Optimal Control Strategies in Ecosystem-Based Fishery Models

Mahir Demir

University of Tennessee, mdemir@vols.utk.edu

Follow this and additional works at: https://trace.tennessee.edu/utk_graddiss

Recommended Citation

https://trace.tennessee.edu/utk_graddiss/5421

This Dissertation is brought to you for free and open access by the Graduate School at Trace: Tennessee Research and Creative Exchange. It has been accepted for inclusion in Doctoral Dissertations by an authorized administrator of Trace: Tennessee Research and Creative Exchange. For more information, please contact trace@utk.edu.
Optimal Control Strategies in Ecosystem-Based Fishery Models

A Dissertation Presented for the
Doctor of Philosophy Degree
The University of Tennessee, Knoxville

Mahir Demir
May 2019
© by Mahir Demir, 2019
All Rights Reserved.
I dedicate this dissertation and completion of the doctorate program to my loving parents, 
Şah Haydar Demir and Remziye Demir. Their unique support, love and faith brought me 
so far in my life. They made my life meaningful.
I would first like to thank my advisor, Dr. Suzanne Lenhart. I cannot thank her enough for her encouragement and guidance during my work. I have learned a great deal from her and I am grateful for the time and effort she dedicated to my success. I am very grateful for her support and advice, without which this dissertation would not have been accomplished. She will be one of my role models in my academic and personal life. I thank her for being a great advisor, her guidance, and for all the opportunities she made possible.

I would like to thank my committee members, Louis Gross, Judy Day, and Tuoc Phan. I deeply respect their opinions and I thank them for supporting my work. I thank Judy Day for introducing me to mathematical biology and for being so supportive and encouraging. I would also like to thank Louis Gross whose help and advice was key to my success.

I acknowledge the generous support of the Turkish Ministry of National Education that made this dissertation possible. I am grateful to the Mathematics Department at the University of Tennessee for supporting me for three years. I thank Henry Simpson, Tadele Mengesha, Xiaobing Feng, Pam Armentrout, Amanda Worsham, and Ben Walker for providing help and support throughout my graduate studies.

Finally, I thank my loving wife, Donem, my daughters, Nermin Alice and Yudum Eva. This would not have been possible without their love and devotion.
“Victory is for those who can say “Victory is mine”. Success is for those who can begin saying “I will succeed” and say “I have succeeded” in the end.”

– Mustafa Kemal Ataturk
Abstract

This dissertation considers the use of food chain models coupled with optimal control theory as a new approach for the problem of implementing ecosystem-based fishery management (EBFM) strategies. We consider the Black Sea anchovy on the southern part of the Black Sea as a case study of the implementation of EBFM. Because of the availability of temporal data, we build our first food chain model using ordinary differential equations to describe the anchovy dynamics, and then build our second food chain model using partial differential equations to include spatial features of the anchovy dynamics. In the study, we use the harvest rate of the anchovy fishery as our control that corresponds to number of fishing fleets.

In the first model, the Black Sea anchovy stock was coupled with a prey and a predator species, using a system of nonlinear differential equations. The objective for the problem is to find the ecosystem-based optimal harvesting strategy that maximizes the discounted net value of the anchovy population with seasonal harvesting. In our numerical simulations, we obtained more profitable harvesting strategies for the southern part of the Black Sea, and also obtained much better structure for the related food web in terms of population biomasses via the optimal control strategy. Furthermore, we discussed the benefits of using food chain models in fishery management, and derived a schedule for ecosystem-based fishery management of the Black Sea anchovy.

In the second model, a spatial food chain model on a bounded domain coupled with optimal control theory examined ecosystem-based harvesting strategies. Our system of nonlinear partial differential equations (PDEs) has logistic growth, movement by diffusion and advection, and Neumann boundary conditions. Numerical simulations were completed to illustrate several scenarios.
# Table of Contents

1 Introduction ............................. 1  
   1.1 Ecosystem Based Fishery Management ....................................... 2  
   1.2 Food Chain Models in Commercial Fishery ..................................... 3  
   1.3 Optimal Control Theory ........................................................... 5  
      1.3.1 Optimal Control Theory for ODE Systems ................................. 5  
      1.3.2 Optimal Control Theory for PDE Systems .................................. 6  
   1.4 Summary of Our Models ............................................................ 7  
   1.5 Biological Key Assumptions of Our Models ...................................... 8  
   1.6 Numerical Approximations and Methods .......................................... 9  

2 ODE Model ..................................... 11  
   2.1 Introduction ................................................................. 11  
   2.2 Model Formulation ............................................................ 16  
   2.3 Positivity and Boundedness of the State Variables ............................. 19  
   2.4 Necessary Conditions ........................................................... 21  
      2.4.1 The Optimality System ............................................. 24  
   2.5 Parameter Estimation ............................................................ 28  
   2.6 Numerical Results ............................................................. 31  
      2.6.1 Harvesting Strategies .............................................. 31  
      2.6.2 Traditional Fishery Management vs Ecosystem Based Fishery Man-
              ageement ......................................................... 40  
      2.6.3 Estimation of the Numbers of Fishing Fleets ............................. 44  
   2.7 Conclusions ................................................................. 48
3  PDE Model

3.1  Introduction and Background ........................................... 51
3.2  Model Formulation .......................................................... 54
3.3  Existence and Uniqueness of State Variables ......................... 60
3.4  Existence of Optimal Control ............................................ 75
3.5  Derivation of the Optimality System ................................... 84
3.6  Numerical Simulations and Parameters ................................. 95
   3.6.1  Numerical Results ................................................... 97
   3.6.2  Imposed Seasonal Optimal Harvesting ............................. 102
   3.6.3  Effect of Movement .................................................. 105
   3.6.4  Comparison of Strategies .......................................... 107
3.7  Conclusions ................................................................. 108

4  Summary and Future Directions ............................................. 110

Bibliography ................................................................. 113

Appendices ................................................................. 121

A  Finite Difference Method ............................................... 122
   A  Forward Finite Difference Method for the State Variables with Explicit Scheme 122
   B  Backward Finite Difference Method for the Adjoint Variables with Explicit Scheme 126

B  Data of the Black Sea Anchovy ......................................... 130

Vita ......................................................................... 132
List of Tables

2.1 Parameter descriptions and units .............................................. 18
2.2 Parameter descriptions and values in the case of constant harvesting strategy. Here $e$ is a scientific notation in MATLAB and it is a shorthand for 10. .................. 30
2.3 Comparison of harvest rates in constant harvesting strategy for the values of $\mu_1 = 30700$, and $\mu_2 = 0.1$. .............................................................. 33
2.4 Comparison of constant and current harvesting strategies for the values of $\mu_1 = 30700$, and $\mu_2 = 0.1$ .............................................................. 35
2.5 Comparison of the three strategies with the assumption of having 50% net profit in the current harvesting strategy. In the fifth row, we use the approximated optimal harvest rate (See Figures 2.11 and 2.12) ................................. 37
2.6 Comparison of the models under the assumption of having 50% net profit in current harvesting strategy. $h_f$ and $h_s$ denote the optimal harvesting rates of food chain model and single equation, respectively. In the case of “Food Chain with $h_s$”, we implement the optimal harvest rate of the single equation to our food chain model to compare results. ......................................................... 44
2.7 The landing of anchovy in the first half and in second half obtained by using our food chain model with the optimal harvesting strategy. Then, by using the non-linear regression model, we obtained the approximated numbers of fishing fleets for the first and second half of the anchovy fishery. The approximated CPUE is about 948 for the first half of the fishery season and 953 for the second half of the fishery season. ................................................................. 47
2.8 Summary of the results in the OC strategy. We also compare the advantageous of using OC strategy by comparing with the current harvest strategy ....... 48
3.1 Parameter description and units. The diffusion coefficients are greater than $\theta > 0$
because of the uniform ellipticity on the diffusion coefficients (See assumption 3.6).

3.2 Parameter descriptions and values obtained from Chapter 2, and justified for our
spatial food chain model. Here $e$ is a scientific notation in MATLAB and it is a
shorthand for $10^{\ldots}$.

B.1 The effort (number of fishing fleets), catch per unit effort (CPUE), and landing
of the Black Sea anchovy in Turkish coast of the Black Sea (STECF, 2017)
List of Figures

