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Sequential Deformations of Hadamard Matrices and Commuting Squares

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Sequential Deformations of Hadamard Matrices and Commuting Squares

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To my mom, for encouraging my curiosity in mathematics. To my dad, for being relentlessly supportive in everything I have done. To my family and friends, for helping me survive graduate school. To my wife, for giving me the confidence to be the best version of myself.

Abstract

In this dissertation, we study analytic and sequential deformations of commuting squares of finite dimensional von Neumann algebras, with applications to the theory of complex Hadamard matrices. The main goal is to shed some light on the structure of the algebraic manifold of spin model commuting squares (i.e., commuting squares based on complex Hadamard matrices), in the neighborhood of the standard commuting square (i.e., the commuting square corresponding to the Fourier matrix). We prove two types of results: Non-existence results for deformations in certain directions in the tangent space to the algebraic manifold of commuting squares (chapters 3 and 4), and finiteness results for commuting squares based on Hadamard matrices with certain symmetries (chapter 5).

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Chapter 1

Introduction

This dissertation aims to gain a better understanding of the real algebraic manifold of complex Hadamard matrices, with applications to the theory of commuting squares and subfactors.

In Chapter 2, we provide all of the necessary background information for the remainder of the discussion. In particular, we discuss the connections between Hadamard matrices, spin model commuting squares, subfactors, and the standard invariant.

In Chapter 3, we focus our attention on the structure of the space of Hadamard matrices near the Fourier matrix. We introduce a set of relations that an analytic family of Hadamard matrices stemming from the Fourier matrix must satisfy. We then shift our attention to sequential deformations of the Fourier matrix and, extending a result from Nicoara and White (2020), prove that a sequence of Hadamard matrices converging to the Fourier matrix must satisfy the analytic relations up to the third order.

In Chapter 4, we present a proof of a result, conjectured in Barros e Sá and Bengtsson (2013), which states that, when three distinct primes divide n, there exist matrices in the tangent space at F_n which are not directions of convergence. This result is not expected to be true when only two distinct primes divide n, since for the case $n = 6$ it is conjectured that an analytic family of Hadamard matrices exists for each direction in the tangent space (see Barros e Sá and Bengtsson (2013) or Tadej and Życzkowski (2006)).

In Chapter 5, we present a new result which generalizes results from Haagerup (2008) and Nicoara and Worley (2019). The result shows that there are only finitely many Hadamard matrices in the algebra $U\mathcal{D}_nU^*$ when U has only non-zero minors.

Chapter 2

Background Information

In this chapter, we will introduce each of the main objects that will be studied in this dissertation - Hadamard matrices, commuting squares, and the standard invariant. Along the way, we will also give a brief but necessary introduction to von Neumann algebras.

2.1 Hadamard Matrices

Hadamard matrices are simple to describe but notoriously difficult to construct, and even basic questions about the structure of the space of Hadamard matrices remain unanswered.

2.1.1 Real Hadamard Matrices

Definition 2.1.1. A *real Hadamard matrix* is a square matrix satisfying two properties:

1. Each entry of the matrix is either 1 or -1.
2. The rows of the matrix are pairwise orthogonal.

Example 2.1.1. The following are Hadamard matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

There are of course several more examples of real Hadamard matrices that have been know for quite some time. In fact, the first appearance of Hadamard matrices came in an 1867 paper by English mathematician James Sylvester in which he showed the existence of a real $n \times n$ Hadamard matrix for all $n = 2^k$ (Sylvester, 1867). However, we are far from a complete classification of real Hadamard matrices. For example, the following question still does not have a complete solution: For which values of n does there exist a real $n \times n$ Hadamard matrix? It is known that if such a matrix exists then either $n = 2$ or n is a multiple of 4. However, the converse of this statement, which asks if there exists a real $n \times n$ Hadamard matrix for every n which is a multiple of 4, remains unresolved. This problem is known as Hadamard's conjecture and has been an open problem for well over 100 years! As of this writing, the smallest value of n for which the conjecture is still unresolved is $n = 668 = 4 \cdot 167$.

2.1.2 Complex Hadamard Matrices

A real Hadamard matrix allows entries of 1 or -1, so the natural generalization, a complex Hadamard matrix, allows entries to be any complex number with modulus 1. However, we choose to modify the definition from above slightly by normalizing the entries to simplify the second condition from above.

Definition 2.1.2. An $n \times n$ *complex Hadamard matrix* U, or simply a *Hadamard matrix*, is a matrix satisfying two properties:

1. Each entry of the matrix has the same modulus.

2. U is unitary: $UU^* = U^*U = I$.

Note that the second condition implies each entry must have modulus $\frac{1}{\sqrt{n}}$. Moving to the complex case, the problem of existence is solved. For each value of n an $n \times n$ Hadamard matrix can be constructed utilizing the nth root of unity $\varepsilon_n = e^{2\pi i/n}$.

Definition 2.1.3. The $n \times n$ *Fourier matrix*, denoted F_n, is constructed by placing ε_n^{ij} in the (i, j)th entry of F_n for all $i, j \in \mathbb{Z}_n$.

Example 2.1.2.

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, F_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

(here ω is the 3rd root of unity, $\omega = \varepsilon_3 = e^{2\pi i/3}$)

At this point, note that if we have a Hadamard matrix H, and we permute the rows or columns of H, the resulting matrix is still a Hadamard matrix. Additionally, if we multiply any row or column of H by a complex number with modulus 1, the resulting matrix is still a Hadamard matrix. For this reason, we often only consider Hadamard matrices up to a notion of equivalence.

Definition 2.1.4. If H_1, H_2 are two $n \times n$ Hadamard matrices, then we say that H_1 is equivalent to H_2 if there exist unitary diagonal matrices D, D' and permutation matrices P, P' such that

$$H_2 = DPH_1P'D'.$$

With this notion of equivalence in mind, we can now begin to talk about the problem of classifying all Hadamard matrices. For $n \leq 5$, Hadamard matrices have been completely classified (Haagerup, 1997). For $n = 2, 3, 5$ the Fourier matrix, F_n, is the only Hadamard matrix (up to equivalence). When $n = 4$, the space becomes more interesting.

Example 2.1.3. For all $|t| = 1$, the following is a Hadamard matrix.

$$U(t) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & it & -1 & -it \\ 1 & -1 & 1 & -1 \\ 1 & -it & -1 & it \end{pmatrix}$$

This family of matrices described above contains all of the equivalence classes of 4×4 Hadamard matrices. Note in particular that $U(1) = F_4$, and $U(-i)$ is the (normalized) real Hadamard matrix from Example 2.1.1.

A complete classification of $n \times n$ Hadamard matrices for any $n > 5$ remains incomplete. Examining the $n = 6$ case, it is conjectured that F_6 is contained in a 4-parameter continuous family of Hadamard matrices. This is an open problem, with applications in quantum information theory (Werner, 2001), on which dozens of papers have been written (see for instance Barros e Sá and Bengtsson (2013), Tadej and Życzkowski (2006)). To further complicate matters, a 6×6 Hadamard matrix discovered in Tao (2004), used to construct a counterexample to a problem in harmonic analysis, is known to be isolated.

Example 2.1.4. Tao (2004). Let $\omega = \varepsilon_3 = e^{2\pi i/3}$

$$T_6 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 \end{pmatrix}$$

It has been shown that the Fourier matrix F_p is isolated for p prime (Petrescu, 1997; Nicoara and White, 2014). It was conjectured by Popa that F_p was the only Hadamard matrix for p prime, as is the case for $p = 2, 3, 5$, but this was shown to be false when a continues family of 7×7 Hadamard matrices was discovered in Petrescu (1997).

Example 2.1.5. Let $\omega = \varepsilon_6 = e^{2\pi i/6}$. The following, first discovered in Petrescu (1997), is a Hadamard matrix for all $|t| = 1$.

$$P(t) = \frac{1}{\sqrt{7}} \begin{pmatrix} \omega t & \omega^4 t & \omega^5 & \omega^3 & \omega^3 & \omega & 1 \\ \omega^4 t & \omega t & \omega^3 & \omega^5 & \omega^3 & \omega & 1 \\ \omega^5 & \omega^3 & \omega \bar{t} & \omega^4 \bar{t} & \omega & \omega^3 & 1 \\ \omega^3 & \omega^5 & \omega^4 \bar{t} & \omega \bar{t} & \omega & \omega^3 & 1 \\ \omega^3 & \omega^3 & \omega & \omega & \omega^4 & \omega^5 & 1 \\ \omega & \omega & \omega^3 & \omega^3 & \omega^5 & \omega^4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2.2 A Brief Introduction to von Neumann Algebras

In this section, we will introduce von Neumann algebras and discuss inclusions of von Neumann algebras known as subfactors. We will start with a definition and characterization of von Neumann algebras. Throughout this section, $\mathcal{H}$ denotes a Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the space of bounded operators on $\mathcal{H}$. For a more detailed introduction to von Neumann Algebras, see Jones (2015).

Theorem 2.2.1 (Bicommutant Theorem)**.** If M is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ with identity, then the following are equivalent:

1. M is a von Neumann Algebra.
2. M is closed in the strong operator topology.
3. M is closed in the weak operator topology.
4. $M = M''$.

This dissertation will mostly deal with finite dimensional von Neumann algebras, which are know to be isomorphic to multimatrix algebras. More precisely:

Proposition 2.2.1. If M is a finite dimensional von Neumann algebra, then there exist $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$M \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Definition 2.2.1. A *factor* is a von Neumann algebra with trivial center.

Factors are the building blocks of von Neumann algebras, so the problem of classifying all von Neumann algebras can be reduced to the problem of classifying factors. Finding a complete classification of factors is a hopeless endeavor, however progress can be made towards classification of subfactors satisfying some nice properties. With this in mind, subfactors are often sorted into a few distinct groups denoted type I, type II_1, type II_∞, and type III. This dissertation will focus on type II_1 factors.

Definition 2.2.2. A factor M is a II_1 *factor* if M is infinite dimensional and has a faithful trace τ. Here a trace τ is a non-zero linear functional $\tau : M \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ and $\tau(x^*x) \geq 0$ for all $x, y \in M$. A trace is faithful if $\tau(x^*x) = 0$ if and only if $x = 0$.

In an attempt to better understand the structure of a von Neumann algebra, one can examine all of the possible ways in which one von Neumann algebra can sit inside of another. These inclusions are called *subfactors* and since Vaughan Jones' groundbreaking paper in 1983 Jones (1983), the study of subfactors has flourished finding connections to knot theory (with the discovery of the Jones polynomial Jones (1987)), representation theory, quantum information theory, ergodic theory (see for example Popa (2021)), and many more.

Definition 2.2.3. A *subfactor* is an inclusion of factors $N \subset M$ such that $1_N = 1_M$. If N and M are II_1 factors, then $N \subset M$ is a II_1 *subfactor.*

In Jones (1983), a process was introduced for constructing a new II_1 factor from any existing II_1 factor. Additionally, this process, known as the basic construction, can be iterated to obtain a tower of II_1 factors.

Definition 2.2.4. For $N \subset M$ a II_1 subfactor, denote $e_N \in \mathcal{B}(L^2(M))$ to be the orthogonal projection $L^2(M) \to L^2(N)$. The *basic construction* is the von Neumann algebra generated by M and e_N denoted $M_1 = \langle M, e_N \rangle$. This process can be iterated to obtain a tower of II_1 factors:

$$N \subset M \subset M_1 \subset M_2 \subset \dots$$

it is not obvious that this construction yields a factor at each iteration, and we again refer the reader to Jones (2015). We end this section by mentioning the famous result of Jones (1983) which gives a surprising restriction on the possible values that the index of an inclusion of II_1 factors can take.

Definition 2.2.5. If $N \subset M$ is a II_1 subfactor then the *index* $[M : N]$ of N in M is the real number $\dim_N(L^2(M))$.

For the definition of $\dim_N(L^2(M))$, see Murray and von Neumann (1937).

Theorem 2.2.2 (Jones (1983))**.** Let $N \subset M$ be a II_1 subfactor then

$$[M : N] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) : n = 3, 4, \dots \right\} \cup [4, \infty)$$

2.3 Commuting Squares

We now shift our attention to the objects known as commuting squares. Commuting squares were introduced by Sorin Popa Popa (1990). They arise naturally in subfactor theory as invariants and construction data for subfactors.

Definition 2.3.1. A commuting square of finite dimensional von Neumann Algebras, P_0, P_{-1}, Q_0, Q_{-1}, is a square of inclusions:

$$\begin{pmatrix} P_{-1} & \subset & P_0 & & \\ \cup & & \cup & , & \tau \\ Q_{-1} & \subset & Q_0 & & \end{pmatrix}$$

satisfying

$$Q_0 \ominus Q_{-1} \perp P_{-1} \ominus Q_{-1}$$

with the inner product on P_0 given by $\langle x, y\rangle = \tau(xy^*)$ where τ is a positive, faithful trace with $\tau(1) = 1$.

Here the symbol "$\ominus$" is defined by $A \ominus B := (B \cap A^{\perp})$. A large class of examples of commuting squares come from finite groups. Before, that example, we will introduce the matrix units, which will be used extensively throughout this dissertation.

