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From Isomorphism Identities to Levy measures of Non-negative Infinitely Divisible Processes

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I am submitting herewith a dissertation written by Vy Nguyen entitled "From Isomorphism Identities to Levy measures of Non-negative Infinitely Divisible Processes." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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From Isomorphism Identities to Lévy Measures of Non-negative Infinitely Divisible Processes

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Dedication

I dedicate this dissertation to my husband, Vinh Nguyen, who always encourages me to pursue my dream and finish my doctoral program.

To my son, Felix Nguyen, who is my daily reminder of what is good in life.

To my mother, Chau Ngo and my sister, Thuy Vy Nguyen, who have raised me to become a strong woman. Their love, support and sacrifices inspire me to complete this dissertation.

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Abstract

In this dissertation, we examine the Lévy measures of non-negative infinitely divisible processes. For a non-negative infinitely divisible process with no drift, Lévy measure is the single most important factor characterizing the process. Understanding the structure of Lévy measure can give more insight about the behavior of the process. However, it is not always easy to describe the Lévy measure of an infinitely divisible process. The descriptions of Lévy measures of squared Bessel processes proposed by Pitman and Yor are examples. It requires deep knowledge from the Ray-Knight theorems and Itô excursion laws to interpret these descriptions. We use isomorphism identities as the main tool to describe the Lévy measure of a non-negative infinitely divisible process. The isomorphism identities that we are interested in connect every non-negative infinitely divisible process to the family of its random translations. It turns out that the Lévy measure of a non-negative infinitely divisible process can be written in term of the laws of its random translations. More precisely, we manage to write the Lévy measure as a series of other Lévy measures which are written in term of the law of random translations. The special technique that we develop to ensure the condition distribution of a non-negative infinitely divisible process being consistently well defined enables us to find the laws corresponds to the component Lévy measures in the series.

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Chapter 1

Introduction

The concept of infinitely divisibility was introduced in 1929 by Bruno de Finetti, and was first studied systematically by the pioneers in this area: Kolmogorov, Lévy and Khintchine. It was Khintchine [14] who gave the first formal definition of an infinitely divisible distribution. It reads: *A distribution of a random variable which for any positive integer n can be represented as a sum of n identically distributed independent random variables is called an infinitely divisible distribution.* Infinitely divisible distribution has fundamental relation with the Central Limit Theorem (CLT) which is one of the most important theorems in statistical mathematics and probability theory. The classical version of the CLT states that under certain conditions, the sum of a large number of independent identically distributed random variables with finite non-zero variances approaches normal distribution, regardless the actual distribution of variables. The power of the classical version of the CLT is that we can apply probabilistic and statistical methods working for normal distribution to many other types of distributions. Can the limit distributions in CLT go beyond the normal distribution? If yes, what are they? The generalized CLT states that under certain conditions, the family of all possible limit distributions of the sum of a large number of random variables is exactly the family of infinitely divisible distributions .

Infinitely divisible distributions are also important because of theirs strong connection with Lévy processes which concern many aspects of modern probability theory and its applications. A Lévy process is a stochastic process $\{X_t, t \geq 0\}$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having the following properties:

a. If $t_0 < t_1 < ... < t_n$ then $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.

b. If $s, t \geq 0$, then $X_{t+s} - X_t$ is equal in distribution to X_s.

c. With probability 1, $t \mapsto X_t$ is right continuous with left limit.

Recall that Brownian motion can be thought of as a stochastic process with independent and stationary increments, whose sample paths are continuous almost surely and its law at any fixed time $t > 0$ is the zero mean normal distribution with variance t. If we replace the normal distribution at fixed time t in the definition of Brownian motion by an infinitely divisible distribution, we get a Lévy process. For a Lévy process X, the law of X_t at time $t = 1$ is infinitely divisible and it is unique for X. The collection of all infinitely divisible distributions is in one-to-one correspondence with the collection of all Lévy processes.

Infinitely divisible distributions play an important role in probability theory and have numerous applications. It is worthy to study its structure thoroughly. A formal definition of an infinitely divisible distribution in $\mathbb{R}^d$ is given as following:

Definition 1.1 (Infinitely Divisible Distribution) *A random vector X in $\mathbb{R}^d$ is said to be infinitely divisible (ID) if for all $n \in \mathbb{N}$, there exist independent identically distributed (i.i.d.) random vectors $Y^{1,n}, ..., Y^{n,n}$ in $\mathbb{R}^d$ such that*

$$X \overset{d}{=} Y^{1,n} + ... + Y^{n,n}, \tag{1.1}$$

where "$\overset{d}{=}$" means equality in distribution.

The concept of infinitely divisibility can be equivalently defined in term of probability measures and characteristic functions as follows:

- A probability measure μ in $\mathbb{R}^d$ is infinitely divisible if for every $n \in \mathbb{N}$ there exists a probability measure μ_n on $\mathbb{R}^d$ such that μ can be expressed as the n-th convolution power of μ_n.
- A characteristic function ϕ is infinitely divisible if and only if for every $n \in \mathbb{N}$ there exists a characteristic function ϕ_n such that $\phi = (\phi_n)^n$.

Example 1.2 Using characteristics functions, it is easy to verify that normal, compound Poisson, geometric, exponential and gamma distributions are infinitely divisible.
Any distribution (not constant) with bounded range is not infinitely divisible. For instance, binomial and uniform distribution are not ID.

The following theorem gives a representation of characteristic functions of all infinitely divisible distributions. It is called the Lévy - Khintchine Formula or the Lévy - Khintchine representation[2].

Theorem 1.3 (Lévy - Khintchine Formula). *Let X be an infinitely divisible vector in $\mathbb{R}^d$. There exists an unique triplet (Σ, ν, b) consisting of a symmetric non-negative definite $d \times d$ matrix Σ, a σ-finite measure ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} 1 \wedge \|x\|^2 \nu dx < \infty$, and a vector $b \in \mathbb{R}^d$ such that for every $u \in \mathbb{R}^d$*

$$\mathbb{E}(\exp(i\langle u, X\rangle)) = \exp\{-\frac{1}{2}\langle u, \Sigma u\rangle + i\langle b, u\rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle\chi(x))\nu dx\} \tag{1.2}$$

where $\chi(x) : \mathbb{R}^d \to \mathbb{R}$ is the cut-off function. Conversely, given a triplet (Σ, ν, b) as above, there exists an infinitely divisible distribution $X \in \mathbb{R}^d$ satisfying the equation (1.2).

The distribution of X is uniquely determined by the triplet (Σ, ν, b) which is called the generating triplet or **Lévy triplet** of X . The measure ν is called the **Lévy measure** and Σ is called the **Gaussian coefficient** of X . We often write $X \sim ID(\Sigma, \nu, b)$. We observe that $ID(\Sigma, 0, 0)$ is a centered Gaussian distribution with covariance matrix Σ and $ID(0, \nu, b)$

is a Poissonian type distribution.

Definition 1.4.(Infinitely Divisible process) *A stochastic process* $\{X_t\}_{t\in T}$ *over some arbitrary set* T *is infinitely divisible (ID) if for any* $t_1, ..., t_n \in T$ *the random vector*

$$(X_{t_1}, ..., X_{t_n})$$

has an infinitely divisible distribution on $\mathbb{R}^n$, $n \geq 1$.

Example 1.5

A Gaussian processes G_t is infinite divisible since for $n \in \mathbb{N}$, and any $t_1, ..., t_n \in T$, $(G_{t_1}, ..., G_{t_n})$ is a Gaussian vector which has infinitely divisible distribution. A Poisson process N_t of parameter λ is infinitely divisible since for $n \in \mathbb{N}$, and any $t_1, ..., t_n \in T$, $(N_{t_1}, ..., N_{t_n})$ has infinitely divisible distribution. In fact, it is easy to verify that all Lévy processes are infinitely divisible.

If the set T from Definition 1.4 is finite, the stochastic process $\{X_t\}_{t\in T}$ is just a random vector and its representation is given by the Lévy Khintchine formula in Theorem 1.3. When T is an uncountable general index set, it is difficult to define Lévy measure for an infinitely divisible process over set T. There are two ways to approach this problem. The first way is proposed by Lee [18] and Maruyama[22]. They defined Lévy measure on the $\sigma-$ring generated by cylindrical subsets of $\mathbb{R}^T \setminus \{0\}$. This approach leads to substantial conceptual and technical difficulty. Rosiński [28] suggested the second way to look at Lévy measure on the canonical path space $(\mathbb{R}^T, \mathcal{B}^T)$, on which the laws of stochastic processes over T are defined. Recall that $\mathbb{R}^T = \{x; x : T \to \mathbb{R}\}$ and $\mathcal{B}^T$ denote the cylindrical σ-algebra of $\mathbb{R}^T$. More details of this approach are discussed in Chapter 2. In this dissertation, we follow the second way.

The **Lévy triplet** (Σ, ν, b), which characterizes the distribution of an infinitely divisible process $\{X_t\}_{t\in T}$, has the following properties: the Gaussian coefficient Σ is a non-negative definite function on $T \times T$, the Lévy measure ν is a measure on $(\mathbb{R}^T, \mathcal{B}^T)$, and drift term

b is a function in $\mathbb{R}^T$. Therefore, for every infinitely divisible process $\{X_t\}_{t\in T}$, there exists mutually independent Gaussian process $G = (G_t)_{t\in T}$ and a Poissonian infinitely divisible process $Y = (Y_t)_{t\in T}$ such that

$$X \stackrel{d}{=} G + Y$$

where the centered Gaussian process G is characterized by a covariance function Σ and the Poisson infinitely divisible process Y is characterized by a Lévy measure ν and the drift function b.

For a non-negative infinitely divisible process with no drift, the Gaussian part Σ and the drift part b equal 0, so that Lévy measure is the single most important factor characterizing the process. Understanding the structure of a Lévy measure can give more insight about the behavior of the process. Given a non- negative infinitely divisible process, one might ask what is its Lévy measure? The answer is not always easy. Even for well known processes, the description of theirs Lévy measures can get really complicated. We look at an example of the process which is the square of one-dimensional Brownian motion starting from 0. By the observation of Shiga and Watanabe[30], this process is infinitely divisible. To describe the Lévy measure of this process, one needs deep knowledge of the Ray-Knight theorems and Itô excursion theory. In fact, this process is the squared Bessel process starting from 0 with dimension 1 which a member of the family called squared Bessel processes that we will examine carefully.

We recall that for $d \geq 0$, a d-dimensional squared Bessel process starting from x is defined as the unique strong solution of the stochastic differential equation:

$$X_t = x + 2\int_0^t \sqrt{X_s}dB_s + dt, \tag{1.3}$$

where B is one dimensional standard Brownian motion. When the dimension d is an integer, we can think about a d-dimensional squared Bessel process starting at x as the square of the distance from the origin of d-dimensional Brownian motion starts at b_0 where $|b_0|^2 = x$. These processes play an important role in financial mathematics because of their strong relation to financial models. Publications related to Bessel processes are numerous. We refer to papers by Shiga-Watanabe[30], Pitman-Yor[23] and the monograph of Revuz-Yor[25].

One of the most important properties of Squared Bessel processes is the additivity property which was observed by Shiga and Watanabe[30]. Following from this property, Squared Bessel processes are infinitely divisible. Denote the Lévy measure of a squared Bessel process with dimension d starting from x by $\nu^{(d,x)}$. By additivity property of squared Bessel processes, we have:

$$\nu^{(d,x)} = d\nu^{(1,0)} + x\nu^{(0,1)}. \tag{1.4}$$

Hence, we only need to describe $\nu^{(1,0)}$ and $\nu^{(0,1)}$. Pitman and Yor [23] have used Ray-Knight and excursion theorems as the main tools to approach this problem. Recall that local time of a process at x up to time t is the accumulated time that process has spent around x up to time t. Ray-Knight theorems describe the law of local time process of a Brownian motion at certain stopping time. Using the second Ray-Knight theorem, one can show that $\nu^{(0,1)}$ is the image of the Itô measure n_+ of positive excursions by the application which associates to an excursion u of its local times: $u \to L^t_\infty(u), t \geq 0$. Using a variant of the first Ray-Knight theorem, the Lévy measure of a squared Bessel process starting from 0 with dimension 1 can be expressed in term of positive *Itô* measure and Lebesgue measure by the application:$(u, s) \to L^{t-s}_\infty(u), t \geq 0$.

We will describe the Lévy measure of a general non-negative infinitely divisible process using isomorphism identities as the main tool. The isomorphism identities that we are interested in are based on random translations. The very first form of these identities was introduced by E.B.Dynkin[6]. In an effort to explain heuristic methods in quantum field theory of K. Symanzik [31], Dynkin established the celebrated Dynkin's Isomorphism Theorem which relates the cumulative local times process of a Markov process to the squares of its associated Gaussian process which occur to be infinitely divisible. In an abstract form, Dynkin's Isomorphism says that the total accumulated local time of the Markov process is an admissible translation of one half of the squared associated Gaussian process. Dynkin's isomorphism can be considered as a special case of more general isomorphism identities, described as follows. Let $\{Y_x\}_{x\in E}$ be a non-negative process. For every a such that $\mathbb{E}(Y_a)>0$, Y_x is infinitely divisible if and only if there exists a non-negative process $(Z_x^{(a)}, x\in E)$ independent of Y such that:

$$Y+Z^{(a)} \text{ has the law of } Y \text{ under } \mathbb{E}(\frac{Y_a}{\mathbb{E}(Y_a)},.). \tag{1.5}$$

This isomorphism connects every non-negative infinitely divisible process Y to a family of non-negative processes $Z^{(a)}$ which can be viewed as random translations of Y. More details of the isomorphism (1.5) and its applications are discussed in Eisenbaum paper[8] and Rosiński paper[28] . We only focus on the use of (1.5) to find the Lévy measure of Y in a series form.

We develop a special technique to ensure the condition distribution of a non-negative infinitely divisible process being consistently well defined which will be used in our description of a Lévy measure. Let $Y=(Y_{x_1},...,Y_{x_n})$ be a non-negative random variable in $\mathbb{R}^n$, and let conditional distribution of $(Y|Y_{x_1}=0)$ be defined as the weak limit μ_0 of probability measure μ_k given by

$$\mu_k(B)=\frac{1}{\mathbb{E}e^{-kY_{x_1}}}\int_{Y\in B}e^{-kY_{x_1}}d\mathbb{P}, \quad B\in\mathcal{B}(\mathbb{R}^n_+), \quad k\geq 1$$

provided such limit exists. It turns out that if $Y=(Y_{x_1},...,Y_{x_n})$ is a non-negative infinitely divisible random variable with Lévy measure ν and zero drift, then the weak limit of $\{\mu_k\}$ exists and the limit distribution μ_0 is infinitely divisible with zero drift and Lévy measure $\nu_1=\mathbf{1}_{\{y_{x_1}=0\}}\nu$.

We can write the Lévy measure of a non-negative infinitely divisible vector $(Y_{x_1},...,Y_{x_n})$ as a series:

$$\nu(dy)=\sum_{i=1}^{n}\tilde{\nu}_i(dy) \tag{1.6}$$

where $\tilde{\nu}_i$ has the following form:

$$\tilde{\nu}_i(dy)=\frac{\mathbb{E}(Y_{x_i})}{y_i}\mathbf{1}_{A_i}(y)\mathscr{L}(Z^{(x_i)})(dy) \tag{1.7}$$

here $A_i = \{y \in \mathbb{R}^n : y_k = 0 \forall k < i \text{ and } y_i > 0\}$. Using the conditioning given above, we identify the law corresponding to the component Lévy measures $\tilde{\nu}_i$.

It follows from [8] and [28] that to describe the Lévy measure ν of a non-negative infinitely divisible process Y it is critical to obtain the law of the family of random translations $Z^{(a)}$. We look at a family of infinitely divisible permanental processes whose random translations have special form. Recall that a permanental process is a positive gamma process whose finite dimensional Laplace transforms are given by a negative power of a determinant. A permanental process is infinitely divisible if and only if it is associated to a Markov process. More details about permanental processes are discussed in Chapter 3. It can be shown that random translations of an infinitely divisible permanental process relate to the local time processes of the Markov process that it is associated with. The example of squared Bessel process starting from 0 with dimension 1 is just a special case of infinitely divisible permanental processes.We can look at the squared Bessel process starting from 0 with dimension 1 as the associated permanental process of a transient Markov process X. By the Dynkin's Isomorphism, a random translation $Z^{(a)}$ of squared Bessel process starting from 0 with dimension 1 is just the total accumulated local times process of X conditioned to start at a and be killed at its last visits to a.

This dissertation is organized in the following way. In Chapter 2, we give a detailed study of Lévy measures on path spaces. In Chapter 3, we present an overview of permanental processes and squared Bessel processes. The definition, general properties of permanental processes and squared Bessel processes are discussed in this chapter. Chapter 4 is devoted to examine the description of Lévy measures of squared Bessel processes using Ray-Knight theorems and excursion laws. In Chapter 5, we look at the isomorphism identities based on random translations whose special case is Dynkin's Isomorphism. Chapter 6 is devoted describe Lévy measures of non-negative infinitely divisible processes in general.

Chapter 2

Lévy measures on path spaces

This Chapter is devoted to study Lévy measures on path spaces. We will show the existence and uniqueness of Lévy measures for every Poissonian infinitely divisible processes. Lévy Khinchine representation for an infinitely divisible process is also discussed. Our standard reference for this Chapter is [28].

2.1 Definitions and preliminaries

There are two natural ways to define Lévy measures for infinitely divisible processes over any set T. The first one is proposed by Lee and Maruyama [18][22]. It defines a Lévy measure on $\sigma-$ring generated by cylindrical subsets of $\mathbb{R}^T\backslash\{0\}$. The second way is proposed by Rosiński [28] which considers a Lévy measure on the canonical path space $(\mathbb{R}^T, \mathcal{B}^T)$, on which the laws of stochastic processes over T are defined . When T is uncountable, the first approach can lead to substantial conceptual and technical difficulties. For that reason, we prefer the second one. We look at Lévy measures as "laws" of processes defined on possibly infinite measure spaces. Here we are using the development from Rosiński paper[28]. Recall that $\mathbb{R}^T = \{x; x : T \mapsto \mathbb{R}\}$ and $\mathcal{B}^T$ will denote the cylindrical (product) $\sigma-$ algebra of $\mathbb{R}^T$. The law of a stochastic process $X = (X_t)_{t\in T}$ is a probability measure μ on $(\mathbb{R}^T, \mathcal{B}^T)$ given by

$$\mu(A) = \mathbb{P}\{\omega : (X_t(\omega))_{t\in T} \in A\}, \quad A \in \mathcal{B}^T,$$

and we write $\mathscr{L}(X) = \mu$.

Definition 2.1 (Lévy measure) *A measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a Lévy measure if the following two conditions hold*

C1. for every $t \in T$ $\quad \int_{\mathbb{R}^T} (|x(t)|^2 \wedge 1)\nu(dx) < \infty$,

C2. for every $A \in \mathcal{B}^T$ $\quad \nu(A) = \nu_*(A \backslash 0_T)$, *where ν_* is the inner measure.*

The first condition can be found in the Lévy Khinchine formula. It is needed for the integral to be well-defined. While the first condition is just a technical one, the second condition ensures "ν does not charge the origin". If T is countable, then $0_T \in \mathcal{B}^T$, so that condition (2) is equivalent to $\nu(0_T) = 0$ which is the usual condition for the uniqueness of ν in the Lévy Khinchine representation. If T is uncountable, $\nu(0_T)$ is undefined since $0_T \notin \mathcal{B}^T$. However, condition (2) still makes sense and it can be shown that there is a unique measure for any infinitely divisible process satisfying this definition.

Remark 2.2 There exists a countable set $T_0 \subset T$ such that

$$\nu\{x \in \mathbb{R}^T : x_{T_0} = 0\} = 0. \tag{2.1}$$

Condition (2) is satisfied because for any $A \in \mathcal{B}^T$

$$\nu(A) \geq \nu_*(A \backslash 0_T) \geq \nu(A \backslash \{x : x_{T_0} = 0\}) = \nu(A).$$

Let $\hat{T} := \{I \subset T : 0 < Card(I) < \infty\}$ be the family of all finite nonempty subsets of the index set T, and $\hat{T}_c := \{J \subset T : J \text{ is nonempty countable}\}$ be the family of all countable nonempty subsets of the index set T.

The next lemma introduces some equivalent conditions to condition (2) which are applicable in concrete situations.

