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LARGE DEVIATIONS FOR SELF INTERSECTION LOCAL TIMES OF ORNSTEIN-UHLENBECK PROCESSES

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To the Graduate Council:

I am submitting herewith a dissertation written by Apostolos Gournaris entitled "LARGE DEVIATIONS FOR SELF INTERSECTION LOCAL TIMES OF ORNSTEIN-UHLENBECK PROCESSES." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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LARGE DEVIATIONS FOR SELF INTERSECTION LOCAL TIMES OF ORNSTEIN-UHLENBECK PROCESSES

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I dedicate this work to my parents, Dimitrios and Athanasia,
and my brother, Theocharis-Panagiotis.

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Abstract

In the area of large deviations, people concern about the asymptotic computation of small probabilities on an exponential scale. The general form of large deviations can be roughly described as:

$$\mathbb{P}\{Y_n \in A\} \approx \exp\{-b_n I(A)\} \qquad (n \to \infty),$$

for a random sequence $\{Y_n\}$, a positive sequence b_n with $b_n \to \infty$, and a coefficient $I(A) \geq 0$. In applications, we often concern about the probability that the random variables take large values, that is we concern about the $\mathbb{P}\{Y_n \geq \lambda\}$, where $\lambda > 0$.

Here, we consider the Ornstein-Uhlenbeck process, study the properties of the local times and self intersection local times of that process, and focus on the large deviations for the self intersection local times of the Ornstein-Uhlenbeck process. A function that we need to study and plays an important role in large deviations is called logarithmic moment generating function.

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Chapter 1

Introduction

The Ornstein – Uhlenbeck process is a stochastic process with applications in financial mathematics and the physical sciences. For instance, its original application in physics was as a model for the velocity of a massive Brownian particle under the influence of friction. Furthermore, it is one of several approaches used to model interest rates, currency exchange rates, and commodity prices stochastically. It is named after Leonard Ornstein and George Eugene Uhlenbeck.

The Ornstein - Uhlenbeck process X_t is defined by the following stochastic differential equation:

$$\mathrm{d}X_t = \mathrm{d}B_t - kX_t\,\mathrm{d}t,$$

where B_t denotes the Brownian motion (more information about the Brownian motion can be found in [14] or in [11], for instance) and k is a positive real number (constant). Here we assume that: $X_0 = 0$. In the second chapter, we apply $It\hat{o}'s$ formula to find the solution of the above equation, and we also compute the expectation and the variance of X_t.

Some of the basic properties of the Ornstein-Uhlenbeck process is that it is a Gaussian and Markov process and it does not have independent increments.

Our goal is to evaluate limits of the following form, because that kind of limits helps us to understand the intensity that the random path of the Ornstein-Uhlenbeck process intersects itself,

$$\Lambda_p(\theta) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{\theta\Big(\int\limits_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\right\},$$

where $\theta > 0$, $L(t,y)$ measures the time that our process X_t spends near to y ($y \in \mathbb{R}$ and $t > 0$) during the period $[0,t]$, and it is called local time, and $\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y$ is the p-multiple self intersection local time that measures the intensity of the path intersections of the process X_t. The function $\Lambda_p(\cdot)$ is called the logarithmic moment generating function.

The study of the self intersection local times of stochastic processes has its own mathematical interest. Apart from that interest, it has been also motivated by Physics. Physicists concern about the geometric shape of the polymer which is often described by a suitable random path. The geometry of the polymer is decided by the intensity that the random path intersects itself.

The logarithmic moment generating function is useful in many applications. For instance, the large deviation principle (which can be found in [3] or in [12], for instance) suggests that under certain conditions we have:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\int\limits_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y \geq \lambda t^p\Big) = -\sup_{\theta>0}\left\{\theta\lambda^{\frac{1}{p}} - \Lambda_p(\theta)\right\},$$

where $\lambda > 0$.

The previous equality means that, the probability that the p-multiple self intersection local time is proportional to the volume of $[0,t]^p$, decreases exponentially (for large t).
The logarithmic moment generating function appears on the right hand side of the previous equality. That is, this function helps us to understand the intensity of the path intersections of a process. This is one of the reasons that motivates us to study the logarithmic moment

generating function.

Here is a summary of what we do in the next chapters:

First we define the local time $L(t, \cdot)$, investigate its existence (Chapter 2), and study some basic properties of local times and self intersection local times, as the continuity (Chapter 3), and the exponential integrability (Chapter 4). Then we provide our large deviation results in Chapter 5. Finally, we include some of our future interests in Chapter 6.

Chapter 2

Local Times

2.1 Definitions

Let $\Big(Z(t), t \in \mathbb{R}^+\Big)$ be a measurable stochastic process taking values in $\mathbb{R}$. For any $t \in \mathbb{R}^+$ the random measure :

$$\mu_t(B) = \int_0^t \mathbb{1}_B(Z_s)\,\mathrm{d}s = \lambda\Big(\{s \in [0,t] : Z(s) \in B\}\Big),$$

where $B \subseteq \mathbb{R}$ and λ is the Lebesgue measure in $\mathbb{R}$, is known as the *occupation measure* of the stochastic process Z_t. The value of $\mu_t(B)$ describes the amount of time in the set $[0,t]$ that the process spends in the set B.

If with probability 1, μ_t is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}$, then we say that the process has a local time over the set $[0,t]$. In particular, the associated density $L(t,x)$ is called local time or occupation density of Z_t. The local time is a version of Radon-Nykodim derivative:

$$L(t,x) = \frac{\mathrm{d}\mu_t}{\mathrm{d}\lambda}(x).$$

(More information about the Radon-Nykodim derivative can be found in [13], for instance). $L(t,x)$ measures the time that Z_t spends near x during the period $[0,t]$. From now we consider that Z_t is our 1 dimensional Ornstein-Uhlenbeck process X_t.

2.2 Existence of local times

Consider the measure:

$$\mu(B) = \int_0^\infty e^{-s} \mathbb{1}_B(X_s)\, \mathrm{d}s = \int_0^\infty e^{-s} \delta_{X_s}(B)\, \mathrm{d}s,$$

where: $\delta_{X_s}(B) = 1$ if $X_s \in B$ and $\delta_{X_s}(B) = 0$ if $X_s \notin B$.
(Note about the notation: We sometimes use e^x and sometimes $\exp\{x\}$ for the exponential function, depending on which notation is more readable on every particular case.)

Notice that:

$$\mu_t(B) \leq e^t \mu(B).$$

Indeed,

$$\begin{aligned}\mu_t(B) &= e^t \int_0^t e^{-t} \mathbb{1}_B(X_s)\, \mathrm{d}s \\ &\leq e^t \int_0^t e^{-s} \mathbb{1}_B(X_s)\, \mathrm{d}s \\ &\leq e^t \int_0^\infty e^{-s} \mathbb{1}_B(X_s)\, \mathrm{d}s.\end{aligned}$$

Hence,

$$\mu_t(B) \leq e^t \mu(B).$$

We know that the measure μ_t is absolute continuous with respect to Lebesgue measure λ if for every measurable set B with $\lambda(B) = 0$ we have $\mu_t(B) = 0$.
We observe that $\mu(B) = 0$ implies $\mu_t(B) = 0$, hence it's enough to show that μ, is absolute continuous with respect to the Lebesgue measure.

By using the Plancherel-Parseval theorem (which can be found in Theorem B.3 Appendix, [3], for instance), it is sufficient to prove that :

$$\int_{-\infty}^{\infty} \mathbb{E}|\hat{\mu}(\lambda)|^2 \,\mathrm{d}\lambda < \infty,$$

where the notation $\hat{\mu}(\cdot)$ refers to the Fourier transform.

We have,

$$\hat{\mu}(\lambda) = \int_0^{\infty} e^{-s} e^{i\lambda X_s} \,\mathrm{d}s,$$

because the Fourier transform of μ is given by:

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} \mu(\mathrm{d}x),$$

and also here we used that:

$$\int_{-\infty}^{\infty} e^{i\lambda x} \delta_{X_s}(\mathrm{d}x) = e^{i\lambda X_s}.$$

Hence,

$$\begin{aligned} |\hat{\mu}(\lambda)|^2 &= \Big(\int_0^{\infty} e^{-t} e^{i\lambda X_t} \,\mathrm{d}t \Big) \Big(\int_0^{\infty} e^{-s} e^{-i\lambda X_s} \,\mathrm{d}s \Big) \\ &= \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} e^{i\lambda (X_t - X_s)} \,\mathrm{d}s \,\mathrm{d}t. \end{aligned}$$

So,

$$\mathbb{E}|\hat{\mu}(\lambda)|^2 = \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} \mathbb{E} e^{i\lambda (X_t - X_s)} \,\mathrm{d}s \,\mathrm{d}t.$$

Note: In the previous step we used Fubini's theorem (which can be found in [7], for instance) to "pass" the expectation inside the integrals.

In order to be able to apply Fubini's theorem and switch the integrals, we need to make sure that the below integral is finite:

$$\int_{\Omega}\int_{0}^{\infty}\int_{0}^{\infty} |e^{-(s+t)}e^{i\lambda(X_t-X_s)}| \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\mathbb{P} \leq \int_{\Omega}\int_{0}^{\infty}\int_{0}^{\infty} e^{-(s+t)}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\mathbb{P} = \int_{\Omega}\mathrm{d}\mathbb{P} = 1 < \infty.$$

We now need to evaluate the expectation: $\mathbb{E}e^{i\lambda(X_t-X_s)}$. By using $It\hat{o}'s$ formula (which can be found in [8] or in [13], for instance), we compute the solution of the stochastic differential equation:

$$\mathrm{d}X_t = \mathrm{d}B_t - kX_t\,\mathrm{d}t$$

as following:

$$X_t = \int_0^t e^{-k(t-u)}\,\mathrm{d}B_u,$$

with $\mathbb{E}(X_t) = 0$ and $\mathbb{V}ar(X_t) = \frac{1}{2k}(1-e^{-2kt})$.
Indeed, we consider the function $g(t,x) = e^{kt}x$ ($x \in \mathbb{R}$, $t > 0$) and applying $It\hat{o}'s$ formula we get:

$$\begin{aligned}\mathrm{d}(e^{kt}X_t) =& ke^{kt}X_t\,\mathrm{d}t + e^{kt}\,\mathrm{d}X_t\\ =& e^{kt}\,\mathrm{d}B_t,\end{aligned}$$

where in the last step we used the above differential equation.
Hence,

$$X_t = \int_0^t e^{-k(t-u)}\,\mathrm{d}B_u.$$

Furthermore,

$$\mathbb{E}(X_t) = 0,$$

because of the properties of the $It\hat{o}'s$ integral [8], and

$$\mathbb{V}ar(X_t) = \mathbb{E}[\int_0^t e^{-k(t-u)}\,\mathrm{d}B_u]^2 = \int_0^t e^{-2k(t-u)}\,\mathrm{d}u = \frac{1}{2k}(1 - e^{-2kt}),$$

where in the last step we used $It\hat{o}'s$ Isometry [8].

