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Characteristic Sets of Matroids

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To the Graduate Council:

I am submitting herewith a dissertation written by Dony Varghese entitled "Characteristic Sets of Matroids." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Dr. Dustin Cartwright, Major Professor

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Characteristic Sets of Matroids

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Doctor of Philosophy

Degree

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Dony Varghese

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This dissertation is dedicated to my family and friends.

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'Why fight when you can negotiate?' - Captain Jack Sparrow

Abstract

Matroids are combinatorial structures that generalize the properties of linear independence. But not all matroids have linear representations. Furthermore, the existence of linear representations depends on the characteristic of the fields, and the linear characteristic set is the set of characteristics of fields over which a matroid has a linear representation. The algebraic independence in a field extension also defines a matroid, and also depends on the characteristic of the fields. The algebraic characteristic set is defined in the similar way as the linear characteristic set.

The linear representations and characteristic sets are well studied. But the algebraic representations and characteristic sets received much less attention, and the possible algebraic characteristic sets are still not completely known. This dissertation is a study of possible pairs of linear-algebraic characteristic sets of matroids.

Furthermore, if a matroid has an algebraic representation over a positive characteristic field, then the matroid can be represented by a particular set of linear matroids in a field of the same characteristic, called Frobenius flock. In this dissertation, we also have studied Frobenius flock representations, and possible flock characteristic sets.

Table of Contents

List of Figures

Notations

$\chi_A(M)$	The algebraic characteristic set of M
$\chi_F(M)$	The flock characteristic set of M
$\chi_L(M)$	The linear characteristic set of M
$\mathbb{E}$	Endomorphism ring
$\mathbb{F}_p$	The field of integers modulo p
$\mathbb{P}$	The set of prime numbers
$\mathbb{R}$	The set of real numbers
$\mathcal{B}$	Set of bases
$\mathcal{I}$	Independent sets
ν	Matroid valuation
F	Frobenius automorphism
M	Matroid
V_α	Frobenius flock

Chapter 1

Introduction

A matroid is a combinatorial structure on a finite set, which Whitney introduced to generalize the concept of linear independence of vectors in a vector space (Whi35). However, not all matroids have linear representations, and the existence of a linear representation can depend on the field. The linear characteristic set, $\chi_L(M)$, of a matroid M is the set of characteristics of fields over which M has a linear representation. The linear characteristic set of a matroid is either a finite set of positive primes or a cofinite set containing 0 (Rad57; Vam75). Conversely, any such set occurs as the linear characteristic set of some matroid (Kah82; Rei).

Similar to the linear independence in a vector space, algebraic independence in a field extension also defines a matroid. For a matroid M on a set E, an algebraic representation over K is a pair (L, ϕ) consisting of a field extension L of K and a map $\phi\colon E \to L$ such that any $I \subseteq E$ is independent in M if and only if the set $\phi(I)$ is algebraically independent over K. If a matroid has a linear representation over a field K, then it also has an algebraic representation over K. Conversely, an algebraic representation over a field of characteristic 0 implies a linear representation over a field of characteristic 0. However, there are matroids with algebraic representations in positive characteristic, but not linear representations. For example, the non-Pappus matroid is algebraic over any field of positive characteristic but never linear.

Similar to linear representations, the algebraic characteristic set of a matroid M, denoted by $\chi_A(M)$, is the set of characteristics of the fields in which a matroid M has algebraic representations. Since the algebraic representations are hard to deal with, the possible algebraic characteristic sets are much less well understood. One of the questions posed by Gordon was the possibility of infinite algebraic characteristic sets which are not cofinite (Gor88). Later, a paper by Evans-Hrushovski provided an example of a matroid that satisfies the question's conditions (EH91). Those are the known infinite algebraic characteristic sets. Another question asked by Gordon was regarding possible cofinite algebraic characteristic sets (Gor88), which are shown to be any cofinite set of primes in Theorem 1.1.

Researchers have studied algebraic representations using linear representations. Lindström constructed a linear representation from an algebraic representation using the derivations in fields and proved that the Lazarson matroids M_p (p a prime) has a singleton algebraic characteristic set (Lin85). Gordon used this technique to give examples of matroids with some special non-singleton finite algebraic characteristic sets (Gor88). Expecting this method to generalize, he raised the question whether matroids with a non-empty finite linear characteristic set had a finite algebraic characteristic set. We answer this in the negative with Theorem 1.1.

Bollen, Draisma, and Pendavingh, inspired by the results of Lindström (Lin85), showed that if a matroid has an algebraic representation over a positive characteristic field, then the matroid can be represented by a particular set of linear matroids in a field of the same characteristic, and they named that particular set a Frobenius flock (BDP18). If a Frobenius flock can represent a matroid, they called it a Frobenius flock representation. Analogous to linear and algebraic representations, we define the flock characteristic set of a matroid M as $\chi_F(M)$, which is the set of characteristics of fields over which M has a Frobenius flock representation.

In this thesis, we are mainly interested in the possible combinations of linear, algebraic, and flock characteristic sets. In other words, given three subsets of $\mathbb{P} \cup \{0\}$, satisfying the constraints on the linear, algebraic, and Frobenius flock characteristic sets noted above, does there exist a matroid with the corresponding characteristic sets? Evans and Hrushovski studied the geometries of field extensions to show that all algebraic representations of particular matroids are equivalent to $\mathbb{E}-$linear representations, where $\mathbb{E}$ is the endomorphism ring of some one-dimensional group G (EH91). We used this result to obtain the main theorem:

Theorem 1.1. *Let $C_L \subseteq C_A \subseteq \mathbb{P} \cup \{0\}$ be finite or cofinite subsets. Suppose either $0 \in C_L, C_A$; and C_L is cofinite, or that $0 \notin C_L, C_A$; and C_L is finite. Then there exists a matroid M such that $\chi_L(M) = C_L, \chi_A(M) = C_A$. Moreover, if C_A is cofinite, then $\chi_F(M) = \mathbb{P}$.*

However, when C_A is not cofinite, the flock characteristic set is undetermined. Here, and throughout the paper, $\mathbb{P}$ denotes the set of all primes. Furthermore, this theorem is a partial converse to the relations between linear and algebraic characteristic sets.

For the flock characteristic set of a matroid, we have a constraint:

Theorem 1.2. *Let M be a matroid. If $0 \in \chi_L(M)$, then $\chi_F(M) = \mathbb{P}$.*

In particular, Theorem 1.2 means that any matroid M with a cofinite linear characteristic set has $\chi_L(M) = \mathbb{P}$. Therefore, Theorems 1.1 and 1.2 together classify all possible combinations of linear, algebraic and Frobenius flock characteristic sets, when the linear characteristic set is cofinite, or equivalently, includes 0.

However, the flock characteristic set can also be finite. We recall that a matroid M is called rigid if it only has trivial valuations (DW92). The main result of (BDP18) is that a Frobenius flock representation M induces a valuation on M, known as the Lindström valuation. Furthermore, they show that if a rigid matroid has an algebraic representation over a field of characteristic p, it has a linear representation over a field of characteristic p. We applied this result to Lazarson and Brylawski matroids and proved they are examples of matroids with finite flock characteristic sets.

Theorem 1.3. *Let p be a prime. Then there exists a matroid M with $\chi_F(M) = \{p\}$. Furthermore, the flock characteristic sets can be any Gordon-Brylawski sets which are determined in (Gor88).*

The thesis is organized as follows. In Chapter 2, we develop the language of matroids and different types of representations. In Chapter 3, we prove the main theorems by constructing matroids and examining examples of matroids. Finally, in Chapter 4, we look at the Frobenius flocks of some particular matroids.

Chapter 2

Preliminaries

We dedicate this chapter to establishing the language of definition and basic properties of matroids and matroid representations.

2.1 Matroids

A matroid can be defined in different but equivalent ways. One way is to define using independence sets as follows.

Definition 2.1. A *matroid* M is a finite set E together with a collection $\mathcal{I}$ of subsets of E, called independent sets of M, such that:

(I1) The empty set is independent.

(I2) Every subset of an independent set is independent.

(I3) If I, J are independent sets with $|I| < |J|$, then there exists $y \in J \backslash I$ such that $I \cup \{y\}$ is independent.

To see matroids are a generalization of linear independence of vectors in a vector space, we can check that these axioms do hold for linearly independent sets of vectors. The first and second axioms are clear. Suppose that the third axiom fails, then for every vector $y \in J$, the set $I \cup \{y\}$ is linearly dependent and y can be written as the linear combination of elements of I. Therefore, $\text{span}(J) \subseteq \text{span}(I)$. But this is not possible since I, J are independent sets with $|I| < |J|$.

A maximal independent set is called a *basis*. The axiom (I3) implies that any two bases of M have the same cardinality, and the *rank* of M is this common cardinality. We denote the set of bases of M by $\mathcal{B}$.

A subset of E not in $\mathcal{I}$ is called dependent. The *circuits* of a matroid are the minimal dependent sets.

Example 2.2. For example, let $E = \{1, 2, 3\}$ and consider the collection of subsets of E, $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then $M = (E, \mathcal{I})$ is a matroid on E.

Let M is a finite set E. The dual matroid M^* of a matroid M is the matroid with set of bases that are complements of the bases of M, i.e. $\mathcal{B}(M^*) = \{E - B | B \in \mathcal{B}(M)\}$. We can also create different matroids by deletion and contraction operations. Let $T \subset E$, then the deletion $M \setminus T$ is restricting M to $E - T$. Which means the independent sets of $M \setminus T$ is $I(M \setminus T) = \{I \subseteq E - T : I \in I(M)\}$. And the contraction is the dual operation of deletion, that is $M/T = (M^* \setminus T)^*$.

We will now define the direct sum of matroids.