Lemma 2.3 *Let ν be a measure on $(\mathbb{R}^T, \mathcal{B}^T)$. The the following conditions are equivalent to condition (2):*

a. for every $T_0 \in \hat{T}_c$ there exists $T_1 \in \hat{T}_c$ such that $T_0 \subset T_1$ and

$$\nu\{x \in \mathbb{R}^T : x_{T_0} = 0\} = \nu\{x \in \mathbb{R}^T : x_{T_0} = 0, x_{T_1} \neq 0\};$$

b. for every $T_0 \in \hat{T}_c$ with $\nu\{x \in \mathbb{R}^T : x_{T_0} = 0\} > 0$ there is $t \notin T_0$ such that

$$\nu\{x \in \mathbb{R}^T : x_{T_0} = 0, x(t) \neq 0\} > 0;$$

c. either $\nu\{x \in \mathbb{R}^T : x_{T_0} = 0\} = 0$ is satisfied for some $T_0 \in \hat{T}_c$ or there is $t \notin T_0$ such that $\nu\{x \in \mathbb{R}^T : x_{T_0} = 0, x(t) \neq 0\} > 0$ is satisfied.

Condition (a) was originally introduced in the lecture notes by Rosiński work[27].

2.2 Lévy Khintchine Representation

Infinitely divisible random variables are characterized by the Lévy - Khintchine Theorem.

Theorem 2.4 (Lévy - Khintchine). *Let X be an infinitely divisible vector in $\mathbb{R}^d$. There exists a unique triplet (Σ, ν, b) consisting of a symmetric non-negative definite $d \times d$ matrix Σ, a σ-finite measure ν on $\mathbb{R}^d$ satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}^d} 1 \wedge \|x\|^2 \nu dx < \infty$, and a vector $b \in \mathbb{R}^d$ such that for every $u \in \mathbb{R}^d$*

$$\mathbb{E}(\exp(i\langle u, X\rangle)) = \exp\{-\frac{1}{2}\langle u, \Sigma u\rangle + i\langle b, u\rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle\chi(x))\nu dx\} \tag{2.2}$$

where $\chi(x) : \mathbb{R}^d \to \mathbb{R}$ is the cut-off function. Conversely, given a triplet (Σ, ν, b) as above, there exists an infinitely divisible distribution $X \in \mathbb{R}^d$ satisfying the equation above.

Different authors might use a slightly different expressions for the Lévy - Khintchine formula depending on how they use the cut-off function. The cut-off function $\chi : \mathbb{R}^d \mapsto \mathbb{R}$ is a bounded measurable function such that $\chi(x) = 1 + o(|x|)$ as $x \to 0$ and $\chi(x) = O(|x|^{-1})$ as $|x| \to \infty$. The most common forms have been used as the cut-off function are $\mathbf{1}_{(0,1)}|x|$, $(1 \vee |x|)^{-1}$ and $(1 + |x|^2)^{-1}$. Such a change in the choice of cut-off function only affect parameter b. We will use the cut-off function $\chi(x) = \mathbf{1}_{(0,1)}|x|$ unless we specify otherwise.

The distribution of X is uniquely determined by the triplet (Σ, ν, b) which is called the generating triplet or **Lévy triplet** of X. The measure ν is the **Lévy measure** and Σ is called the **Gaussian coefficient** of X . We often write $X \sim ID(\Sigma, \nu, b)$. We observe that $ID(\Sigma, 0, 0)$ is a centered Gaussian distribution with covariance matrix Σ, $ID(\Sigma, 0, b)$ is a Gaussian distribution with mean b and covariance matrix Σ; and $ID(0, \nu, 0)$ is a compound Poisson distribution.

Let $X = (X_t)_{t \in T}$ be an infinitely divisible process, so that for every non-empty finite subset of T, $I \in \hat{T}$, the random vector X_I is infinitely divisible in $\mathbb{R}^I$. By theorem 2.4, there exists a unique Lévy triplet (Σ_I, ν_I, b_I) satisfies (2.2).

From the uniqueness of the triplet (Σ_I, ν_I, b_I), the following consistency conditions hold: for every $I, J \in \hat{T}$ with $I \subset J$

1. Σ_J restricted to $I \times I$ equals Σ_I,

2. b_J restricted to I equals b_I,

3. $\nu_J \circ \pi_{IJ}^{-1} = \nu_I$ on $\mathcal{B}_0^I$,

where $\pi_{IJ} : \mathbb{R}^J \mapsto \mathbb{R}^I$ is the natural projection from $\mathbb{R}^J$ onto $\mathbb{R}^I$. By the Kolmogorov Extension Theorem, there exists mutually independent centered Gaussian process $G = (G_t)_{t \in T}$ and a Poissonian infinitely divisible process $Y = (Y_t)_{t \in T}$ such that

$$X \stackrel{d}{=} G + Y.$$

For every $I \in \hat{T}$, $G_I \sim N(0, \Sigma_I)$, and

$$\mathbb{E}(\exp(i\langle u, Y_I\rangle)) = \exp\{i\langle b_I, u\rangle + \int_{\mathbb{R}^I} (e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle\chi(y))\nu_I dy\}, \quad u \in \mathbb{R}^I. \tag{2.3}$$

The covariance function Σ of G restricted to $I \in \hat{T}$ is Σ_I and there exists $b : T \mapsto \mathbb{R}$ which restricted to I equals b_I.

Definition 2.5 *A family $\{\nu_I : I \in \hat{T}\}$ of finite dimensional Lévy measures is consistent if it satisfies condition (3) of the consistency conditions above.*

Remark 2.6 A consisted family of finite dimensional Lévy measures is not necessarily a projective system of measures.

Theorem 2.7(Rosiński,[28]) *Let $Y = (Y_t)_{t\in T}$ be a Poissonian infinitely divisible process as in (2.3). Then there exist unique Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ and a shift function $b \in \mathbb{R}^T$ such that for every $I \in \hat{T}$ and $u \in \mathbb{R}^I$*

$$\mathbb{E}(\exp(i\langle u, Y_I\rangle)) = \exp\{i\langle b_I, u\rangle + \int_{\mathbb{R}^T} \big(e^{i\langle u,y_I\rangle} - 1 - i\langle u, y_I\rangle\chi(y_I)\big)\nu dy\}. \tag{2.4}$$

Therefore, for any consistent system of Lévy measures $\{\nu_I : I \in \hat{T}\}$ there exists a unique Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ such that

$$\nu \circ \pi_I^{-1} = \nu_I \quad on \quad \mathcal{B}_0^I, I \in \hat{T}. \tag{2.5}$$

Furthermore, ν is the smallest among all measures ρ such that $\rho \circ \pi_I^{-1} = \nu_I \quad on \quad \mathcal{B}_0^I, I \in \hat{T}$.

Proof The proof of this theorem can be found in [28] under the proof of Therorem 2.8.

This theorem shows the existence and uniqueness of Lévy measures for every Poissonian infinitely divisible processes. It enables us to write the representation for an infinitely divisible process.

Corollary 2.8 (Lévy Khintchine representation) *Let $X = (X_t)_{t\in T}$ be an infinitely divisible process. Then there exist a unique triplet (Σ, ν, b) consisting of a non-negative definite function Σ on $T \times T$, a Lévy measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ and a function $b \in \mathbb{R}^T$ such that for every $I \in \hat{T}$ and $u \in \mathbb{R}^I$*

$$\mathbb{E}(\exp(i\langle u, X_I\rangle))$$
$$= \exp\{-\frac{1}{2}\langle u, \Sigma_I u\rangle + i\langle b_I, u\rangle + \int_{\mathbb{R}^T}(e^{i\langle u,x_I\rangle} - 1 - i\langle u, x_I\rangle\chi(x_I))\nu dx\}, \tag{2.6}$$

where Σ_I is the restriction of Σ to $I \times I$. (Σ, ν, b) is called the generating triplet of X. Conversely, given a generating triplet (Σ, ν, b) as above, there exists an infinitely divisible process $X = (X_t)_{t\in T}$ satisfying (2.6).

It is useful to think about Lévy measures on path spaces as“laws” of processes defined on possibly infinite measure spaces. The following concept was introduced in [28]

Definition 2.9 (Representations of Lévy measures) *Let $\{\nu_I : I \in \hat{T}\}$ be a consistent family of finite dimensional Lévy measures, which extends uniquely to a Lévy measure ν by Theorem 2.7. A collection of measurable functions $V = (V_t)_{t\in T}$ defined on a measure space $(S, \mathcal{S}, n)$ is said to be a representation of ν if for every $I \in \hat{T}$*

$$n\{V_I \in B\} = \nu_I(B), \quad \textit{for every} \quad B \in \mathcal{B}_0^I.$$

A representation V is called exact if $n\circ V^{-1} = \nu$ or, equivalently, if $n\circ V^{-1}$ satisfies condition 2 in the definition of Lévy measure. Here V is viewed as a function from S into $\mathbb{R}^T$ given by $V(s)(.) = V_{(.)}(s)$.

The difference between representations and exact representations of Lévy measures is a technical one, as we can see in the following Lemma.

Lemma 2.10 *Any representation of a Lévy measure, defined on a σ-finite measure space, can be modified to an exact representation by restricting it to a smaller domain.*

Remark 2.11 The process $V_t(x) = x(t), x \in \mathbb{R}^T, t \in T$ is an exact representation of ν on $(S, \mathcal{S}, n) = (\mathbb{R}^T, \mathcal{B}^T, \nu)$. However, such representation is too general. It does not give much of information about the Lévy measure. Therefore, we are seeking more specific representations on richer structures, such as Borel spaces.

Example 2.12 (Lévy processes) Let $Y = (Y_t)_{t \geq 0}$ be a Poissonian Lévy process determined by $\mathbb{E}e^{iuY_t} = e^{tK(u)}$, where K is the cumulant function given by

$$K(u) = \int_{\mathbb{R}} (e^i ux - 1 - iu[[x]])\rho(dx) + iuc.$$

For every $I = \{t_1, ..., t_n\}$ with $0 \leq t_1 < ... < t_n$, and $a = (a_1, ..., a_n) \in \mathbb{R}^I$,

$$\mathbb{E} \exp i \sum_{k=1}^{n} a_k Y_{t_k} = \exp\Big\{ \sum_{k=1}^{n} K(u_k) \Delta t_k \Big\},$$

where $\Delta t_k = t_k - t_{k-1}$, $u_k = \sum_{j=k}^{n} a_j$, and $t_0 = 0$. Therefore, the Lévy measure ν_1 of X_I is given by

$$\nu_I(B) = \sum_{k=1}^{n} \int_{\mathbb{R}} \mathbf{1}_B(vx_k)\rho(dv)\Delta t_k, \quad B \in \mathcal{B}^n$$

where $x_k \in \mathbb{R}^n$, $x_k = (0, ..., 0, 1, ..., 1)$, $k = 1, ..., n$. The first $(k-1)$ elements of x_k are 0, the remaining are 1. Define $V = (V_t)_{t \in T}$ on the half plan $\mathbb{R}_+ \times \mathbb{R}$ equipped with a measure $\lambda \otimes \rho$ given by

$$V_t(r, v) = \mathbf{1}_{t \geq r} v, \quad (r, v) \in \mathbb{R}_+ \times \mathbb{R}$$

where λ denotes the Lebesgue measure. Then V is a representation of the Lévy measure ν of Y. Let I be a finite set of indices , for any $B \in \mathcal{B}_I^{\mathcal{I}}$ we have

$$\begin{aligned}(\lambda \otimes \rho) V_I \in B &= \int_{\mathbb{R}} \int_0^\infty \mathbf{1}_B(V_I(r,v)) dr \rho(dv) \\ &= \int_{\mathbb{R}} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \mathbf{1}_B(\mathbf{1}_{\{t_1 \geq r\}} v, ..., \mathbf{1}_{t \geq r} v) dr \rho(dv) \\ &= \int_{\mathbb{R}} \sum_{k=1}^n \mathbf{1}_B(0, ..., 0, v, ..., v) \rho(dv) \Delta t_k = \nu_I(B)\end{aligned}$$

For $T_0 = \mathbb{N}$, we have

$$(\lambda \otimes \rho)\{(r,v) : \mathbf{1}_{\{n \geq r\}} r = 0 \forall n \in \mathbb{N}\} = 0.$$

Thus, V is an exact representation of ν.

Example 2.13(General Compound Poisson Processes) Let $V = \{V_t\}_{t \in T}$ be a stochastic process and let ζ be a Poisson random variable with mean θ. Let $\{V^{(n)}\}_{n \in \mathbb{N}}$ be a sequence of independent copies of V and independent of ζ. Then,

$$Y_t = \sum_{n=1}^{\zeta} V_t^{(n)}$$

is a Poissonian infinitely divisible process such that for every $I \in \hat{T}$, $a \in \mathbb{R}^I$,

$$\mathbb{E} \exp i \sum_{t \in I} a_t Y_t = \exp\{\theta \mathbb{E}(e^{i \sum_{t \in I} a_t V_t} - 1)\} = \exp\{\int_{\mathbb{R}^I} (e^{i<a,y>} - 1) \nu_I(dy)\}.$$

$V = \{V_t\}_{t \in T}$ is a representation the Lévy measure ν of Y on $(\Sigma, \mathcal{F}, \theta\mathbb{P})$.

Chapter 3

Permanental processes and Squared Bessel processes

In this Chapter, we focus on permanental processes which serve as generalization of the squared centered Gaussian processes. We are going to explore the definition of permanental processes, the connections between a permanental process and the local times of a Markov process and the characterization of an infinitely divisible permanental process. We also introduce the squared Bessel processes and their important properties. The standard references for this Chapter are [10][21][17].

3.1 Definition and Existence of Permanental Processes

A permanental process with parameter space E is a positive Gamma process whose finite dimensional Laplace transforms are given by a negative power of a determinant[10].

Definition 3.1(Permanental Process) *A real-valued positive process* $(Y_x, x \in E)$ *is a permanental process if for every* $(\alpha_1, \alpha_2, .., \alpha_n)$ *in* $\mathbb{R}_+^n$ *and every* $(x_1, x_2, ..., x_n)$ *in* E^n*, its finite-dimensional Laplace transforms satisfy*

$$\mathbb{E}\big(\exp\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i Y_{x_i}\}\big) = |I + \alpha U|^{-1/\beta}, \tag{3.1}$$

where I is the $n \times n$-identity matrix, α is the diagonal matrix $diag(\alpha_i))1 \leq i \leq n$ and $U = (u(x_i, x_j))_{1\leq i,j\leq n}$, β is a fixed positive number and for an $n \times n$ matrix A, $|A|$ is the determinant of A.
Process (Y_x) is called permanental process with kernel $(U(x, y), x, y \in E)$ and index β.

Let $(\phi_x)_{x\in E}$ be a centered Gaussian process with covariance $U = (u(x_i, x_j))_{1\leq i,j\leq n}$, then the finite dimensional Laplace transforms of the process $(\phi_x^2)_{x\in E}$ is given by:

$$\mathbb{E}\big(\exp\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\phi_{x_i}^2\}\big) = |I + \alpha U|^{-1/2}.$$

Thus $(\phi_x^2)_{x\in E}$ is a permanental process with index $\beta = 2$ and symmetric positive kernel U.

Remark 3.2 : The right hand side of (3.1) is not unique with respect to matrix U. If P is a diagonal matrix with non-zero entries, then

$$|I + \alpha U| = |I + \alpha P^{-1}UP| = |I + \alpha P^{-1}U^T P|.$$

In general, the right hand side of (3.1) might not be a Laplace transform. The natural question is in which conditions in term of $U = (u(x_i, x_j))_{1\leq i,j\leq n}$ and β that (3.1) makes sense, or that the right hand side of (3.1) is a Laplace transform. Vere-Jones [32] has answered the question by establishing the necessary and sufficient condition on U for the existence of Y. They are based on the following definition.

Definition 3.3 *For any $n \times n$ matrix M:*

$$det_\beta M = \sum_{\sigma\in\mathcal{S}_n}\beta^{n-\nu(\sigma)}\prod_{i=1}^{n}M_{i,\sigma(i)},$$

where $\mathcal{S}_n$ is the symmetric group of order n and $\nu(\sigma)$ is the number of cycles of σ. For every multi-index $k = (k_1, k_2, ..., k_n)$, $M(k)$ denotes the derived $|k| \times |k|$-matrix (where $|k| = k_1 + k_2 + ... + k_n$) obtained from M by selecting the first row and column k_1 times, the

second k_2 times,..., the n^{th} k_n times. For $\beta > 0$, a matrix M is said to be β-positive definite if for all possible derived matrices $M(k)$, $det_\beta(M(k)) \geq 0$
Note that $det_{-1}M = |M|$ the determinant of M, and $det_1(M) = Perm(M)$.

The next proposition is Proposition 4.5 in [32].

Proposition 3.4 *A permanental vector $(Y_{x_i}, 1 \leq i \leq n)$ corresponding to $U = (u(x_i,x_j))_{1\leq i,j\leq n}$ and index β exists if and only if for every $r > 0$:*

- $|I + rG| > 0$,
- *set $Q_r = U(I + rU)^{-1}$, then $Q_r(k)$ is β-positive definite for every k in $\mathbb{N}^n$.*

It is impossible to very the proposition 3.3 except in a very special case when U is the potential density of a transient Markov process that we are going to discuss in section 3.2.

3.2 Permanental processes associated to Markov processes

Let (X_t) be a transient Markov process on a nice measure space $(E, \mathcal{E})$, where $\mathcal{E}$ is the Borel sets of E, with α-potential density $u^\alpha(x, y)$. See appendix A.2 for more details of Markov transition semigroup and potential densities.

Theorem 3.5 (Eisenbaum and Kaspi,[10])*For every $\beta > 0$, there exists a positive process $(Y_x, x \in E)$ such that for every $(\alpha_1, \alpha_2, ..., \alpha_n)$ in $\mathbb{R}^n_+$ and every $(x_1, x_2, ..., x_n)$ in E^n*

$$\mathbb{E}\Big(\exp\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i Y_{x_i}\}\Big) = |I + \alpha U|^{-1/\beta}$$

where I is the $n \times n$-identity matrix, α is the diagonal matrix $diag(\alpha_i))1 \leq i \leq n$ and $U = (u(x_i, x_j))_{1\leq i,j\leq n}$.

This theorem says that for every $\beta > 0$, there exists a permanental process with index β and with kernel is the 0-potential density function of X.

Proof of Theorem 3.5: see [10], theorem 3.1 for the proof.
The proof of this theorem use the fact that U^{-1} is an M-matrix, then all the real eigenvalues of U^{-1} are positive. In addition, the matrix $Q_r = U(I + rU)^{-1}$ is a resolvent matrix hence all its entries are non-negative. It follows that Q_r is β-positive definite for all $\beta > 0$. Therefore, the theorem follows Proposition 3.4.

When $\beta = 2$, for every fixed $x \in E$, Y_x has the law of squared centered Gaussian variable with a variance equal to $u(x, x)$. If moreover the 0-potential function is symmetric, Y is the square of a centered Gaussian process with a covariance equal to $u(x, y), x, y \in E$. This process has been noted and studied by many authors [5][7][21]. This Gaussian process is referred as "Gaussian process associated" to X.

Definition 3.6 *In the general case, we call the process Y the permanental process with index β associated to X.*

The existence of an associated permanental process with index β for every $\beta > 0$ implies the property of infinite divisibility of these process.

Let $(L_t^x, x \in E, t \geq 0)$ be local time process of X_t and $(u(x, y), (x, y) \in E \times E)$ be its 0-potential density function (or Green function) so that it satisfies $u(x, y) = \mathbb{E}_x(L_\infty^y)$. Let a be an element of E such that $u(a, a) > 0$. We define the probability $\tilde{\mathbb{P}}^a$ as follows:

$$\tilde{\mathbb{P}}^a|_{\mathcal{F}_t} = \frac{u(X_t, a)}{u(a, a)} \mathbb{P}^a|_{\mathcal{F}_t}.$$

Under $\tilde{\mathbb{P}}^a$, the process X starts at a and is killed at its last visit to a. Expectation with respect to $\tilde{\mathbb{P}}^a$ is denoted by $\tilde{\mathbb{E}}^a$.

Let $Y_x, x \in E$ be the permanental process with index 2 associated to the Markov process X, defined on a probability space unrelated with X. The following theorem relates the law of Y and the law of L^x_∞ under $\tilde{\mathbb{P}}^a$.

Theorem 3.7 (Eisenbaum and Kaspi,[10])*For every $a \in E$ such that $u(a,a) > 0$, for every functional F on the space of measurable functions from E to $\mathbb{R}$, we have*

$$\mathbb{E}\tilde{\mathbb{E}}^a(F(L^x_\infty + \frac{1}{2}Y_x; x \in E)) = \mathbb{E}(\frac{Y_a}{u(a,a)}F(\frac{1}{2}Y_x; x \in E)). \tag{3.2}$$

Proof of Theorem 3.7 see [10], Theorem 3.3 for details of the proof.

Now, let X be a recurrent Markov process with a state space E. For $a \in E$, define $T_a = \inf\{t \geq 0 : X_t = a\}$ and $\tau_r = \inf\{t \geq 0 : L^a_t > r\}$. Let S_θ be an exponential time with parameter θ, independent of X. Then X killed at T_a and X killed at τ_{S_θ} are both transient Markov processes. We denote by ϕ and ψ their respective associated permanental processes with index 2. We have the following identity for the process $(L^x_{\tau_r}, x \in E)$[11].