Thus,

$$\mathbb{E}[X_t - X_s] = 0.$$

Furthermore,

$$\begin{aligned}\mathbb{E}[X_t - X_s]^2 =& \mathbb{E}[\int_0^t e^{-k(t-u)}\,\mathrm{d}B_u - \int_0^s e^{-k(s-u)}\,\mathrm{d}B_u]^2\\ =& \frac{1}{2k}(1 - e^{-2kt})\\ &+ \frac{1}{2k}(1 - e^{-2ks}) - \frac{1}{k}(e^{-k|t-s|} - e^{-k(t+s)}).\end{aligned}$$

Indeed, by using the $It\hat{o}'s$ Isometry:

$$\mathbb{E}[\int_0^t e^{-k(t-u)}\,\mathrm{d}B_u]^2 = \frac{1}{2k}(1 - e^{-2kt}),$$

and

$$\mathbb{E}[\int_0^s e^{-k(s-u)}\,\mathrm{d}B_u]^2 = \frac{1}{2k}(1 - e^{-2ks}).$$

Also for the expectation:

$$\mathbb{E}[(\int_0^t e^{-k(t-u)}\,\mathrm{d}B_u)(\int_0^s e^{-k(s-u)}\,\mathrm{d}B_u)],$$

if $s < t$ the above expectation is equal to:

$$\begin{aligned}\mathbb{E}[(\int_0^s e^{-k(t-u)}\,\mathrm{d}B_u + \int_s^t e^{-k(t-u)}\,\mathrm{d}B_u)(\int_0^s e^{-k(s-u)}\,\mathrm{d}B_u)] =& e^{-k(t+s)}E[\int_0^s e^{ku}\,\mathrm{d}B_u]^2\\ =& \frac{1}{2k}\Big(e^{-k(t-s)} - e^{-k(t+s)}\Big),\end{aligned}$$

where here we used that:

$$\mathbb{E}\Big(\int_s^t e^{-k(t-u)}\,\mathrm{d}B_u\Big)\Big(\int_0^s e^{-k(s-u)}\,\mathrm{d}B_u\Big) = 0,$$

since we can separate the integrals because of independence and by using the properties of the $It\hat{o}'s$ integral [8] the above expectation is zero.

If $s > t$ then it equals to:

$$\frac{1}{2k}\Big(e^{-k(s-t)} - e^{-k(t+s)}\Big).$$

Thus,

$$\mathbb{E}[(\int_0^t e^{-k(t-u)}\,\mathrm{d}B_u)(\int_0^s e^{-k(s-u)}\,\mathrm{d}B_u)] = \frac{1}{2k}\Big(e^{-k|t-s|} - e^{-k(t+s)}\Big).$$

Hence, we computed that:

$$\mathbb{E}[X_t - X_s] = 0,$$

and

$$\mathbb{V}ar[X_t - X_s] = \frac{1}{2k}(1 - e^{-2kt}) + \frac{1}{2k}(1 - e^{-2ks}) - \frac{1}{k}(e^{-k|t-s|} - e^{-k(t+s)}).$$

So,

$$\mathbb{E}|\hat{\mu}(\lambda)|^2 = \int_0^\infty \int_0^\infty e^{-(s+t)} e^{-\frac{1}{2k}\lambda^2[1-e^{-k|t-s|}+e^{-k(t+s)}-\frac{1}{2}(e^{-2kt}+e^{-2ks})]} \, \mathrm{d}s \, \mathrm{d}t.$$

Here we used that since our process X_t is Gaussian then every finite linear combination of random variables of our process follows one dimensional Normal distribution and hence the characteristic function is given by:

$$\mathbb{E}e^{i\lambda(X_t - X_s)} = e^{i\lambda\mathbb{E}[X_t - X_s] - \frac{1}{2}\lambda^2 \mathbb{V}ar[X_t - X_s]}.$$

If we show that the integral :

$$\int_{-\infty}^{\infty} \mathbb{E}|\hat{\mu}(\lambda)|^2 \, \mathrm{d}\lambda = \int_0^\infty \int_0^\infty e^{-(s+t)} \int_{-\infty}^{\infty} e^{-\frac{1}{2k}\lambda^2[1-e^{-k|t-s|}+e^{-k(t+s)}-\frac{1}{2}(e^{-2kt}+e^{-2ks})]} \, \mathrm{d}\lambda \, \mathrm{d}s \, \mathrm{d}t$$

is finite, then the local time of the one dimensional Ornstein-Uhlenbeck process exists.

We integrate first with respect to λ. By completing the square the above integral is equal to:

$$C \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} \, \mathrm{d}s \, \mathrm{d}t,$$

where

$$\Phi(t,s) = 1 - e^{-k|t-s|} + e^{-k(t+s)} - \frac{1}{2}(e^{-2kt} + e^{-2ks}) = 1 - e^{-k|t-s|} - \frac{1}{2}(e^{-kt} - e^{-ks})^2,$$

and C is a positive constant.

$$\int\limits_{-\infty}^{\infty} \mathbb{E}|\hat{\mu}(\lambda)|^2 \,\mathrm{d}\lambda = C \int\limits_0^{\infty}\int\limits_0^{\infty} \frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} \,\mathrm{d}s\,\mathrm{d}t = C \int\limits_0^{\infty}\int\limits_0^{t} \Big[\frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} + \int\limits_t^{\infty} \Big[\frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}}\Big] \,\mathrm{d}s\,\mathrm{d}t.$$

We observe that the function Φ gets the value zero if and only if $t = s$, and by using the theory of asymptotic functions, $\sqrt{\Phi(t,s)} \sim \sqrt{|t-s|}$ "behaves" as the $\sqrt{|t-s|}$ when $|t-s| \to 0$. Hence,

$$\int\limits_0^{t} \frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} \,\mathrm{d}s$$

is a simple improper integral, and the same is true for the below integral:

$$\int\limits_t^{\infty} \frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} \,\mathrm{d}s.$$

Consider now the two following integrals:

$$\int\limits_0^{\infty}\int\limits_0^{t} \frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} \,\mathrm{d}s\,\mathrm{d}t,$$

and

$$\int\limits_0^{\infty}\int\limits_t^{\infty} \frac{e^{-(s+t)}}{\sqrt{\Phi(t,s)}} \,\mathrm{d}s\,\mathrm{d}t.$$

Convergence of both of the above two integrals implies the existence of the local time. To that end, we compute the integral:

$$\int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_{-\infty}^{\infty} e^{-(s+t)} e^{-\frac{1}{2}\lambda^2|t-s|} \,\mathrm{d}\lambda\,\mathrm{d}s\,\mathrm{d}t.$$

On the one hand, we integrate with respect to λ and the integral is equal to:

$$C_1 \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{|t-s|}} \, \mathrm{d}s \, \mathrm{d}t,$$

where C_1 is a positive constant.

One the other hand, we use Fubini's - Tonelli's heorem and we integrate first with respect to s, t and finally to λ.

(Since the function that appears in above triple integral is non-negative, there is no need to check its absolute integrability in order to apply Fubini's theorem.)

We have:

$$\begin{aligned}
\int_{-\infty}^\infty \int_0^\infty \int_0^\infty e^{-(s+t)} e^{-\frac{1}{2}\lambda^2|t-s|} \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}\lambda = & \int_{-\infty}^\infty \int_0^\infty \int_0^t e^{-(s+t)} e^{-\frac{1}{2}\lambda^2|t-s|} \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}\lambda \\
& + \int_{-\infty}^\infty \int_0^\infty \int_t^\infty e^{-(s+t)} e^{-\frac{1}{2}\lambda^2|t-s|} \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}\lambda \\
& = \int_{-\infty}^\infty \int_0^\infty \frac{1}{-1+\frac{1}{2}\lambda^2} (e^{-2t} - e^{-t(1+\frac{1}{2}\lambda^2)}) \, \mathrm{d}t \, \mathrm{d}\lambda \\
& + \int_{-\infty}^\infty \int_0^\infty \frac{1}{1+\frac{1}{2}\lambda^2} e^{-2t} \, \mathrm{d}t \, \mathrm{d}\lambda \\
& = \int_{-\infty}^\infty \frac{1}{-1+\frac{1}{2}\lambda^2} [\frac{1}{2} - \frac{1}{1+\frac{1}{2}\lambda^2}] \, \mathrm{d}\lambda + \int_{-\infty}^\infty \frac{1}{\lambda^2+2} \, \mathrm{d}\lambda \\
& = \int_{-\infty}^\infty \frac{2}{\lambda^2+2} \, \mathrm{d}\lambda \\
& \leq 2\pi.
\end{aligned}$$

We conclude that,

$$C_1 \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{|t-s|}} \, \mathrm{d}s \, \mathrm{d}t \leq C_1 2\pi < \infty.$$

Hence,:

$$\int_0^\infty \int_0^t \frac{e^{-(s+t)}}{\sqrt{|t-s|}} \, \mathrm{d}s \, \mathrm{d}t \leq \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{|t-s|}} \, \mathrm{d}s \, \mathrm{d}t < \infty,$$

and

$$\int_0^\infty \int_t^\infty \frac{e^{-(s+t)}}{\sqrt{|t-s|}} \, \mathrm{d}s \, \mathrm{d}t \leq \int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{|t-s|}} \, \mathrm{d}s \, \mathrm{d}t < \infty.$$

As a result:

$$\int_{-\infty}^{\infty} \mathbb{E}|\hat{\mu}(\lambda)|^2 \, \mathrm{d}\lambda < \infty,$$

which proves the existence of the local time for the one dimensional Ornstein-Uhlenbeck process.

Question:

Does the local time of the d-dimensional ($d \geq 2$) Ornstein-Uhlenbeck process exist?

Answer:

We believe that such local time does not exist. Our belief comes from the following:

Consider the occupation measure:

$$\mu_t(B) = \int_0^t \mathbb{1}_B(X_s) \, \mathrm{d}s,$$

where X_s is a d-dimensional ($d \geq 2$) Ornstein–Uhlenbeck process and B is a Borel set $B \subseteq \mathbb{R}^d$.

As in the 1-dimensional case we need to investigate the convergence of the following integral:

$$\int_{\mathbb{R}^d} \mathbb{E}|\hat{\mu}(\lambda)|^2 \,\mathrm{d}\lambda = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} e^{-(s+t)} \mathbb{E} e^{i\lambda\cdot(X_t - X_s)} \,\mathrm{d}\lambda \,\mathrm{d}s \,\mathrm{d}t$$
$$= \int_0^\infty \int_0^\infty e^{-(s+t)} \int_{\mathbb{R}^d} e^{-\frac{1}{2k}|\lambda|^2[1-e^{-k|t-s|}+e^{-k(t+s)}-\frac{1}{2}(e^{-2kt}+e^{-2ks})]} \,\mathrm{d}\lambda \,\mathrm{d}s \,\mathrm{d}t.$$

By following similar calculations as in the 1-dimensional case we see that the above integral diverges. Hence, from now on our Ornstein - Uhlenbeck process X_t is one dimensional.

Chapter 3

Continuity of local times

Here we encounter an obstacle: Notice that for each $t \in \mathbb{R}^+$, the notation $L(t, \cdot)$ represents an equivalent class of the densities. For this reason, it is not quite valid at this point to talk about, $L(t, 0)$, for example. However, if there is a member $\tilde{L}(t, y)$ among the equivalent class of the functions $L(t, y)$, which is continuous in y , then for each t the function $\tilde{L}(t, y)$ is well defined.
An effective way to install this continuity is to use Kolmogorov continuity theorem.

Kolmogorov continuity theorem: For any compact subset of $\mathbb{R}^+ \times \mathbb{R}$, Q , there are $m \geq 1$, $n > 1$, $n < m$ and $c_m > 0$ such that:

$$\mathbb{E}|L(t, y) - L(t, z)|^m \leq c_m|y - z|^n,$$

for every $(t, y), (t, z) \in Q$.

