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The abc conjecture

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To the Graduate Council:

I am submitting herewith a thesis written by Jeffrey Paul Wheeler entitled "The abc conjecture." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Pavlos Tzermias, Major Professor

We have read this thesis and recommend its acceptance:

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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David F Anderson

S. B. Mulay

Acceptance for the Council:

Vice Provost and Dean of Graduate Studies

The abc Conjecture

A Thesis
Presented for the
Master of Science Degree
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Jeffrey Paul Wheeler
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Dedication

This thesis is dedicated to

My Parents

whose loving sacrifices

ensured me the best possible education.

"...for we were born only yesterday and know nothing, and our days on earth are but a shadow."

Job 8:9 (NIV)

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Abstract

The problem of the abc Conjecture is stated and various consequences are established. Other known consequences are stated without proof. Topics supporting belief that the abc Conjecture is true are discussed. The idea of good abc triples is defined and all known good triples are stated. Some computational computer work verifying these values is discussed. This is the first time that such brute force computations have been published.

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Chapter 1

Introduction

On its own the abc Conjecture merits much admiration. As is often the case with some of the more intriguing problems of Number Theory, the abc Conjecture is easy to state but yet difficult to verify. Unlike most other Number Theory problems, though, this conjecture has many fascinating applications; one of which is a version of one of the subject's most celebrated problems.

Pierre de Fermat (1601 – 1665) stated his "Last Theorem" [1] in the margin of his copy of Diophantus's *Arithmetica* in 1637. In one of the boldest claims by one of the brightest individuals in the history of mathematics, Fermat wrote that he had a proof, but that he did not have enough room to write it in the margin. It is very likely that his proof was incomplete. Nonetheless, his innocent enough statement incited hundreds of capable (and not so capable) individuals into feverish work for over three and one-half centuries. These individuals made great accomplishments in mathematics; the development of Modern Algebra being one of the foremost. This intriguing chapter in mathematics' history came to a close in 1993 with the work of Andrew Wiles. The significance of this is best summarized in a comment by John Fraleigh regarding Wiles' proof of Fermat's Last Theorem: *"One wonders, with the pace of science today, whether any mathematician could now make a mathematical conjecture whose status (true, false, or undecidable) could not be established, despite intense effort by the best mathematicians, for another* 350 *years".*

Though it may be the case that the abc Conjecture is one such conjecture, it is too early to tell. As we will see, much has been accomplished, yet the conjecture's certainty or uncertainty is not in sight. The interesting connection, though, is that the abc Conjecture implies a weaker (yet significant) form of Fermat's Last Theorem (see Conjecture 3.1).

The abc Conjecture was posed in 1985 by both J. Oesterlé and D.W. Masser. Oesterlé was motivated by a conjecture of Szpiro regarding elliptic curves. A little later, Masser was motivated by considering an analogous statement over $\mathbb{Z}$ of Mason's Theorem for polynomials. We will see both Szpiro's conjecture (Conjecture 3.4) and Mason's Theorem (Theorem 2.1).

[1] $x^n + y^n = z^n$ has no nontrivial solutions in $\mathbb{Z}$ for $n \geq 3$.

Chapter 2

The Problem Stated

2.1 The abc Conjecture

First we begin with a defintion:

Definition 2.1 (The radical of a positive integer).
*For $n \in \mathbb{P}$, suppose $n = {p_1}^{e_1} \cdots {p_k}^{e_k}$ where the p_i's are distinct prime numbers and the e_i's are positive integers. We then define the **radical** of n to be:*

$$r(n) = p_1 \cdots p_k \text{ with } r(1) = 1.$$

In other words, $r(n)$ is the greatest square-free factor of n.

Now we concern ourselves with the hypotheses. We will be considering non-trivial triples of integers (a, b, c) such that $a + b = c$ and $gcd(a, b, c) = 1$. Obviously, any sum of the form $a + b = c$ can be rearranged so that $a, b, c > 0$, hence we will assume all elements of our triples are positive.

Oesterlé originally stated the conjecture in the form

$$L = L(a, b, c) = \frac{\log \max(|a|, |b|, |c|)}{\log r(abc)} = \frac{\log c}{\log r(abc)}$$

and considered whether the L's are bounded. We will consider this more extensively in Chapter 5.1.

Masser refined the statement into its more common form, namely: for each $\varepsilon > 0$ there exists a positive universal constant $\mu(\varepsilon)$ [1] such that

$$\max(|a|, |b|, |c|) = c \leq \mu(\varepsilon) r(abc)^{1+\varepsilon}.$$

We now state two Lemmas that will be repeatedly quite useful.

Lemma 2.1.
Under the hypotheses of the abc Conjecture, r is a muliplicative function [2].

[1] The literature commonly refers to this constant as $C(\varepsilon)$.

[2] By definition of a multiplicative function, it is already the case that the elements involved have gcd = 1. The redundancy is for emphasis.

Proof.
Obvious. □

Lemma 2.2.
For all $n \in \mathbb{P}$, $r(n) \leq n$.

Proof.
Obvious □

It is worthwhile to emphasize the importance of the ε in Masser's version of the abc Conjecture. We will do this by using an example developed by Wojtek Jastrzebowski and Dan Spielman as reported by Serge Lang [8]. We show that there does not exist a μ such that $c \leq \mu \cdot r(abc)$ for all a, b, and c meeting the hypotheses.

For an example, consider $a_n = 3^{2^n} - 1, b_n = 1$, and $c_n = 3^{2^n}$ where $n \in \mathbb{P}$. Note that the values meet the conditions of the hypotheses of the abc Conjecture. First,

Claim 2.1.
$2^n | (3^{2^n} - 1)$

Proof.
For $n = 1$, $2 | (3^2 - 1)$.
Assume true for k, i.e. $2^k | (3^{2^k} - 1)$. So

$$
\begin{aligned}
3^{2^{k+1}} - 1 &= 3^{2^k \cdot 2} - 1 && (2.1)\\
&= (3^{2^k})^2 - 1 && (2.2)\\
&= (3^{2^k} - 1)(3^{2^k} + 1) \qquad \text{(difference of two squares)} && (2.3)
\end{aligned}
$$

Since

$$2^k | (3^{2^k} - 1) \qquad \text{(induction hypothesis)} \qquad (2.4)$$

and

$$2 | (3^{2^k} + 1) \qquad \text{(viz., } 3^{2^k} + 1 \text{ is even)} \qquad (2.5)$$

then

$$2^{k+1} | (3^{2^{k+1}} - 1). \qquad (2.6)$$

Hence, by induction, the claim is established. □

Proposition 2.1.
The ε in the abc Conjecture is essential.

Proof.
For contradiction, assume there exists μ such that $c_n \leq \mu \cdot r(a_n b_n c_n)$ for the above conditions.
So

$$\begin{aligned}
\max(|a_n|, |b_n|, |c_n|) &= 3^{2^n} && && (2.7)\\
&\leq \mu \cdot r(a_n b_n c_n) && \text{(by assumption)} && (2.8)\\
&= \mu \cdot r([3^{2^n} - 1] \cdot 1 \cdot 3^{2^n}) && && (2.9)\\
&= \mu \cdot 3 \cdot r(3^{2^n} - 1) && \text{(by Lemma 2.1)} && (2.10)\\
&= \mu \cdot 3 \cdot r\left(2^n \cdot \frac{3^{2^n} - 1}{2^n}\right) && \text{(by Claim 2.1)} && (2.11)\\
&\leq \mu \cdot 3 \cdot 2 \cdot \left(\frac{3^{2^n} - 1}{2^n}\right) && \text{(by Lemma 2.2)} && (2.12)
\end{aligned}$$

(One may regard the fraction in statement (2.12) as the product of all factors of $3^{2^n} - 1$ different from 2).

Multiplying both sides by 2^n and dividing by 3^{2^n}:

$$2^n \leq \mu \cdot 6 \cdot \frac{3^{2^n} - 1}{3^{2^n}}. \tag{2.13}$$

Letting $n \to \infty$ the inequality fails and we get the contradiction establishing the necessity of the ε.

□

Before we close this section, we state a remark that the reader may find useful while contemplating later material.

Remark 2.1.
In the abc Conjecture, $\mu(\varepsilon)$ varies inversely with the choice of ε.

2.2 The Polynomial Analogue of the abc Conjecture

Before we consider some of the consequences of the abc Conjecture, let us take a look at one of the conjecture's influences. Recall from the Introduction that Mason's Theorem inspired Masser. Hence we shall state Mason's Theorem.

First, a definition:

Definition 2.2 (The radical of a polynomial).
Let $p(t)$ be a polynomial whose coefficients belong to an algebraically closed field of characteristic 0. Put

$\mathbf{n_0(p)}$ = *the number of distinct zeros of* $p(t)$.

In other words, $n_0(p)$ *counts the zeros of* $p(t)$ *by giving them each a multiplicity of one.*

With this definition, we may now state:

Theorem 2.1 (Mason's Theorem). [3]
Let $a(t)$, $b(t)$, *and* $c(t)$ *be polynomials whose coefficients belong to an algebraically closed field of characteristic* 0. *Suppose* $a(t)$, $b(t)$, *and* $c(t)$ *are relatively prime and that* $a(t) + b(t) = c(t)$. *Then*

$$\max \deg\{a(t), b(t), c(t)\} \leq n_0(a(t) \cdot b(t) \cdot c(t)) - 1.$$

Proof.
We have

$$a + b = c \tag{2.14}$$

Dividing both sides by c:

$$\frac{a}{c} + \frac{b}{c} = 1 \tag{2.15}$$

Putting $f = \frac{a}{c}$ and $g = \frac{b}{c}$, we have:

$$f + g = 1 \tag{2.16}$$

Differentiating we get:

$$f' + g' = 0 \tag{2.17}$$

Rewrite as:

$$\frac{f'}{f} \cdot f + \frac{g'}{g} \cdot g = 0 \tag{2.18}$$

$\therefore$

$$\frac{g'}{g} \cdot g = -\frac{f'}{f} \cdot f \tag{2.19}$$

So

$$\frac{g}{f} = \frac{-\frac{f'}{f}}{\frac{g'}{g}} \tag{2.20}$$

Observe that $a = f \cdot c$ and $b = g \cdot c$, hence

$$\frac{b}{a} = \frac{g}{f} \tag{2.21}$$

---

[3]It is essential that the reader realizes the similarities between Mason's Theorem and Masser's version of the abc Conjecture.