Corollary 3.8 *Let X be a recurrent Markov process. For $a \in E$ and every functional F on measurable function from E to $\mathbb{R}$, we have*

$$\mathbb{E}\mathbb{E}^a\big(F(L^x_{\tau_r} + \frac{1}{2}\phi_x; x \in E)\big) = \mathbb{E}\big(F(\frac{1}{2}\psi_x; x \in E)|\psi_a = r\big)$$

Besides, we have :$(\psi_x; x \in E|\psi_a = 0) = (\phi_x; x \in E)$

Proof of Corollary 3.8:see [10], Corollary 3.6 for details of the proof.

3.3 Characterization of the infinitely divisible permanental processes

The natural question is when a permanental process has the property of infinite divisibility? From the previous section, we know that a permanental process associated to a Markov process is infinitely divisible. But, is that the only case? This section is devoted to answer these questions. And the answer is: a permanental process is infinitely divisible if and only if it is associated to a Markov process.

If a permanental process with index $\beta > 0$ is infinitely divisible then the permanental process with the same kernel and index 2 is also infinitely divisible. Hence, on this section we will consider the case $\beta = 2$.

Definition 3.9 *A $n \times n$ matrix A is an $M-$matrix if and only if*

- $A_{ij} \leq 0$ *for* $i \neq j$
- *A is non-singular and $A^{-1} \geq 0$(i.e. $A^{-1}_{ij} \geq 0$ for every i, j).*

Lemma 3.10 *Let $(G_{i,j}, 1 \leq i, j \leq n)$ be a real non-singular $n \times n$ matrix. There exists a positive infinitely divisible random vector $(Y_1, Y_2, ..., Y_n)$ such that for every $(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n_+$,*

$$\mathbb{E}(\exp\{-\frac{1}{2}\sum_{i=1}^{n} \alpha_i Y_i\}) = |I + \alpha G|^{-1/2} \tag{3.3}$$

if and only if , there exists a signature matrix S such that $SG^{-1}S$ is an $M-$matrix.

If U satisfies (3.3), the its real eigenvalues must be positive.

Proof of Lemma 3.10 see [10], Lemma 4.2 for the proof.

Theorem 3.11 (Eisenbaum and Kaspi,[10])*Let $(G_{i,j}, 1 \leq i, j \leq n)$ be a real non-singular $n \times n$ matrix. There exists a positive infinitely divisible random vector $(Y_1, Y_2, ..., Y_n)$ such* *that (3.3) is satisfied, if and only if*

$$G(i,j) = d(i)u(i,j)d(j) \tag{3.4}$$

for every (i,j)*, where* d *is a function on* $\{1,2,...,n\}$ *and* u *is the Green function of a Markov process.*

See [10], Theorem 4.3 for the proof.

The property (3.4) is equivalent to the following property[9]

$$G(i,j) = d^{-1}(i)u(i,j)d(j) \tag{3.5}$$

where d is a function on $\{1,2,...,n\}$ and u is the Green function of a Markov process. Note that $|I+\alpha G| = |I+\alpha U|$. It is saying that Y is a infinitely divisible permanental vector associated to a Markov process.

Theorem 3.12 (Eisenbaum and Kaspi,[10])*Let* $(k(x,y), x, y \in E)$ *be a jointly continuous function on* $E \times E$ *such that* $k(x,x) > 0$ *for every* $x \in E$*. There exists a positive infinitely divisible process* $(Y_x, x \in E)$ *such that for every* $(\alpha_1, \alpha_2, ..., \alpha_n)$ *in* E^n*,*

$$\mathbb{E}(\exp\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i Y_i\}) = |I+\alpha K|^{-1/2} \tag{3.6}$$

where $K = (k(x_i.x_j))_{1\leq i,j\leq n}$
if and only if

$$k(x,y) = d(x)g(x,y)d(y) \tag{3.7}$$

where d *is a positive function and* g *the Green function of a Markov process.*

See [10], Theorem 4.4 for the proof.

Corollary 3.13 *Let $(k(x,y), x, y \in E)$ be a jointly continuous function on $E \times E$ such that $k(x,x) > 0$ for every $x \in E$. Let $(Y_x, x \in E)$ be a process such that for every $(\alpha_1, \alpha_2, ..., \alpha_n)$ in $\mathbb{R}_+^n$ and every $(x_1, x_2, ..., x_n)$ in E^n,*

$$\mathbb{E}(\exp\{-\frac{1}{2}\sum_{i=1}^{n} \alpha_i Y_i\}) = |I + \alpha K|^{-1/2} \tag{3.8}$$

where $K = (k(x_i.x_j))_{1 \le i,j \le n}$

Then $Y_x, x \in E$ is infinitely divisible if and only if it is associated to a Markov process.

3.4 Squared Bessel Processes

Definition 3.14 (Squared Bessel Process) *For any real number $\delta \geq 0$ and $x \geq 0$ the square of δ-dimensional Bessel processes started at x, $X = \{X_t\}_{t \geq 0}$ is defined as the unique solution to the SDE*

$$X_t = x + 2\int_0^t \sqrt{X_s} dW_s + \delta t, \tag{3.9}$$

where W is one dimensional standard Brownian motion. We will denote X as $BESQ^\delta(x)$.

When the dimension δ is an integer, $BESQ^\delta(x)$ is the square of the distance from the origin of δ-dimensional Brownian motion starts at B_0 where $|B_0|^2 = x$. Let B be that Brownian motion. By the Ito's formula, we have

$$|B_t|^2 = |B_0|^2 + 2(\sum_{i=1}^{n} \int_0^t B_s^{(i)} dB_s^{(i)}) + \delta t, \tag{3.10}$$

For $\delta \geq 1$, $|B_t|$ is almost surely positive for positive t, we can consider the process:

$$\beta_t = \sum_{i=1}^{n} \int_0^t \frac{B_s^{(i)}}{|B_s|} dB_s^{(i)}$$

which is a one dimensional Brownian motion since $< \beta, \beta >_t = t$. Hence, the equation (3.10) can be written in the form of (3.16) by taking $X_t = |B_t|^2$.

The squared Bessel processes become interesting due to the important observation of Shiga and Watanabe [30].

Theorem 3.15(Shiga and Watanabe [30]) *Let X and X' be independent stochastic processes with X a $BESQ^{\delta}(x)$ and X' a $BESQ^{\delta'}(x')$, $\delta, \delta' > 0$. Then $X + X'$ is a $BESQ^{\delta+\delta'}(x + x')$.*

Proof: We provide the proof for its simplicity and beauty. The original proof can be found in [25], chapter 11, theorem 1.2.

Proof. X satisfies (3.16) while X' satisfies

$$X'_t = x' + 2\int_0^t \sqrt{X'_s}dW'_s + \delta' t$$

where W' is an one dimensional Brownian motion independent of W. The sum $Y_t = X_t + X'_t$ satisfies

$$Y_t = x + x' + 2\int_0^t (\sqrt{X_s}dW_s + \sqrt{X'_s}dW'_s) + \delta t + \delta' t, \qquad (3.11)$$

Set

$$\beta_t = \int_0^t \frac{\sqrt{X_s}dW_s + \sqrt{X'_s}dW'_s}{\sqrt{Y_s}}.$$

Then β_t is a one dimensional Brownian motion since $< \beta, \beta >_t = t$. The integral in (3.18) becomes $\int_0^t \sqrt{Y_s}d\beta_s$ and Y_t is a $BESQ^{\delta+\delta'}(x + x')$. □

Corollary 3.16(Shiga and Watanabe[30]) *Squared Bessel Processes are infinitely divisible*

Let X_t be a $BESQ^{\delta}_x$. For every $n \in \mathbb{N}$, there exist i.i.d processes $Y_t^{(j)}$ which are $BESQ^{\delta/n}_{x/n}$, $j = 1, .., n$ such that $X \overset{d}{=} Y^{(1)} + ... + Y^{(n)}$.

Proposition 3.17 *If X is a $BESQ^{\delta}_{(x)}$, then for any $c > 0$, the process $c^{-1}X_{ct}$ is a $BESQ^{\delta}_{(x/c)}$.*

Remarks 3.18

- For $\delta \geq 3$, the process $BESQ^{\delta}$ is transient.

- For $\delta \leq 2$, the process $BESQ^\delta$ is recurrent.
- For $\delta \geq 2$, the set $\{0\}$ is polar which means $P(T_0 < \infty) = 0$ where $T_0 = \inf\{s : X_s = 0\}$.
- For $\delta \leq 1$, the set $\{0\}$ is reached a.s.
- For $\delta = 0$, $\{0\}$ is an absorbing point.

Chapter 4

Lévy measures of Squared Bessel Processes

This Chapter is devoted to examine the description of Lévy measures of squared Bessel processes using Ray-Knight theorems and excursion laws.

4.1 Ray-Knight Theorems

There are two Ray-Knight theorems. They describe the relationship of Brownian's local times and other independent Brownian motions. The first Ray-Knight theorem is related to $T_0 = \inf\{t : W_t = 0\}$ where W_t is a Brownian motion. It describes the relationship of the Brownian local times $\{L^r_{T_0}, r \in \mathbb{R}_+\}$ and two independent Brownian motions. This theorem was provided independently by Ray[24] and Knight[16]. There are many versions of the First Ray-Knight theorem. We now introduce the version used by Marcus and Rosen which can be found in Theorem 2.6.3 page 52 of [21].

Let B_t $\bar{B}_t$ be two independent standard Brownian motions. Then, the process $\{B_t^2 + \bar{B}_t^2 : t \in \mathbb{R}_+\}$ and is a two dimensional squared Bessel process.

Definition 4.1 (Terminal times) A stopping time T is called a terminal time if for every t

$$T > t \Rightarrow T = t + T \circ \theta_t \quad a.s, \tag{4.1}$$

Set

$$u_T(x,y) = \mathbb{E}^x(L_T^y), \tag{4.2}$$

It is obvious that T_0-the first time that a Brownian motion hits 0 is a terminal time. In the next step, we want to evaluate $u_{T_0}(x,y)$ for the Brownian motion.

Lemma 4.2 *Let $\{L_t^x, (x,t) \in \mathbb{R}^1 \times \mathbb{R}_+\}$ be the local times of Brownian motion and let u_{T_0} be given by (2.3). Then*

$$u_{T_0}(x,y) = \begin{cases} 2(|x| \wedge |y|) & xy > 0 \\ 0 & xy \leq 0 \end{cases}, \tag{4.3}$$

Proof of Lemma 4.2 See [21], page 42 for the proof.

Lemma 4.3 *Let T be a terminal time with potential density (A.2)$u_T(x,y)$ as defined above. Let Σ be the matrix with elements $\Sigma_{i,j} = u_T(x_i, x_j), i,j = 1,2,...,n$. Let Λ be the matrix with elements $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$. For all $\lambda_1, ..., \lambda_n$ sufficiently small and $1 \leq l \leq n$,*

$$\mathbb{E}^{x_l} exp\Big(\sum_{i=1}^{n} \lambda_i L_T^{x_i}\Big) = \frac{det(I - \hat{\Sigma}\Lambda)}{det(I - \Sigma\Lambda)}, \tag{4.4}$$

where

$$\hat{\Sigma}_{i,j} = \Sigma_{i,j} - \Sigma_{l,j}, \quad i,j = 1, ..., n., \tag{4.5}$$

These equation also hold when T is replace by $\tau_A(\lambda)$, for any CAF, $A = \{A_t, t \in \mathbb{R}_+\}$, and $\mathbb{E}^x$ is replaced by $\mathbb{E}^x_\lambda$.

Proof of Lemma 4.3 See page 49 [21] for the proof.

Theorem 4.4(First Ray-Knight Theorem) *Let $x > 0$. Then under $P^x \times P_{B,\bar{B}}$*

$$\{L^r_{T_0} + (B^2_{r-x} + \bar{B}^2_{r-x})\mathbb{1}_{r\geq x} : r \in \mathbb{R}_+\} = \{B^2_r + \bar{B}^2_r : r \in \mathbb{R}_+\}, \tag{4.6}$$

Equivalently, under P^x, $L^r_{T_0}$ between 0 and x has the law of a second dimensional squared Bessel process $Y = \{Y_r : 0 \leq r \leq x\}$ with $Y_0 = 0$, and then proceeds from x as a $0-th$ dimensional squared Bessel process $Z = \{Z_r : x \leq r < \infty\}$ with $Z_x = Y_x$, where Z also has the property that, conditioned on Y_x, it is independent of Y.

Proof of Theorem 4.4 See page 52 of [21] for the proof.

Remark 4.5 There are several important variants of the First Ray-Knight Theorem, among which is the following

> $\{L^r_\infty(|B| + L^0); r \geq 0\}$ *has the law of second order squared Bessel process starting at 0 .*

The second Ray-Knight Theorem is related $\tau(t) = \inf\{s : L^0_s > t\}$ the right continuous inverse of the local time at 0 of a standard Brownian motion. Heuristically, it is the amount of time it takes for the local time at 0 equal to t. The second Ray-Knight Theorem describes the behavior of the Brownian motion local time $L^x_{\tau(t),x\geq 0}$ in term of squares of standard Brownian motion and Brownian motion starting from $\sqrt{t}$. We use the version of the second Ray-Knight theorem proposed by Marcus and Rosen which can be found in Theorem 2.7.1 of [21].

Lemma 4.6 *Let W be a Brownian motion $(\Omega, \mathcal{F}, \mathcal{F}_t, P^x)$ and let $\{L^y_t, (y,t) \in \mathbb{R}^1 \circ \mathbb{R}_+\}$ be the local time of W. Let λ be an exponential random variable with mean $1/\alpha$ which is* *independent of* $(\Omega, \mathcal{F}, \mathcal{F}_t, P^x)$. *Then*

$$u_{\tau(\lambda)}(x, y) = \mathbb{E}^x_\lambda(L^y_{\tau(\lambda)}) = u_{T_0}(x, y) + 1/\alpha \tag{4.7}$$

where $P^x_\lambda := P^x \times P_\lambda$.

Proof of Lemma 4.6 See page 47 of [21] for the proof.

Theorem 4.7 (Second Ray-Knight Theorem) *Let* $t > 0$. *The under the measure* $P^0 \times P_B$,

$$\{L^x_{\tau(t)} + B^2_x; x \geq 0\} = \{(B_x + \sqrt{t})^2; x \geq 0\}, \tag{4.8}$$

on $C(\mathbb{R}_+)$ *where* $\{B_x; x \geq 0\}$ *is a real-valued Brownian motion starting at* 0, *independent of the original Brownian motion (that is, the Brownian motion with local time* $L^r_{\tau(t)}$*). Equivalently, under* P^0, $\{L^r_{\tau(t)}; r \geq 0\}$ *has the law of a* $0 - th$ *order squared Bessel process starting at* t.

Proof of Theorem 4.7 See page 53,54 and 55 of [21] for the proof.

The first and second Ray-Knight theorems are stated in terms of Bessel processes because this is how they often appear in the literature[25].

4.2 Excursion Laws

4.2.1 Poisson Point Processes

In order to define Poisson Point Processes, we first need to introduce the notion of Poisson random measures.

Definition 4.8 (Poisson random measures) Let $(E, \mathcal{E}, \mu)$ be a measurable space with μ a σ-finite measure. A Poisson random measure (PRM) with intensity measure μ is a family of random variables $M = \{M(A), A \in \mathcal{E}\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ such that

- If $B \in \mathcal{E}$ is such that $\mu(B) < \infty$, then $M(B) \sim$Poisson$(\mu(B))$, i.e. $P(M(B) = k) = \frac{\mu(B)^k}{k!} exp\{-\mu(B)\}$, for k=0,1,2...

  If $\mu(B) = \infty$, then $M(B) = \infty$ a.s.

- Let $\{B_i\}$ be a sequence of pairwise disjoint sets of $\mathcal{E}$, $\{M(B_i)\}$ forms a sequence of independent random variables such that

$$\sum_{i=1}^{\infty} M(B_i) = M(\bigcup_{i=1}^{\infty} B_i) \text{ a.s.}$$

Properties of Poisson random measures

- **Superposition property**: Let $\{\mu_n\}$ be a sequence of $\sigma-$finite measures, and $\{M^{(n)}\}$ are independent PRMs with intensity measures $\{\mu_n\}$. If $\mu = \sum_{n \geqslant 1} \mu_n$ is also a $\sigma-$finite measure, then $M = \sum_{n \geqslant 1} M^{(n)}$ is a PRM with intensity measure μ.

- **Splitting property**: Let M be a PRM on $(E, \mathcal{E})$ with intensity μ and $\{B_n\}$ a sequence of pairwise disjoint sets of $\mathcal{E}$, then the restrictions $\{M|_{B_n}\}$ are independent PRMs with intensity $\{\mu(. \cap B_n)\}$.

- **Image property**: Let $f : (E, \mathcal{E}) \to (G, \mathcal{G})$ be a measurable function, μ a $\sigma-$finite measure on $(E, \mathcal{E})$ and γ the image measure of μ by f. If M is a PRM on $(E, \mathcal{E})$ with intensity μ, and γ is also $\sigma-$finite, and if we define $M \circ f^{-1}(C) = M(f^{-1}(C))$ for $C \in \mathcal{G}$. Then $M \circ f^{-1}$ is a PRM with intensity measure γ.

One can construct Poisson measures as follows [29]. First, suppose $\mu(E) < \infty$ and define the probability measure $\rho(B) = \frac{\mu(B)}{\mu(E)}$ for measurable set B in $\mathcal{E}$. Let $\{\xi_n\}$ be a sequence of iid random elements on $(E, \mathcal{E})$ with law $\rho(.) = \frac{\mu(.)}{\mu(E)}$, and N be a Poisson random variable with parameter $\mu(E)$ which is independent of $\{\xi_n\}$. Then the random measure

$$M = \sum_{i=1}^{N} \delta_{\xi_i}$$

where δ_x stands for the Dirac point mass at $x \in E$, is a Poisson measure with intensity ν. The random measure M is a counting measure for any $B \in \mathcal{E}$

$$M(B) = card\{i \leq N : \xi_i \in B\}$$

When μ is $\sigma-$finite, we can construct a Poisson random measure using the splitting property and superposition property.

The following formula connects the expectation of a function of Poisson random measure to the integral involving its intensity measure which can be found in [15].

Proposition 4.9 (Campbell's formula) *Let $f : E \to \mathbb{R}_+$ be a measurable function and M a Poisson random measure with intensity measure μ. Let's us define*

$$\langle M, f \rangle = \int_E f(x) M(dx) \tag{4.9}$$

then

$$\mathbb{E}\big(\exp\{-\langle M, f \rangle\}\big) = \exp\{-\int_E (1 - e^{-f(x)})\mu(dx)\} \tag{4.10}$$

Let M be a Poisson random measure on $[0, \infty) \times E$ with intensity measure $\lambda \times \mu$, where λ is the Lebesgue measure on $[0, \infty)$ and μ is a σ finite measure on E

Lemma 4.10 *Almost surely, for all $t \geqslant 0$,*

$$M(\{t\} \times E) = 0 \quad or \quad 1 \tag{4.11}$$

If $M(\{t\} \times E) = 1$ there exists one and only one point $\triangle_t \in E$ such that

$$M \mid_{\{t\}\times E} = \delta_{(t,\triangle_t)}$$

If $M(\{t\} \times E) = 0$ then we define $\triangle_t = \partial$ where ∂ is an isolated additional point.

Definition 4.11 (Poisson Point Process) The process defined by $\triangle = (\triangle_t, t \geqslant 0)$ is a Poisson point process with characteristic measure μ.

Exponential Formula Let f be a measurable function on $E \cup \partial$ with $f(\partial) = 0$ and

$$\int_E |1 - e^{f(x)}|\mu(dx) < \infty$$

We have for every $t \geq 0$

$$\mathbb{E}\big(\exp -\{\sum_{0\leq s\leq t} f(\triangle(s))\}\big) = \exp\{-\int_0^t \int_E \Big(1 - e^{-f(x)}\Big)\mu(dx)ds\} \tag{4.12}$$

Proof Exponential formula comes directly from the Campbell's formula.

Lemma 4.12 *Let $B \in \mathcal{E}$ such that $0 < \mu(B) < \infty$. The first entrance time of $\triangle$ into B, $T_B = \inf\{t \geq 0 : \triangle_t \in B\}$ is a stopping time. Moreover,T_B has an exponential distribution with parameter $\mu(B)$. And the random variable $\triangle_{T_B}$ is independent of T_B and has the law $\mu(. \cap B)/\mu(B)$.*

4.2.2 Excursions

In this subsection, we would like to investigate the distribution of the intervals of time during which a Markov process makes an excursion away from a given point. Here we follow the construction from the work of Bertoin[2]. We will work with the Skorokhod space of *càdlàg* (right continuous and left limit)paths. Let an isolated point ∂ be the cemetery point and

$\zeta = \inf\{t : \omega(t) = \partial\}$ be the life time of a strong Markov process X_t. We consider a set of paths which are right continuous on $[0, \infty)$, have left limit on $(0, \infty)$ and stay at the cemetery point after life time.

$$\Omega' = \mathcal{D}\big([0, \infty), \mathcal{E} \cup \{\partial\}\big).$$

Let $\mathcal{L} = \{t : X_t = 0\}$ be the zero set of X. Since X is right continuous, if $t_n \in \mathcal{L}$ and $t_n \downarrow t$ then $t \in \mathcal{L}$. It means every point of $cl(\mathcal{L})\backslash\mathcal{L}$ is isolated from the right. Since $cl(\mathcal{L})^c$ is open, it is a countable union of disjoint open intervals. It follows that $\mathcal{L}^c$ is a countable union of disjoint open intervals of the form (u, v) or $[u, v)$. Every of such interval is associated with an excursion of X.