Definition 2.3. Let $M = (E, \mathcal{I})$ and $M' = (E', \mathcal{I}')$ are matroids, then the *direct sum* $M \oplus M'$ is the matroid $(E \cup E', \{I \cup I' : I \in \mathcal{I}, I' \in \mathcal{I}'\})$.

Now, we shall describe the matroid valuations. We will be using it to describe Frobenius flock representations. The matroid valuations were introduced by Dress and Wenzel (DW92).

Definition 2.4. Let M be a matroid with set of bases $\mathcal{B}$. Then a *valuation* of M is a function $\nu : \mathcal{B} \to \mathbb{R} \cup \{\infty\}$ such that for all $B, B' \in \mathcal{B}$ and $i \in B \setminus B'$, there exists $j \in B' \setminus B$ such that

$$\nu(B) + \nu(B') \geq \nu\left((B \setminus \{i\}) \cup \{j\}\right) + \nu\left((B' \setminus \{j\}) \cup \{i\}\right)$$

Every matroid has a *trivial valuation* defined by,

$$\nu(B) = \begin{cases} 0 & \text{if } B \in \mathcal{B} \\ \infty & \text{if } B \notin \mathcal{B} \end{cases}$$

2.2 Linear representations

After Whitney (Whi35) provided examples of matroids that are vectors in a vector space, the question was does there exist representation for any matroid as vectors in a vector space?

The linear representation of a matroid M is defined as:

Definition 2.5. Let M be a matroid. Then a *linear representation* of M is a function ϕ that maps E to a vector space V over some field K, such that a subset I of E is independent if and only if $\phi(I)$ is linearly independent.

Example 2.6. Let $E = \{1, 2, 3\}$ and M be the matroid on E with the independent sets, $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Consider the map

$$1 \longleftrightarrow \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}, 2 \longleftrightarrow \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}, 3 \longleftrightarrow \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix}$$

then this is a linear representation, because the matroid independence of M is the same as the vector independence of vectors.

The linear representations of a matroid can depend on the field characteristic.

Definition 2.7. Let M be a matroid. The *linear characteristic set* $\chi_L(M)$ is the set of field characteristics p, such that M is linearly representable over some field of characteristic p.

The linear characteristic set $\chi_L(M) \subseteq \mathbb{P} \cup \{0\}$, where, $\mathbb{P}$ is the set of prime numbers. But not all subsets of $\mathbb{P} \cup \{0\}$ can be the linear characteristic set of a matroid. In particular, they are finite or co-finite.

Proposition 2.8. *Let $\chi_L(M)$ be the linear characteristic set of a matroid M. Then either*

- $0 \in \chi_L(M)$ *and* $\chi_L(M)$ *is cofinite (Rad57); or*
- $0 \notin \chi_L(M)$ *and* $\chi_L(M)$ *is finite (Vam75).*

Here, $\chi_L(M)$ cofinite means complement of $\chi_L(M)$ is a finite set of primes.

The question of possible linear characteristic sets has been completely answered by Khan and Reid.

Proposition 2.9.

- *Every cofinite linear characteristic set (necessarily including 0) is realizable (Rei).*
- *All finite linear characteristic sets (necessarily excluding 0) are realizable (Kah82).*

Let $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ be two matroids with the linear characteristic sets $\chi_L(M_1)$ and, $\chi_L(M_2)$ respectively. We can make a new linear characteristic set from these, using the following lemma.

Lemma 2.10. $\chi_L(M_1 \oplus M_2) = \chi_L(M_1) \cap \chi_L(M_2)$

Proof. We can assume that E_1 and E_2 are disjoint. Let V and W be the linear spaces that represent M_1 and, M_2 respectively. By taking direct sum of vector spaces, it is easy to prove that if $p \in \chi_L(M_1) \cap \chi_L(M_2)$, then $p \in \chi_L(M_1 \oplus M_2)$. The $M_1 \oplus M_2$ is a matroid on $E_1 \cup E_2$ and since E_1 and E_2 are disjoint, M_i for $(i = 1, 2)$ is a restriction of $M_1 \oplus M_2$ into E_i for $(i = 1, 2)$ respectively. Then conversely, if $p \in \chi_L(M_1 \oplus M_2)$, then $p \in \chi_L(M_i)$ for $(i = 1, 2)$ since M_i is a restriction of $M_1 \oplus M_2$. □

2.3 Algebraic representations

Similar to the linear independence in a vector space, algebraic independence in field extensions also defines a matroid. Let K be a field and L an extension field of K. Elements $a_1, \ldots, a_n \in L$ are called algebraically independent over K if there exists no nonzero polynomial $f \in K[x_1, \ldots, x_n]$ such that $f(a_1, ..., a_n) = 0$. Algebraic independence satisfies the matroid independence axioms.

Definition 2.11. Let M be a matroid on a finite set E. An *algebraic representation* of M over K is a pair (L, ϕ) consisting of a field extension L of K and a map $\phi : E \to L$ such that any $I \subseteq E$ is independent in M if and only if the set $\phi(I)$ is algebraically independent over K.

Similar to linear representations, we can define the algebraic characteristic set of a matroid M.

Definition 2.12. Let M be a matroid. The *algebraic characteristic set* $\chi_A(M)$ the set of field characteristics p, such that M is algebraically representable over some field of characteristic p.

Although the algebraic representations seem straight forward, the possible algebraic characteristic sets are not well studied. But we have the following relationships between linear representations and algebraic representations.

Proposition 2.13. *Let $M = (E, \mathcal{I})$ be a matroid. If M is linearly representable over a field K, then M is algebraically representable over K.*

Proof. Let the rank of $M = (E, \mathcal{I})$ to be r. We have a linear representation $\phi : E \to V$, where V is a vector space over K. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis of M and let $v_i = \phi(e_i)$ for all $1 \leq i \leq r$. Then $\{v_1, v_2, \ldots, v_r\}$ is a basis for V. Now let $\{t_1, t_2, \ldots, t_r\}$ be independent transcendentals over K. Then for $e \in E$, the vector $\phi(e) = \sum_{i=1}^{r} a_i v_i$ where $a_i \in K$, which is a unique representation of vector. Then let us define $\psi(e) = \sum_{i=1}^{r} a_i t_i$, which is a well-defined function from $\phi : E \to K(t_1, \ldots, t_n)$.

Now let $\{b_1, b_2, \ldots, b_r\}$ be a basis of M. Then, there exists a $r \times r$ invertible matrix A over K such that:

$$(\phi(b_1)\phi(b_2)\cdots\phi(b_r)) = (v_1 v_2 \cdots v_r)\, A$$

Thus

$$(\psi(b_1)\psi(b_2)\cdots\psi(b_r)) = (t_1 t_2 \cdots t_r)\, A$$

and so,

$$(\psi(b_1)\psi(b_2)\ldots\psi(b_r))\, A^{-1} = (t_1 t_2 \cdots t_r)\,.$$

Hence, each of $\{t_1, t_2, \ldots, t_r\}$ is algebraically dependent on $\{\psi(b_1), \psi(b_2), \ldots, \psi(b_r)\}$. But $\{t_1, t_2, \ldots, t_r\}$ is a basis for the algebraic matroid and, therefore, $\{\psi(b_1), \psi(b_2), \ldots, \psi(b_r)\}$ is also a basis.

Similarly, the dependent sets are mapped to dependent sets. So, this is an algebraic representation of M. □

But the converse of the Proposition 2.13 is not true. There are matroids algebraically representable over finite field K, but not linearly representable over K.

Example 2.14. The non-Pappus matroid is a rank three matroid obtained from the Pappus configuration, with nine points and eight lines. The nine points are the nine elements of the matroid and the collinear points on these eight straight lines are the dependent sets for the non-Pappus matroid. The non-Pappus matroid is not linearly representable over any field. But, the algebraic characteristic set is the set of primes.

To see that, create a matrix N whose linear dependences are the same as the matroid dependences. Since the rank of the non-Pappus matroid is 3, we may take N to be a 3×8 matrix. Consider the points $\{a, c, d\}$, it is a basis, so we may take the first three columns of N to be an identity matrix. We can determine the coordinates for the remaining elements of the matroid. Then we have

$$\begin{matrix} a & c & d & f & h & b & e & g & i \end{matrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & x & 1 & x & z \\ 0 & 0 & 1 & 1 & 1 & 0 & y & xy & y \end{pmatrix}$$

where x and y are indeterminate and $z = x + y - yx$.

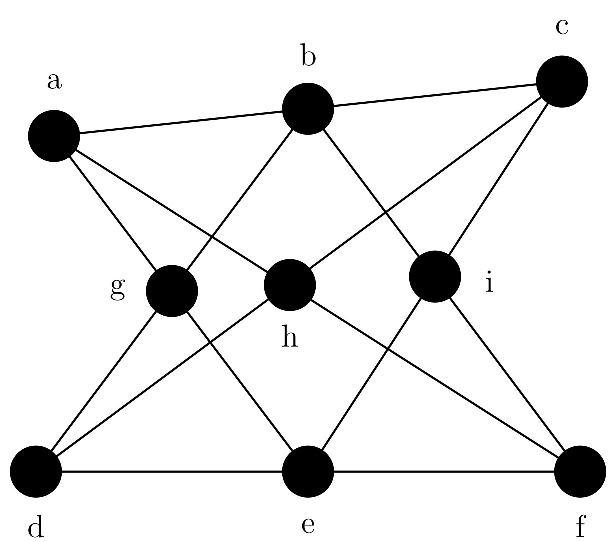

Figure 2.1: The Non Pappus Matroid

Now the set $\{g, h, i\}$ will be dependent set if the determinant of the following sub-matrix is 0

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & x & z \\ 1 & xy & y \end{vmatrix} = 0$$

But this determinant is $xy - yx$ and it is 0 means $xy = yx$. For linear representations, we work on a field and there the multiplication is commutative. So by this contradiction, the non-Pappus matroid is not linearly representable over any field. But the non-Pappus matroid is algebraically representable over a division ring of any characteristic with two non-commuting indeterminates (Ing71).