Substituting (2.20) into (2.21)

$$\frac{b}{a} = \frac{-\frac{f'}{f}}{\frac{g'}{g}} \tag{2.22}$$

Now suppose $R(t)$ is a rational function with ρ_i representing the **distinct** roots of the numerator and denominator. Then

$$R(t) = \prod_i (t - \rho_i)^{q_i} \text{ where the } q_i \in \mathbb{Z} \tag{2.23}$$

$$\text{Notice: } q_i \text{ is the multiplicity of the root } \rho_i \text{ where } q_i \begin{cases} > 0 & \text{if } t - \rho_i \text{ is in the numerator} \\ < 0 & \text{if } t - \rho_i \text{ is in the denominator} \end{cases} \tag{2.24}$$

Thus

$$R'(t) = \sum_i q_i \cdot \frac{R(t)}{t - \rho_i} \tag{2.25}$$

Hence

$$\frac{R'(t)}{R(t)} = \sum_i \frac{q_i}{t - \rho_i} \tag{2.26}$$

The advantage of (2.26) is that the multiplicity of each distinct root is now exactly one. Now suppose

$$a(t) = \prod_i (t - \alpha_i)^{m_i}, \ b(t) = \prod_j (t - \beta_j)^{n_j}, \text{ and } c(t) = \prod_k (t - \gamma_k)^{r_k} \tag{2.27}$$

Then by (2.22) and (2.26),

$$\frac{b}{a} = -\frac{\frac{f'}{f}}{\frac{g'}{g}} = -\frac{\sum_i \frac{m_i}{t-\alpha_i} - \sum_k \frac{r_k}{t-\gamma_k}}{\sum_j \frac{n_j}{t-\beta_j} - \sum_k \frac{r_k}{t-\gamma_k}} \tag{2.28}$$

A common denominator for the numerator and denominator of (2.28) is (since a, b, and c are relatively prime)

$$D(t) := \prod_i (t - \alpha_i) \cdot \prod_j (t - \beta_j) \cdot \prod_k (t - \gamma_k) \tag{2.29}$$

where

$$\deg(D(t)) = n_0(abc) \tag{2.30}$$

Now we make the observation that

$$\deg\left(\frac{f'}{f}\right) = -\infty, \text{ if } b \equiv 0; \ \deg\left(\frac{g'}{g}\right) = -\infty, \text{ if } a \equiv 0; \tag{2.31}$$

and

$$\deg\left(\frac{f'}{f}\right) = \deg\left(\frac{g'}{g}\right) = -1 \text{ if } a \text{ nor } b \equiv 0. \tag{2.32}$$

Hence

$$\deg\left(D \cdot \frac{f'}{f}\right) \text{ and } \deg\left(D \cdot \frac{g'}{g}\right) \leq n_0(abc) - 1. \tag{2.33}$$

(Note that in (2.31), (2.32), and (2.33) we needed the hypothesis that the polynomials have coefficients in a field of characteristic 0. This will also be used in (2.37), (2.38), and (2.39).)

By (2.22) we get

$$\frac{b}{a} = \frac{-D \cdot \frac{f'}{f}}{D \cdot \frac{g'}{g}} \tag{2.34}$$

Hence

$$-a \cdot \left(D \cdot \frac{f'}{f}\right) = b \cdot \left(D \cdot \frac{g'}{g}\right) \tag{2.35}$$

Since (a,b) = 1

$$a \mid \left(D \cdot \frac{g'}{g}\right) \tag{2.36}$$

Thus by (2.33)

$$\deg(a) \leq n_0(abc) - 1 \tag{2.37}$$

A similar argument yields

$$\deg(b) \leq n_0(abc) - 1 \tag{2.38}$$

As well

$$\deg(c) \leq \max\{\deg(a), \deg(b)\} \tag{2.39}$$

So, by (2.37), (2.38), and (2.39):

$$\max\deg\{a(t), b(t), c(t)\} \leq n_0(a(t) \cdot b(t) \cdot c(t)) - 1. \tag{2.40}$$

$\square$

Having established Mason's Theorem we get

Corollary 2.1 (Fermat's theorem for polynomials).
Let $x(t)$, $y(t)$, and $z(t)$ be relatively prime polynomials whose coefficients belong to an algebraically closed field of characteristic 0 such that at least one of them has degree > 0. Then

$$x(t)^n + y(t)^n = z(t)^n$$

has no solution for $n \geq 3$.

Proof.
By Mason's Theorem we have

$$\deg(x(t)^n) = n \cdot \deg(x(t)) \leq \deg(x(t)) + \deg(y(t)) + \deg(z(t)) - 1.$$

By successively replacing the $x(t)$ on the LHS with $y(t)$ and $z(t)$ and summing we get

$$n[\deg(x(t) + \deg(y(t)) + \deg(z(t))] \leq 3[\deg(x(t)) + \deg(y(t)) + \deg(z(t))] - 3$$

This is an obvious contradiction for $n \geq 3$. □

Remark 2.2.
Fermat's theorem for polynomials fails if char $p > 0$.

For an example, let $f(x) = x + 1$, $g(x) = x$, and $h(x) = 1$ be polynomials whose coefficients are in a field of char $p > 0$. Then $f(x)^p = g(x)^p + h(x)^p$.

Chapter 3

Consequences of the abc Conjecture

3.1 Specific Consequences

This chapter states some of the conjecture's fascinating implications. We begin with one of the more interesting ones. As stated in the Introduction, the abc Conjecture implies a weaker form of Fermat's Last Theorem. This is due to the $\mu(\varepsilon)$.

Conjecture 3.1 (The Asymptotic Fermat Problem). [1]
Then there exists $N \in \mathbb{Z}$ such that for $n > N$,

$$x^n + y^n = z^n,$$

where $gcd(x, y, z) = 1$, has only trivial solutions in the integers.

Theorem 3.1.
The abc Conjecture implies the Asymptotic Fermat Problem.

Notation ($x(t) \ll y(t)$).
We will use the symbol $\ll$ to mean the following:
For functions $x(t)$ and $y(t)$

$$x(t) \ll y(t) \text{ means } \exists\, C \in \mathbb{R},\ C > 0 \text{ such that } x(t) \leq C \cdot y(t) \text{ for all } t.$$

Another way to state this is that in big oh notation $x(t) \ll y(t)$ means $x(t) = O(y(t))$.

Proof of Theorem 3.1.
Again, we may make the appropriate rearrangements in the sum so that all integers are positive.
By the abc Conjecture:

$$|\, x^n \,| \leq \mu\left(\frac{\varepsilon}{3}\right) \cdot r(xyz)^{1+\frac{\varepsilon}{3}} \ll |\, xyz \,|^{1+\frac{\varepsilon}{3}} \tag{3.1}$$

$$|\, y^n \,| \leq \mu\left(\frac{\varepsilon}{3}\right) \cdot r(xyz)^{1+\frac{\varepsilon}{3}} \ll |\, xyz \,|^{1+\frac{\varepsilon}{3}} \tag{3.2}$$

[1] In the wake of Andrew Wiles' accomplishment, this conjecture may be labeled a corollary or, more aptly, an academic corollary.

and

$$|z^n| \leq \mu\left(\frac{\varepsilon}{3}\right) \cdot r(xyz)^{1+\frac{\varepsilon}{3}} \ll |xyz|^{1+\frac{\varepsilon}{3}}. \tag{3.3}$$

Hence

$$|x^n| \cdot |y^n| \cdot |z^n| = |xyz|^n \ll (|xyz|^{1+\frac{\varepsilon}{3}})^3 = |xyz|^{3+\varepsilon}. \tag{3.4}$$

Thus for $|xyz| > 1$ we get n bounded. Otherwise, $|xyz| \leq 1$ and at least one of the integers must be 0. □

It is worthwhile to note the role of the $\mu(\varepsilon)$ in the previous proof. In particular, our choice of ε determines the value of the N.

The abc Conjecture also implies the following classical conjecture. Before we state the conjecture, we establish the necessary definition.

Definition 3.1 (Wieferich Condition).
*A prime $p \in \mathbb{Z}$ satisfies the **Wieferich Condition** iff $2^{p-1} \not\equiv 1 \bmod p^2$.*

Conjecture 3.2 (Infinity of Primes Satisfying the Wieferich Condition).
There exist infinitely many primes p satisfying the Wieferich Condition.

Theorem 3.2.
The abc Conjecture implies that an infinity of primes satisfy the Wieferich Condition.

It will be helpful to employ the set $S := \{p \mid p \text{ is prime and } 2^{p-1} \not\equiv 1 \bmod p^2\}$. But before we prove the theorem, we first establish the following claim:

Claim 3.1.
Let $n \in \mathbb{P}$ and p be a prime such that $2^n \equiv 1 \bmod p$ but $2^n \not\equiv 1 \bmod p^2$. Then $p \in S$.

Proof of Claim 3.1.
Put $d = ord(2)$ in $U(\mathbb{Z}/p\mathbb{Z})$ where $|U(\mathbb{Z}/p\mathbb{Z})| = p-1$ Hence $d \mid (p-1)$ and $d \mid n$.
Write $n = dr$ for some $r \in \mathbb{Z}$. So $2^n \not\equiv 1 \bmod p^2 \Rightarrow 2^d \not\equiv 1 \bmod p^2$.
Now write $p-1 = dm$ for some $m \in \mathbb{Z}$. $d \leq p-1$, $\therefore m < p-1 < p$. Also, p prime $\Rightarrow (m,p) = 1$.
Since $d = ord(2)$ in $U(\mathbb{Z}/p\mathbb{Z})$, $2^d \equiv 1 \bmod p$.
Hence $\exists\, k \in \mathbb{Z}$ such that $pk = 2^d - 1 \Leftrightarrow 2^d = 1 + pk$.
Since $2^d \neq 1 mod p^2$, it follows that $p \nmid k$.
So $2^{p-1} = (2^d)^m \equiv 1^m \bmod p = 1 \bmod p$.
But

$$\begin{aligned} 2^{p-1} = 2^{dm} = (2^d)^m = (1+pk)^m &= \binom{m}{0} 1^m (pk)^0 + \binom{m}{1} 1^{m-1}(pk)^1 + \underbrace{\sum_{i=2}^{m} \binom{m}{i} 1^{m-i} (pk)^i}_{\text{divisible by } p^2} \\ &\equiv 1 + mpk (mod p^2) \\ &\not\equiv 1 (mod p^2) \qquad (\textit{since } p \nmid k \textit{ and } p \nmid m) \end{aligned}$$

$\therefore p \in S$ and the claim is established. □

The following proof is due to Silverman.

Proof of Theorem 3.2.
Suppose S is finite. Write $2^n - 1 = u_n v_n$ where $\forall\, p_i \mid u_n$, $p_i \in S$ and each $p_k \mid v_n$, $p_k \notin S$. S finite $\Rightarrow u_n$ is bounded. Suppose $p \mid v_n$. By the claim, $2^n \equiv 1 \bmod p^2$, i.e. $p^2 \mid (2^n - 1)$. $\therefore\ p^2 \mid u_n v_n$ (since $2^n - 1 = u_n v_n$). But $p \nmid u_n$, $\therefore\ p^2 \mid v_n$.
Since $(2^n - 1) + 1 = 2^n$, by the abc Conjecture

$$\begin{aligned}
\mid 2^n - 1 \mid = u_n v_n &\leq \mu(\varepsilon) \cdot r(u_n v_n)^{1+\varepsilon} && (3.5)\\
&\leq \mu(\varepsilon) \cdot (u_n {v_n}^{\frac{1}{2}})^{1+\varepsilon} && (3.6)\\
&\ll (u_n {v_n}^{\frac{1}{2}})^{1+\varepsilon} && (3.7)\\
&= {u_n}^{1+\varepsilon} \cdot {v_n}^{\frac{1+\varepsilon}{2}} && (3.8)\\
&\ll {v_n}^{\frac{1+\varepsilon}{2}}. && (3.9)
\end{aligned}$$

Therefore

$$u_n v_n \ll {v_n}^{\frac{1+\varepsilon}{2}}. \tag{3.10}$$

Multiplying both sides by ${v_n}^{\frac{-1-\varepsilon}{2}} \cdot {u_n}^{-1}$

$${v_n}^{\frac{1-\varepsilon}{2}} \ll {u_n}^{-1}. \tag{3.11}$$

Hence

$$\begin{aligned}
v_n &\ll {u_n}^{\frac{2}{\varepsilon-1}} && (3.12)\\
&\Rightarrow \text{ a finite number of } v_n && (3.13)\\
&\text{contradiction as } n \to \infty. && (3.14)
\end{aligned}$$

□

Regarding the Wieferich Condition, there are only two known exceptions. Moreover, by the Lang-Trotter conjectures, the probibilty that $2^{p-1} \equiv 1 + pk (mod p^2)$ for a fixed residue class k mod p should be $O(\frac{1}{p})$. Hence, for fixed x, the number of primes $p \leq x$ such that $2^{p-1} \equiv 1 + pk(mod p^2)$ should be $O(\sum_{p<x} \frac{1}{p}) = O(\log\log x)$; i.e. most primes should satisfy the Wieferich Condition.

