Predicting Surface Heat Flux and Temperature via a Pulse-Echo Acoustic Transducer for Inverse Heat Conduction Applications

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I am submitting herewith a thesis written by Kevin Jiju Mathew entitled "Predicting Surface Heat Flux and Temperature via a Pulse-Echo Acoustic Transducer for Inverse Heat Conduction Applications." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mechanical Engineering.

Jay Frankel, Major Professor

We have read this thesis and recommend its acceptance:

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Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)
Predicting Surface Heat Flux and Temperature via a Pulse-Echo Acoustic Transducer for Inverse Heat Conduction Applications

A Thesis Presented for the
Master of Science
Degree
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Kevin Jiju Mathew
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ABSTRACT

Practical heat transfer situations rise where in-depth measurements must be used to predict a transient surface temperature or heat flux history. These occurrences are especially evident and necessary when a surface is exposed to a harsh thermal or chemical environment as the surface mounted sensor would most likely fail or lose its integrity over time. Unlike direct or forward problems, where the boundary condition is specified and the task is determining the temperature distribution, the reversed analysis produces numerous undesirable mathematical features. In particular, a well-posed process becomes ill-posed during this reversal. Any small error in the measurement leads to dramatic error amplification of the inverse prediction. This thesis describes an alternative measurement technique based on ultrasonic interferometry. Classically, in-depth thermocouples are used that require holes to be drilled into the sample. For the proposed sensor scenario, the sensor is mounted onto the back-side (passive side) of the sample and an ultrasonic pulse is released and timed (round-trip) in the sensor that produces the pulse. This time-of-flight measurement, using a pulse-echo arrangement, can be correlated to either surface temperature or heat flux. Regularization, a mathematical approach for stabilizing ill-posed problems, is introduced based on a future-time concept. In this approach, a family of predictions is produced based on the chosen regularization parameter. The most challenging problem associated with inverse problems is the identification of the optimal prediction. For the present study, a thermal phase plane is utilized to provide a qualitative view that explicitly shows instability and over-smoothing of the transient surface condition based on the regularization parameter. For a quantitative measure or metric, cross-correlation is described and its corresponding phase plane is used for estimating the optimal prediction, i.e., identification of the optimal regularization parameter. A numerical study is illustrated demonstrating the methodology and its accuracy for reconstructing the surface boundary condition.
TABLE OF CONTENTS

Chapter One Introduction ........................................................................................................ 1
  1.1: Opening Remarks ............................................................................................................. 1
Chapter Two Input and data generation .................................................................................. 2
  2.1: Introduction ....................................................................................................................... 2
  2.2: Heat Equation and Auxiliary Conditions ........................................................................ 2
  2.3: Chosen Surface Heat fluxes for Inverse Study ................................................................. 3
  2.4: Temperature Distribution ................................................................................................. 4
  2.5: Time-of-flight ................................................................................................................... 6
  2.6: Simulating Noisy Data ...................................................................................................... 11
Chapter Three traditional Inverse analysis using future time method .................................. 13
  3.1: Introduction ....................................................................................................................... 13
  3.2: Regularization .................................................................................................................. 13
  3.3: Heat Flux .......................................................................................................................... 13
    3.3.1: Family of Predictions Using Perfect Data ................................................................. 17
    3.3.2: Family of Predictions Using Real-Life (Noisy) Data .................................................... 20
Chapter Four preconditioned Inverse analysis using future time method .............................. 25
  4.1: Introduction ....................................................................................................................... 25
    4.1.1: Preconditioning ............................................................................................................ 25
  4.2: Heat Flux .......................................................................................................................... 25
  4.3: Case A: n = 1/2 ................................................................................................................ 26
    4.3.1: Family of Predictions Using Noisy Data (Case 2) ....................................................... 27
    4.3.2: Isolating the Optimal Prediction .................................................................................. 28
    4.3.3: Root-Mean Square Error (RMSE) .............................................................................. 30
  4.4: Case B: n = 3/4 ................................................................................................................ 31
    4.4.1: Family of Predictions Using Noisy Data (Case 2) ....................................................... 33
    4.4.2: Isolating the Optimal Prediction .................................................................................. 33
    4.4.3: Root-Mean Square Error (RMSE) .............................................................................. 36
  4.5: Case C: n = 1 .................................................................................................................... 37
    4.5.1: Family of Predictions Using Noisy Data .................................................................... 38
    4.5.2: Isolating the Optimal Prediction .................................................................................. 39
    4.5.3: Root-Mean Square Error (RMSE) .............................................................................. 42
Chapter Five conclusion .......................................................................................................... 44
  5.1 Conclusions ....................................................................................................................... 44
  5.2 Future Work ....................................................................................................................... 44
References .................................................................................................................................. 46
Appendices ................................................................................................................................ 48
Appendix A ................................................................................................................................. 49
Appendix B ................................................................................................................................. 55
Appendix C: ............................................................................................................................... 59
Appendix D: ............................................................................................................................... 63
Appendix E: ............................................................................................................................... 65
Vita ............................................................................................................................................. 68
LIST OF TABLES

Table 2.1: Thermophysical properties of stainless steel 304 ........................................ 5
Table 3.1: Root-mean-square experimental error values for traditional inverse analysis 24
Table 4.1: Root-mean-square experimental error values for preconditioned inverse
analysis, case one .............................................................................................................. 31
Table 4.2: Root-mean-square experimental error values for preconditioned inverse
analysis, case two .......................................................................................................... 37
Table 4.3: Root-mean-square experimental error values for preconditioned inverse
analysis, Case C using noisy T.o.F data generated for Case 2....................................... 43
LIST OF FIGURES

Figure 2.1: Input flux case 2: single gauss.............................................................. 3
Figure 2.2: Exact reduced temperature distribution at various locations within the sample .......................................................... 6
Figure 2.3: Test sample subjected to the boundary conditions with the acoustic sensor mounted onto the passive side [1]......................................................... 7
Figure 2.4: Time-of-flight vs time for the input flux case of single Gauss function (Case2) .......................................................... 10
Figure 2.5: Time-of-flight data calculated from heat flux and surface temperature to demonstrate the validity of the governing equations (Case 2)................................. 11
Figure 2.6: Time-of-flight data calculated from heat flux and perturbed 1 percent of maximum value to simulate extreme real-life scenario (Case 2)................................. 12
Figure 3.1: Inverse reconstruction with perfect data (Case 2)........................................ 17
Figure 3.2: Phase-plane analysis for the inverse analysis with perfect measurement data (Case 2) ........................................................................ 19
Figure 3.3: Derivative of cross-correlation coefficients vs cross-correlation coefficient for traditional inverse analysis with perfect measurement data (Case 2) ...................... 20
Figure 3.4: Traditional inverse reconstruction with noisy data .................................................. 21
Figure 3.5: Phase-plane analysis for the traditional inverse analysis with noisy measurement data........................................................................................................................................... 22
Figure 3.6: Derivative of cross-correlation coefficients vs cross-correlation coefficient for traditional inverse analysis with noisy measurement data ...................... 23
Figure 4.1: Preconditioned inverse reconstruction, Case A (n = 0.5), with noisy data (Case 2) ........................................................................ 28
Figure 4.2: Phase-plane analysis for the preconditioned inverse analysis, case one, with noisy measurement data........................................................................................................................................... 29
Figure 4.3: Derivative of cross-correlation coefficients vs cross-correlation coefficient for preconditioned inverse analysis, case one, with noisy measurement data............. 30
Figure 4.4: Preconditioned inverse reconstruction, case B, with noisy data (case 2)....... 33
Figure 4.5: Phase-plane analysis for the preconditioned inverse analysis, case B, with noisy measurement data (Case 2) ........................................................................ 34
Figure 4.6: Derivative of cross-correlation coefficients vs cross-correlation coefficient for preconditioned inverse analysis, Case B, with noisy measurement data (Case 2) ... 35
Figure 4.7: Isolating the optimal prediction for preconditioned inverse analysis, case B, with noisy measurement data (Case 2) ........................................................................ 36
Figure 4.8: Preconditioned inverse reconstruction, case three, with noisy data ............ 39
Figure 4.9: Phase-plane analysis for the preconditioned inverse analysis, case C, with noisy measurement data (Case 2) ........................................................................ 40
Figure 4.10: Derivative of cross-correlation coefficients vs cross-correlation coefficient for preconditioned inverse analysis, case C, with noisy measurement data (Case 2) 41
Figure 4.11: Isolating the optimal prediction for preconditioned inverse analysis, case C, with noisy data (Case 2) ........................................................................ 42
1.1: Opening Remarks

Quantities such as temperature distribution in a sample and heat flux are important parameters of interest within the hypersonic and heat transfer community. However, predicting surface heat flux and temperature in harsh thermal environments requires the use of inverse heat conduction analysis that removes the need for surface mounted instrumentation. In-depth measurements protect the integrity of the sensor from harsh or caustic environments. Inverse analysis generally utilizes in-depth measurements that are then mathematically projected to the surface based on the classical (parabolic) heat equation [1]. However, in-depth measurements yield to an “ill-posed” analysis and thus necessitate regularization [2-4]. In-depth measurements add additional layers of complexity as the exact probe locations and sensor properties are often estimated. The analysis becomes even more cumbersome as sampling rate is increased. New measurements methods are required to be developed to estimate the surface heat flux and temperature based on external measurements.