Definition 2.3.2. The matrix unit $e_{i,j}$ for $i, j \in S$, where S is a finite indexing set, is the matrix in $M_{|S|}(\mathbb{C})$ which has 1 in the $(i, j)^{\text{th}}$ entry, and 0 elsewhere. These matrices satisfy the following multiplicative relation for $i, j, k, l \in S$:

$$e_{i,j}e_{k,l} = \begin{cases} e_{i,l} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

Example 2.3.1. Let G be a finite group of order n, let $\mathcal{D}_n$ denote the $n \times n$ diagonal matrices, and set $u_g = \sum_{h\in G} e_{h,g^{-1}h}$ for all $g \in G$. If $\mathbb{C}[G] := \mathit{span}\{u_g : g \in G\}$, then the following is a commuting square:

$$\mathfrak{C}_G := \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & & \\ \cup & & \cup & , & \frac{1}{n}\text{Tr} \\ \mathbb{C} & \subset & \mathbb{C}[G] & & \end{pmatrix}.$$

Note that $\mathbb{C}[G]$ is the *left regular representation* of the group algebra of G over the complex numbers.

A special case, which will play a leading role in this dissertation, comes when we set $G = \mathbb{Z}_n$. One can check that the matrices in $\mathbb{C}[\mathbb{Z}_n]$ are those which are constant on each diagonal. Such matrices are know as *circulant matrices.*

Definition 2.3.3. The $n \times n$ *circulant matrices*, denoted $\mathcal{C}_n$, are the collection of $n \times n$ matrices $C = (c_{i,j})$ which satisfy $c_{i,j} = c_{k,l}$ if $i - j \equiv k - l(n)$.

Example 2.3.2. The 4×4 circulant matrices are given by

$$\mathcal{C}_4 = \left\{ \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_3 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_0 \end{pmatrix} : x_0, x_1, x_2, x_3 \in M_n(\mathbb{C}) \right\}$$

Additionally, since $\mathbb{Z}_n$ is an abelian group, $\mathbb{C}[\mathbb{Z}_n] = \mathcal{C}_n$ forms an n dimensional commutative subalgebra of $M_n(\mathbb{C})$ (a maximal abelian *-subalgebra or MASA) and thus must be isomorphic to $\mathcal{D}_n$. This isomorphism is realized by conjugating the diagonal matrices with the Fourier matrix, F_n.

Proposition 2.3.1.

$$\mathcal{C}_n = \mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*.$$

Proof. Here we will show that $\mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*$. We use the Fourier basis of $\mathcal{D}_n$ given by $f_k := \sum_{i \in \mathbb{Z}_n} \varepsilon^{ik}$ for each $k \in \mathbb{Z}_n$ and check that

$$\frac{1}{n} F_n f_{-k} F_n^* = \sum_{i \in \mathbb{Z}_n} e_{i,i-k} = u_k.$$

Since F_n is unitary and $\mathcal{D}_n$ and $\mathbb{C}[\mathbb{Z}_n]$ both have dimension n (as a vector space), we must have that $\mathbb{C}[\mathbb{Z}_n] = F_n \mathcal{D}_n F_n^*$. □

With this particular example in mind it is natural to ask which MASA's, when placed in the bottom right corner, form a commuting square. Alternatively, for which unitary matrices U is the following a commuting square:

$$\mathfrak{C}(U) = \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & & \\ \cup & & \cup & , & \frac{1}{n}\mathrm{Tr} \\ \mathbb{C} & \subset & U\mathcal{D}_n U^* & & \end{pmatrix} ?$$

The answer, as the reader may have guessed, provides the connection between commuting squares and Hadamard matrices. Such commuting squares are known as *spin model* commuting squares.

Proposition 2.3.2. For a unitary matrix U, the following are equivalent

1. $\mathfrak{C}(U)$ is a commuting square.

2. U is a Hadamard matrix.

3. For all $d, d' \in \mathcal{D}_n$, $\tau(UdU^*d') = \tau(d)\tau(d')$.

Proof. Before we start the proof we will do this computation once: for $d, d' \in \mathcal{D}_n$ with $d = \sum_{k \in \mathbb{Z}_n} d_k e_{k,k}$ and $d' = \sum_{k' \in \mathbb{Z}_n} d'_k e_{k',k'}$ we have

$$\begin{aligned} UdU^*d' &= \left(\sum_{i,j \in \mathbb{Z}_n} u_{i,j} e_{i,j}\right) \cdot \left(\sum_{k \in \mathbb{Z}_n} d_k e_{k,k}\right) \cdot \left(\sum_{i',j' \in \mathbb{Z}_n} \overline{u_{i',j'}} e_{j',i'}\right) \cdot \left(\sum_{k' \in \mathbb{Z}_n} d'_{k'} e_{k',k'}\right) \\ &= \sum_{i,j,i' \in \mathbb{Z}_n} u_{i,j} d_j \overline{u_{i',j}} d'_{i'} e_{i,i'} \end{aligned}$$

so taking the trace,

$$\begin{aligned} \tau(UdU^*d') &= \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} u_{i,j} d_j \overline{u_{i,j}} d'_i \\ &= \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} |u_{i,j}|^2 d_j d'_i. \end{aligned}$$

(1) $\Rightarrow$ (2). Suppose $\mathfrak{C}(U)$ is a commuting square. To proceed we will need to use the Fourier basis for the diagonal matrices $f_k := \sum_{i \in \mathbb{Z}_n} \varepsilon^{ik} e_{i,i}$. Note that $f_k \in \mathcal{D}_n \ominus \mathbb{C}$ and $Uf_kU^* \in UDU^* \ominus \mathbb{C}$, for all $k \neq 0$. So for all $k \neq 0$ and $l \neq 0$ we have, using the commuting square relation, that

$$0 = \tau(Uf_kU^*f_{-l}) = \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} \varepsilon^{jk} |u_{i,j}|^2 \varepsilon^{-il}.$$

Note also that for $k = 0$ and $l \neq 0$ (and vice versa) we have $f_k = I$ so

$$\tau(U f_k U^* f_{-l}) = \tau(U U^* f_{-l}) = \tau(f_l) = 0$$

if $k = 0$ and $l = 0$ we have

$$\tau(U f_k U^* f_l) = \tau(I) = 1.$$

Now if we denote $V = (|u_{i,j}|^2)_{i,j \in \mathbb{Z}_N}$ note that we have

$$\tau(U f_k U^* f_{-l}) = (F_n V F_n^*)_{k,l}$$

so the above computations are equivalent to

$$e_{0,0} = F_n V F_n^* \;\Rightarrow\; F_n^* e_{0,0} F_n = V.$$

Thus we have for each $i, j \in \mathbb{Z}_n$

$$V_{i,j} = (F_n^* e_{0,0} F_n)_{i,j} = \frac{1}{n}$$

so we have $|u_{i,j}| = \frac{1}{\sqrt{n}}$ for all $i, j \in \mathbb{Z}_n$.

(2) $\Rightarrow$ (3) This follows immediately from the computation done at the beginning of the proof:

$$\tau(U d U^* d') = \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} |u_{i,j}|^2 d_i d'_j = \left(\frac{1}{n} \sum_{i \in \mathbb{Z}_n} d_i \right) \left(\frac{1}{n} \sum_{j \in \mathbb{Z}_n} d'_j \right) = \tau(d)\tau(d').$$

(3) $\Rightarrow$ (1) If $U d U^* \in U \mathcal{D}_n U^* \ominus \mathbb{C}$ and $d' \in \mathcal{D}_n \ominus \mathbb{C}$ then we have

$$\tau(U d U^* d') = \tau(d)\tau(d') = 0.$$

Thus $U \mathcal{D}_n U^* \ominus \mathbb{C} \perp \mathcal{D}_n \ominus \mathbb{C}$. $\square$

Consider the commuting square stemming from the Hadamard matrix H:

$$\mathfrak{C}(H) = \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & & \\ \cup & & \cup & , & \frac{1}{n}\mathrm{Tr} \\ \mathbb{C} & \subset & H\mathcal{D}_nH^* & & \end{pmatrix}.$$

The top row is an inclusion of finite dimensional von Neumann algebras with a trace, so we can apply the basic construction to the top row:

$$\mathcal{D}_n \subset M_n(\mathbb{C}) \subset P_1 = \langle M_n(\mathbb{C}), e_{\mathcal{D}_n}\rangle.$$

Additionally, the trace extends to P_1 by

$$\tau(xe_{\mathcal{D}_n}y) = \frac{1}{n}\tau(xy) \text{ for all } x, y \in M_n(\mathbb{C}).$$

Similarly, we can adjoin $e_{\mathcal{D}_n}$ to $H\mathcal{D}_nH^*$ on the bottom row and note that, since $H\mathcal{D}_nH^* \subset M_n(\mathbb{C})$, we have that

$$\mathbb{C} \subset H\mathcal{D}_nH^* \subset Q_1 = \langle H\mathcal{D}_nH^*, e_{\mathcal{D}_n}\rangle \subset P_1.$$

We can iterate this process and if we set P_H to be the weak closure of $\bigcup_{i\geq 1} P_i$, and Q_H to be the weak closure of $\bigcup_{i\geq 1} Q_i$. Note that P_H, Q_H are infinite dimensional, and the trace extends to P_H and Q_H so we have that $Q_H \subset P_H$ is a II_1 subfactor. Additionally, this subfactor has index n. So from every Hadamard matrix, we can construct a II_1 subfactor with index n.

2.4 The Standard Invariant

A problem of major interest in subfactor theory is determining whether or not a pair of subfactors are isomorphic, and a powerful tool used to accomplish this goal is known as the standard invariant.

Definition 2.4.1. For $N \subset M$ an inclusion of II_1 factors, the *standard invariant* is the sequence of commuting squares of finite dimensional von Neumann algebras

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \dots \end{array}$$

We note that all squares of inclusions in the standard invariant are commuting squares of finite dimensional von Neumann Algebras.

Chapter 3

Deformations of the Fourier Matrix

To motivate this section, consider the following example:

Example 3.0.1. For all $|t| = 1$, $U(t)$ is a Hadamard matrix where

$$U(t) := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & it & -1 & -it \\ 1 & -1 & 1 & -1 \\ 1 & -it & 1 & it \end{pmatrix}$$

Note that $U(1) = F_4$, so this gives us an analytic family of Hadamard matrices passing through the Fourier matrix F_4. This section, we will focus on gaining a better understanding of the structure of the space Hadamard matrices near the "nicest" possible point, the Fourier matrix F_n. To do this, we ask from which directions it is possible to approach F_n with an analytic family of Hadamard matrices. We will refer to the set of possible directions of convergence as the tangent space at F_n. Previous work of Nicoară and White (2017) has calculated the dimension of the tangent space and shown the existence of a basis for the tangent space such that each basis element yields an analytic family of Hadamard matrices approaching F_n along that direction. It remains unknown, however, whether or not one can approach F_n from every direction in the tangent space. This is because if $d_1, d_2 \in M_n(\mathbb{C})$ are linearly independent directions of convergence, is not necessarily true that $d_1 + d_2$ is also a direction of convergence. A clear understanding of exactly when one can approach F_n from each direction in the tangent space is desired. For example, it is known that F_6 has 4 linearly independent directions of convergence, and it is conjectured that these 4 directions combine to form a 4 parameter analytic of Hadamard matrices containing F_6. This relates to a major open problem in quantum information theory on which dozens of papers have been written (see Werner (2001); Barros e Sá and Bengtsson (2013); Tadej and Życzkowski (2006)). In this chapter, we find a set of restrictions that an element a of the tangent space needs to satisfy.

3.1 Analytic Deformations

During this section, we will discuss the relations which must be satisfied in order for an analytic family of Hadamard matrices passing through F_n to exist. Let $\{H_t\}, t \in \mathbb{R}$, be an analytic family of Hadamard matrices such that $H_t \to F_n$ as $t \to 0$ (with $H_t \neq F_n$ when $t \neq 0$) where F_n is the $n \times n$ Fourier matrix. Since F_n is unitary, this is equivalent to $U_t \to I$ where $U_t = H_t F_n^*$. Since U_t is unitary, for all t we have that $U_t U_t^* = I$. Furthermore, for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$ we must have (since $F_n p F_n^* \in \mathcal{D}_n$ and H_t is Hadamard, see Proposition 2.3.2)

$$\tau(U_t p U_t^* q) = \tau(H_t F_n^* p F_n H_t^* q) = \tau(H_t F_n^* p F_n H_t^*)\tau(q) = \tau(p)\tau(q) = \tau(pq).$$

Note that the last equality follows from the fact that F_n is a Hadamard matrix with $\mathcal{C}_n = F_n \mathcal{D}_n F_n^*$. In light of these two facts, we introduce the following bilinear continuous functions on $M_n(\mathbb{C}) \times M_n(\mathbb{C})$:

$$f_0(x, y) = xy$$

$$f^{p,q}(x, y) = \tau(xpyq) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Proposition 3.1.1. If $\mathcal{F} = \{f^{p,q} : p \in \mathcal{C}_n, q \in \mathcal{D}_n\} \cup \{f_0\}$, then H_t is an analytic family of Hadamard matrices passing through F_n if and only if there exists an analytic family of matrices U_t satisfying $f(U_t, U_t^*) = f(I, I)$ for all $f \in \mathcal{F}$.