Conjecture 3.3 (Hall's Original Conjecture).
Let u, v be relatively prime[2] nonzero integers such that $u^3 - v^2 \neq 0$. Then

$$\mid u^3 - v^2 \mid \gg \mid u \mid^{\frac{1}{2}-\varepsilon}.$$

Theorem 3.3.
The abc Conjecture implies Hall's Original Conjecture.

The following proof is due to Lang.

[2] Originally the assumption that u and v be relatively prime was not made. This is remedied by removing any common factor and then proceeding as dictated in the proof.

Proof.
Note that we could equivalently state that $v^2 = u^3 + t,\ t \in \mathbb{Z}$, has a bound for t.[3] In particular, the abc Conjecture would imply that $|u| \ll |t|^{2+\varepsilon}$. We prove a more general statement:

Fix nonzero $a, b \in \mathbb{Z}$ and let $m, n \in \mathbb{P}$ be such that $mn > m + n$. Put

$$a \cdot u^m + b \cdot v^n = k. \tag{3.15}$$

Fix $\varepsilon' > 0$. By the abc Conjecture

$$|u|^m \ll |uv \cdot r(k)|^{1+\varepsilon'}. \tag{3.16}$$

A similar argument yields

$$|v|^n \ll |uv \cdot r(k)|^{1+\varepsilon'}. \tag{3.17}$$

Without loss of generality, now suppose

$$|a \cdot u^m| \leq |b \cdot v^n|. \tag{3.18}$$

Then

$$|u| \ll |v|^{\frac{n}{m}}. \tag{3.19}$$

Subsituting into (3.17)

$$|v|^n \ll |v^{1+\frac{n}{m}} \cdot r(k)|^{1+\varepsilon'} = |v|^{(1+\frac{n}{m})(1+\varepsilon')} \cdot r(k)^{1+\varepsilon'}. \tag{3.20}$$

Hence

$$|v|^{n-(\frac{m+n}{m})(1+\varepsilon')} \ll r(k)^{1+\varepsilon'}. \tag{3.21}$$

Thus

$$|v| \ll r(k)^{\frac{m(1+\varepsilon')}{mn-(m+n)(1+\varepsilon')}} \ll k^{\frac{m(1+\varepsilon')}{mn-(m+n)(1+\varepsilon')}}. \tag{3.22}$$
$$\tag{3.23}$$

(Note that we needed Lemma 2.2, namely: $r(k) \leq k$.)

By (3.19),

$$|u| \ll r(k)^{\frac{m(1+\varepsilon')}{mn-(m+n)(1+\varepsilon')} \cdot \frac{n}{m}}. \tag{3.24}$$
$$= r(k)^{\frac{n(1+\varepsilon')}{mn-(m+n)(1+\varepsilon')}}. \tag{3.25}$$
$$\ll k^{\frac{m(1+\varepsilon')}{mn-(m+n)(1+\varepsilon')}}. \tag{3.26}$$

[3] Note that the abc Conjecture, if true, improves Baker's bound for this situation.

Having established the general case, we may establish the implication of Hall's Conjecture. Pick ε such that $\varepsilon = \frac{12\varepsilon'}{1-5\varepsilon'}$, i.e. $\varepsilon' = \frac{\varepsilon}{12+5\varepsilon}$. Put $m = 3$ and $n = 2$.
By (3.26),

$$|u| \ll k^{\frac{2+2\varepsilon'}{1-5\varepsilon'}} = k^{2+\frac{12\varepsilon'}{1-5\varepsilon'}}. \tag{3.27}$$

Thus

$$|u|^{\frac{1}{2}-\frac{12\varepsilon'}{1-5\varepsilon'}} \ll k^{(2+\frac{12\varepsilon'}{1-5\varepsilon'})(\frac{1}{2}-\frac{12\varepsilon'}{1-5\varepsilon'})} = k^{[1-\frac{3}{2}\cdot\frac{12\varepsilon'}{1-5\varepsilon'}-(\frac{12\varepsilon'}{1-5\varepsilon'})^2]}. \tag{3.28}$$

Substituting for ε'

$$|u|^{\frac{1}{2}-\varepsilon} \ll k^{(1-\frac{3}{2}\cdot\varepsilon-\varepsilon^2)} < k. \tag{3.29}$$

$\square$

Recall from the Introduction that Oesterlè was inspired by a conjecture of Szpiro. Hence we shall consider Szpiro's Conjecture. First, some preliminaries.

Since we are considering fields of characteristic 0 we may assume that our elliptic curves have Weierstrass equations of the form

$$E : y^2 = x^3 - ux + v \tag{3.30}$$

where $u, v \in \mathbb{Z}$. Given this we identify the *disciminant of E*, namely

$$\Delta = 16(4u^3 - 27v^2)$$

and $D := 4u^3 - 27v^2$ is the discrimanant of the cubic polynomial. Also, we will want to indentify the *conductor of E*, namely for prime $p \in \mathbb{Z}$

$$c(E) := \prod_p p^{f_p}$$

where

$$f_p = \begin{cases} 0 & \text{if the reduction of E is non-singular} \\ 1 & \text{if the reduction of E is multiplicative} \\ 2+\delta_p & \text{if the reduction of E is additive} \end{cases}$$

and δ_p is a bounded constant independent of the curve with $\delta_p = 0$ if $p \geq 5$.

Before we continue, it is important to observe that

$$r(D) \leq c(E). \tag{3.31}$$

Conjecture 3.4 (Original Szpiro Conjecture).
Assuming a Weierstrass equation with D the discriminant of the cubic polynomial and c(E) the conductor of the equation, then

$$|D| \ll r(D)^{6+\varepsilon} \ll c(E)^{6+\varepsilon}.$$

(Note that Szpiro did not include the notion of r(D) in his Conjecture.)

Theorem 3.4.
The abc Conjecture implies the Original Szpiro Conjecture.

Proof.
Fix $\varepsilon > 0$ and put $\varepsilon'' = \frac{1}{3}\varepsilon$. We have

$$4u^3 - 27v^2 = D.$$

By the abc Conjecture (in particular, our proof of Hall's Conjecture)

$$|u|^3 \ll [(r(D))^{2+\varepsilon''}]^3 \qquad \text{by (3.27) and noting that } r(D) < D \tag{3.32}$$

and

$$|v|^2 \ll [(r(D)^{3+\varepsilon''}]^2 \qquad \text{by (3.22)} \tag{3.33}$$

Hence

$$|D| \ll r(D)^{6+\varepsilon} \ll c(E)^{6+\varepsilon} \qquad \text{by (3.31)} \tag{3.34}$$

□

Remark 3.1.
The abc Conjecture is equivalent to Szpiro's Original Conjecture.

For the proof of the opposite implication, see [8].

3.2 Futher Consequences

In this section we list further consequences of the abc Conjecture without proof. For further information, see [9].

Definition 3.2 (Brown Pairs).
Pairs of integers satisfying Brocard's Problem $n! + 1 = m^2$ are called **Brown Pairs.**

Theorem 3.5.
The abc Conjecture implies that there exist only finitely many Brown Pairs.

The proof of this is in [11].

For the interested reader, the above problem has been generalized to the number of integer solutions of the equations $(x!)^n + 1 = y^m$ (see [10]) and $x! + B^2 = y^2$ for arbitrary B (see [3]).

Definition 3.3 (Powerful Numbers).
For $n \in \mathbb{P}$, n is said to be a ***powerful number*** *if for every prime p dividing n, p^2 divides n.*

Erdös refers to theses numbers as **k-ful numbers** where the k plays the role of the 2 in the above definition.

Conjecture 3.5 (*Erdös – Mollin – Walsh Conjecture*).
There do no exist three consecutive powerful integers.

Theorem 3.6.
The abc Conjecture implies that the set of triples of consecutive powerful integers is finite.

Conjecture 3.6 (Mordell's Conjecture). [4]
Any curve of genus larger than 1 defined over a number field K has only finitely many rational points in K.

The following is due to Elkies [4].

Theorem 3.7.
The abc Conjecture for number fields implies the Mordell Conjecture over an arbitrary number field.

In [5] it is established that the abc Conjecture with an explict constant $\mu(\varepsilon)$ would give explicit bounds on the heights of rational points in Mordell's Conjecture.

Theorem 3.8 (Roth's Theorem).
Fix $\varepsilon > 0$. For every algebraic number α, the diophantine inequality

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}}$$

has only finitely many solutions.

In 1994, E. Bombieri [1] proved that the abc Conjecture implies a stronger version of Roth's Theorem:

Theorem 3.9.
The abc Conjectue implies that, for the conditions of Roth's Theorem,

$$|\alpha - \frac{p}{q}| > \frac{1}{q^{2+k}}$$

for all but a finite number of fractions $\frac{p}{q}$ in reduced form, where $k = C(\alpha) \cdot (\log q)^{-\frac{1}{2}} \cdot (\log\log q)^{-1}$ for some constant $C(\alpha)$ depending only upon α.

For the above, the reader may also see [5].

The following is due to Granville [6]:

Theorem 3.10 (Squarefree Values of Polynomials).
The abc Conjecture implies that for a polynomial $F(x)$ with integer coefficients, no repeated roots, and content = 1, $F(n)$ is squarefree for infinitely many integers n.

In closing, we mention that the abc Conjecture also gives a way of counting squarefree values of polynomials, implies that the Dirichlet L-function has no Siegel zeros, and gives bounds for the order of the Tate-Shafarevich group. Many more implications are given in [9].

[4] This is now a theorem after the work of G. Faltings (1984).

Chapter 4

Evidence for the abc Conjecture

In this chapter, a theorem of C.L. Stewart and Kunrui Yu establishing a weak form of the abc Conjecture is discussed.

4.1 Preliminaries

For the following, let p be a prime number and put

$$q = \begin{cases} 2 & \text{if } p > 2 \\ 3 & \text{if } p = 2. \end{cases} \tag{4.1}$$

As well put

$$\alpha_0 = \begin{cases} \zeta_4 & \text{if } p > 2 \\ \zeta_6 & \text{if } p = 2, \end{cases} \tag{4.2}$$

where ζ_m has the usual meaning $e^{\frac{2\pi i}{m}}$ for $m \in \mathbb{P}$. Put $K = \mathbb{Q}(\alpha_0)$ and let $D = \Omega \cap K$, i.e. D is the ring of algebraic integers in K (*Note:* since K is a cyclotomic field, $D \equiv \mathbb{Z}[\zeta_0]$). For $c = x + iy \in \mathbb{C}$, $|c| = \sqrt{x^2 + y^2}$. Let $\alpha_1, \ldots, \alpha_n \in D$ such that $|\alpha_i| \leq A_i$ for $1 \leq i \leq n$ where each $A_i \geq 4$. Put

$$A = \max_{1 \leq i \leq n} A_i.$$

Let $b_1, \ldots, b_n$ be rational integers (i.e. in $\mathbb{Z}$) such that $|b_i| \leq B$ where B is a fixed integer ≥ 3. For $\alpha \in K \backslash \{0\}$, since D is a Dedekind domain the fractional ideal $(\alpha)D$ can be written as a unique product of prime ideals in D, i.e. $(\alpha)D = \wp_1^{e_{\wp_1}} \cdot \cdots \cdot \wp_g^{e_{\wp_g}}$. Define $\mathrm{ord}_{\wp_i} \alpha = e_{\wp_i}$. This is the ramification index of $\wp_i$. Let $f_\wp$ be the residue class degree of $\wp$. Lastly, put $\Theta = {\alpha_1}^{b_1} \cdots {\alpha_n}^{b_n} - 1$.