It has been demonstrated that a non-intrusive method can be implemented that uses ultrasonic pulse setting and the time-of-flight (T.o.F) can be retrieved from the instrumentation. [5-9]. In this context, ultrasonic refers to acoustic waves composed of frequencies greater than 20 $kHz$. There are three common instrumentation arrangements that are used to measure T.o.F.: 1) “through transmission” which places the transmitter and receiver in opposition; 2) “angle beam” or also known as “pitch-catch” method, which uses one sensor but requires non-normal surface interactions; and “pulse-echo” method, which uses one transmitter-receiver sensor placed normal to the surface [1]. The sensor of choice for this study is pulse-echo as is the most common sensor arrangement and easier to study in an experimental setting.
CHAPTER TWO
INPUT AND DATA GENERATION

2.1: Introduction

In this section, a forward heat conduction problem is produced for generating artificial data for the later inverse heat conduction simulation process.

2.2: Heat Equation and Auxiliary Conditions

Consider the transient, one-dimensional, constant property, transient heat equation given as [11]

$$\frac{1}{\alpha} \frac{\partial \theta}{\partial t} (x, t) = \frac{\partial^2 \theta}{\partial x^2} (x, t), \quad x \in [0, w], \quad t \geq 0$$  \hspace{1cm} 2.1.a

subject to the boundary conditions

$$q''(0, t) = -k \frac{\partial \theta}{\partial x} (0, t) = q_s''(t) = ?$$  \hspace{1cm} 2.1.b

$$q''(w, t) = -k \frac{\partial \theta}{\partial x} (w, t) = q_w''(t), \quad t \geq 0$$  \hspace{1cm} 2.1.c

and initial condition

$$\theta (x, 0) = 0, \quad x \in [0, w]$$  \hspace{1cm} 2.1.d

For the inverse problem, $q_s''(t) = ?$ is sought. This represents the net (conductive) heat flux. For setting up the data, $q_s''(t)$ is known. The next section describes the surface heat fluxes chosen for the simulation studies.
2.3: Chosen Surface Heat fluxes for Inverse Study

To show the flexibility and adaptability of the method, three input flux cases were chosen. Case one consisted of a condition where the input flux resembled a step function with amplitude of $A_0 = 100 \frac{W}{cm^2}$. While the option for net heat flux at $x = 0$ is often challenging to produce in real-life scenarios, it was considered because it can be easily modeled and represents a challenging reconstruction due to discontinuities (on-off).

Case two, for the input net heat flux, was a Gaussian function with amplitude of $A_0 = 100 \frac{W}{cm^2}$ as shown in Figure 2.1. This case was chosen as it resembles a typical atmospheric maneuver for a high-speed flight vehicle. Case three is a double Gaussian function reflecting that of a complex maneuver that a flight vehicle might encounter during a long-term gliding event. The data from these input conditions will be used in the inverse analysis to accurately reconstruct surface heat flux. Case featuring the single Gauss will be the input flux case that is extensively described throughout this work.

Figure 2.1: Input flux case 2: single gauss
2.4: Temperature Distribution

The temperature distribution in a sample is also of great interest within the hypersonic community. Let $\theta(x, t)$ be defined as the reduced temperature and given as $T(x, t) = \theta(x, t) + T_0$. Assuming $q''(0, t)$ is known, the exact temperature distribution at any point $x$ in space within the sample, see Appendix A for derivation, can be obtained as

$$
\theta(x, t) = \frac{\alpha}{k} \sum_{m=0}^{\infty} \frac{\psi_m(x)}{N_m} \int_{u=0}^{t} q''(u)e^{-\alpha\lambda_m^2(t-u)} \, du, \quad x \in [0,w], \quad t \geq 0 \quad 2.2.a
$$

where $\psi_m(x)$ is the $m^{th}$ eigenfunction defined as

$$
\psi_m(x) = \cos(\lambda_m x), \quad m = 0, 1, ..., \infty \quad 2.2.b
$$

and where $\lambda_m$ defines the $m^{th}$ eigenvalue as

$$
\lambda_m = \frac{m\pi}{w}, \quad m = 0, 1, ..., \infty \quad 2.2.c
$$

with the $m^{th}$ normalization integral, $N_m$, defined as

$$
N_m = \begin{cases} w, & m = 0 \\ \frac{w}{2}, & m = 1, 2, ..., \infty \end{cases} \quad 2.2.d
$$

The temperature profile at any location in the slab subject to some prescribed surface heat flux condition is now available. Figure 2.2 shows the temperature histories produced by the previously described single Gauss heat flux. Table 2.1 contains the thermophysical properties used for this simulation (stainless steel 304) [12-15].
Table 2.1: Thermophysical properties of stainless steel 304

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity, $k$</td>
<td>14.7 (W/m-K)</td>
</tr>
<tr>
<td>Density, $\rho$</td>
<td>6861 (kg/m$^3$)</td>
</tr>
<tr>
<td>Specific heat, $C_p$</td>
<td>571.345 (J/kg-K)</td>
</tr>
<tr>
<td>Speed of sound at room temperature</td>
<td>5750 (m/s)</td>
</tr>
<tr>
<td>Coefficient of linear expansion, $\beta$</td>
<td>$17.6 \times 10^{-6}$ (m/m-K)</td>
</tr>
</tbody>
</table>

Using a Gaussian function as the input case for surface heat flux, it can be observed, from Figure 2.2, that the maximum reduced surface temperature (active side) at $x = 0$ is roughly 195°C. For the single region problem defined in Figure 2.3, the sensor (passive) side must remain cool enough to not corrupt or damage the sensor (max 50°C). From Figure 2.2, the temperature at the back boundary defined as $x=w$ (passive side) where the acoustic transducer is located, is roughly 0°C. This simple analysis allows for using this type of sensor for the heat flux and time span indicated without damage concerns to the sensor.

Should there be need for these instruments be used in scenarios where the back boundary temperature exceeds the recommended maximum value, adding an insulator (buffer layer between sample material and transducer) material such as quartz can then allow for those applications to be considered.
2.5: Time-of-flight

In this work, a non-intrusive “pulse-echo” (P.E.) transmitter-receiver acoustic transducer [7-8], as shown below in Fig. 2.3, is attached to the back boundary of a stainless steel sample with a thickness of one inches (i.e. \( w = 2.54 \) cm)
The P.E. acoustic transducer used in this analysis sends a longitudinal/compressional wave from the passive side to the active side and the reflected signal is collected at the passive side using the receiver [1]. The speed of sound in solids depends on various mechanical properties such as Young’s modulus, bulk modulus, shear modulus, and density for elastic materials. These mechanical properties are usually temperature dependent. The local speed of sound can be estimated using these properties and expressed as a function of reduced temperature, $\theta(x, t)$. From Frankel and Bottländer [1], this relationship is

$$\int_{x=0}^{w} \theta(x, t) \, dx = \lambda_o \big[ \bar{G}(t) - G_0 \big]$$ \hspace{1cm} 2.3.a

$$\lambda_o = \frac{c(T_0)}{2(\beta_0 \left[ \frac{1}{c(T_0)} \right] \frac{dc}{dT} |_{T=T_0})}$$ \hspace{1cm} 2.3.b
where \( G(t) \approx \tilde{G}(t) \) is the time-of-flight variable, \( \lambda_o \) is called the acoustic parameter, \( c(T_0) \) is the speed of sound evaluated at the initial condition \( T_0 \), \( \beta_0 \) is the linear thermal expansion coefficient at the initial condition \( T_0 \). As pointed out by Frankel and Bottländer [1], \( \tilde{G}(t) \) approximates \( G(t) \) due to series truncation.

The uniqueness of this work revolves around the use of time-of-flight (T.o.F.) as the single and only input required to determine both the surface heat flux and surface temperature. During many practical applications, this will be the only data that is measured using the P.E. acoustic transducer. This measured data form, unlike locally measured values obtained by heat flux or temperature gauges, is obtained as a global averaged value within the sample. Thermocouples are often located in-depth (away from the active boundary) and since signal penetration is a function of time, there is a physical delay associated with this measurement technique. This delay in measurement, while using a TC, can be substantially lessened when using an acoustic transducer as the measurement device.

As mentioned above, the P.E. configuration directly measures T.o.F which can be correlated to heat flux and surface temperature through an energy balance. Using Eq. (2.1.a) and a forward solution of the heat equation, which assumes that \( q''(0, t) \) is known, we can manipulate the analysis and estimate the heat flux from the T.o.F data as [1]

\[
\int_{u=0}^{t} q_s''(u)k_q(t-u)du = \lambda_o \left( \frac{k}{\alpha} \right) \Delta \tilde{G}(t)
\]

where the convolution kernel, \( k_q(t-u) \) in Eq. (2.4.a) is defined as

\[
k_q(t-u) = 1
\]

Here \( \lambda_o \) is the acoustic parameter [1] defined in Eq. (2.3.b), \( k \) is the thermal conductivity, and \( \alpha \) is the thermal diffusivity of the material. The derivation of Eq. (2.4.a) is worked out in extensive detail in Appendix B.
Similarly, it can be shown that if the interest is in seeking surface temperature (instead of heat flux as shown in Eq. (2.4.a)), then the measurement equation becomes

$$\int_{u=0}^{t} \theta(0,u)k_T(t-u)du = f(t)$$  \hspace{1cm} 2.5.a

where the convolution kernel, $k_T(t-u)$ in Eq. (2.5.a) is defined as

$$k_T(t-u) = \frac{1}{\sqrt{t-u}} \sum_{j=0}^{\infty} (-1)^j \left\{ e^{\left(-\frac{(2w_j)^2}{4\alpha(t-u)}\right)} - e^{\left(-\frac{(2w(j+1))^2}{4\alpha(t-u)}\right)} \right\}$$  \hspace{1cm} 2.5.b

with the resulting forcing function, $f(t)$ defined as

$$f(t) = \frac{\sqrt{\pi} \lambda_o}{\sqrt{\alpha}} [G(t) - G_0]$$  \hspace{1cm} 2.5.c

where $G_0$ is the T.o.F at the uniform initial condition. The derivations of Eq. (2.5.a), Eq. (2.5.b), and Eq. (2.5.c) are worked out in extensive detail in Appendix C.