The excursion interval (g, d) is an open interval which is maximal between open intervals on which $X_t \neq 0$. The left end point of the excursion interval $g \in cl(\mathcal{L})$, the right end point $d \in cl(\mathcal{L}) \cup \{\infty\}$, and $l = d - g$ is the length of the excursion interval.

The excursion of X away from 0 is the piece of path of the type $(X_{g+t} : 0 \leq t \leq d - g)$ corresponding to each excursion interval (g, d). For each $h > 0$, denote by U^h the set of excursions with length $l > h$ and by $U = \cup_{h>0} U^h$ the space of excursions. The number of excursions in U^h is denoted as κ_h.

Now put $\hat{h} = \inf\{h > 0; k_h = 0 \quad a.s\}$. For any $h \in (0, \hat{h})$, we have $\kappa_h \geq 1 \quad a.s.$, and we may define n_h as the distribution of the first excursion in U^h. It means n_h is the probability measure on U^h corresponding to the law of the process $\{X_{g_1(h)+t}; 0 \leq t < l_1(h)\}$ under P. Here $g_1(h)$,$l_1(h)$ are the left end point and the right end point, respectively, of the first excursion interval with length $l > h$.

The next result shows how n_h can be combined in to a single measure n on U, so call the excursion law of X[13]. Let $n(.|A) = n(. \cap A)/n(A)$ whenever $0 < n(A) < \infty$.

Lemma 4.13 (Excursion Law, Itô) *There exist a measure n on U such that $n(U^h) \in (0, \infty)$ and $n_h = n(.|U^h)$ for every $h \in (0, \hat{h})$. Furthermore n is unique up to a normalization, and it is bounded iff the recurrence time is a.s. positive.*

Proof of Lemma 4.13 The proof can be found at page 435 of [13].

In order to define the excursion process, we need to introduce the inverse local time.

The inverse Local time The local time L of a Markov process has its right-continuous inverse

$$\tau(t) = \inf\{s \geq 0 : L_s^0 > t\}, \quad t \geq 0$$

and its left-continuous inverse

$$\tau(t^-) = \inf\{s \geq 0 : L_s^0 \geq t\}, \quad t \geq 0$$

Remark 4.14

- For every $t \geq 0$,$\tau(t)$ and $\tau(t^-)$ are stopping times.
- The process $\tau(t)$ is increasing, right-continuous and adapted to the filtration $(\mathcal{F}_{\tau(t)})$.
- We have a.s for all $t > 0$,

  $$\tau(L_t^0) = \inf\{s > t : X_s = 0\}$$

  and

  $$\tau\big((L_t^0)^{-1}\big) = \sup\{s < t : X_s = 0\}$$

In the next step, we will introduce the **excursion process** of X. Let $U_\partial = U \cup \{\partial\}$ where ∂ is a cemetery point. The excursion process $e = \{e_t : t \geq 0\}$ of X takes the values on U_∂ is defined as:

$$e_t = \begin{cases} \big(X_{s+\tau(t^-)} : 0 \leq s < \tau(t) - \tau(t^-)\big) & if \quad \tau(t^-) < \tau(t) \\ \partial \quad \text{otherwise.} \end{cases} \tag{4.13}$$

The following theorem describes the excursion process due to Itô [12].

Theorem 4.15 (Itô)

- *If 0 is recurrent, the excursion process e_t is a Poisson point process with characteristic measure n.*
- *If 0 is transient, then $e = (e_t, 0 \leq t \leq L(\infty))$ is a Poisson point process with characteristic measure n, stopped at the first point in U^∞, the space of excursions with infinite length.*

Proof of Itô theorem: See [2], page 118 for the proof.

Proposition 4.16 *If f is a $B(R_+) \otimes U$ measurable such that*

$$\int_0^\infty \int |f(s,u)| n(du) ds < \infty \text{ and } f \geq 0$$

then

$$\mathbb{E}\Big(exp\{-\sum_{0<s\leq x} f(s, e_s)\}\Big) = exp\{-\int_0^x \int \Big(1 - e^{-f(s,u)}\Big) n(du) ds\} \tag{4.14}$$

Proof This Proposition comes directly from the exponential formula.

The Excursion process of Brownian Motion

Let W be the Wiener space, P be the Wiener measure and $\mathcal{F}$ be the Borel $\sigma-$ field. For $w \in W$, we set:

$$R(w) = \inf\{t > 0 : w(t) = 0\}.$$

The space U is the set of these functions w such that $0 < R(w) < \infty$ and $w(t) = 0$ for every $t \geq R(w)$. A point δ is cemetery as usual. An excursion of Brownian motion either lies above or below the time axis, and we will call U_+ and U_- as corresponding subsets of

U. The restrictions of the Itô measure n on U_+ and U_- are n_+ and n_-, respectively.

Starting for a Brownian motion B, we can define the excursion process of B. Conversely, if the excursion process is known, we may recover B.

Proposition 4.17 *We have*

$$\tau(t)(w) = \sum_{s \le t} R(e_s(w)) \qquad \tau(t^-)(w) = \sum_{s<t} R(e_s(w))$$

and

$$B_t = e_s(t - \tau(s^-)) \quad \textit{if} \quad \tau(s^-) \le t \le \tau(s).$$

Proof. The first two formulas are consequences of the fact that $\tau(t) = \sum_{s \le t}(\tau(s) - \tau(s^-))$. For the third one, we observe that if $\tau(s^-) \le t \le \tau(s)$ for some s, then by the definition of the excursion process

$$e_s(u) = B_{u+\tau(s^-)} \quad \text{for} \quad 0 \le u < \tau(s) - \tau(s^-)$$

or

$$B_t = e_s(t - \tau(s^-))$$

□

4.3 Representation of Lévy measures of squared Bessel Processes

Denote the Lévy measure of a squared Bessel process with dimension d starting from x by $\nu^{(d,x)}$. By additivity property of squared Bessel processes observed by Shiga and Watanabe, we have:

$$\nu^{(d,x)} = d\nu^{(1,0)} + x\nu^{(0,1)}. \tag{4.15}$$

Hence, it is sufficient to describe only Lévy measures of $BESQ^0(1)$ and $BESQ^1(0)$. The description of these Lévy measures on $C(\mathbf{R}_+)$ are found in Pitman and Yor[23], Mansuy and Yor [19].

Representation of Lévy measure of $BESQ^0(1)$

Let U_+ be the positive excursion space of Brownian motion and n_+ be the Itô measure on U_+. The Lévy measure ν of $BESQ^0(x)$

$$\nu = (x \otimes n_+) \circ S^{-1}, \tag{4.16}$$

where $S_t(u) = L^t_\infty(u)$, $u \in U_+$.

Then Lévy measure of $BESQ^0(1)$ has the form: $\nu^{(0,1)} = (n_+) \circ S^{-1}$.

Proof

Proof. By the second Ray-Knight Theorem: $(L^t_{\tau_x}(B), t \geq 0) = BESQ^0_x$ in law. Let Y_t be the square of a zero dimensional Bessel process starting from x.

We have

$$\begin{aligned} L^t_{\tau_x}(B) = \int_0^{\tau_x} \delta_t(B_r)dr = \sum_{0<s\leq x} \int_{\tau_{s-}}^{\tau_s} \delta_t(B_r)dr \\ = \sum_{0<s\leq x} \int_0^{R(e_s)} \delta_t(e_s(r))dr. \end{aligned} \tag{4.17}$$

Then,

$$\begin{aligned} \mathbb{E}\Big(\exp\{-\int_0^\infty Y_t\rho(dt)\}\Big) =& \mathbb{E}\Big(\exp\{-\int_0^\infty L^t_{\tau_x}(B)\rho(dt)\}\Big) \\ =& \mathbb{E}\Big(\exp\{-\sum_{0<s\leq x}\int_0^\infty \int_0^{R(e_s)} \delta_t(e_s(r))dr\rho(dt)\}\Big) \end{aligned} \tag{4.18}$$

Where ρ is a $\sigma-$finite measure on $\mathbb{R}_+$. For examples, $\rho(dt) = f(t)dt$ or $\rho(dt) = \sum_{i=1}^{n} \lambda_i \delta_{t_i}(dt)$.

Let $F(s, e_s) = \int_0^\infty \int_0^{R(e_s)} \delta_t(e_s(r)) dr \rho(dt)$, then

$F(s, u) = \int_0^\infty \int_0^{R(u)} \delta_t(u(r)) dr \rho(dt) = \int_0^\infty L^t_{R(u)}(u)\rho(dt)$

Apply the proposition 4.16, we have :

$$\begin{aligned}
\mathbb{E}\Big(\exp\{-\int_0^\infty Y_t \rho(dt)\}\Big) =& \mathbb{E}\Big(\exp\{-\sum_{0<s\leq x} \int_0^\infty \int_0^{R(e_s)} \delta_t(e_s(r)) dr \rho(dt)\}\Big) \\
=& \exp\Big\{ -\int_0^x \int_U \Big(1 - \exp(-\int_0^\infty L^t_{R(u)}(u)\rho(dt))\Big) n(du) ds\Big\} \\
=& \exp\Big\{ -x \int_U \Big(1 - e^{-\int_0^\infty L^t_{R(u)}(u)\rho(dt)}\Big) n(du)\Big\} \\
=& \exp\Big\{ -x \int_{U_+} \Big(1 - e^{-\int_0^\infty L^t_{R(u)}(u)\rho(dt)}\Big) n_+(du)\Big\}
\end{aligned} \tag{4.19}$$

Since $L^t_{R(u)} = 0$ when $t > 0$ and $u \in U_-$.

Therefore, $BESQ^0(x)$ is an infinitely divisible process with the triplet $(0, \nu, 0)$ and

$$\nu = (x \cdot n_+) \circ S^{-1},$$

where $S_t(u) = L^t_\infty(u)$, $u \in U_+$.

Or, we can say that the representation of Lévy measure ν is $S_t(u) = L^t_\infty(u)$ on the measure space $(U_+, \mathcal{U}, x \cdot n_+)$. □

It is easy to see that the Lévy measure of $BESQ^0(1)$ has the form: $\nu^{(0,1)} = (n_+) \circ S^{-1}$.

Representation of Lévy measure of $BESQ^1(0)$

Let U_+ be the positive excursion space of Brownian motion and n_+ be the Itô measure on U_+. The Lévy measure ν of $BESQ^2(0)$ has the form:

$$\nu = (2\lambda \otimes n_+) \circ V^{-1},$$

where $V_t(s,u) = L_\infty^{t-s}(u)$, $u \in U_+$.

Then Lévy measure of $BESQ^1(0)$ has the form: $\nu^{(1,0)} = (\lambda \otimes n_+) \circ V^{-1}$.

Proof

Proof. By the variance of the First Ray-Knight Theorem: $L_\infty^t(|B| + L^0) = BESQ_0^2$ in law.

Let Y_t be the square of a two dimensional Bessel process starting from 0.

We have:

$$\begin{aligned} L_\infty^t(|B| + L^0) &= \int_0^\infty \delta_t(|B_r| + L_r^0)dr = \sum_{0<s<\infty} \int_{\tau_{s-}}^{\tau_s} \delta_t(|B_r| + L_r^0)dr \\ &= \sum_{0<s<\infty} \int_0^{R(e_s)} \delta_t(|e_s(r)| + s)dr \end{aligned} \tag{4.20}$$

Let $G(s,e_s) = \int_0^\infty \int_0^{R(e_s)} \delta_t(|e_s(r)| + s)dr\rho(dt)$,

then $G(s,u) = \int_0^\infty \int_0^{R(u)} \delta_t(|u(r)| + s)dr\rho(dt) = \int_0^\infty L_{R(u)}^{t-s}(|u|)\rho(dt)$.

$$\begin{aligned} \mathbb{E}\Big(\exp\{-\int_0^\infty Y_t\rho(dt)\}\Big) =& \mathbb{E}\Big(\exp\{-\int_0^\infty L_\infty^t(|B| + L^0)\rho(dt)\}\Big) \\ =& \mathbb{E}\Big(\exp\{-\sum_{0<s\le x}\int_0^\infty \int_0^{R(e_s)} \delta_t(|u(r)| + s)dr\rho(dt)\}\Big) \\ =& \mathbb{E}\Big(\exp\{-\sum_{0<s<\infty} G(s,e_s)\}\Big) \\ =& \exp\Big\{-\int_0^\infty \int_U \Big(1 - \exp(-\int_0^\infty L_{R(u)}^{t-s}(|u|)\rho(dt))\Big)n(du)ds\Big\} \end{aligned} \tag{4.21}$$

$$= \exp\Big\{-\int_0^\infty \int_{U_+} \Big(1 - e^{(-\int_0^\infty L_{R(u)}^{t-s}(u)\rho(dt))}\Big)n_+(du)ds$$

$$- \int_0^\infty \int_{U_-} \Big(1 - e^{(-\int_0^\infty L_{R(u)}^{t-s}(-u)\rho(dt))}\Big) n_-(du)ds \Big\}$$

$$= \exp\Big\{ -2 \int_0^\infty \int_{U_+} \Big(1 - e^{(-\int_0^\infty L_{R(u)}^{t-s}(u)\rho(dt))}\Big) n_+(du)ds \Big\} \qquad (4.22)$$

Therefore, $BESQ^2(0)$ is an infinitely divisible process with the triplet $(0, \nu, 0)$ and

$$\nu = (2\lambda \otimes n_+) \circ V^{-1},$$

where $V_t(s,u) = L_\infty^{t-s}(u)$, $u \in U_+$.

Or, we can say that the representation of Lévy measure ν is $V_t(s,u) = L_\infty^{t-s}(u)$ on the measure space $(\mathbb{R}_+ \otimes U_+, \mathcal{B} \otimes \mathcal{U}, 2\lambda \otimes n_+)$. □

It is easy to see that the Lévy measure of $BESQ^1(0)$ has the form: $\nu^{(1,0)} = (\lambda \otimes n_+) \circ V^{-1}$.

Chapter 5

Isomorphism Identities

In this Chapter, we investigate isomorphism identities based on random translations. The very first form of these identities is the celebrated Dynkin's Isomorphism. We first provide the preliminaries of the $h-$ transforms which plays an important role in isomorphism identities.

5.1 The h-Transforms

The h-Transforms

The "h-transforms" is an important class of transformations of Markov processes. It is also known as "h-path processes" or "superharmonic transforms". J.D.Doob [4]was the first one introduced the "h-transforms" in his study of the boundary limits of Brownian motion. An intuitive interpretation of the "h-transforms" are: they have to do with conditioning the process on its behavior at its lifetime.

We are going to look at a motivating example which is found at page 178 of [1]. Let D be a domain in $\mathbb{R}^d$ and let X_t be a Brownian motion killed on exiting the domain. We would like to give a precise meaning to the intuitive notion of Brownian motion conditioned to exit the domain at a certain point. Let h be a positive harmonic function in D (i.e., h is C^2 in D, and $\triangle h = 0$ there) and suppose that h is 0 everywhere on the boundary of D except at one point z. By the Markov property at time t, we have:

$$\mathbb{P}^x(X_t \in dy | X_{\tau_D=z}) = \frac{\mathbb{P}^x(X_t \in dy, X_{\tau_D=z})}{\mathbb{P}^x(X_{\tau_D=z})}$$
$$= \frac{\mathbb{P}^x(X_t \in dy)\mathbb{P}^y(X_{\tau_D} = z)}{\mathbb{P}^x(X_{\tau_D=z})} \tag{5.1}$$

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$.

If $p_t^0(x, dy)$ represents the probability that Brownian motion started at x and killed on leaving D is in dy at time t, we then have the probability for Brownian motion conditioned to exit D at z having the form as $\frac{h(y)}{h(x)} p_t^0(x, dy)$.

As before, let E_∂ be a locally compact metric space E with an isolated "cemetery point" ∂ adjoined. Let $\{X_t, t \in T\}$ be a strong Markov process the state space E_∂ and transition semigroup (P_t). A function h is invariant with respect to X if $P_t h(x) = h(x)$ for all t and x. If h is invariant, then by the Markov property we have

$$\mathbb{E}^x\big(h(X_t)|\mathcal{F}_s\big) = \mathbb{E}^x\big(h(X_{t-s} \circ \theta_s | \mathcal{F}_s)\big) = \mathbb{E}^{X_s} h(X_{t-s})$$
$$= P_{t-s} h(X_s) = h(X_s). \tag{5.2}$$

We observe that for each x, $h(X_t)$ is a martingale with respect to $\mathbb{P}^x$. Conversely, if $h(X_t)$ is a martingale with respect to $\mathbb{P}^x$ for all x,

$$P_t h(x) = \mathbb{E}^x h(X_t) = h(x) \tag{5.3}$$

by the definition of martingale, and so h is invariant.

Remark 5.1: The invariant functions with respect to a Brownian motion killed on leaving a domain are the harmonic function.

Let h be a non-negative invariant function for a strong Markov process X_t with transition probabilities $\{p_t(x, y)\}$ and transition semigroup (P_t), and let $E_h = \{x : 0 < h(x) < \infty\}$.

Here we follow the construction of the h-transforms of Chung and Walsh in [3].

Definition 5.2

$$p_t^h(x,y) = \begin{cases} \frac{h(y)}{h(x)} p_t(x,y), & \text{if } x \in E_h \\ 0, & \text{if } x \in E - E_h \end{cases} \tag{5.4}$$

Remark 5.3:

1. We have

$$P_t^h f(x) = \int_E p_t^h(x,y) f(y) dm(y) = \begin{cases} \int_E \frac{h(y)}{h(x)} p_t(x,y) f(y) dm(y), & \text{if } x \in E_h \\ 0, & \text{if } x \in E - E_h \end{cases}$$

$$= \begin{cases} \frac{1}{h(x)} P_t(fh)(x), & \text{if } x \in E_h \\ 0, & \text{if } x \in E - E_h \end{cases} = \frac{\mathbb{1}_{E_h}(x)}{h(x)} P_t(fh)(x) \tag{5.5}$$

   where we make the convention that $0.\infty = 0$

2. We have $p_t^h(x, E - E_h) = 0$ for all x and $t \geq 0$.

   If $x \in E - E_h$, this is true by (5.4).

   If $x \in E_h$, $P_t h(x) = h(x) < \infty$, so

$$p_t^h(x, \{h = \infty\}) = \frac{1}{h(x)} \int_{\{h=\infty\}} p_t(x,y) h(y) dm(y) \leq \frac{1}{h(x)} P_t h(x) = 1.$$

   But the integral can only be 0 or ∞, hence $p_t^h(x, \{h = \infty\}) = 0$. Thus,

$$p_t^h(x, E - E_h) = p_t(x, h = 0) = \frac{1}{h(x)} \int_{\{h=0\}} p_t(x,y) h(y) dm(y) = 0.$$

Proposition 5.4 *(P_t^h) is a Markov semigroup on E.*

Proof: see [3], proposition 11.5 page 322 for the proof.

Let $A \in \mathcal{F}_t$, then the probability measure corresponds to (P_t^h) is

$$\mathbb{P}^{x/h}(A) = \int_A \frac{h(X_t(\omega))}{h(x)} p_t(x, X_t(\omega)) d\omega = \mathbb{E}^x(\frac{h(X_t)}{h(X_0)}; A) \tag{5.9}$$

Let $M_t = \frac{h(X_t)}{h(X_0)}$, then because of the invariance property of function h, M_t is a non-negative continuous martingale with $M_0 = 1$, as long as $h(x) > 0$. We also observe that $\mathbb{P}^{x/h}$ gives more mass to paths where $h(X_t)$ is big and less where it is small.

Take some functional F which depends only on the path up to time t, we have:

$$\mathbb{E}^{x/h}(F) = \int_\Omega F(\omega)\mathbb{P}^{x/h}(d\omega) = \int_\Omega F(\omega)\frac{h(X_t(\omega))}{h(x)}\mathbb{P}^x(d\omega) = \frac{1}{h(x)}\mathbb{E}^x(Fh(X_t)) \tag{5.10}$$

The h-transform for a strong Markov process X when $h(x) = \frac{u(x,0)}{u(0,0)}$

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ be a strong symmetric Markov process with transition semigroup $\{P_t : t \geq 0\}$ and continuous 0-potential density $u(x, y)$. Let 0 be a fixed element in E and assume that

$$h(x) = \mathbb{P}^x(T_0 < \infty) = \frac{u(x, 0)}{u(0, 0)} > 0 \tag{5.11}$$

for all $x \in E$.