Given the above, we now state some essential preliminaries. These are stated without proof; the curious reader may see [12].

Lemma 4.1.
If $[K({\alpha_0}^{1/q}, \ldots, {\alpha_n}^{1/q}) : K] = q^{n+1}, ord_\wp \alpha_j = 0$ *for* $j = 1, \ldots, n$, *and* $\Theta \neq 0$, *then*

$$ord_\wp \theta < (c_1 n)^n p^2 \cdot \log B \cdot \log\log A \cdot \log A_1 \cdot \; \cdots \; \cdot \log A_n$$

where c_1 *is an effectively computable postive number.*

Lemma 4.2.
For $\alpha_1, \ldots, \alpha_n \in \mathbb{P}$, if $[\mathbb{Q}(\alpha_1{}^{1/2}, \ldots, \alpha_n{}^{1/2}) : \mathbb{Q}] = 2^n$ and $b_1 \cdot \log \alpha_1 + \cdots + b_n \cdot \log \alpha_n \neq 0$, then

$$|b_1 \cdot \log \alpha_1 + \cdots + b_n \cdot \log \alpha_n| > exp(-c_2 n)^n \log B (\log\log A)^2 \log A_1 \cdots \log A_n,$$

where c_2 is an effectively computable positive number.

Lemma 4.3.
Let α_1, α_2, ..., α_n be prime numbers with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Then

$$[\mathbb{Q}(\alpha_1{}^{\frac{1}{2}}, \alpha_2{}^{\frac{1}{2}}, \cdots, \alpha_n{}^{\frac{1}{2}}) : \mathbb{Q}] = 2^n.$$

Let $q = 2$ and $\alpha_0 = \zeta_4$ or $q = 3$ and $\alpha_0 = \zeta_6$ as well put $K = \mathbb{Q}(\alpha_0)$. Then

$$[K(\alpha_0{}^{\frac{1}{q}}, \alpha_1{}^{\frac{1}{q}}, \ldots, \alpha_n{}^{\frac{1}{q}}) : K] = q^{n+1}$$

except when $q = 2$, $\alpha_0 = \zeta_4$, and $\alpha_1 = 2$ and in this case

$$[K(\alpha_0{}^{\frac{1}{2}}, (1+i)^{\frac{1}{2}}, \alpha_2{}^{\frac{1}{2}}, \ldots, \alpha_n{}^{\frac{1}{2}}) : K] = 2^{n+1}.$$

Lemma 4.4.
Let $p_1 = 2$, p_2, be the sequence of prime numbers in increasing order. Then $\exists$ an effectively computable constant $c_3 > 0$ such that for every positive integer r we have

$$\prod_{j=1}^{r} \frac{p_j}{\log p_j} > \left(\frac{r+3}{c_3}\right)^{r+3}.$$

4.2 The Evidence

Theorem 4.1.
There exists an effectively computable constant k such that for all a, b, and $c \in \mathbb{P}$ with $(a, b, c) = 1$, $c > 2$, and $a + b = c$

$$\log c < r(abc)^{\frac{2}{3} + \frac{k}{\log\log r(abc)}}$$

The following proof is due to Stewart and Yu.

Proof.
Let $c_4, c_5, \ldots$ denote effectively computable positive constants. Without loss of generality suppose $a \leq b$. Since $a + b = c$, $gcd(a, b, c) = 1$, and $c \geq 2$, it follows that $a < b < c$ and that $r(abc) \geq 6$. Write

$$a = p_1{}^{e_1} \cdots p_t{}^{e_t},\ b = q_1{}^{f_1} \cdots q_u{}^{f_u},\ and\ c = s_1{}^{g_1} \cdots s_v{}^{g_v},$$

where $p_1, \ldots, p_t, q_1, \ldots, q_u, s_1, \ldots, s_v$ are distinct primes with $t \geq 0, u \geq 1, v \geq 1$, and $e, f, g \in \mathbb{P}$. Denote the largest prime dividing a by p_a except when $a = 1$ and in this situation simply put $p_a = 1$. Similarly denote the largest primes dividing b and c by p_b and p_c respectively. Then for any prime p

$$\max\{\text{ord}_p a,\ \text{ord}_p b,\ \text{ord}_p c\} \leq \frac{\log c}{\log 2}. \tag{4.3}$$

Observe that

$$\log c = \sum\nolimits_{p|c} (\mathrm{ord}_p c \cdot \log p) \leq (\max_{p|c}\{\mathrm{ord}_p c\}) \cdot \log r(abc). \tag{4.4}$$

Since $(a,b) = (a,c) = (b,c) = 1$, for each prime p dividing c,

$$\mathrm{ord}_p c = \mathrm{ord}_p \left(\frac{c}{-b}\right) = \mathrm{ord}_p \left(\frac{a}{-b} - 1\right) \leq \mathrm{ord}_p(\left(\frac{a}{b}\right)^4 - 1). \tag{4.5}$$

We now estimate

$$\mathrm{ord}_p(\left(\frac{a}{b}\right)^4 - 1) = \mathrm{ord}_p({p_1}^{4e_1} \cdots {p_t}^{4e_t} \cdot {q_1}^{-4f_1} \cdots {q_u}^{-4f_u} - 1)$$

for each prime p dividing c. We do this by employing Lemma 4.1.

Put $\Theta = (\frac{a}{b})^4 - 1$. If $p = 2$, we put $K = \mathbb{Q}(\zeta_6)$, while if $p > 2$ we put $K = \mathbb{Q}(\zeta_4)$. Define q and α_0 as in statements (4.1) and (4.2). Now let $\wp$ be a prime ideal of the ring of algebraic integers of K lying above the prime p. Thus we have

$$\mathrm{ord}_p \Theta \leq \mathrm{ord}_\wp \Theta.$$

For n in Lemma 4.1, put $n = t+u$. As well let $\alpha_1, \ldots, \alpha_n$ be the primes $p_1, \ldots, p_t, q_1, \ldots, q_u$ arranged in increasing order, except in the case when $p > 2$ and $\alpha_1 = 2$. In this situation, take $\alpha_1 = 1+i$ instead of $\alpha_1 = 2$ and note that $2^4 = (1+i)^8$. Since $p|c$ and $(a,c) = (b,c) = 1$ we have $\mathrm{ord}_p \alpha_i = 0$ for $i = 1, \ldots, t+u$. Clearly $\Theta \neq 0$. Thus, by Lemma 4.3,

$$[K({\alpha_0}^{\frac{1}{q}}, {\alpha_1}^{\frac{1}{q}}, \ldots, {\alpha_{t+u}}^{\frac{1}{q}}) : K] = q^{t+u+1}.$$

Now put

$$B = \max\{8e_1, \ldots, 8e_t, 8f_1, \ldots, 8f_u\}.$$

So, by (4.3),

$$B \leq 8 \cdot \frac{\log c}{\log 2}.$$

Hence by Lemma 4.1

$$\mathrm{ord}_p c \leq \mathrm{ord}_\wp \Theta < (c_4 \cdot (t+u))^{t+u} \cdot p^2 \cdot \log\log c \cdot \log\log r(abc) \cdot \prod\nolimits_{p|ab} \log p. \tag{4.6}$$

Similarly if $p|b$ then, by considering $\mathrm{ord}_p((\frac{c}{a})^4 - 1)$, we have

$$\mathrm{ord}_p b < (c_5 \cdot (t+v))^{t+v} \cdot p^2 \cdot \log\log c \cdot \log\log r(abc) \cdot \prod\nolimits_{p|ac} \log p \tag{4.7}$$

and if $p|a$ then, by considering $\mathrm{ord}_p((\frac{c}{b})^4 - 1)$, we also have

$$\mathrm{ord}_p a < (c_6 \cdot (u+v))^{u+v} \cdot p^2 \cdot \log\log c \cdot \log\log r(abc) \cdot \prod\nolimits_{p|bc} \log p. \tag{4.8}$$

It follows from (4.4) and (4.6) that

$$\frac{\log z}{\log\log c} < (c_5 \cdot (t+u))^{t+u} \cdot {p_c}^2 \cdot \prod\nolimits_{p|ab} \log p \cdot (\log r(abc))^2. \tag{4.9}$$

Since $b > \frac{c}{2}$ and $c \geq 3$,

$$\log b > \log c - \log 2 > \frac{\log c}{4}. \tag{4.10}$$

But (4.4) holds if we replace c by b. So from (4.7)

$$\frac{\log c}{4 \log\log z} < (c_5 \cdot (t+v))^{t+v} \cdot {p_b}^{x} \cdot \prod\nolimits_{p|ac} \log p \cdot (\log r(abc))^2. \tag{4.11}$$

Now either $a > \sqrt{b}$ or $a \leq \sqrt{b}$. Hence

$$\begin{cases} \text{for } a > \sqrt{b}, & \log a \geq \frac{1}{2} \cdot \log b > \frac{\log c}{8} \\ \text{or } a \leq \sqrt{b}, & \log(\frac{a+b}{b}) = \log(1+\frac{a}{b}) < \log\left(1+\frac{1}{\sqrt{b}}\right) < \frac{1}{\sqrt{b}} < \frac{\sqrt{2}}{\sqrt{c}}. \end{cases} \tag{4.12}$$

In the former case, we use (4.4) with c replaced by a together with (4.8) to conclude that

$$\frac{\log c}{8 \log\log c} < (c_6 \cdot (u+v))^{u+v} \cdot {p_a}^2 \cdot \prod\nolimits_{p|bc} \log p \cdot (\log r(abc))^2. \tag{4.13}$$

In the latter case,

$$0 < \log\frac{c}{b} = \log\left(\frac{a+b}{b}\right) = g_1 \cdot \log s_1 + \ldots g_v \cdot \log s_v - f_1 \cdot \log q_1 - \cdots - f_u \cdot \log q_u. \tag{4.14}$$

By Lemma 4.3 we may use Lemma 4.2 to obtain a lower bound for $\log\frac{c}{b}$. Comparing this with the upper bound given by (4.12) we again obtain (4.13) with c_6 replaced by c_7. Put $\rho = u+t+v$. From (4.9), (4.11), (4.13), we deduce that

$$\left(\frac{\log c}{4 \log\log c}\right)^3 < (c_8 \cdot \rho)^{2\rho} \cdot (p_a p_b p_c)^2 \cdot \left(\prod\nolimits_{p|abc} \log p\right)^2 \cdot (r(abc))^6. \tag{4.15}$$

By Lemma 4.4,

$$\left(\frac{\rho}{c_9}\right)^\rho < \prod_{i=1}^{\rho-3} \frac{p_i}{\log p_i} < 2 \cdot \prod_{\substack{p|abc \\ p \neq p_a, p_b, p_c}} \frac{p}{\log p}, \tag{4.16}$$

with the usual convention that the empty product is 1.