At this junction, T.o.F data are numerically generated for the present investigation. The major contribution of this study is to demonstrate a viable methodology for estimating surface heat flux and temperature. Algebraic manipulations to Eq. (2.4.a) and Eq. (2.5.a) yield the expression which can be solved to numerically generate the T.o.F data. Once the expression for T.o.F is obtained, it can be discretized and the definite integral can be numerically approximated using a trapezoidal rule. To validate the governing equations, Eq. (2.4.a) and Eq. (2.5.a) were both numerically solved as they should yield the same T.o.F data. Figure 2.4, estimated using Eq. (2.4.a), shows the T.o.F behavior as a function of time for the prescribed input flux condition.
Similarly, Figure 2.5 shows an overlay of T.o.F data estimated using Eq. (2.5.a). The results support the hypothesis that the T.o.F data should be exactly the same when calculated from surface temperature or heat flux. The slight discrepancy could be the result of the numerical method implemented to estimate the results. Results are assumed to be “perfect data” as errors are not yet introduced. Here, the sampling rate (for demonstration purposes) was set to a value that is producible in a data acquisition system. Hence, a continual decrease in the time step (for convergence) was not performed as the heat flux generated T.o.F. calculations were both operationally less and deemed to contain less numerical discretization errors.
Figure 2.5: Time-of-flight data calculated from heat flux and surface temperature to demonstrate the validity of the governing equations (Case 2)

2.6: Simulating Noisy Data

Data generated up to this junction are assumed perfect, but that condition is rarely obtainable in real-world data collection which normally involves random and bias errors. Numerically articulated noise should have these components, but bias errors are normally removed whenever possible. Hence, this study will only consider random errors. Thus, to simulate real-life scenarios, the perfect data generated from above was randomly perturbed about the maximum value per Figure 2.6. Perturbing about the maximum value was selected as this an extreme case, if the inverse analysis can produce stable predictions to this highly noisy condition, then it can be concluded that this approach is compatible to handle complex scenarios.
Figure 2.6: Time-of-flight data calculated from heat flux and perturbed 1 percent of maximum value to simulate extreme real-life scenario (Case 2)
CHAPTER THREE
TRADITIONAL INVERSE ANALYSIS USING FUTURE TIME
METHOD

3.1: Introduction

Inverse problems are prevalent in all branches of physics and engineering. The use of surface mounted sensors is often discouraged in many applications as they may not survive a harsh thermal environment. Examples include development of thermal protection systems (TPS) used in the aerospace industry; study of advanced high-temperature materials; and, in the understanding fire and fusion technology. This limitation can be resolved by using in-depth or backside sensors. These sensor orientations protect the integrity of the probe when they are subjected to hostile conditions to predict surface conditions such as heat flux or temperature. This approach, however, leads to an analysis that is highly ill posed as small errors from the collected discontinuous data leads to dramatic error amplification in the prediction of either surface heat flux or temperature [16].

3.2: Regularization

Equations (2.4.a) and (2.5.a) are first kind Volterra integral equations and as such they require careful analysis to find stable/regularized and accurate predictions. Many regularization methods are available including Singular-Value Decomposition (SVD), Tikhonov Regularization, Digital Filtering, Future time method etc [16]. These methods can produce predictions to some level of accuracy. In this study, inverse analysis is performed through future time method. The crucial part of inverse problem is to scientifically and methodically acquire the optimal regularization parameter; This method is further explained in detail in this chapter.

3.3: Heat Flux

As mentioned earlier, Eq. (2.4.a) is a first kind Volterra integral equation [1]. To stabilize this highly ill-posed equation, a regularization parameter, $\gamma$ must be interjected
into the formulation for stabilizing the proposed numerical method for resolving the surface boundary condition. For the present work, we introduce the notation of future time which serves as the regularization parameter. To begin, we let $t \rightarrow t + \gamma$ into Eq. (2.4a) to obtain

$$\int_{u=0}^{t+\gamma} q_s''(u)k_q(t + \gamma - u)du = \lambda_q \Delta \tilde{G}(t + \gamma), \quad t \geq 0 \tag{3.1.a}$$

where $\lambda_q$ is defined as

$$\lambda_q = \lambda_o \left( \frac{k}{\alpha} \right) \tag{3.1.b}$$

Separating the “forward time region” in the integral representation yields

$$\lambda_q \Delta \tilde{G}(t + \gamma) = \int_{u=0}^{t} q_s''(u)k_q(t + \gamma - u)du$$

$$+ \int_{u=t}^{t+\gamma} q_s''(u)k_q(t + \gamma - u)du \tag{3.1.c}$$

As specified above, future-time will help stabilize the analysis by holding the heat flux fixed in the future time interval, $\in [t, t + \gamma]$. This transforms the ill-posed first kind Volterra integral equation to a well-posed second kind Volterra integral equation; namely

$$\lambda_q \Delta \tilde{G}(t + \gamma) \cong \int_{u=0}^{t} q_s''(u)k_q(t + \gamma - u)du$$

$$+ q_s''(t) \int_{u=t}^{t+\gamma} k_q(t + \gamma - u)du \tag{3.1.d}$$

Next, we introduce the first in a series of approximations as
\[ q_s''(u) \cong q_s''(t), u \in [t, t + \gamma] \]  \hspace{1cm} 3.1.e

Next, we move from the continuous time domain to the discrete time domain in Eq. (3.1.d), by letting \( t \rightarrow t_i \) and hence \( \gamma \rightarrow \gamma_m \)

\[
\lambda_q \Delta \tilde{G}(t_i + \gamma_m) \cong \\
\int_{u=0}^{t_i} q_s''(u)k_q(t_i + \gamma_m - u)du + q_s''(t_i) \int_{u=t_i}^{t_i+\gamma_m} k_q(t_i + \gamma_m - u)du \\
i = 1,2, \ldots, N - mM_f, m = [1,2,3,4,5,6,7] 
\]

where \( M_f \) is a scalar multiplying factor, in this study \( M_f = 5 \), to reasonably scale the future-time parameter, \( \gamma_m \), defined as

\[
\gamma_m = mM_f \Delta t 
\]  \hspace{1cm} 3.1.g

and \( t_i \) in Eq. (3.1.f) is defined as

\[
t_i = (i - 1)\Delta t, \quad i = 1,2, \ldots, N 
\]  \hspace{1cm} 3.1.h

where \( N \) equals the total number of data collected and \( \Delta t \) is defined as

\[
\Delta t = \frac{t_{max}}{N - 1} 
\]  \hspace{1cm} 3.1.i

The definite integrals defined in Eq. (3.1.f) can be approximated using a single panel approximation (i.e., trapezoidal) as
\[\lambda_q \Delta \tilde{G}(t_i + \gamma_m) \equiv \sum_{j=1}^{i-1} \int_{u=t_{ij}}^{t_{ij+1}} q''_s(u)k_q(t_i + \gamma_m - u)du + q''_s(t_i)C_{\gamma m}, \quad 3.1.j\]

\[i = 1, 2, \ldots, N\]

where \(C_{\gamma m}\) is defined as

\[C_{\gamma m} = \int_{u=t_i}^{t_i + \gamma_m} k_q(t_i + \gamma_m - u)du \quad 3.1.k\]

Next, release the last panel in the numerical approximation to create additional stability. This leads to the intermediate formulation

\[\lambda_q \Delta \tilde{G}(t_i + \gamma_m) \equiv \sum_{j=1}^{i-2} \int_{u=t_{ij}}^{t_{ij+1}} q''_s(u)k_q(t_i + \gamma_m - u)du \quad 3.1.1\]

\[+ q''_s(t_i) \int_{u=t_{i-1}}^{t_i} k_q(t_i + \gamma_m - u)du + q''_s(t_i)C_{\gamma m}\]

After some additional algebraic manipulations, we obtain

\[\tilde{q}''_s(t_i) = \frac{\lambda_q \Delta \tilde{G}(t_i + \gamma_m) - \sum_{j=1}^{i-2} \int_{u=t_{ij}}^{t_{ij+1}} \tilde{q}''_s(u)k_q(t_i + \gamma_m - u)du}{C_{\gamma m} + \int_{u=t_{i-1}}^{t_i} k_q(t_i + \gamma_m - u)du} \quad 3.1.m\]

where \(\tilde{q}''_s(t_i) \equiv q''_s(t_i), i = 1, 2, \ldots, M.\)

As evident from the final data reduction equation, the present form requires little numerical processing for resolving the heat flux than many traditional approaches.
3.3.1: Family of Predictions Using Perfect Data

As mentioned earlier, inverse analysis provides a family of predictions based on the regularization parameter’s value. A key to quality estimation is the ability to identify the optimal prediction that minimizes the error (which is unknown). As error in the input increases, the inverse analysis becomes highly ill-posed. To illustrate this concept, consider the traditional inverse analysis with perfect data (no noise. There are nearly an infinite number of values that can be used as the future time parameter. In this study, seven such values were considered, $m = [1,2,3,4,5,6,7]$. The goal is to extract the value of the regularization parameter that can produce the minimum error. Thus, extracting this optimal value represents the challenge for all inverse methods. Figure 3.1 displays the family of heat flux predictions for Case 2. It is evident that for large $\gamma$ that over-smoothing effects are observed.