2.1 The location of the Black Sea, obtained by wikipedia Wikipedia (2007). . . . . . 12
2.2 Landing of the Black Sea anchovy on Turkish coasts (dashed) and in the Black Sea. 13
2.3 The flow diagram of consumption in our food chain model. . . . . . . . . . . . 16
2.4 Landing of the Black Sea anchovy with data (red) and simulation (blue) when the harvest rate, $h = 0.48$ in constant harvesting strategy. . . . . . . . . . . . . . . 31
2.5 Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when the harvest rate, $h = 0.48$ in constant harvesting strategy. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green). . . . . . . . . . . . . . . . . 32
2.6 Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when the harvest rate, $h = 0.4$ in constant harvesting strategy. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green). [-10pt] . . . . . . . . . . . . . . . . 33
2.7 Annual landing of the Black Sea anchovy with data (red) and simulation (blue) in current harvesting strategy with relative error 0.25. . . . . . . . . . . . . . . . . . . . 34
2.8 Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when average harvest rate is about 0.48 in the current harvesting strategy. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green). . . . . . . . . . . . . . . . . 36
2.9 Landing of the Black Sea anchovy on the southern part with OC case, $h_{max} = 0.4$. 38
2.10 Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when the optimal harvesting strategy applied with $h_{max} = 0.4$. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green). . . . . . . . . . . . . . . . . 38
2.11 The harvest rates for the Black Sea anchovy in the optimal harvesting strategy for the first three years with $h_{max} = 0.4$. We start the harvesting in first three months and then stop harvesting the system in the rest of the year. . . . . . . . . . . . . . . . . . . . . 39
2.12 The approximated harvest rates for the Black Sea anchovy in the optimal harvesting strategy, $h_{\text{max}} = 0.4$ and in the first half of the fishing season, $h = 0.335$ for the first three years. ................................................................. 40
2.13 Annual landing of the Black Sea anchovy with constant harvest rate, 0.5, intrinsic growth rate, 0.2, and carrying capacity, 275,000 tonnes in the case of parameter estimation for the single species model. .............................. 41
2.14 Left plot: Biomass of the Black Sea anchovy for the single species model. Right plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) in the optimal harvesting strategy. ......................................................... 42
2.15 LHS: Optimal harvesting strategy for the single equation, RHS: Optimal harvesting strategy for our food chain model. ............................................................ 43
2.16 Nonlinear regression between CPUE and the landing of the anchovy population depending on the data from 1985 to 2000, and the data from 2013 to 2016 (See Table B.1 in Appendix B). ............................................... 46
3.1 The location of the Black Sea (obtained from Gucu et al. (2017)) .............. 52
3.2 The flow diagram of consumption in our food chain model. ....................... 55
3.3 Initial Biomass of anchovy (blue), jellyfish (red), and zooplankton (green). 97
3.4 Spatial dynamics of anchovy population without harvesting and with different diffusion rates ................................................................. 98
3.5 Dynamics of anchovy population without harvesting, which is integrated over space at each time step with different diffusion rates (for 12 months). .... 99
3.6 Optimal harvesting rate applied for the entire year with diffusion rates, $D_i = 0.01$ and $b_i = 0$ for $i = 1, 2, 3$. The total landing is 376,170 tonnes, and the discounted net profit is $J = 284,590$ tonnes. ................................. 99
3.7 Optimal harvesting rate applied for the entire year with diffusion rates, $D_i = 0.05$ and $b_i = 0$ for $i = 1, 2, 3$. The total landing is 330,060 tonnes, and the discounted net profit is $J = 213,840$ tonnes. ................................. 100
3.8 Optimal harvesting rate applied for the entire year with diffusion rates, \( D_i = 0.1 \) and \( b_i = 0 \) for \( i = 1, 2, 3 \). The total landing is 278,080 tonnes, and the discounted net profit is \( J = 175,620 \) tonnes.

3.9 Dynamics of populations without harvesting in the case of diffusion rates, \( D_i = 0.05 \) for \( i = 1, 2, 3 \).

3.10 Optimal harvesting rate applied for the entire year with different initial conditions, diffusion rates, \( D_i = 0.1 \) and advection rates \( b_i = 0 \) for \( i = 1, 2, 3 \). We also reduced the cost by reducing \( \mu_1 \) from 20000 to 500. In the left plot, we used true initial biomass, and we increased the initial to \( 5e^5 \) in the right plot to show the effects of having different initial conditions.

3.11 Imposed seasonal optimal harvesting rates applied for 3 months with diffusion rate, \( D_i = 0.05 \) and advection rate, \( b_i = 0 \) for \( i = 1, 2, 3 \). The total landing is 306,550 tonnes, and the discounted net profit is \( J = 202,600 \) tonnes.

3.12 Population biomasses integrated over space at each time step. Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green).

3.13 Optimal harvest rates for five years.

3.14 Stock dynamics of the anchovy population and seasonal optimal harvesting strategy for five consecutive years with almost zero diffusion and \( b_i = 0 \) for \( i = 1, 2, 3 \).

3.15 Imposed seasonal optimal harvesting rate applied for 3 months with diffusion rates, \( D_i = 0.05 \) and \( b_i = 0.05 \) for \( i = 1, 2, 3 \). The total landing is 299,900 tonnes, and the discounted net profit is \( J = 197,340 \) tonnes.

3.16 Imposed seasonal optimal harvesting rate applied for 3 months with diffusion rates, \( D_i = 0.05 \) and \( b_i = 0.1 \) for \( i = 1, 2, 3 \). The total landing is 281,900 tonnes, and the discounted net profit is \( J = 183,120 \) tonnes.
3.17 Comparison of imposed seasonal optimal harvest rates and the constant harvest rate applied for three months with $D_1 = 0.08$ and $b_1 = 0.2$, $b_2 = 0.15$, and $b_3 = 0.1$. The left plots show the dynamics of populations when we apply the optimal harvesting strategy with the maximum harvest rate $0.4$, and the right plots show the dynamics of populations when we apply the constant harvesting strategy with $h = 0.4$. 
Chapter 1

Introduction

Mathematical modeling is a very useful tool to analyze biological and ecological systems using a variety of analytical, numerical, and graphical techniques. Mathematical models have been used in many different areas. For example, in ecology, mathematical models play an important role to understand dynamics of populations. This dissertation utilizes this tool to investigate population dynamics. We especially investigate the Black Sea anchovy population and its fishery. The Black Sea is an inland semi-enclosed sea in Europe, and the anchovy is the most dominant species in the Black Sea ecosystem and in its commercial fishery.

The Black Sea anchovy has been experiencing overfishing for a couple of decades, however overfishing is not the only stressor on the Black Sea anchovy population probably the biggest environmental effect on the Black Sea anchovy stock took place at the end of the 1980s due to food competition with an invasive jellyfish, *Mnemiopsis leidyi*, and predation by the jellyfish on the anchovy larvae and eggs (*Oguz et al. (2008)*). Because of these two main pressures on the anchovy population, the harvesting of the Black Sea anchovy has almost stopped in the northern part of the Black Sea. The anchovy fishery in the northern part is not profitable anymore as a commercial fishery, and the anchovy fishery depends almost solely on southern part of the Black Sea (*Gucu et al. (2017)*).
Fortunately, the pressure of jellyfish on the anchovy population has reduced by an other jellyfish, *Beroe ovata* that was introduced in the Black Sea ecosystem right after *Mnemiopsis leidyi* broke out. But overfishing is still one of the main issues in anchovy fishing on the southern part of the Black Sea. To reduce the pressure of overfishing, it is important to come up with ecosystem based fishery management strategies to reduce the harvesting pressure.

In this dissertation, we will analyze the dynamics of the Black Sea anchovy and the effects of these stressors on the anchovy population. Then, we will address these problems implementing ecosystem based management strategies. We will build food chain models by using ordinary differential equations (ODEs) and partial differential equations (PDEs). We use a ODE system to represent the dynamical system of the anchovy population since our data is yearly in time and not dependent on space. Nevertheless, ODE models are not helpful to illustrate spatial features of a system. That is why we will also use a PDE model to analyze spatial features of the anchovy dynamics.

Therefore, to come up with ecosystem based management strategies and analyze the effect of these two stressors on the anchovy population, we build food chain models to investigate the Black Sea anchovy dynamics and to analyze effects of the anchovy fishing on the Black Sea food web. We also use optimal control theory (OCT) to find optimal harvesting rates that maximize anchovy landing under some ecosystem friendly constraints.

### 1.1 Ecosystem Based Fishery Management

The ecosystem-based fisheries management (EBFM) focuses on the whole food web and the ecosystem of a target fish population to obtain a healthy, sustainable, and diverse ecosystem. EBFM was developed to move beyond single species management by incorporating ecosystem considerations for the sustainable utilization of marine resources. On the other hand, traditional fisheries management has focused on single species sustainability for commercially valuable species (*Trochta et al. (2018)*). Single-species management can be quite successful
(Quinn and Collie (2005)), but often ignores important ecosystem considerations such as species interactions, bycatch, changes in ecosystem structure, and gear impacts on habitat (Link (2002)). Too often traditional fishery management has failed to take a precautionary approach to maintain and protect sustainable fisheries, biodiversity, and marine ecosystem function (Lauck et al. (1998)).

Therefore, there is an increasingly growing interest in EBFM approaches. One of the well known approaches is the use of marine protected areas (MPA), which are areas with no harvest in seas, oceans, or large lakes (Ozturk et al. (2017), Neubert (2003), Moberg et al. (2015), and FAO (2011)). Another useful approach for the implementation of EBFM is a stochastic viability approach. The viability approach relies on mathematical models derived from the theory of dynamic systems control under constraints. Within this generic framework, the ecoviability framework (also termed co-viability) specifically focuses on the ecological and economic viability of exploited ecosystems including fisheries and marine resources (Doyen et al. (2012) and Doyen et al. (2017)). See the book by Lara and Doyen (2008) for a detailed introduction to the viability approach. Besides, these intervention methods, another way to protect fish populations from overexploitation is to use optimal control theory as a tool with food chain models to suggest management policies for natural renewable food resources. In the dissertation, we will use the food web approach that is the best option for the Black Sea anchovy ecosystem since we want directly to see the effects of predator-prey relationships on the anchovy fishing. This approach is also very useful to investigate the effects of anchovy fishing on the anchovy food web when we apply different harvesting strategies.

1.2 Food Chain Models in Commercial Fishery

A commercial fishery harvests fish populations for profit by using advanced fishing fleets. It offers valuable food sources for humans across the world. Therefore, there is significantly growing interest in the management of fish stocks. Mathematical models help us to understand the dynamics of these fish stocks. Many harvesting models for fishery management have been analyzed by using a single equation (Burgess et al. (2017a), Burgess
et al. (2017b), Salilew (2016), and Kot (2001)). One such fishery model uses ordinary differential equations (ODEs) by Schaefer (1991) in the form:

\[
\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qEN
\]

\[
\frac{dE}{dt} = k(pqEN - cE),
\]

(1.1)

where \(N\) is the stock level, \(E\) is the level of fishing effort, \(r\) is the intrinsic growth rate of the stock, \(K\) is the carrying capacity of the stock, \(q\) is the catchability, \(p\) is the price per fish, \(c\) is the cost per unit effort, and \(k\) is a proportionality constant. The first equation contains (logistic) growth term of the stock and harvesting term, which is proportional to both the stock level and the level of fishing effort. The second equation has economics interpretation rather than ecological, and it states that the rate of change of fishing effort is proportional to profit, with \(pqEN\) as the revenue and \(cE\) as the costs (Kot (2001)).