We are going to construct the process X conditioned to hit 0 and die at its last exist from 0 which turns out to be the h-transform of X. This construction follows the work of Marcus and Rosen which can be found in 3.9 of [21]. Let

$$\mathcal{L} := \sup\{s | X_s = 0\} \tag{5.12}$$

with $\sup \emptyset = 0$, and

$$\zeta := \inf\{t > 0 | X_t = \partial\}. \tag{5.13}$$

ζ is the "death time" of X.

We assume that Ω is the space of right continuous E_∂-valued functions $\{\omega(t), t \in [0,\infty)\}$ such that $\omega(t) = \partial$ for all $t \geq \zeta(\omega)$. Furthermore, we assume that $\mathcal{L}(\omega) < \infty$ for all $\omega \in \Omega$ and that $\mathcal{L}(\omega)$ is a left limit point of zeros of $\omega(t)$ on $\{\mathcal{L}(\omega) > 0\}$.

We define the killing operator $k_\mathcal{L} : \Omega \mapsto \Omega$ as

$$k_\mathcal{L}(\omega)(s) = \begin{cases} \omega(s), & \text{if } s < \mathcal{L}(\omega) \\ \partial, & \text{if } s \geq \mathcal{L}(\omega). \end{cases} \tag{5.14}$$

Let $\Omega_h = k_\mathcal{L}(\Omega)$. We then have

$$\Omega_h = \{\omega \in \Omega | \zeta(\omega) = \mathcal{L}(\omega)\}. \tag{5.15}$$

Note that $\{\mathcal{L} > 0\} = \{T_0 < \infty\}$ and, more generally, $\{\mathcal{L} > s\} = \theta_s\{T_0 < \infty\}$. For any $f \in \mathcal{B}(E_\partial)$,

$$f(X_t) \circ k_\mathcal{L} := f(k_\mathcal{L}(X(t))) = f(X_t)\mathbb{1}_{t<\mathcal{L}} + f(\partial)\mathbb{1}_{t\geq\mathcal{L}}, \tag{5.16}$$

which implies that $f(X_t) \circ k_\mathcal{L}$ is $\mathcal{F}$ measurable. Thus, for any $a < b$, we have $k_\mathcal{L}^{-1}(\{a \leq f(X_t) \leq b\}) = (f(X_t) \circ k_\mathcal{L})^{-1}[a,b] \in \mathcal{F}$. Since the set of the form $\{a \leq f(X_t) \leq b\}$ generate $\mathcal{F}^0$, it follows that $k_\mathcal{L}^{-1} : \mathcal{F}^0 \mapsto \mathcal{F}$.

We can define the probability measure $\{\bar{\mathbb{P}}^x; x \in E\}$ on $(\Omega_h, \mathcal{F}^0)$ by setting

$$\bar{\mathbb{P}}^x(A) = \frac{1}{h(x)}\mathbb{P}^x\big(k_\mathcal{L}^{-1}(A) \cap \{\mathcal{L} > 0\}\big) \qquad A \in \mathcal{F}^0. \tag{5.17}$$

Then, we have

$$\bar{\mathbb{E}}^x(F) = \frac{1}{h(x)}\mathbb{E}^x(F \circ k_\mathcal{L}\mathbb{1}_{\mathcal{L}>0}) \tag{5.18}$$

for any positive $\mathcal{F}^0$ measurable function F. (5.17) is just the special case of (5.18) in the case $F = \mathbb{1}_A$ with $A \in \mathcal{F}^0$.

Consider $\bar{\mathbb{E}}^x(F)$ for $F = f_1(X_{t_1})...f_n(X_{t_n})$ with $t_1 < ... < t_n$, where $f_i \in \mathcal{B}(S)$, $i = 1, ..., n$ and recall our convention that functions on E are extended to E_∂ by setting $f_i(\partial) = 0$. Thus

$$F \circ k_{\mathcal{L}} = F \mathbb{1}_{\{\mathcal{L} > t_n\}} = F \mathbb{1}_{\{\mathcal{L} > 0\}} \circ \theta_{t_n} \tag{5.19}$$

Following from (5.18),(5.19) and the Markov property for X that

$$\begin{aligned}\bar{\mathbb{E}}^x(F) =& \frac{1}{h(x)} \mathbb{E}^x(F \mathbb{1}_{\{\mathcal{L} > t_n\}}) \\ =& \frac{1}{h(x)} \mathbb{E}^x(F \mathbb{P}^{X_{t_n}}(\mathcal{L} > 0)) \\ =& \frac{1}{h(x)} \mathbb{E}^x(F h(X_{t_n})).\end{aligned} \tag{5.20}$$

Using the abbreviation $F_{n-1} = f_1(X_{t_1)}...f_{n-1}(X_{t_{n-1}})$ in (5.20) we have

$$\begin{aligned}\bar{\mathbb{E}}^x(F) =& \frac{1}{h(x)} \mathbb{E}^x(F h(X_{t_n})) \\ =& \frac{1}{h(x)} \mathbb{E}^x(F_{n-1} f_n(X_{t_n}) h(X_{t_n})) \\ =& \frac{1}{h(x)} \mathbb{E}^x(F_{n-1} \mathbb{E}^{X_{t_{n-1}}} f_n(X_{t_n - t_{n-1}}) h(X_{t_n - t_{n-1}})) \\ =& \frac{1}{h(x)} \mathbb{E}^x(F_{n-1} h(X_{t_{n-1}}) \bar{\mathbb{E}}^{X_{t_{n-1}}} \{f_n(X_{t_n - t_{n-1}})\} \\ =& \bar{\mathbb{E}}^x \big(F_{n-1} \bar{\mathbb{E}}^{X_{t_{n-1}}} \{f_n(X_{t_n - t_{n-1}})\}\big).\end{aligned} \tag{5.21}$$

Using (5.21) for the first equality and (5.22) for the second, for any $t_{n-1} < t_n$ and $f_n \in \mathcal{B}(E)$ we have

$$\begin{aligned}\bar{\mathbb{E}}^x(f_n(X_{t_n}) | \mathcal{F}^0_{t_{n-1}}) =& \bar{\mathbb{E}}^{X_{t_{n-1}}}(f_n(X_{t_n - t_{n-1}})) \\ =& \frac{1}{h(X_{t_{n-1}})} \mathbb{E}^{X_{t_{n-1}}}(f_n(X_{t_n - t_{n-1}}) h(X_{t_n - t_{n-1}})) \\ =& \frac{1}{h(X_{t_{n-1}})} \mathbb{P}_{t_n - t_{n-1}} f h(X_{t_{n-1}}).\end{aligned} \tag{5.22}$$

Using (5.22), we can verify the following lemma.

Lemma 5.5 $\bar{X} = (\Omega_h, \mathcal{F}^0, \mathcal{F}^0_t, X_t, \theta_t, \bar{\mathbb{P}}^x)$ *is a right continuous simple Markov process with* *transition semigroup* $\{P_t^h; t \geq 0\}$.

It turns out that process $\bar{X}$ which is X conditioned to hit 0 and die at its last exit from 0 is the h-transform of X with $h(x) = \frac{u(x,0)}{u(0,0)}$. The probability measure $\bar{\mathbb{P}}$ is $\mathbb{P}^{x/h}$.

Let $\bar{U}^\alpha$ be the α-potential of $\bar{X}$. Then, for any $f \in \mathcal{B}(E)$

$$\begin{aligned}\bar{U}^\alpha f(x) &= \int_0^\infty e^{-\alpha t} P_t^h f(x) dt \\ &= \frac{1}{h(x)} \int_0^\infty e^{-\alpha t} P_t f h(x) dt \\ &= \frac{1}{h(x)} U^\alpha f h(x) \\ &= \frac{1}{h(x)} \int u^\alpha(x,y) h(y) f(y) dm(y)\end{aligned} \tag{5.23}$$

Then $\bar{X}$ has α-potential density $\frac{1}{h(x)} u^\alpha(x,y)h(y)$. It is easy to see that α-potential densities of $\bar{X}$ is continuous on $E \times E$ for each α. The problem we have here is that α-potential of $\bar{X}$ is not symmetric which means that $\bar{X}$ is not strongly symmetric with respect to the reference measure m. In the next step, we are going to change the reference measure in order to have symmetric α-potential densities.

Let$\mathcal{F}^h, \mathcal{F}_t^h$ denote the standard augmentation of $\mathcal{F}^0, \mathcal{F}_t^0$ under $\{\mathbb{P}^{x/h}; x \in E\}$ and let $\tilde{X} = (\Omega_h, \mathcal{F}^h, \mathcal{F}_t^h, X_t, \theta_t, \mathbb{P}^{x/h})$. We define the measure $\tilde{m}$ as $\tilde{m}(dy) := h^2(y)dm(y)$. Then,$\tilde{X}$ is a strong Markov process and has α-potential densities

$$\tilde{u}^\alpha(x,y) := \frac{u^\alpha(x,y)}{h(x)h(y)} \tag{5.24}$$

with respect to the $\tilde{m}(dy)$. It is obvious that the potential density $\tilde{u}^\alpha$ is symmetric with respect to $\tilde{m}$.

In fact, $\bar{X}$ and $\tilde{X}$ are the same Markov process. They simply have different α-potentials with respect to different reference measures. They are both called the h-transform of X. $\tilde{X}$

has a local time $\tilde{L} = \{\tilde{L}_t^y, (y,t) \in E \times \mathbb{R}_+\}$ that satisfies

$$\mathbb{E}^{x/h}\big(\int_0^\infty e^{-\alpha t} d\tilde{L}_t^y\big) = \tilde{u}^\alpha(x,y). \tag{5.25}$$

We know that $\bar{L}_t^y := g(y)\tilde{L}_t^y$ is also a local time of $\tilde{X}$ at y, as long as $g(y) > 0$ and

$$\mathbb{E}^{x/h}\big(\int_0^\infty e^{-\alpha t} d\bar{L}_t^y\big) = g(y)\tilde{u}^\alpha(x,y). \tag{5.26}$$

Let $g(y) = h^2(y) > 0$. Then

$$\bar{L}_t^y := h^2(y)\tilde{L}_t^y \qquad \forall t \in \mathbb{R}_+ \tag{5.27}$$

and

$$\mathbb{E}^{x/h}\big(\int_0^\infty e^{-\alpha t} d\bar{L}_t^y\big) = h^2(y)\tilde{u}^\alpha(x,y) = \frac{1}{h(x)} u^\alpha(x,y)h(y) \tag{5.28}$$

which is the $\alpha-$potential density of $\bar{X}$.

Remark 5.6 The role of the killing operator in the definition of probability measure $\mathbb{P}^{x/h}$ on $(\Omega_h, \mathcal{F}^0)$ justifies our interpretation of $\tilde{X}$ as the paths of X conditioned to hit 0 and die on their last exit from 0. For this reason $\mathbb{P}^{x/h}$ is often written as $\mathbb{P}^{x,0}$.

5.2 The Dynkin Isomorphism Theorems

In this section, we are going to explore the relationship between a strong symmetric Markov process and its associated mean zero Gaussian process G (the Gaussian process with covariance is the 0-potential of X). This relationship is described in several isomorphism theorems that relate the Markov local times and squares of G.

One of them is the isomorphism theorem due to E. B. Dynkin[6]. It relates local times of the h-transform of X to the squares of G. Interestingly, Dynkin Isomorphism theorem has its roots in mathematical physics. It is built from an effort to explain heuristic methods in quantum field theory of K. Symanzik[31].

There are many different ways to prove the Dynkin Isomorphism. We find the combinatoric one is beautiful since it only use combinatorial arguments and calculation of moments. The combinatorial approach is also used to prove other isomorphisms such as Eisenbaum Isomorphism and Generalized Second Ray-Knight Theorem. In this section we provide the proof, both for its intrinsic interest and because understanding them may be fruitful.

We discuss several basic Gaussian moment formulas[21] which serve as the main ingredients in combinatorial proof. In the following formulas $G = \{G_x, x \in E\}$ is a centered Gaussian process with covariance function C.

Formula 1

$$\mathbb{E}\Big(\prod_{i=1}^{n} G_{x_i}\Big) = \sum_{p\in\mathcal{R}_n} \prod_{(i_1,i_2)\in p} C(x_{i_1}, x_{i_2}) \tag{5.29}$$

where $\mathcal{R}_n$ is the set of pairing p of the indices $[1, n]$, and the product runs over all pairs in p.

Proof: See [26], page 6 for the original proof.

Case 1: n is odd. Then $\mathcal{R}_n$ is empty. Formula (5.1) is true since the left hand side is zero by symmetry.

Case 2: n is even.

The characteristic function of Gaussian vector $(G_{x_1}, ..., G_{x_n})$ where $x_1, ..., x_n \in E$ has the form:

$$\mathbb{E}\exp\Big(i\sum_{j=1}^{n} z_j G_{x_j}\Big) = \exp\Big(-\sum_{j,k=1}^{n} z_j z_k C(x_j, x_k)/2\Big). \tag{5.30}$$

We take the derivative of (5.2) with respect to z_1 and then set $z_1 = 0$ to obtain:

$$i\mathbb{E}\Big(G_{x_1}\exp(i\sum_{j=2}^{n} z_j G_{x_j})\Big) = \Big(-\sum_{k=2}^{n} z_k C(x_1, x_k)\Big)\exp\Big(-\sum_{j,k=2}^{n} z_j z_k C(x_j, x_k)/2\Big). \tag{5.31}$$

We then take the derivative of (6.13) with respect to z_2 and then set $z_2 = 0$ to obtain:

$$-\mathbb{E}\big(G_{x_1}G_{x_2}\exp(i\sum_{j=3}^{n} z_jG_{x_j})\big)$$

$$= \big(-\sum_{k=3}^{n} z_kC(x_1,x_k)\big)\big(-\sum_{k=3}^{n} z_kC(x_2,x_k)\big)\exp\big(-\sum_{j,k=3}^{n} z_jz_kC(x_j,x_k)/2\big)$$

$$-C(x_1,x_2)\exp\big(-\sum_{j,k=3}^{n} z_jz_kC(x_j,x_k)/2\big). \tag{5.32}$$

By continuing this process with $z_3, ..., z_n$, we obtain the left hand side

$$\mathbb{E}\big(G_{x_1}G_{x_2}..G_{x_n}\big)$$

and the right hand side

$$\sum_{p\in\mathcal{R}_n}\prod_{(i_1,i_2)\in p} C(x_{i_1},x_{i_2})$$

which prove formula 1. □

Formula 2

$$\mathbb{E}\big(\prod_{i=1}^{n} G_{x_i}^2\big) = \sum_{A_1\cup...\cup A_j=[1,n]}\prod_{l=1}^{j} 2^{|A_l|-1}cy(A_l), \tag{5.33}$$

where the sum is over all (unordered) partition $A_1\cup ... \cup A_j$ *of* $[1,n]$ *and, if we have* $A_l = \{l_1, l_2, ..., l_{|A_l|}\}$ *the the cycle function* $cy(A_l)$ *is defined as*

$$cy(A_l) = \sum_{\pi\in\mathcal{P}^{\odot}_{|A_l|}} C(x_{l_{\pi(1)}},x_{l_{\pi(2)}})...C(x_{l_{\pi(|A_l|)}},x_{l_{\pi(1)}}), \tag{5.34}$$

where $\mathcal{P}^{\odot}_k$ *denotes the set of permutation of* $[1,k]$ *on the circle. Note that* $(1,2,3)$, $(3,1,2)$ *and* $(2,3,1)$ *are considered to be the same permutation* $\pi\in\mathcal{P}^{\odot}_3$.

Proof: See [26], page 6 for the original proof.

We are going to use the first formula to explore $\mathbb{E}\big(\prod_{i=1}^{n} G_{x_i} G_{x_i^{'}}\big)$. Eventually, we will set $x_i = x_i^{'}$ to get the left hand side of (5.34). We consider the paring of $2n$ indices $\{x_i\}_{i=1}^{n} \cup \{x_i^{'}\}_{i=1}^{n}$. First, let consider x_1. If x_1 is paired with $x_1^{'}$, we set $A_1 = \{1\}$ in which case $cy(A_1) = C(x_1, x_1^{'})$ and eventually equals to $C(x_1, x_1)$. This set $A_1 = \{1\}$ is giving a factor $2^{|A_1|-1} cy(A_1)$.

Let $D^{(1)}$ be the pair contains x_1. If x_1 is paired with either x_i or $x_i^{'}$ with $i \neq 1$, set $\pi(1) = i$ and define $(y_{\pi(1)}, z_{\pi(1)})$ to be $(x_i, x_i^{'})$ if x_1 is paired with x_i, but $(x_i^{'}, x_i)$ if x_1 is paired with $x_i^{'}$. It follows that $D^{(1)} = \{x_1, y_{\pi(1)}\}$. Next, let $D^{(2)}$ be the pair contains $z_{\pi(1)}$. If $z_{\pi(1)}$ is paired with either x_j or $x_j^{'}$, set $\pi(2) = j$ and define $(y_{\pi(2)}, z_{\pi(2)})$ to be $(x_j, x_j^{'})$ if $z_{\pi(1)}$ is paired with x_j, but $(x_j^{'}, x_j)$ if $z_{\pi(1)}$ is paired with $x_j^{'}$. $D^{(2)} = \{z_{\pi(1)}, y_{\pi(2)}\}$. We continue this process to get $D^{(1)}, ..., D^{(l)}$ until we get to $D^{(l+1)}$, the pair contains $z_{\pi(l)}$ and $x_1^{'}$. We set $A_1 = \{1, \pi(1), ..., \pi(l)\}$. The set of pairs $\{D^{(1)}, ..., D^{(l+1)}\}$ is giving a factor

$$\sum C(x_1, y_{\pi(1)}) C(z_{\pi(1)}, y_{\pi(2)}) ... C(z_{\pi(l)}, x_1^{'}) \tag{5.35}$$

where the sum is over all permutations of $\{\pi(1), ..., \pi(l)\}$ and over all ways of assigning $(x_{\pi(i)}, x_{\pi(i)}^{'})$ to $(y_{\pi(i)}, z_{\pi(i)})$. There are 2^l ways to make these assignments. Set $y_{\pi(i)} = z_{\pi(i)} = x_{\pi(i)}$, (5.35) can be written as

$$2^l \sum_{\pi \in \mathcal{P}_{\{\pi(1), ..., \pi(l)\}}} C(x_1, x_{\pi(1)}) C(x_{\pi(1)}, x_{\pi(2)}) ... C(x_{\pi(l)}, x_1).$$

Observe that by adding 1 to a permutation π of $\{\pi(1), ..., \pi(l)\}$ we get a permutation of A_1 on circle and $2^{|A_1|-1} = 2^l$. It follows that the set $A_1 = \{1, \pi(1), ..., \pi(l)\}$ is giving a factor

$$2^{|A_1|-1} \sum_{\pi \in \mathcal{P}^{\odot}_{|A_1|}} C(x_1, x_{\pi(1)}) C(x_{\pi(1)}, x_{\pi(2)}) ... C(x_{\pi(l)}, x_1) = 2^{|A_1|-1} cy(A_1).$$

Continue with some x_k where $k \notin A_1$ we will get (5.5). $\square$

It is useful to write formula 2 as the following form:

$$\mathbb{E}\big(\prod_{i=1}^{n} G_{x_i}^2/2\big) = \sum_{A_1\cup...\cup A_j=[1,n]} \prod_{l=1}^{j} \frac{1}{2} cy(A_l), \tag{5.36}$$

Formula 3

$$\mathbb{E}\big(G_a G_b \prod_{i=1}^{n} G_{x_i}^2/2\big) = \sum_{A\subseteq[1,n]} ch(A;a,b) \sum_{A_1\cup...\cup A_j=[1,n]\backslash A} \prod_{l=1}^{j} \frac{1}{2} cy(A_l) \tag{5.37}$$

where the sum is over all (unordered) partitions $A_1 \cup ... \cup A_j = [1,n]\backslash A$ *and if* $A = \{l_1, l_2, ..., l_{|A|}\}$ *then the chain function* $ch(A;a,b)$ *is defined as*

$$ch(A;a,b) = \sum_{\pi\in\mathcal{P}_{|A|}} C(x_a, x_{l_{\pi(1)}}) C(x_{l_{\pi(1)}}, x_{l_{\pi(2)}})...C(x_{l_{\pi(1)}}, x_b) \tag{5.38}$$

where $\mathcal{P}_k$ *denotes the set of permutations of* $[1,k]$*.*

Proof: See [26], page 7 for the original proof.

To see this we use the previous procedure but start with the pair contains a and end with the pair contain b. We have the set of pairs $\{D^{(1)}, ..., D^{(l+1)}\}$ with $l \le n$ where $D^{(1)} = \{a, y_{\pi(1)}\}$, $D^{(2)} = \{z_{\pi(1)}, y_{\pi(2)}\}$ ... and $D^{(l+1)} = \{z_{\pi(l)}, b\}$.