However, for fields of characteristic zero, we have the following result.

Proposition 2.15. *If* $0 \in \chi_A(M)$*, then* $0 \in \chi_L(M)$*. Here both* $\chi_L(M)$ *and* $\chi_A(M)$ *are cofinite.*

2.4 Frobenius flock representations

The algebraic representations are not easy to study. Therefore, the possible algebraic characteristic sets are still undetermined. Bollen, Draisma, and, Pendavingh showed that if a matroid has an algebraic representation over a positive characteristic field, then the matroid can be represented by a particular set of linear matroids in a field of the same characteristic. They named that particular set a Frobenius flock.

The Frobenius flocks are a special case of linear flocks. We begin by reviewing the definitions from (BDP18).

For the definition of the linear flock, we need notations and definitions of deletion and contraction for vector spaces. Let E be a finite set and K a field. For $v \in K^E$, and $I \subseteq E$, define $v_I \in K^I$ be the restriction of v to the coordinates indexed by I and for a linear subspace $V \subseteq K^E$ and $I \subseteq E$ define deletion and contraction to be

$$V \setminus I = \left\{ v_{E\setminus I} \mid v \in V \right\} \text{ and } V/I = \left\{ v_{E\setminus I} \mid v \in V, v_I = 0 \right\},$$

respectively, both of which are subspaces of K^{E-I}. Since $V \setminus I$ is the projection of V to K^I, and $V/(E-I)$ is the kernel of that projection, the rank-nullity theorem implies that $\dim V \setminus I + \dim V/(E-I) = \dim V$. It is also easy to see that when applied to disjoint sets, deletion, and contraction commute with each other, and also that multiple deletions or contractions can be combined.

Each vector space $V \subseteq K^E$ defines a matroid whose bases are the sets B such that $V \setminus (E-B) = K^B$. We denote it by $M(V)$. The deletion and contraction of vector spaces are closely related to deletion and contraction of matroids. For instance, for any $I \subseteq E$, $M(V/I) = M/I$ and $M(V \setminus I) = M \setminus I$.

Now suppose that ϕ is an automorphism of K. Then for any $v \in K^E$ we can define an action of ϕ^{-1} coordinatewise:

$$\phi^{-1} v = \left(\phi^{-1} \left(v_i \right) \right)_{i \in E}$$

and for a vector space $V \in K^E$, we have $\phi^{-1}V = \{\phi^{-1}v \mid v \in V\}$, which is also a vector space.

Following (Bol18, Def. 4.1), a *ϕ-linear flock* of E over K is defined to be a map $V\colon \alpha \to V_\alpha$ which assigns a d-dimensional linear subspace $V_\alpha \subseteq K^E$ to each $\alpha \in \mathbb{Z}^E$, such that

(LF1) $V_\alpha/i = V_{\alpha+e_i} \setminus i$ for all $\alpha \in \mathbb{Z}^E$ and $i \in E$; and

(LF2) $V_{\alpha+\mathbf{1}} = \phi^{-1}V_\alpha$ for all $\alpha \in \mathbb{Z}^E$.

Here e_i is the ith unit vector in $\mathbb{Z}^n$ and $\mathbf{1} \in \mathbb{Z}^n$ is the vector whose entries are all 1. If $\phi = F$, where $F : x \to x^p$ is the Frobenius map, then we call F-linear flock as *Frobenius flock* (BDP18, Sec. 4).

For each $\alpha \in \mathbb{Z}^n$, the vector space V_α defines a matroid $M(V_\alpha)$ whose bases are the d-element sets B such that $V \setminus (E - B) = K^B$. The union of these sets of bases, for all $\alpha \in \mathbb{Z}^n$, is also a matroid, which we call the support matroid of V_α (BDP18, Lem. 17).

Definition 2.16. Let M be a matroid. If there exists a Frobenius flock V_α with support matroid M, then V_α is a *Frobenius flock representation* of M.

Similar to linear and algebraic representations, the Frobenius flock representations also depend on the field characteristics. So, we can define the flock characteristic set as follows.

Definition 2.17. Let M be a matroid. The *flock characteristic set* $\chi_F(M)$ is the set of field characteristics p, such that M is Frobenius flock representable over some field of characteristic p.

A Frobenius flock is a set of vector spaces over a field of positive characteristics. Therefore, $\chi_F(M) \subseteq \mathbb{P}$ for any matroid M.

The Frobenius flocks were introduced to study algebraic representations. They are related by the following theorem proved in (BDP18, Thm. 34).

Theorem 2.18. *Let K be an algebraically closed field of characteristic $p > 0$, E be a finite set and M be a matroid on E. If M has an algebraic representation over K, then there exists a Frobenius flock V_α over K with support matroid M.*

Hence, combining all the relations between linear, algebraic and flock characteristic sets, we have $\chi_L(M) \setminus \{0\} \subseteq \chi_A(M) \setminus \{0\} \subseteq \chi_F(M)$ for any matroid M.

Chapter 3

Characteristic Sets

3.1 Specified characteristic sets

In this section, we will use a lemma of Evans and Hrushovski to construct matroids with specified characteristic sets. Evans and Hrushovski constructed algebraic realizations of matroids using matrices of endomorphisms over one-dimensional groups. Moreover, they showed that for certain matroids, all algebraic realizations are equivalent to realizations by such matrices. The following appears as (EH91, Lem. 3.4.1):

Lemma 3.1. *Let Φ be a collection of equations in the variables $x_0, ..., x_n$ including the equations:*

$$x_0 = 0, x_1 = 1, x_i \neq x_j \, (where \, i \neq j)$$

together with equations of the form:

$$x_i = x_j + x_k\,(\textit{where}\,j, k \neq 0, k \neq i \neq j), x_i = x_j \cdot x_k\,(\textit{where}\,i, j, k \neq 0, 1\,\textit{and}\,k \neq i \neq j).$$

Then there exists a matroid M such that M is linearly representable if and only if there exist distinct values for $x_0, \ldots, x_n$ in F which simultaneously satisfy every equation in Φ. Moreover, M has an algebraic realization over a field K if and only if there exists a linear representation of M over the division ring generated by the ring of endomorphisms of a 1-dimensional algebraic group G over a field of the same characteristic as K.

From now on, we will refer to systems of equations satisfying the conditions of Lemma 3.1 to mean the form in the first paragraph. When we talk about solutions to such a system in a division ring Q, we will always mean an assignment of *distinct* values from Q for each of the variables, such that all equations are satisfied.

The connection between algebraic realizations and endomorphisms of 1-dimensional groups form a specific construction of algebraic matroids. The main result of (EH91) is to show that under certain conditions, algebraic matroids are equivalent to ones from the 1-dimensional group construction, which is used to prove the necessity direction of the algebraic realization in Lemma 3.1.

For classifying the algebraic realizations of a matroid as in Lemma 3.1, it is helpful to recall the classification of one-dimensional algebraic groups. Over an algebraically closed field K, any one-dimensional algebraic groups G is isomorphic to the additive group $G_a = (K, +)$, the multiplicative group $G_m = (K^*, \cdot)$, or an elliptic curve over

K. If characteristic of K is 0, then their endomorphism rings are K, $\mathbb{Z}$, and either $\mathbb{Z}$ or an order in an imaginary quadratic number field, respectively. If characteristic of K is $p > 0$, then the endomorphism ring of of G_m is still $\mathbb{Z}$, but the endomorphisms of G_a are instead isomorphic to the non-commutative ring of p-polynomials, denoted $K[F]$. In addition to the same possibilities as characteristic 0, the endomorphism ring of an elliptic curve in positive characteristic may be an order in a quaternion ring.

Lemma 3.2. *Let $n > 1$ be an integer. Then there exists a system of equations Φ_n, satisfying the conditions in Lemma 3.1, whose variables include y_i for $1 \leq i \leq n+1$, and w, with the following properties: First, any solution in a division ring to the system of equations Φ_n satisfies $y_i = t^i$ and $w = t^{n+1} + nt^{n-1} + (n-1)t^{n-2}$, for some value of t such that $t^{n-1} + t^{n-2}$ is non-zero. Second, for any field F, there exists a finite set S which contains elements, all algebraic over the prime subfield of F, such that for any $t \notin S$, there exists a solution to Φ_n with $y_1 = t$.*

Proof. We define Φ_n in terms of the variables $x_0, x_1, y_1, \ldots, y_{n+1}$, $z_1, \ldots, z_{n-1}$, $w_1, \ldots, w_{2n-2} = w$ by the following equations:

$x_0 = 0$	$y_2 = y_1 . y_1$	$z_1 = y_1 + x_1$	$w_1 = y_3 + y_1$
$x_1 = 1$	$y_3 = y_2 \cdot y_1$	$z_2 = z_1 \cdot y_1$	$w_2 = w_1 + z_1$
	$\vdots$	$z_3 = z_2 \cdot y_1$	$w_3 = w_2 \cdot y_1$
	$y_n = y_{n-1} \cdot y_1$	$\vdots$	$w_4 = w_3 + z_2$
	$y_{n+1} = y_n \cdot y_1$	$z_{n-1} = z_{n-2} \cdot y_1$	$w_5 = w_4 \cdot y_1$
			$w_6 = w_5 + z_3$

$$\vdots$$

$$w_{2n-3} = \quad w_{2n-2} \cdot y_1$$

$$w_{2n-2} = w_{2n-1} + z_{n-1}$$

If we let t denote the value of y_1, then any solution to these equations in a division ring must satisfy:

$$\begin{array}{llll}
x_0 = 0 & y_1 = t & z_1 = t + 1 & w_1 = t^3 + t \\
x_1 = 1 & y_2 = t^2 & z_2 = t\,(t + 1) & w_2 = t^3 + 2t + 1 \\
 & y_3 = t^3 & z_3 = t^2\,(t + 1) & w_3 = t^4 + 2t^2 + t \\
 & \vdots & \vdots & w_4 = t^4 + 3t^2 + 2t \\
 & y_n = t^n & z_{n-1} = t^{n-2}\,(t + 1) & w_5 = t^5 + 3t^3 + 2t^2 \\
 & y_{n+1} = t^{n+1} & & w_6 = t^5 + 4t^3 + 3t^2 \\
 & & & \vdots \\
 & & & w_{2n-3} = t^{n+1} + (n-1)\,t^{n-1} \\
 & & & \qquad + (n-2)t^{n-2} \\
 & & & w_{2n-2} = t^{n+1} + nt^{n-1} \\
 & & & \qquad + (n-1)\,t^{n-2}
\end{array}$$

This proves the first claim, using w as shorthand for the variable w_{2n-2}.