Proof. The forward direction of this proof is outlined above. For the reverse direction, first note that U_t is unitary since

$$U_t U_t^* = f_0(U_t, U_t^*) = f_0(I, I) = I.$$

Since U_t, F_n are unitary, H_t is unitary for all t. Additionally, we have for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$ that

$$\tau(U_t p U_t^* q) = f^{p,q}(U_t, U_t^*) = f^{p,q}(I, I) = \tau(pq).$$

Using this fact, for all $q, q' \in \mathcal{D}_n$ we have

$$\tau(H_t q H_t^* q') = \tau(U_t F_n q F_n^* U_t^* q) = \tau(F_n q F_n^* q) = \tau(F_n q F_n^*)\tau(q) = \tau(H_t q H_t^*)\tau(q).$$

Note that we used the fact that $F_n q F_n^* \in \mathcal{C}_n$, and in the last equality we introduced $U_t U_t^* = I$ into the first trace and used the property of the trace to rearrange. □

If $U_t \to I$ is such an analytic family, we may write for $a_j \in M_n(\mathbb{C})$ with $a_0 = I$

$$U_t = \sum_{j=0}^{\infty} a_j t^j \text{ and } U_t^* = \sum_{j=0}^{\infty} a_j^* t^j$$

Proposition 3.1.2. If $f(U_t, U_t^*) = f(I, I)$ for all $f \in \mathcal{F}$ then we must have:

$$f(a_j, I) + f(I, a_j^*) = -\sum_{k=1}^{j-1} f(a_{j-k}, a_k^*)$$

for all $j \geq 1$.

Proof.

$$
\begin{aligned}
& f(U_t, U_t^*) = f(I, I) \\
\iff\ & f\left(\sum_{l=0}^{\infty} a_l t^l, \sum_{k=0}^{\infty} a_k^* t^k\right) = f(I, I) \\
\iff\ & \sum_{j=1}^{\infty}\left(\sum_{l+k=j} f(a_l, a_k^*)\right) t^j = 0 \\
\iff\ & \sum_{j=1}^{\infty}\left(\sum_{k=0}^{j} f(a_{j-k}, a_k^*)\right) t^j = 0 \\
\iff\ & \sum_{k=0}^{j} f(a_{j-k}, a_k^*) = 0 \text{ for all } j \geq 1. \\
\iff\ & f(a_j, I) + f(I, a_j^*) = -\sum_{k=1}^{j-1} f(a_{j-k}, a_k^*)
\end{aligned}
$$

□

We prefer to think of the function $f_0(x, y) = xy$ as a normalizing function, and we wish to spend more of our time focusing on the functions $\mathcal{F}' := \{f^{p,q}(x, y) = \tau(xpyq) : p \in \mathcal{C}_n, q \in \mathcal{D}_n\}$. For this reason we now separate into two conditions. First, if we do have $f_0(U_t, U_t^*) = f(I, I)$ then we have for all j that

$$a_j + a_j^* = -\sum_{k=1}^{j-1} a_{j-k} a_k^*$$

so solving for a_j^* and plugging into the relation for $f \in \mathcal{F}'$, we obtain

$$f(a_j, I) - f(I, a_j) = \sum_{k=1}^{j-1} f(I, a_{j-k} a_k^*) - f(a_{j-k}, a_k^*)$$

$$\Rightarrow \tau(a_j[p, q]) = \sum_{k=1}^{j-1} \tau([p, a_{j-k}] a_k^* q)$$

for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$.

For the first case, $j = 1$, we see that a_1 must satisfy $a_1 + a_1^* = 0$ and $\tau(a_1[p, q]) = 0$ for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$. This is equivalent to requiring $a_1 \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ with a normalization $a_1^* = -a_1$. Now we define the following space, which could be thought of as the set of directions in which it is possible to approach F_n with an analytic family of Hadamard matrices.

Definition 3.1.1.

$$A = \{a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp : a^* = -a\}.$$

So to calculate an upper bound on the number of possible "directions" in which you can approach F_n, one need only calculate the dimension of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$. Additionally, this idea was extended in Nicoara and White (2014) to calculate the number of possible "directions" of convergence of non-equivalent Hadamard matrices. In Nicoară (2006); Petrescu (1997) it was shown that the first order relations are strong enough to show that F_n is isolated among non-isomorphic Hadamard matrices whenever n is prime. It is now natural to ask if the higher order relations give a further restriction on the dimension of this space. Equivalently, are directions of convergence in A which violate higher order relations?

Definition 3.1.2. For $a_1 \in A$, if there exist $a_2, \ldots, a_j$ satisfying for all $2 \leq l \leq j$

$$a_l + a_l^* = -\sum_{k=1}^{l-1} a_{l-k} a_k^*$$

and

$$\tau(a_l[p, q]) = \sum_{k=1}^{l-1} \tau([p, a_{l-k}] a_k^* q)$$

then we say a_1 satisfies the jth order relations.

In Nicoara and White (2020) it was shown that every $a \in A$ satisfies the second order relations. In other words, the second order relation gives no further restriction on the number of possible directions from which you can approach F_n. It was conjectured by Barros e Sá and Bengtsson (2013), and will be shown to be true here that, when n is divisible by at least three distinct primes, there are some $a \in A$ that do not satisfy the third order relations. In this case then, the third order relations do give a restriction on the directions which it is possible to approach F_n with an analytic family of Hadamard matrices.

Definition 3.1.3. For $a_1 \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, if there exist $a_2, \ldots, a_j$ satisfying for all $1 \leq l \leq j$

$$\tau(a_l[p,q]) = \sum_{k=1}^{l-1} \tau([p, a_{l-k}]a_k^* q)$$

then we say a_1 satisfies the jth order trace relations.

To prove this result, we will construct $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ that violates the third order trace relations. So we first need to show that, in some sense, the jth order relations are equivalent to the jth order trace relations, at least up to $j = 3$.

3.1.1 Second Order Relations

To begin, we will show that if we have $a \in A$, then to show that a satisfies the 2nd order relations it is sufficient to show that a satisfies the 2nd order trace relations.

Proposition 3.1.3. Suppose $a \in A$. a satisfies the second order relations if and only if a satisfies the second order trace relations.

Proof. The second order trace relations are less restrictive than the second order relations, so we need only show that, when a satisfies the second order trace relations, a in fact satisfies the second order relations. Suppose a satisfies the 2nd order trace relations, so we have a_2' satisfying for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$

$$\tau(a_2'[p,q]) = \tau([p,a]a^*q).$$

Taking the adjoint of this relation, replacing p with p^* and q with q^* (note $\mathcal{C}_n$ and $\mathcal{D}_n$ are closed under $*$), and adding the relations together we get

$$\begin{aligned}\tau((a_2' + a_2'^*)[p,q]) &= \tau([p,a]a^*q) + \tau(a[p,a^*]q) \\ &= -\tau(aa^*[p,q]).\end{aligned}$$

Rearranging we see that

$$\tau((a_2' + a_2'^* + aa^*)[p, q]) = 0.$$

So if we set $a_2 = a_2' - \frac{a_2' + a_2'^* + aa^*}{2}$, then we have both

$$\tau(a_2[p, q]) = \tau(a_2'[p, q]) = \tau([p, a]a^* q)$$

and

$$\begin{aligned} a_2 + a_2^* &= \left(a_2' - \frac{a_2' + a_2'^* + aa^*}{2}\right) + \left(a_2'^* - \frac{a_2'^* + a_2' + aa^*}{2}\right) \\ &= -aa^*. \end{aligned}$$

Thus a satisfies the second order relations. □

Earlier, we mentioned that we would like to show the equivalence of the jth order trace relations and the jth order relations. The previous proposition shows that, if we know $a \in A$ then we need only check the trace relation to show that a in fact satisfies the full second order relation. In the following proposition, we describe what we mean when we say the two relations are equivalent:

Proposition 3.1.4. Every $a \in A$ satisfies the 2nd order relations if and only if every $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the 2nd order trace relations.

Proof. For the reverse direction, note that, since $A \subset [\mathcal{C}_n, \mathcal{D}_n]^\perp$, every $a \in A$ satisfies the 2nd order trace relations, so applying Proposition 3.1.3, we have that every $a \in A$ satisfies the 2nd order relations.

Now suppose that each $a \in A$ satisfies the 2nd order relations, and note that any element in $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ can be decomposed as $a = ia' + a''$ where $a', a'' \in A$ by setting

$$a' = \frac{a + a^*}{2i}, \; a'' = \frac{a - a^*}{2}.$$

For this reason, the trace relation for a becomes

$$\begin{aligned}\tau([p,a]a^*q) &= \tau([p,ia'+a''](ia'+a'')^*q)\\ &= \tau([p,a']a'^*q) + i(\tau([p,a']a''^*q) + \tau([p,a'']a'^*q)) + \tau([p,a'']a''^*q).\end{aligned}$$

From our assumption, we know that there exist a_2', a_2'' such that

$$\begin{aligned}\tau(a_2'[p,q]) &= \tau([p,a']a'^*q)\\ \tau(a_2''[p,q]) &= \tau([p,a'']a''^*q).\end{aligned}$$

So the following proposition will complete the proof. □

Proposition 3.1.5. If every $a \in A$ satisfies the second order relation then for every pair, $a', a'' \in A$, there exists x such that

$$\tau(x[p,q]) = \frac{1}{2}\left(\tau([p,a']a''^*q) + \tau([p,a'']a'^*q)\right).$$

Proof. Let $a = a' + a'' \in A$. Let a_2, a_2', a_2'' satisfy

$$\begin{aligned}\tau(a_2[p,q]) &= \tau([p,a]a^*q)\\ \tau(a_2'[p,q]) &= \tau([p,a']a'^*q)\\ \tau(a_2''[p,q]) &= \tau([p,a'']a''^*q).\end{aligned}$$

Set $x = \frac{a_2 - a_2' - a_2''}{2}$ then we have

$$\begin{aligned}\tau(x[p,q]) &= \frac{1}{2}\left(\tau([p,a]a^*q) - \tau([p,a']a'^*q) - \tau([p,a'']a''^*q)\right)\\ &= \frac{1}{2}\left(\tau([p,a']a''^*q) + \tau([p,a'']a'^*q)\right).\end{aligned}$$

□

We now recall the main result of Nicoara and White (2020).

Theorem 3.1.1. Every a satisfying the first order relations satisfies the second order relations.

Since we now know that every $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, satisfies the 2nd order trace relations, we would like to be able to write down, in some concrete way, for each $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ an element a_2 that satisfies the 2nd order trace relation. In light of Proposition 3.1.5, we can do slightly better than that.

Theorem 3.1.2. There exists a bilinear continuous map

$$\varphi_2 : [\mathcal{C}_n, \mathcal{D}_n]^\perp \times [\mathcal{C}_n, \mathcal{D}_n]^\perp \to [\mathcal{C}_n, \mathcal{D}_n]$$

satisfying for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

$$\tau(\varphi_2(a', a'')[p, q]) = \frac{1}{2}\left(\tau([p, a']a''^* q) + \tau([p, a'']a'^* q)\right).$$

Proof. From Theorem 3.1.2, Proposition 3.1.4, and Proposition 3.1.5 for every pair $(a', a'') \in [\mathcal{C}_n, \mathcal{D}_n]^\perp \times [\mathcal{C}_n, \mathcal{D}_n]^\perp$ there exists x such that

$$\tau(x[p, q]) = \frac{1}{2}\left(\tau([p, a']a''^* q) + \tau([p, a'']a'^* q)\right).$$

Define $\varphi_2(a', a'')$ to be the projection of x onto $[\mathcal{C}_n, \mathcal{D}_n]$. This map is well defined since if x' is any other such element, then $\tau((x - x')[p, q]) = 0$ which means $x - x' \perp [\mathcal{C}_n, \mathcal{D}_n]$. □

3.1.2 Third Order Relations

From the previous section, for any $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ we can find a particular $a_2 = \varphi_2(a, a)$ that satisfies the second order trace relation. In what follows, Propositions 3.1.6 and 3.1.8 will show that any choice of element a_2 satisfying the second order (trace) relation is sufficient for showing that a_3 satisfies the third order (trace) relations. In particular, we may use $a_2 = \varphi_2(a, a)$ when we want to show that a satisfies the third order trace relations.