Let $A(\mathcal{D}) = \{\pi(1), ..., \pi(l)\}$ and $\mathcal{D} = \{D^{(1)}, ..., D^{(l+1)}\}$. $\mathcal{D}$ is a pairing of the $2l+2$ elements $\{x_i\}_{i\in C(\mathcal{D})}$, $\{x_i^{'}\}_{i\in C(\mathcal{D})}$, a and b. Let $B(\mathcal{D}) = \{1,2,...,n\}/C(\mathcal{D})$ and $\mathcal{F} = \{F^{(1)}, ..., F^{(l-1)}\}$ is a pairing of the set of $2(n-l)$ indices consisting of $\{x_i\}_{i\in B(\mathcal{D})}$ and $\{x_i^{'}\}_{i\in B(\mathcal{D})}$. Using the first formulas, we have

$$\mathbb{E}\big(G_a G_b \prod_{i=1}^{n} G_{x_i} G_{x_i^{'}}/2\big) = \frac{1}{2^n} \sum_{A\cup B=\{1,...,n\}} \Big(\sum_{\text{parings of}\{x_i\}_{i\in B}\cup\{x_i^{'}\}_{i\in B}} C(F^1)...C(F^{|B|})\Big)$$

$$\times \sum C(a, y_{\pi(1)}) C(z_{\pi(1)}, y_{\pi(2)}) ... C(z_{\pi(i)}, y_{\pi(i+1)}) ... C(z_{\pi(|A|)}, b) \tag{5.39}$$

where the last sum is over all permutation $(\pi(1), ..., \pi(|A|))$ of A, and over all ways to assigning $(x_{\pi(i)}, x'_{\pi(i)})$ to $(y_{\pi(i)}, z_{\pi(i)})$. There are $2^{|A|}$ ways to make these assignments. If we set $y_i = z_i = x'_i = x_i$, the last sum in (6.21) is

$$2^{|A|} \sum_{\pi \in \mathcal{P}_{|A|}} C(a, x_{\pi(1)}) C(x_{\pi(1)}, x_{\pi(2)}) ... C(x_{\pi(i)}, x_{\pi(i+1)}) ... C(x_{\pi(|C|)}, b) \tag{5.40}$$

where the sum is over all permutation π of A.

Using the first formula, we have

$$\sum_{\text{parings of} \{x_i\}_{i \in B} \cup \{x'_i\}_{i \in B}} C(F^1) ... C(F^{|B|}) = \mathbb{E}\big(\prod_{i \in B} G_{x_i} G_{x'_i}\big) \tag{5.41}$$

Therefore, setting $x_i = x'_i$ in (5.39) and using (5.40) and (5.41), we have

$$\mathbb{E}\big(G_a G_b \prod_{i=1}^{n} G_{x_i}^2 / 2\big)$$

$$= \sum_{A \cup B = \{1, ..., n\}} \mathbb{E}\big(\prod_{i \in B} \frac{G_{x_i}^2}{2}\big) \sum_{\pi \in \mathcal{P}_{|A|}} C(a, x_{\pi(1)}) C(x_{\pi(1)}, x_{\pi(2)}) ... C(x_{\pi(i)}, x_{\pi(i+1)}) ... C(x_{\pi(|A|)}, b)$$

Using (5.36) and (5.37) we have (5.38). □

Theorem 5.7 (Dynkin Isomorphism Theorem) *Let X be a strong symmetric Markov process with continuous 0-potential density $u(x, y)$. Let 0 denote a fixed element of E. Assume that*

$$h(x) = \mathbb{P}^x(T_0 < \infty) = \frac{u(x, 0)}{u(0, 0)} > 0 \tag{5.42}$$

for all $x \in E$. Let $\tilde{X}$ denote the h-transform of X as described in section 2 and let $\bar{L} = \{\bar{L}_t^y; (y, t) \in E \times \mathbb{R}_+\}$ denote the local time of $\tilde{X}$, normalized so that

$$\mathbb{E}^{x,0}\big(\bar{L}_\infty^y\big) = \frac{u(x, y) h(y)}{h(x)} \tag{5.43}$$

Let $G = \{G_y; y \in E\}$ denote the mean zero Gaussian process with covariance $u(x,y)$. Then, for any countable subset $D \in E$,

$$\{\bar{L}^y_\infty + \frac{1}{2}G_y^2; y \in D, \mathbb{P}^{x,0} \times \mathbb{P}_G\} \overset{law}{=} \{\frac{1}{2}G_y^2; y \in D, \frac{G_x G_0}{u(0,x)}\mathbb{P}_G\} \tag{5.44}$$

Equivalently, for all $x, x_1, ..., x_n$ in E and bounded measurable function F on $\mathbb{R}^n_+$, for all n,

$$\mathbb{E}^{x,0}\mathbb{E}_G\big(F(\bar{L}^{x_i}_\infty + \frac{1}{2}G^2_{x_i})\big) = \mathbb{E}_G\{\frac{G_x G_0}{u(0,x)}F(\frac{1}{2}G^2_{x_i})\} \tag{5.45}$$

Here we use the notation $F(f(x_i)) := F(f(x_1)...f(x_n))$.

Proof of Dynkin Isomorphism Theorem The combinatorial proof is given both for its intrinsic interest and because understanding them may be fruitful. The proof can be found in section 8.3 of [21].

Proof. We first show that

$$\mathbb{E}^{x,0}\mathbb{E}_G\big(\prod_{i=1}^n(\bar{L}^{x_i}_\infty + \frac{1}{2}G^2_{x_i})\big) = \mathbb{E}_G\{\frac{G_x G_0}{u(0,x)}\prod_{i=1}^n(\frac{1}{2}G^2_{x_i})\} \tag{5.46}$$

for any $x_1, ..., x_n \in E$, not necessarily distinct.

Using formula 3 (5.9), we have:

$$\mathbb{E}\big(G_x G_0 \prod_{i=1}^n \frac{G^2_{x_i}}{2}\big)$$

$$= \sum_{B\cup C=\{1,...,n\}} \mathbb{E}\big(\prod_{i\in B}\frac{G^2_{x_i}}{2}\big) \sum_{\pi\in\mathcal{P}_{|C|}} C(X, x_{\pi(1)})C(x_{\pi(1)}, x_{\pi(2)})...C(x_{\pi(i)}, x_{\pi(i+1)})...C(x_{\pi(|C|)}, 0)$$

$$= \sum_{B\cup C=\{1,...,n\}} \mathbb{E}_G\big(\prod_{i\in B}\frac{G^2_{x_i}}{2}\big) \sum_{\pi\in\mathcal{P}_{|C|}} u(x, x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)})...u(x_{\pi(i)}, x_{\pi(i+1)})...u(x_{\pi(|C|)}, 0) \tag{5.47}$$

The left hand side of (5.46) is

$$\mathbb{E}^{x,0}\mathbb{E}_G\big(\prod_{i=1}^{n}(\bar{L}_\infty^{x_i}+\frac{1}{2}G_{x_i}^2)\big)=\sum_{B\cup C=\{1,\dots,n\}}\mathbb{E}_G\big(\prod_{i\in B}\frac{G_{x_i}^2}{2}\big)\mathbb{E}^{x,0}\big(\prod_{i\in C}\bar{L}_\infty^{x_i}\big) \tag{5.48}$$

In order to prove (5.46), we need to prove:

$$u(0,x)\mathbb{E}^{x,0}\big(\prod_{i\in C}\bar{L}_\infty^{x_i}\big)=\sum_{\pi\in\mathcal{P}_{|C|}}u(x,x_{\pi(1)})u(x_{\pi(1)},x_{\pi(2)})...u(x_{\pi(i)},x_{\pi(i+1)})...u(x_{\pi(|C|)},0) \tag{5.49}$$

From Section 2, we have $\tilde{X}$ is a strong symmetric Markov process with respect to the reference measure $\tilde{m}dy=h^2(y)dm(y)$ and has 0-potential density

$$\tilde{u}(x,y)=\frac{u(x,y)}{h(x)h(y)} \tag{5.50}$$

From (5.38), $\tilde{X}$ has a local time $\tilde{L}=\{\tilde{L}_t^y,(y,t)\in E\times\mathbb{R}_+\}$ that satisfies

$$\mathbb{E}^{x,0}\big(\tilde{L}_t^y\big)=\tilde{u}(x,y). \tag{5.51}$$

From Lemma 4.4 and (4.8), we have

$$\mathbb{E}^{x,0}\big(\prod_{i\in C}\tilde{L}_\infty^{x_i}\big)=\sum_{\pi\in\mathcal{P}_{|C|}}\tilde{u}(x,x_{\pi(1)})\tilde{u}(x_{\pi(1)},x_{\pi(2)})...\tilde{u}(x_{\pi(|C|-1)},x_{\pi(|C|)}) \tag{5.52}$$

Using (6.50) and (6.60), we have

$$\mathbb{E}^{x,0}\big(\prod_{i\in C}\frac{\bar{L}_\infty^{x_i}}{h^2(x_i)}\big)=\sum_{\pi\in\mathcal{P}_{|C|}}\frac{u(x,x_{\pi(1)})}{h(x)h(x_{\pi(1)})}\frac{u(x_{\pi(1)},x_{\pi(2)})}{h(x_{\pi(1)})h(x_{\pi(2)})}...\frac{u(x_{\pi(|C|-1)},x_{\pi(|C|)})}{h(x_{\pi(|C|-1)})h(x_{\pi(|C|)})}.$$

Which is equivalent to

$$\mathbb{E}^{x,0}\big(\prod_{i\in C}\bar{L}_\infty^{x_i}\big)=\sum_{\pi\in\mathcal{P}_{|C|}}\frac{h(x_{\pi(|C|)})}{h(x)}u(x,x_{\pi(1)})u(x_{\pi(1)},x_{\pi(2)})...u(x_{\pi(|C|-1)},x_{\pi(|C|)})$$

$$=\sum_{\pi\in\mathcal{P}_{|C|}}\frac{1}{u(x,0)}u(x,x_{\pi(1)})u(x_{\pi(1)},x_{\pi(2)})...u(x_{\pi(|C|-1)},x_{\pi(|C|)})u(x_{\pi(|C|)},0).$$

Then (5.49) is proved which establishes (5.46).

Now, let $x_1, ..., x_n$ be fixed and let μ_1 and μ_2 be the measures on $\mathbb{R}^n_+$ defined by

$$\int F(.)d\mu_1 = \mathbb{E}^{x,0}\mathbb{E}_G\big(F(\bar{L}^{x_1}_\infty + \frac{1}{2}G^2_{x_1}, ..., \bar{L}^{x_n}_\infty + \frac{1}{2}G^2_{x_n})\big) \tag{5.53}$$

and

$$\int F(.)d\mu_2 = \mathbb{E}_G\{\frac{G_x G_0}{u(0,x)}F(\frac{1}{2}G^2_{x_1}, ..., \frac{1}{2}G^2_{x_n})\} \tag{5.54}$$

for all bounded measurable functions F on $\mathbb{R}^n_+$. The measure μ_1 is determined by its characteristic function

$$\varphi_1(\lambda_1, ..., \lambda_n) = \mathbb{E}^{x,0}\mathbb{E}_G\big(\exp(i\sum_{i=1}^{n}\lambda_i(\bar{L}^{x_i}_\infty + \frac{1}{2}G^2_{x_i}))\big). \tag{5.55}$$

For $\lambda_1, ..., \lambda_n$ fixed, $\varphi(\lambda_1, ..., \lambda_n)$ is determined by the distribution of the real valued random variable $\xi = \sum_{i=1}^{n}\lambda_i(\bar{L}^{x_i}_\infty + \frac{1}{2}G^2_{x_i})$. The measure μ_1 is uniquely determined by the moments of ξ or, equivalently, by the terms in the left-hand side of (5.46).

Let

$$\varphi_2(\lambda_1, ..., \lambda_n) = \mathbb{E}_G\big(\frac{G_x G_0}{u(0,x)}\exp\big(i\sum_{i=1}^{n}\lambda_i G^2_{x_i}\big)\big). \tag{5.56}$$

By (5.46) and the above argument , we have $\varphi_1(\lambda_1, ..., \lambda_n) = \varphi_2(\lambda_1, ..., \lambda_n)$. Hence $\mu_1 = \mu_2$ and (5.45) is proved. □

Theorem 3.7 which relates the local time process of a transient Markov process and its associated permanental process can be viewed as the Dynkin Isomorphism for permanental processes.

5.3 Isomorphism Identities

Inspired by Dynkin Isomorphism, Rosiński [28] has established some general identities for infinitely divisible processes using their random translations.

Theorem 5.8(Rosiński[28]) *Let* $(X_t, t \in T)$ *be an infinitely divisible process having a* $\sigma-$*finite Lévy measure* ν*. Let* $(Z_t, t \in T)$ *be a process independent of* X *such that the law of* Z*,* $\mathscr{L}(Z)$*is absolutely continuous with respect to* ν*. Then* $\mathscr{L}(X+Z)$ *is absolutely continuous with respect to* $\mathscr{L}(X)$*. Hence, there exists a measurable functional* $g : \mathbb{R}^E \to \mathbb{R}_+$ *such that for any measurable functional* $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}F\big((X_t + Z_t)_{t\in T}\big) = \mathbb{E}\{F\big((X_t)_{t\in T}\big); g(X)\} \tag{5.57}$$

(5.57) can also be viewed as

$$X + Z \text{ has the law of } X \text{ under } \mathbb{E}(g(X), .).$$

Remark 5.9:

- From (5.57), we have the law of $(X+Z)$ and of X are isometric with respect to different probability measures. This is why the results of this kind are named "Isomorphism theorems" or "Isomorphism identities".
- The processes Z can be viewed as random translation of X.

Identity (5.57) can be applied in two different ways. One can obtain the properties of Z through the properties of X via isomorphism (5.57). Follow this direction, Marcus and Rosen obtained many results for local times of Markov processes using Dynkin isomorphism theorem [20][21]. The other way to apply this isomorphism is to derive information about X from Z. This direction is much more difficult.

We would like to know more about function g in (5.57). Unfortunately, it is not always specified. Below, we give another more explicit form of g.

Proposition 5.10. *Let $Y = (Y_t)_{t\in T}$ be a Poissonian infinitely divisible process with Lévy measure ν and a shift function b. Let N be a Poisson random measure on $(\mathbb{R}^T, \mathcal{B}^T)$ having intensity measure ν. Then the process $\tilde{Y} = (\tilde{Y}_{t\in T})$ given by*

$$\tilde{Y}_t = \int_{\mathbb{R}^T} y(t)\{N(dy) - \chi(y(t))\nu(dy)\} + b(t) \tag{5.58}$$

has the same distribution as Y. $\tilde{Y}$ will be called a canonical spectral representation of Y.

Theorem 5.11(Rosiński [28]) *Let $X = (X_t)_{t\in T}$ be an infinitely divisible process given by*

$$X = G + Y$$

where $G = (G_t)_{t\in T}$ is a centered Gaussian process independent of a Poissonian process $Y = (Y_t)_{t\in T}$ having a σ-finite measure ν and given by its canonical spectral representation

$$\tilde{Y}_t = \int_{\mathbb{R}^T} y(t)\{N(dy) - \chi(y(t))\nu(dy)\} + b(t)$$

where N is a Poisson random measure with intensity ν. Let $Z = (Z_t)_{t\in T}$ be an arbitrary process independent of N.

(a) Suppose that $\mathscr{L}(Z) << \nu$ and let $q := \frac{d\mathscr{L}(Z)}{d\nu}$ be the Randon-Nikodym derivative of $\mathscr{Z}$ with respect to ν. Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}F\big((X_t + Z_t)_{t\in T}\big) = \mathbb{E}\{F\big((X_t)_{t\in T}\big); N(q)\} \tag{5.59}$$

where

$$N(q) = \int_{\mathbb{R}^T} q(x)N(dx).$$

Conversely, for any F as above,

$$\mathbb{E}\{F\big((X_t)_{t\in T}\big); N(q) > 0\} = \mathbb{E}F\big((X_t + Z_t)_{t\in T}(N(q) + q(Z))^{-1}\big). \tag{5.60}$$

Therefore, if $\nu\{x : q(x) > 0\} = \infty$, then $\mathscr{L}(X + Z)$ and $\mathscr{X}$ are equivalent.

(b) *Suppose that $\mathscr{L}(Z) << \nu + \delta_{0_T}$ and let $q := \frac{d\mathscr{L}(Z)}{d(\nu+\delta_{0_T})}$ be the Randon-Nikodym derivative of $\mathscr{Z}$ with respect to $\nu + \delta_{0_T}$. Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$*

$$\mathbb{E}F\big((X_t + Z_t)_{t\in T}\big) = \mathbb{E}\{F\big((X_t)_{t\in T}\big); N(q) + q(0_T)\} \tag{5.61}$$

Conversely, for any F as above,

$$\mathbb{E}\{F\big((X_t)_{t\in T}\big); N(q)+q(0_T) > 0\} = \mathbb{E}F\big((X_t+Z_t)_{t\in T}(N(q)+q(Z)+q(0_T)\mathbb{1}_{U^c}(Z))^{-1}\big). \tag{5.62}$$

where U is any set such that $0_T \in U \in \mathscr{B}^T$ and $\nu(U) = 0$. Therefore, $\mathscr{L}(X + Z)$ and $\mathscr{X}$ are equivalent if $q(0_T) > 0$ or $\nu\{x : q(x) > 0\} = \infty$

Proof The proof of theorem 5.8 and 5.11 can be found in Section 6 of [28].

Eisenbaum[8] has established a lemma that enlarge even more point of view on the Isomorphism Identities.

Lemma 5.12: *Let $(Y_x, x \in E)$ be a positive process. Then, Y is infinitely divisible if and only if for every a such that $\mathbb{E}(Y_a) > 0$, there exists a process $(Z_x^{(a)}, x \in E)$ independent of Y such that*

$$Y + Z^{(a)} \text{ has the law of } Y \text{ under } \mathbb{E}(\frac{Y_a}{\mathbb{E}(Y_a)}, .) \tag{5.63}$$

Proof of Lemma 5.12 Since this lemma gives an isomorphism that plays an important role in our description of the Lévy measure of a non-negative infinitely divisible process, we will provide the full proof for completeness. See [8], lemma 3.1 for the original proof.

Proof. If Y is infinitely divisible then for every $x = (x_1, ..., x_n) \in E^n$, there exists ν_x a Lévy measure of $\mathbb{R}^n$ such that $\int_{\mathbb{R}_+^n} (1 \wedge |y|)\nu_x(dy) < \infty$ and for every $\alpha = (\alpha_1, ..., \alpha_n)$ in $\mathbb{R}_+^n$

$$\mathbb{E}(e^{-\sum_{i=1}^n \alpha_i Y_{x_i}}) = \exp\{-\int_{\mathbb{R}_+^n} (1 - e^{-\sum_{i=1}^n \alpha_i y_i})\nu_x(dy)\} \tag{5.64}$$

Differentiate both sides of (5.64) with respect to α_i, we have:

$$\mathbb{E}\big(Y_{x_i} e^{-\sum_{i=1}^n \alpha_i Y_{x_i}}\big) = \exp\{-\int_{\mathbb{R}_+^n} (1 - e^{-\sum_{i=1}^n \alpha_i y_i})\nu_x(dy)\} \int_{\mathbb{R}_+^n} y_i e^{-\sum_{i=1}^n \alpha_i y_i} \nu_x(dy)$$

Or

$$\mathbb{E}\big(\frac{Y_{x_i}}{\mathbb{E}(Y_{x_i})} e^{-\sum_{i=1}^n \alpha_i Y_{x_i}}\big) = \mathbb{E}(e^{-\sum_{i=1}^n \alpha_i Y_{x_i}}) \int_{\mathbb{R}_+^n} \frac{y_i}{\mathbb{E}(Y_{x_i})} e^{-\sum_{i=1}^n \alpha_i y_i} \nu_x(dy)$$

it follows that there exists a process $Z^{(x_i)}$ independent of Y such that:

$$Y + Z^{(x_i)} \text{ has the law of } Y \text{ under } \mathbb{E}(\frac{Y_{x_i}}{\mathbb{E}(Y_{x_i})}, .).$$

Note that the law of vector $(Z_{x_1}^{(x_i)}, ..., Z_{x_n}^{(x_i)})$ is $\frac{y_i}{\mathbb{E}(Y_{x_i})}\nu_x(dy)$.