![Figure 3.1: Traditional inverse reconstruction with perfect data (Case 2)](image)
3.3.1.1: Isolating the Optimal Prediction

The uniqueness of this study is selecting the optimum regularization parameter from the provided set of choices. To visually and mathematically select the optimum value, this study applies a combination of phase-plane and cross-correlation analyses to the prediction family in order to estimate the optimal regularization parameter and hence the optimal heat flux prediction. Figure 3.2 represents the phase plane analysis for the reconstruction represented in Figure 3.1. Phase plane analysis presents a visual aid in identifying the optimal prediction. When the phase-plane prediction, for fixed regularization parameter, begins to form a pattern or shape, it can be assumed that one is near the optimal prediction.

Figure 3.3 shows the relationship between derivative of the cross-correlation coefficients, \( \dot{\rho} \), plotted with respect to the cross-correlation coefficient \( \rho \). Cross-correlation is the measure of similarity between two series data streams. This is also known as a sliding dot product or sliding inner product [16]. Normalized expression for cross-correlation can be defined as

\[
\rho_{1,2}(j) = \frac{1}{N} \frac{\sum_{n=0}^{N-1} x_1(n)x_2(n)}{\left[ \sum_{n=0}^{N-1} x_1^2(n) \sum_{n=0}^{N-1} x_2^2(n) \right]^{\frac{1}{2}}}
\]

3.2
Figure 3.2: Phase-plane analysis for the inverse analysis with perfect measurement data (Case 2)

Cross-correlation provides a mathematical basis or metric on how to identify the optimal regularization parameter. The phase plane presentation displayed in Fig. 3.2 provides a visual aid that is highly helpful. As evident by Figures 3.2 and Fig. 3.3 below, phase plane and cross-correlation analyses do not strongly aid in finding the optimal future time prediction when in the presence of perfect measurement data.
3.3.2: Family of Predictions Using Real-Life (Noisy) Data

To simulate more physically correct conditions, random noise (approximately one percent) is added to the T.o.F data sampled at a frequency of 100 Hz. The random noise added is one percent of the maximum value. This condition was considered as the worst-case scenario. Inverse analysis is performed using Eq. (3.1.m). Figure 3.4 displays the heat flux predictions based on the displayed future-time parameters. That is, small future-time parameters show instability while large future time parameters show over-smoothing effects.
3.3.2.1: Isolating the Optimal Prediction (Noise)

Identifying optimal predictions from a family of inverse reconstructions is not trivial as the error in the calculations adds additional complications. Figure 3.5 shows the phase plane of traditional reconstructions involving noisy data. Figure 3.5 does not display the formation of a clear pattern. This is indicative of insufficient filtering somewhere in the methodology. The phase plane and cross-correlation tools previously described may have difficulties at the present level of analysis.
Figure 3.5: Phase-plane analysis for the traditional inverse analysis with noisy measurement data

To further illustrate the limitations of the traditional or pedestrian formulation, consider Figure 3.6 which plots the time derivative cross-correlation coefficients versus the cross-correlation coefficients. While the results are moving to the top-right corner as expected, this method shows no correlation between the families of predictions. As a result, the optimal prediction cannot be isolated.
Figure 3.6: Derivative of cross-correlation coefficients vs cross-correlation coefficient for traditional inverse analysis with noisy measurement data

3.3.2.2: Error Analysis (Root-Mean Square Error, RMSE)

Root-mean-square experimental error for the traditional formulation provides a qualitative view on the prevalence of error in this methodology. Again, in real experiments, the RMSE does not exist. Table 3.1 highlights these values and shows the basic effect of an inverse analysis. That is, the RMSE decreases with increasing regularization parameter up to a point but then the NMSE increases with increasing regularization parameter. Finding a method that identifies this minimum values is key to inverse analysis.
Table 3.1: Root-mean-square experimental error values for traditional inverse analysis

<table>
<thead>
<tr>
<th>M</th>
<th>$\gamma_m(s)$</th>
<th>Traditional analysis (W/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>139299.2670</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>80124.8238</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>57237.1894</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>46254.3707</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>41012.4123</td>
</tr>
<tr>
<td>6</td>
<td>0.3</td>
<td>39126.1807</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>39381.3406</td>
</tr>
</tbody>
</table>
CHAPTER FOUR
PRECONDITIONED INVERSE ANALYSIS USING FUTURE TIME
METHOD

4.1: Introduction

As demonstrated in Chapter 3, when attempting to perform inverse analysis on measurement data with noise, the traditional or pedestrian formulation fails to define a strong method for identifying the optimal regularization parameter based on the proposed phase-plane and cross-correlation tools. This chapter aims to propose an alternative integral formulation for prior to using the identification tools (phase plane and cross correlation) described in Chapter 3. It will be demonstrated that preconditioning the formulation permits a means for identifying the optimal regularization parameter using the tools of Chapter 3. Further, the numerical implementation still remains simple and intuitive.

4.1.1: Preconditioning

Several attractive features will be demonstrated for the newly proposed preconditioned method for inverse analysis. The soon to be described preconditioner will act as a parameter-free, low-pass filter providing enough information to utilize phase plane and cross correlation for identifying the optimal future time parameter.

4.2: Heat Flux

The preconditioner concept will be developed in the context of heat flux per the formulation proposed in Eq. (2.4.a). This equation, once converted to the frequency domain using Laplace Transform, is multiplied by $\frac{1}{s^n}$ to get the desired amount of filtering. As is evidenced from this study, the filtering operation is applied to the governing equation rather than to the measured T.o.F. data. In this way, the equality sign is retained in the conservation of energy. The value of $n$, or the attenuation factor, determines the amount of filtering that is applied to the equation.
4.3: Case A: n = 1/2

To begin, take the Laplace Transform to Eq. (2.4.a) as

\[ \mathcal{L}\{\Delta G(t)\} = \frac{1}{\lambda_q} \mathcal{L} \left\{ \int_{u=0}^{t} 1 \cdot q''_s(u) \, du \right\} \]

After Laplace Transform is applied, the above equation becomes

\[ \lambda_q \Delta \hat{G} = \hat{q''_s}(s) \cdot \frac{1}{s} \]

Next, multiply both sides by \( \frac{1}{s^{(1/2)}} \) to obtain

\[ \frac{\lambda_q \Delta \hat{G}}{s^{(1/2)}} = \frac{\hat{q''_s}(s)}{s^{(3/2)}} \]

Taking the inverse Laplace Transform of Eq. (4.2) ([17] p. 1022 Eq. (29.3.4) and Eq. (29.3.5)) yields

\[ \lambda_q \int_{u=0}^{t} \frac{\Delta G(u)}{\sqrt{\pi(t-u)}} \, du = \int_{u=0}^{t} q''_s(u) \frac{2\sqrt{t-u}}{\sqrt{\pi}} \, du, \quad t \geq 0 \]

For the case of \( n = \frac{1}{2} \), the filtered equation takes the form of Eq. (4.3). The future-time method is applied to Eq. (4.3). The final data reduction equation can be expressed simply as

\[ \hat{q''_s}(t_i) = \frac{A - B}{C + D}, \quad i = 2,3,\ldots, N - mMf \]
where the function $A$ is defined as

$$A = \frac{\lambda_q}{2} \sum_{j=1}^{i+mMf-1} \int_{u=t_j}^{t_{j+1}} \Delta \tilde{G}(u) \sqrt{(t_i + \gamma_m - u)} du$$

4.4.b

function $B$ is defined as

$$B = \sum_{j=1}^{i-2} \int_{u=t_j}^{t_{j+1}} q''_s(u) \sqrt{(t_i + \gamma_m - u)} du$$

4.4.c

function $C$ is defined as

$$C = \int_{u=t}^{t+y} \sqrt{(t_i + \gamma_m - u)} du$$

4.4.d

And function $D$ is defined as

$$D = \int_{u=t_{i-1}}^{t_i} \sqrt{(t_i + \gamma_m - u)} du$$

4.4.e

4.3.1: Family of Predictions Using Noisy Data (Case 2)

Similar to the traditional or pedestrian formulation, this analysis was performed on noisy data sampled at a sampling frequency at 100 Hz. The preconditioned inverse analysis was performed for reconstructing the surface heat flux using Eq. (4.4.a). Figure 4.1 shows the effect on the heat flux predictions using the previously defined future time parameters. The jump toward stability is highlighted when using this value of $n=1/2$ in the preconditioner.
Figure 4.1: Preconditioned inverse reconstruction, Case A \((n = 0.5)\), with noisy data (Case 2)

