{1}{2}\phi_2^2+\phi_2=q^{-3}-q+q^2+q^4-q^9+\cdots$

Vita

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