For the second claim, we need to show that there exists a t outside of a finite set with a solution to Φ_n. To show that, let's consider the above solution to Φ_n as polynomials in t and let P be the set of those polynomials. Now consider the set S, which include the roots of equations of the form $p - q = 0$, $p \neq q$ for all $p, q \in P$. Now, we need to show that for all $p, q \in P$, $p - q$ are non-zero polynomials. We check cases by the degree of t. The elements with degree 1 are y_1 and z_1. The difference between y_1 and z_1 is 1, so they are distinct. Similarly, degree 2 elements are y_2 and z_2, their difference is t so they are distinct. For $3 \leq i \leq n$, the elements with degree i are $t^i, t^i + t^{i-1}, t^i + (i-2)\, t^{i-2} + (i-3)t^{i-3}$ and $t^i + (i-1)\, t^{i-2} + (i-2)t^{i-3}$. The difference between these terms are either a monic polynomial, $(i-2)\, t^{i-2} + (i-3)t^{i-3}$ or $(i-1)\, t^{i-2} + (i-2)t^{i-3}$. The terms $(i-2)\, t^{i-2} + (i-3)t^{i-3}$ and $(i-1)\, t^{i-2} + (i-2)t^{i-3}$ are not zero because a prime cannot divide consecutive integers. So degree i elements are distinct for $3 \leq i \leq n$. The elements with degree $n+1$ are y_{n+1}, w_{2n-3}, and w_{2n-2}. The difference between these terms are either a monic polynomial, $(n-1)\, t^{n-1} + (n-2)\, t^{n-2}$, or $nt^{n-1} + (n-1)\, t^{n-2}$. These are not zero since because a prime cannot divide consecutive integers and n is not zero in K. Then S is a finite set of elements, all algebraic over the prime subfield of F and for any t outside of the finite set, each of the variables in the solution to Φ_n with $y_1 = t$ will be distinct. □

Proposition 3.3. *Let C be a finite set of primes. Then there exists a matroid M such that $\chi_L(M) = \chi_A(M) = C$.*

Proof. Let n be the product of the primes in C. We use the system Φ_n from Lemma 3.2 and add the equation $y_{n+1} = w + y_{n-2}$. Now, use Lemma 3.1 to construct a matroid

M. If Φ_n has a solution in a division ring Q, then by Lemma 3.2, there exists a $t \in Q$ such that $y_{n+1} = t^{n+1}$ and $w + y_{n_2} = t^{n+1} + nt^{n-1} + nt^{n-2}$. Moreover, $t^{n-1} + t^{n-2}$ is non-zero in Q, so for these two quantities to be equal, n must be zero, which means that the characteristic of Q is contained in C. Then, $\chi_L(M) \subset C$. Also, since the endomorphism ring of a 1-dimensional group can only have positive characteristic if the field of definition has the same characteristic, then $\chi_A(M) \subset C$.

On the other hand, for any infinite field K whose characteristic is contained C, we can choose $t \in K$ outside a finite set and have a solution to Φ_n with $y_i = t^i$. Furthermore, because $n = 0$ in K, this will also be a solution with the additional equation, showing that $\chi_L(M) \supset C$ and completing the proof of the proposition. □

Lemma 3.4. *Let C be the union of $\{0\}$ and a cofinite set of primes. Then there exists a set of equations Φ_C, satisfying the set of constraints in Lemma 3.1 such that if Φ_C has a solution over a division ring, then the characteristic of the division ring is contained in C, and, conversely if F is any infinite field whose characteristic is contained in C, then Φ_C has a solution in F.*

Proof. Let $q_1, \ldots, q_m$ be the primes not in C. Set $n = q_1 q_2 \cdots q_m$ and consider Φ_n from Lemma 3.2. We will construct a system of equations Φ_C, by adding a variable v and the equation $v = w + y_{n-2}$ to Φ_n.

If Q is a division ring of characteristic not in C, then by Lemma 3.2, for any solution in Q, there exists a $t \in Q$ such that $y_{n-2} = t^{n-2}$ and $w = t^{n+1} - t^{n-2}$, because $n = 0$ in Q. Moreover, $v = t^{n+1}$. Therefore, the variables are not distinct, because $y_{n+1} = t^{n+1}$. So, if Φ_C has a solution with distinct values over a division ring, then the characteristic of the division ring is contained in C.

Conversely, if F is a field of characteristic in C, there is a solution in $F(t)$ with $y_i = t^i$ by Lemma 3.2 and by setting $v = t^{n+1} + nt^{n-1} + nt^{n-2}$. By Lemma 3.2, all the variables Φ_n are distinct. Similar to the proof in that lemma, v does not coincide with any of the variables used in Φ_n because it has a different degree in t than all except y_{n+1}, w_{2n-3}, and w_{2n-2}. The differences between v and each of these are a polynomial with leading coefficient n, 1, and 1 respectively, so they are distinct elements of $F(t)$, because n is non-zero in F. Therefore, Φ_C has a solution in $F(t)$. □

Proposition 3.5. *Let C be the union of $\{0\}$ and a cofinite set of primes. Then there exists a matroid M such that $\chi_L(M) = C$ and $\chi_A(M) = \{0\} \cup \mathbb{P}$.*

Proof. Let Φ_C be the system of equations from Lemma 3.4. Then, use Lemma 3.1 to construct a matroid M. By these two lemmas, M is realizable over any infinite field of characteristic contained in C and not realizable over any field of characteristic not contained in C. Therefore, $\chi_L(M) = C$. In particular, M is realizable over $\mathbb{Q}$, which is the field of fractions of the endomorphism ring of G_m, so M is algebraically realizable over any field. □

Proposition 3.6. *Let C be the union of $\{0\}$ and a cofinite set of primes. Then there exists a matroid M with $\chi_L(M) = \chi_A(M) = C$.*

Proof. We start with the system Φ_C as in Lemma 3.4, to which we add the variables u_1, u_2, and u_3 and the equations $u_2 = u_1 \cdot u_1$, $u_3 = u_2 \cdot u_1$, and $x_1 = 1 = u_3 + u_1$ to get Φ. Let M be a matroid constructed from this system according to Lemma 3.1. Any solution to Φ must in a division ring of characteristic 0 must satisfy $u_1^3 + u_1 - 1 = 0$. This polynomial is irreducible in $\mathbb{Q}$, so the value u_1 takes must be degree three over $\mathbb{Q}$. However, the ring of endomorphisms of G_m or an elliptic curve is contained in either the rationals, a quadratic number field, or a quaternion algebra over $\mathbb{Q}$, and all elements of these rings have degree at most 2 over $\mathbb{Q}$. Therefore, any algebraic realization of M must come from the algebraic group G_a, whose endomorphism ring has the same characteristic as the field of definition. Then, by Lemma 3.4, the characteristic of any division ring having solutions to Φ, and thus to Φ_C must be contained in C, and thus $\chi_A(M) \subset C$.

On the other hand, we want to show that $\chi_L(M) \supset C$. Let K be an algebraically closed field whose characteristic is contained in C. Let u_1 be any root of the polynomial $u_1^3 + u_1 - 1$ so long as $u_1 \neq -1$ (which is only possible in characteristic 3). Then, set $u_2 = u_1^2$, and $u_3 = u_1^3$, and we claim that 0, 1, u_1, u_2, and u_3 are distinct. We consider the possible equalities:

(i) If u_1, u_2, or u_3 is zero, then $u_1 = 0$ and so $0 = 1$ which is not possible.

(ii) If $u_1 = 1$ then, $2 = 1$ which is also not possible.

(iii) If $u_2 = 1$ then u_1 must be ± 1. We've shown that $u_1 = 1$ is not possible, and we assume that $u_1 \neq -1$.

(iv) If $u_3 = 1$ then $u_1 + 1 = 1$. So, $u_1 = 0$ which is again not possible.

(v) If $u_1 = u_2$ or $u_2 = u_3$, then $u_1 = 1$, which we've already shown to be impossible.

(vi) If $u_1 = u_3$, then $u_2 = 1$, which we've also already shown to be impossible.