However, these models ignore the interactions between species in food webs, and do not give any information on the effects of fishing on related food webs. On the other hand, food chain models are very useful to examine the effects of predator-prey relationships, and the effects of a fishery on related food webs. In many applications, particularly those related to aquatic systems, food chain models with three trophic levels are often good representations of the food web in an ecosystem (Gwaltney et al. (2004)). This work uses the Rosenzweig-MacArthur model, which is given by the following set of differential equations:

\[
\frac{dx_1}{dt} = x_1 \left[r\left(1 - \frac{x_1}{K}\right) - \frac{a_2x_2}{b_2 + x_1}\right],
\]

\[
\frac{dx_2}{dt} = x_2 \left[e_2\frac{a_2x_1}{b_2 + x_1} - \frac{a_3x_3}{b_3 + x_2} - d_2\right],
\]

\[
\frac{dx_3}{dt} = x_3 \left[e_3\frac{a_3x_2}{b_3 + x_2} - d_3\right],
\]

where, \(x_1, x_2\) and \(x_3\) are the population stocks of three different species in the food chain, and these species are referred to as the prey \((x_1)\), predator \((x_2)\) and super-predator \((x_3)\) in the
system. Also, $r$ and $K$ are the prey growth rate and the prey carrying capacity, respectively, and $a_i, b_i, e_i$ and $d_i$, $i = 2, 3$, are the maximum predation rate, half saturation constant, conservation factor to growth from consumption, and the death rate of the predator ($i = 2$) and the super-predator ($i = 2$).

1.3 Optimal Control Theory

Optimal control is a very useful tool in mathematics applied in many different fields, including aerospace, robotics, bio-engineering, economics, finance, and control of dynamical systems in ecology and biology. It helps in making decisions by choosing control variables that maximize or minimize a goal subject to dynamical systems. The dynamical systems can be defined using ODEs, PDEs, stochastic differential equations, discrete equations, integro-difference equations or a combination of above equations. See the book by Lenhart and Workman (2007) for a detailed introduction to optimal control theory, especially with application to biological problems. In the study, we will discuss the theory of optimal control when applied to ordinary differential equations and partial differential equations.

1.3.1 Optimal Control Theory for ODE Systems

Optimal control theory for ODE systems, was developed by Lev Semenovic Pontryagin and his co-workers in the late 1950s (Pontryagin et al. (1967)). Pontryagin’s key idea was the introduction of the adjoint variables to attach the differential equations to the objective functional (like a Lagrange multiplier attaching a constraint for optimization of a function). Then he and his co-workers converted the problem of finding an optimal control to maximize (minimize) the objective functional subject to dynamic equations (with initial conditions) to maximizing (minimizing) the Hamiltonian with respect to the control at each time. This work resulted in Pontryagin’s Maximum Principle, for the optimal control of finite dimensional problems governed by ordinary differential equations (ODEs).

In optimal control problems, variables are divided into three classes: state variables, adjoint variables, and control variables. The rate of change in the state variables is governed by first order differential equations. The trajectories of the state variables are steered directly
by the control variables. The adjoint variable, the state variable and the optimal control
characterization form an optimality system. We then use the optimality system to maximize
or minimize an objective functional that achieves a desired goal. See the book by Lenhart
and Workman (2007) for detailed explanations of optimal control theory for ODE system.

1.3.2 Optimal Control Theory for PDE Systems

The optimal control theory for PDEs was developed by Lions (1971). Even though some
similarities occur, there is no full generalization of Pontryagin’s Maximum Principle to PDEs.
See the book by Li and Yong (1995) for some known cases of Maximum Principles for PDEs.
To obtain optimality systems for PDEs, we will discuss the following steps for an optimal
control problem with a PDE (Lenhart and Workman (2007) and Hackbusch (1978)):

1. After setting up a PDE in a weak solution space with a control in a specified set and
an objective functional, proving existence of an optimal control is a first step.

2. To derive the necessary conditions, we need to differentiate the maps:
\( h \) (control) \( \rightarrow J(h) \) (Objective functional) as well as \( h \) (control) \( \rightarrow x(h) \) (state) in
the following weak sense,
\[
\frac{x(h + \epsilon l) - x(h)}{\epsilon} \rightharpoonup \psi \quad \text{as } \epsilon \to 0
\]

where for all \( h \in A \) (Admissible class for control variables) and \( l \in L^\infty \) such that
\( (h + \epsilon l) \in A \), and we call \( \psi \) as the “sensitivity” function.

3. The “sensitivity” function is the directional derivative of the control-to-state map. The
sensitivity function solves a PDE, which is linearized version of the state PDE.

4. By using the operator associated with the sensitivity PDE, we obtain the adjoint
operator associated with the adjoint PDE together with boundary conditions and
transversality condition.
5. Then, differentiate the objective functional \( J(h) \) with respect to the control, \( h \) at the optimal control.

6. Finally, use the adjoint PDE and the sensitivity PDE in this derivative of \( J \) to obtain the explicit characterization of an optimal control.

1.4 Summary of Our Models

In Chapter 2, we build a food chain model for the anchovy fishery in the Black Sea using ODEs. We give background for our problem formulation and emphasize our goal in the objective functional. Then, we prove the positivity, existence and uniqueness of state solution. Next, we prove existence of an optimal control and derive the necessary conditions that an optimal control must satisfy. Moreover, we obtain the optimality system, which consists of the state system, the adjoint system and the characterization of optimal control. We first estimate the parameters of our model by using the annual landing data of the Black Sea anchovy on the southern part obtained from STECF (2017). Then we present numerical solutions to the optimality system for some bioeconomically interesting scenarios under the consideration of sustainable ecosystem, and we compare the three main strategies for fishery management of the Black Sea anchovy. After that, we estimate the number of fishing fleets by using our model outputs. Finally, we compare the results from TFM and EBFM in optimal control case, and show the benefits of using food chain models together with optimal control theory.

In Chapter 3, we build a food chain model with spatial features using PDEs. We first give background for our problem formulation and then formulate our model with biologically feasible boundary and initial conditions, and present our goal in the objective functional for an optimal fishery. Then, we prove the positivity, existence and uniqueness of the weak solution for our PDE system by using Banach Fixed Theorem. Next, we first prove existence of an optimal control and then demonstrate the necessary conditions that an optimal control must satisfy. Moreover, we obtain the optimality system, which consists of the state
equations, the adjoint equations, and the characterization of optimal control. Finally, we present numerical solutions to our optimality system for some (ecosystem friendly) scenarios, and then we discuss and compare these scenarios for fishery management of the Black Sea anchovy.

1.5 Biological Key Assumptions of Our Models

In the dissertation, we focused on key species that have significant effects on anchovy dynamics. We consider the jellyfish, *Mnemiopsis leidyi*, as a main predator of the anchovy population. We also have other predators of the anchovy in the Black Sea ecosystem, but we will not directly include their effect in the study since their abundance is low in the Black Sea. Anchovy is especially a primary prey species for harbor porpoises and common dolphins, and of secondary importance to bottlenose dolphins, and lack of prey availability coincided with mass mortality events of all three cetaceans in the past, suggesting their health was compromised (Birkun et al. (2014)). Therefore, we will just consider *Mnemiopsis leidyi* as a main predator of the anchovy population, and the effects of other predators will be hidden in logistic growth term in our models.

Since zooplankton populations are main food sources for the anchovy and jellyfish population, we will include the direct effect of the zooplankton in our models. Thus, we will consider these three populations as key populations. We also will consider the effects of jellyfish’s predators (mainly effects of *Beroe ovata*, another jellyfish) by adding an extra term in the models. The effects of other species in the food web will be represented in logistic growth terms with intrinsic growth rates and carrying capacities. In our study, the carrying capacities of anchovy and zooplankton will be larger than initial biomasses of anchovy and zooplankton populations since we want to reduce the effect of carrying capacities in our study. That is why we chose the carrying capacities as almost double of the initial biomass for the anchovy and zooplankton populations. On the other hand, the carrying capacity of jellyfish will be smaller than the initial biomass of the jellyfish since the main preys of jellyfish are anchovy and zooplankton. The main contribution to jellyfish biomass will come
from predation of zooplankton and anchovy populations.

In the study, we will consider the southern part of the Black Sea, since the anchovy fishery mainly has taken place on the southern part of the Black Sea, which is mostly the Turkish coast. That is why we will use the landing data of Turkish Coast during the study. In the ODE model, even if we do not include the migration of the anchovy population, we assume that the anchovy population use all the Black Sea in off-seasons, and enter the system right before the fishery seasons. In the PDE model, we consider advection and diffusion of the species in our food chain model. Even if the anchovy population first move to northwest right after the fishery season, and move to southern part when temperature drops, and finally move from the west coast to the east coast during the fishery season, we will just assume that the anchovy population moves from the west coast to the east coast in the PDE model in the application of one year. It does not make sense in long term since we do not include the migration of the anchovy population. That is why we will set our time domain as 12 months in the case of advection.