Thus, by(4.15),

$$\left(\frac{\log c}{4 \log\log c}\right)^3 < {c_{10}}^\rho \cdot (r(abc))^2 \cdot (\log r(abc))^{12}. \tag{4.17}$$

Again by Lemma 4.4 we have

$${c_{10}}^\rho < (r(abc))^{\frac{c_{11}}{\log\log r(abc)}}, \tag{4.18}$$

and the result now follows from (4.17). □

Recently the authors improved this estimate. In [13], the better estimate

$$c < exp(c_{11} \cdot (r(abc))^{\frac{1}{3}} \cdot (\log r(abc))^3$$

where c_{11} is an effectively computatble positive constant is established. The method employed to improve the estimate is a $p-$adic linear independence measure for logarithms of algebraic numbers. This result, due to Yu, is an ultrametric analog of an Archimedean measure due to E.M. Matveev.

A second estimate is also established. In particular, if a, b, c are relatively prime positive integers such that $a + b = c$ and $c > 2$, then

$$c < exp(p' \cdot (r(abc))^{c_{12} \cdot \frac{\log\log\log r_*(abc)}{\log\log r(abc)}},$$

where c_{12} is an effectively computable constant, $r_*(abc) = \max\{r(abc), 16\}$, and $p' = \min\{p_a, p_b, p_c\}$.

Chapter 5

Good Triples Associated with the abc Conjecture

In this chapter, we consider the notion of good triples. We will also state the known good triples.

5.1 Preliminaries

Recall Oesterlé's version of the abc Conjecture, namely, under the appropriate hypotheses, he considered

$$L = L(a,b,c) = \frac{\log \max(|a|,|b|,|c|)}{\log r(abc)} = \frac{\log c}{\log r(abc)}$$

and asked if these L's have a bound. It is this form of the abc Conjecture that we will be using for the topics of this chapter.

Theorem 5.1.
The abc Conjecture holds iff $\limsup\{L\} \leq 1$.

Proof.
$(\Rightarrow)$
Assume the abc Conjecture. So

$$\begin{aligned} L(abc) &= \frac{\log \max\{|a|,|b|,|c|\}}{\log r(abc)} && (5.1)\\ &\leq \frac{\log[\mu(\varepsilon)\cdot r(abc)^{1+\varepsilon}]}{\log r(abc)} \text{ by the abc Conjecture} && (5.2)\\ &= \frac{\log \mu(\varepsilon)}{\log r(abc)} + 1 + \varepsilon. && (5.3)\end{aligned}$$

Fix $\varepsilon > 0$. Put $k = \mu(\varepsilon)$.

$$\text{We want } \frac{\log k}{\log r(abc)} \leq \varepsilon \text{ for all but finitely many triples } (a,b,c) \tag{5.4}$$

$$\Leftrightarrow \log r(abc) \geq \frac{\log k}{\varepsilon} \tag{5.5}$$

$$\Leftrightarrow r(abc) \geq M \ := \ e^{\frac{\log k}{\varepsilon}}. \tag{5.6}$$

This holds since, by the hypotheses of the abc Conjecture, there exist only finitely many (a,b,c)'s such that $r(a,b,c) \leq M$.

($\Leftarrow$)
Suppose $\limsup\{L\} \leq 1$. This is true iff

$$\limsup\left(\frac{\log c_n}{\log r(a_n b_n c_n)}\right) \leq 1 \tag{5.7}$$

$$\Rightarrow \frac{\log c_n}{\log r(a_n b_n c_n)} \leq 1+\varepsilon \text{ for } n \text{ large.} \tag{5.8}$$

Then for $n > N$ for some $N \in \mathbb{Z}$:

$$c_n \leq r(a_n b_n c_n)^{1+\varepsilon}. \tag{5.9}$$

Choose constants $\mu_1(\varepsilon)$, $\mu_2(\varepsilon)$, ..., $\mu_N(\varepsilon)$ such that

$$c_i \leq \mu_i(\varepsilon) \cdot r(abc)^{1+\varepsilon} \text{ for all } i \tag{5.10}$$

Let

$$\mu(\varepsilon) = \max_{1 \leq i \leq N}\{\mu_i(\varepsilon)\} \tag{5.11}$$

Thus

$$c_n \leq \mu(\varepsilon) \cdot r(a_n b_n c_n)^{1+\varepsilon} \text{ for all } n. \tag{5.12}$$

$\square$

Recall from Proposition 2.1 our choices for a_n, b_n, and c_n, namely:

$$a_n = 3^{2^n} - 1,\ b_n = 1, \text{ and } c_n = 3^{2^n}.$$

So, for these values

$$L_n = \frac{\log 3^{2^n}}{\log r(3^{2^n} - 1 \cdot 1 \cdot 3^{2^n})} \tag{5.13}$$

$$= \frac{\log 3^{2^n}}{\log 3 + \log r(3^{2^n} - 1)} \tag{5.14}$$

$$\geq \frac{\log 3^{2^n}}{\log 3 + \log 2 \cdot r(\frac{3^{2^n}-1}{2^n})} \tag{5.15}$$

$$\geq \frac{2^n \log 3}{\log 3 + \log 2 + \log(3^{2^n} - 1) - \log 2^n} \qquad \text{since } r(3^{2^n}) \leq 2 \cdot \left(\frac{3^{2^n}-1}{2^n}\right). \tag{5.16}$$

So

$$L_n \geq \frac{2^n \log 3}{\log 3 + \log(3^{2^n} - 1) - (n-1)\cdot \log 2}. \tag{5.17}$$

Thus for $n = 3$:

$$L_3 \geq \frac{8 \cdot \log 3}{\log 3 + \log(3^8 - 1) - 2\cdot \log 2} \tag{5.18}$$

$$\approx 1.255203.... \tag{5.19}$$

In particular, $L_3 > 1$.

It is easy to see that the fraction on the RHS of inequality (5.17) increases as n gets large. Hence there are infinitely many triples (a_n, b_n, c_n) such that $L_n > 1$.
We have just shown

Theorem 5.2.
The abc Conjecture holds iff $\limsup\{L_n\} = 1$.

5.2 Good Triples

Definition 5.1 (Good Triple).
For the abc Conjecture, we say that a triple (a, b, c) *is a* ***good triple*** *if* $L > 1.4$.

So, by Theorem 5.2 we get

Corollary 5.1.
If the abc Conjecture holds, there are only finitely many good triples.

The following is the list (Table 5.1) of known good abc triples as of January 2, 2002:

Table 5.1: Known Good abc Triples

No.	L	a	b	c	Discoverer(s)
1.	1.622912	2	$3^{10}\cdot 109$	23^5	E.R.
2.	1.625991	11^2	$3^2\cdot 5^6\cdot 7^3$	$2^{21}\cdot 23$	B.W.
3.	1.623490	$19\cdot 1307$	$7\cdot 29^2\cdot 31^8$	$2^8\cdot 3^{22}\cdot 5^4$	Je.B. & Ju.B.
4.	1.580756	283	$5^{11}\cdot 13^2$	$2^8\cdot 3^8\cdot 17^3$	Je.B. & Ju.B., A.N.
5.	1.567887	1	$2\cdot 3^7$	$5^4\cdot 7$	B.W.
6.	1.547075	7^3	3^{10}	$2^{11}\cdot 29$	B.W.
7.	1.544434	$7^2\cdot 41^2\cdot 311^3$	$11^{16}\cdot 13^2\cdot 79$	$2\cdot 3^3\cdot 5^{23}\cdot 953$	A.N.
8.	1.536714	5^3	$2^9\cdot 3^{17}\cdot 13^2$	$11^5\cdot 17\cdot 31^3\cdot 137$	H.R. & P.M.
9.	1.522699	$13\cdot 19^6$	$2^{30}\cdot 5$	$3^{13}\cdot 11^2\cdot 31$	A.N.
10.	1.522160	$3^{18}\cdot 23\cdot 2269$	$17^3\cdot 29\cdot 31^8$	$2^{10}\cdot 5^2\cdot 7^{15}$	A.N.
11.	1.502839	239	$5^8\cdot 17^3$	$2^{10}\cdot 37^4$	Je.B. & Ju.b., A.N.
12.	1.497621	$5^2\cdot 7937$	7^{13}	$2^{18}\cdot 3^7\cdot 13^2$	B.W.
13.	1.492432	$2^2\cdot 11$	$3^2\cdot 13^{10}\cdot 17\cdot 151\cdot 4423$	$5^9\cdot 139^6$	A.N.
14.	1.491590	73	$2^{13}\cdot 7^7\cdot 941^2$	$3^{16}\cdot 103^2\cdot 127$	A.N.
15.	1.489245	2^{24}	$11^7\cdot 19\cdot 29^2$	$3^{11}\cdot 5^3\cdot 7^3\cdot 41$	A.N.
16.	1.488865	11^2	$3^9\cdot 13$	$2^{11}\cdot 5^3$	B.W.
17.	1.482910	37	2^{15}	$3^8\cdot 5$	B.W.
18.	1.481322	$5^{14}\cdot 19$	$2^5\cdot 3\cdot 7^{13}$	$11^7\cdot 37^2\cdot 353$	A.N.
19.	1.474450	1	$3^{18}\cdot 7$	$2^3\cdot 11\cdot 23\cdot 53^3$	A.N.
20.	1.474137	7^2	$2^{10}\cdot 11\cdot 53^2$	$3^4\cdot 5^8$	Je.B. & Ju.B., A.N.
21.	1.471298	$3^4\cdot 199$	11^8	$2^3\cdot 5^7\cdot 7^3$	Je.B. & Ju.B., A.N.
22.	1.465676	$17^4\cdot 67$	$2^{19}\cdot 137^4$	$3^{15}\cdot 5^3\cdot 13\cdot 89^2$	H.R. & P.M.
23.	1.465520	7^{12}	$2^{14}\cdot 67^3\cdot 461$	$3^{13}\cdot 11\cdot 19^4$	A.N.
24.	1.461924	$2^7\cdot 5^2$	$7^6\cdot 41$	13^6	B.W.