Now choose t to be transcendental. Then it is not any roots of the equations $x^2 = 1$, $x^3 = 1$ and, $x^3 + x = 1$. Then u_i's are different from the variables in solution to Φ_n. So, Φ has a solution in K. Therefore $C \subset \chi_L(M)$, which completes the proof of the proposition. □

Proposition 3.7. *Let C be a finite set of primes. Then there exists a matroid M with $\chi_L(M) = C$ and $\chi_A(M) = \chi_F(M_P) = \mathbb{P}$.*

Proof. Let n be the product of the primes in C. Consider the system of equations Φ consisting of Φ_n from Lemma 3.2 together with additional variables $u_1, \ldots, u_8$ and the equations:

$$u_3 = u_2 + x_1$$

$$u_4 = u_1 \cdot u_3$$

$$u_5 = u_2 \cdot u_1$$

$$u_6 = u_5 + w$$

$$u_7 = u_6 + y_{n-2}$$

$$u_8 = u_4 + y_{n+1}$$

$$u_8 = u_7 + u_1$$

Let M be the matroid related Φ by Lemma 3.1. If we have any solution to Φ in a division ring Q, then there exists $t \in Q$ such that $y_i = t^i$ and $w = t^{n+1} + nt^{n-2} + (n-1)t^{n-2}$ by Lemma 3.2. If we let a and b be the values of u_1 and u_2, respectively. Then, the other variables satisfy:

$$u_3 = b + 1$$

$$u_4 = ab + a$$

$$u_5 = ba$$

$$u_6 = ba + t^{n+1} + nt^{n-1} + (n-1)t^{n-2}$$

$$u_7 = ba + t^{n+1} + nt^{n-1} + nt^{n-2}$$

$$u_8 = ab + a + t^{n+1}$$

$$= ba + a + t^{n+1} + nt^{n-1} + nt^{n-2}$$

If Q is commutative, then $ab = ba$ and so the last equation implies that $nt^{n-1} + nt^{n-2} = 0$. Since $t^{n-1} + t^{n-2}$ is non-zero by Lemma 3.2, then $n = 0$, which means that the characteristic of a commutative field which has solutions to Φ must be contained in C.

Conversely, let $K = \mathbb{F}_p(a, b, t)$, where $p \in C$ and consider the solution formed by setting $y_i = t^i$, $u_1 = a$, $u_2 = b$, and assigning the other variables as above. Then the variables $u_1, \ldots, u_8$ are distinct polynomials. Moreover, the variables u_i are not contained in $\mathbb{F}_p(t)$, whereas all the variables used by the system Φ_n are contained in $\mathbb{F}_p(t)$, so these are also distinct.

Finally, we want to show that M is algebraically realizable over the field $\overline{\mathbb{F}}_p$ for any prime p. Since M is linearly realizable when $p \in C$, it is sufficient to consider the case when $p \notin C$, so n is non-zero. We give an algebraic realization by finding a solution to Φ over the division ring $\overline{\mathbb{F}}_p(F)$ coming from the endomorphism ring of G_a. We first choose $\alpha \in \overline{\mathbb{F}}_p \setminus \mathbb{F}_{p^{n-1}} \setminus \mathbb{F}_{p^{n-2}}$. Thus, $\alpha^{p^{n-1}} - \alpha$ and $\alpha^{p^{n-2}} - \alpha$ are non-zero, so we set $\beta = (\alpha^{p^{n-1}} - \alpha)^{-1}$ and $\gamma = (\alpha^{p^{n-2}} - \alpha)^{-1}$. Then, let $y_1 = F$, $u_1 = \beta F^{n-1} + \gamma F^{n-2}$,

$u_2 = n\alpha$, and the other variables as:

$$
\begin{aligned}
u_3 &= n\alpha + 1 \\
u_4 &= (n\beta\alpha^{p^{n-1}} + \beta)F^{n-1} + (n\gamma\alpha^{p^{n-2}} + \gamma)F^{n-2} \\
&= (n\alpha\beta + n + \beta)F^{n-1} + (n\alpha\gamma + n + \gamma)F^{n-2} \\
u_5 &= n\alpha\beta F^{n-1} + n\alpha\gamma F^{n-2} \\
u_6 &= F^{n+1} + (n\alpha\beta + n)F^{n-1} + (n\alpha\gamma + n - 1)F^{n-2} \\
u_7 &= F^{n+1} + (n\alpha\beta + n)F^{n-1} + (n\alpha\gamma + n)F^{n-2} \\
u_8 &= F^{n+1} + (n\alpha\beta + n + \beta)F^{n-1} + (n\alpha\gamma + n + \gamma)F^{n-2}
\end{aligned}
$$

All of these are distinct values in $\overline{\mathbb{F}}_p(F)$ and satisfy the equations in Φ. Moreover, they are distinct from the variables used in Φ_n, because those all lie in the subfield $\mathbb{F}_p(F)$. □

Proof of Theorem 1.1. We first suppose that C_A is finite, which implies that $C_L \subset C_A$ is also finite and that neither C_A nor C_L contains 0. By Proposition 3.7, there exists a matroid M_1 such that $\chi_L(M_1) = C_L$ and $\chi_A(M_1) = \mathbb{P}$. By Proposition 3.3, there exists another matroid M_2 such that $\chi_L(M_2) = \chi_A(M_2) = C_A$. Since the characteristic set of a direct sum is the intersection of the characteristic sets, $\chi_L(M_1 \oplus M_2) = C_L$ and $\chi_A(M_1 \oplus M_2) = C_A$.

Now suppose that C_A is cofinite. Then, C_L may be either finite or cofinite, and $0 \in C_A, C_L$ if and only if C_L is cofinite. Then by Proposition 3.6, there exists a matroid M_1 such that $\chi_L(M_1) = \chi_A(M_1) = C_A \cup \{0\}$. Moreover, by Theorem 1.2, whose proof doesn't use anything in this section, $\chi_F(M_1) = \mathbb{P}$. By either Proposition 3.5 or 3.7, there exists a matroid M_2 such that $\chi_L(M_2) = C_L$ and either $\chi_A(M_2) = \mathbb{P} \cup \{0\}$ (if C_L is cofinite) or $\chi_A(M_2) = \mathbb{P}$ (if C_L is finite). Because the Frobenius flock characteristic set contains the algebraic characteristic set, $\chi_F(M_2) = \mathbb{P}$. Again, the characteristic sets of a direct sum are the intersections of the characteristic sets, so $\chi_L(M_1 \oplus M_2) = C_L$, $\chi_A(M_1 \oplus M_2) = C_A$, and $\chi_F(M_1 \oplus M_2) = \mathbb{P}$. For the Frobenius flock characteristic set of a direct sum, this follows from Theorems 4.11, 4.13, and 4.18 from (Bol18). □

3.2 Stretching Frobenius flocks

In this section, we prove Theorem 1.2, establishing the existence of Frobenius flocks for any matroid which is linear over a field of characteristic 0.

Lemma 3.8. *Let V_α be a ϕ-linear flock over a field K. Suppose that ψ is an automorphism of K such that $\psi^m = \phi$, then there exists a ψ-linear flock V'_β where $V'_{m\alpha} = V_\alpha$ for all $\alpha \in \mathbb{Z}^n$, and whose support matroid is the same as the support matroid of V_α.*

Proof. Let $\beta \in \mathbb{Z}^n$, and write $\beta = m\alpha + (r_1, \ldots, r_n)$ where $0 \leq r_i < m$ and $\alpha \in \mathbb{Z}^n$. We define the sets $I_{<j} = \{i : r_i < j\}$, $I_{>j} = \{i : r_i > j\}$ and $I_j = \{i : r_i = j\}$.

Now let us define the K-vector space

$$V'_\beta = \bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k},$$

and we claim that as β ranges over all elements of $\mathbb{Z}^n$, V'_β defines a ψ-linear flock. Note that a term $\phi^k V_\alpha / I_{>k} \setminus I_{<k}$ in the definition of V'_β is a subspace of K^{I_k} and so the direct sum gives a vector subspace of K^E via the isomorphism $K^E \cong \oplus_{k=0}^{m-1} K^{I_k}$. Also, $\beta = m\alpha$, meaning that $r_i = 0$ for all i, then only the $k = 0$ summand of the definition of V'_β is non-trivial, and this shows that $V'_{m\alpha} = V_\alpha$.

As noted above, the rank-nullity theorem implies that $d = \dim V_\alpha = \dim V/I_{>0} + \dim V \setminus I_0$. By induction, and because the sets I_j partition E, $d = \sum_{k=0}^{m-1} \dim V_\alpha / I_{>k} \setminus I_{<k}$, which implies that $\dim V'_\beta = d$

We check the axiom (LF2) of a linear flock first. Consider

$$\beta + \mathbf{1} = m\alpha' + (r'_1, \ldots, r'_n)$$

and if we define $\alpha' = \alpha + e_{I_{m-1}}$, $I'_j = \{i : r'_i = j\} = I_{j-1}$ for $1 \leq j \leq m-1$ and $I'_0 = I_{m-1}$, then similarly to the decomposition $\beta' = m\alpha' + (r'_1, \ldots, r'_n)$, where $r_i = j$ if and only if $i \in I'_j$. In addition, we also define $I'_{<k} = \bigcup_{j<k} I'_j$ and $I'_{>k} = \bigcup_{j>k} I'_j$, which means that $I'_{<k} = I_{<k-1} \cup I_{m-1}$ and $I'_{>k} = I_{>k-1} - I_{m-1}$, where $-$ denotes the set difference, to distinguish it from matroid deletion.

For $I \subseteq E$, the following generalization holds in analogy with Lemma 9 of (BDP18),

(LF1') $V_\alpha / I = V_{\alpha + e_I} \setminus I$ for all $\alpha \in \mathbb{Z}^n$ and $I \subseteq \{1, 2, \ldots, n\}$ where $e_I = \sum_{i \in I} e_i$.