1.6 Numerical Approximations and Methods

To solve optimality systems numerically, there are many techniques and methods such as Pseudospectral methods (Garg et al. (2010)), Radau Pseudospectral methods (Garg et al. (2011)), and Forward-Backward Sweep Method (Hackbusch (1978)). We use the Forward-Backward Sweep Method to approximate the solutions of our optimality system, in Chapter 2 and Chapter 3. The optimality system, with initial conditions for the state systems and final time conditions for the adjoint systems, is solved using an iteration method. Due to the structure of the optimality system, we use a forward sweep method to solve state systems, and use a backward sweep method to solve adjoint systems, numerically. For systems of ODEs, we use a fourth-order Runge Kutta method, or ode45 to solve the optimality system. For systems of PDEs, we use an explicit finite difference method to solve the state and adjoint systems.
An iterative forward-backward sweep method (Lenhart and Workman (2007)) is described below:

1. Initiate a guess for the control variable to start the iteration.

2. Given the initial conditions, the state system is solved forward in time, and then given the final time conditions, the adjoint system is solved backward in time by using the outputs of the state system in each time step.

3. Then, the control variable is updated using a convex combination of the previous control value and the new control value determined using the control characterization.

4. Steps 1-3 are repeated until successive values of all states, adjoints, and control(s) are sufficiently close, i.e. there is convergence of the optimality system (See the article by Hackbusch (1978) for convergence of the algorithm).
Chapter 2

ODE Model

2.1 Introduction

Interest is growing in the management of natural renewable resources due to its importance for sustainable ecosystems and for their economic value. One of the well-known examples of renewable resources is marine fish populations, which are valuable food sources for humans across the world. However, many marine populations are severely overfished (Hilborn (2012)). Because of the effects of overfishing on fish populations, many native species and ecosystems are under pressure of degradation and destruction. One of the species suffering from overfishing is the Black Sea anchovy. The Black Sea is an inland semi-enclosed sea connected to the Marmara Sea with a narrow passageway (Figure 2.1), and the anchovy is the most dominant species for the ecosystem and for the commercial fish industry. Anchovy has a crucial role in the Black Sea pelagic food web as a prey and predator of some species, and it is also an important consumer of zooplankton, especially when the anchovy stock is large (Daskalov et al. (2007)).

The Black Sea anchovy has been experiencing overfishing for a couple of decades. The first signs of overfishing for the Black Sea anchovy appeared after 1984, when anchovy shoals were difficult to find and the fishery enterprise incurred losses (Shlyakhov et al. (1990)). The fishing effort and mortality also dropped subsequently because of decreasing profitability of fishing (European Parliament Committees (2010)). During the collapse phase, the size and
age structure of the catch shifted toward a predominance of small, immature individuals (Gucu (1997)). In 1995-2005, the stock partially recovered and annual catches increased to 300,000 – 400,000 tonnes (European Parliament Committees (2010)), but in the last decade, the annual catch of anchovy dropped between 200,000 – 300,000 tonnes (STECF (2017)). Now commercial fishery of the Black Sea anchovy has almost stopped in most of the northern part of the Black Sea. The size of the anchovy population in the northern part has decreased because of overfishing (Oguz (2017) and Oguz et al. (2012)), and so the harvesting of the Black Sea anchovy population is not profitable anymore in the most of the northern part for commercial purposes. That is why the Black Sea anchovy fishery now depends mostly on the southern part of the Black Sea (Gucu et al. (2017)), which mainly consists of the Turkish coasts. But, we now have the same overfishing problem in the southern part, and the system is unstable and highly sensitive. Therefore, it becomes necessary and essential to come up with an ecosystem-based fishery management strategy for the Black Sea anchovy fishery on the southern part.

![Map of the Black Sea](image.png)

**Figure 2.1:** The location of the Black Sea, obtained by wikipedia Wikipedia (2007).
Unfortunately overfishing is not the only stressor on the Black Sea anchovy population, probably the biggest environmental effect on the Black Sea anchovy stock took place in the end of the 1980s due to food competition with an invasive jellyfish, *Mnemiopsis leidyi*, and predation by the jellyfish on the anchovy larvae and eggs (Oguz et al. (2008)). This huge and aggressive invasion occurred in 1988-1989, and the catastrophic reduction of the Black Sea anchovy stock (See Figue 2.2) in the late 1980s was due to the combined action of two factors: overfishing and the *Mnemiopsis leidyi* invasion (Grishin et al. (2007)). The absolute loss of the anchovy catch between 1989 and 1992 due to the massive invasion of the *Mnemiopsis leidyi* can be approximately estimated as 1 million tonnes (European Parliament Committees (2010)). Due to the absence of predators to control its abundance, *Mnemiopsis leidyi* spread suddenly through all the Black Sea and contributed to the collapse of the Black Sea fisheries through food competition and direct predation on fish larvae (European Parliament Committees (2010)). The abundance of *Mnemiopsis leidyi* were gradually decreased by one of its predators, another jellyfish, *Beroe ovata*, which was introduced in the Black Sea for the biocontrol of the *Mnemiopsis leidyi* (Kideys (2002)). The Black Sea ecosystem has seen some recovery over the last 10 - 15 years, due to the decrease of the invasive *Mnemiopsis leidyi* population (Oguz et al. (2009)).

![Figure 2.2: Landing of the Black Sea anchovy on Turkish coasts (dashed) and in the Black Sea.](image)

13
To include these stressors on the Black Sea anchovy population, factors in the environment that affect the anchovy population should be considered in deciding management strategies. Therefore, we build a model, which represents the effects of these factors on the anchovy population. Dynamics models can be good representations of such systems to consider predator-prey relations, environmental effects, and human effects on marine systems. That is why dynamical food chain models will help us to not only see effects of commercial fishery on the corresponding food web, but also to obtain better strategies. Thus, using the ecosystem based modeling techniques will play a crucial role for an optimal and sustainable fishery management. There have been some studies investigating different methods to protect fish populations and conserve marine ecosystems, such as no-take marine reserves (marine protected areas from fishery), and limitation on fishing seasons, as well as the implementation of landing quotas (FAO (2011), Ozturk et al. (2017), and STECF (2017)). Besides, these intervention methods, another way to protect fish populations from overexploitation is to use optimal control theory as a tool with food chain models to suggest management policies for natural renewable food resources. In many cases, no-take marine reserve areas are natural results of maximizing yield in fishery management models (Neubert (2003), Joshi et al. (2009), and Kelly et al. (2016)).

Many fishery models involving bioeconomics of target fishery and optimal harvesting used ordinary differential equations (ODEs) (Clark (1990), Kot (2001), Burgess et al. (2017b), Moberg et al. (2015), and Sahoo and Poria (2015)). Clark’s work provided a foundation for using optimal control theory in ODEs as a tool in fishery management. Also, some work has been done by using food chain models with three trophic levels. However, most of the studies investigated predator-prey relations and corresponding stability analysis (Ali and Chakravarty (2015) and Abrams and Roth (1994)). In many applications, particularly those related to aquatic systems, food chain models with three trophic levels are often good representations of the food web in an ecosystem (Gwaltney et al. (2004)). Even if fish populations live in diverse and complex ecosystems, including three trophic levels in a model could be helpful to understand interactions in a sustainable fishery. Many harvesting models for a fishery management have been analyzed by using a single equation (Burgess et al.
(2017a), Burgess et al. (2017b), Salilew (2016), and Kot (2001)). However, in our study, we use a food chain model with three trophic levels to track the effects of the fishery on the corresponding food web. We then couple the food chain model with optimal control theory to obtain an optimal fishery management for the Black Sea anchovy.

In the study, we have three main goals. One of them is to obtain ecologically and economically beneficial (optimal) fishery management strategies for the Black Sea anchovy harvest in the southern part of the Black Sea by using annual landing data obtained from the Black Sea Assessment of Scientific, Technical and Economic Committee for Fishery (STECF (2017)). Then estimate the number of fishing fleets that take place in the anchovy fishery by using statistical tools. In the estimation of fishing fleets for our harvesting strategies, we use the data of catch per unit effort (CPUE), the annual numbers of fishing fleets used in the anchovy fishery, and the annual landing of anchovy (these data obtained from STECF (2017)). Our last goal is to compare the results of TFM and EBFM, and then discuss the benefits of using food chain models with optimal control theory.

In the next section, we give background for our problem formulation and emphasize our goal in the objective functional. In Section 3, we prove the positivity, existence and uniqueness of state solution. In section 4, we prove existence of an optimal control and derive the necessary conditions that an optimal control must satisfy. Moreover, we obtain the optimality system, which consists of the state system, the adjoint system and the characterization of optimal control. In Section 5, we first estimate the parameters of our model by using the annual landing data of the Black Sea anchovy on the southern part. Then we present numerical solutions to the optimality system for some bioeconomically interesting scenarios under the consideration of sustainable ecosystem, and we compare the three main strategies for fishery management of the Black Sea anchovy. After that, we estimate the number of fishing fleets by using our model outputs. Finally, we compare the results from TFM and EBFM in optimal control case, and show the benefits of using food chain models together with optimal control theory.
2.2 Model Formulation

We use a food chain model with three trophic levels to represent the behavior of the food web system, which consists of anchovy, $A$, zooplankton, $Z$, as a prey of anchovy, and jellyfish, $P$ as a predator of both anchovy and zooplankton. Each compartment has units of biomass (tonnes) in the southern part of the Black Sea and has logistic growth. In this study, we harvest the Black Sea anchovy $A$, and the harvest term, $h(t)A(t)$, is proportional to the anchovy biomass and the harvest rate, $h(t)$, which represents the amount of anchovy biomass taken by using purse seine fishing vessels in commercial fishery on the southern part of the Black Sea at time $t$. The Figure 2.3 shows the consumption of each compartment in our model.