Conversely, assume that for every a, there exists a process $Z^{(a)}$ satisfying (5.63). By computing the law of Y under $\mathbb{E}(Y_a Y_b, .)$, applying the above formula twice, we see that for every couple (a, b) of E, we must have:

$$\mathbb{E}(Y_a)\mathbb{E}(Z_b^{(a)} F(Z_x^{(a)})) = \mathbb{E}(Y_b)\mathbb{E}(Z_a^{(b)} F(Z_x^{(b)})) \tag{5.65}$$

Or

$$Z^{(a)} \text{ under } \mathbb{E}(\mathbb{E}(Y_a)Z_b^{(a)}, .) \text{ has the same law as } Z^{(b)} \text{ under } \mathbb{E}(\mathbb{E}(Y_b)Z_a^{(b)}, .).$$

To lighten the writing, we set $x_1 = a$. We have

$$\frac{\frac{\partial}{\partial \alpha_1}\mathbb{E}(e^{-\sum_{i=1}^n \alpha_i Y_{x_i}})}{\mathbb{E}(e^{-\sum_{i=1}^n \alpha_i Y_{x_i}})} = -\mathbb{E}(e^{-\sum_{i=1}^n \alpha_i Z_{x_i}^{(a)}})\mathbb{E}(Y_{x_1})$$

and hence

$$\mathbb{E}(e^{-\sum_{i=1}^{n}\alpha_i Y_{x_i}})$$

$$= \mathbb{E}(e^{-\sum_{i=1}^{n}\alpha_i Y_{x_i}})\big|_{\alpha_1=0} \exp\big(- \mathbb{E}(Y_{x_1})\mathbb{E}(\frac{1-e^{-\alpha_1 Z_{x_1}^{(a)}}}{Z_{x_1}^{(a)}} e^{-\sum_{i=2}^{n}\alpha_i Z_{x_i}^{(a)}})\big) \tag{5.66}$$

We now use (5.66) and an induction argument to end our proof. For $n = 1$ it follows immediately from (5.64) that $\mathbb{E}(e^{-\alpha_1 Y_{x_1}}) = \exp(-\int_0^\infty (1-e^{-\alpha_1 y_1})\nu_x(dy_1))$ where $\nu_x(dy_1) = \frac{\mathbb{E}(Y_{x_1})}{y_1}\mathbb{P}(Z_{x_1}^{(a)} \in dy_1)$.

Assume now that the law of $(Y_{x_1}, Y_{x_2}, ..., Y_{x_{n-1}})$ is given by

$$\mathbb{E}(e^{-\sum_{i=1}^{n-1}\alpha_i Y_{x_i}}) = \exp\{-\int_{\mathbb{R}_+^{n-1}} (1-e^{-\sum_{i=1}^{n-1}\alpha_i y_i})\nu_x(dy)\}$$

with $\nu_x(dy) = \frac{\mathbb{E}(Y_{x_1})}{y_1}\mathbb{P}(Z_x^{(a)} \in dy)$.

By (5.66) $\nu_x(dy) = \int_{\mathbb{R}_+} \frac{\mathbb{E}(Y_{x_n})}{y_n}\mathbb{P}(Z_x^{(x_n)} \in dy, Z_{x_n}^{(x_n)} \in dy_n)$ for every x_n distinct from $x_1, x_2, ..., x_{n-1}$.

Using (5.66), we obtain:

$$\mathbb{E}(e^{-\sum_{i=1}^{n}\alpha_i Y_{x_i}}) = \exp\{-\int_{\mathbb{R}_+^{n-1}} (1-e^{-\sum_{i=2}^{n}\alpha_i y_i})\nu_x(dy)\}$$

$$\times \exp\{-\int_{\mathbb{R}_+^{n}} (e^{-\sum_{i=2}^{n}\alpha_i y_i} - e^{-\sum_{i=1}^{n})\alpha_i y_i})\frac{\mathbb{E}(Y_{x_1})}{y_1}\mathbb{P}(Z_x^{(a)} \in dy_1 dy_2 ... dy_n)\}$$
$$= \exp\{-\int_{\mathbb{R}_+^{n}} (1-e^{-\sum_{i=1}^{n}\alpha_i y_i})\nu_x(dy)\frac{\mathbb{E}(\psi_{x_1})}{y_1}\mathbb{P}(Z_x^{(a)} \in dy_1 dy_2 ... dy_n)\}.$$

□

Dynkin's isomorphism theorem is just a special case of (5.63) when the positive infinitely divisible process is the squared of Gaussian process which is associated to a transient Markov process. Lemma 5.12 connects every infinitely divisible process $(\psi_x, x \in E)$ to a family of its random translation $Z_x^{(a)}, x \in E$.

Chapter 6

Lévy measures of non-negative Infinitely Divisible processes

In this Chapter we will explore the general expression of Lévy measures of nonnegative infinitely divisible processes using an Isomorphism Theorem based on random translations. We also look at the special case of infinitely divisible permanental processes whose random translations have a very special form.

6.1 Conditional Distribution

The concept of conditional distribution will be used in our description of Lévy measures of non-negative infinitely divisible processes, hence we first need to clarify what do we mean by the conditional distribution $(Y|Y_{x_1} = 0)$ given that Y_x is a non-negative stochastic process. We begin with conditional probability of events.

Definition 6.1 *Given* $(\Omega, \mathcal{F}, \mathbb{P})$*, for sets* $A, B \in \mathcal{F}$*, such that* $\mathbb{P}(B) > 0$*, the conditional probability of* A *given that* B *has occurred is defined as*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(B)}.$$

We need to generalize the definition so as to be able to handle the conditioning by random variables. Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $A \in \mathcal{F}$. If $B \in \mathcal{B}$ is such that $\mathbb{P}(X \in B) > 0$, then as above, the conditional probability of A given $X \in B$, is defined by

$$\mathbb{P}(A|X \in B) = \frac{\mathbb{P}(A, X \in B)}{\mathbb{P}(X \in B)}.$$

Following by these definition, if $\mathbb{P}(Y_{x_1} = 0) > 0$, we have no trouble define the condition distribution $(Y|Y_{x_1} = 0)$. It has the law $\mathscr{L}(Y|Y_{x_1} = 0)(dy) = \mathbb{P}(Y \in dy|Y_{x_1} = 0)$. We now have to deal with the case $\mathbb{P}(Y_{x_1} = 0) = 0$ or even cases when $\mathscr{L}(Y_{x_1})$ does not have an atom at 0.

Let $Y = (Y_{x_1}, ..., Y_{x_n})$ be a random vector in $\mathbb{R}_+^n$. Since the conditional distribution of $(Y|Yx_1 = 0)$ is defined only $\mathscr{L}(Y_{x_1})$-a.s, its value can be set arbitrarily when $\mathbb{P}(Y_{x_1} = 0) = 0$. To assign such value consistently, including cases when $\mathscr{L}(Y_{x_1})$ does not have an atom at 0, we propose the following definition.

Definition 6.2 *Let $Y = (Y_{x_1}, ..., Y_{x_n})$ be a non-negative random vector. The conditional distribution of $Y|Y_{x_1} = 0$ is defined as the weak limit μ_0 of probability measures μ_k given by*

$$\mu_k(B) = \frac{1}{\mathbb{E}e^{-kY_{x_1}}} \int_{Y \in B} e^{-kY_{x_1}} d\mathbb{P}, \quad B \in \mathcal{B}(\mathbb{R}_+^n), \quad k \geq 1 \tag{6.1}$$

provided such limit exists.

If $\mathbb{P}(Y_{x_1} = 0) > 0$, the limit distribution μ_0 exists and for every $B \in \mathcal{B}$, $\mu_0(B) = \lim_{k\to\infty} \mu_k(B) = \mathbb{P}(Y \in B|Y_{x_1} = 0)$, where the right hand side is defined in the usual way. We would like to show that the conditional distribution of $Y|Y_{x_1} = 0$ is well defined for any infinitely divisible random variable in $\mathbb{R}_+^n$.

Proposition 6.3 *Let $Y = (Y_{x_1}, ..., Y_{x_n})$ be a non-negative infinitely divisible random vector with Lévy measure ν and zero drift. Then the conditional distribution of $(Y|Y_{x_1} = 0)$ is infinitely divisible with zero drift and Lévy measure ν_1 given by $\nu_1(dy_1, ..., dy_n) =$*

$\mathbf{1}_{\{y_1=0\}}(y)\nu(dy_1,...,dy_n)$.

Moreover, the conditional distribution of $(Y|Y_{x_1}=...=Y_{x_i}=0)$ *is infinitely divisible with zero and Lévy measure* $\mathbf{1}_{\{y_1=..=y_i=0\}}(y)\nu(dy_1,...,dy_n)$.

Proof of Proposition 6.3

Proof. Recall that μ_0 is the weak limit of probability measures μ_k if

$$\int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n a_i y_i}\mu_k(dy) \longrightarrow \int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n a_i y_i}\mu_0(dy).$$

Using the definition 6.2, we have:

$$\begin{aligned}\int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n a_i y_i}\mu_k(dy) &= \int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n a_i Y_{x_i}}\frac{e^{-kY_{x_1}}}{\mathbb{E}e^{-kY_{x_1}}}\mathbb{P}(Y\in dy)\\ &=\mathbb{E}\frac{e^{-(k+a_1)Y_{x_1}}e^{\sum_{i=2}^n a_i Y_{x_1}}}{\mathbb{E}e^{-kY_{x_1}}} = \frac{\mathbb{E}e^{-(k+a_1)Y_{x_1}}e^{\sum_{i=2}^n a_i Y_{x_i}}}{\mathbb{E}e^{-kY_{x_1}}}\\ &=\frac{\exp\{\int_{\mathbb{R}_+^n}(e^{-(k+a_1)y_1}e^{\sum_{i=2}^n a_i y_i}-1)\nu(dy_1...dy_n)\}}{\exp\{\int_{\mathbb{R}_+^n}(e^{-ky_1}-1)\nu(dy_1,...,dy_n)\}}\\ &=\exp\{\int_{\mathbb{R}_+^n} e^{-ky_1}(e^{\sum_{i=2}^n a_i y_i}-1)\nu(dy_1...dy_n)\}.\end{aligned}$$

Take k to infinity, we have:

$$\int_{\mathbb{R}_+^n} e^{-\sum_{i=1}^n a_i y_i}\mu_k(dy) \longrightarrow \exp\{\int_{\mathbb{R}_+^n}(e^{-\sum_{i=1}^n a_i y_i}-1)\mathbf{1}_{\{y_1=0\}}\nu(dy_1,...,dy_n)\}$$

the existence of this limit shows that μ_0 exists, and it if the law of and infinitely divisible vector in $\mathbb{R}_+^n$ with Lévy measure $\mathbf{1}_{\{y_1=0\}}\nu(dy_1,...,dy_n)$. □

6.2 Lévy measures of non-negative infinitely divisible processes

Denote by $\mathcal{M}(E,\mathbb{R}_+)$ the set of measurable paths from E in to $\mathbb{R}_+$. When E is a separable metric space w.r.t some metric d, a real valued process $(Y(x), x\in E)$ is stochastically continuous is for every $\epsilon > 0$ and every a in E

$$\lim_{x \to a} \mathbb{P}\big(|Y(x) - Y(a)| > \epsilon\big) = 0$$

as x approaches to a with respect to the metric d.

In the lemma below, we express a Lévy measure of the non-negative infinitely divisible process as a series of others Lévy measures which can be written in term of the laws of admissible random translations. We begin with the case of random vectors. The first part of this lemma is just the Lemma 4.6 of [28].

Lemma 6.4 *Let $Y = (Y_{x_1}, ..., Y_{x_n})$ be a non-negative vector with $m_i = \mathbb{E}(Y_{x_i}) \in (0, \infty)$. Then, the following are equivalent*

(i) Y is infinitely divisible .

(ii) For every $i \leq n$, there exists a vector of non-negative random variables $Z^{(x_i)} = (Z^{(x_i)}_{x_1}, ..., Z^{(x_i)}_{x_n})$ independent of Y such that

$$Y + Z^{(x_i)} \text{ has the law of } Y \text{ under } \mathbb{E}(\frac{Y_{x_i}}{m_i}, .). \tag{6.4}$$

Moreover, if (ii) holds, Y has Lévy measure ν of the form:

$$\nu = \sum_{i=1}^{n} \tilde{\nu}_i \tag{6.5}$$

where $\tilde{\nu}_i$ has the form:

$$\tilde{\nu}_i(dy) = \mathbf{1}_{A_i}(y) \frac{m_i}{y_i} \mathscr{L}(Z^{(x_i)})(dy) \tag{6.6}$$

and $A_i = \{y \in \mathbb{R}^n : y_1 = 0, .., y_{i-1} = 0, y_i > 0\}$ are disjoint. The drift of Y is given as :

$$c = (m_1 \mathbb{P}(Z^{x_1}_{x_1} = 0), ..., m_n \mathbb{P}(Z^{x_n}_{x_n} = 0)). \tag{6.7}$$

Furthermore, if Y has zero drift $\tilde{\nu}_i$ is the Lévy measure of the non-negative infinitely divisible process $V^{(1,..,i)}$ satisfying

$$(Y|Y_{x_1} = ... = Y_{x_{i-1}} = 0) = (Y|Y_{x_1} = .. = Y_{x_{i-1}} = Y_{x_i} = 0) + V^{(1,..,i)}. \tag{6.8}$$

Proof of the Lemma

Proof. We follow with some necessary modifications, arguments from [28], Lemma 4.6. Assume that Y is infinitely divisible, then for every $x = (x_1, ..., x_n) \in E^n$ and and for every $\alpha = (\alpha_1, ..., \alpha_n)$ in $\mathbb{R}^n_+$, the Laplace transform of Y has the form:

$$\phi(\alpha_1, ..., \alpha_n) = \mathbb{E}(e^{-\sum_{i=1}^n \alpha_i Y_{x_i}}) = \exp\{-\sum_{i=1}^n \alpha_i c_i - \int_{\mathbb{R}^n_+} (1 - e^{-\sum_{i=1}^n \alpha_i y_i})\nu(dy)\} \tag{6.9}$$

where $c_i \geq 0$. We have

$$\frac{\partial}{\partial \alpha_i}\phi(\alpha_1, ..., \alpha_n) = -\phi(\alpha_1, ..., \alpha_n)\{c_i + \int_{R^n_+} y_i e^{-\sum_{j=1}^n \alpha_j y_j}\nu(dy)\}. \tag{6.10}$$

Set $\alpha_1 = ... = \alpha_n = 0$, we have $m_i = \mathbb{E}(Y_{x_i}) = c_i + \int_{R^n_+} y_i \nu(dy)$. Let $Z^{(x_i)}$ be a vector independent of Y whose distribution is given by:

$$\mathscr{L}(Z^{(x_i)}) = \frac{c_i}{m_i}\delta_{(0,..,0)}(dy) + \frac{y_i}{m_i}\nu(dy).$$

Using (6.10), we have

$$\mathbb{E}\{\exp\{-\sum_{i=1}^n \alpha_i Y_{x_i}\}; \frac{Y_{x_i}}{m_i}\} = \mathbb{E}\{\exp\{-\sum_{j=1}^n \alpha_j (Y_{x_j} + Z^{(x_i)}_{x_j})\}\},$$

It follows that there exists a process $Z^{(x_i)}$ independent of Y such that:

$$Y + Z^{(x_i)} \text{ has the law of } Y \text{ under } \mathbb{E}(\frac{Y_{x_i}}{\mathbb{E}(Y_{x_i})}, .).$$

Now assume that for every $i \leq n$, there exists a vector of non-negative random variables $Z^{(x_i)} = (Z^{(x_i)}_{x_1}, ..., Z^{(x_i)}_{x_n})$ independent of Y satisfies 6.4, we will prove that Y is infinitely divisible with Lévy measure and drift given by (6.6) and 6.7. By computing the law of Y under $\mathbb{E}(Y_{x_i}Y_{x_j}, .)$, applying (6.4) twice we get for any bounded measurable functional $F : \mathbb{R}^n \mapsto \mathbb{R}$ and $i, j \leq n$

$$m_i\mathbb{E}(F(Z^{(x_i)})Z^{x_i}_{x_j}) = m_j\mathbb{E}(F(Z^{(x_j)})Z^{x_j}_{x_i}). \tag{6.11}$$

Taking $F(y) = \mathbf{1}_{(y_i=0)}$ in (6.11) we get:

$$m_i\mathbb{E}(\mathbf{1}_{\{Z^{(x_i)}_{x_i}=0\}}Z^{x_i}_{x_j}) = m_j\mathbb{E}(\mathbf{1}_{\{Z^{(x_j)}_{x_i}=0\}}Z^{x_j}_{x_i}).$$

This implies that for every $i, j \leq n$,

$$\{Z^{(x_i)}_{x_j} > 0\} \subset \{Z^{(x_i)}_{x_i} > 0\} a.s. \tag{6.12}$$

Applying (6.11) to $F(y) = y_i^{-1}\mathbf{1}_{(y_i>0)}y_j^{-1}\mathbf{1}_{(y_j>0)}$, and taking into account (6.12) we obtain:

$$m_i\mathbb{E}(F(Z^i)\mathbf{1}_{\{Z^{(x_i)}_{x_j}>0\}}(Z^{x_i}_{x_i})^{-1}) = m_j\mathbb{E}(F(Z^j)\mathbf{1}_{\{Z^{(x_j)}_{x_i}>0\}}(Z^{x_j}_{x_j})^{-1}). \tag{6.13}$$

For $n = 1$, from (6.4) we get

$$\mathbb{E}(e^{-\alpha Y}Y) = m\mathbb{E}e^{-\alpha(Y+Z)} = m\mathbb{E}e^{-\alpha Y}\mathbb{E}e^{-\alpha Z},$$

which yields

$$\frac{d}{d\alpha}\log\mathbb{E}(e^{-\alpha Y}) = -m\mathbb{E}e^{-\alpha Z}.$$

Therefore

$$\mathbb{E}(e^{-\alpha Y}) = \exp\{-m\mathbb{E}\int_0^\alpha(\mathbf{1}_{Z=0} + e^{-sZ}\mathbf{1}_{Z>0})ds\}$$

or

$$\mathbb{E}(e^{-\alpha Y}) = \exp\{-\alpha m\mathbb{P}(Z=0) - m\int_0^\infty(1 - e^{-\alpha y})\mathbf{1}_{(y>0)}y^{-1}\mathscr{L}(Z)(dy)\},$$

which show our claim.

We proceed by induction. Assuming (ii), suppose that (i) and (6.5), (6.6), (6.7) hold for $n-1$. Since $\tilde{Y} := Y_{\{x_1,\dots,x_{n-1}\}}$ and $\tilde{Z}^{(x_1)} := Z^{(x_i)}_{\{x_1,\dots,x_{n-1}\}}$ satisfy (ii) for $n-1$ in the place of n, by the induction hypothesis $\tilde{Y}$ is infinitely divisible with the Laplace transform

$$\tilde{\phi}(\alpha_1,...,\alpha_{n-1}) = \exp\{-\sum_{i=1}^{n-1}\alpha_i\tilde{c}_i - \int_{\mathbb{R}_+^{n-1}}(1-e^{-\sum_{i=1}^{n-1}\alpha_i y_i})\tilde{\nu}(dy)\},$$

where

$$\tilde{c} = (m_1\mathbb{P}(Z^{x_1}_{x_1}=0),...,m_{n-1}\mathbb{P}(Z^{x_{n-1}}_{x_{n-1}}=0)).$$

and

$$\tilde{\nu}(dy) = \sum_{k=1}^{n-1}\mathbf{1}_{\{y_1=...=y_{k-1}=0,y_k>0\}}m_k y_k^{-1}\mathscr{L}(\tilde{Z}^{(x_k)})(dy).$$

Let $\phi(\alpha_1,...,\alpha_n) = \mathbb{E}\exp\{-\sum_{i=1}^n\alpha_i Y_i\}$. Proceeding as before, we get

$$\frac{\partial}{\partial\alpha_n}\log\phi(\alpha_1,...,\alpha_n) = m_n\mathbb{E}\exp\{-\sum_{i=1}^n\alpha_i Z^{x_n}_{x_i}\}.$$

Hence

$$\phi(\alpha_1,...,\alpha_n) = \tilde{\phi}(\alpha_1,...,\alpha_{n-1})\exp\{-m_n\mathbb{E}(e^{-\sum_{i=1}^{n-1}\alpha_i Z^{x_n}_{x_i}}\int_0^{\alpha_n}e^{-sZ^{x_n}_{x_n}}ds)\}. \tag{6.14}$$

Notice that $Z^{(x_n)}_{x_i}=0$ a.s on the set $Z^{(x_n)}_{x_n}=0$ by (6.12). Therefore, the exponent of the last term on the right hand side of (6.14) can be written as:

$$-m_n\mathbb{E}\{\alpha_n\mathbf{1}_{\{Z^{(x_n)}_{x_n}=0\}} + e^{-\sum_{i=1}^{n-1}\alpha_i Z^{(x_n)}_{x_i}}(1-e^{-\alpha Z^{(x_n)}_{x_n}})\mathbf{1}_{\{Z^{(x_n)}_{x_n}>0\}}(Z^{(x_n)}_{x_n})^{-1}\}$$

$$= -\alpha_n c_n - m_n\sum_{k=1}^n\mathbb{E}\{e^{-\sum_{i=1}^{n-1}\alpha_i Z^{(x_n)}_{x_i}}(1-e^{-\alpha Z^{(x_n)}_{x_n}})\mathbf{1}_{\{Z^{(x_n)}_{x_1}=...=Z^{(x_n)}_{x_{k-1}}=0,Z^{(x_n)}_{x_n}>0\}}(Z^{(x_n)}_{x_n})^{-1}\}$$

which, after applying (6.13) and noticing that the term $Z_{x_n}^{(x_k)} > 0$ can be replace by $Z_{x_k}^{(x_k)} > 0$ gives us

$$= -\alpha_n c_n - \sum_{k=1}^{n} m_k \mathbb{E}\{e^{-\sum_{i=1}^{n-1}\alpha_i Z_{x_i}^{(x_k)}}(1 - e^{-\alpha Z_{x_n}^{(x_k)}})\mathbf{1}_{\{Z_{x_1}^{(x_k)}=\ldots=Z_{x_{k-1}}^{(x_k)}=0, Z_{x_k}^{(x_k)}>0\}}(Z_{x_k}^{(x_k)})^{-1}\}$$

$$= -\alpha_n c_n - \sum_{k=1}^{n} m_k \int_{\mathbb{R}_+^n}(e^{-\sum_{i=1}^{n-1}\alpha_i y_i} - e^{-\sum_{i=1}^{n}\alpha_i y_i})\mathbf{1}_{\{y_1=\ldots=y_{k-1}=0, y_k>0\}} y_k^{-1}\mathscr{L}(Z^{(x_k)})(dy)$$

$$= -\alpha_n c_n + \int_{R_+^n}(1 - e^{-\sum_{i=1}^{n-1}\alpha_i y_i})\nu(dy) - \int_{R_+^n}(1 - e^{-\sum_{i=1}^{n}\alpha_i y_i})\nu(dy).$$

Substituting the above into (6.14) show that Y is infinitely divisible and has the Lévy measure and drift term as in (6.5)(6.6) and 6.7.