Proof:

First we consider the Fourier transform of the local time:

$$\hat{L}(t, \lambda) = \int_{\mathbb{R}} e^{i\lambda y} L(t, y)\, \mathrm{d}y,$$

and then we express the local time in terms of its Fourier transform:

$$\begin{aligned}
L(t,y) =&\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda y}\hat{L}(t,\lambda)\,\mathrm{d}\lambda\\
=&\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda y}\int_{\mathbb{R}} e^{i\lambda y}L(t,y)\,\mathrm{d}y\,\mathrm{d}\lambda.
\end{aligned}$$

Hence we have (for $m \geq 2$ even integer, and for every $(t,y),(t,z) \in Q$, where Q is a compact subset of $\mathbb{R}^+ \times \mathbb{R}$):

$$\begin{aligned}
&\mathbb{E}[L(t,y)-L(t,z)]^m\\
=&\frac{1}{(2\pi)^m}\mathbb{E}\Big[\int_{\mathbb{R}} e^{-i\lambda y}\int_{\mathbb{R}} e^{i\lambda y}L(t,y)\,\mathrm{d}y\,\mathrm{d}\lambda - \int_{\mathbb{R}} e^{-i\lambda z}\int_{\mathbb{R}} e^{i\lambda z}L(t,z)\,\mathrm{d}z\,\mathrm{d}\lambda\Big]^m\\
=&\frac{1}{(2\pi)^m}\mathbb{E}\Big[\int_{\mathbb{R}} e^{-i\lambda y}\int_{[0,t]} e^{i\lambda X_s}\,\mathrm{d}s\,\mathrm{d}\lambda - \int_{\mathbb{R}} e^{-i\lambda z}\int_{[0,t]} e^{i\lambda X_s}\,\mathrm{d}s\,\mathrm{d}\lambda\Big]^m\\
=&\frac{1}{(2\pi)^m}\mathbb{E}\Big[\int_{\mathbb{R}}(e^{-i\lambda y}-e^{-i\lambda z})\int_{[0,t]} e^{i\lambda X_s}\,\mathrm{d}s\,\mathrm{d}\lambda\Big]^m\\
\leq&\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\prod_{n=1}^{m}|e^{-i\lambda_n y}-e^{-i\lambda_n z}|\int_{[0,t]^m}\mathbb{E}\, e^{i\sum_{j=1}^m \lambda_j X_{s_j}}\,\mathrm{d}s_1 ...\,\mathrm{d}s_m\,\mathrm{d}\lambda_1 ...\,\mathrm{d}\lambda_m.
\end{aligned}$$

(In the last step, we applied Fubini's-Tonelli's theorem to switch the expectation with the integrals, in a similar way we used it in chapter 2, that's why the proof is omitted here).

Since the Ornstein-Uhlenbeck process is a Gaussian process, any finite linear combination of its variables (such as: $\sum_{j=1}^m \lambda_j X_{s_j}$) follows an one dimensional Normal distribution.

Hence the characteristic function gives us the below expectation:

$$\mathbb{E}\, e^{i\sum_{j=1}^m \lambda_j X_{s_j}} = e^{i\mathbb{E}(\sum_{j=1}^m \lambda_j X_{s_j})-\frac{1}{2}\mathbb{V}ar(\sum_{j=1}^m \lambda_j X_{s_j})}.$$

Thus,

$$\mathbb{E}[L(t,y)-L(t,z)]^m$$

$$\leq \frac{1}{(2\pi)^m}\int\limits_{\mathbb{R}^m}\prod_{n=1}^{m}|e^{-i\lambda_n y}-e^{-i\lambda_n z}|\int\limits_{[0,t]^m} e^{-\frac{1}{2}\mathbb{V}ar(\sum_{j=1}^{m}\lambda_j X_{s_j})}\,\mathrm{d}s_1 ... \mathrm{d}s_m\,\mathrm{d}\lambda_1 ... \mathrm{d}\lambda_m.$$

Our goal is to make the term $|y-z|^n$ (where $n<m$) appear on the right hand side of the above inequality. The above product helps us to create that term.
Indeed, by using the elementary inequality:

$$|e^{iu}-1|\leq 2^{1-\delta}|u|^{\delta},$$

for every real number u and $0<\delta<\frac{1}{2}$, we get:

$$|e^{-i\lambda_n y}-e^{-i\lambda_n z}|\leq 2^{1-\delta}|\lambda_n|^{\delta}|y-z|^{\delta},$$

because:

$$|e^{-i\lambda_n y}-e^{-i\lambda_n z}| =|e^{-i\lambda_n y}||1-e^{-i\lambda_n(z-y)}|\leq 2^{1-\delta}|\lambda_n|^{\delta}|y-z|^{\delta}.$$

<u>Note:</u> Comment about the proof of: $|e^{iu}-1|\leq 2^{1-\delta}|u|^{\delta}$.
We have:

$$|e^{iu}-1|=|e^{iu}-1|^{1-\delta}|e^{iu}-1|^{\delta}\leq 2^{1-\delta}|e^{iu}-1|^{\delta},$$

so it is enough to show that: $|e^{iu}-1|\leq|u|$, which holds because:

$$|e^{iu}-1|^2=2-2\cos u\leq u^2.$$

(the last inequality is true because if we consider $f(u)=u^2+2\cos u-2$ for every real number u, and study its properties, then f is always non-negative).

As a result we have:

$$
\begin{aligned}
&\mathbb{E}[L(t,y)-L(t,z)]^m \\
\\
\leq& C_{1,m}|y-z|^{m\delta}\int\limits_{\mathbb{R}^m}\prod_{n=1}^{m}|\lambda_n|^{\delta}\int\limits_{[0,t]^m} e^{-\frac{1}{2}\mathbb{V}ar(\sum_{j=1}^m \lambda_j X_{s_j})}\,\mathrm{d}s_1\dots\mathrm{d}s_m\,\mathrm{d}\lambda_1\dots\mathrm{d}\lambda_m \\
=& C_{2,m}|y-z|^{m\delta}\int\limits_{\mathbb{R}^m}\prod_{n=1}^{m}|\lambda_n|^{\delta}\int\limits_{[0,t]^m_<} e^{-\frac{1}{2}\mathbb{V}ar(\sum_{j=1}^m \lambda_j X_{s_j})}\,\mathrm{d}s_1\dots\mathrm{d}s_m\,\mathrm{d}\lambda_1\dots\mathrm{d}\lambda_m \\
\leq& C_{2,m}|y-z|^{m\delta}\int\limits_{[0,t]^m_<}\int\limits_{\mathbb{R}^m}\prod_{n=1}^{m}|\lambda_n-\lambda_{n+1}|^{\delta} e^{-\frac{1}{2}\mathbb{V}ar(\sum_{j=1}^m \lambda_j (X_{s_j}-X_{s_{j-1}}))}\,\mathrm{d}\lambda_1\dots\mathrm{d}\lambda_m\,\mathrm{d}s_1\dots\mathrm{d}s_m,
\end{aligned}
$$

where $C_{1,m}, C_{2,m}$ are positive constants that depend on m ($C_{1,m}=\frac{1}{(2\pi)^m}2^{m(1-\delta)}$ and $C_{2,m}=m!\,C_{1,m}$) and in the last step we used change of variables ($\lambda_{m+1}=0$). The notation $[0,t]^m_<$ means that $0<s_1<\dots<s_m<t$.

Also, in the last step we switched the order of the integrals. Notice that the function that appears inside the two m-multiple integrals

$$
\prod_{n=1}^{m}|\lambda_n|^{\delta}e^{-\frac{1}{2}\mathbb{V}ar(\sum_{j=1}^m \lambda_j X_{s_j})},
$$

is non-negative. In general, when we apply Fubini's theorem we need to check that the previous function is absolute integrable, but there is no need to check that in case it is non-negative according to Fubini's-Tonelli's theorem for non-negative functions [7].

Our goal is to show that the above double integral is finite. In order to be able to integrate we need somehow to move the λ_j's outside the variance. Under the assumption of independence of increments, we could easily do it by using the definition of the variance. However, as we mentioned earlier the Ornstein-Uhlenbeck process does not have independent increments, thus if we used the definition of the variance, we would have another term that relates to the covariance, which makes the calculations very difficult.

Hence, we follow a different path and in order to control the above variance we use the following formula:

$$\mathbb{V}ar(\sum_{j=1}^{m} \lambda_j (X_{s_j} - X_{s_{j-1}})) \geq \frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})} \frac{1}{m} \sum_{j=1}^{m} {\lambda_j}^2 \mathbb{V}ar(X_{s_j} - X_{s_{j-1}}),$$

which is true for Gaussian random variables with mean zero, [1], and where R is the determinant of the covariance of the random variables: X_{s_1}, X_{s_2},...,X_{s_m} ,

$$R = det\Big(Cov(X_{s_1}, ..., X_{s_m})\Big).$$

By using that formula we have this "nice" separation between the λ_j's and the variance. In order to be able to integrate with respect to λ's, we would like to eliminate the first quotient that appears on the right hand side of the above formula. For this, we use some techniques about locally nondeterministic processes that developed in [1].

More specifically, we can bound the above quotient from below by a positive number D_m that depends on m, that is:

$$\frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})} \geq D_m > 0.$$

One way to do this is the following:

In [1], they study in detail properties of stochastic processes such as local nondeterminism. They mention that a stochastic process is called locally nondeterministic if a future observation is relatively unpredictable, on a basis of a finite set of observations from the immediate past. They also provide some sufficient and necessary conditions under which a stochastic process is locally nondeterministic. One lemma that appears in [1] mentions that:

A stochastic process X_t is locally nondeterministic on a set $J = [0, t]_<$ if and only if:

$$\liminf_{c \to 0^+,\, t_m - t_1 \leq c} \frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})} > 0,$$

for every $m \geq 2$, where $0 < t_1 < \cdots < t_m < t$.

Hence, in order to control the quotient:

$$\frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})},$$

from below, we need to show that our Ornstein-Uhlenbeck process is locally nondeterministic. This result has been proved and can be found in [10], for instance.

Another way to prove that:

$$\frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})} \geq D_m > 0,$$

is to evaluate this quotient.

Indeed:

$$\frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})} = \frac{det\Big(Cov(X_{s_1}, ..., X_{s_m})\Big)}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})}$$

$$= \frac{\mathbb{V}ar(X_{s_1})\mathbb{V}ar(X_{s_2}|X_{s_1}) \cdots \mathbb{V}ar(X_{s_m}|X_{s_{m-1}}, ..., X_{s_1})}{\mathbb{V}ar(X_{s_1})\mathbb{V}ar(X_{s_2} - X_{s_1}) \cdots \mathbb{V}ar(X_{s_m} - X_{s_{m-1}})}$$

$$= \frac{\mathbb{V}ar(X_{s_2}|X_{s_1}) \cdots \mathbb{V}ar(X_{s_m}|X_{s_{m-1}})}{\mathbb{V}ar(X_{s_2} - X_{s_1}) \cdots \mathbb{V}ar(X_{s_m} - X_{s_{m-1}})},$$

where in the last step we used the fact that our process is Markovian.

We have already computed that:

$$\begin{aligned}
\mathbb{V}ar(X_{s_i} - X_{s_{i-1}}) &= \frac{1}{2k}(1 - e^{-2ks_i}) + \frac{1}{2k}(1 - e^{-2ks_{i-1}}) - \frac{1}{k}(e^{-k(s_i - s_{i-1})} - e^{-k(s_i + s_{i-1})}) \\
&= \frac{1}{k}\Big(1 - e^{-k(s_i - s_{i-1})} + e^{-k(s_i + s_{i-1})} - \frac{1}{2}(e^{-2ks_i} + e^{-2ks_{i-1}})\Big) \\
&= \frac{1}{k}\Big(1 - e^{-k(s_i - s_{i-1})} - \frac{1}{2}(e^{-ks_i} - e^{-ks_{i-1}})^2\Big),
\end{aligned}$$

for $i = 2, ..., m$.

We now compute the conditional variances of the numerator: (for $j = 2, ..., m$)

$$\begin{aligned}
&\mathbb{V}ar(X_{s_j}|X_{s_{j-1}}) = \mathbb{E}\Big\{(X_{s_j} - E(X_{s_j}|X_{s_{j-1}}))^2|X_{s_{j-1}}\Big\} \\
&= \mathbb{E}\Big\{\int\limits_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \int\limits_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u \\
&- \mathbb{E}\Big[\Big(\int\limits_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \int\limits_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|X_{s_{j-1}}\Big]\Big\}^2 \\
&= \mathbb{E}\Big\{\int\limits_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big\}^2 \\
&= \mathbb{E}\Big(\int\limits_{s_{j-1}}^{s_j} e^{-2k(s_j-u)}\,\mathrm{d}u\Big) \\
&= \frac{1}{2k}\Big(1 - e^{-2k(s_j-s_{j-1})}\Big).
\end{aligned}$$

<u>Note:</u> Explanations about the previous calculations.