Proposition 3.1.6. If $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the third order trace relations then for every a_2 such that

$$\tau(a_2[p,q]) = \tau([p,a]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n$$

there exists a_3 such that

$$\tau(a_3[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Proof. Since a satisfies the third order relations we have a_2' and a_3' satisfying the respective trace relations. Note that $a_2 - a_2' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ so there exists $y \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ such that $a_2' = a_2 + y$. Plugging in this relation we have

$$\begin{aligned}\tau(a_3'[p,q]) &= \tau([p,a]a_2'^*q) + \tau([p,a_2']a^*q) \\ &= \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) + \tau([p,a]y^*q) + \tau([p,y]a^*q) \\ &= \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) + 2\tau(\varphi_2(a,y)[p,q]).\end{aligned}$$

$a_3 = a_3' - 2\varphi_2(a,y)$ satisfies the desired relation. □

This next result should be compared to Proposition 3.1.3, as it allows us to avoid checking the third order normalization condition to show that a particular element satisfies the third order relations. In fact, the arguments in Propositions 3.1.3 and 3.1.7 can be generalized for the jth order relations.

Proposition 3.1.7. a satisfies the third order relations if and only if a satisfies the third order trace relations.

Proof. If a satisfies the third order relations then a must satisfy the third order trace relations. Now suppose a satisfies the third order trace relations, then there exist a_2 such that (since a satisfies the second order trace relations and thus the second order relations from Proposition 3.1.3)

$$a_2 + a_2^* = -aa^*$$

$$\tau(a_2[p,q]) = \tau([p,a]a^*q).$$

From the previous Proposition 3.1.6, we have a_3' such that

$$\tau(a_3'[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q).$$

Take the adjoint of this relation, (replace p with p^* and q with q^*) and adding the two relations together we get

$$\begin{aligned}\tau((a_3' + a_3'^*)[p,q]) &= \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) + \tau(a_2[p,a^*]q) + \tau(a[p,a_2^*]q) \\ &= -\tau(aa_2^*[p,q]) - \tau(a_2a^*[p,q]).\end{aligned}$$

Rearranging shows

$$\tau((a_3' + a_3'^* + aa_2^* + a_2a^*)[p,q]) = 0.$$

So if we set $a_3 = a_3' - \frac{a_3'+a_3'^*+aa_2^*+a_2a^*}{2}$ we have both

$$\tau(a_3[p,q]) = \tau(a_3'[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q)$$

and

$$\begin{aligned}a_3 + a_3^* &= \left(a_3' - \frac{a_3' + a_3'^* + aa_2^* + a_2a^*}{2}\right) + \left(a_3'^* - \frac{a_3'^* + a_3' + aa_2^* + a_2a^*}{2}\right) \\ &= -aa_2^* - a_2a^*.\end{aligned}$$

Thus a satisfies the third order relations. □

Proposition 3.1.8. If a satisfies the third order relations, then for every a_2 satisfying

$$a_2 + a_2^* = -aa^*$$

$$\tau(a_2[p,q]) = \tau([p,a]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n$$

there exists a_3 such that

$$a_3 + a_3^* = -aa_2^* - a_2a^*$$

$$\tau(a_3[p,q]) = \tau([p,a]a_2^*q) + \tau([p,a_2]a^*q) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Now we are ready for the "equivalence" result for the third order relations.

Proposition 3.1.9. Every $a \in A$ satisfies the third order relations if and only if every $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the third order trace relations.

Proof. If each $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the third order trace relations, then, since $A \subset [\mathcal{C}_n, \mathcal{D}_n]^\perp$, each $a \in A$ satisfies the third order trace relations. From Proposition 3.1.7, then each $a \in A$ must satisfy the third order trace relations.

Now suppose each $a \in A$ satisfies the third order relations. Again, we start with the fact that each element $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ may be written as $a = ia_0 + a_1$ with $a_0, a_1 \in A$. From our assumption, there exist a_3', a_3'' such that for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

$$\tau(a_3'[p,q]) = \tau([p,a_0]\varphi_2(a_0,a_0)^*q) + \tau([p,\varphi_2(a_0,a_0)]a_0^*q)$$

$$\tau(a_3''[p,q]) = \tau([p,a_1]\varphi_2(a_1,a_1)^*q) + \tau([p,\varphi_2(a_1,a_1)]a_1^*q).$$

Note also that we have for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$ that

$$\tau(\varphi_2(a,a)[p,q]) = \tau([p,a]a^*q).$$

Then, expanding, this expression becomes

$$\begin{aligned}\tau([p,a]\varphi_2(a,a)^*q) + \tau([p,\varphi_2(a,a)]a^*q) &= \sum_{j,k,l=0}^{1} i^{j+k+l}(\tau([p,a_j]\varphi_2(a_k,a_l)^*q) + \tau([p,\varphi_2(a_k,a_l)]a_j^*q)) \\ &= \sum_{t=0}^{3} i^t \sum_{j+k+l=t} \tau([p,a_j]\varphi_2(a_k,a_l)^*q) + \tau([p,\varphi_2(a_k,a_l)]a_j^*q).\end{aligned}$$

If we set

$$s_t = \sum_{j+k+l=t} \tau([p, a_j]\varphi_2(a_k, a_l)^* q) + \tau([p, \varphi_2(a_k, a_l)]a_j^* q)$$

then note that

$$s_0 = \tau(a_3'[p, q])$$

$$s_3 = \tau(a_3''[p, q])$$

so to complete this proof we need to show that for any pair $a_0, a_1 \in A$, there exists x_1, x_2 such that

$$s_1 = \tau(x_1[p, q])$$

$$s_2 = \tau(x_2[p, q]).$$

Set $\tilde{a} = a_0 + a_1$, then there exists $\tilde{a}_3$ such that for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

$$\begin{aligned}\tau(\tilde{a}_3[p, q]) &= \tau([p, \tilde{a}]\varphi_2(\tilde{a}, \tilde{a})^* q) + \tau([p, \varphi_2(\tilde{a}, \tilde{a})]\tilde{a}^* q) \\ &= \sum_{t=0}^{3} s_t \\ &= \tau(a_3'[p, q]) + \tau(a_3''[p, q]) + s_1 + s_2.\end{aligned}$$

Now set $\tilde{a}' = a_0 - a_1$, then there exists $\tilde{a}_3'$ such that for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

$$\begin{aligned}\tau(\tilde{a}_3'[p, q]) &= \tau([p, \tilde{a}']\varphi_2(\tilde{a}', \tilde{a}')^* q) + \tau([p, \varphi_2(\tilde{a}', \tilde{a}')]\tilde{a}'^* q) \\ &= \sum_{t=0}^{3} (-1)^t s_t \\ &= \tau(a_3'[p, q]) - \tau(a_3''[p, q]) - s_1 + s_2.\end{aligned}$$

Adding/ subtracting these relations show that $x_1 = \frac{\tilde{a}_3 - \tilde{a}_3'}{2} - a_3''$ and $x_2 = \frac{\tilde{a}_3 + \tilde{a}_3'}{2} - a_3'$ satisfy the desired relation. □

To conclude this section, lets take note of what has been accomplished. The goal is to construct an element $a \in A$ that does not satisfy the third order relations. From what has been shown up to this point, we need only identify $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ that does not satisfy the third order trace relations. Additionally, note that, for a given a, there may be many a_2 satisfying

$$\tau(a_2[p,q]) = \tau([p,a]a^*q),$$

but we have shown that we need only show that the third order trace relations are violated for a specified a_2 (this is the contrapositive of Proposition 3.1.6). In particular, we may choose $a_2 = \varphi_2(a,a)$ from Theorem 3.1.2.

3.2 Sequential Deformations

To obtain the relations in the previous section, we assumed that we had an analytic family of Hadamard matrices converging to F_n. In this section, we assume only that we have a sequence of Hadamard matrices converging to F_n and, surprisingly, we obtain the same relations; at least up to the third order.

In this section, we will show that when we have a sequence of Hadamard matrices converging to F_n we find, by taking "derivatives" of the sequence, that the sequence must yield a direction of convergence (as described in Nicoară (2006)) $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfying the same relations that a direction of convergence of an analytic family must satisfy. It was shown in Nicoară (2006) that the first order relations hold for sequences, and in Nicoara and White (2014) it was shown that the second order relations hold for sequences. In this section we will include both of those results, and present a new result that shows the third order relations hold for sequences.

Let $\{H_k\},\ k \geq 0$, be a sequence of Hadamard matrices such that $H_k \to F_n$. Similar to in section 2, if we set $U_k = H_k F_n^*$, then we have that $U_k \to I$. We now recall the bilinear continuous functions on $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ from 3.1,

$$f_0(x, y) = xy$$

$$f^{p,q}(x, y) = \tau(xpyq) \text{ for all } p \in \mathcal{C}_n, q \in \mathcal{D}_n.$$

Proposition 3.2.1. If $\mathcal{F} = \{f^{p,q} : p \in \mathcal{C}_n, q \in \mathcal{D}_n\} \cup \{f_0\}$, then H_k is a sequence of Hadamard matrices passing through F_n if and only if for $U_k := H_k F_n^*$ we have $f(U_k, U_k^*) = f(I, I)$ for all $f \in \mathcal{F}$.

We now make an additional assumption that $U_k \neq I$ for any k. If we set $t_k = \|U_k - I\|$, then we have that $t_k \neq 0$ and $t_k \to 0$. Set $a_k := \dfrac{U_k - I}{t_k}$ then, from the compactness of the unit ball, by restricting to a subsequence we may assume that there exists $a \in M_n(\mathbb{C})$ such that $a_k \to a$. We formally define a direction of convergence as in Nicoară (2006):

Definition 3.2.1. Let $\{U_k\}_{k \geq 0}$ be a sequence of unitary matrices in $M_n(\mathbb{C})$ such that $U_k \to I$ as $k \to \infty$ and $U_k \neq I$ for all k. $a \in M_n(\mathbb{C})$ is a *direction of convergence* of $\{U_k\}$ if a subsequence of $a_k := \dfrac{U_k - I}{\|U_k - I\|}$ converges to a. Note that, since $\|a_k\| = 1$ for all k, we are guaranteed that some subsequence converges.

Now we restrict ourselves to a subsequence of a_k satisfying $a_k \to a$ so that a is a direction of convergence. Rearranging $a_k = \dfrac{U_k - I}{t_k}$, we have $U_k = t_k a_k + I$ and applying Proposition 3.2.1 we must have for all $f \in \mathcal{F}$ that

$$f(t_k a_k + I, t_k a_k^* + I) = f(I, I).$$

Expanding this relation becomes

$$t_k^2 f(a_k, a_k^*) + t_k(f(I, a_k^*) + f(a_k, I)) = 0.$$

Dividing both sides by t_k and taking a limit as $k \to \infty$ we see that a must satisfy

$$f(a, I) + f(I, a^*) = 0.$$

Proposition 3.2.2. If a is a direction of convergence of $\{U_k\}$, then a satisfies the first order relations.

Continuing this train of thought, as it was done in Nicoara (2011), let

$$b_k = \frac{\frac{U_k - I}{t_k} - a}{t_k} = \frac{a_k - a}{t_k}.$$

Since $a_k \to a$, we have that $t_k b_k \to 0$. In contrast to the first order relation, we now have no control over $\|b_k\|$, so we cannot assume that b_k converges (or has a convergent subsequence). However, we do still have that $t_k b_k \to 0$ since $a_k \to a$. Rearranging this relation gives $t_k^2 b_k + t_k a + I = U_k$, so applying Proposition 3.2.1, we have

$$f(t_k^2 b_k + t_k a + I, t_k^2 b_k^* + t_k a^* + I) = f(I, I).$$

Expanding this relation, we have

$$t_k^4 f(b_k, b_k^*) + t_k^3 (f(b_k, a^*) + f(a, b_k^*)) + t_k^2 (f(b_k, I) + f(I, b_k^*)) + t_k (f(a, I) + f(I, a^*)) = 0.$$

From Proposition 3.2.2, we have that $f(a, I) + f(I, a^*) = 0$, so dividing this relation by t_k^2 and taking a limit we have (using $t_k b_k \to 0$)

$$\lim_{k\to\infty} f(b_k, I) + f(I, b_k^*) = -f(a, a^*).$$

So it seems like in the limit, the b_k satisfy the relation we want, but we cannot guarantee the b_k converge. Applying $f_0(x, y) = xy$, we do have that

$$\lim_{k\to\infty} b_k + b_k^* = -aa^*.$$

Plugging this relation into the relation for $f^{p,q}(x, y) = \tau(xpyq)$ we have for all $p \in \mathcal{C}_n$, $q \in \mathcal{D}_n$

$$\lim_{k\to\infty} \tau(b_k[p, q]) = \tau([p, a]a^* q).$$

Now if we write $b_k = b'_k + b''_k$ where $b'_k \in [\mathcal{C}_n, \mathcal{D}_n]$ and $b''_k \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ this relation becomes

$$\lim_{k\to\infty} \tau(b'_k[p,q]) = \tau([p,a]a^*q).$$

Since this holds for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$, we must have that $\tau(b'_k x)$ converges for all $x \in [\mathcal{C}_n, \mathcal{D}_n]$ which means b'_k converges weakly to some b'. Since we are working in a finite dimensional Hilbert space, we also have that $b'_k \to b'$ an any norm. Thus there exists $b' \in [\mathcal{C}_n, \mathcal{D}_n]$ such that for all $p \in \mathcal{C}_n$ and $q \in \mathcal{D}_n$

$$\tau\left(b'[p,q]\right) = \tau([p,a]a^*q) = \lim_{k\to\infty} \tau(b'_k[p,q]).$$

Note this implies $b'_k \to b'$ and a satisfies the second order trace relation. Applying Proposition 3.1.3, then we have that a satisfies the second order relations.