4.3.2: Isolating the Optimal Prediction

Although this case was successful in reducing the error bandwidth in the final ill-posed reconstruction, the main objective of any inverse analysis is to isolate the optimal prediction from the family of predictions. Figure 4.2 displays the resulting phase plane analysis for Case A using the data generated by Case 2 heat flux. The attenuation factor for this case was not effective in isolating the optimal prediction. Figure 4.3 further solidifies this claim as the cross-correlation phase plane plot fails to show strong correlation between the families of predictions.
Figure 4.2: Phase-plane analysis for the preconditioned inverse analysis, case A, with noisy measurement data (Case 2)
Figure 4.3: Derivative of cross-correlation coefficients vs cross-correlation coefficient for preconditioned inverse analysis, case A, with noisy measurement data

4.3.3: Root-Mean Square Error (RMSE)

The root-mean-square error for the preconditioned inverse analysis provides insight on how effective the parameter free preconditioner is in reducing errors. The resulting analysis still contained non-negligible error, but when compared to Table 3.1, there is a significant improvement.
Table 4.1: Root-mean-square experimental error values for preconditioned inverse, case A, analysis, (Case 2)

<table>
<thead>
<tr>
<th>M</th>
<th>$\gamma_m(s)$</th>
<th>Preconditioned analysis (W/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>97777.5012</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>36903.4372</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>23587.2131</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>23552.8086</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>28009.3971</td>
</tr>
<tr>
<td>6</td>
<td>0.3</td>
<td>33706.4007</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>39730.9350</td>
</tr>
</tbody>
</table>

4.4: Case B: $n = 3/4$

First, we apply the Laplace Transform to Eq. (2.3.a) to get in the form represented by Eq. (4.1.b). Next, we multiply both sides of Eq. (4.1.b) by $\frac{1}{S^{(3/4)}}$ to obtain

\[
\frac{\lambda_q \Delta G}{S^{(3/4)}} = \overline{q''_s}(s) \cdot \frac{1}{S^{(7/4)}}
\]

Taking the inverse Laplace Transform of Eq. (4.5) produces

\[
\lambda_q \int_{u=0}^{t} \frac{\Delta G(u)}{\sqrt{t-u} \Gamma \left(\frac{3}{4}\right)} du = \int_{u=0}^{t} q''_s(u) \frac{(t-u)^{3/4}}{\Gamma \left(\frac{7}{4}\right)} du, \quad t \geq 0
\]

where the $\Gamma(x)$ is the gamma function defined as ([17] p. 255 Eq. (6.1.1))

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt \quad \Re(x) > 0
\]
For this case, \( n = \frac{3}{4} \), the preconditioned measurement equation takes the form of Eq. (4.6.a). Following our previously outlined procedure for the regularization process based on the future time, it can be shown to produce

\[
\tilde{q}_s''(t_i) = \frac{A - B}{C + D}, \quad i = 2, 3, \ldots, N - mMf \quad 4.7.a
\]

where the function \( A \) is defined as

\[
A = \lambda_q \sum_{j=1}^{i+mMf-1} \int_{u=t_j}^{t_{j+1}} \Delta \tilde{G}(u) \frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{t_i + \gamma_m - u}} \, du \quad 4.7.b
\]

function \( B \) is defined as

\[
B = \sum_{j=1}^{i-2} \int_{u=t_j}^{t_{j+1}} q_s''(u) \frac{(t_i + \gamma_m - u)^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)} \, du \quad 4.7.c
\]

function \( C \) is defined as

\[
C = \int_{u=t}^{t+\gamma} \frac{(t_i + \gamma_m - u)^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)} \, du \quad 4.7.d
\]

Function \( D \) is defined as

\[
D = \int_{u=t_{i-1}}^{t_i} \frac{(t_i + \gamma_m - u)^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)} \, du \quad 4.7.e
\]
4.4.1: Family of Predictions Using Noisy Data (Case 2)

Similar to Case A, this analysis was performed on noisy data sampled at a sampling frequency at 100 Hz. As expected, the preconditioned analysis was effective in filtering the equation and thus was successful in producing a stable reconstruction. Figure 4.4 displays substantial improvement when compared to the traditional or pedestrian formulation.

![Graph](image)

Figure 4.4: Preconditioned inverse reconstruction, case B, with noisy data (case 2)

4.4.2: Isolating the Optimal Prediction

Phase-plane analysis for Case B is given in Fig. 4.5. This figure demonstrates that as the future-time-parameter increases, a shape forms indicative of near-sufficient physical smoothing. This proves that the preconditioned inverse analysis is conducive to generating a means for estimating the optimal regularization parameter by phase plane and cross correlation principles.
Figure 4.5: Phase-plane analysis for the preconditioned inverse analysis, case B, with noisy measurement data (Case 2)

Figure 4.6 displays the cross-correlation, phase-plane plot indicating an apparent correlation amongst the family of predictions as the future-time-parameter increases. Using this estimation, the optimal prediction can be isolated. To isolate the optimal prediction, based on the cross-correlation, phase-plane plot, the best prediction is located as both $\rho \to 1$ and $\dot{\rho} \to 0.75$. Figure 4.7 highlights the optimal heat flux prediction using highly noisy time-of-flight data.
Figure 4.6: Derivative of cross-correlation coefficients vs cross-correlation coefficient for preconditioned inverse analysis, Case B, with noisy measurement data (Case 2)
Figure 4.7: Isolating the optimal prediction for preconditioned inverse analysis, case B, with noisy measurement data (Case 2)

4.4.3: Root-Mean Square Error (RMSE)

The root-mean-square error for the preconditioned inverse analysis, Case B, provides quantitative insight on how effective the parameter free preconditioner was on the noisy time-of-flight data in forming the heat flux approximation for specified future-time parameter. The attenuation factor was not large enough to effectively reduce the error completely. However, there is a significant improvement in the overall error reduction as indicated by comparing Table 3.1 to Table 4.1.
Table 4.2: Root-mean-square experimental error values for preconditioned inverse, Case 
B, analysis, (Case 2)

<table>
<thead>
<tr>
<th>M</th>
<th>$y_m(s)$</th>
<th>Preconditioned analysis (W/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>156854.3569</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>54475.9163</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>34245.2470</td>
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<tr>
<td>4</td>
<td>0.2</td>
<td>30455.7480</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>32056.2453</td>
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<tr>
<td>6</td>
<td>0.3</td>
<td>35657.5229</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>40092.5262</td>
</tr>
</tbody>
</table>

4.5: Case C: $n = 1$

As before, we apply the Laplace Transform to Eq. (2.4.a) to get in the form 
represented by Eq. (4.1.b). Next, we multiply both sides of Eq. (4.1.b) by $\frac{1}{s}$ to obtain

$$\frac{\lambda_q \Delta \hat{G}}{s} = \frac{q''_s(s)}{s^2} \cdot \frac{1}{s^2}, \quad \Re(s) > 0$$

4.8

Take the inverse Laplace Transform of Eq. (4.8) above to obtain ([17] p.1022 Eq. (29.3.3))

$$\lambda_q \int_{u=0}^{t} \Delta G(u) du = \int_{u=0}^{t} q''_s(u)(t-u) du, \quad t \geq 0$$

4.9

For this case, $n = 1$, the filtered equation takes the form of Eq. (4.9). Apply the future-
time method to this equation by letting $t \rightarrow t + \gamma$. The final data reduction equation can 
be expressed generally as

$$\tilde{q}''_s(t_i) = \frac{A - B}{C + D}, \quad i = 2,3, ..., N - mMf$$

4.10.a
where the function A is defined as

\[
A = \lambda_q \sum_{j=1}^{i+mMf-1} \int_{u=t_j}^{t_{j+1}} \Delta \tilde{G}(u) du
\]

4.10.b

function B is defined as

\[
B = \sum_{j=1}^{i-2} \int_{u=t_j}^{t_{j+1}} q''_s(u)(t_i + \gamma_m - u) du
\]

4.10.c

function C is defined as

\[
C = \int_{u=t}^{t+\gamma} (t_i + \gamma_m - u) du
\]

4.10.d

Function D is defined as

\[
D = \int_{u=t_{i-1}}^{t_i} (t_i + \gamma_m - u) du
\]

4.10.e

4.5.1: Family of Predictions Using Noisy Data

Similar to Cases A, B, this analysis was also performed using noisy time-of-flight data sampled at a sampling frequency at 100Hz. The preconditioned inverse analysis, for case C, was performed to the measurement data for reconstructing the surface heat flux using Eq. (4.10.a). Figure 4.8 displays the resulting family of heat flux predictions over the indicated values of the regularization parameters. When comparing with the results from the traditional inverse analysis, Cases A and B using the preconditioned method, Case C has further success in reducing the error bandwidth. The attenuation factor for
Case C, $n = 1$, provides sufficient parameter free filtering to modeled system for enabling further accuracy.