First of all, as we saw in the second chapter:

$$X_{s_j} = \int\limits_0^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u.$$

Then, we used some properties of the conditional expectation (that can be found in [7], for instance). We have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int\limits_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \int\limits_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|X_{s_{j-1}}\Big] \\
&= \mathbb{E}\Big[\Big(\int\limits_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|X_{s_{j-1}}\Big] + \mathbb{E}\Big[\Big(\int\limits_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|X_{s_{j-1}}\Big] \\
&= \int\limits_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \mathbb{E}\Big[\int\limits_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big],
\end{aligned}$$

because when we are conditioning with respect to $X_{s_{j-1}}$ the

$$X_{s_{j-1}} = \int_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u$$

is "known" (we mean that it is the same as the $X_{s_{j-1}}$), and the

$$\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u$$

is independent from $X_{s_{j-1}}$ (since: $0 < s_{j-1} < s_j$).

Finally, from the properties of $It\hat{o}'s$ integral (see [8] for instance)

$$E\Big[\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big] = 0,$$

and from $It\hat{o}'s$ Isometry (see [8] for instance)

$$\mathbb{E}\Big\{\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big\}^2 = \mathbb{E}\Big(\int_{s_{j-1}}^{s_j} e^{-2k(s_j-u)}\,\mathrm{d}u\Big).$$

Thus,

$$\frac{R}{\prod_{j=1}^m \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})}$$

$$= \frac{\frac{1}{(2k)^{m-1}}\big(1 - e^{-2k(s_2-s_1)}\big)\cdots\big(1 - e^{-2k(s_m-s_{m-1})}\big)}{\frac{1}{k^{m-1}}[1 - e^{-k(s_2-s_1)} - \frac{1}{2}(e^{-ks_2} - e^{-ks_1})^2]\cdots[1 - e^{-k(s_m-s_{m-1})} - \frac{1}{2}(e^{-ks_m} - e^{-ks_{m-1}})^2]}.$$

Let's focus on one of the terms of the previous quotient:

$$\frac{1 - e^{-2k(s_i-s_{i-1})}}{1 - e^{-k(s_i-s_{i-1})} - \frac{1}{2}(e^{-ks_i} - e^{-ks_{i-1}})^2},$$

where $i = 2, 3, ..., m$.

Notice that,

$$1 - e^{-k(s_i - s_{i-1})} - \frac{1}{2}(e^{-ks_i} - e^{-ks_{i-1}})^2 \leq 1 - e^{-k(s_i - s_{i-1})} \ .$$

Hence,

$$\frac{1 - e^{-2k(s_i - s_{i-1})}}{1 - e^{-k(s_i - s_{i-1})} - \frac{1}{2}(e^{-ks_i} - e^{-ks_{i-1}})^2} \geq \frac{1 - e^{-2k(s_i - s_{i-1})}}{1 - e^{-k(s_i - s_{i-1})}} \geq 1.$$

As a result,

$$\frac{R}{\prod_{j=1}^{m} \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})}$$

$$= \frac{\frac{1}{(2k)^{m-1}}(1 - e^{-2k(s_2 - s_1)}) \cdots (1 - e^{-2k(s_m - s_{m-1})})}{\frac{1}{k^{m-1}}[1 - e^{-k(s_2 - s_1)} - \frac{1}{2}(e^{-ks_2} - e^{-ks_1})^2] \cdots [1 - e^{-k(s_m - s_{m-1})} - \frac{1}{2}(e^{-ks_m} - e^{-ks_{m-1}})^2]}$$

$$\geq \frac{1}{2^{m-1}} = D_m > 0.$$

Thus our expectation becomes:

$$\mathbb{E}[L(t, y) - L(t, z)]^m$$

$$\leq C_{2,m}|y - z|^{m\delta} \int_{[0,t]^m_<} \int_{\mathbb{R}^m} \prod_{n=1}^{m} |\lambda_n - \lambda_{n+1}|^{\delta} e^{-\frac{D_m}{2m} \sum_{j=1}^{m} {\lambda_j}^2 \mathbb{V}ar(X_{s_j} - X_{s_{j-1}})} \, \mathrm{d}\lambda_1 ... \, \mathrm{d}\lambda_m \, \mathrm{d}s_1 ... \, \mathrm{d}s_m$$
$$\leq C_{2,m}|y - z|^{m\delta} \int_{[0,t]^m_<} \int_{\mathbb{R}^m} \prod_{n=1}^{m} |\lambda_n - \lambda_{n+1}|^{\delta} e^{-D_{1,m} \sum_{j=1}^{m} {\lambda_j}^2 (1 - e^{-2k(s_j - s_{j-1})})} \, \mathrm{d}\lambda_1 ... \, \mathrm{d}\lambda_m \, \mathrm{d}s_1 ... \, \mathrm{d}s_m,$$

where $D_{1,m} = \frac{D_m}{2m}$ and in the last step we used that:

$$\mathbb{V}ar(X_{s_j} - X_{s_{j-1}}) \geq \frac{1}{2k}\Big(1 - e^{-2k(s_j - s_{j-1})}\Big),$$

because:

$$\begin{aligned}\mathbb{V}ar(X_{s_j} - X_{s_{j-1}}) &= \mathbb{E}(X_{s_j} - X_{s_{j-1}})^2 \\ &\geq \mathbb{E}(X_{s_j} - \mathbb{E}(X_{s_j}|\mathcal{F}_{s_{j-1}}))^2.\end{aligned}$$

(where $\mathcal{F}_{s_{j-1}} = \sigma\{X_s : s \leq s_{j-1}\}$. By using this notation we refer to the σ-algebra that is generated from our process X_s when $s \leq s_{j-1}$.)

and also:

$$\begin{aligned}&\mathbb{E}(X_{s_j} - E(X_{s_j}|\mathcal{F}_{s_{j-1}}))^2 \\ &= \mathbb{E}\Big\{ \int_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u - \\ &- \mathbb{E}\Big[\Big(\int_0^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|\mathcal{F}_{s_{j-1}}\Big]\Big\}^2 \\ &= \mathbb{E}\Big\{ \int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big\}^2 \\ &= \mathbb{E}\Big(\int_{s_{j-1}}^{s_j} e^{-2k(s_j-u)}\,\mathrm{d}u\Big) \\ &= \frac{1}{2k}\Big(1 - e^{-2k(s_j - s_{j-1})}\Big).\end{aligned}$$

<u>Note:</u> Explanations about the previous calculations.

First of all, as we saw in the second chapter:

$$X_{s_j} = \int_0^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u.$$

Then, we used some properties of the conditional expectation (that can be found in [7], for instance). We have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{0}^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|\mathcal{F}_{s_{j-1}}\Big] \\
&= \mathbb{E}\Big[\Big(\int_{0}^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|\mathcal{F}_{s_{j-1}}\Big] + \mathbb{E}\Big[\Big(\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big)\Big|\mathcal{F}_{s_{j-1}}\Big] \\
&= \int_{0}^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u + \mathbb{E}\Big[\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big],
\end{aligned}$$

because when we are conditioning with respect to $\mathcal{F}_{s_{j-1}}$ the

$$X_{s_{j-1}} = \int_{0}^{s_{j-1}} e^{-k(s_j-u)}\,\mathrm{d}B_u$$

is "known" (we mean that it belongs to the σ-algebra of $\mathcal{F}_{s_{j-1}}$ that we defined above), and the

$$\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u$$

is independent from $\mathcal{F}_{s_{j-1}}$ (since: $0 < s_{j-1} < s_j$).

Finally, from the properties of $It\hat{o}'s$ integral (see [8] for instance)

$$E\Big[\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big] = 0,$$

and from $It\hat{o}'s$ Isometry (see [8] for instance)

$$\mathbb{E}\Big\{\int_{s_{j-1}}^{s_j} e^{-k(s_j-u)}\,\mathrm{d}B_u\Big\}^2 = \mathbb{E}\Big(\int_{s_{j-1}}^{s_j} e^{-2k(s_j-u)}\,\mathrm{d}u\Big).$$

Using the triangle inequality (for $0 < \delta < \frac{1}{2}$),

$$\prod_{n=1}^{m} |\lambda_n - \lambda_{n+1}|^{\delta} \leq \prod_{n=1}^{m} (|\lambda_n|^{\delta} + |\lambda_{n+1}|^{\delta}) \leq \sum_{j_1,\ldots,j_m=0}^{2} \prod_{n=1}^{m} |\lambda_n|^{\delta_{j_n}},$$

where $\delta_{j_n} = 0$ if $j_n = 0$, $\delta_{j_n} = \delta$ if $j_n = 1$ and $\delta_{j_n} = 2\delta$ if $j_n = 2$, for $n = 1, \cdots, m$

and the above sum means:

$$\sum_{j_1,\ldots,j_m=0}^{2} \prod_{n=1}^{m} |\lambda_n|^{\delta_{j_n}} = \sum_{j_1=0}^{2} \sum_{j_2=0}^{2} \cdots \sum_{j_m=0}^{2} \prod_{n=1}^{m} |\lambda_n|^{\delta_{j_n}}.$$

Notice,

$$\prod_{n=1}^{m} |\lambda_n|^{\delta_{j_n}} \leq \prod_{n=1}^{m} (1 \vee |\lambda_n|)^{\delta_{j_n}} \leq \prod_{n=1}^{m} (1 \vee |\lambda_n|)^{2\delta},$$

where $a \vee b = max\{a, b\}$.

Thus,

$$\prod_{n=1}^{m} |\lambda_n - \lambda_{n+1}|^{\delta} \leq 3^m \prod_{n=1}^{m} (1 \vee |\lambda_n|)^{2\delta}.$$

As a result,

$$\mathbb{E}[L(t,y) - L(t,z)]^m$$

$$\leq C_{3,m} |y-z|^{m\delta} \int_{[0,t]^m_<} \int_{\mathbb{R}^m} \prod_{n=1}^{m} (1 \vee |\lambda_n|)^{2\delta} e^{-D_{1,m} \sum_{j=1}^{m} {\lambda_j}^2 (1 - e^{-2k(s_j - s_{j-1})})} \, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_m \, \mathrm{d}s_1 \ldots \mathrm{d}s_m,$$

where $C_{3,m} = 3^m \, C_{2,m}$.