Proposition 3.2.3. If a is a direction of convergence of $\{U_k\}$, then a satisfies the second order relations.

We now will continue along the same line of reasoning and show a new result: a direction of convergence of $\{U_k\}$ must satisfy the third order relations. Since a satisfies the second order relations, there exists b such that, for all $f \in \mathcal{F}$,

$$f(b, I) + f(I, b^*) = -f(a, a^*).$$

Furthermore, this implies that $\tau(b[p,q]) = \tau(b'[p,q])$, so there exists $b'' \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ such that $b = b' + b''$. Note now that we have $b'_k \to b'$, but we do not have that $b_k \to b$. Set

$$c_k = \frac{\frac{\frac{U_k - I}{t_k} - a}{t_k} - b}{t_k} = \frac{\frac{a_k - a}{t_k} - b}{t_k} = \frac{b_k - b}{t_k}$$

which after rearranging gives

$$U_k = c_k t_k^3 + t_k^2 b + t_k a + I.$$

Note now that $t_k^2 c_k \to 0$, and that $t_k c_k = b_k - b$. Now for all $f \in \mathcal{F}$ we have

$$f(U_k, U_k^*) = f(I, I)$$

so we now expand this relation. Because a satisfies the second order relations, the terms corresponding to t_k and t_k^2 cancel, so we divide both sides of the relation by t_k^3 to obtain on the left hand side:

$$t_k^3 f(c_k, c_k^*) + t_k^2(f(b, c_k^*) + f(c_k, b^*)) + t_k(f(c_k, a^*) + f(a, c_k^*) + f(b, b^*)) + f(c_k, I) + f(I, c_k^*),$$

and on the right hand side:

$$= -f(a, b^*) - f(b, a^*).$$

Taking the limit as $k \to \infty$ (using $t_k^2 c_k \to 0$) this becomes:

$$\lim_{k\to\infty} t_k^3 f(c_k, c_k^*) + t_k(f(c_k, a^*) + f(a, c_k^*)) + f(c_k, I) + f(I, c_k^*) = -f(a, b^*) - f(b, a^*).$$

We can decompose each $c_k = c_k' + c_k''$ where $c_k' \in [\mathcal{C}_n, \mathcal{D}_n]$ and $c_k'' \in [\mathcal{C}_n, \mathcal{D}_n]^{\perp}$. Note that we have some convergence for $c_k' \in [\mathcal{C}_n, \mathcal{D}_n]$; since $t_k c_k' = b_k' - b'$, we know that $t_k c_k' \to 0$. Using this fact, this limit simplifies to

$$\lim_{k\to\infty} - t_k^3 \tau([p, c_k'']c_k''^* q) - t_k(\tau([p, c_k'']a^* q) + \tau([p, a]c_k''^* q)) + \tau(c_k[p, q]) = \tau([p, a]b^* q) + \tau([p, b]a^* q).$$

Applying $f_0(x, y) = xy$ this expression becomes:

$$\lim_{k\to\infty} t_k^3 c_k'' c_k''^* + t_k(c_k'' a^* + a c_k''^*) + c_k + c_k^* = -ab^* - ba^*.$$

Plugging in this relation to the functions $f^{p,q}(x, y) = \tau(pxqy)$ for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$, this expression can be rearranged to become

$$\lim_{k\to\infty} - t_k^3 \tau([p, c_k'']c_k''^* q) - t_k(\tau([p, c_k'']a^* q) + \tau([p, a]c_k''^* q)) + \tau(c_k[p, q]) = \tau([p, a]b^* q) + \tau([p, b]a^* q).$$

Now note that $a, c_k'' \in [\mathcal{C}_n \mathcal{D}_n]^\perp$, so using Theorem 3.1.2, we have for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

$$\tau(\varphi_2(c_k'', c_k'')[p,q]) = \tau([p, c_k'']c_k''^* q)$$
$$\tau(\varphi_2(c_k'', a)[p,q]) = \frac{1}{2}\left(\tau([p, c_k'']a^* q) + \tau([p,a]c_k''^* q)\right).$$

So if we set $\tilde{c_k} = c_k - t_k^3 \varphi_2(c_k'', c_k'') - 2t_k \varphi_2(c_k'', a)$, we obtain

$$\lim_{k\to\infty} \tau(\tilde{c_k}[p,q]) = \tau([p,a]b^* q) + \tau([p,b]a^* q).$$

In the same way as for the second order relations, this shows that the right hand side defines a linear functional on $[\mathcal{C}_n, \mathcal{D}_n]$ which implies there exists $c' \in [\mathcal{C}_n, \mathcal{D}_n]$ such that

$$\tau(c'[p,q]) = \tau([p,a]b^* q) + \tau([p,b]a^* q).$$

Thus a satisfies the third order trace relations, and by Proposition 3.1.7, a satisfies the third order relations.

Proposition 3.2.4. If a is a direction of convergence of $\{U_k\}$, then a satisfies the third order relations.

Chapter 4

Hadamard Matrices near F_n

In this chapter, we will prove that, when three distinct primes divide n, there exist matrices in the tangent space at F_n which are not directions of convergence. We will do this by constructing an element $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ that does not satisfy the third order trace relations. By Proposition 3.1.7 and 3.1.9, the existence of such an element shows that the third order relations are not satisfied. I.e. there is some direction of convergence in $A = \{a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp : a = -a^*\}$ from which we cannot approach F_n. Additionally, Proposition 3.1.6 allows us to work with any a_2 so long as a_2 satisfies $\tau(a[p,q]) = \tau([p,a]a^*q)$. In the previous chapter, we showed that the relations that an analytic family of Hadamard matrices containing F_n must satisfy are exactly the same as the relations a sequence of Hadamard matrices converging to F_n must satisfy. This means that there is some direction of convergence in $A = \{a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp : a = -a^*\}$ from which we cannot approach F_n even with a sequence of Hadamard matrices.

4.1 Tangent Space at F_n

Our goal is to construct a concrete element of the tangent space of F_n, $[\mathcal{C}_n, \mathcal{D}_n]^\perp$, which does not satisfy the third order trace relations, so it is necessary that we are able to describe a nice basis of $[\mathcal{C}_n, \mathcal{D}_n]^{\perp}$. First, we set a basis to use for $\mathcal{C}_n$ and $\mathcal{D}_n$. For $i \in \mathbb{Z}_n$ define

$$q_i := e_{i,i}$$

$$p_i := \sum_{j \in \mathbb{Z}_n} e_{j,j-i}.$$

$$\mathcal{D}_n = span\{q_i : i \in \mathbb{Z}_n\},\ \mathcal{C}_n = span\{p_i : i \in \mathbb{Z}_n\}.$$

Since $\mathcal{C}_n = \mathbb{C}\mathbb{Z}_n$, the multiplication of elements in $\mathcal{C}_n$ is characterized by the group structure of $\mathbb{Z}_n$. I.e. $p_i p_j = p_{i+j}$. Multiplication of diagonal matrices, is straightforward as well. So we wish to develop a notation that takes full advantage of the structure of both $\mathcal{D}_n$ and $\mathcal{C}_n$.

Definition 4.1.1. For $h, g, k \in \mathbb{Z}_n$, define $M(\langle h \rangle + g, k)$ to be the matrix with 1's on the kth diagonal corresponding to the coset $\langle h \rangle + g$ and 0's elsewhere. More formally,

$$M(\langle h \rangle + g, k) = \sum_{i \in \langle h \rangle + g} e_{i,i-k}.$$

Note then that for $h, g, k, h', g', k' \in \mathbb{Z}_n$ we have

$$\begin{aligned} M(\langle h \rangle + g, k) \cdot M(\langle h' \rangle + g', k') &= \sum_{i \in \langle h \rangle + g} e_{i,i-k} \sum_{j \in \langle h' \rangle + g'} e_{j,j-k'} \\ &= \sum_{i \in \langle h \rangle + g \cap \langle h' \rangle + g' + k} e_{i,i-k-k'} \\ &= M(\langle h \rangle + g \cap \langle h' \rangle + g' + k, k + k'). \end{aligned}$$

$$
\begin{aligned}
M(\langle h\rangle + g, k)^* &= \left(\sum_{i\in\langle h\rangle+g} e_{i,i-k}\right)^* \\
&= \sum_{i\in\langle h\rangle+g} e_{i-k,i} \\
&= \sum_{j\in\langle h\rangle+g-k} e_{j,j+k} \\
&= M(\langle h\rangle + g - k, -k).
\end{aligned}
$$

These matrices are nice for several reasons. First, they are "easy" to multiply together; the multiplication boils down to a multiplication of two diagonal matrices (intersecting two cosets) combined with a multiplication of two circulant matrices (addition mod n). Second, they naturally describe the spaces with which we are concerned. A basis for the circulant matrices is given by

$$
\begin{aligned}
p_i &= \sum_{k\in\mathbb{Z}_N} e_{k,k-i} = M(\langle 1\rangle, i) \\
q_i &= e_{i,i} = M(\langle 0\rangle + i, 0) \\
[p_i, q_j] &= M(\langle 0\rangle + j + i, i) - M(\langle 0\rangle + j, i)
\end{aligned}
$$

Continuing this trend, they are particularly useful for describing a basis of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$. This is a reformulation of a result from Nicoara and White (2014).

Proposition 4.1.1. For $h, g \in \mathbb{Z}_n$, the distinct matrices

$$
a(h,g) = \sum_{i\in\langle h\rangle+g} e_{i,i-h} = M(\langle h\rangle + g, h)
$$

form a basis for $[\mathcal{C}_n, \mathcal{D}_n]^\perp$.

Proof. The fact that the set of distinct $a(h,g)$ form a linearly independent set follows from the fact that no pair have a non-zero entry in the same position.

Let $S = span\{a(h,g) : h, g \in \mathbb{Z}_n\}$. Since we have for all $h, g, i, j \in \mathbb{Z}_N$

$$\begin{aligned}\tau(a(h,g)[p_i, q_j]) &= \tau\left(M\left(\langle h\rangle + g, h\right) \cdot \left(M\left(\langle 0\rangle + j + i, i\right) - M\left(\langle 0\rangle + j, i\right)\right)\right) \\ &= \tau\left(M\left(C_1, h+i\right)\right) - \tau(M\left(C_2, h+i\right)).\end{aligned}$$

First note that the trace will always be zero unless $h + i \equiv 0(n)$, so replacing i with $-h$ we have that

$$\begin{aligned}C_1 &= \langle h\rangle + g \bigcap \langle 0\rangle + j - h \\ C_2 &= \langle h\rangle + g \bigcap \langle 0\rangle + j.\end{aligned}$$

Note that $j \in \langle h\rangle + g$ if and only if $j - h \in \langle h\rangle + g$, so we have that

$$\tau(a(h,g)[p_{-h}, q_j]) = |C_1| - |C_2| = 0.$$

This shows that $S \subset [\mathcal{C}_n, \mathcal{D}_n]^{\perp}$.

To complete the proof now, we need to show that for $a = (a_{k,k-l})_{k,l \in \mathbb{Z}_n} \in [\mathcal{C}_n, \mathcal{D}_n]^{\perp}$ we have that $a_{k,k-l} = a_{k',k'-l}$ whenever $k \equiv k'(l)$. Suppose $k = k'(l)$. In general, if $a \in [\mathcal{C}_n, \mathcal{D}_n]^{\perp}$ then we have for all $i, j \in \mathbb{Z}_n$ that

$$\begin{aligned}0 &= \tau(a[p_i, q_j]) \\ &= a_{j,j+i} - a_{j-i,j}.\end{aligned}$$

Replacing i with l and j with $rl + k$, we have

$$a_{rl+k, rl+k+l} = a_{(r-1)l+k, (r-1)l+k+l}.$$

Letting r range from $1 \leq r \leq n$, we see that

$$a_{k,k+l} = a_{l+k,l+k+l} = a_{2l+k,2l+k+l} = \cdots = a_{nl+k,nl+k+l}.$$

If $k' \equiv k(l)$, then $k' = tl + k$ for some $t \in \mathbb{Z}_n$. Thus $a_{k,k+l} = a_{k',k'+l}$, so any element of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$ may be written as a sum of $a(h,g)$, or $[\mathcal{C}_n, \mathcal{D}_n]^\perp \subset S$ which completes the proof. □

Now that we have a useful basis for $[\mathcal{C}_n, \mathcal{D}_n]^\perp$, we will go through the main result in Nicoara and White (2020) that shows that the second order relations always hold. Recall from the previous sections, in particular Proposition 3.1.3, that it will be sufficient to show that every $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the 2nd order trace relation.