Then, we have

$$V'_{\beta+\mathbf{1}} = \bigoplus_{k=0}^{m-1} \psi^k V_{\alpha'} / I'_{>k} \setminus I'_{<k} \qquad \text{by definition of } V'$$

$$= \left(\bigoplus_{k=1}^{m-1} \psi^k V_{\alpha + e_{I_{m-1}}} / \left(I_{>k-1} \setminus I_{m-1} \right) \setminus \left(I_{<k-1} \cup I_{m-1} \right) \right)$$

$$\oplus V_{\alpha + e_{I_{m-1}}} / I_{<m-1} \qquad \text{by the above identities}$$

$$= \left(\bigoplus_{k=1}^{m-1} \psi^k V_\alpha / I_{>k-1} \setminus I_{<k-1} \right) \oplus V_{\alpha + \mathbf{1}} \setminus I_{<m-1} \qquad \text{by (LF1')}$$

$$= \left(\bigoplus_{k=1}^{m-1} \psi^k V_\alpha / I_{>k-1} \setminus I_{<k-1} \right) \oplus \phi V_\alpha \setminus I_{<m-1} \qquad \text{by (LF2)}$$

$$= \left(\bigoplus_{k=1}^{m-1} \psi^k V_\alpha / I_{>k-1} \setminus I_{<k-1} \right) \oplus \psi \cdot \psi^{m-1} V_\alpha \setminus I_{<m-1} \qquad \text{because } \phi = \psi^m$$

$$= \psi V_\beta' \qquad \text{by definition of } V_\beta'$$

This completes the proof of (LF2).

Now we consider the axiom (LF1), which says that $V_\beta / i = V_{\beta + e_i} \setminus i$. We first consider the case when $i \notin I_{m-1}$ and let $j = r_i$, so that $i \in I_j$. Therefore, the vector $\beta + e_i$ can be written as $m\alpha + (r_1', \ldots, r_n')$, where $r_1', \ldots, r_n' < m$ and $r_k' = r_k$ unless $k = i$ in which case $r_i' = r_i + 1$. Then, if $I'_{<k} = \{i : r_i' < k\}$ and $I'_{>k} = \{i : r_i' > k\}$, as usual, then $I'_{<j+1} = I_{<j+1} - \{i\}$ and $I'_{>j} = I_{>j} \cup \{i\}$, but other than these two exceptions, $I'_{<k} = I_{<k}$ and $I'_{>k} = I_{>k}$. Therefore, the definition of V' gives us:

$$V_{\beta+e_i}' = \left(\bigoplus_{\substack{k=0 \\ k \neq j, j+1}}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k} \right) \oplus \left(\psi^j V_\alpha / \left(I_{>j} \cup \{i\} \right) \setminus I_{<j} \right)$$

$$\oplus \left(\psi^{j+1} V_\alpha / I_{>j+1} \setminus \left(I_{<j+1} - \{i\} \right) \right).$$

The deletion of the ith component only affects the summand contained in K^{E_j}, which is the last summand, so by combining the deletions:

$$V'_{\beta+e_i} \setminus i = \left(\bigoplus_{\substack{k=0 \\ k\neq j,j+1}}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k} \right) \oplus \left(\psi^j V_\alpha / \left(I_{>j} \cup \{i\} \right) \setminus I_{<j} \right)$$

$$\oplus \left(\psi^{j+1} V_\alpha / I_{>j+1} \setminus I_{<j+1} \right)$$

$$= \left(\bigoplus_{\substack{k=0 \\ k\neq j}}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k} \right) \oplus \left(\psi_J V_\alpha / (I_{>j} \cup \{i\}) \setminus I_{<j} \right)$$

$$= \left(\bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k} \right) / \{i\} = V'_\beta / \{i\},$$

because the contraction of $\{i\}$ only affects the $k = j$ summand. This completes the proof of (LF1) when $i \notin I_{m-1}$.

Now suppose that $i \in I_{m-1}$. In this case $\beta + e_i = m\alpha' + (r'_1, \ldots, r'_n)$ where $\alpha' = \alpha + e_i$, $I'_k = \{i : r'_i = k\} = I_k$ for $k \neq 0, m-1$, $I'_{m-1} = I_{m-1} \setminus \{i\}$ and $I'_0 = I_0 \cup \{i\}$. Then,

$$\begin{aligned}
V'_{\beta+e_i} &= \left(\bigoplus_{k=1}^{m-2} \psi^k V_{\alpha+e_i} / \left(I_{>k} - \{i\} \right) \setminus \left(I_{<k} \cup \{i\} \right) \right) \oplus \left(V_{\alpha+e_i} / \left(I_{>0} - \{i\} \right) \right) \\
&\quad \oplus \left(\psi^{m-1} V_{\alpha+e_i} \setminus \left(I_{<m-1} \cup \{i\} \right) \right) \\
V'_{\beta+e_i} \setminus i &= \left(\bigoplus_{k=1}^{m-2} \psi^k V_{\alpha+e_i} / \left(I_{>k} \setminus \{i\} \right) \setminus \left(I_{<k} \cup \{i\} \right) \right) \oplus \left(V_{\alpha+e_i} / \left(I_{>0} - \{i\} \right) \setminus \{i\} \right) \\
&\quad \oplus \left(\psi^{m-1} V_{\alpha+e_i} \setminus \left(I_{<m-1} \cup \{i\} \right) \right) \\
V'_{\beta+e_i} \setminus i &= \left(\bigoplus_{k=1}^{m-2} \psi^k V_\alpha / I_{>k} \setminus I_{<k} \right) \oplus \left(V_\alpha / I_{>0} \right) \oplus \left(\psi^{m-1} V_\alpha \setminus I_{<m-1} / \{i\} \right) \\
&\qquad \text{(by (LF1) in each summand)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k}\right) / \{i\} \\
&= V_\beta^{'} / i,
\end{aligned}
$$

which completes the proof of (LF1) and thus that V'_β is a matroid flock.

Finally, we want to show that the support matroids of V_α and V'_β are the same. Since $V'_{m\alpha} = V_\alpha$, any basis of the support matroid of V_α will also be a basis of the support matroid of V'_β. For the converse, we suppose that β is any coordinate in $\mathbb{Z}^n$ and B is any subset of E. Then, with α, $I_{>k}$, and $I_{<k}$ as before,

$$
V'_\beta = \bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{>k} \setminus I_{<k}
$$

and

$$
V'_\beta / (E - B) = \bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{>k} / (I_k - B) \setminus I_{<k}.
$$

The deletion of a vector space always contains the contraction of the same set, and thus,

$$
\begin{aligned}
V'_\beta / (E - B) &\subset \bigoplus_{k=0}^{m-1} \psi^k V_\alpha / I_{>k} / (I_k - B) / (I_{<k} - B) \setminus (I_{<k} \cap B) \\
&= \bigoplus_{k=0}^{m-1} \psi^k V_\alpha / (E - B) / (I_{>k} \cap B) \setminus (I_{<k} \cap B).
\end{aligned}
$$

However, this last expression is the same construction that was used to make V'_β, but applied to $V_\alpha/(E - B)$. Therefore, its vector space dimension is the same as that of $V_\alpha/(E - B)$, which, by the containment, implies that $\dim V'_\beta/(E - B) \leq$

$\dim V_\alpha/(E-B)$. If B is a basis of the support matroid of V'_β, then $\dim V'_\beta = |B|$, which means that B is also a basis of the support matroid of V_α. This concludes the proof that V_α and V'_β have the same support matroids. □

Using this Lemma 3.8, we can prove Theorem 1.2.

Proof of Theorem 1.2. By (Ing71), if $0 \in \chi_L(M)$, then M has a representation over a finite extension of rationals (a number field K). Let $\mathcal{O}_K$ be the ring of integers in the number field K. Then, using Going-up theorem (Mar18, Thm 20) for any prime p, there exists a prime ideal $\mathfrak{P} \subset \mathcal{O}_K$ such that $\mathfrak{P} \cap \mathbb{Z} = (p)$. By (Mar18, Thm 14), $\mathcal{O}_K$ is a Dedekind domain and if $\mathfrak{I}$ is any non-zero ideal in $\mathcal{O}_K$, then $\mathcal{O}_K/\mathfrak{I}$ is finite. So $\mathfrak{P}$ is a maximal ideal and $\mathcal{O}_K/\mathfrak{P}$ is a finite field. The containment of $\mathbb{Z}$ in $\mathcal{O}_K$ induces a ring-homomorphism $\mathbb{Z} \to \mathcal{O}_K/\mathfrak{P}$, and the kernel is $\mathfrak{P} \cap \mathbb{Z} = (p)$. So, we obtain an embedding $\mathbb{F}_p \to \mathcal{O}_K/\mathfrak{P}$. Then $\mathcal{O}_K/\mathfrak{P}$ is an extension of finite degree over $\mathbb{F}_p$. Thus, $\mathcal{O}_K/\mathfrak{P} \cong \mathbb{F}_{p^n}$ for some n. Also, any localization of a Dedekind domain at a non-zero prime ideal is a discrete valuation ring (DF99, Thm 15, Ch 16). Then $(\mathcal{O}_K)_{\mathfrak{P}} = (\mathcal{O}_K \setminus \mathfrak{P})^{-1} \mathcal{O}_K$ is a discrete valuation ring with the maximal ideal $\mathfrak{P}\,(\mathcal{O}_K)_{\mathfrak{P}}$. So, there exists a valuation $\nu : K^* \to \mathbb{Z}$ with the valuation ring $(\mathcal{O}_K)_{\mathfrak{P}}$. Also, $(\mathcal{O}_K)_{\mathfrak{P}}/\mathfrak{P}\,(\mathcal{O}_K)_{\mathfrak{P}}$ is the field of fractions of $\mathcal{O}_K/\mathfrak{P}$, then $(\mathcal{O}_K)_{\mathfrak{P}}/\mathfrak{P}\,(\mathcal{O}_K)_{\mathfrak{P}} \cong \mathcal{O}_K/\mathfrak{P} \cong \mathbb{F}_{p^n}$.