![Flow Diagram](image)

**Figure 2.3:** The flow diagram of consumption in our food chain model.
Given a control $h$, the corresponding state variables, $A(h)$, $P(h)$, and $Z(h)$ satisfy the following state system:

\[
\frac{dA}{dt} = r_1A(1 - \frac{A}{K_1}) + m_0AZ - m_1PA - hA \\
\frac{dP}{dt} = r_2P(1 - \frac{P}{K_2}) + m_2PA + m_3PZ - m_6P \\
\frac{dZ}{dt} = r_3Z(1 - \frac{Z}{K_3}) - m_4AZ - m_5PZ
\]  

(2.1)

with the initial conditions:

\[A(0) = A_0, \quad P(0) = P_0, \quad Z(0) = Z_0\]  

(2.2)

where all the coefficients and initial conditions are positive and bounded. The terms $m_0AZ$, $m_1PA$, $m_2PA$, $m_3PZ$, $m_4AZ$, and $m_5PZ$ represent interaction terms between the species. For example, the term $m_1PA$ is a decay term for the anchovy population, and $m_2PA$ is a growth term for the jellyfish population. The term $m_6P$ comes from the predator–prey relation between *Mnemiopsis Leidyi* and its predators, like *Beroe Ovata*, in the Black Sea food web. Since we want to keep the system of ODEs simple enough for the study, we add the term, $m_6P$ instead of adding one more equation in the system (2.1).

Since the directions of consumption are important in terms of the biomass flow between trophic levels, we need to consider the following relationships: $m_4 > m_0$, $m_1 > m_2$, and $m_5 > m_3$. For example, in the predator-prey relation between the anchovy and zooplanton populations, the anchovy population will consume more than one unit of zooplankton to increase its size one unit, that is why we have the relation $m_4 > m_0$. 

17
Table 2.1: Parameter descriptions and units

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Descriptions</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>Intrinsic growth rate of anchovy, $A$</td>
<td>days$^{-1}$</td>
</tr>
<tr>
<td>$r_2$</td>
<td>Intrinsic growth rate of jellyfish, $P$</td>
<td>days$^{-1}$</td>
</tr>
<tr>
<td>$r_3$</td>
<td>Intrinsic growth rate of zooplankton, $Z$</td>
<td>days$^{-1}$</td>
</tr>
<tr>
<td>$K_1$</td>
<td>Carrying capacity of anchovy, $A$</td>
<td>Tonnes</td>
</tr>
<tr>
<td>$K_2$</td>
<td>Carrying capacity of jellyfish, $P$</td>
<td>Tonnes</td>
</tr>
<tr>
<td>$K_3$</td>
<td>Carrying capacity of zooplankton, $Z$</td>
<td>Tonnes</td>
</tr>
<tr>
<td>$m_0$</td>
<td>Growth rate of $A$ due to predation of $Z$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Consumption rate of $A$ due to its predator $P$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Growth rate of $P$ due to predation of $A$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>Growth rate of $P$ due to predation of $Z$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>Consumption rate of $Z$ due to its predator $A$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_5$</td>
<td>Consumption rate of $Z$ due to its predator $P$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_6$</td>
<td>Consumption rate of $P$ due to its predators,</td>
<td>days$^{-1}$</td>
</tr>
</tbody>
</table>

Assuming $\Omega = \bigcup_{i=1}^n [a_i,b_i]$ with $n$ denoting the number of years for fishing seasons, the interval $[a_i,b_i]$ represents the fishery seasons for $i = 1, 2, \ldots, n$. Thus, we have a seasonal fishery on the set $\Omega$, and the control, $h$, is zero on the set $[0,T] \setminus \Omega$. The fishery season lasts about 3 months in commercial fishery of the Black Sea anchovy. In this commercial fishery of the Black Sea anchovy, the main targets are big fish schools to reduce the cost of the fishery. The anchovy population aggregates and creates big fish schools when the ambient temperature drops to 16 - 18$^\circ$C, which mostly corresponds to the middle of October, or the beginning of November (Gucu et al. (2017)). Therefore, during the study, we consider the fishery season as 3 months (November - January).

Our objective functional for system (2.1) – (2.2) is

$$J(h) = \int_0^T e^{-\alpha t}(phA - \mu_1 h - \mu_2 h^2)dt = \int_\Omega e^{-\alpha t}(phA - \mu_1 h - \mu_2 h^2)dt$$ (2.3)
where $J(h)$ is the discounted net value of the Black Sea anchovy fishing, $h$ is the harvest rate (the control variable), and $e^{-\alpha t}$ denotes the discount rate with interest rate $\alpha$. The term $e^{-\alpha t}phA$ represents the revenue of the fishery with price $p$, and $e^{-\alpha t}(\mu_1 h + \mu_2 h^2)$ represents the cost of the fishery. This cost term is nonlinear in $h$, but the coefficient $\mu_2$ is taken small enough to have a small effect on numerical calculations. In this study, for convenience, we take $p = 1$ monetary unit. Our purpose is to find an optimal control, $h^*$ in $A$ such that

$$J(h^*) = \sup_{h \in A} J(h)$$

where $A$ is the class of admissible controls such that

$$A = \{h : [0, T] \rightarrow [0, M] \mid h=0 \text{ on } [0, T] \setminus \Omega \text{ and } h \text{ Lebesque measurable} \}.$$

### 2.3 Positivity and Boundedness of the State Variables

In this part, we will show the positivity of our state variables, and then we will show that the state variables are uniformly bounded. These results will be used later in the proofs of the existence and uniqueness of the optimal control, $h^*$.

**Theorem 2.1.** Given solutions of the state system for $A$, $P$, and $Z$ with initial conditions (2.2), there exists constants $C_1$, $C_2$, $C_3 > 0$ such that

$$0 < A(t) \leq C_1$$

$$0 < P(t) \leq C_2$$

for all $t \in [0, T]$

$$0 < Z(t) \leq C_3$$

**Proof.** Let us first show that $0 < Z(t) \leq C_3$ for all $t \in [0, T]$.

$$\frac{dZ}{dt} = r_3Z(1 - \frac{Z}{K_3}) - m_4AZ - m_5PZ$$
We use the integration factor to show $Z(t) > 0$ for all $t \in [0, T]$, but firstly we need to substitute $Z = \frac{1}{\hat{Z}}$ to obtain the following linear equation:

$$ \frac{d\hat{Z}}{dt} = -(r_3 - m_4 A - m_5 P)\hat{Z} + \frac{r_3}{K_3} $$

Letting $\phi(A, P) = r_3 - m_4 A - m_5 P$, we get the following linear form:

$$ \frac{d\hat{Z}}{dt} + \phi(A, P)\hat{Z} = \frac{r_3}{K_3} $$

When we multiply both side of the previous equation by the integral factor $\mu = e^{\int_0^t \phi(A, P)ds}$ and take the integral over the interval $t \in [0, T]$, we will obtain the following

$$ \hat{Z}(t)e^{\int_0^t \phi(A, P)ds} = \hat{Z}_0 e^{\int_0^t \phi(A, P)ds} + \int_0^t e^{\int_0^s \phi(A, P)ds} \frac{r_3}{K_3} > 0 $$

since the initial biomass $\hat{Z}_0$, $r_3$, $K_3$, and the exponential function are positive, we can obtain $\hat{Z}(t) > 0$. Thus, it follows that

$$ Z(t) > 0 \quad \text{for all } t \in [0, T]. $$

Similarly, we can get $P > 0$ and $A > 0$ for all $t \in [0, T]$. Now, let us first show $Z(t)$ has an upper bound over the interval $[0, T]$. Since all the coefficients are positive defined, and the states are positive, we can get the following inequality as

$$ \frac{dZ}{dt} = r_3 Z(1 - \frac{Z}{K_3}) - m_4 AZ - m_5 PZ \leq r_3 Z(1 - \frac{Z}{K_3}) \leq r_3 Z $$

When we arrange the above inequality and take the integral from 0 to $t$, where $t \in [0, T]$ and $T$ in $\mathbb{R}$, we will obtain

$$ \int_0^t \frac{dZ}{Z} \leq \int_0^t r_3 ds $$
After solving the integral, we got

\[ Z(t) \leq Z_0 e^{r_3 t} \quad \text{for all } t \in [0, T]. \]

which gives by letting \( C_3 = Z_0 e^{r_3 T} \) that

\[ 0 < Z(t) \leq C_3 \quad \text{for } t \in [0, T]. \]

Similarly, we can bound \( A(t) \) by using \( 0 < Z(t) \leq C_3 \), and then bound \( P(t) \) over the interval \( t \in [0, T] \) as

\[ 0 < A(t) \leq C_1 \]
\[ 0 < P(t) \leq C_2 \]

\[ \Box \]

### 2.4 Necessary Conditions

In this section, we will prove that there exists an optimal control for the ODE system (2.1). Then, using Pontryagin’s Maximum Principle, we will derive the necessary conditions that an optimal control and its corresponding states must satisfy.