Table 5.1 continued on next page

continued from previous page

No.	L	a	b	c	Discoverer(s)
25.	1.459425	$5^{11} \cdot 31 \cdot 191$	$2^{8} \cdot 7^{13} \cdot 89 \cdot 859^{2}$	$3^{30} \cdot 13^{4} \cdot 277$	K.V.
26.	1.457794	$5^{12} \cdot 17^{2} \cdot 31^{2} \cdot 1699$	$23^{14} \cdot 29$	$2^{19} \cdot 3^{2} \cdot 11 \cdot 13^{10} \cdot 47$	A.N.
27.	1.457790	$3^{6} \cdot 5^{12}$	$2^{16} \cdot 13 \cdot 59^{4}$	$7^{11} \cdot 47 \cdot 113$	A.N.
28.	1.457482	$3 \cdot 109 \cdot 131^{4}$	$5^{22} \cdot 89$	$2^{3} \cdot 11^{2} \cdot 19^{5} \cdot 97^{4}$	T.S.
29.	1.457066	$3^{2} \cdot 5^{2}$	$2^{4} \cdot 17^{3} \cdot 31^{4}$	$7^{10} \cdot 257$	A.N.
30.	1.456203	$2^{25} \cdot 19$	$3 \cdot 5^{15} \cdot 1033$	$11^{7} \cdot 13^{3} \cdot 47^{2}$	A.N.
31.	1.455673	1	$2^{5} \cdot 3 \cdot 5^{2}$	7^{4}	B.W.
32.	1.455126	$3^{2} \cdot 11^{6}$	2^{35}	$19^{5} \cdot 13883$	Je.B. & Ju.B.
33.	1.455024	$23^{2} \cdot 31^{5}$	$2^{25} \cdot 7 \cdot 109^{3}$	$3^{19} \cdot 5^{2} \cdot 19^{2} \cdot 29$	T.S.
34.	1.454435	$7^{8} \cdot 2707$	$2^{10} \cdot 5^{10} \cdot 29^{3}$	$3^{18} \cdot 11^{4} \cdot 43$	T.S. & A.R.
35.	1.453343	13^{6}	$2 \cdot 3^{4} \cdot 7^{4} \cdot 11^{9} \cdot 23$	$5^{7} \cdot 103^{4} \cdot 2399$	A.N.
36.	1.452613	$2^{19} \cdot 13 \cdot 103$	7^{11}	$3^{11} \cdot 5^{3} \cdot 11^{2}$	B.W.
37.	1.451344	$3^{5} \cdot 7$	$5^{6} \cdot 67$	2^{20}	Je.B. & Ju.B., A.N.
38.	1.450858	$3^{5} \cdot 7^{3}$	$2^{13} \cdot 23^{3} \cdot 59$	$5^{3} \cdot 19^{6}$	Je.B. & Ju.B.
39.	1.450026	1	$3^{3} \cdot 5^{3} \cdot 7^{7} \cdot 23$	$2^{13} \cdot 11^{4} \cdot 13 \cdot 41$	A.N.
40.	1.449651	1	$3 \cdot 5^{5} \cdot 47^{2}$	$2^{18} \cdot 79$	G.F.
41.	1.447977	$11^{2} \cdot 43$	$5^{9} \cdot 7^{2} \cdot 13^{4} \cdot 97$	$2^{3} \cdot 3 \cdot 73^{7}$	A.N.
42.	1.447743	89	$7 \cdot 11^{8}$	$2^{20} \cdot 3^{3} \cdot 53$	A.N.
43.	1.447591	3^{17}	$2^{21} \cdot 5^{6} \cdot 23 \cdot 7993$	$47^{2} \cdot 307^{5}$	T.S.
44.	1.446873	409^{4}	$2^{21} \cdot 11^{5} \cdot 17 \cdot 19 \cdot 397$	$3^{5} \cdot 7^{5} \cdot 13^{9}$	T.S.
45.	1.446246	$3^{2} \cdot 5^{7} \cdot 79$	$2^{29} \cdot 13$	$11^{7} \cdot 19^{2}$	A.N.
46.	1.445064	$2 \cdot 13^{2}$	5^{8}	$3 \cdot 19^{4}$	Je.B. & Ju.B., A.N.
47.	1.444596	$3^{11} \cdot 5^{8} \cdot 4229$	$17^{5} \cdot 23^{5} \cdot 31^{3}$	$2^{32} \cdot 7^{2} \cdot 109^{3}$	T.S. & A.R.
48.	1.444199	$2^{19} \cdot 263$	$83 \cdot 167^{5}$	$5^{4} \cdot 29^{7}$	H.R. & P.M.
49.	1.443502	$2^{2} \cdot 11^{4} \cdot 17$	$5^{17} \cdot 13577$	$3^{4} \cdot 23^{9} \cdot 71$	A.N.
50.	1.443307	1	$2^{12} \cdot 5^{3}$	$3^{5} \cdot 7^{2} \cdot 43$	B.W.
51.	1.443284	$3^{2} \cdot 19^{3}$	5^{11}	$2^{17} \cdot 373$	Je.B. & Ju.B., A.N.
52.	1.442014	$2^{5} \cdot 11^{2} \cdot 19^{9}$	$5^{15} \cdot 37^{2} \cdot 47$	$3^{7} \cdot 7^{11} \cdot 743$	A.N.
53.	1.441814	$3^{16} \cdot 23^{2}$	$2^{13} \cdot 29^{2} \cdot 37^{3}$	$5^{9} \cdot 11^{4} \cdot 13$	A.N.
54.	1.441519	$7^{3} \cdot 29^{5} \cdot 151^{2}$	$2^{4} \cdot 5^{16} \cdot 97 \cdot 919$	$3^{27} \cdot 13^{4}$	A.N.
55.	1.441441	31^{3}	$2 \cdot 17 \cdot 41^{5}$	$3 \cdot 5^{7} \cdot 7^{5}$	A.N.
56.	1.440969	$3^{4} \cdot 23^{2}$	31^{5}	$2^{15} \cdot 5^{3} \cdot 7$	Je.B. & Ju.B., A.N.
57.	1.440264	$2^{35} \cdot 7^{2} \cdot 17^{2} \cdot 19$	$3^{27} \cdot 107^{2}$	$5^{15} \cdot 37^{2} \cdot 2311$	A.N.
58.	1.439063	1	$2^{4} \cdot 3^{7} \cdot 547$	$5^{8} \cdot 7^{2}$	B.W.
59.	1.438357	1	$19 \cdot 509^{3}$	$2^{19} \cdot 3^{4} \cdot 59$	Je.B. & Ju.B.
60.	1.436180	$2 \cdot 13^{5}$	$7^{6} \cdot 173^{2}$	$3^{13} \cdot 47^{2}$	A.N.
61.	1.435006	$2^{10} \cdot 7$	5^{7}	$3^{8} \cdot 13$	B.W.
62.	1.433956	$11^{9} \cdot 43$	$2^{4} \cdot 23^{6} \cdot 47 \cdot 277^{2}$	$5^{14} \cdot 7^{2} \cdot 13^{4}$	A.N.
63.	1.433464	$2^{5} \cdot 3^{18}$	$5^{6} \cdot 7^{10} \cdot 23^{2}$	$11^{9} \cdot 691 \cdot 1433$	A.N.
64.	1.433452	$5^{3} \cdot 8111$	$19^{12} \cdot 29$	$2^{19} \cdot 3^{3} \cdot 17^{4} \cdot 233^{2}$	A.N.
65.	1.433043	31^{2}	$3^{5} \cdot 5^{9}$	$2^{5} \cdot 23^{4} \cdot 53$	Je.B. & Ju.B., A.N.
66.	1.432904	2^{21}	$7^{6} \cdot 17 \cdot 8209^{2}$	$5^{12} \cdot 743^{2}$	A.N.
67.	1.432143	$3^{17} \cdot 67$	$7^{7} \cdot 11^{3} \cdot 227^{2} \cdot 547$	$2^{14} \cdot 5^{7} \cdot 17^{6}$	T.S.
68.	1.431815	$61^{4} \cdot 149$	$2^{23} \cdot 13 \cdot 29^{5}$	$3^{8} \cdot 5 \cdot 7^{4} \cdot 73^{4}$	T.S.
69.	1.431623	$17^{4} \cdot 79^{3} \cdot 211$	$2^{29} \cdot 23 \cdot 29^{2}$	5^{19}	A.N.
70.	1.431260	$2^{27} \cdot 7^{5}$	$3^{26} \cdot 11 \cdot 19 \cdot 139$	$5^{2} \cdot 13^{6} \cdot 43^{4} \cdot 179$	A.N.
71.	1.431183	2^{11}	$3^{9} \cdot 7^{3} \cdot 11^{3} \cdot 19$	$29 \cdot 277^{4}$	K.V.
72.	1.431092	$2^{9} \cdot 19^{2}$	$59^{6} \cdot 73$	$3^{3} \cdot 5^{7} \cdot 7^{2} \cdot 31^{3}$	A.N.
73.	1.430418	193	$2 \cdot 5^{6} \cdot 19^{2} \cdot 1193^{2}$	$3^{9} \cdot 13^{8}$	A.N.
74.	1.430176	$3^{6} \cdot 7^{2} \cdot 13 \cdot 127^{2}$	$2^{38} \cdot 61 \cdot 137$	$5^{11} \cdot 19^{6}$	Je.B. & Ju.B.
75.	1.429873	$2^{9} \cdot 37 \cdot 97^{5}$	$5^{5} \cdot 7 \cdot 89^{7}$	$3^{20} \cdot 17^{4} \cdot 3323$	A.N.
76.	1.429552	$3^{9} \cdot 29$	$7^{6} \cdot 43^{2}$	$2^{24} \cdot 13$	A.N.
77.	1.429007	3^{21}	$7^{2} \cdot 11^{6} \cdot 199$	$2 \cdot 13^{8} \cdot 17$	A.N.
78.	1.428908	73^{2}	$2^{11} \cdot 11^{4} \cdot 13^{3}$	$3^{11} \cdot 5^{5} \cdot 7 \cdot 17$	Je.B. & Ju.B.
79.	1.428402	$5^{14} \cdot 11$	$3^{6} \cdot 7^{5} \cdot 13^{2} \cdot 251$	$2^{21} \cdot 23^{4}$	A.N.
80.	1.428323	11	$7^{3} \cdot 167^{2}$	$2 \cdot 3^{14}$	Je.B. & Ju.B., A.N.
81.	1.427566	73	$11^{5} \cdot 157^{2}$	$2^{2} \cdot 3^{10} \cdot 7^{5}$	Je.B. & Ju.B., A.N.
82.	1.427488	61^{4}	$2^{20} \cdot 41^{3} \cdot 83^{2}$	$3^{22} \cdot 5 \cdot 19 \cdot 167$	A.N.
83.	1.427115	3^{10}	$7^{8} \cdot 23$	$2^{9} \cdot 509^{2}$	A.N.
84.	1.426753	31	$2^{5} \cdot 5^{10} \cdot 19^{2}$	$3 \cdot 7^{5} \cdot 11^{3} \cdot 41^{2}$	Je.B. & Ju.B., A.N.
85.	1.426565	3	5^{3}	2^{7}	B.W.
86.	1.423381	$5^{2} \cdot 11$	$13^{3} \cdot 1483^{2}$	$2^{29} \cdot 3^{2}$	Je.B. & Ju.B., A.N.
87.	1.422083	$17 \cdot 19^{4}$	$3^{3} \cdot 5^{10} \cdot 7^{2} \cdot 29^{3}$	$2^{13} \cdot 13^{7} \cdot 613$	K.V.
88.	1.421828	$2^{4} \cdot 59$	$5^{12} \cdot 19$	$3^{3} \cdot 11^{2} \cdot 17^{5}$	Je.B. & Ju.B., A.N.
89.	1.421575	5^{7}	$11^{5} \cdot 13^{2}$	$2^{15} \cdot 7^{2} \cdot 17$	A.N.
90.	1.421371	$67 \cdot 263^{3}$	$3^{10} \cdot 5^{9} \cdot 23^{3}$	$2^{10} \cdot 7^{6} \cdot 13^{2} \cdot 41^{3}$	T.S.
91.	1.421008	$2^{9} \cdot 37^{3} \cdot 89$	$3^{9} \cdot 5^{9} \cdot 31$	103^{6}	A.N.
92.	1.420437	$7^{8} \cdot 19$	$2^{15} \cdot 5^{2} \cdot 37^{2}$	$3 \cdot 17^{7}$	A.N.
93.	1.420320	3^{13}	$2^{21} \cdot 5^{4} \cdot 199^{2}$	$7^{8} \cdot 83^{2} \cdot 1307$	A.N.
94.	1.420232	$2^{14} \cdot 3^{10} \cdot 43 \cdot 461$	$11^{5} \cdot 29^{4} \cdot 83 \cdot 397^{2}$	5^{26}	T.S.
95.	1.420036	23^{3}	$3^{9} \cdot 5^{7} \cdot 31$	$2^{7} \cdot 7^{3} \cdot 13 \cdot 17^{4}$	A.N.
96.	1.419292	$19^{4} \cdot 37^{2}$	$3^{4} \cdot 5^{14} \cdot 79$	$2^{8} \cdot 31^{5} \cdot 73^{2}$	A.N.
97.	1.418919	7^{2}	$2^{17} \cdot 181^{2}$	$3^{8} \cdot 809^{2}$	Je.B. & Ju.B., A.N.
98.	1.418233	$13 \cdot 3499$	2^{39}	$3^{4} \cdot 5^{11} \cdot 139$	Je.B. & Ju.B.
99.	1.417633	$5^{6} \cdot 1609$	$2^{9} \cdot 3^{14} \cdot 13^{3}$	1523^{4}	Je.B. & Ju.B.
100.	1.416793	$3^{9} \cdot 43^{3}$	$5^{13} \cdot 5323$	$2^{7} \cdot 7^{3} \cdot 23^{6}$	A.N.