By (BCD22, Lem 3.5), using this valuation, there exists a linear flock with trivial automorphism over a finite field $\mathbb{F}_{p^n}$.

Now consider the inverse Frobenius automorphism $F^{-1} : x \mapsto x^{-p}$ of $\mathbb{F}_{p^n}$, then F^{-n} is the trivial automorphism. Then, using Lemma 3.8 with $m = n$ and $\psi = F^{-1}$

, we have M has a Frobenius flock representation over a field of characteristic p. So we have $\chi_F(M) = \mathbb{P}$. $\square$

3.3 Finite Frobenius flock characteristic sets

Definition 3.9. Consider a set of primes $\{p_1, p_2, \ldots, p_k\}$ and let $n = p_1 \cdots p_k + 1$ and $s = [\log_2 n]$. For $0 \leq i \leq s$, set $b_i = \left[n/2^{(s-i+1)}\right]$. Then $b_0 = 0, b_1 = 1, b_2 = 2$ or 3 and in general, $b_i = 2b_{i-1}$ or $2b_{i-1} + 1$. The *Brylawski matrix* N_n is the matrix:

$$
\begin{array}{cccccccccccccc}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & \cdots & & & & & \\
\end{array}
$$
$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & & 1 & 0 & & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & \cdots & 2 & 1 & \cdots & 2 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & & b_i & b_i & & b_s & b_s
\end{pmatrix}
$$

We call the set of primes $\{p_1, p_2, \ldots, p_k\}$ a *Gordon-Brylawski set*, if the residues $b_0, b_1, \ldots, b_s$ all differ by at least two modulo each prime p_i (except for b_0 and b_1; perhaps b_1 and b_2) (Gor88).

Proposition 3.10. *Let N_n be the Brylawski matrix where $n = p_1 \cdots p_k + 1$ and M_n be the matroid which is linearly represented over $\mathbb{F}_{p_1}$ by the matrix N_n, then $\chi_F(M_n) \subseteq \{p_1, \ldots, p_k\}$. If $\{p_1, p_2, \ldots, p_k\}$ is a Gordon-Brylawski set, then $\chi_F(M_n) = \{p_1, \ldots, p_k\}$.*

Proof. Assume that M_n has a Frobenius flock representation over a field K of characteristic p. Let A be the first four columns of Brylawski matrix, then it is isomorphic to $U_{3,4}$, which is rigid by (BDP18, Lemma 53). Then by (Bol18, Lemma 3.27), there exist $\alpha \in \mathbb{Z}^E$ such that $M_\alpha = M(\mathcal{V}_\alpha)$ contains columns of A as a circuit. Then we can show that $\mathcal{V}_\alpha$ equals the row space of the Brylawski matrix N_n over $\mathbb{F}_p$. Let B be the matrix representing V_α and c_i denote the i^{th} column of B. Inductively, as the first four elements form a circuit, we may row-reduce the matrix B such a way that the columns corresponding to this circuit are as in the matrix N_n. Since $\{v_{1,}v_2, v_5\}$ is a circuit, then 3^{rd} entry in c_5 is 0 and $\{v_{3,}v_4, v_5\}$ is a circuit, then all non-zero entries of c_5 are same. Then by the column scaling we get that the c_5 is the 5^{th} column of the matrix N_n. For i^{th} element of the Brylawski matroid N_n, there are exactly 2 circuits of the form $\{v_i, v_j, v_k\}$. where $j, k < i$. Therefore, the i^{th} column of the B can be written as the linear combination of those 2 corresponding columns in B and, the i^{th} column of B can be scaled to make the i^{th} column of N_n. Hence M_α is linearly represented by $\mathbb{F}_p$ by Brylawski matrix.

The sub-determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & b_s \end{vmatrix} = 2b_s - 1 = n - 1 = p_1 \cdots p_k,$$

where $s = [\log_2 n]$ and $b_s = \left[n/2^{(s-i+1)}\right]$. Hence these three columns are dependent over characteristic p_1. Therefore $p = p_i$ for some i, so $\chi_F(M_n) \subseteq \{p_1, \ldots, p_k\}$. If the set of primes $\{p_1, p_2, \ldots, p_k\}$ is a Gordon-Brylawski set, then by (Gor88, Theorem 5), we have $\chi_L(M_n) = \{p_1, \ldots, p_k\}$, therefore $\chi_F(M_n) = \{p_1, \ldots, p_k\}$. □

Proof of Theorem 1.3. The Lazarson matroid (4.1) M_p where p is a prime, has singleton flock characteristic set (BDP18, Theorem 56). The Brylawski matroid N_n has finite non-singleton flock characteristic set if the conditions in Definition 3.9 are met. □

3.4 Infinite algebraic characteristic sets

One of the problems Gordon asked in (Gor88) is the possibility of infinite algebraic characteristic sets that are not cofinite. We have attempted to answer this question in this section.

Proposition 3.11. *Let $n > 1$ be an integer. Then there exists a matroid M such that $\chi_A(M) = \{p \text{ prime } : p \not\equiv 1 \mod n\}$.*

Proof. We define a system of equations Φ_n satisfying the conditions in Lemma 3.1, in terms of the variables $x_0, x_1, y_1, \cdots, y_{n-1}, z_1, z_2, z_3$ by the following equations:

$$x_0 = 0, x_1 = 1, y_2 = y_1 \cdot y_1, y_3 = y_2 \cdot y_1, \cdots, y_{n-1} = y_{n-2} \cdot y_1,$$

$$y_{n-1} \cdot y_1 = x_1, z_2 = y_1 \cdot z_1 z_3 = z_1 \cdot y_1.$$

If we let t denote the value of y_1 and s denote the value of z_1 , then in any solution to these equations in a division ring must satisfy:

$$x_0 = 0$$

$$x_1 = 1$$

$$y_1 = t$$

$$y_2 = t^2$$

$$y_3 = t^3$$

$$\vdots$$

$$y_{n-1} = t^{n-1}$$

$$1 = t^n$$

$$z_2 = st$$

$$z_3 = ts$$

Now, use Lemma 3.1 to construct a matroid M. Since a solution to Φ_n will have distinct values for each variable. Moreover, the distinctness implies the inequality $s \cdot t \neq t \cdot s$ and, the non-commutativity implies that the $\chi_L(M) = \emptyset$.

Furthermore, we want show $\chi_A(M) = \{p \text{ prime } : p \not\equiv 1 \mod n\}$. Assume that Φ_n has a solution in a division ring generated by the ring of endomorphisms of a 1-dimensional algebraic group over a field of characteristic p. If n divides $p - 1$, then the polynomial $t^n = 1$ splits over $\mathbb{F}_p$. So, $t \in \mathbb{F}_p$, and t will commute with all elements. But since $st \neq ts$, this case is not possible. So, this contradiction implies $\chi_A(M) \subseteq \{p \text{ prime } : p \not\equiv 1 \mod n\}$.

On the other hand, if n doesn't divide $p-1$, then choose t to be a primitive $n-$th root in $\overline{\mathbb{F}_p}$, but not in $\mathbb{F}_p$, then t^i are different for $0 \leq i \leq n-1$. Also choose $s = F$. Then $ts = tF$ and $st = Ft = t^pF \neq ts$. So, there exists a solution to Φ_n in $\overline{\mathbb{F}}_p(F)$. Then we have $\chi_A(M) = \{p \text{ prime } : p \not\equiv 1 \mod n\}$. □

Now we look at the set of primes p with $p \not\equiv 1 \mod n$. Let A be an infinite set of primes, then the proportion of primes in A is A is $\lim_{m\to\infty} \frac{|\{p \in A; p < m\}|}{|\{p \in \mathbb{P}; p < m\}|}$. By Dirichlet's theorem on arithmetic progressions, for all a such that $gcd(a, n) = 1$, the set $\{a + kn, 0 < k \in \mathbb{Z}\}$ contains infinitely many prime numbers, and each has $\frac{1}{\varphi(n)}$ proportion of primes, where φ is the Euler's totient function. Then the set of primes p with $p \not\equiv 1 \mod n$ has $\frac{\varphi(n)-1}{\varphi(n)}$ proportion of primes.

We can take the intersection of characteristic sets by taking the direct sum of matroids. Then we have the following question. Let $r \in \mathbb{Q}$ with $0 \leq r \leq 1$, can we find a matroid M with algebraic characteristic set with the proportion of primes as r?

Chapter 4

Frobenius Flock Representations

4.1 Lazarson matroids

Consider the following matrix.

$$\begin{array}{ccccccccc} x_0 & x_1 & & x_n & z & y_0 & y_1 & & y_n \end{array}$$
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & & \vdots & 1 & 1 & 0 & & 1 \\ \vdots & & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & & \cdots & 1 & 1 & 1 & 1 & & 0 \end{pmatrix} \tag{4.1}$$

Definition 4.1. The Lazarson matroid $M_n = (E, \mathcal{B})$, with the set of bases $\mathcal{B}$ of (4.1) over $\mathbb{Q}$ without $\{y_0, \ldots, y_n\}$.

In (BDP18, Thm. 56), the authors showed that $\chi_F(M_p) = \{p\}$ for p prime and for a composite number $\chi_F(M_n) = \{p \text{ prime } : p|n\}$. We show that if n is a composite number, then the Lazarson matroid M_n does not have any algebraic representation, furthermore, it does not have any Frobenius flock representation.