Lemma 4.1.1. For $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, the second order trace relation holds if and only if

$$\sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j}[p_i, q_j] = 0 \quad \Rightarrow \quad \sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j} \tau([p_i, a]a^* q_j) = 0.$$

Proof. For the forward direction, note that the second order trace relation holds implies there exists $x \in [\mathcal{C}_n, \mathcal{D}_n]$ such that

$$\tau(x[p,q]) = \tau([p,a]a^*q)$$

for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$. So if $\sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j}[p_i, q_j] = 0$ then

$$\sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j} \tau([p_i, a]a^* q_j) = \sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j} \tau(x[p_i, q_j]) = 0.$$

For the reverse direction, note that the given condition is equivalent to requiring the linear map

$$\theta : [\mathcal{C}_n, \mathcal{D}_n] \to \mathbb{C}$$

given by

$$\theta\left(\sum_{i,j} \alpha_{i,j}[p_i, q_j]\right) = \sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j} \tau([p_i, a]a^* q_j)$$

is well defined. Thus $\tau([p,a]a^*q)$ defines a linear functional on $[\mathcal{C}_n, \mathcal{D}_n]$, so by the Riesz representation theorem, there exists $x \in [\mathcal{C}_n, \mathcal{D}_n]$ such that, in particular,

$$\tau(x[p,q]) = \tau([p,a]a^*q)$$

for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$. $\square$

In light of the previous lemma, we need to show that for every $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, $\tau([p,a]a^*q)$ defines a linear functional on $[\mathcal{C}_n, \mathcal{D}_n]$, the difficult step of course showing that the given map is well-defined. To do so we need to find a basis for the matrices $a = (\alpha_{i,j})_{i,j\in\mathbb{Z}_n}$ such that $\sum_{i,j} \alpha_{i,j}[p_i, q_j] = 0$.

Proposition 4.1.2. The distinct matrices

$$t(h,g) = \sum_{j\in\langle h\rangle+g} e_{h,j}$$

form a basis for the set $T := \left\{ t \in M_n(\mathbb{C}) \middle| \sum_{i,j\in\mathbb{Z}_n} t_{i,j}[p_i, q_j] = 0 \right\}$.

Proof. Again note that they clearly form a linearly independent set since the no two matrices have a nonzero entry on the same position. To show that the span of the $t(h,g)$ is contained in T see that we have for all h, g:

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}_n} t(h,g)_{i,j}[p_i, q_j] &= \sum_{j\in\langle h\rangle+g} [p_h, q_j] \\ &= [M(\langle 1\rangle, h), M(\langle h\rangle + g, 0)] \\ &= M(\langle h\rangle + g + h, h) - M(\langle h\rangle + g, h) = 0.\end{aligned}$$

Now to complete the proof we need to show that for $t \in T$ we have $t_{k,l} = t_{k,l'}$ whenever $l \equiv l'(k)$. If $t \in T$ we have for all $i, j \in \mathbb{Z}_n$

$$0 = \sum_{i,j\in\mathbb{Z}_n} t_{i,j}[p_i, q_j] = \sum_{i,j\in\mathbb{Z}_n} t_{i,j}e_{j+i,j} - t_{i,j}e_{j,j-i}.$$

Now make the substitutions $k = i$ and $j = l - k$ in the first sum, and the substitutions $k = i$ and $j = l$ in the second sum, we see that for all $l, k \in \mathbb{Z}_n$ we have

$$t_{k,l-k} = t_{k,l}.$$

Replacing l with $rk + l$ for $1 \leq r \leq n$, we get

$$t_{k,l} = t_{k,k+l} = t_{k,2k+l} = \cdots = t_{k,nk+l}.$$

If $l \equiv l'(k)$, then $l' = sk + l$ for some $k \in \mathbb{Z}_n$. This shows that T is spanned by the $t(h, g)$ which completes the proof. $\square$

Before we prove the result for the second order relations, we write out a lemma which is the main computation necessary to show the result. We do it in slightly more generality so that we may use this computation in other places.

Lemma 4.1.2. If $l' - l \equiv 0(\gcd(j, j'))$ then for all $h, g, k, k' \in \mathbb{Z}_n$ we have

$$\tau\left([p_h, M(\langle j\rangle + k, l)]\, M(\langle j'\rangle + k', l')^* M(\langle h\rangle + g, 0)\right) = 0.$$

Proof. Multiplying out the left hand side:

$$\tau(M(C_1, l - l' + h)) - \tau(M(C_2, l - l' + h)).$$

The trace of each will always be zero unless $l - l' + h \equiv 0(n)$, so we replace h with $l' - l$, the left hand side now becomes:

$$|C_1| - |C_2|$$

where

$$C_1 = \langle j\rangle + k + l' - l \bigcap \langle j'\rangle + k' \bigcap \langle l' - l\rangle + g$$
$$C_2 = \langle j\rangle + k \bigcap \langle j'\rangle + k' \bigcap \langle l' - l\rangle + g$$

If C_1 and C_2 are both nonempty, then they are cosets of the same subgroup which means they would have the same size. So we now just need to show that $C_1 \neq \emptyset$ if and only if $C_2 \neq \emptyset$. Using the Chinese Remainder Theorem, C_1 is nonempty if and only if

$$\begin{aligned} k + l' - l - k' &\equiv 0(\gcd(j, j')) \\ k + l' - l - g &\equiv 0(\gcd(j, l' - l)) \\ k' - g &\equiv 0(\gcd(j', l' - l)). \end{aligned}$$

Using the assumption, $l' - l \equiv 0(\gcd(j, j'))$ and the fact that $l' - l \equiv 0(\gcd(j, l' - l))$, these equations are equivalent to

$$\begin{aligned} k - k' &\equiv 0(\gcd(j, j')) \\ k - g &\equiv 0(\gcd(j, l' - l)) \\ k' - g &\equiv 0(\gcd(j', l' - l)). \end{aligned}$$

Now see that these are exactly the relations that imply $C_2 \neq \emptyset$. □

Now we complete the proof for the main result of Nicoara and White (2020).

Theorem 4.1.1. Every $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the second order trace relations.

Proof. Using Propositions 4.1.1, 4.1.2 and Lemma 4.1.1 we need to show that for all $h, g, j, k, j', k' \in \mathbb{Z}_n$

$$\begin{aligned} 0 &= \sum_{r,s \in \mathbb{Z}_n} t(h, g)_{r,s} \tau([p_r, a(j, k)] a(j', k')^* q_s) \\ &= \sum_{s \in \langle h \rangle + g} \tau([p_h, a(j, k)] a(j', k')^* q_s) \\ &= \tau\left([p_h, M(\langle j \rangle + k, j)] M(\langle j' \rangle + k', j')^* M(\langle h \rangle + g, 0)\right). \end{aligned}$$

Since $j' - j \equiv 0(\gcd(j, j'))$ (for any pair $j, j' \in \mathbb{Z}_n$), the result follows from Lemma 4.1.2. □

In fact, we have proven a stronger statement:

Theorem 4.1.2. For every pair $a_1, a_2 \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, there exists x such that, for every $p \in \mathcal{C}_n, q \in \mathcal{D}_n$,

$$\tau(x[p,q]) = \tau([p, a_1]a_2^* q).$$

4.2 Solving the Second Order Relations

At this point, we have shown that every element $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ satisfies the second order trace relations. The goal of this section is to, for a specified $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, find a particular b such that $\tau(b[p,q]) = \tau([p,a]a^*q)$. More generally, with 4.1.2 in mind, we want to specify $a_1, a_2 \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ and find a corresponding b such that $\tau(b[p,q]) = \tau([p,a_1]a_2^*q)$ for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$. It will not be necessary to solve this relation in full generality, so we will instead focus on pairs of basis elements of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$ of the form $a(h,0), a(g,0)$. For ease of notation, through the next two sections, we will let $a_g = a(g,0)$ for any $g \in \mathbb{Z}_n$.

Theorem 4.2.1. If we set

$$b(h,g) = \sum_{r \in \mathbb{Z}_n} b_r M\left(\langle 0 \rangle + r, g-h\right)$$

with $b_r \in \mathbb{C}$ satisfying for all $r \in \mathbb{Z}_n$

$$b_r - b_{r+h-g} = \tau([p_{g-h}, a_h]a_g^* q_r)$$

then for all $p \in \mathcal{C}_n, q \in \mathcal{D}_n$

$$\tau(b(h,g)[p,q]) = \tau([p,a_h]a_g^* q).$$

Proof. It is not immediately clear that such b_r exist, but this follows from the fact that there exists some b satisfying the desired relation (Theorem 4.1.2). We will perform this calculation first for clarity. Theorem 4.1.2 guarantees there exists b such that for all $i, j \in \mathbb{Z}_n$,

$$\tau(b[p_i, q_j]) = \tau([p_i, a_h]a_g^* q_j).$$

We may write $b = \sum_{r,k \in \mathbb{Z}_n} b_{r,k} M\left(\langle 0 \rangle + r, k\right)$, and if we set $i = g - h$, then for all $j \in \mathbb{Z}_n$ the above relation becomes

$$b_{j,h-g} - b_{j+h-g,h-g} = \tau([p_{g-h}, a_h] a_g^* q_j).$$

This shows that constants $b_r \in \mathbb{C}$ as described in the Theorem statement exist.

Let $b(h,g)$ be as described in the Theorem statement. Since the relation we wish to show is linear in p and q, it suffices to show the relation holds for the basis elements. I.e. for all $i, j \in \mathbb{Z}_n$

$$\tau(b(h,g)[p_i, q_j]) = \tau([p_i, a_h] a_g^* q_j).$$

Additionally, note that both the left and right hand sides will be 0 if $i \neq g - h$ (the non-zero entries won't land on the main diagonal), so we need only show that they are equal when $i = g - h$.

$$\begin{aligned}
\tau(b(h,g)[p_{g-h}, q_j]) &= \tau\left(\sum_{r \in \mathbb{Z}_n} b_r M\left(\langle 0 \rangle + r, g - h\right)\right)[p_{g-h}, q_j] \\
&= b_j - b_{j+h-g} \\
&= \tau([p_{g-h}, a_h] a_g^* q_j).
\end{aligned}$$

□

Lemma 4.2.1. $a \in [\mathcal{C}_n, \mathcal{D}_n]^{\perp}$ satisfies the third order trace relation if and only if for every b satisfying

$$\tau(b[p,q]) = \tau([p,a]a^* q)$$

we have

$$\sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j} [p_i, q_j] = 0 \;\Rightarrow\; \sum_{i,j \in \mathbb{Z}_n} \alpha_{i,j} \left(\tau([p_i, b] a^* q_j) + \tau([p_i, a] b^* q_j)\right) = 0.$$

Proof. For the forward direction, if a satisfies the third order trace relation, then (by Proposition 3.1.6) for any b satisfying

$$\tau(b[p,q]) = \tau([p,a]a^*q)$$

there exists c such that

$$\tau(c[p,q]) = \tau([p,b]a^*q) + \tau([p,a]b^*q).$$

So if $\sum_{i,j\in\mathbb{Z}_n} \alpha_{i,j}[p_i,q_j] = 0$ then

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}_n} \alpha_{i,j}\left(\tau([p_i,b]a^*q_j) + \tau([p_i,a]b^*q_j)\right) &= \sum_{i,j\in\mathbb{Z}_n} \alpha_{i,j}\tau(c[p_i,q_j]) \\ &= \tau\left(c\sum_{i,j\in\mathbb{Z}_n} \alpha_{i,j}[p_i,q_j]\right) = 0.\end{aligned}$$

For the reverse direction, note that the condition is equivalent to requiring that the third order trace relation is a well defined linear functional on $[\mathcal{C}_n, \mathcal{D}_n]$. So by the Riesz representation theorem, there exists c satisfying

$$\tau(c[p,q]) = \tau([p,b]a^*q) + \tau([p,a]b^*q).$$

□

4.3 The Third Order Relations Fail when Three Distinct Primes divide n

We are now ready to prove the main result. We aim to show that, when n is divisible by at least three primes, then the third order relations are not satisfied for every a in the tangent space of F_n. In order to do this, from Proposition 3.1.9, it suffices to show that there exists

$a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$ which does not satisfy the third order trace relations. Once we choose a specific $a \in [\mathcal{C}_n, \mathcal{D}_n]^\perp$, using the basis of $[\mathcal{C}_n, \mathcal{D}_n]^\perp$ described in Proposition 4.1.1, to work with, we will suppose a satisfies the third order trace relations. Additionally, we will use the specific b from Theorem 4.2.1 which satisfies

$$\tau(b[p,q]) = \tau([p,a]a^*q).$$

Note that from Proposition 3.1.6 it is sufficient to show that the third order relations do not hold for a specific choice of b. Now, from Lemma 4.2.1, a satisfies the third order trace relations if and only if

$$\sum_{i,j\in\mathbb{Z}_n} \alpha_{i,j}[p_i,q_j] = 0 \ \Rightarrow \sum_{i,j\in\mathbb{Z}_n} \alpha_{i,j}\left(\tau([p_i,b]a^*q_j) + \tau([p_i,a]b^*q_j)\right) = 0.$$

From Proposition 4.1.2, for $u, v \in \mathbb{Z}_n$, the distinct matrices

$$t(u,v) = \sum_{j\in\langle u\rangle+v} e_{u,j}$$

form a basis of the set of matrices which make the left hand side of the above statement 0. If we can show that the right hand side is non-zero for a choice of $t(u,v)$, then we have shown that a does not satisfy the third order trace relations, and that there must exist elements in the tangent space of F_n that do not satisfy the third order relations.