Figure 4.8: Preconditioned inverse reconstruction, case C, with noisy data (Case 2)

4.5.2: Isolating the Optimal Prediction

Figure 4.9 clearly demonstrates the onset of a pattern indicative of optimality. That is, the cusp of stability (smallest future time parameter that produces a pattern) is seen when $n = 1$. Visually, $m = 6,9$ seems to indicate a near optimal heat flux prediction.
Figure 4.9: Phase-plane analysis for the preconditioned inverse analysis, case C, with noisy measurement data (Case 2)

Figure 4.10 shows a strong correlation amongst the family of predictions as the future-time-parameter increases. Here, the future time parameter in the range 0.2 and 0.25 seconds narrows the optimal values. Figure 4.11 highlights the optimal heat flux prediction for $m=4 \ (\gamma_4=0.2s)$ and $m=5 \ (\gamma_5=0.25s)$. These results appear superior to Cases A,B.
Figure 4.10: Derivative of cross-correlation coefficients vs cross-correlation coefficient for preconditioned inverse analysis, case C, with noisy measurement data (Case 2)
Figure 4.11: Isolating the optimal prediction for preconditioned inverse analysis, case C, with noisy data (Case 2)

4.5.3: Root-Mean Square Error (RMSE)

The root-mean-square error for the preconditioned inverse analysis, Case C, shows the effectiveness of the proposed scheme. Table 4.3 highlights the RMSE of the heat flux over increasing values of the future time parameter.
Table 4.3: Root-mean-square experimental error values for preconditioned inverse analysis, Case C using noisy T.o.F data generated for Case 2

<table>
<thead>
<tr>
<th>M</th>
<th>$\gamma_m$ (s)</th>
<th>Preconditioned analysis (W/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>197518.5914</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>55166.3315</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>29137.8589</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>24259.0857</td>
</tr>
<tr>
<td>5</td>
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<td>25952.0497</td>
</tr>
<tr>
<td>6</td>
<td>0.3</td>
<td>29705.2411</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>34141.3362</td>
</tr>
</tbody>
</table>
CHAPTER FIVE
CONCLUSION

5.1 Conclusions

Inverse applications in engineering often rely on in-depth measurement techniques due to the surface exposure to harsh thermal or chemical environments. Analysis of the collected data, when mathematically projected to predict surface conditions such as surface heat flux or temperature, are often ill-posed as any small error in the measurement leads to dramatic error amplification of the inverse prediction. This limitation leads to challenging mathematical analysis that generally relies upon filtering the measured data before analysis. As demonstrated in this study, traditional or pedestrian inverse analysis is often ineffective in isolating the optimal predictions, a key objective in any inverse analysis. This study applied a parameter free preconditioner to the governing functional equation for effectively producing a formulation conducive to identifying the optimal regularization parameter based on phase-plane analysis and cross-correlation principles. As the reported results suggest, this methodology was highly successful in isolating the optimal regularization parameter and thereby producing a representative and accurate prediction.

5.2 Future Work

The methodology described in this thesis can be further generalized for estimating the surface temperature. This is also a fundamental property required in many engineering studies. It is expected that a similar Volterra integral equation (but in surface temperature) can be formulated possessing a complex kernel, shown in detail in Appendix D and E. The approach taken here should be applicable to this output requirement.

An experiment should be developed using Stainless Steel 304 (due to its ultrasonic characterization being well defined). The major issues lie in the actual instrumentation where synchronization is required between the input heat flux and
measured time-of-flight. Most purchasable T.o.F. experimental set ups do not allow for this. This is needed for forming a benchtop and benchmark investigation.
REFERENCES


[14] Livschitz, B. G.: Physikalische Eigenschaften der Metalle u


APPENDIX A

Derivation of the exact temperature distribution for a finite plate (non-homogeneous linear heat equation)

Consider the transient, one-dimensional, constant property heat equation in reduced temperature given as [11]

\[
\frac{1}{\alpha} \frac{\partial \theta}{\partial t}(x, t) = \frac{\partial^2 \theta}{\partial x^2}(x, t), \quad x \in [0, w], \quad t \geq 0 \tag{A.1.a}
\]

subject to the boundary conditions

\[
q''(0, t) = -k \frac{\partial \theta}{\partial x}(0, t) = q_s''(t) \tag{A.1.b}
\]

\[
q''(w, t) = -k \frac{\partial \theta}{\partial x}(w, t) = q_w''(t), \quad t \geq 0 \tag{A.1.c}
\]

and initial condition

\[
\theta(x, 0) = 0, \quad x \in [0, w] \tag{A.1.d}
\]

To begin the analysis, consider the corresponding homogeneous system to Eq. (A.1.a) as

\[
\frac{1}{\alpha} \frac{\partial \theta_H}{\partial t}(x, t) = \frac{\partial^2 \theta_H}{\partial x^2}(x, t), \quad x \in [0, w], \quad t \geq 0 \tag{A.2.a}
\]

subject to the boundary conditions

\[
q''(0, t) = -k \frac{\partial \theta_H}{\partial x}(0, t) = 0 \tag{A.2.b}
\]
\[ q''(w, t) = -k \frac{\partial \theta_H}{\partial x}(w, t) = 0, \quad t \geq 0 \quad \text{A.2.c} \]

and initial condition

\[ \theta_H(x, 0) = 0, \quad x \in [0, w] \quad \text{A.2.d} \]

To solve Eq. (A.2.a), apply the method of separation of variables and assume solution exists in the form

\[ \theta_H(x, t) = X(x)\Gamma(t) \quad \text{A.3} \]

Substitute Eq. (A.3) into the heat equation (Eq. A.2.a) and associated boundary conditions, Eq. (A.2.b) and Eq. (A.2.c). The heat equation becomes

\[ \frac{1}{\alpha}X(x)\dot{\Gamma}(t) = X''(x)\Gamma(t) \quad \text{A.4} \]

Upon separating

\[ \frac{1}{\alpha} \frac{\dot{\Gamma}(t)}{\Gamma(t)} = \frac{X''(x)}{X(x)} = \sigma, \quad \sigma = \begin{cases} \lambda^2 \\ 0 \\ -\lambda^2 \end{cases} \quad \text{A.5} \]

Apply to boundary conditions to obtain

\[ X'(0)\Gamma(t) = X'(w)\Gamma(t) = 0 \quad \text{A.6} \]

If \( \Gamma(t) = 0 \) for all \( t > 0 \), the solution becomes trivial and hence is not allowed. Therefore, the eigenvalue problem (EVP) takes the form
\[ X''(x) - \sigma X(x) = 0, \quad x \in [0, w] \]  

Subject to the separated boundary conditions

\[ X'(0) = X'(w) = 0 \]

Case 1: \( \sigma = \lambda^2 \)
Solution to the EVP becomes

\[ X(x) = A \cosh(\lambda x) + B \sinh(\lambda x) \]

\[ X'(x) = \lambda (A \sinh(\lambda x) + B \cosh(\lambda x)) \]

Apply the boundary conditions to the solution above to calculate the unknowns, \( A \) and \( B \).

\[ X'(0) = 0 = B \lambda, \quad B = 0 \]

\[ X'(w) = 0 = A \lambda \sinh(\lambda w), \quad A = 0 \text{ since } \sinh(z) \neq 0 \]

Case 1, \( \sigma = \lambda^2 \), only produces the trivial solution. Thus \( \sigma \neq \lambda^2 \)

Case 2: \( \sigma = 0 \)
Solution to the EVP becomes

\[ X(x) = A(x) + B \]

\[ X'(x) = A \]

Apply the boundary conditions to the solution above to calculate the unknown, \( A \)
\[ X'(0) = 0 = A \quad \text{A.9.c} \]

Thus for case 2, the solution becomes
\[ X_0(x) = B_0 \psi_0(x) \quad \text{A.10.a} \]
where the eigenvalue is
\[ \sigma = \lambda_0 = 0 \quad \text{A.10.b} \]
the eigenfunction is defined as
\[ \psi_0(x) = 1 \quad \text{A.10.c} \]
and the normalization integral becomes
\[ N_0 = \int_{x=0}^{L} \psi_0^2(x) \, dx = w, \quad m = 0 \quad \text{A.10.d} \]

Case 3: \( \sigma = -\lambda^2 \)

Solution to the EVP becomes
\[ X(x) = A \cos(\lambda x) + B \sin(\lambda x) \quad \text{A.11.a} \]
\[ X'(x) = \lambda (-A \sin(\lambda x) + B \cos(\lambda x)) \quad \text{A.11.b} \]

Apply the boundary conditions to the solution above to calculate the unknowns, \( A \) and \( B \).
\[ X'(0) = 0 = B \lambda, \quad B = 0 \quad \text{A.11.c} \]
\[ X'(w) = 0 = -A\lambda \sin(\lambda w) \quad \text{A.11.d} \]

If \( A, \lambda = 0 \) the solution will result in a trivial solution similar to case 1, thus, \( \sin(\lambda w) \) must equal zero. Therefore, for case 3, the solution becomes

\[ X_m(x) = A_m \psi_m(x) \quad \text{A.12.a} \]

where the eigenvalues are

\[ \lambda_m = \frac{m\pi}{w}, \ m = 0,1,2, ... \quad \text{A.12.b} \]

the eigenfunctions are defined as

\[ \psi_m(x) = \cos(\lambda_m x), \ m = 0,1,2, ... , \ x \in [0,w] \quad \text{A.12.c} \]

and the normalization integrals are

\[ N_m = \int_{x=0}^{L} \psi_0^2(x) \, dx = \frac{w}{2}, \ m = 1,2, ... \quad \text{A.12.d} \]