Proposition 4.2. *Let M_n be the Lazarson matroid where n is a composite number then $\chi_F(M_n) = \emptyset$.*

Proof. Assume that M_n has a Frobenius flock representation over a field of characteristic p and ν be the associated valuation. Then by Lemma 55 and Theorem 56 of (BDP18), there exists an $\alpha \in \mathbb{Z}^E$ such that $M_\alpha^\nu = (E, \mathcal{B}_\alpha^\nu)$ has a linear representation by (4.1) over $\mathbb{F}_p$ where

$$\mathcal{B}_\alpha^\nu = \left\{ B \in \mathcal{B} : \sum_{i \in B} \alpha_i - \nu(B) = \max \left\{ \sum_{i \in B} \alpha_i - \nu(B) : B \in \mathcal{B} \right\} \right\}.$$

We can do shifting on ν i.e., $\nu'(B) = \nu(B) - \sum_{i \in B} \alpha_i$ and translating i.e. $\nu''(B) = \nu'(B) - \min \nu'$, we get $\mathcal{B}_\alpha^\nu = \mathcal{B}_0^{\nu_0} = \{B \in \mathcal{B} : v_0(B) = 0\}$ for some valuation ν_0. Let $\mathcal{B}'$ be the set of bases (4.1) for over $\mathbb{F}_p$. Then the valuation ν_0 must satisfy the following,

$$\nu_0(B) = \begin{cases} 0 & \text{if } B \in \mathcal{B}'; \\ > 0 & \text{if } B \in \mathcal{B} \backslash \mathcal{B}'. \end{cases}$$

Now consider the following sets,

$$\begin{aligned} B_1 &= \left\{ x_{i_1}, x_{i_2}, \ldots, x_{i_{(n+1)-p}}, y_{i_1}, y_{j_2}, \ldots, y_{j_p} \right\} \\ B_2 &= \{ x_{i_1}, y_{i_1}, y_{j_2}, \ldots, y_{j_n} \} \end{aligned}$$

Here all the indices are distinct, then $B_1, B_2 \in \mathcal{B}$ and $\det B_1 = \pm(p-1)$ and $\det B_2 = \pm(n-1)$. So both $B_1, B_2 \in \mathcal{B}'$. Now by the definition of valuation, for any $a \in B_1 \setminus B_2$, there exists a $b \in B_2 \setminus B_1$ so that

$$0 = \nu_0(B_1) + \nu_0(B_2) \geq \nu_0\left((B_1 \cup \{b\}) \setminus \{a\}\right) + \nu_0\left((B_2 \cup \{a\}) \setminus \{b\}\right). \tag{4.2}$$

Now take $a = x_{i_2} \in B_1 \setminus B_2$, then $b \in \left\{y_{j_{p+1}}, y_{j_{p+2,}}, \ldots, y_{j_n}\right\}$, without loss of generality, take $b = y_{j_{p+1}}$. Then

$$(B_1 \cup \{b\}) \setminus \{a\} = \left\{x_{i_1}, \ldots, x_{i_{(n+1)-p}}, y_{i_1}, y_{j_2}, \ldots, y_{j_p}, y_{j_{p+1}}\right\} \in \mathcal{B}.$$

But $\det((B_1 \cup \{b\}) \setminus \{a\}) = \pm p$, so $((B_1 \cup \{b\}) \setminus \{a\}) \notin \mathcal{B}'$. Then $\nu_0\left((B_1 \cup \{b\}) \setminus \{a\}\right) > 0$. But then the inequality (4.2) is not satisfied. So M_n, doesn't have any Frobenius flock representation. □

4.2 Frobenius flock representation of matroids

The Frobenius flock representations are helpful to study algebraic representations of matroids. But the Frobenius flock of a matroid is not yet well understood. Bollen, Draisma, and Pendavingh studied the Frobenius flock representations of special matroids (BDP18). Particularly, rational matroids (matroids which are linear over $\mathbb{Q}$), rigid matroids (all valuations of those matroids are trivial) and, Lazarson matroids. In this section, we study the Frobenius flock representations of the non-Fano matroid and the non-Pappus matroid.

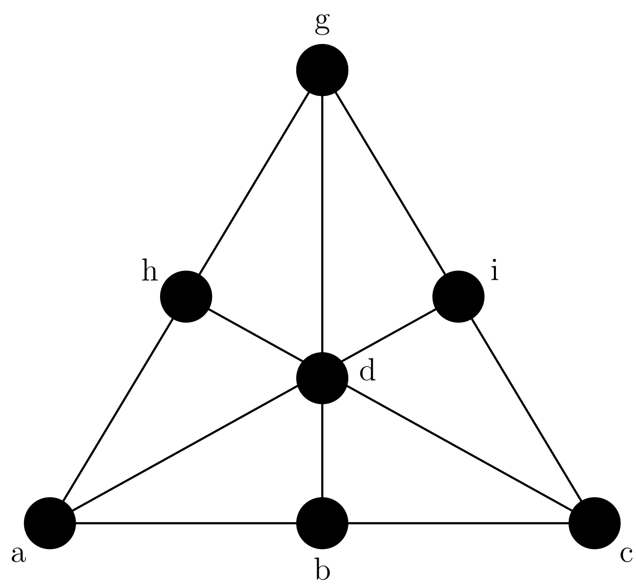

Figure 4.1: The Non Fano Matroid

The *non-Fano matroid* is a rank-three matroid obtained from the Fano plane. The seven points are the seven elements of the matroid and the points on these six straight lines are the dependent sets for the non-Fano matroid. The *Fano matroid* is the matroid on the same seven points, six lines and, additionally, b, h and, i are collinear, therefore $\{b, h, i\}$ is a dependent set. So, $\{b, h, i\}$ is a basis of the non-Fano matroid but a dependent set of the Fano matroid.

The linear characteristic set of the non-Fano matroid all field characteristics but 2. But, the algebraic characteristic set contains all field characteristics, and so does the flock characteristic set. Furthermore, we have this observation about the non-Fano matroid and the Fano matroid. In the following propositions, $\mathcal{B}(M)$ denotes the bases of the matroid M.

Proposition 4.3. *Let M be the non-Fano matroid, $M^{'}$ the Fano matroid and $\mathcal{B}(M) - \mathcal{B}(M') = \{B'\}$. Let M_α be a Frobenius flock representation of M over a field of characteristic 2. There exists β and γ in $\mathbb{Z}^7$ such that $M_\beta = M'$, M_γ is a matroid with $B' \in \mathcal{B}(M_\gamma)$ and $\gamma - \beta = 0 + \sum_{i \in B'} k \cdot e_i$ for some $0 < k \in \mathbb{Z}$.*

Proof. By Lemma 55 and Theorem 56 of (BDP18), there exists a $\beta \in \mathbb{Z}^E$ such that $M_\beta = M'$. We can translate the flock to make $\beta = 0$ and let the associated valuation be ν. Then $\sup\{\sum_{i \in \mathcal{B}} \beta_i - \nu(B)\} = 0$ and bases with 0 valuations are exactly bases of the Fano matroid.

$$\nu(B) = \begin{cases} 0 & \text{if } B \neq B' \\ k & \text{if } B = B' \end{cases}$$

where $0 < k \in \mathbb{Z}$. Now for ν from above, consider $\gamma = \beta + \sum_{i \in B'} k \cdot e_i$, then $\sup\{\sum_{i \in \mathcal{B}} \gamma_i - \nu(B)\} = 2k$ and $\sum_{i \in \mathcal{B}} \gamma_i - \nu(B') = 2k$. Then we have $B' \in \mathcal{B}(M_\gamma)$.

□

Similar to the non-Fano matroid, the non-Pappus matroid is also a rank-three matroid obtained from the Pappus configuration (Figure 2.1). In the *Pappus matroid* the points g, h and, i are collinear, therefore the set $\{g, h, i\}$ is a dependent set. So, $\{g, h, i\}$ is a basis of the non-Pappus matroid but a dependent set of the Pappus matroid. The non-Pappus matroid is not linearly representable over any field. But, the algebraic characteristic set is the set of primes (Example 2.14), and so is the flock characteristic set.

Proposition 4.4. *Let M be the non-Pappus matroid, M' the Pappus matroid and $\mathcal{B}(M) - \mathcal{B}(M') = \{B'\}$. Let M_α be a Frobenius flock representation of M over a field of characteristic p. There there exists β and γ in $\mathbb{Z}^9$ such that $M_\beta = M'$, M_γ is a matroid with $B' \in \mathcal{B}(M_\gamma)$ and $\gamma - \beta = 0 + \sum_{i \in B'} k \cdot e_i$ for some $0 < k \in \mathbb{Z}$.*

Proof. Consider the points $\{a, c, d, f\}$. Then it is isomorphic to $U_{3,4}$ there exist $\beta \in \mathbb{Z}^E$ such that M_β contains columns of $\{a, c, d, f\}$ as a circuit. Then we can find the coordinates for the 9 points of $\mathcal{V}_\beta$. This gives the matrix:

$$\begin{array}{ccccccccc} a & c & d & f & h & b & e & g & i \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & x & 1 & x & z \\ 0 & 0 & 1 & 1 & 1 & 0 & y & xy & y \end{pmatrix}$$

where x and y are indeterminate and $z = x + y - yx$. Then, there exists a $\beta \in \mathbb{Z}^9$ such that $M_\beta = M^{'}$. We can translate the flock to make $\beta = 0$ and let the associated valuation be ν. Then $\sup\{\sum_{i\in\mathcal{B}} \beta_i - \nu\,(B)\} = 0$ and bases with 0 valuations are exactly bases of the Pappus matroid.

$$\nu(B) = \begin{cases} 0 & \text{if } B \neq B' \\ k & \text{if } B = B' \end{cases}$$

where $0 < k \in \mathbb{Z}$. Now for ν from above, consider $\gamma = \beta + \sum_{i\in B'} k \cdot e_i$, then $\sup\{\sum_{i\in\mathcal{B}} \gamma_i - \nu\,(B)\} = 2k$ and $\sum_{i\in\mathcal{B}} \gamma_i - \nu\,(B') = 2k$. Then we have $B' \in \mathcal{B}(M_\gamma)$.

□

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