**Theorem 2.2 (Existence of an Optimal Control).** *There exists an optimal control \( h^* \) in the class of admissible controls \( \mathcal{A} \), which maximizes the objective functional \( J(h) \) subject to our state system (2.1) with initial conditions (2.2).*

*Proof.* Suppose \( h^* \) is an optimal control with corresponding states \( A^*, P^*, \) and \( Z^* \) of the system (2.1)–(2.2). Using Pontryagin’s Maximum Principle, we form the Hamiltonian:
\[ H = e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2) + \lambda_A \left[ r_1 A - \frac{r_1}{K_1} A^2 + m_0 AZ - m_1 PA - hA \right] \\
+ \lambda_P \left[ r_2 P - \frac{r_2}{K_2} P^2 + m_2 PA + m_3 PZ - m_6 P \right] \\
+ \lambda_Z \left[ r_3 Z - \frac{r_3}{K_3} Z^2 - m_4 AZ - m_5 PZ \right]. \quad (2.4) \]

From PMP, we obtain the existence of the adjoint functions satisfying

\[
\frac{d\lambda_A}{dt} = -\frac{\partial H}{\partial A} = -\left( e^{-\alpha t} h + \lambda_A [r_1 - 2\frac{r_1}{K_1} A + m_0 Z - m_1 P - h] + m_2 \lambda_P P - m_4 \lambda_Z Z \right) \\
\frac{d\lambda_P}{dt} = -\frac{\partial H}{\partial P} = -\left( -m_1 \lambda_A A + \lambda_P [r_2 - 2\frac{r_2}{K_2} P + m_2 A + m_3 Z - m_6] - m_5 \lambda_Z Z \right) \quad (2.5) \\
\frac{d\lambda_Z}{dt} = -\frac{\partial H}{\partial Z} = -\left( m_0 A \lambda_A + m_3 \lambda_P P + \lambda_Z [r_3 - 2\frac{r_3}{K_3} Z - m_4 A - m_5 P] \right).
\]

together with the transversality conditions, \( \lambda_A(T) = 0, \lambda_P(T) = 0, \lambda_Z(T) = 0 \). Now, let us obtain a characterization of optimal control \( h^* \), by differentiating the Hamiltonian with respect to the optimal control, \( h \).

\[
\frac{\partial H}{\partial h} = e^{-\alpha t}(A - \mu_1 - 2\mu_2 h) - \lambda_A A \quad \text{on} \quad [0, T]
\]

In order to maximize the Hamiltonian with respect to the control, \( h \) at \( h^* \), we will have 3 cases as follow:

When \( \frac{\partial H}{\partial h} = 0 \) at time \( t \), the Hamiltonian, \( H \) will obtain its maximum value in the interval \( 0 \leq h^*(t) \leq M \). Then

\[
\frac{\partial H}{\partial h} = e^{-\alpha t}(A - \mu_1 - 2\mu_2 h) - \lambda_A A = 0 \quad \text{at} \quad h^*
\]
implies
\[ 0 \leq h^*(t) = \frac{A(1 - e^{\alpha t} \lambda A) - \mu_1}{2\mu_2} \leq M \quad \text{on} \quad [0, T] \]

When \( \frac{\partial H}{\partial h} < 0 \) at time \( t \), then the Hamiltonian, \( H \) has its maximum value at \( h^*(t) = 0 \),
\[ e^{-\alpha t} (A - \mu_1) - \lambda A < 0 \quad \text{at} \quad h^*, \]
and
\[ A[1 - e^{\alpha t} \lambda A] - \mu_1 < 0. \]

By dividing both side by \( 2\mu_2 \), we get
\[ \frac{A(1 - e^{\alpha t} \lambda A) - \mu_1}{2\mu_2} < 0 = h^*(t) \quad \text{on} \quad [0, T] \]

When \( \frac{\partial H}{\partial h} > 0 \) at time \( t \), then the Hamiltonian, \( H \) has its maximum value at \( h^*(t) = M \),
\[ e^{-\alpha t} (A - \mu_1 - 2\mu_2 M) - \lambda A > 0 \quad \text{at} \quad h^*, \]
and
\[ A(1 - e^{\alpha t} \lambda A) - \mu_1 > 2\mu_2 M \]

which implies
\[ \frac{A(1 - e^{\alpha t} \lambda A) - \mu_1}{2\mu_2} > M = h^*(t) \quad \text{on} \quad [0, T] \]
By maximizing the Hamiltonian with respect to the control, we obtain a characterization of the optimal control as

\[ h^*(t) = \min \left\{ M, \max \left\{ 0, \frac{A^*(1 - e^{at}A^*) - \mu_1}{2\mu_2} \right\} \right\} \quad \text{on } [0, T] \]

2.4.1 The Optimality System

In this part, we will use Pontryagin’s Maximum Principle (Pontryagin et al. (1967)) to obtain the optimality system and derive the characterization of an optimal control. In the proof of following Theorem 2.3, we will maximize the Hamiltonian of the optimal control problem by using Pontryagin’s Maximum Principle (PMP).

**Theorem 2.3.** Given an optimal control \( h^* \) and the state solutions \( A^*, P^*, \) and \( Z^* \) of the system (2.1), there exist adjoint variables \( \lambda_A, \lambda_P \) and \( \lambda_Z \) with corresponding to \( A^*, P^*, \) and \( Z^* \), respectively, which satisfy the following equations:

\[
\begin{align*}
\frac{d\lambda_A}{dt} &= - \left( e^{-at}h + \lambda_A[r_1 - 2 \frac{r_1}{K_1}A + m_0Z - m_1P - h] + m_2\lambda_PP - m_4\lambda_ZZ \right) \\
\frac{d\lambda_P}{dt} &= - \left( - m_1\lambda_AA + \lambda_P[r_2 - 2 \frac{r_2}{K_2}P + m_2A + m_3Z - m_6] - m_5\lambda_ZZ \right) \\
\frac{d\lambda_Z}{dt} &= - \left( m_0A\lambda_A + m_3\lambda_PP + \lambda_Z[r_3 - 2 \frac{r_3}{K_3}Z - m_4A - m_5P] \right)
\end{align*}
\]

with the transversality conditions:

\[
\lambda_A(T) = 0, \quad \lambda_P(T) = 0, \quad \lambda_Z(T) = 0.
\] (2.7)

Furthermore, a characterization of optimal control \( h^* \) is given by

\[ h^*(t) = \min \left\{ M, \max \left\{ 0, \frac{A^*(1 - e^{at}A^*) - \mu_1}{2\mu_2} \right\} \right\} \quad \text{on } [0, T]. \] (2.8)
Proof. Suppose $h^*$ is an optimal control with corresponding states $A^*$, $P^*$, and $Z^*$ of the system (2.1)–(2.2). Using Pontryagin’s Maximum Principle, we form the Hamiltonian:

$$H = e^{-\alpha t} (hA - \mu_1 h - \mu_2 h^2) + \lambda_A \left[ r_1 A - \frac{r_1}{K_1} A^2 + m_0 AZ - m_1 PA - hA \right]$$

$$+ \lambda_P \left[ r_2 P - \frac{r_2}{K_2} P^2 + m_2 PA + m_3 PZ - m_6 P \right]$$

$$+ \lambda_Z \left[ r_3 Z - \frac{r_3}{K_3} Z^2 - m_4 AZ - m_5 PZ \right]. \quad (2.9)$$

From PMP, we obtain the existence of the adjoint functions satisfying

$$\frac{d\lambda_A}{dt} = -\frac{\partial H}{\partial A} = -\left( e^{-\alpha t} h + \lambda_A [r_1 - 2 \frac{r_1}{K_1} A + m_0 Z - m_1 P - h] + m_2 \lambda_P P - m_4 \lambda_Z Z \right)$$

$$\frac{d\lambda_P}{dt} = -\frac{\partial H}{\partial P} = -\left( -m_1 \lambda_A A + \lambda_P [r_2 - 2 \frac{r_2}{K_2} P + m_2 A + m_3 Z - m_6] - m_5 \lambda_Z Z \right) \quad (2.10)$$

$$\frac{d\lambda_Z}{dt} = -\frac{\partial H}{\partial Z} = -\left( m_0 A \lambda_A + m_3 \lambda_P P + \lambda_Z [r_3 - 2 \frac{r_3}{K_3} Z - m_4 A - m_5 P] \right).$$

together with the transversality conditions, $\lambda_A(T) = 0$, $\lambda_P(T) = 0$, $\lambda_Z(T) = 0$. Now, let us obtain a characterization of optimal control $h^*$, by differentiating the Hamiltonian with respect to the optimal control, $h$.

$$\frac{\partial H}{\partial h} = e^{-\alpha t} (A - \mu_1 - 2 \mu_2 h) - \lambda_A A \quad \text{on} \quad [0, T]$$

In order to maximize the Hamiltonian with respect to the control, $h$ at $h^*$, we will have 3 cases as follow:

1. When $\frac{\partial H}{\partial h} = 0$ at time $t$, the Hamiltonian, $H$ will obtain its maximum value in the interval $0 \leq h^*(t) \leq M$. Then
\[
\frac{\partial H}{\partial h} = e^{-\alpha t}(A - \mu_1 - 2\mu_2 h) - \lambda_A A = 0 \quad \text{at } h^*
\]

implies

\[
0 \leq h^*(t) = \frac{A(1 - e^{\alpha t}\lambda_A) - \mu_1}{2\mu_2} \leq M \quad \text{on } [0, T]
\]

2. When \(\frac{\partial H}{\partial h} < 0\) at time \(t\), then the Hamiltonian, \(H\) has its maximum value at \(h^*(t) = 0\), and

\[
e^{-\alpha t}(A - \mu_1) - \lambda_A A < 0 \quad \text{at } h^*
\]

implies

\[
A[1 - e^{\alpha t}\lambda_A] - \mu_1 < 0.
\]

By dividing both side by \(2\mu_2\), we get

\[
\frac{A(1 - e^{\alpha t}\lambda_A) - \mu_1}{2\mu_2} < 0 = h^*(t) \quad \text{on } [0, T]
\]

3. When \(\frac{\partial H}{\partial h} > 0\) at time \(t\), then the Hamiltonian, \(H\) has its maximum value at \(h^*(t) = M\), and

\[
e^{-\alpha t}(A - \mu_1 - 2\mu_2 M) - \lambda_A A > 0 \quad \text{at } h^*
\]

implies

\[
A(1 - e^{\alpha t}\lambda_A) - \mu_1 > 2\mu_2 M.
\]
which implies

\[
\frac{A(1 - e^{\alpha t} \lambda_A) - \mu_1}{2\mu_2} > M = h^*(t) \text{ on } [0, T]
\]

By maximizing the Hamiltonian with respect to the control, we obtain a characterization of the optimal control as

\[
h^*(t) = \min \left\{ M, \max \left\{ 0, \frac{A^*(1 - e^{\alpha t} \lambda_A^*) - \mu_1}{2\mu_2} \right\} \right\} \text{ on } [0, T]
\]

**Remark 3.1.1.** Our optimality system, consists of the state system (2.1)–(2.2), the adjoint system (2.6)–(2.7), and the characterization of the optimal control, \( h^* \) (2.8). Since the adjoint system is linear in \( \lambda_A, \lambda_P, \) and \( \lambda_Z \) with bounded coefficients, the solutions are bounded for all \( t \in [0, T] \). Therefore, solutions of our optimality system is bounded for all \( t \in [0, T] \). This results in the existence and uniqueness of the optimality system for sufficiently small final time \( T \) together with bounded coefficients, initial conditions, and transversality conditions.