Theorem 4.3.1. Let n be a positive integer with at least three distinct prime divisors. Let $A = [\mathcal{C}_n, \mathcal{D}_n]^\perp$ denote the tangent space to the manifold of complex Hadamard matrices, at the Fourier matrix F_n. Then there exists an element $a \in A$ with $\|a\| = 1$, such that a is not a direction of convergence for any sequence of Hadamard matrices converging to F_n. In particular, there does not exist an analytic family of complex Hadamard matrices of the form $U_t = F_n(I + ta + \dots)$ $(|t| < \delta)$.

Proof. If three distinct primes divide n, we may write $n = xyz$ with $x, y, z > 1$ relatively prime. Additionally, we may assume $x, y \neq 2$ and $x \neq 3$. Let h_0, h_1, h_2 be the unique elements in $\mathbb{Z}_n$ satisfying:

$$\begin{array}{ccccc} h_0 \equiv 1(x) & , & h_1 \equiv -1(x) & , & h_2 \equiv 0(x) \\ h_0 \equiv 0(y) & , & h_1 \equiv -1(y) & , & h_2 \equiv 0(y) \\ h_0 \equiv 1(z) & , & h_1 \equiv 0(z) & , & h_2 \equiv 1(z) \end{array}$$

Set $a = a_{h_0} + a_{h_1} + a_{h_2}$ and set $u = h_0 + h_1 - h_2$, and note that u satisfies

$$u \equiv 0(x), \quad u \equiv -1(y), \quad u \equiv 0(z).$$

Note now that we have the following:

$$\langle h_0 \rangle = \langle y \rangle \, , \; \langle h_1 \rangle = \langle z \rangle \, , \; \langle h_2 \rangle = \langle xy \rangle \, , \; \langle u \rangle = \langle xz \rangle \, .$$

Let $b(h_i, h_j)$ be the element described in Theorem 4.2.1 satisfying

$$\begin{aligned} \tau(b(h_i, h_j)[p, q]) &= \tau([p, a_{h_i}]a^*_{h_j} q) \\ b(h_i, h_j) &= \sum_{r \in \mathbb{Z}_N} b^{i,j}_r M\left(\langle 0 \rangle + r, h_j - h_i\right) \\ b^{i,j}_r - b^{i,j}_{r+h_j-h_i} &= \tau([p_{h_j - h_i}, a_i]a^*_j q_r). \end{aligned}$$

Now note that if we set $b = \sum_{i,j} b(h_i, h_j)$, then we have, since $b(h_i, h_j)$ is linear in each component,

$$\tau(b[p, q]) = \tau([p, a]a^* q).$$

From Lemma 4.2.1, we now need to show that for some $v \in \mathbb{Z}_n$ (note that we fixed $u \in \mathbb{Z}_n$ above),

$$\begin{aligned}
0 &\neq \sum_{i,j\in\mathbb{Z}_n} t(u,v)_{i,j}(\tau([p_i,b]a^*q_j) + \tau([p_i,a]b^*q_j)) \\
&= \sum_{j\in\langle u\rangle+v} (\tau([p_u,b]a^*q_j) + \tau([p_u,a]b^*q_j)) \\
&= \tau([p_u,b]a^*M(\langle u\rangle+v,0))) + \tau([p_u,a]b^*M(\langle u\rangle+v,0)) \\
&= \sum_{j,k,l=0}^{2} \tau([p_u,b(h_j,h_k)]a^*_{h_i}M(\langle u\rangle+v,0))) + \tau([p_u,a_{h_i}]b(h_j,h_k)^*qM(\langle u\rangle+v,0))).
\end{aligned}$$

At this point, we will only assume that $v \notin \langle z\rangle$, and later we will choose a specific $v \in \mathbb{Z}_n$ (of course still satisfying $v \notin \langle z\rangle$). Expanding the commutators, this is the sum of 108 terms, each term is the trace of a product of 4 matrices. The following 3 lemmas greatly simplify this expression by showing that all but one of these terms are 0.

Lemma 4.3.1. If i,j,k are not distinct, then

$$\tau([p_u,b(h_j,h_k)]a^*_{h_i}M(\langle u\rangle+v,0))) = 0.$$

Proof. It is important to note that each of the matrices in each term have non-zero entries on exactly one diagonal, and so the resulting matrix will can only have non-zero entries on exactly one diagonal. The trace will be 0 unless the product lands on the main diagonal. More formally,

$$p_u b(h_j,h_k)a^*_{h_i}M(\langle u\rangle+v,0) = M(C_1, u+h_k-h_j-h_i),$$

$$b(h_j,h_k)p_u a^*_{h_i}M(\langle u\rangle+v,0) = M(C_2, u+h_k-h_j-h_i).$$

Regardless of whether the cosets C_1, C_2 are non-empty, the trace of such matrices will be always be 0 unless $u+h_k-h_j-h_i \equiv 0(n)$. We will now show that, based on our choices of

u, h_0, h_1, h_2, this cannot happen.

First, suppose $i = j = k$, then for the sum to satisfy the relation we must have $h_i \equiv u(n)$. Since we don't have this relation for any $0 \leq i \leq 2$, we have reached a contradiction. Furthermore, if $i = k$ or $j = k$ then we have the same conclusion. Now suppose $j = i$, then we have 6 cases, and we list an equivalence that shows $u + h_k - 2h_i \not\equiv 0(n)$ for each case (note that $g \equiv 0(n)$ if and only if $g \equiv 0(x)$, $g \equiv 0(y)$, and $g \equiv 0(z)$):

$$k = 0, i = 1: \; u + h_0 - 2h_1 \equiv 1(z)$$

$$k = 0, i = 2: \; u + h_0 - 2h_2 \equiv 1(x)$$

$$k = 1, i = 0: \; u + h_1 - 2h_0 \equiv -2(y)$$

$$k = 1, i = 2: \; u + h_1 - 2h_2 \equiv -1(x)$$

$$k = 2, i = 0: \; u + h_2 - 2h_0 \equiv -1(z)$$

$$k = 2, i = 1: \; u + h_2 - 2h_1 \equiv 1(z)$$

For $k = 1, i = 0$ we used the assumption that $y \neq 2$. □

Lemma 4.3.2. If we don't have $i = 0$, $j = 1$, $k = 2$, then

$$\tau([p_u, b(h_j, h_k)]a^*_{h_i} M(\langle u \rangle + v, 0))) = 0.$$

Proof. From the previous lemma, we may assume i, j, k are distinct. Again, for this product to land on the main diagonal we need $u + h_k - h_j - h_i \equiv 0(n)$. Note that this relation stays the same swapping i and j, so first we will show that k must be equal to 2. If $k = 0$, then we have (recall, $u = h_0 + h_1 - h_2$)

$$u + h_0 - h_1 - h_2 \equiv 0(n) \;\Rightarrow\; 2h_2 \equiv 2h_0(n),$$

If $k = 1$, then we have

$$u + h_1 - h_0 - h_2 \equiv\ 0(n) \Rightarrow\ 2h_1 \equiv 2h_2(n).$$

Both give a contradiction (for the second case we used $x, y \neq 2$), so we must have $k = 2$. The last case to deal with then is the case where $i = 1$, $j = 0$, and $k = 2$. In this case, note that the product

$$\begin{aligned} a_{h_1}^* \cdot M\left(\langle xz\rangle + v, 0\right) &= M\left(\langle z\rangle, h_1\right) \cdot M\left(\langle xz\rangle + v, 0\right) \\ &= M\left(\langle z\rangle \bigcap \langle xz\rangle + v - h_1, -h_1\right). \end{aligned}$$

This coset is non-empty if and only if $v - h_1 \in \langle z\rangle$. Since $h_1 \in \langle z\rangle$, this is equivalent to requiring $v \in \langle z\rangle$, but we have chosen $v \notin \langle z\rangle$. □

The previous two lemmas simplify the first half of the expression down to a single expression that needs to be evaluated. The next lemma deals with the second half of the sum.

Lemma 4.3.3. For all $0 \leq i, j, k \leq 2$,

$$\tau([p_u, a_{h_i}]b(h_j, h_k)^* M\left(\langle u\rangle + v, 0\right))) = 0.$$

Proof. Again, we will rely on the fact that the product won't land on the main diagonal, and hence the trace will be 0, unless $u + h_i + h_k - h_j = 0(n)$, and we again recall the fact that $u = h_0 + h_1 - h_2$. In the cases $j = k$ or $i = k$ we get that $u \equiv -h_l(n)$, which is a contradiction regardless of the value of $0 \leq l \leq 2$. Now suppose $i = j$, again we must check 6 cases, and we list an equivalence that shows $u + 2h_i - h_k \not\equiv 0(n)$:

$$i = 0, k = 1:\ 2h_0 - h_1 \equiv 3(x)$$

$$i = 0, k = 2:\ 2h_i - h_j \equiv 1(z)$$

$$i = 1, k = 0: \ 2h_i - h_j \equiv -1(z)$$

$$i = 1, k = 2: \ 2h_i - h_j \equiv -1(z)$$

$$i = 2, k = 0: \ 2h_i - h_j \equiv 1(z)$$

$$i = 2, k = 1: \ 2h_i - h_j \equiv 1(x).$$

For $i = 0, k = 1$, we used that $x \neq 3$. Now we know that the entries must be distinct, and note that h_j and h_i are interchangeable. We have 3 cases left to check. Again, we list an equivalence that shows $u + h_i + h_j - h_k \not\equiv 0(n)$

$$k = 0 : u + h_1 + h_2 - h_0 \equiv -2(x)$$

$$k = 1 : u + h_0 + h_2 - h_1 \equiv 2(x)$$

$$k = 2 : u + h_0 + h_1 - h_2 \equiv 1(x)$$

In the first two cases we used that $x \neq 2$. □

Now we finally return to the main proof. Lemmas 4.3.1, 4.3.2, and 4.3.3 show that there is only term which is not guaranteed to be 0:

$$\tau([p_u, b(h_1, h_2)]a^*_{h_0} M(\langle u \rangle + v, 0))).$$

In fact we will show that this term is non-zero, which will complete the proof. Let's first look back at the definition of the relevant $b(h_1, h_2)$, and note that we have chosen h_2, h_1 so that $h_2 - h_1 \equiv 1(n)$.

$$\tau(b(h_1, h_2)[p, q]) = \tau([p, a_{h_1}]a^*_{h_2} q)$$
$$b(h_1, h_2) = \sum_{r \in \mathbb{Z}_n} b^{1,2}_r M(\langle 0 \rangle + r, 1)$$
$$b^{1,2}_r - b^{1,2}_{r-1} = \tau([p_1, a_{h_1}]a^*_{h_2} q_r).$$

For ease of notation, we replace $b_r^{1,2} = b_r$ for the remainder of the proof. Expanding, this expression becomes

$$\tau([p_u, b(h_1, h_2)]a_{h_0}^* M(\langle u\rangle + v, 0)) = \sum_{r\in\mathbb{Z}_n} b_r \tau(M(C_{r+u}, 0)) - b_r \tau(M(C_r, 0))$$

Where

$$C_r = \langle 0\rangle + r \bigcap \langle y\rangle \bigcap \langle u\rangle + v.$$

Not that the unique value of r for which the coset C_r is nonempty is vh_0. This is because $vh_0 \equiv v(x)$, $vh_0 \equiv 0(y)$, and $vh_0 \equiv v(z)$. Using this fact, and substituting $r \mapsto r - u$ in the first term leaves us with the expression:

$$\begin{aligned}\sum_{r\in\mathbb{Z}_n} b_r \tau(M(C_{r+u}, 0)) - b_r \tau(M(C_r, 0)) &= \sum_{r\in\mathbb{Z}_n} (b_{r-u} - b_r)|C_r| \\ &= b_{vh_0 - u} - b_{vh_0}.\end{aligned}$$

We now wish to use the property $b_r - b_{r-1} = \tau([p_1, a_{h_1}]a_{h_2}^* q_r)$, so we add and subtract the appropriate terms to obtain:

$$\begin{aligned}b_{vh_0-u} - b_{vh_0} &= -\sum_{s=0}^{u-1} b_{vh_0-s} - b_{vh_0-s-1} \\ &= -\sum_{s=0}^{u-1} \tau([p_1, a_{h_1}]a_{h_2}^* q_{vh_0-s}) \\ &= -\sum_{s=0}^{u-1} \tau(M\left(C_s^1, 0\right)) - \tau(M\left(C_s^2, 0\right)).\end{aligned}$$

Where

$$C_s^1 = \langle z\rangle + h_2 \bigcap \langle xy\rangle \bigcap \langle 0\rangle + vh_0 - s$$
$$C_s^1 = \langle z\rangle \bigcap \langle xy\rangle \bigcap \langle 0\rangle + vh_0 - s$$

Now see that C_s^1, C_s^2 are non-empty for exactly one value of s, which we will denote s_v^1, s_v^2 respectively. C_s^1 is non-empty if and only if $s_v^1 = vh_0 - h_2$ and C_s^2 is non-empty if and only if $s_v^2 = vh_0$. To finish the proof, we will now specify a value for v (still satisfying $v \notin \langle z \rangle +$). We have two cases.