We integrate first with respect to lambdas:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^m} \prod_{n=1}^{m} (1 \vee |\lambda_n|)^{2\delta} e^{-D_{1,m}\sum_{j=1}^{m} {\lambda_j}^2 (1-e^{-2k(s_j - s_{j-1})})} \, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_m \\
&= \prod_{n=1}^{m} \int\limits_{\mathbb{R}} (1 \vee |\lambda|)^{2\delta} e^{-D_{1,m}\lambda^2 (1-e^{-2k(s_n - s_{n-1})})} \, \mathrm{d}\lambda \\
&= \prod_{n=1}^{m} \Big\{ \int\limits_{-1}^{1} e^{-D_{1,m}\lambda^2 (1-e^{-2k(s_n - s_{n-1})})} \, \mathrm{d}\lambda + \int\limits_{|\lambda|>1} |\lambda|^{2\delta} e^{-D_{1,m}\lambda^2 (1-e^{-2k(s_n - s_{n-1})})} \, \mathrm{d}\lambda \Big\} \\
&\leq \prod_{n=1}^{m} \Big\{ 2 + \int\limits_{\mathbb{R}} |\lambda|^{2\delta} e^{-D_{1,m}\lambda^2 (1-e^{-2k(s_n - s_{n-1})})} \, \mathrm{d}\lambda \Big\} \\
&= \prod_{n=1}^{m} \Big\{ 2 + \Big(\int\limits_{0}^{\infty} \lambda^{\delta - \frac{1}{2}} e^{-D_{1,m}\lambda} \, \mathrm{d}\lambda \Big) (1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}} \Big\} \\
&\leq \prod_{n=1}^{m} \Big\{ 2(1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}} + \frac{\Gamma(\delta + \frac{1}{2})}{D_{1,m}^{\delta + \frac{1}{2}}} (1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}} \Big\} \\
&\leq \prod_{n=1}^{m} \Big\{ \Big(2 + \frac{\Gamma(\delta + \frac{1}{2})}{D_{1,m}^{\delta + \frac{1}{2}}} \Big) (1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}} \Big\} \\
&= \Big(2 + \frac{\Gamma(\delta + \frac{1}{2})}{D_{1,m}^{\delta + \frac{1}{2}}} \Big)^m \prod_{n=1}^{m} (1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}[L(t,y) - L(t,z)]^m \\
&\leq C_{4,m} |y - z|^{m\delta} \int\limits_{[0,t]^m_<} \prod_{n=1}^{m} (1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}} \, \mathrm{d}s_1 \ldots \mathrm{d}s_m,
\end{aligned}$$

where $C_{4,m} = \Big(2 + \frac{\Gamma(\delta + \frac{1}{2})}{D_{1,m}^{\delta + \frac{1}{2}}} \Big)^m C_{3,m}$, and $\Gamma(\cdot)$ denotes the Gamma function.

We observe that:

$$\begin{aligned}
\int\limits_{[0,t]^m_<} \prod_{n=1}^{m} (1 - e^{-2k(s_n - s_{n-1})})^{-\delta - \frac{1}{2}} \, \mathrm{d}s_1 \ldots \mathrm{d}s_m &= \int\limits_{[0,t]^m} \prod_{n=1}^{m} (1 - e^{-2ku_n})^{-\delta - \frac{1}{2}} \, \mathrm{d}u_1 \ldots \mathrm{d}u_m \\
&= \Big(\int\limits_{0}^{t} (1 - e^{-2ku})^{-\delta - \frac{1}{2}} \, \mathrm{d}u \Big)^m.
\end{aligned}$$

The last integral is finite if we choose $0 < \delta < \frac{1}{2}$, which completes the proof!

(For the computations of this Chapter we used some ideas from [2].)

Chapter 4

Exponential Integrability of self intersection local times

The following function plays an important role when it comes to large deviations and it is called logarithmic moment generating function:

$$\Lambda_p(\theta) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\right\},$$

where $\theta > 0$ and the random quantity here is the $\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y$, which is called the p-multiple self intersection local time as we will see in the next chapter.

In Chapter 5 we will evaluate the above limit, but before that we need to make sure that the above expectation is finite.
For this we use Girsanov Theorem, which can be found in [9] or in [15] for instance, and allows us to "move" from one process to another one, and thus to use some known results from the Brownian Motion.

We consider the Doléans-Dade exponential:

$$M_t = \exp\left\{k\int_0^t X_s\,\mathrm{d}B_s - \frac{k^2}{2}\int_0^t X_s^2\,\mathrm{d}s\right\}.$$

From Girsanov theorem a new probability measure can be defined such that:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t,$$

and we have the following notations:

Under $\mathbb{P}:$ X is Ornstein-Uhlenbeck process and B is Brownian Motion.

Under $\mathbb{Q}:$ X is a Brownian motion and B satisfies: $\mathrm{d}B_t = \mathrm{d}X_t + kX_t\,\mathrm{d}t$.

Hence we have:

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\\ &= \mathbb{E}^{\mathbb{Q}}\Big\{\exp\Big\{-k\int_0^t X_s\,\mathrm{d}X_s - \frac{k^2}{2}\int_0^t X_s^2\,\mathrm{d}s\Big\}\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big\}.\end{aligned}$$

Under the measure $\mathbb{Q}$, X_t is a Brownian motion, thus from $It\hat{o}'s$ formula we get:

$$\int_0^t X_s\,\mathrm{d}X_s = \frac{X_t^2 - t}{2}\quad .$$

Indeed, we consider the function $g(x) = x^2, x \in \mathbb{R}$, and applying $It\hat{o}'s$ formula we have:

$$\mathrm{d}(X_t^2) = 2X_t\,\mathrm{d}X_t + \mathrm{d}t,$$

which is the differential form of the previous equation.

As a result:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}} \exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\} \\
&= \mathbb{E}^{\mathbb{Q}}\Big\{\exp\Big\{-k\frac{X_t^2-t}{2}-\frac{k^2}{2}\int_0^t X_s^2\,\mathrm{d}s\Big\}\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big\} \\
&= \exp\Big(\frac{kt}{2}\Big)\mathbb{E}^{\mathbb{Q}}\Big\{\exp\Big\{-k\frac{X_t^2}{2}-\frac{k^2}{2}\int_0^t X_s^2\,\mathrm{d}s\Big\}\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big\} \\
&\leq \exp\Big(\frac{kt}{2}\Big)\mathbb{E}^{\mathbb{Q}} \exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\} < \infty,
\end{aligned}
$$

where we used the exponential integrability of the p-multiple self intersection local time of the Brownian motion [3].

Chapter 5

Large Deviations

In this chapter we provide the large deviation results for the self intersection local times of the Ornstein-Uhlenbeck process.

Let's focus first on the self intersection local times. Consider positive times $t_1, t_2, \cdots, t_p$ with $p \geq 2$ such that $t_k \neq t_j$ when $k \neq j$. We are interested in the number of the cases such that:

$$X_{t_1} = X_{t_2} = \cdots = X_{t_p},$$

where X is our one dimensional Ornstein-Uhlenbeck process.
In other words, we are interested in the number of the cases that the process intersects itself. The notion of intersection local times have been introduced to measure the intensity of the path intersections of random processes.
There are several ways to construct the self intersection local time. We now briefly describe two of them that appear in [3].

The first of them suggests that we define:

$$\beta_\epsilon(A) = \int_{\mathbb{R}} \int_A \prod_{j=1}^{p} p_\epsilon(X_{s_j} - x) \, \mathrm{d}s_1 \cdots \mathrm{d}s_p dx,$$

where

$$p_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon}},$$

and A is compact subset of $(\mathbb{R}^+)^p_<$. (For instance, A can take the form of: $[0,t]^p_<$ where a point $(s_1, s_2, \cdots, s_p)$ that belongs to that set satisfies: $0 < s_1 < s_2 < \cdots < s_p < t$).

From [3] we know that there exists a measure $\beta(\cdot)$ on $(\mathbb{R}^+)^p_<$ such that for any compact subset of $(\mathbb{R}^+)^p_<$ of the form $A_1 \times A_2 \times \cdots \times A_p$ we have:

$$\lim_{\epsilon \to 0^+} \mathbb{E}|\beta_\epsilon(A_1 \times A_2 \times \cdots \times A_p) - \beta(A_1 \times A_2 \times \cdots \times A_p)|^m = 0,$$

for all $m > 0$.

The above random measure $\beta(\cdot)$ is called the p-multiple self intersection local time of our process X.

The second method suggests that we can define the self intersection local time with the help of the local time.

Indeed, symbolically we can express the local time as:

$$L(t,x) = \int_0^t \delta_x(X_s)\,\mathrm{d}s,$$

where $\delta_x(\cdot)$ represents the Dirac function defined by the property that:

$$\int_{\mathbb{R}} f(y)\delta_x(y) = f(x),$$

for every bounded and continuous function, $f(\cdot)$, on $\mathbb{R}$.

We consider the following local time:

$$L(t,x) = \int_{[0,t]^p} \delta_x(X_{s_1} - X_{s_2}, \cdots, X_{s_{p-1}} - X_{s_p})\,\mathrm{d}s_1 ... \mathrm{d}s_p,$$

and define the p-multiple self intersection local time as $L(t,0)$.

In other words, the self intersection local time can be viewed as a local time multi-parameter process.

Note: The above two definitions of the self intersection local time are equivalent!

Proof of Note:

Consider:

$$h(x_1,\cdots,x_{p-1}) = \int\limits_{\mathbb{R}} p_1(-x)\prod_{j=1}^{p-1} p_1\Big(\sum_{k=j}^{p-1} x_k - x\Big)\,\mathrm{d}x,$$

and let:

$$h_\epsilon(x_1,\cdots,x_{p-1}) = \epsilon^{-(p-1)}h(\epsilon^{-1}x_1,\cdots,\epsilon^{-1}x_{p-1}).$$

Then,

$$\begin{aligned}
&\int\limits_{[0,t]} h_\epsilon(X_{s_1}-X_{s_2},\cdots,X_{s_{p-1}}-X_{s_p})\,\mathrm{d}s_1\cdots\mathrm{d}s_p\\
&= \epsilon^{-(p-1)}\int\limits_{[0,t]}\int\limits_{\mathbb{R}} p_1(-x)\prod_{j=1}^{p-1} p_1\Big(\epsilon^{-1}\sum_{k=j}^{p-1}(X_{s_k}-X_{s_{k+1}})-x\Big)\,\mathrm{d}x\,\mathrm{d}s_1\cdots\mathrm{d}s_p\\
&= \int\limits_{[0,t]}\int\limits_{\mathbb{R}} p_\epsilon(-x)\prod_{j=1}^{p-1} p_\epsilon\Big(\sum_{k=j}^{p-1}(X_{s_k}-X_{s_{k+1}})-x\Big)\,\mathrm{d}x\,\mathrm{d}s_1\cdots\mathrm{d}s_p\\
&= \int\limits_{[0,t]}\int\limits_{\mathbb{R}} p_\epsilon(-x)\prod_{j=1}^{p-1} p_\epsilon\Big((X_{s_j}-X_{s_p})-x\Big)\,\mathrm{d}x\,\mathrm{d}s_1\cdots\mathrm{d}s_p\\
&= \int\limits_{[0,t]}\int\limits_{\mathbb{R}} \prod_{j=1}^{p} p_\epsilon(X_{s_j}-x)\,\mathrm{d}x\,\mathrm{d}s_1\cdots\mathrm{d}s_p,
\end{aligned}$$

or

$$\int\limits_{\mathbb{R}^{p-1}} h_\epsilon(x)L(t,x)\,\mathrm{d}x = \beta_\epsilon([0,t]),$$

and we get the result by letting $\epsilon \to 0^+$.

Finally, according to [3], we can express the p-multiple self intersection local time in terms of the L^p norm.

Indeed,

$$\beta_\epsilon([0,t_1] \times \cdots \times [0,t_p]) = \int_{\mathbb{R}} \prod_{j=1}^{p} \int_{\mathbb{R}} p_\epsilon(y-x) L(t_j, y) \, \mathrm{d}y dx,$$

and if we consider $t_1 = t_2 = \cdots = t_p$ and let $\epsilon \to 0^+$ we have:

$$\beta([0,t] \times \cdots \times [0,t])_< = \frac{1}{p!} \int_{\mathbb{R}} L^p(t,x) \, \mathrm{d}x.$$

Our goal is to evaluate the following limit:

$$\Lambda_p(\theta) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ \theta \Big(\int_{-\infty}^{\infty} L^p(t,y) \, \mathrm{d}y \Big)^{\frac{1}{p}} \right\},$$

where $\theta > 0$ and Λ_p is the logarithmic moment generating function.