The optimality system consists of the state system, the adjoint system, and the characterization of the optimal control, \( h^* \), is given by in the following form:

\[
\frac{dA^*}{dt} = r_1 A^*(1 - \frac{A^*}{K_1}) + m_0 A^*Z^* - m_1 P^*A^* - h^*A^*
\]

\[
\frac{dP^*}{dt} = r_2 P^*(1 - \frac{P^*}{K_2}) + m_2 P^*A^* + m_3 P^*Z^* - m_6 P^*
\]

\[
\frac{dZ^*}{dt} = r_3 Z^*(1 - \frac{Z^*}{K_3}) - m_4 A^*Z^* - m_5 P^*Z^*
\]
\[
\begin{align*}
\frac{d\lambda_A}{dt} &= -\left( e^{-\alpha t} h + \lambda_A [r_1 - 2 \frac{r_1}{K_1} A^* + m_0 Z^* - m_1 P^* - h] + m_2 \lambda_P P^* - m_4 \lambda_Z Z^* \right) \\
\frac{d\lambda_P}{dt} &= -\left( -m_1 \lambda_A A^* + \lambda_P [r_2 - 2 \frac{r_2}{K_2} P^* + m_2 A^* + m_3 Z^* - m_6] - m_5 \lambda_Z Z^* \right) \\
\frac{d\lambda_Z}{dt} &= -\left( m_0 A^* \lambda_A + m_3 \lambda_P P^* + \lambda_Z [r_3 - 2 \frac{r_3}{K_3} Z^* - m_4 A^* - m_5 P^*] \right) \\
h^*(t) &= \min \left\{ M, \max \left\{ 0, \frac{A^*(1 - e^{\alpha t} \lambda_A^* - \mu_1)}{2\mu_2} \right\} \right\} \text{ on } [0, T]
\end{align*}
\]

together with initial conditions,

\[
A^*(0) = A_0, \quad P^*(0) = P_0, \quad Z^*(0) = Z_0
\]

and transversality conditions:

\[
\lambda_A(T) = 0, \quad \lambda_P(T) = 0, \quad \lambda_Z(T) = 0.
\]

### 2.5 Parameter Estimation

In the section, we estimate the parameters of our model by using the annual landing data of anchovy population, which is obtained by the Black Sea Assessment of Scientific, Technical and Economic Committee for Fishery 2017 (STECF (2017)).

The dynamics of anchovy population specifically depends on the Black Sea food web, and so our model is specific to the corresponding food web. That is why we need to estimate all the parameters by using the data obtained from the system of the Black Sea food web. We use the landing data of Black Sea anchovy on the southern part of the Black Sea to estimate the parameters of our model. We used the last 14 years’ landing data from 2003 to 2016.
from STECF (2017) since it represents the current harvesting strategy on the southern part of the Black Sea.

In order to estimate the parameters, we use Ordinary Least Squares (OLS) method, minimizing the sum of the squares of the differences between the observed annual landing data and those predicted by our food chain model as

$$\min(Y) = \min \left( \sum_{i=1}^{n} \left[ \frac{\text{Data}(i) - \text{Model}(i)}{\sum_{i=1}^{n} \text{Data}(i)^2} \right]^2 \right) \tag{2.12}$$

where $n$ is the number of data points, $\text{Data}(i)$ represents the yearly landing data that is the amount of fish taken from the system in each fishing season, and $\text{Model}(i)$ represents the simulated annual landing from our model. Actually we used relative error version of the OLS, and it gives us an opportunity to see how well our model represents the behavior of the system.

We used ode45 solver to get numerical outcomes of the model, and used fmincon from Optimization Toolbox of MATLAB. By using fmincon, we find a minimum of a constrained nonlinear multivariable function $Y$ subject to boundary of our parameters as in (2.12) subject to all parameters in Table 2.2 constrained on the interval $[0,1]$, and as well as the relationships between $m_4 > m_0$, $m_1 > m_2$, and $m_5 > m_3$ obtained by considering positions of the species in the food pyramid of the Black Sea.

We give the values of estimated parameters in the Table 2.2, and throughout the section we will use these parameter values except for $h$ (will vary) to produce all the numerical solutions and outcomes. We estimated 10 parameters and the constant harvest rate, $h$ using 14 data points with time unit days.
Table 2.2: Parameter descriptions and values in the case of constant harvesting strategy. Here $e$ is a scientific notation in MATLAB and it is a shorthand for 10.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Descriptions</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>Initial biomass of anchovy, $A$</td>
<td>$1.8e^5$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$P_0$</td>
<td>Initial biomass of jellyfish, $A$</td>
<td>$1.5e^4$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>Initial biomass of zooplankton, $A$</td>
<td>$1.9e^4$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$r_1$</td>
<td>Intrinsic growth rate of anchovy, $A$</td>
<td>0.3</td>
<td>Estimated</td>
</tr>
<tr>
<td>$r_2$</td>
<td>Intrinsic growth rate of jelly fish, $P$</td>
<td>0.75</td>
<td>Estimated</td>
</tr>
<tr>
<td>$r_3$</td>
<td>Intrinsic growth rate of zooplankton, $Z$</td>
<td>0.9</td>
<td>Estimated</td>
</tr>
<tr>
<td>$K_1$</td>
<td>Carrying capacity of anchovy, $A$</td>
<td>$3.5e^5$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$K_2$</td>
<td>Carrying capacity of jelly fish, $P$</td>
<td>$7.5e^3$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$K_3$</td>
<td>Carrying capacity of zooplankton, $Z$</td>
<td>$4e^4$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$m_0$</td>
<td>Growth rate of $A$ due to predation of $Z$</td>
<td>$1.4e^{-6}$</td>
<td>Estimated</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Consumption rate of $A$ due to its predator $P$</td>
<td>$0.66e^{-5}$</td>
<td>Estimated</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Growth rate of $P$ due to predation of $A$</td>
<td>$4.95e^{-6}$</td>
<td>Estimated</td>
</tr>
<tr>
<td>$m_3$</td>
<td>Growth rate of $P$ due to predation of $Z$</td>
<td>$5.7e^{-6}$</td>
<td>Estimated</td>
</tr>
<tr>
<td>$m_4$</td>
<td>Consumption rate of $Z$ due to its predator $A$</td>
<td>$0.2e^{-5}$</td>
<td>Estimated</td>
</tr>
<tr>
<td>$m_5$</td>
<td>Consumption rate of $Z$ due to its predator $P$</td>
<td>$1e^{-5}$</td>
<td>Estimated</td>
</tr>
<tr>
<td>$m_6$</td>
<td>Consumption rate of $P$ due to its predators,</td>
<td>0.2</td>
<td>Estimated</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>Coefficient of linear part of the cost function</td>
<td>30700</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>Coefficient of quadratic part of the cost function</td>
<td>0.1</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>The interest rate of the discount rate</td>
<td>0.01</td>
<td>Assumed</td>
</tr>
<tr>
<td>$h$</td>
<td>Harvest rate</td>
<td>0.48</td>
<td>Estimated</td>
</tr>
</tbody>
</table>
2.6 Numerical Results

We used the population-specific parameters given in Table 2.2 to obtain all the outcomes in this section. We first present three main harvesting strategies for ecosystem based fishery management on the southern part of the Black Sea, and then discuss why ecosystem based fishery management (focusing by using food chain models) is better than using traditional based fishery management (focusing on just one species in target fishing).

2.6.1 Harvesting Strategies

The main purpose of the study is to obtain an ecosystem-based optimal fishery management strategies for the Black Sea anchovy population on the southern part of the Black Sea. Having the goal in mind, we will discuss three main management strategies, which are categorized as constant harvesting strategy, current harvesting strategy and optimal harvesting strategy. Each type of strategy is explained below.

Constant Harvesting Strategy

In the harvesting strategy, we use a constant harvest rate for each fishing season from 2003 to 2016 on the southern part of the Black Sea. We estimated our model parameters using this strategy.

![Figure 2.4: Landing of the Black Sea anchovy with data (red) and simulation (blue) when the harvest rate, $h = 0.48$ in constant harvesting strategy.](image)
When estimating the parameters in Table 2.2, the constant harvest rate is also estimated as 0.48. This makes sense when we look at the literature (Oguz (2017), Gucu et al. (2017), STECF (2017), and Bilgin et al. (2016)). These studies state that the Black Sea anchovy population has been overfished and the harvest rate must be reduced.