Case 1: Suppose $h_1 < u$. Set $v = 1$, then

$$s_1^1 = h_0 - h_2 = u - h_1$$
$$s_1^2 = h_0 = u + 1.$$

Now we have that $s_1^1 \in \{0, \ldots, u-1\}$ and $s_1^2 \notin \{0, \ldots, u-1\}$, which implies

$$-\sum_{s=0}^{u-1} \tau(M\left(C_s^1, 0\right)) - \tau(M\left(C_s^2, 0\right)) = -1 \neq 0.$$

Case 2: Suppose $h_1 > u$. Set $v = h_2$, then

$$s_{h_2}^1 = 0$$
$$s_{h_2}^2 = h_2 = h_1 + 1.$$

Again we have that $s_{h_2}^1 \in \{0, \ldots, u-1\}$ and $s_{h_2}^2 \notin \{0, \ldots, u-1\}$ (note that $h_1 + 1 \neq 0$ since $h_1 \not\equiv 0(n)$), which implies

$$-\sum_{s=0}^{u-1} \tau(M\left(C_s^1, 0\right)) - \tau(M\left(C_s^2, 0\right)) = -1 \neq 0.$$

This completes the proof. □

Remark 4.3.1. When n has only 2 distinct prime divisors, the same result is not expected to be true in general. In particular for $n = 6$ there is evidence that analytic parametric families of Hadamard matrices exist along any direction in the tangent space, which has dimension 4 (see Barros e Sá and Bengtsson (2013), Tadej and Życzkowski (2006)).

Chapter 5

A Finiteness Result for Hadamard Matrices

In this chapter we prove a finiteness result for complex Hadamard matrices which have certain symmetries (and their corresponding commuting squares). Specifically, we show that if U is a unitary matrix such that all minors of U are non-zero then the maximal abelian $*$-subalgebra UDU^* of $M_n(\mathbb{C})$ contains only finitely many complex Hadamard matrices, up to equivalence. This result generalizes results from Haagerup (2008) and Nicoara and Worley (2019).

Throughout this chapter, for a specified unitary matrix $U \in M_n(\mathbb{C})$ and $x = (x_0, \ldots, x_{n-1}) \in \mathbb{C}^n$, let $D_x = \sum_{k \in \mathbb{Z}_n} x_k e_{k,k}$ and let $\hat{x}_{i,j}$ denote the $(i,j)^{\text{th}}$ entry of the matrix $\sqrt{n} U D_x U^*$.

Proposition 5.0.1. Let $x \in \mathbb{C}^n$. If UD_xU^* is a Hadamard matrix, then UD_xU^* is equivalent to a Hadamard matrix of the form $UD_{x'}U^*$ where $x'_0 = 1$.

Proof. If UD_xU^* is Hadamard, then UD_xU^* is unitary, which implies $|x_k| = 1$ for all $k \in \mathbb{Z}_n$. Now see that UD_xU^* is equivalent to $\overline{x_0} I U D_x U^*$. $\square$

Proposition 5.0.2. If UD_xU^* is a Hadamard matrix with $x_0 = 1$ then, if we set $y_k = \overline{x_k}$, then (x, y) is a solution to the set of equations

$$x_0 = y_0 = 1, \; x_k y_k = 1, \; \hat{x}_{k,0}\hat{y}_{0,k} = 1 \text{ for all } k \in \mathbb{Z}_n.$$

Proof. By design, $x_0 = y_0 = 1$. Since UD_xU^* is Hadamard, UD_xU^* is unitary, which implies D_x is unitary, which implies $|x_k| = 1$ for all k. Thus for all $k \in \mathbb{Z}_n$

$$x_k y_k = x_k \overline{x_k} = x_k \frac{1}{x_k} = 1.$$

Now, since UD_xU^* is Hadamard each entry has modulus $\frac{1}{\sqrt{n}}$ which means $|x\hat{_{k,0}}| = 1$ for all $k \in \mathbb{Z}_n$. Now note that we have the relationship, $(UD_xU^*)^* = UD_yU^*$ which implies $\hat{y}_{0,k} = \overline{\hat{x}_{k,0}}$ so we have for all $k \in \mathbb{Z}_n$

$$x_{k,0}\hat{y}_{0,k} = x_{k,0}\overline{x_{k,0}} = 1.$$

□

For $x \in \mathbb{C}^n$ define $supp(x)$ to be the subset of $\mathbb{Z}_n$ such that $x_k \neq 0$ for all $k \in supp(x)$ and $x_k = 0$ for all $k \notin supp(x)$.

Theorem 5.0.1. Suppose U is an $n \times n$ unitary matrix with the additional condition that every minor of U is non-zero, i.e. $\det(U)_{K\times L} \neq 0$ for any subsets $K, L \subset \mathbb{Z}_n$ with $|K| = |L| \geq 1$. If $0 \neq x \in \mathbb{C}^n$, then $\forall t \in \mathbb{Z}_n$

$$|supp(x)| + |supp(\hat{x}_t)| \geq n + 1$$

where $\hat{x}_t = \{\hat{x}_{k,t}\}_{k\in\mathbb{Z}_n}$ is the t^{th} column of $\sqrt{n}UD_xU^*$.

Proof. Suppose $0 \neq x \in \mathbb{C}^n$ such that

$$|supp(x)| + |supp(\hat{x}_t)| \leq n.$$

Set $L = supp(x)$, then $|L| \geq 1$ (since $x \neq 0$) and

$$|\mathbb{Z}_n \setminus supp(\hat{x}_t)| = n - |supp(\hat{x}_t)| \geq |supp(x)| = |L|.$$

So we may choose $K \subset \mathbb{Z}_n \setminus supp(\hat{x}_t)$ such that $|K| = |L|$. Since $K \subset \mathbb{Z}_n \setminus supp(\hat{x}_t)$, we have for all $k \in K$

$$0 = \hat{x}_{k,t} = \sqrt{n} \sum_{i \in \mathbb{Z}_n} u_{k,i} x_i u^*_{i,t} = \sqrt{n} \sum_{i \in L} u_{k,i} x_i u^*_{i,t}.$$

Let $T \subset \mathbb{Z}_n$ such that $t \in T$ and $|T| = |L| = |K|$. Now see that the (k,t)th entry of $\sqrt{n} U D_x U^*$ is

$$(\sqrt{n}(U)_{K \times L}(D_x)_{L \times L}(U^*)_{L \times T})_{k,t} = \sqrt{n} \sum_{i \in L} u_{k,i} x_i u^*_{i,t}.$$

As this holds for all $k \in K$, the t^{th} column of $\sqrt{n}(U)_{K \times L}(D_x)_{L \times L}(U^*)_{L \times T}$ consists only of zeros. This implies

$$0 = \det\left(\sqrt{n}(U)_{K \times L}(D_x)_{L \times L}(U^*)_{L \times T}\right) = \sqrt{n}\det(U)_{K \times L} \cdot \det(D_x)_{L \times L} \cdot \det(U^*)_{L \times T}.$$

But this is a contradiction since $\det(U)_{K \times L} \neq 0$, $\det(U^*)_{L \times T} \neq 0$, and $\det(D_x)_{L \times L} = \prod_{l \in L} x_l \neq 0$ since $L = supp(x)$. □

Proposition 5.0.3. Suppose U is an $n \times n$ unitary matrix such that all minors of U are non-zero. Then for $x \neq 0$

$$|supp(x)| + |supp(\hat{x}^s)| \geq n + 1$$

where $\hat{x}^s = \{\hat{x}_{s,l}\}_{l \in \mathbb{Z}_n}$ is the s^{th} row $\sqrt{n} U D_x U^*$.

Proof. This follows from the the previous proposition and fact that the s^{th} row of $\sqrt{n} U D_x U^*$ is the s^{th} column of $\sqrt{n} U D_{\overline{x}} U^*$. □

Theorem 5.0.2. If U is a unitary matrix such that all minors of U are non-zero, then there are only finitely many solutions to the following set of equations

$$x_0 = y_0 = 1, \qquad x_k y_k = 1, \qquad \hat{x}_{k,0}\hat{y}_{0,k} = 1 \qquad \text{for all } k \in \mathbb{Z}_n.$$

Proof. Suppose there are infinitely many $z = (x, y) \in \mathbb{C}^n \times \mathbb{C}^n$ that satisfy the equations. Since this is an algebraic variety, and compact algebraic varieties over $\mathbb{C}$ are finite, we must have that this closed set of solutions is unbounded. So let $\{z^{(m)}\} = \{(x^{(m)}, y^{(m)})\}$ be a sequence such that

$$\lim_{m\to\infty} \|z^{(m)}\|_2 = \infty.$$

Now note that

$$\|x^{(m)}\|_2^2 \|y^{(m)}\|_2^2 = \left(1 + \sum_{i=1}^{n-1} |x_i|^2\right)\left(1 + \sum_{i=1}^{n-1} |y_i|^2\right) \geq \|z^{(m)}\|_2^2 - 1$$

which implies $\|x^{(m)}\|_2 \|y^{(m)}\|_2 \to \infty$. Now we set

$$r^{(m)} := \frac{x^{(m)}}{\|x^{(m)}\|_2}, \qquad s^{(m)} := \frac{y^{(m)}}{\|y^{(m)}\|_2}.$$

Since $\|s^{(m)}\|_2 = \|r^{(m)}\|_2 = 1$ for all m, this sequence is contained in a compact set, so by passing to a subsequence we may set

$$r := \lim_{m\to\infty} r^{(m)}, \qquad s := \lim_{m\to\infty} s^{(m)}.$$

Now see that for all $k \in \mathbb{Z}_n$ we have

$$r_k s_k = \lim_{m\to\infty} r_k^{(m)} s_k^{(m)} = \lim_{m\to\infty} \frac{x_k^{(m)} y_k^{(m)}}{\|x^{(m)}\|_2 \|y_k^{(m)}\|_2} = \lim_{m\to\infty} \frac{1}{\|x^{(m)}\|_2 \|y^{(m)}\|_2} = 0$$

$$\hat{r}_{k,0}\hat{s}_{0,k} = \lim_{m\to\infty} \hat{r}_{k,0}^{(m)} \hat{s}_{0,k}^{(m)} = \lim_{m\to\infty} \frac{\hat{x}_{k,0}^{(m)} \hat{y}_{0,k}^{(m)}}{\|x^{(m)}\|_2 \|y^{(m)}\|_2} = \lim_{m\to\infty} \frac{1}{\|x^{(m)}\|_2 \|y^{(m)}\|_2} = 0.$$

This implies that

$$supp(r) \cap supp(s) = \emptyset \quad \Rightarrow \quad |supp(r)| + |supp(s)| \leq n$$
$$supp(\hat{r}_0) \cap supp(\hat{s}^0) = \emptyset \quad \Rightarrow \quad |supp(\hat{r}_0)| + |supp(\hat{s}^0)| \leq n.$$

Combining this with Theorem 5.0.1 Proposition 5.0.3, we obtain

$$2n + 2 \leq |supp(r)| + |supp(s)| + |supp(\hat{r}_0)| + |supp(\hat{s}^0)| \leq 2n.$$

This is a contradiction. □

Theorem 5.0.3. If U is a unitary matrix such that all minors of U are non-zero, then there are only finitely many Hadamard matrices in the algebra UDU^* up to equivalence.

Remark 5.0.1. In Haagerup (2008), it is shown that there are only finitely many circulant $p \times p$ Hadamard matrices for p prime. Equivalently, the algebra $\mathcal{C}_p = F_p \mathcal{D}_p F_p^*$ has only finitely many Hadamard matrices. In Nicoara and Worley (2019), it is shown that there are finitely many $(p+1) \times (p+1)$ circulant core Hadamard matrices. Circulant core matrices are of the form

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x_0 & x_1 & \dots & x_{p-1} \\ 1 & x_{p-1} & x_0 & \dots & x_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & x_2 & \dots & x_0 \end{pmatrix}.$$

Both results rely on a result from Chebatarov in 1926, proven also in Tao (2004), that all minors of the Fourier matrix F_p are non-zero when p is prime.

Remark 5.0.2. If a random unitary U is chosen, then with probability 1 all minors will be nonzero. Thus this theorem applies to a almost every unitary matrix U.

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Vita

Shuler Hopkins grew up in Blountville, Tennessee, and attended high school at Sullivan Central High School. After high school Shuler attended King University in Bristol, Tennessee, where he obtained Bachelor of Science degrees in both Mathematics and Physics. Shuler decided to pursue a graduate degree in Mathematics at the University of Tennessee, Knoxville. He was always most interested in the areas of abstract algebra and analysis, and his research area landed in the intersection of these to areas in the field know as Operator Algebras. His research has mostly focused on gaining a better understanding of the structure of the space Hadamard matrices. After graduation, Shuler will take on a position as a visiting assistant professor at Sewanee: The University of the South. Shuler is excited to start a new chapter alongside his wife Elizabeth in Sewanee.