We show that:

$$\Lambda_p(\theta) = \sup_{g \in \mathcal{F}_1} \left[\theta \Big(\int_{-\infty}^{\infty} (g(x))^{2p} \pi(\mathrm{d}x) \Big)^{\frac{1}{p}} - \frac{1}{2} \int_{-\infty}^{\infty} (g^{'}(x))^2 \pi(\mathrm{d}x) \right],$$

where:

$$\mathcal{F}_1 = \{ g : \int_{-\infty}^{\infty} g^2(x) \pi(\mathrm{d}x) = 1 \ , \ \int_{-\infty}^{\infty} (g^{'}(x))^2 \pi(\mathrm{d}x) < \infty \},$$

and $\quad \pi(\mathrm{d}x) = \exp(-kx^2) \, \mathrm{d}x.$

5.1 Lower Bound

We consider a function f, bounded continuous on $\mathbb{R}$ and $q > 1$ the conjugate of p (that is: $\frac{1}{p} + \frac{1}{q} = 1$) such that:

$$\int_{-\infty}^{\infty} |f(x)|^q \, \mathrm{d}x = 1.$$

Then, by using Holder inequality we have:

$$\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}} \geq \int_{-\infty}^{\infty} f(y)L(t,y)\,\mathrm{d}y = \int_0^t f(X_s)\,\mathrm{d}s.$$

Hence, Feynman-Kac formula gives us:

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}$$
$$\geq \sup_{g\in\mathcal{F}_1} \Big[\theta \int_{-\infty}^{\infty} f(x)g^2(x)\pi(\mathrm{d}x) - \frac{1}{2}\int_{-\infty}^{\infty} (g^{'}(x))^2\pi(\mathrm{d}x)\Big].$$

Taking supremum over f we conclude:

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}$$
$$\geq \sup_{g\in\mathcal{F}_1} \Big[\theta\Big(\int_{-\infty}^{\infty} (g(x))^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}} - \frac{1}{2}\int_{-\infty}^{\infty} (g^{'}(x))^2\pi(\mathrm{d}x)\Big].$$

Here we used the following Feynman-Kac formula from [3]:

Feynman-Kac formula:

For any bounded, continuous function f on $\mathbb{R}$, we have:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\Big(\int_0^t f(X_s)\,\mathrm{d}s\Big) = \sup_{g\in\mathcal{F}_1} \Big[\int_{-\infty}^{\infty} f(x)g^2(x)\pi(\mathrm{d}x) - \frac{1}{2}\int_{-\infty}^{\infty} (g^{'}(x))^2\pi(\mathrm{d}x)\Big],$$

where

$$\mathcal{F}_1 = \{g : \int_{-\infty}^{\infty} g^2(x)\pi(\mathrm{d}x) = 1 \ , \ \int_{-\infty}^{\infty} (g^{'}(x))^2\pi(\mathrm{d}x) < \infty\}.$$

Moreover, in the last step we used Holder's inequality and Hahn-Banach Theorem from [4] (Unit 4, Corollary 2).

<u>Note:</u> We explain here how we used Holder's inequality and Hahn-Banach Theorem in order to show:

$$\sup_f \Big[\theta \int_{-\infty}^{\infty} f(x)g^2(x)\pi(\mathrm{d}x) - \frac{1}{2}\int_{-\infty}^{\infty} (g^{'}(x))^2\pi(\mathrm{d}x)\Big]$$
$$= \theta\Big(\int_{-\infty}^{\infty} (g(x))^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}} - \frac{1}{2}\int_{-\infty}^{\infty} (g^{'}(x))^2\pi(\mathrm{d}x).$$

For the one direction:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)g^2(x)\pi(\mathrm{d}x) &= \int_{-\infty}^{\infty} f(x)e^{-\frac{1}{q}kx^2}g^2(x)e^{-\frac{1}{p}kx^2}\,\mathrm{d}x \\ &\leq \Big(\int_{-\infty}^{\infty} |f(x)|^q\pi(\mathrm{d}x)\Big)^{\frac{1}{q}}\Big(\int_{-\infty}^{\infty} |g(x)|^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_{-\infty}^{\infty} |g(x)|^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}},\end{aligned}$$

because:

$$\int_{-\infty}^{\infty} |f(x)|^q\pi(\mathrm{d}x) \leq \int_{-\infty}^{\infty} |f(x)|^q\,\mathrm{d}x = 1.$$

For the other direction: By using Hahn-Banach Theorem (which can be found in [4], for instance) there exists function f_0 in $L^{q}(\mathbb{R})$ with $\int_{-\infty}^{\infty}|f_0(x)|^q\pi(\mathrm{d}x) = 1$ such that:

$$\Big(\int_{-\infty}^{\infty}|g(x)|^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}} = \int_{-\infty}^{\infty} g^2(x)f_0(x)\pi(\mathrm{d}x)$$
$$= \int_{-\infty}^{\infty} g^2(x)f(x)\pi(\mathrm{d}x) + \int_{-\infty}^{\infty} g^2(x)(f_0(x)-f(x))\pi(\mathrm{d}x)$$
$$< \epsilon + \int_{-\infty}^{\infty} g^2(x)f(x)\pi(\mathrm{d}x),$$

where f is a continuous bounded function on $\mathbb{R}$. Notice that here we used the fact that the functions that are bounded and continuous, are dense in the unit sphere of $L^q(\mathbb{R})$ (the unit sphere of $L^q(\mathbb{R})$ contains functions from $L^q(\mathbb{R})$ whose L^q norm is equal to one). That means, for every function f_0 of the unit sphere of $L^q(\mathbb{R})$, there exists a sequence of continuous bounded functions that converges to f_0 in L^q norm. Finally, by letting ϵ goes to zero we get what we want.

5.2 Upper Bound

First we prove the exponential tightness for the local time. In particular we show that:
For every $L > 0$ there exists a compact set $K \subset L^p(\mathbb{R})$ such that:

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(\frac{1}{t}L(t,\cdot) \notin K) \leq -L,$$

The proof of this exponential tightness is based on the following Lemmas and the Frechet - Kolmogorov Theorem.

Lemma 1

(a). For every $L > 0$, for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$\mathbb{P}\Big(\sup_{|h|<\delta}\frac{1}{t^p}\int_{\mathbb{R}} |L(t,x+h)-L(t,x)|^p\,\mathrm{d}x \geq \epsilon\Big) \leq e^{-Lt},$$

where $L(t,x)$ denotes here the local time of the Ornstein-Uhlenbeck process.

(b).For every $L>0$ there exists $D>0$ such that:

$$\mathbb{P}\Big(\int\limits_{\mathbb{R}}\Big(\frac{L(t,x)}{t}\Big)^{p}\,\mathrm{d}x\geq D\Big)\leq e^{-Lt},$$

where $L(t,x)$ denotes here the local time of the Ornstein-Uhlenbeck process.

Proof

We only prove (a). since we follow a similar path for (b).

(a). Let $L>0$ and $\epsilon>0$. Consider $L^{'}=L+\frac{k}{2}$. Then, there exists $\delta>0$ such that:

$$\mathbb{P}\Big(\sup_{|h|<\delta}\frac{1}{t^{p}}\int\limits_{\mathbb{R}}|L(t,x+h)-L(t,x)|^{p}\,\mathrm{d}x\geq\epsilon\Big)\leq e^{-L^{'}t},$$

where $L(t,x)$ denotes here the Brownian local time.

Also, by Girsanov theorem we write:

$$\mathbb{P}\Big(\sup_{|h|<\delta}\frac{1}{t^{p}}\int\limits_{\mathbb{R}}|L(t,x+h)-L(t,x)|^{p}\,\mathrm{d}x\geq\epsilon\Big)$$

(L(t,x) denotes here the local time of O-U process)

$$=\mathbb{E}^{\mathbb{P}}\mathbf{1}\Big\{\sup_{|h|<\delta}\frac{1}{t^{p}}\int\limits_{\mathbb{R}}|L(t,x+h)-L(t,x)|^{p}\,\mathrm{d}x\geq\epsilon\Big\}$$

(L(t,x) denotes here the local time of O-U process)

$$=\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big\{-k\frac{X_t^2-t}{2}-\frac{k^2}{2}\int\limits_{0}^{t}X_s^2\,\mathrm{d}s\Big\}\mathbf{1}\Big\{\sup_{|h|<\delta}\frac{1}{t^{p}}\int\limits_{\mathbb{R}}|L(t,x+h)-L(t,x)|^{p}\,\mathrm{d}x\geq\epsilon\Big\}\Big]$$

(L(t,x) denotes here the Brownian local time)

$$\leq e^{\frac{kt}{2}}\mathbb{E}^{\mathbb{Q}}\Big[\mathbf{1}\Big\{\sup_{|h|<\delta}\frac{1}{t^{p}}\int\limits_{\mathbb{R}}|L(t,x+h)-L(t,x)|^{p}\,\mathrm{d}x\geq\epsilon\Big\}\Big]$$

(L(t,x) denotes here the Brownian local time)

$$\leq e^{\frac{kt}{2}}e^{-L^{'}t}$$

$$=e^{-Lt},$$

where we used the same calculations as in Chapter 4 in order to get the first inequality.

Lemma 2

For every $L > 0$, for every $\epsilon > 0$ there exists $M > 0$ such that the local time of the Ornstein-Uhlenbeck process satisfies:

$$\mathbb{P}\Big(\int_{|x|\geq M} L^p(t,x)\,\mathrm{d}x \geq \epsilon t^p \Big) \leq 2e^{-Lt}.$$

Proof

(Before we start the proof we want to mention that this Lemma is not true for the Brownian Motion).

Here we use the following result that is true for the Brownian local time:

For every $L > 0$ there exists $N > 0$ such that:

$$\mathbb{P}\Big(\sup_x L(t,x) \geq Nt \Big) \leq e^{-Lt}.$$

Let $L > 0$ and $\epsilon > 0$ and $L(t,x)$ be the local time of the Ornstein-Uhlenbeck process.

(M will be determined later on.)

We have:

$$\begin{aligned}
&\mathbb{P}\Big(\int_{|x|\geq M} L^p(t,x)\,\mathrm{d}x \geq \epsilon t^p \Big) \\
&\leq \mathbb{P}\Big(\sup_x L^{p-1}(t,x) \int_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \epsilon t^p \Big) \\
&= \mathbb{P}\Big(\sup_x L^{p-1}(t,x) \int_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \epsilon t^p, \sup_x L(t,x) \leq Nt \Big) \\
&+ \mathbb{P}\Big(\sup_x L^{p-1}(t,x) \int_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \epsilon t^p, \sup_x L(t,x) \geq Nt \Big) \\
&\leq \mathbb{P}\Big(\int_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \frac{\epsilon}{N^{p-1}} t \Big) + \mathbb{P}\Big(\sup_x L(t,x) \geq Nt \Big).
\end{aligned}$$

Thus, by using Girsanov theorem in the same way as we did in Lemma 1:

$$\mathbb{P}\Big(\int\limits_{|x|\geq M} L^p(t,x)\,\mathrm{d}x \geq \epsilon t^p\Big) \leq \mathbb{P}\Big(\int\limits_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \frac{\epsilon}{N^{p-1}}t\Big) + e^{-Lt}.$$

In addition:

$$\int\limits_{|x|\geq M} L(t,x)\,\mathrm{d}x \leq \frac{1}{M}\int\limits_0^t |X_s|\,\mathrm{d}s.$$

(<u>Note:</u> The previous inequality is true since:

$$\begin{aligned}\int\limits_{|x|\geq M} L(t,x)\,\mathrm{d}x &= \int\limits_{|x|\geq M} |x|\frac{1}{|x|}L(t,x)\,\mathrm{d}x\\ &\leq \frac{1}{M}\int\limits_{\mathbb{R}} |x|L(t,x)\,\mathrm{d}x\\ &= \frac{1}{M}\int\limits_0^t |X_s|\,\mathrm{d}s,\end{aligned}$$

where in the last step we used a result from [3], that is the property that defines the local time:

$$\int\limits_{\mathbb{R}} f(x)L(t,x)\,\mathrm{d}x = \int\limits_0^t f(X_s)\,\mathrm{d}s,$$

for every continuous f on $\mathbb{R}$ and where $t > 0$.)