Table 5.1 continued on next page

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No.	L	a	b	c	Discoverer(s)
101.	1.416438	$41^4 \cdot 33941$	$3^{12} \cdot 19^7$	$2^{23} \cdot 5^9 \cdot 29$	Je.B. & Ju.B.
102.	1.416078	$3^{22} \cdot 37 \cdot 204749$	$2^8 \cdot 5^{27}$	$13^2 \cdot 31^6 \cdot 103^4 \cdot 113$	A.N.
103.	1.416051	$3 \cdot 5^4 \cdot 599$	$11 \cdot 23^8$	$2^{22} \cdot 59^3$	Je.B. & Ju.B., A.N.
104.	1.415633	$2^{46} \cdot 23$	$3^9 \cdot 5^5 \cdot 11^7 \cdot 31^2 \cdot 43$	$19^{11} \cdot 59 \cdot 7207$	A.N.
105.	1.415561	7^3	$5^{13} \cdot 181$	$2^4 \cdot 3 \cdot 11 \cdot 13^2 \cdot 19^5$	A.N.
106.	1.415273	$3 \cdot 23^4$	$5^{13} \cdot 31$	$2 \cdot 7^4 \cdot 199^3$	H.R. & P.M.
107.	1.415090	$2^6 \cdot 5^2 \cdot 7^{13} \cdot 13^2 \cdot 463$	$3^4 \cdot 43^{12}$	$11^{12} \cdot 389^2 \cdot 6841$	A.N.
108.	1.414503	$3^{11} \cdot 5^4$	$7 \cdot 11^6 \cdot 43$	$2^{17} \cdot 17^3$	X.G.
109.	1.414352	$3^7 \cdot 5^{14} \cdot 7^2$	$2^{51} \cdot 11^2$	$29^5 \cdot 73 \cdot 419^2 \cdot 1039$	A.N.
110.	1.413698	$2^6 \cdot 5 \cdot 137$	3^{14}	13^6	Je.B. & Ju.B., A.N.
111.	1.413279	5^2	$3^7 \cdot 13^3$	$2^8 \cdot 137^2$	Je.B. & Ju.B., A.N.
112.	1.413166	$3^6 \cdot 157^3 \cdot 283$	23^{10}	$2^{30} \cdot 5^2 \cdot 11^2 \cdot 13$	Je.B. & Ju.B., A.N.
113.	1.412893	$13 \cdot 733$	$3^9 \cdot 5^5 \cdot 89^5$	$2^{19} \cdot 7^2 \cdot 31^5 \cdot 467$	K.V.
114.	1.412681	5	3^{11}	$2^{10} \cdot 173$	B.W.
115.	1.411682	79^3	$3^6 \cdot 7 \cdot 11 \cdot 13^5$	$2^{18} \cdot 43^3$	A.N.
116.	1.411615	$3 \cdot 13^2 \cdot 1049$	$2^{39} \cdot 29^2 \cdot 107$	$19^3 \cdot 139^6$	Je.B. & Ju.B., A.N.
117.	1.410830	$13 \cdot 29^4$	$3 \cdot 7^{10} \cdot 19^4$	$2^5 \cdot 5 \cdot 43^2 \cdot 139^4$	A.N.
118.	1.410683	$67^2 \cdot 2399$	$3^{13} \cdot 107^3$	$2^6 \cdot 5^{15}$	Je.B. & Ju.B.
119.	1.410044	$2^{13} \cdot 3^{13} \cdot 11^3$	$13 \cdot 29 \cdot 43^6 \cdot 673$	$5^{20} \cdot 17$	A.N.
120.	1.409742	5^{12}	$2^2 \cdot 3^{21} \cdot 43^2 \cdot 52859$	$7^{10} \cdot 13^3 \cdot 17^2 \cdot 151^2$	A.N.
121.	1.408973	7^2	83^5	$2^2 \cdot 3^{12} \cdot 17 \cdot 109$	Je.B. & Ju.B., A.N.
122.	1.408866	2^{12}	$3^{15} \cdot 19^2 \cdot 73^2 \cdot 3343$	$5 \cdot 41^3 \cdot 193^5$	A.N.
123.	1.408577	$2 \cdot 7 \cdot 11 \cdot 13^6$	$23 \cdot 43^4 \cdot 449^4$	$3^{16} \cdot 53^4 \cdot 97^2$	T.S. & A.R.
124.	1.407787	$2^2 \cdot 13$	$7^3 \cdot 41^5 \cdot 181$	$3^{14} \cdot 5 \cdot 67^3$	A.N.
125.	1.407404	$3^2 \cdot 233$	$23^7 \cdot 293^2$	$2^{15} \cdot 5^2 \cdot 13^5 \cdot 31^2$	A.N.
126.	1.407208	241	$2^{12} \cdot 3^4 \cdot 5^6 \cdot 1181$	$11^8 \cdot 13^4$	Je.B. & Ju.B.
127.	1.407051	$3^9 \cdot 163$	$2^3 \cdot 11^6 \cdot 17$	5^{12}	Je.B. & Ju.B., A.N.
128.	1.406524	7^9	$3^2 \cdot 5^7 \cdot 13^3$	$2^{16} \cdot 19^2 \cdot 67$	N.E. & J.K.
129.	1.406420	$2^{19} \cdot 367^3$	$5^{17} \cdot 197 \cdot 281$	$13^2 \cdot 251^6$	A.N.
130.	1.406097	$2^{16} \cdot 41 \cdot 71$	$3^{15} \cdot 7^2$	19^7	A.N.
131.	1.406080	$13^5 \cdot 19^3$	$2 \cdot 11^{12} \cdot 1123 \cdot 76081$	$3^{38} \cdot 397$	A.N.
132.	1.406079	$5 \cdot 7^2$	$13^2 \cdot 43^3$	$2^{11} \cdot 3^8$	Je.B. & Ju.B., A.N.
133.	1.405785	13^3	$2^9 \cdot 37^2$	$3^2 \cdot 5^7$	A.N.
134.	1.405443	$2^{24} \cdot 3^5$	$5 \cdot 19^5 \cdot 59^2$	$7^{10} \cdot 167$	A.N.
135.	1.404484	631	$2^{26} \cdot 5 \cdot 29^2$	$3^3 \cdot 7^{10} \cdot 37$	A.N.
136.	1.404264	1	$3^9 \cdot 7^2 \cdot 197$	$2^7 \cdot 5^7 \cdot 19$	A.N.
137.	1.403980	$5^{12} \cdot 227$	$2^8 \cdot 3 \cdot 7^3 \cdot 23^7 \cdot 41$	$11 \cdot 19^5 \cdot 67^5$	A.N.
138.	1.403958	$3^9 \cdot 103$	$2^8 \cdot 11^2 \cdot 13^5 \cdot 41^2 \cdot 47$	$5^{14} \cdot 53^3$	A.N.
139.	1.403482	$3^3 \cdot 13$	$2^5 \cdot 11 \cdot 19^2 \cdot 73^3$	$5^2 \cdot 7^{11}$	A.N.
140.	1.402864	$5 \cdot 67^3 \cdot 127^2 \cdot 19219$	$13^{18} \cdot 37 \cdot 277$	$2 \cdot 3^{15} \cdot 7^2 \cdot 31^{10}$	A.N.
141.	1.402737	$3^4 \cdot 19 \cdot 61 \cdot 173^2$	$2^{44} \cdot 7^{10}$	$5^2 \cdot 149^4 \cdot 503 \cdot 929^3$	K.V.
142.	1.402183	$3^{12} \cdot 5^6$	$7^9 \cdot 31^2$	$2^9 \cdot 11^5 \cdot 571$	A.N.
143.	1.401993	$3 \cdot 5^{14} \cdot 199$	$7^2 \cdot 11^5 \cdot 17^4 \cdot 41$	$2^{30} \cdot 13^4$	A.N.
144.	1.401979	$2^{33} \cdot 5$	$3^9 \cdot 7^6 \cdot 31^2 \cdot 97$	$11^2 \cdot 19^3 \cdot 127^4$	A.N.
145.	1.401419	$3^{10} \cdot 5^4 \cdot 401$	$13^6 \cdot 47^3$	$2^{29} \cdot 31^2$	T.S. & A.R.
146.	1.401291	2^{22}	$7 \cdot 67^7 \cdot 137$	$3 \cdot 5^4 \cdot 13^5 \cdot 353^2$	T.S.
147.	1.401261	$3^6 \cdot 11^2 \cdot 47 \cdot 359^2$	17^{13}	$2^{21} \cdot 5^4 \cdot 2749^2$	K.V.
148.	1.401156	$2^{29} \cdot 7$	$3^2 \cdot 31^2 \cdot 73^4 \cdot 349$	$5^{15} \cdot 53^2$	A.N.
149.	1.400812	$23^4 \cdot 71^2$	$7^{14} \cdot 1231$	$2^4 \cdot 5^2 \cdot 11^2 \cdot 29^7$	A.N.
150.	1.400588	13^4	$17^6 \cdot 463$	$2^{21} \cdot 73^2$	A.N.
151.	1.400317	$2^{14} \cdot 3^{13} \cdot 5$	$7 \cdot 29^6 \cdot 71^2$	$11^9 \cdot 13^2 \cdot 53$	A.N.
152.	1.400262	$5^{18} \cdot 6359$	$3^2 \cdot 47^6 \cdot 73^3$	$2^7 \cdot 19^{10} \cdot 79$	A.N.

Discovers of the Known Good abc Triples

Initials	Name(s)
Je.B. & Ju.B.	Jerzy Browkin and Juliusz Brzezinski
G.F.	Gerhard Frey
N.E. & J.K.	Noam Elkies and Joe Kanapka
A.N.	Abderrahmane Nitaj
H.R. & P.M.	Herman te Riele and Peter Montgomery
E.R.	Eric Reyssat
T.S. & A.R.	Traugott Schulmeiss and Andrej Rosenheinrich
B.W.	Benne M.M. de Weger
K.V.	Kees Visser
X.G.	Xiao Gang

5.3 Computations Regarding Good Triples

It seems that the values of the good triples displayed in Table 5.1 were discovered by means of various algorithms. Hence a brute force approach was taken to confirm that the stated values were indeed all possible good triples over a particular interval. Initially a program was written in Matlab but was found to be too inefficient. With the aid of Joel Mejeur (now with the Department of Defense) and Michael Saum (University of Tennessee, Knoxville), a program was written in C and then run in parallel (using MPI) on a cluster of between 24 and 30 Intel 450 MHz Pentium III computers. Initially the program checked for good triples over the intervals $1 \leq a \leq 100,000$ and $a \leq b \leq 100,000$. Running time for this case was approximately four and one-half days. Note that runs covering even larger intervals are underway and results will be summarized in a future paper.