After obtaining the eigensets, apply the integral transform, Eq. (A.13.a) to Eq. (A.1.a) and the result into the inversion formula below, Eq. (A.13.b),

\[ \bar{T}_m = \int_{x=0}^{L} \psi_m(x)T(x,t) \, dx, \ m = 0,1,2, ... , \ t \geq 0 \quad \text{A.13.a} \]

\[ T(x,t) = \sum_{m=0}^{\infty} \frac{\psi_m(x)}{N_m} \bar{T}_m, \ x \in [0,w], \ t \geq 0 \quad \text{A.13.b} \]
The temperature distribution within the sample can be expressed as

\[ T(x, t) = \frac{\alpha}{k} \sum_{m=0}^{\infty} \frac{\psi_m(x)}{N_m} \int_{u=0}^{t} e^{-\alpha \lambda_m^2 (t-u)} q''(u) du, \]

\[ x \in [0, w], \quad t \geq 0 \]
APPENDIX B

Derivation of the Surface Heat Flux Equation for Acoustics

To begin, consider a non-intrusive “pulse-echo” (P.E.) transmitter-receiver acoustic transducer, as shown below in Fig. 2.3, is attached to the back boundary of a stainless steel sample with a thickness of one inches (i.e. \( w = 1 \) inches). The travel time, \( G \), or the time-of-flight, T.o.F., for this P.E. configuration is defined as

\[
G = \frac{2w}{c}
\]  

where \( c \) is the speed of sound in that medium. Eq. (B.1) can be discretized to account for elemental or piecewise distribution as

\[
G(t) \approx \sum_{i=1}^{N} G_i(t) = 2 \sum_{i=1}^{N} \frac{\Delta x}{c[T(x_i, t)]} \quad t \geq 0
\]  

for sufficiently large \( N \). To account for piecewise elemental thermal expansion [1] the expression becomes

\[
\bar{w}(t) = w\{1 + \beta_0[T(t) - T_0]\} \quad t \geq 0
\]  

where \( \beta_0 \) is the linear thermal expansion coefficient evaluated at the initial condition \( T_0 \). Substituting the expression for elemental expansion, the T.o.F can be expressed as

\[
G(t) \approx \sum_{i=1}^{N} G_i(t) = 2 \sum_{i=1}^{N} \frac{\Delta x[1 + \beta_0[T(x_i, t) - T_0]]}{c[T(x_i, t)]} \quad t \geq 0
\]
If the linear expansion is considered to be zero, $\beta \rightarrow 0$, Eq. (B.2) can be recovered.

Reference T.o.F, $G_0$, T.o.F at initial temp $T(x,0) = T_0$ can be defined as

$$G_0 = \frac{2w}{c(T_0)} = 2 \sum_{i=1}^{N} \frac{\Delta x}{c[T(x_i,0)]} = 2 \sum_{i=1}^{N} \frac{\Delta x}{c[T_0]} \quad \text{B.5}$$

Let the difference between source on and reference condition be

$$G(t) - G_0 \approx \sum_{i=1}^{N} G_i(t) = 2 \sum_{i=1}^{N} \frac{\Delta x}{c[T(x_i,t)]} + \frac{\Delta x}{c[T_0]} \quad \text{B.6}$$

Take the limit as $N \rightarrow \infty$ (or $\Delta x \rightarrow 0$) and invoke the Reimann sum ([19] p.9) to produce

$$G(t) - G_0 = 2 \int_{x=0}^{w} \left( \frac{1 + \beta_0[T(x,t) - T_0]}{c[T(x,t)]} - \frac{1}{c[T_0]} \right) dx \quad \text{B.7}$$

Upon further manipulation, refer to [1] for more details, the T.o.F can be related to reduced temperature as

$$\int_{x=0}^{w} \theta(x,t) = \frac{c[T_0](G(t) - G_0)}{2[\beta_0 - \frac{1}{c[T_0]} \frac{dc}{dT} |_{T_0}] \alpha} \quad \text{B.8}$$

Now, consider the transient, one-dimensional, constant property heat equation in reduced temperature given as [11]

$$\frac{1}{\alpha} \frac{\partial \theta}{\partial t}(x,t) = \frac{\partial^2 \theta}{\partial x^2}(x,t), \quad x \in [0,w], \quad t \geq 0 \quad \text{B.9.a}$$

subject to the boundary conditions
\[ q''(0, t) = -k \frac{\partial \theta}{\partial x}(0, t) = q_s''(t) \]  
B.9.b

\[ q''(w, t) = -k \frac{\partial \theta}{\partial x}(w, t) = q_w''(t), \quad t \geq 0 \]  
B.9.c

and initial condition

\[ \theta(x, 0) = 0, \quad x \in [0, w] \]  
B.9.d

Integrate Eq. (B.1.a) over the entire space to obtain

\[ \frac{1}{\alpha} \int_{x=0}^{w} \frac{\partial \theta}{\partial t}(x, t) \, dx = \int_{x=0}^{w} \frac{\partial^2 \theta}{\partial x^2}(x, t) \, dx \]  
B.10

Using Leibniz’s rule separate differentiation and integration on the LHS and integrate the RHS to obtain

\[ \frac{1}{\alpha} \frac{d}{dt} \int_{x=0}^{w} \theta(x, t) \, dx = \frac{\partial \theta}{\partial x} \bigg|_0^w \]  
B.11.a

Evaluate the limits to obtain

\[ \frac{1}{\alpha} \frac{d}{dt} \int_{x=0}^{w} \theta(x, t) \, dx = \frac{\partial \theta}{\partial x}(w, t) - \frac{\partial \theta}{\partial x}(0, t) \]  
B.11.b

Multiply RHS by \( \frac{k}{k} \) to obtain
\[
\frac{1}{\alpha} \frac{d}{dt} \int_{x=0}^{w} \theta(x,t) dx = \frac{1}{k} \left[ k \frac{\partial \theta}{\partial x} (w,t) - k \frac{\partial \theta}{\partial x} (0,t) \right]
\]

B.12

Recall, Fourier’s law in reduced temperature as

\[
q''(x,t) = -k \frac{\partial \theta}{\partial x}(x,t)
\]

B.13

Assume the back, passive, boundary to be adiabatic

\[
\frac{\partial \theta}{\partial x}(w,t) = 0
\]

B.14

Plug Eq. (B.5) into Eq. (B.4) to obtain

\[
\frac{k}{\alpha} \frac{d}{dt} \int_{x=0}^{w} \theta(x,t) dx = q''(0,t)
\]

B.15

Substituting Eq. (B.8) for the integral expression in Eq. (B.15) yields

\[
q''(0,t) = \lambda_0 \left( \frac{k}{\alpha} \frac{d\tilde{G}(t)}{dt} \right)
\]

B.16.a

Where \( \lambda_0 \) is defined as

\[
\lambda_0 = \frac{c[T_0]}{2[\beta_0 - \frac{1}{c[T_0]} \frac{dc}{dT}|_{T_0}] T_0}
\]

B.16.b

Equation B.16.a is the relationship for the surface heat flux to T.o.F.
APPENDIX C
Derivation of the Surface Temperature Equation for Acoustics

From Frankel and Bottländer [1] obtain Eq. 22 as

\[ \hat{\theta} (0, s) = \lambda_0 M_0(s) \hat{H}(s), \quad \Re(s) > 0 \quad \text{C.1.a} \]

where \( M_0(s) \) is defined as

\[ M_0(s) = \frac{\cosh \left( \frac{s}{\alpha} w \right)}{\sqrt{\Re(s) > 0} \int_{x'=0}^{w} \cosh \left( \frac{s}{\alpha} (w - x') \right) dx} \quad \text{C.1.b} \]

and \( \hat{H}(s) \) is defined as

\[ \hat{H}(s) = [ \hat{G}(0, s) - \left( \frac{G_0}{s} \right) ] \quad \text{C.1.c} \]

Simplify \( M_0(s) \) further to obtain

\[ M_0(s) = \frac{\sqrt{s}}{\alpha} \frac{\cosh \left( \frac{s}{\alpha} w \right)}{\sinh \left( \frac{s}{\alpha} w \right)} \quad \Re(s) > 0 \quad \text{C.1.d} \]

Eq. C.1.a can be re-written as

\[ \hat{\theta} (0, s) = \lambda_0 \hat{H}(s) \frac{\sqrt{s}}{\alpha} \frac{\cosh \left( \frac{s}{\alpha} w \right)}{\sinh \left( \frac{s}{\alpha} w \right)} \quad \text{C.2} \]
Note for Eq. (C.2), as \( M_0(s) \to \infty, s \to \infty \) and as such needs to be mathematically re-written to an equivalent exponential function to achieve stability. Sine and cosine hyperbolic functions can be equivalently represented as exponential function as [17]

\[
\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{C.3.a}
\]

\[
\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{C.3.b}
\]

Eq. (C.2) can be written as

\[
\hat{\theta}(0, s) \sqrt{\frac{\alpha}{s}} \left[ \frac{\frac{s}{\sqrt{\alpha \omega}} e^{\frac{s}{\sqrt{\alpha \omega}}} - e^{-\frac{s}{\sqrt{\alpha \omega}}}}{e^{\frac{s}{\sqrt{\alpha \omega}}} + e^{-\frac{s}{\sqrt{\alpha \omega}}}} \right] = \lambda_0 \hat{H}(s) \quad \text{C.4}
\]