Hence, by using Markov inequality:

$$\begin{aligned}\mathbb{P}\Big(\int\limits_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \frac{\epsilon}{N^{p-1}}t\Big) &\leq \mathbb{P}\Big(\int\limits_0^t |X_s|\,\mathrm{d}s \geq \frac{\epsilon}{N^{p-1}}tM\Big)\\ &\leq e^{-\frac{\epsilon}{N^{p-1}}tM} E^{\mathbb{P}}\Big(e^{\int_0^t |X_s|\mathrm{d}s}\Big),\end{aligned}$$

and we can bound the last expectation by:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}} e^{\int_0^t |X_s| \mathrm{d}s} \\
&\leq \mathbb{E}^{\mathbb{Q}} \Big\{ e^{-k\int_0^t X_s \mathrm{d}X_s - \frac{k^2}{2}\int_0^t X_s^2 \mathrm{d}s} e^{\int_0^t |X_s| \mathrm{d}s} \Big\} \\
&\leq e^{\frac{kt}{2}} \mathbb{E}^{\mathbb{Q}} \Big\{ e^{\int_0^t (|X_s| - \frac{k^2}{2} X_s^2) \mathrm{d}s} \Big\} \\
&\leq e^{(\frac{k}{2} + \frac{1}{2k^2})t}.
\end{aligned}$$

As a result,

$$\mathbb{P}\Big(\int\limits_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \frac{\epsilon}{N^{p-1}} t \Big) \leq e^{-(\frac{\epsilon}{N^{p-1}}M - \frac{k}{2} - \frac{1}{2k^2})t}.$$

By choosing sufficiently large M, that is M that satisfies: $\frac{\epsilon}{N^{p-1}}M - \frac{k}{2} - \frac{1}{2k^2} > L$ we get:

$$\mathbb{P}\Big(\int\limits_{|x|\geq M} L(t,x)\,\mathrm{d}x \geq \frac{\epsilon}{N^{p-1}} t \Big) \leq e^{-Lt}.$$

Finally, we proved that for every $L > 0$, for every $\epsilon > 0$ there exists $M > 0$ (M that satisfies: $\frac{\epsilon}{N^{p-1}}M - \frac{k}{2} - \frac{1}{2k^2} > L$) such that:

$$\mathbb{P}\Big(\int\limits_{|x|\geq M} L^p(t,x)\,\mathrm{d}x \geq \epsilon t^p \Big) \leq e^{-Lt} + e^{-Lt} = 2e^{-Lt}.$$

Construction of the compact set

Let $L > 0$ and consider a sequence ϵ_j such that $\epsilon_j \downarrow 0$. Then, Lemmas 1 and 2 imply that: There exist $\delta_j > 0$ such that:

$$\mathbb{P}\Big(\sup_{|h|<\delta_j} \frac{1}{t^p} \int\limits_{\mathbb{R}} |L(t,x+h) - L(t,x)|^p \,\mathrm{d}x \geq \epsilon_j \Big) \leq e^{-jLt},$$

there exist $M_j > 0$ such that:

$$\mathbb{P}\Big(\int\limits_{|x|\geq M_j} \Big(\frac{L(t,x)}{t}\Big)^p \mathrm{d}x \geq \epsilon_j \Big) \leq 2e^{-jLt},$$

and there exist $D_j > 0$ such that:

$$\mathbb{P}\Big(\int_{\mathbb{R}} \Big(\frac{L(t,x)}{t}\Big)^p \,\mathrm{d}x \geq D_j\Big) \leq e^{-jLt}.$$

Consider the following set:

$$\begin{aligned} E_L = & \bigcap_{j=1}^{\infty} \Big\{ f \in L^p(\mathbb{R}) : \sup_{|h|<\delta_j} \int_{\mathbb{R}} |f(x+h) - f(x)|^p \,\mathrm{d}x < \epsilon_j \Big\} \bigcap \\ & \bigcap_{j=1}^{\infty} \Big\{ f \in L^p(\mathbb{R}) : \int_{|x|\geq M_j} |f(x)|^p \,\mathrm{d}x < \epsilon_j \Big\} \bigcap \bigcap_{j=1}^{\infty} \Big\{ f \in L^p(\mathbb{R}) : \int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x < D_j \Big\} \\ = & \bigcap_{j=1}^{\infty} (A_j \cap B_j \cap C_j). \end{aligned}$$

<u>**Theorem**</u> *(Frechet - Kolmogorov)*

Let K be a subset of $L^p(\mathbb{R})$ where $p \in [1, \infty)$. K is relatively compact if and only if:
(a).

$$\sup_{f \in K} \int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x < \infty.$$

(b). For every $\epsilon > 0$ there is some $\delta > 0$ such that for every $f \in K$ and $h \in \mathbb{R}$ with $|h| < \delta$:

$$\int_{\mathbb{R}} |f(x+h) - f(x)|^p \,\mathrm{d}x < \epsilon \ .$$

(c). For every $\epsilon > 0$ there is some $M > 0$ such that for every $f \in K$

$$\int_{|x|>M} |f(x)|^p \,\mathrm{d}x < \epsilon.$$

By applying *Frechet-Kolmogorov Theorem*, we get that E_L is relatively compact. Then its closure is compact in $L^p(\mathbb{R})$. Let $K_L = \bar{E}_L$. In addition:

$$\begin{aligned}
\mathbb{P}\Big(\frac{1}{t}L(t,\cdot) \notin K_L\Big) &\leq \mathbb{P}\Big(\frac{1}{t}L(t,\cdot) \notin E_L\Big) \\
&\leq \mathbb{P}\Big(\frac{1}{t}L(t,\cdot) \in \bigcup_{j=1}^{\infty}(A_j^c \cup B_j^c \cup C_j^c)\Big) \\
&\leq \sum_{j=1}^{\infty}\Big\{\mathbb{P}\Big(\frac{1}{t}L(t,\cdot) \in A_j^c\Big) + \mathbb{P}\Big(\frac{1}{t}L(t,\cdot) \in B_j^c\Big) + \mathbb{P}\Big(\frac{1}{t}L(t,\cdot) \in C_j^c\Big)\Big\} \\
&\leq \sum_{j=1}^{\infty}\Big(2e^{-jLt} + 2e^{-jLt}\Big) \\
&= 4e^{-Lt}\frac{1}{1-e^{-Lt}}.
\end{aligned}$$

To summarize:

For any $L > 0$ we can find a compact subset, K_L, in $L^p(\mathbb{R})$ such that:

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(\frac{1}{t}L(t,\cdot) \notin K_L) \leq -L.$$

We now finish the proof of the upper bound.

Let a positive real number L. Then there exists a compact set $K \subset L^p(\mathbb{R})$ such that:

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(\frac{1}{t}L(t,\cdot) \notin K) \leq -L.$$

We write:

$$\begin{aligned}
\mathbb{E}\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\} &= \mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\in K\}}\Big] \\
&\quad + \mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\notin K\}}\Big].
\end{aligned}$$

For each $g \in K$, by Hahn-Banach theorem, [4] (in a similar way we used it in the proof of the lower bound) there exists a bounded and continuous function f such that:

$$\Big(\int_{-\infty}^{\infty} |g(x)|^p \, \mathrm{d}x\Big)^{\frac{1}{p}} < \epsilon + \int_{-\infty}^{\infty} f(x)g(x) \, \mathrm{d}x,$$

where $\int_{-\infty}^{\infty} |f(x)|^q \, \mathrm{d}x = 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

By finite cover theorem, there are bounded and continuous functions $f_1, ..., f_m$ with $\int_{-\infty}^{\infty} |f_i(x)|^q \, \mathrm{d}x = 1$ for every $i = 1, ..., m$ such that:

$$\Big(\int_{-\infty}^{\infty} |g(x)|^p \, \mathrm{d}x\Big)^{\frac{1}{p}} < \epsilon + \max_{1 \le i \le m} \int_{0}^{t} f_i(x)g(x) \, \mathrm{d}x.$$

Hence,

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y) \, \mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\in K\}}\Big] &\le \mathbb{E}\Big[\exp\Big\{\theta\epsilon t + \theta \max_{1 \le i \le m} \int_0^t f_i(X_s) \, \mathrm{d}s\Big\}\Big] \\
&= \exp\Big\{\theta\epsilon t\Big\}\mathbb{E}\Big[\max_{1 \le i \le m} \exp\Big\{\theta \int_0^t f_i(X_s) \, \mathrm{d}s\Big\}\Big] \\
&\le \exp\Big\{\theta\epsilon t\Big\} \sum_{i=1}^{m} \mathbb{E} \exp\Big\{\theta \int_0^t f_i(X_s) \, \mathrm{d}s\Big\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty} L^p(t,y) \, \mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\in K\}}\Big] \\
&\le \theta\epsilon + \sup_{g \in \mathcal{F}_1} \Big[\theta\Big(\int_{-\infty}^{\infty} (g(x))^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}} - \frac{1}{2}\int_{-\infty}^{\infty} (g^{'}(x))^2 \pi(\mathrm{d}x)\Big],
\end{aligned}$$

and by letting $\epsilon \to 0^+$ we get:

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\in K\}}\Big]$$
$$\leq \sup_{g\in\mathcal{F}_1}\Big[\theta\Big(\int_{-\infty}^{\infty}(g(x))^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}}-\frac{1}{2}\int_{-\infty}^{\infty}(g^{'}(x))^2\pi(\mathrm{d}x)\Big].$$

We now focus on the second term and apply Holder's inequality:

$$\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\notin K\}}\Big]$$
$$\leq \Big(\mathbb{E}\Big[\exp\Big\{2\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{P}(\frac{1}{t}L(t,\cdot)\notin K)\Big)^{\frac{1}{2}}.$$

So,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\notin K\}}\Big]$$
$$\leq \limsup_{t\to\infty}\frac{1}{2t}\log\mathbb{E}\Big[\exp\Big\{2\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]+\limsup_{t\to\infty}\frac{1}{2t}\log\mathbb{P}(\frac{1}{t}L(t,\cdot)\notin K)$$
$$\leq \limsup_{t\to\infty}\frac{1}{2t}\log\mathbb{E}\Big[\exp\Big\{2\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]-\frac{1}{2}L.$$

As a result,

$$\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\notin K\}}\Big]\longrightarrow 0\ \ ,$$

by letting $L\to\infty$.

Here we used that:

$$\limsup_{t\to\infty}\frac{1}{2t}\log\mathbb{E}\Big[\exp\Big\{2\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]$$

$$\leq \limsup_{t\to\infty}\frac{1}{2t}\log e^{\frac{kt}{2}}\mathbb{E}\Big[\exp\Big\{2\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]$$

$$= \frac{k}{4}+\limsup_{t\to\infty}\frac{1}{2t}\log\mathbb{E}\Big[\exp\Big\{2\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]<\infty,$$

where the first limit contains the self intersection local time of the O-U process and the other limits of the Brownian motion.

Thus, we conclude:

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\Big]$$

$$\leq \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Big[\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}\mathbf{1}_{\{\frac{1}{t}L(t,\cdot)\in K\}}\Big]$$

$$\leq \sup_{g\in\mathcal{F}_1}\Big[\theta\Big(\int_{-\infty}^{\infty}(g(x))^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}}-\frac{1}{2}\int_{-\infty}^{\infty}(g^{'}(x))^2\pi(\mathrm{d}x)\Big].$$

To summarize, by combining the lower and upper bounds we have:

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{\theta\Big(\int_{-\infty}^{\infty}L^p(t,y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big\}$$

$$= \sup_{g\in\mathcal{F}_1}\Big[\theta\Big(\int_{-\infty}^{\infty}(g(x))^{2p}\pi(\mathrm{d}x)\Big)^{\frac{1}{p}}-\frac{1}{2}\int_{-\infty}^{\infty}(g^{'}(x))^2\pi(\mathrm{d}x)\Big].$$

Chapter 6

Future Plans

The area of large deviations has many applications. For instance, large deviations results can be used to prove existence and uniqueness of solutions of stochastic partial differential equations. This is something that we would definitely love to study in future. Another subject we would like to study in future is the mutual intersection local times, for which we have some results such as the following results which is about the existence.