Now, we need to show that $\tilde{\nu}_i(dy) = \mathbf{1}_{\{y_1=\ldots=y_{i-1}=0, y_i>0\}}\frac{m_i}{y_i}\mathscr{L}(Z^{(x_i)})(dy)$ is the Lévy measure of a random vector satisfy (6.8).

Claim:

$$\mathbf{1}_{\{y_i>0\}}(y)\nu(dy) = \mathbf{1}_{\{y_i>0\}}\frac{\mathbb{E}(Y_{x_i})}{y_i}\mathscr{L}(Z^{(x_i)})(dy). \tag{6.15}$$

Proof of the Claim: We have

$$\mathscr{L}(Z^{(x_i)})(dy) = \frac{c_i}{m_i}\delta_{(0,..,0)}(dy) + \frac{y_i}{m_i}\nu(dy).$$

or

$$\mathbf{1}_{\{y_i>0\}}(y)\mathscr{L}(Z^{(x_i)})(dy) = \mathbf{1}_{\{y_i>0\}}(y)\frac{c_i}{m_i}\delta_{(0,..,0)}(dy) + \mathbf{1}_{\{y_i>0\}}(y)\frac{y_i}{m_i}\nu(dy).$$

It is easy to see that the first term on the right hand side is zero which yields the claim.

Applying (6.15) for $i = 1$, we have

$$\tilde{\nu}_1(dy) = \mathbf{1}_{\{y_1>0\}}(y)\frac{m_1}{y_1}\mathscr{L}(Z^{(x_1)})(dy) = \mathbf{1}_{\{y_1>0\}}(y)\nu(dy).$$

It follows that

$$\nu = \mathbf{1}_{\{y_1=0\}}\nu + \tilde{\nu}_1.$$

From proposition 6.3, $\mathbf{1}_{\{y_1=0\}}\nu$ is the Lévy measure of $(Y|Y_{x_1} = 0)$. Let $V^{(1)}$ be the random vector with Lévy measure $\tilde{\nu}_1$, then

$$Y = (Y|Y_{x_1} = 0) + V^{(1)}.$$

We can write $\tilde{\nu}_i$ as

$$\tilde{\nu}_i(dy) = \mathbf{1}_{\{y_1=\ldots=y_{i-1}=0,y_i>0\}}(y)\frac{m_i}{y_i}\mathscr{L}(Z^{(x_i)})(dy) = \mathbf{1}_{\{y_1=\ldots=y_{i-1}=0\}}(y)\mathbf{1}_{\{y_i>0\}}\frac{m_i}{y_i}(y)\mathscr{L}(Z^{(x_i)})(dy).$$

Applying (6.15) , we get

$$\tilde{\nu}_i = \mathbf{1}_{\{y_1=\ldots=y_{i-1}=0\}}\mathbf{1}_{\{y_i>0\}}\nu.$$

It follows that

$$\mathbf{1}_{\{y_1=\ldots=y_{i-1}=0\}}\nu = \mathbf{1}_{\{y_1=\ldots=y_{i-1}=0\}}\mathbf{1}_{\{y_i=0\}}\nu + \tilde{\nu}_i.$$

Equivalently, $\tilde{\nu}_i$ is the Lévy measure of the random vector $V^{(1,..,i)}$ satisfy:

$$(Y|Y_{x_1} = 0, .., Y_{x_{i-1}} = 0) = (Y|Y_{x_1} = 0, .., Y_{x_{i-1}} = 0, Y_{x_i} = 0) + V^{(1,..,i)}.$$

□

The following result complements Proposition 4.7 in [28].

Theorem 6.5 *Let $(Y_x, x \in E)$ be non-negative process with $m_x = \mathbb{E}Y_x < \infty$ for every $x \in E$. Then, Y is infinitely divisible if and only if for every a such that $\mathbb{E}(Y_a) > 0$, there* *exists a process* $(Z_x^{(a)}, x \in E)$ *independent of* Y *such that*

$$Y + Z^{(a)} \text{ has the law of } Y \text{ under } \mathbb{E}(\frac{Y_a}{m_a}, .).$$

If in addition Y *is separable in probability with a separant* $E_0 = \{x_i : i \in \mathbb{N}\}$ *then the Lévy measure* ν *of* Y *is of the form*

$$\nu = \sum_{i \geq 1} \tilde{\nu}_i$$

where $\tilde{\nu}_i$ *has the form:*

$$\tilde{\nu}_i(dy) = \mathbf{1}_{A_i}(y) \frac{m_{x_i}}{y_{x_i}} \mathscr{L}(Z^{(x_i)})(dy)$$

where $A_i = \{y \in \mathbb{R}^E : y_1 = 0, .., y_{i-1} = 0, y_i > 0\}$*, the the drift is given by*

$$c = (m_x \mathbb{P}(Z_x^{(x)} = 0))_{x \in E}.$$

Furthermore, if Y *has zero drift ,*$\tilde{\nu}_i$ *is the Lévy measure of the non-negative infinitely divisible process* $V^{(1,..,i)}$ *satisfying:*

$$(Y | Y_{x_1} = 0, .., Y_{x_{i-1}} = 0) = (Y | Y_{x_1} = 0, .., Y_{x_{i-1}} = 0, Y_{x_i} = 0) + V^{(1,..,i)}.$$

We observe that the family of random translations play a very important role in the description of Lévy measure of a non-negative infinitely divisible process. We now look at the case of infinitely divisible permanental processes whose random translations have a very special form .

We consider a transient Markov process X with state space E, admitting 0-potential densities $(u(x,y), (x,y) \in E \times E)$ w.r.t. a σ- finite reference measure m and a local time process $(L_t^x, x \in E, t \geq 0)$ which is normalized to satisfy $\mathbb{E}_x(L_\infty^y) = u(x,y)$. We define the probability $\tilde{\mathbb{P}}^a$ as follows:

$$\tilde{\mathbb{P}}^a|_{\mathcal{F}_t} = \frac{u(X_t, a)}{u(a,a)} \mathbb{P}^a|_{\mathcal{F}_t}.$$

Under $\tilde{\mathbb{P}}^a$, the process X starts at a and is killed at its last visit to a. In fact, $\tilde{\mathbb{P}}^a$ is the probability measure of the $h-$ transform of X_t when $h(x) = \frac{u(x,a)}{u(a,a)}$. Expectation with respect to $\tilde{\mathbb{P}}^a$ is denoted by $\tilde{\mathbb{E}}^a$.

We have shown in Chapter 3 that for any $\beta > 0$ there exists an infinitely divisible permanental process with kernel $u(x,y)$ index β. It can be also shown that a permanental process is infinitely divisible if and only if it admits for kernel the $0-$potential densities of a transient Markov process.

Let $(Y_x, x \in E)$ be the permanental process with index 2 associated to the Markov process X. Theorem 3.7 says that for every $a \in E$ such that $u(a,a) > 0$, we have

$$\frac{1}{2}Y + L_\infty^{(a)} \text{has the law of } \frac{1}{2}Y \text{ under } \mathbb{E}(\frac{\frac{1}{2}Y_a}{\mathbb{E}(\frac{1}{2}Y_a)}, .)$$

where $L_\infty^{(a)}$ is the local times process of X conditioned to start at a and killed at its last visit to a. The random translations family of the infinitely divisible process $\frac{1}{2}Y$ is $\{L_\infty^{(a)}\}$.

In Chapter 4, we describe the Lévy measure of the squared Bessel process starting from 0 with dimension 1 using Ray-Knight theorem and excursion laws. Here we look at the Lévy measure of the squared Bessel process starting from 0 with dimension 1 from a different angle: the laws of its random translations family.

When X is a Brownian motion starting from $a > 0$ and dying at the first time of hitting 0, then 0- potential function has the form:

$$u_{T_0}(x,y) = \begin{cases} 2(|x| \wedge |y|) & xy > 0 \\ 0 & xy \leq 0 \end{cases},$$

It implies that the associated Gaussian process of X is $\sqrt{2}B_x$ and the associated permanental process with index 2 is $2B_x^2$. Let Y_x be the associated permanental process with index 2, then $\frac{1}{2}Y_x$ is B_x^2 which is the squared Bessel process starting from 0 with dimension 1.

We have the admissible transition process $Z^{(a)}$ is just the local time process of process X conditioned to start at a and killed at its last visit to a.

For $x \geq a$, the process $Z_x^{(a)}$ is just $L_{T_0}^x$ of a Brownian motion under P^a. By the first Ray -Knight theorem, we have:

- When $x = a$: $Z_a^{(a)} = B_a^2 + \bar{B}_a^2$ which has an exponential law with parameter $\frac{1}{2a}$.
- When $x > a$: $Z_x^{(a)}$ is a 0 dimensional squared Bessel process starting from an exponential law with parameter $\frac{1}{2a}$.

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Appendices

A Some Basic Definitions

A.1 Markov Process

Definition(Markov Process) *The Markov process* (X_t, P^x) *is a stochastic process*

$$X : [0, \infty) \times \Omega \to E$$

and a family of probability measures $\{P^x : x \in E\}$ *on* $\Omega, \mathcal{F}$ *satisfying the following:*

1. *For each* t*,* X_t *is* $\mathcal{F}_t$ *measurable.*
2. *For each* t *and each Borel subset* A *of* E*, the map* $x \to P^x(X_t \in A)$ *is Borel measurable.*
3. *For* $0 \leq s, t$*, each Borel subset* A *of* E *and each* $x \in E$ *we have*

$$P^x(X_{t+s} \in A|\mathcal{F}_s) = P^{X_s}(X_t \in A).$$

Remark

- Condition 3 is called the Markov property. The general version of the Markov property is

$$\mathbb{E}^x(Y \circ \theta_s|\mathcal{F}_s) = \mathbb{E}^{X_s}(Y) \quad P^x - a.s$$

  where Y is bounded and measurable with respect to $\mathcal{F}_\infty$ and $\{\theta_s\}$ is the shift operator by $s \geq 0$.

- The strong Markov property has the form

$$\mathbb{E}^x(Y \circ \theta_T|\mathcal{F}_T) = \mathbb{E}^{X_T}(Y) \quad P^x - a.s$$

  where T is a finite stopping time.

A.2 Transition probability, Potential density

Definittion Markov transition function The collection $\{P_t(.,.), 0 \leq t < \infty\}$ is a homogeneous Markov transition function (or semigroup) on $(E, \mathcal{E})$ iff $\forall 0 \leq s, t < \infty$ we have

a) $\forall x \in E$: $A \to P_t(x, A)$ is a probability measure on $\mathcal{E}$,

b) $\forall A \in \mathcal{E}$: $x \to P_t(x, A)$ is $\mathcal{E}$-measurable;

c) $\forall x \in E, \forall A \in \mathcal{E}$

$$P_{s+t}(x, A) = \int_E P_s(x, dy) P_t(y, A). \tag{.1}$$

When all the measures $P_t(x, .)$ have densities with respect to one given reference measure, $p_t(x, y)$ is transition density and is defined as $P_t(x, dy) = \mathbb{P}^x(X_t \in dy)$.

For a measurable function f, we have

$$P_t f(x) = \int_E f(y) P_t(x, dy) = \int_E f(y) p_t(x, y) dy. \tag{.2}$$

For $\alpha > 0$, the α-potential of X is defined by

$$U^\alpha(x, C) = \mathbb{E}_x\left(\int_0^\infty e^{-\alpha t} \mathbf{1}_C(X_t) dt\right) = \int_0^\infty e^{-\alpha t} P_t(x, C) dt \quad C \in \mathcal{E}. \tag{.3}$$

When applied to bounded measurable functions

$$U^\alpha f(x) = \int_E U^\alpha(x, dy) f(y)$$

it becomes an invertible operator whose inverse is given by $\alpha I - \mathcal{A}$, where I is the identity operator and $\mathcal{A}$ is the infinitesimal generator of the semigroup (P_t). When all the measures $U^\alpha(x, .)$ have densities with respect to one given reference measure, $u^\alpha(x, y)$ is the potential densities.

$$u^\alpha(x, y) = \int e^{-\alpha t} p_t(x, y) dt. \tag{.4}$$

The 0-potential is denoted shortly as $u(x, y)$.

A.3 Continuous Additive Functional

Definition 5.2 (Continuous Additive functional) Let X_t be a Markov process on $(\Omega, \mathcal{F}, \mathcal{F}_t)$. A family $A = A_t; t \geq 0$ of random variables on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is called a continuous additive functional(CAF) of X_t if

1. $t \to A_t$ is almost surely continuous and nondecreasing, with $A_0 = 0$ and $A_t = A_\zeta$ for all $t \geq \zeta$. Here ζ is referred as the "death time" of X.

2. A_t is $\mathcal{F}_t$ measurable.

3. $A_{t+s} = A_t + A_s \circ \theta_t$ for all $s, t \in \mathbb{R}_+$ a.s.

A.4 Local time

Definition 5.3 (Local time) The local time of the process X_t is denoted as $\{L_t^y; (t, y) \in \mathbb{R}_+ \times \mathcal{S}\}$ and is defined by

$$L_t^y = \lim_{\epsilon \to 0} \int_0^t f_{\epsilon,y}(X_r) dr, \tag{.5}$$

where $f_{\epsilon,y}$ is an approximate δ-function at y. That is $f_{\epsilon,y}$ is an non negative function supported in $B(y, \epsilon)$ with $\int f_{\epsilon,y}(x)dx = 1$. It is easy to see that $L_0^y = 0$, L_t^y is almost surely continuous and non-decreasing in t, and has the additive property

$$L_{t+s}^y = L_t^y + L_s^y \circ \theta_t. \tag{.6}$$

The following formulas describe the relation of the local time, transition density and α-potential density of a process X_t

$$\mathbb{E}^x(L_t^y) = \int_0^t p_s(x, y) ds = u(x, y), \tag{.7}$$

$$u^\alpha(x, y) = \mathbb{E}^x(\int_0^\infty e^{-\alpha t} dL_t^y), \tag{.8}$$

$$u^{\alpha}(x,y) = \mathbb{E}^x(e^{-\alpha T_y})u^{\alpha}(y,y), \tag{.9}$$

where $T_y = \inf\{s : X_s = y\}$.

A.5 Gaussian Random Variable, Gaussian Vector, Gaussian Process

Gaussian random variable

A real valued random variable X is called a Gaussian random variable if it has characteristic function of the form:

$$\mathbb{E}\exp(i\lambda X) = \exp\big(im\lambda - \frac{\sigma^2\lambda^2}{2}\big) \tag{.10}$$

where m and σ are some real numbers. Taking the first derivative of (.10) with respect to λ and the setting $\lambda = 0$, we have

$$\mathbb{E}(X) = m. \tag{.11}$$

Similarly with the second derivative of (.10), we get:

$$Var(X) = \sigma^2. \tag{.12}$$

Gaussian Vector

A random vector $X = (X_1, X_2, ..., X_n) \in \mathbb{R}^n$ is a Gaussian random vector if (y, X) is a Gaussian random variable for each $y \in \mathbb{R}^n$. It means the characteristic function of X having the form:

$$\theta_X(y) \overset{def}{=} \mathbb{E}\exp(i(y,X)) = \exp\big(i\mathbb{E}((y,X)) - \frac{Var((y,X))}{2}\big) \tag{.13}$$

for each $y \in \mathbb{R}^n$. Let $m = (m_1, m_2, ..., m_n)$. Setting

$$\mathbb{E}X_j = m_j \quad \text{and} \quad \mathbb{E}(X_j - m_j)(X_k - m_k) = C_{j,k}. \tag{.14}$$

Then, (.13) becomes

$$\theta_X(y) = \exp\big(imy^t - \frac{yCy^t}{2}\big) \tag{.15}$$

where $C = \{C_{j,k}\}_{j,k=1}^n$ is a symmetric $n \times n$ matrix with real components. m is called the mean vector, or the mean, and C is called the covariance matrix of X. The mean vector and the covariance matrix of a Gaussian vector determine its distribution.

We can always eliminate the mean vector by subtracting m from X. In this chapter, we will focus on Gaussian vectors with the mean vector $\mathbb{E}(X) = m = 0$. The characteristic function of those Gaussian vector has the form

$$\theta_X(y) = \mathbb{E}\exp(i(y, X)) = \exp\big(-\frac{yCy^t}{2}\big) = \exp\big(-\frac{(y, Cy)}{2}\big). \tag{.16}$$

Here, the entries of of the $n \times n$ matrix $C = \{C_{j,k}\}_{j,k=1}^n$ are

$$C_{j,k} = \mathbb{E}(X_j X_k) \quad 1 \leq j, k \leq n \tag{.17}$$

Since $\mathbb{E}(y, X)^2 = (y, Cy)$, C is positive definite. It follows that the covariance matrix of a $\mathbb{R}^\ltimes$ Gaussian vector is a symmetric positive definite $n \times n$ matrix. In the next step, we are going to show that conversely, any symmetric positive definite $n \times n$ matrix B is the covariance matrix of some Gaussian random vector in $\mathbb{R}^n$.

Claim: If B is a real symmetric positive definite $n \times n$ matrix, then there exists a symmetric matrix A such that $B = A^2$.

Proof of the claim:

Since B is real symmetric, there exists an orthogonal matrix P and a diagonal matrix D such that

$$B = PDP^T$$

where the diagonal entries of D are eigenvalues of B and the column vectors of P are the eigenvectors of eigenvalues of B.

Since B is positive definite, all of its eigenvalues are non-negative. We can define matrix $D^{1/2}$ as a matrix with entries $D^{1/2}_{i,j} = \sqrt{D_{i,j}}$.

Let $A = PD^{1/2}P^T$, then $B = A^2$. $\square$

Let X be a vector whose components are independent standard normal random variables. The covariance of X is the identity matrix I. We define a Gaussian random vector Z by $Z = AX$. The characteristic function of Z has the form

$$\mathbb{E}\exp(i(y, Z)) = \mathbb{E}\exp(i(A^t y, X)) = \exp\big(-(A^t y, IA^t y)/2\big) = \exp\big(-(y, AIA^t y)/2\big).$$

Since A is symmetric, $AIA^t = A^2 = B$. It follows that Z is a Gaussian vector with covariance matrix B.

Gaussian Processes

If S is a general set, a stochastic process $G = \{G_x, x \in S\}$ is called a Gaussian process on S if for any n and any $x_1, ..., x_n \in S$, $(G_{x_1}, ..., G_{x_n})$ is a Gaussian vector. The function

$$C(x, y) = \mathbb{E}(G_x, G_y), \quad x, y \in S \tag{.18}$$

on $S \times S$ is the covariance function of G. By the Kolmogorov's extension theorem, there is a correspondence between Gaussian processes and symmetric positive definite functions on $S \times S$.

Example : Let $S = R^1_+$, and let $C(s, t) = s \wedge t = \int \mathbb{1}_{[0,s]}(x)\mathbb{1}_{[0,t]}(x)dx$. We have

$$\sum_{i,j=1}^{n} C(s_i, s_j) y_i y_j = \int \big(\sum_{i=1}^{n} y_i \mathbb{1}_{[0,s_i]}(x)\big)^2 dx. \geq 0 \tag{.19}$$

It follows that $C(s,t)$ is positive definite. Then there exists a Gaussian process $B = \{B_s, s \in R^1_+\}$ with covariance function C. In fact, B is a Brownian motion in R^1_+.

Vita

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