Factor out \( e^{\frac{\sqrt{\omega}}{\alpha}} \) from numerator and denominator to obtain

\[
\hat{\theta}(0, s) \sqrt{\frac{\alpha}{s}} \left[ \frac{\frac{s}{\sqrt{\alpha \omega}} e^{\frac{s}{\sqrt{\alpha \omega}}} - e^{-\frac{s}{\sqrt{\alpha \omega}}}}{e^{\frac{s}{\sqrt{\alpha \omega}}} + e^{-\frac{s}{\sqrt{\alpha \omega}}}} \right] = \lambda_0 \hat{H}(s) \quad \text{C.5}
\]

The expression above can be simplified using the geometric series ([19] p.24) as

\[
\hat{\theta}(0, s) \sqrt{\frac{\alpha}{s}} \left( 1 - e^{-2\frac{s}{\sqrt{\alpha \omega}}} \right) \sum_{j=0}^{\infty} (-1)^j e^{-2j\frac{s}{\sqrt{\alpha \omega}}} = \lambda_0 \hat{H}(s) \quad \text{C.6}
\]

Distribute the term in parenthesis and manipulate into an invertible form as
\[ \hat{\vartheta}(0, s)\sqrt{\alpha}\left\{ \sum_{j=0}^{\infty} (-1)^j \left[ \frac{e^{-(a_{1,j})\sqrt{s}}}{\sqrt{s}} - \frac{e^{-(a_{2,j})\sqrt{s}}}{\sqrt{s}} \right] \right\} = \lambda_0 \hat{H}(s) \tag{C.7.a} \]

where \( a_{1,j} \) is defined as

\[ a_{1,j} = \frac{2jw}{\sqrt{\alpha}} \tag{C.7.b} \]

and \( a_{2,j} \) is defined as

\[ a_{2,j} = \frac{2(j+1)w}{\sqrt{\alpha}} \tag{C.7.c} \]

Take the inverse Laplace Transform of Eq. (C.7.a) ([17] p. 1020 and p.1026) and write in general form as

\[ \int_{u=0}^{t} \theta(0, u)k_T(t - u)du = f(t) \tag{C.8.a} \]

where the convolution kernel, \( k_T(t - u) \) in Eq. (C.8.a) is defined as

\[ k_T(t - u) = \frac{1}{\sqrt{t - u}} \sum_{j=0}^{\infty} (-1)^j \left\{ e^{\frac{-(2wj)^2}{4\alpha(t-u)}} - e^{\frac{-(2wj+1)^2}{4\alpha(t-u)}} \right\} \tag{C.8.b} \]

with the resulting forcing function, \( f(t) \) in Eq. (C.8.a) defined as

\[ f(t) = \frac{\sqrt{\pi}\lambda_0}{\sqrt{\alpha}} [G(t) - G_0] \tag{C.8.c} \]
Equation B.8.a is the relationship for the surface temperature to T.o.F.
APPENDIX D

Derivation of the Surface Temperature Equation for traditional inverse analysis

As mentioned in chapter 3, Eq. (2.5.a) is a first kind Volterra integral equation. To estimate this highly ill-posed equation, a future time parameter, \( \gamma \), is introduced for stabilizing the numerical method by holding surface temperature fixed for a prescribed forward time interval which dynamically varies as time progresses. To begin the traditional inverse analysis using the introduced future time method, consider the governing Eq. (2.5.a) with the corresponding kernel Eq. (2.5.b) and the forcing function, defined in Eq. (2.5.c) and apply the method of future-time to obtain

\[
\frac{\lambda_0}{\sqrt{\alpha}} \Delta \tilde{G}(t + \gamma) \equiv \int_{u=0}^{t} \theta(u) k_T(t + \gamma - u) du \\
+ \theta(t) \int_{u=t}^{t+\gamma} k_T(t + \gamma - u) du
\]

After following the future time methodology outlined in chapter 3 for heat flux, the final data reduction equation to traditionally reconstruct surface temperature can be generally expressed as

\[
\theta(0, t_i) = \frac{A - B}{C + D}, \quad i = 2, 3, \ldots, N - mMf
\]

where the function \( A \) is defined as

\[
A = \frac{\lambda_0}{\sqrt{\alpha}} \Delta \tilde{G}(t_i + \gamma_m)
\]

function B is defined as
\[ B = \sum_{j=1}^{i-2} \int_{u=t_j}^{t_{j+1}} \theta(u) k_T(t_i + \gamma_m - u) du \]  \hspace{1cm} \text{D.2.c}

Function C is defined as

\[ C = \int_{u=t}^{t+\gamma} k_T(t_i + \gamma_m - u) du \]  \hspace{1cm} \text{D.2.d}

And function D is defined as

\[ D = \int_{u=t_{i-1}}^{t_i} k_T(t_i + \gamma_m - u) du \]  \hspace{1cm} \text{D.2.e}
APPENDIX E

Derivation of the Surface Temperature Equation for preconditioned inverse analysis (n=1)

Similar to the preconditioned inverse analysis of heat flux, surface temperature can also be estimated using the parameter free filter method introduced in this chapter 4. This filter aids in smoothing the input measured data, resulting in a more stable analysis when compared to the traditional.

To begin the preconditioner analysis, apply Laplace Transform to Eq. (2.5.a) to obtain

\[
\frac{\lambda_0}{\sqrt{\alpha}} \Delta \hat{G} = \hat{\theta}(s) \hat{k}_T(s) \quad \text{E.1.a}
\]

where the transformed kernel, \( \hat{k}_T(s) \), is defined as ([18] p. 471 Eq. (4)

\[
\hat{k}_T(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{s}} \left( e^{-\frac{(2wm)^2}{4\alpha t}} - e^{-\frac{(2w(m+1))^2}{4\alpha t}} \right) \quad \text{E.1.b}
\]

To apply the filter to the above transformed equation, multiply both sides by \( \frac{1}{s} \) to obtain

\[
\frac{1}{s} \frac{\lambda_0}{\sqrt{\alpha}} \Delta \hat{G} = \frac{1}{s} \hat{\theta}(s) \hat{k}_T(s) \quad \text{E.2}
\]

Take the inverse Laplace Transform of Eq. (4.15) above to obtain

\[
\frac{\lambda_0}{\sqrt{\alpha}} \int_{u=0}^{t} \Delta G(u) du = \int_{u=0}^{t} \theta(0,u) M_T(t-u) du \quad \text{E.3.a}
\]

Where the preconditioned kernel \( M_T \) is defined as
\[ M_T(t - u) = \sum_{m=0}^{\infty} (-1)^m \left\{ \left( \frac{2\sqrt{t - u}}{\sqrt{\pi}} e^{\frac{- (2wm)^2}{4\alpha(t-u)}} \right) \right. \\
\left. - \frac{(2wm)^2}{\alpha} \text{erfc}\left( \frac{\alpha}{2\sqrt{t-u}} \right) \right\} \]

E.3.b

After applying the future time method. The final data reduction equation can be expressed generally as

\[ \theta(0, t_i) = \frac{A - B}{C + D}, \quad i = 2, 3, \ldots, N - mmf \]

E.4.a

where the function \( A \) is defined as

\[ A = \frac{\lambda_0}{\sqrt{\alpha}} \sum_{j=1}^{i+mmf-1} \Delta \tilde{G}(u) du \]

E.4.b

function \( B \) is defined as

\[ B = \sum_{j=1}^{i-2} \int_{u=t_j}^{t_{j+1}} \theta_{\gamma_m,N}(0, t_n) M_T(t_i + \gamma_m - u) du \]

E.4.c

function \( C \) is defined as

66
\[ C = \int_{u=t}^{t+\gamma} M_T(t_i + \gamma_m - u) du \]  
\[ \text{E.4.d} \]

and function D is defined as

\[ D = \int_{u=t_{i-1}}^{t_i} M_T(t_i + \gamma_m - u) du \]  
\[ \text{E.4.e} \]
VITA

Kevin was born and brought up in the Indian city known as the “Venice of the East.” There he attended schooling until the age of 13 before immigrating to the United States in 2008 with his family. Even at a young age, he discovered his passion for science and engineering which was further solidified by the coursework he was exposed to in high school. He enrolled in the University of Tennessee at Knoxville (UTK) to pursue a bachelor’s degree in Aerospace Engineering with a minor in Material Science and Engineering. At UTK, he became involved in multiple research projects in topics including powder X-ray diffractometry to study thermal properties of advanced ceramic-ceramic composites, and inverse heat conduction for aerospace applications which produced many journal articles and conference talks. After graduation, he interned at Collins Aerospace, a United Technologies Corporation company, as a Project Engineer. There he was exposed to the industry side of Aerospace Engineering which revolved around product manufacturing and quality control based on strict Aerospace standards. Wanting to further his education, he enrolled for a master’s program at UTK in Mechanical Engineering. There he worked under Dr. Jay I. Frankel in his inverse heat transfer laboratory. Fascinated by the area of heat conduction, Kevin decided to pursue this area for his master’s thesis; His research includes determining surface heat flux and temperature distribution using acoustic measurements. Upon graduation, Kevin is moving to Detroit, MI to join General Motors as a Controls Engineer.