Let $X_1(t), ..., X_p(t)$ be independent d-dimensional Ornstein-Uhlenbeck processes that satisfy:

$$\mathrm{d}X_i(t) = \mathrm{d}B(t) - kX_i(t)\,\mathrm{d}t,$$

where $i = 1, ..., p$, $k > 0$ and $p > 1, d \geq 1$ integers. $B(t)$ denotes here a d-dimensional Brownian motion.

For each $t = (t_1, ..., t_p) \in (\mathbb{R}^+)^p$ define:

$$\mu_t(B) = \int_0^{t_1} \cdots \int_0^{t_p} \mathbf{1}_B\Big(X_1(s_1) - X_2(s_2), ..., X_{p-1}(s_{p-1}) - X_p(s_p)\Big)\,\mathrm{d}s_1 \cdots \mathrm{d}s_p,$$

where $B \subset \mathbb{R}^{d(p-1)}$.

We investigate whether the above measure is absolutely continuous with respect to the

Lebesgue measure on $\mathbb{R}^{d(p-1)}$.

As in the self intersection local times, it is enough to check the absolutely continuity of the below measure:

$$\mu(B) = \int_0^\infty \cdots \int_0^\infty e^{-(s_1+\cdots+s_p)} \mathbf{1}_B\Big(X_1(s_1) - X_2(s_2), ..., X_{p-1}(s_{p-1}) - X_p(s_p)\Big)\,\mathrm{d}s_1 \cdots \mathrm{d}s_p.$$

To this end we need to check the convergence of the below integral:

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{E}|\hat{\mu}(\lambda_1, ..., \lambda_{p-1})|^2 \,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{p-1}.$$

First we compute the Fourier transform:

$$\begin{aligned}
\hat{\mu}(\lambda_1, ..., \lambda_{p-1}) &= \int_0^\infty \cdots \int_0^\infty e^{-(s_1+...+s_p)} e^{i\lambda_1\cdot(X_1(s_1)-X_2(s_2))+\cdots+\lambda_{p-1}\cdot(X_{p-1}(s_{p-1})-X_p(s_p))}\,\mathrm{d}s_1 \cdots \mathrm{d}s_p \\
&= \int_0^\infty \cdots \int_0^\infty e^{-(s_1+...+s_p)} e^{i\sum_{j=1}^p (\lambda_j - \lambda j-1)\cdot X_j(s_j)}\,\mathrm{d}s_1 \cdots \mathrm{d}s_p \\
&= \prod_{j=1}^p \int_0^\infty e^{-s} e^{i(\lambda_j - \lambda j-1)\cdot X_j(s)}\,\mathrm{d}s.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\hat{\mu}(\lambda_1, ..., \lambda_{p-1})|^2 &= \hat{\mu}(\lambda_1, ..., \lambda_{p-1}) \cdot \overline{\hat{\mu}(\lambda_1, ..., \lambda_{p-1})} \\
&= \prod_{j=1}^p \int_0^\infty e^{-s} e^{i(\lambda_j - \lambda j-1)\cdot X_j(s)}\,\mathrm{d}s \cdot \prod_{j=1}^p \int_0^\infty e^{-s} e^{-i(\lambda_j - \lambda j-1)\cdot X_j(s)}\,\mathrm{d}s \\
&= \prod_{j=1}^p \int_0^\infty \int_0^\infty e^{-(s+t)} e^{i(\lambda_j - \lambda j-1)\cdot(X_j(t)-X_j(s))}\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\mathbb{E}|\hat{\mu}(\lambda_1,...,\lambda_{p-1})|^2\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_{p-1}\\
&=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\prod_{j=1}^{p}\int_0^\infty\int_0^\infty e^{-(s+t)}e^{-\frac{1}{2k}|\lambda_j-\lambda_{j-1}|^2[1-e^{-k|t-s|}-\frac{1}{2}(e^{-kt}-e^{-ks})^2]}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_{p-1}\\
&=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\int_{(\mathbb{R}^+)^{2p}}e^{-\sum_{j=1}^p(t_j+s_j)}e^{-\frac{1}{2k}\sum_{j=1}^p|\lambda_j-\lambda_{j-1}|^2C_j}\,\mathrm{d}s_1\cdots\mathrm{d}s_p\,\mathrm{d}t_1\cdots\mathrm{d}t_p\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_{p-1}\\
&=\int_{(\mathbb{R}^+)^{2p}}e^{-\sum_{j=1}^p(t_j+s_j)}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}e^{-\frac{1}{2k}\sum_{j=1}^p|\lambda_j-\lambda_{j-1}|^2C_j}\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_{p-1}\,\mathrm{d}s_1\cdots\mathrm{d}s_p\,\mathrm{d}t_1\cdots\mathrm{d}t_p,
\end{aligned}$$

where $C_j=1-e^{-k|t_j-s_j|}-\frac{1}{2}(e^{-kt_j}-e^{-ks_j})^2$.

<u>d=1</u>

Let's bound first the following integral:

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}e^{-\frac{1}{2k}\sum_{j=1}^p|\lambda_j-\lambda_{j-1}|^2C_j}\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_{p-1}.$$

Notice that:

$$\int_{\mathbb{R}}e^{-\frac{1}{2k}[\lambda_1^2C_1+(\lambda_2-\lambda_1)^2C_2]}\,\mathrm{d}\lambda_1\le\left(\frac{\pi k}{C_1}\right)^{\frac{1}{4}}\left(\frac{\pi k}{C_2}\right)^{\frac{1}{4}},$$

where we used Holder's inequality.

By keep doing the same we get that:

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}e^{-\frac{1}{2k}\sum_{j=1}^p|\lambda_j-\lambda_{j-1}|^2C_j}\,\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_{p-1}\le\left(\frac{\pi k}{C_1}\right)^{\frac{1}{4}}\left(\frac{\pi k}{C_2}\right)^{\frac{1}{4}}\left(\frac{2\pi k}{C_3}\right)^{\frac{1}{2}}\cdots\left(\frac{2\pi k}{C_{p-1}}\right)^{\frac{1}{2}}\left(\frac{2\pi k}{C_p}\right)^{\frac{1}{2}}.$$

Hence,

$$\int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}} \int\limits_{(\mathbb{R}^+)^{2p}} e^{-\sum_{j=1}^{p}(t_j+s_j)} e^{-\frac{1}{2k}\sum_{j=1}^{p}(\lambda_j-\lambda_{j-1})^2 C_j} \,\mathrm{d}s_1 \cdots \mathrm{d}s_p \,\mathrm{d}t_1 \cdots \mathrm{d}t_p \,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{p-1}$$
$$\leq 2^{\frac{p-2}{2}} (\pi k)^{\frac{p-1}{2}} \int\limits_{(\mathbb{R}^+)^{2p}} \frac{1}{{C_1}^{\frac{1}{4}} {C_2}^{\frac{1}{4}} {C_3}^{\frac{1}{2}} \cdots {C_p}^{\frac{1}{2}}} e^{-\sum_{j=1}^{p}(t_j+s_j)} \,\mathrm{d}s_1 \cdots \mathrm{d}s_p \,\mathrm{d}t_1 \cdots \mathrm{d}t_p$$
$$= \int\limits_0^\infty \int\limits_0^\infty \frac{1}{C_1^{\frac{1}{4}}} e^{-(t_1+s_1)} \,\mathrm{d}s_1 \,\mathrm{d}t_1 \int\limits_0^\infty \int\limits_0^\infty \frac{1}{C_1^{\frac{1}{4}}} e^{-(t_2+s_2)} \,\mathrm{d}s_2 \,\mathrm{d}t_2 \int\limits_0^\infty \int\limits_0^\infty \frac{1}{C_3^{\frac{1}{2}}} e^{-(t_3+s_3)} \,\mathrm{d}s_3 \,\mathrm{d}t_3 \cdots$$
$$\cdots \int\limits_0^\infty \int\limits_0^\infty \frac{1}{C_p^{\frac{1}{2}}} e^{-(t_p+s_p)} \,\mathrm{d}s_p \,\mathrm{d}t_p < \infty,$$

which is finite as we saw in Chapter 2.

For any other dimensions $d > 1$ we do not bound the integral but we evaluate it:

$$\int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \int\limits_{(\mathbb{R}^+)^{2p}} e^{-\sum_{j=1}^{p}(t_j+s_j)} e^{-\frac{1}{2k}\sum_{j=1}^{p}|\lambda_j-\lambda_{j-1}|^2 C_j} \,\mathrm{d}s_1 \cdots \mathrm{d}s_p \,\mathrm{d}t_1 \cdots \mathrm{d}t_p \,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{p-1}.$$

We have:

$$\int\limits_{\mathbb{R}^d} e^{-\frac{1}{2k}C_1|\lambda_1|^2} e^{-\frac{1}{2k}C_2|\lambda_2-\lambda_1|^2} \,\mathrm{d}\lambda_1 = \Big(\frac{2\pi k}{C_1+C_2}\Big)^{\frac{d}{2}} e^{-\frac{1}{2k}(C_2+\frac{C_2^2}{C_1+C_2})|\lambda_2|^2}.$$

By following the same process we find that:

$$\int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \int\limits_{(\mathbb{R}^+)^{2p}} e^{-\sum_{j=1}^{p}(t_j+s_j)} e^{-\frac{1}{2k}\sum_{j=1}^{p}|\lambda_j-\lambda_{j-1}|^2 C_j} \,\mathrm{d}s_1 \cdots \mathrm{d}s_p \,\mathrm{d}t_1 \cdots \mathrm{d}t_p \,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{p-1}$$

$$= \int\limits_{(\mathbb{R}^+)^{2p}} e^{-\sum_{j=1}^{p}(t_j+s_j)} \frac{1}{(C_1+C_2)^{\frac{d}{2}}} \frac{1}{(C_2+\frac{C_2^2}{C_1+C_2}+C_3)^{\frac{d}{2}}} \cdots$$

$$\cdots \frac{1}{\Bigg(C_{p-1}+C_p+\frac{\frac{C_{p-1}^2}{\cdots}}{C_2+\frac{C_2^2}{C_1+C_2}}\Bigg)^{\frac{d}{2}}} \,\mathrm{d}s_1 \,\mathrm{d}t_1 \,\mathrm{d}s_2 \,\mathrm{d}t_2 \cdots \mathrm{d}s_p \,\mathrm{d}t_p.$$

For $p = 2$ for instance, our integral becomes:

$$\int\limits_{(\mathbb{R}^+)^4} e^{-(t_1+s_1)-(t_2+s_2)} \frac{1}{(C_1+C_2)^{\frac{d}{2}}} \, \mathrm{d}s_1 \, \mathrm{d}t_1 \, \mathrm{d}s_2 \, \mathrm{d}t_2,$$

by using a similar method as in the second chapter (which includes the theory of asymptotic functions) we conclude that in order to investigate the convergence of the above integral we need to investigate the convergence of the following integral:

$$\int\limits_{\mathbb{R}^d} \left(\frac{2}{|\lambda|^2+2} \right)^2 \mathrm{d}\lambda.$$

The last integral is finite if and only if d is less than 4.

We observe that when $p = 2$ the mutual intersection local time of the Ornstein-Uhlenbeck process exists if and only if: $p(d-2) < d$, which is the same as the condition that someone needs to check when it comes to the existence of the mutual intersection local time of the Brownian motion. ([5] , [6]).

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Vita

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