Results (Good Triples for $1 \leq a, b \leq 100,000$ (Initial Run)).

$a = 1,$	$b = 2400,$	$L = 1.455673$	*(No. 31)*	(5.20)
$a = 1,$	$b = 4374,$	$L = 1.567887$	*(No. 5)*	(5.21)
$a = 3,$	$b = 125,$	$L = 1.426565$	*(No. 85)*	(5.22)
$a = 37,$	$b = 32768,$	$L = 1.482910$	*(No. 17)*	(5.23)
$a = 343,$	$b = 59049,$	$L = 1.547075$	*(No. 6)*	(5.24)
$a = 7168,$	$b = 78125,$	$L = 1.435006$	*(No. 61)*	(5.25)

(No. ·) refers to the number in Table 5.1.

It is worthwhile to point out that (5.22) is the good abc triple with the smallest c value.

Results (Further Good Triples for $1 \leq a, b \leq 1,000,000$ (Further Run)). [1]

$a = 5,$	$b = 177147,$	$L = 1.412681$	*(No. 114)*	(5.26)
$a = 1,$	$b = 512000,$	$L = 1.443307$	*(No. 50)*	(5.27)
$a = 121,$	$b = 255879,$	$L = 1.488865$	*(No. 16)*	(5.28)
$a = 338,$	$b = 390625,$	$L = 1.445064$	*(No. 46)*	(5.29)
$a = 2197,$	$b = 700928,$	$L = 1.405785$	*(No. 133)*	(5.30)

5.4 Program Listings

5.4.1 abc-mpi.c

```
/*--------------------------------------------------------------------------*/
/* abc-mpi.c                                                                */
/*                                                                          */
/* Written by J. Mejeur (May 2002)                                          */
/* Revised by M. Saum (July 2002)                                           */
/*--------------------------------------------------------------------------*/

#include <stdlib.h>
#include <stdio.h>
#include <math.h>
#include <mpi.h>
#include <getopt.h>
#include <sys/time.h>

```

[1]This is a work in process. With some improvements we have hope of extending the ranges to 10,000,000. The interested reader may also see [7] regarding similar unpublished work by Joe Kanapka.

```
struct timeval t_start, t_stop;
struct timezone tz_dummy;

int get_primes(int n, unsigned int *primes);
inline unsigned int get_r(unsigned int n, unsigned int *primes, int num_primes);
unsigned int gcd(unsigned int a, unsigned int b);
double diff_time(struct timeval *, struct timeval *);

int main(int argc, char **argv) {
    unsigned int a, b, c, i;
    double *rs;
    unsigned int num_primes;
    unsigned int max=10000, min;
    unsigned int *primes;
    double L;
    char filename[256];
    FILE *fp=NULL;

    int count=0;
    int rank=0, size;

    /* Set up MPI communication */
    MPI_Comm world;

    MPI_Init(&argc, &argv);
    world = MPI_COMM_WORLD;
    MPI_Comm_rank(world, &rank);
    MPI_Comm_size(world, &size);

    /* Get command line options */
    min=0;
    while(1) {
        c=getopt(argc, argv, "l:m:");
        if(c==-1)
            break;
        switch(c) {
        case 'l':
            min=(unsigned int) atoi(optarg);
            break;
        case 'm':
            max=(unsigned int) atoi(optarg);
            break;
        }
    }

    /* Prime numbers calculated up to 2*max + 1   */
    /* to ensure that prime factorization can     */
    /* occur with c (=a+b).                       */
    /* Primes calculated only on master (rank=0)  */
    /* and sent to all slave processors via MPI.  */
    if(rank==0) {
        gettimeofday(&t_start,&tz_dummy);
        primes=calloc(2*max+1, sizeof(unsigned int));

        printf("Generating list of primes\n");
        fflush(stdout);
        num_primes=get_primes(2*max+1, primes);
        printf("Found %d primes.\n", num_primes);
        fflush(stdout);

        for(a=1;a<size;a++) {
            MPI_Send(&num_primes, 1, MPI_INT, a, 100, world);
            MPI_Send(primes, num_primes, MPI_UNSIGNED, a, 101, world);
        }

    } else {
        MPI_Recv(&num_primes, 1, MPI_INT, 0, 100, world, MPI_STATUS_IGNORE);
        primes=calloc(num_primes, sizeof(unsigned int));
        MPI_Recv(primes, num_primes, MPI_UNSIGNED, 0, 101, world, MPI_STATUS_IGNORE);
    }

    /* rs array contains log of radical for each number 1..2*max+1  */
    /* rs array calculated on each processor.  No need to send via  */
    /* MPI, as send traffic could be very large.  A good assumption */
    /* is that all processors participating in the MPI VM are of    */
    /* same order of magnitude speed wise.                          */
    rs = calloc(2*max+1,sizeof(double));
    for(i=1;i<=2*max+1;i++)
        rs[i-1] = log(get_r(i,primes,num_primes));

    if(rank==0) {
        sprintf(filename, "abc.out.%d", rank);
        fp=fopen(filename, "w");
        gettimeofday(&t_stop,&tz_dummy);
        fprintf(fp,"Time to gen primes and send = %g\n", diff_time(&t_start,&t_stop));
        fflush(fp);
    }

    /* Main loop.  Each processor starts with different a, increments */
    /* by number of processors each time.                             */
    for(a=rank+1;a<=max;a+=size) {
        if(rank==0) {
            fprintf(fp,"rank(%d): working on a=%d\n",rank, a);
```

```
            fflush(fp);
        }

        count++;
        if(a<=min)
            for(b=min;b<max;b++) {
                /* No need to calculate gcd if both a,b divisible by same small prime */
                if( !( ( !(a%2) && !(b%2) ) || ( !(a%3) && !(b%3) ) || ( !(a%5) && !(b%5) ) || ( !(a%7) && !(b%7) ) ) ) {
                    if(gcd(a, b) == 1) {
                        c=a+b;

                        L=log((double)c) / (rs[a-1]+rs[b-1]+rs[c-1]);

                        if(L > 1.4) {
                            printf("([%d]:%d,%d,%f)\n", rank,a, b, L);
                            fflush(stdout);
                        }
                    }
                }
            }


        if(a>min)
            for(b=a;b<=max;b++) {
                /* No need to calculate gcd if both a,b divisible by same small prime */
                if( !( ( !(a%2) && !(b%2) ) || ( !(a%3) && !(b%3) ) || ( !(a%5) && !(b%5) ) || ( !(a%7) && !(b%7) ) ) ) {
                    if( gcd(a, b) == 1) {
                        c = a+b;

                        L=log((double)c) / (rs[a-1]+rs[b-1]+rs[c-1]);

                        if(L > 1.4) {
                            printf("([%d]:%d,%d,%f)\n", rank,a, b, L);
                            fflush(stdout);
                        }
                    }
                }
            }

    }

    /* Ensure all slave processors are done computing */
    MPI_Barrier(world);

    if(rank==0) {
        gettimeofday(&t_stop,&tz_dummy);
        fprintf(fp,"\nTotal Time = %g\n", diff_time(&t_start,&t_stop));
        fclose(fp);
        printf("\n\n");
    }

    /* MPI Cleanup and shutdown */
    MPI_Finalize();

    return 0;
}
```

5.4.2 util.c

```
/*--------------------------------------------------------------------------*/
/* util.c                                                                   */
/*                                                                          */
/* Written by J. Mejeur (May 2002)                                          */
/* Revised by M. Saum (July 2002)                                           */
/*--------------------------------------------------------------------------*/

#include <stdlib.h>
#include <stdio.h>
#include <math.h>

unsigned int gcd(unsigned int a, unsigned int b) {
    /* recursive routine to calculate gcd(a,b) */

    int r;

    r = (b%a);
    if(r==0)
        return a;
    return gcd(r, a);
}

unsigned int get_primes(int n, unsigned int *primes) {
    /* Routine taken and altered from Octave's list_primes.m */

    unsigned int count;
    int i, p;
    int a, d;
    int is_prime, is_unknown;

    if (n==2) {
        primes[0]=2;
```

```
            realloc(primes, sizeof(unsigned int));
            return 1;
        }

        primes[0]=2;
        primes[1]=3;

        count=2;
        i=3;
        p=5;

        while(p<n) {

            is_prime = 1;
            is_unknown = 1;

            d = 3;
            while(is_unknown) {
                a = floor ( p / (float)d);
                if ( a <= d ) {
                    is_unknown = 0;
                }
                if ( (a*d) == p) {
                    is_prime = 0;
                    is_unknown = 0;
                }
                d = d + 2;
            }

            if (is_prime) {
                primes[count] = p;
                count++;
            }
            p+=2;
        }

        realloc( primes, count*sizeof(unsigned int));

        return count;
}

inline unsigned int get_r(unsigned int n, unsigned int *primes, int num_primes) {
    /* Check the list of primes to see if it is good.
     * If k still equals 1, the number must have been
     * prime, therefore set k to just be the number */

    int i;
    unsigned int k;

    k=1;
    for(i=0;i<num_primes;i++) {
        if(primes[i] > n)
            break;
        else if ( (n%primes[i])==0 )
            k=k*primes[i];
    }

    if(k==1) {
        k=n;
    }

    return k;
}
/*--------------------------------------------------------------------------*/
/* FUNCTION diff_time(t1,t2)                                                */
/*  Returns a decimal value (in secs) of elapsed time between t1 and t2 */
/*--------------------------------------------------------------------------*/
double diff_time(struct timeval * t_1, struct timeval * t_2) {
    double diff;

    diff = (double) ((t_2->tv_sec+t_2->tv_usec/1.0E6)
                     - (t_1->tv_sec+t_1->tv_usec/1.0E6));

    return (diff);
}
```

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Vita

Jeffrey Paul Wheeler was born April 22, 1968 in Wheeling, West Virginia. He graduated from Linsly School in May 1986. In May 1990, Jeffrey graduated from Miami University in Oxford, Ohio earning a Bachelor of Arts degree in Mathematics with a minor in Social Work. After graduation, Jeffrey took a position as a lecturer at Belmont Technical College in St. Clairsville, Ohio. He also served in an administrative position. In August 1998, Jeffrey entered the graduate program at the University of Tennessee, Knoxville. During his four years, Jeffrey held the position of graduate teaching associate. In each of those four years, Jeffrey was a finalist for the Dorthea and Edgar Eaves Teaching Award and was recipient of the award in the academic year 2001.

Upon a return to Miami Universtiy, Jeffrey met the woman who was to be his wife. The couple married June 3, 2000 in Chagrin Falls, Ohio and resided in Knoxville, Tennessee. Jeffrey's wife, Jamie, earned her MBA from the University of Tennessee, Knoxville in May 2002 while Jeffrey earned his master's degree in August 2002. Currently the couple resides in Memphis, Tennessee where Jamie is employed by FedEx and Jeffrey is pursuing a doctorate degree in Mathematics at the University of Memphis.