The simulated landing data in the strategy is very consistence in long term because of the constant harvest rate (See Figure 2.4). This also results in almost a periodic oscillation in the biomass of species, which offers a sustainable anchovy fishery (See Figure 2.5). But the biomass of the Black Sea anchovy goes down below 50,000 tonnes at the end of each fishing season, which can be risky for the anchovy population when any outbreak of jellyfish occur in the system. That is why it is good to reduce the constant harvest rate to have better structure in terms of population biomasses.

We want to find an upper bound for the harvest rate, $h$ to obtain a sustainable and productive ecosystem. Therefore, we decreased the current harvest rate for the Black Sea anchovy fishing to see whether or not we can obtain more productive and sustainable structure of food web in terms of population bio-masses in anchovy fishery on the southern part of the Black Sea. We decreased the harvest rate by 0.4 and it resulted in not only more sustainable ecosystem (See Figure 2.6), but also resulted in much better discounted net profit for the anchovy fishery on the southern part of the Black Sea when we compare with the harvest rate, 0.48 (See Table 2.3).
Table 2.3: Comparison of harvest rates in constant harvesting strategy for the values of $\mu_1 = 30700$, and $\mu_2 = 0.1$.

<table>
<thead>
<tr>
<th>Harvesting Rates</th>
<th>Landing (Tonnes) (Data 2,865,392)</th>
<th>D. Net Cost</th>
<th>D. Net Profit (Tonnes)</th>
<th>Comparison (1,718,700 $\rightarrow$ 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=0.48$</td>
<td>3,243,400</td>
<td>1,063,100</td>
<td>1,718,700</td>
<td>1</td>
</tr>
<tr>
<td>$h=0.4$</td>
<td>3,226,600</td>
<td>884,060</td>
<td>1,882,100</td>
<td>1.10</td>
</tr>
</tbody>
</table>

Figure 2.6: Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when the harvest rate, $h = 0.4$ in constant harvesting strategy. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green).

Reducing the harvest rate reduces the fishing pressure on anchovy population. This results in a better structure of the anchovy population in terms of its biomass since the anchovy population stays around 50,000 tonnes at the end of each fishing seasons, instead of going below 50,000 tonnes in the case of the harvest rate, 0.48. We increase the discounted net profit by 10% for the Black Sea anchovy fishery by reducing the cost of the harvesting (See Table 2.3). As easily can be seen from the Table 2.3, when we decrease the harvesting effort from 0.48 to 0.4, the total discounted net value for 14 years increases from 1,718,700 tonnes to 1,882,100 tonnes even if we land less fish.

For an ecosystem-based fishery management strategy of the Black Sea anchovy, it is important to keep the harvesting effort between 0.35 and 0.4 to not only have more productive
and sustainable anchovy ecosystem in the Black Sea food web, but also to have more profit from the fishery in the long term using our food chain model with constant harvest rates.

**Current Harvesting Strategy**

In the harvesting strategy, we used the parameter values in Table 2.2 except for the constant harvest rate, 0.48, and then estimated different yearly constant harvest rates for each fishing season by using the OLS method. When we estimate harvest rates for each year, from the first five set of harvest rates, which give the best fit for the strategy, we picked the set of harvest rates, whose average is close to 0.48. We will compare the constant harvest strategy with this current harvest strategy. The strategy of using different yearly harvest rates (current harvest strategy) gives similar landing results with the landing data of the anchovy fishery (See Table 2.4). The relative error of this fit is about 0.25, which implies our model captures 75% of variation and information about the current harvesting strategy (See Figure 2.7). Therefore, we use the strategy as a representation of the recent harvesting strategy for the Black Sea anchovy fishing on the southern part.

![Figure 2.7](image-url)

**Figure 2.7:** Annual landing of the Black Sea anchovy with data (red) and simulation (blue) in current harvesting strategy with relative error 0.25.
Table 2.4: Comparison of constant and current harvesting strategies for the values of \( \mu_1 = 30700 \), and \( \mu_2 = 0.1 \)

<table>
<thead>
<tr>
<th>Harvesting Strategies</th>
<th>Landing (Tonnes) (Data 2,865,392)</th>
<th>D. Net Cost</th>
<th>D. Net Profit (Tonnes)</th>
<th>Comparison (1,415,200 → 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>2,870,900</td>
<td>1,063,330</td>
<td>1,415,200</td>
<td>1</td>
</tr>
<tr>
<td>Constant, ( h=0.48 )</td>
<td>3,243,400</td>
<td>1,063,100</td>
<td>1,718,700</td>
<td>1.21 (21%)</td>
</tr>
<tr>
<td>Constant, ( h=0.4 )</td>
<td>3,226,600</td>
<td>884,060</td>
<td>1,882,100</td>
<td>1.33 (33%)</td>
</tr>
</tbody>
</table>

Total landing of the Black Sea anchovy on the southern part is 2,865,392 tonnes for 14 years. When we apply the current harvesting strategy for 14 years, total landing is 2,870,900 tonnes, which is very close to the actual landing of the Black Sea anchovy. Since we do not know the actual discounted net profit, we assume that the discounted net profit of anchovy fishing corresponds about 50\% of landing of the Black Sea anchovy. Thus we chose the coefficient of cost function \( \mu_1 \) as \( \mu_1 = 30700 \) to obtain about 50\% discounted net profit of landing for Black Sea anchovy in this current strategy (See the first row of the Table 2.4, the discounted net profit is almost half of the landing). This assumption also will help us to compare the discounted net profits of current harvesting strategy with other two strategies.

When we apply the current harvesting strategy for the Black Sea anchovy, the structure of food web in terms of biomass of species will not stay balanced and sustainable for most of the years since the harvesting strategy is not consistence (See Figure 2.8). In most of the fishing seasons, the biomass of the Black Sea anchovy goes down to 50,000 tonnes at the end of the fishing seasons, which is a sign of overfishing of the Black Sea anchovy, and it also results in less reproduction and biomass for the next fishing seasons since the small populations in terms of biomass will produce less new individuals for the next fishing season. When we look at the later years, especially in 2016, even if the biomass of the Black Sea anchovy does not go down to 50,000 tonnes at the end of the previous fishing season, the fishermen of the Black Sea anchovy could not have a good fishery season even though the system offered more fish to be harvested at that year. It may be because of the violation of fishery season in commercial fishery. In commercial fishery of the Black Sea anchovy, the
Figure 2.8: Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when average harvest rate is about 0.48 in the current harvesting strategy. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green).

Main targets are big fish schools, and the anchovy population aggregates and creates big fish schools when the ambient temperature drops to 16 - 18°C, which mostly corresponds to the middle or end of October (Gucu et al., 2017). But, some fishermen and fleets owners may start the fishery season a little bit early to catch more anchovy. This situation may violate the process of aggregation of big fish schools, and may result in small fish schools and landing at that year.

When we compare the current harvesting strategy with the constant harvesting strategies for constant harvest rates 0.4 and 0.48, we see that the constant harvesting strategies offer more landing and much better discounted net profit than the current harvesting strategy (See Table 2.4). For example, we obtain %33 more profit for the constant harvest rate, 0.4.

Optimal Harvesting Strategy

Now, we use optimal control theory to maximize the discounted net profit of the Black Sea anchovy. We use the same parameter values obtained in the constant harvest strategy (See Table 2.2), and then apply the forward-backward sweep method for the state equations, the adjoint equations, and the optimal control characterization to solve the optimality system numerically (Lenhart and Workman (2007) and Hackbusch (1978)). To obtain more productive ecosystem, we want the anchovy biomass to be greater than 50,000 tonnes at the end of the each fishing season. In this way, we avoid overfishing of the Black Sea anchovy.
and obtain a sustainable and reproductive ecosystem. We take the maximum harvest rate as \( h = 0.4 \) to ensure having at least 50,000 tonnes anchovy biomass at the end of the each fishing season. We do not directly put the state constraint in the optimal control problem to have at least 50,000 tonnes anchovy biomass at the end of the each fishing season. Instead, we vary the maximum harvest rate to be able to get at least 50,000 tonnes anchovy biomass at the end of the each fishing season. Applying the optimal control strategy, we obtain much better net profit (See Table 2.5).

Although the average harvest rate of optimal harvesting strategy is less than 0.4 in each fishing season, we catch more fish and reduce the discounted net cost of anchovy fishery by applying the optimal harvesting strategy (See Table 2.5). These combined effects result in more discounted net profit for the fishery. In this strategy, we catch more fish in the long term since the biomass of anchovy population in optimal harvesting strategy is always above the 50,000 tonnes at the end of the each fishery season (See Figure 2.10), and so the anchovy population will increase its size by producing more new individuals for the next fishery season. Also, reducing cost of the fishery by applying the optimal harvesting strategy contributes the increase in the discounted net profit (See Table 2.5).

**Table 2.5:** Comparison of the three strategies with the assumption of having 50% net profit in the current harvesting strategy. In the fifth row, we use the approximated optimal harvest rate (See Figures 2.11 and 2.12)

<table>
<thead>
<tr>
<th>Harvesting Strategies</th>
<th>Landing (Tonnes) (Data 2,865,392)</th>
<th>D. Net Cost</th>
<th>D. Net Profit (Tonnes)</th>
<th>Comparison (1,415,200 ( \rightarrow ) 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>2,870,900</td>
<td>1,063,330</td>
<td>1,415,200</td>
<td>1</td>
</tr>
<tr>
<td>Constant, ( h=0.48 )</td>
<td>3,243,400</td>
<td>1,063,100</td>
<td>1,718,700</td>
<td>1.21 (21%)</td>
</tr>
<tr>
<td>Constant, ( h=0.4 )</td>
<td>3,226,600</td>
<td>884,060</td>
<td>1,882,100</td>
<td>1.33 (33%)</td>
</tr>
<tr>
<td>Optimal, ( h_{max} = 0.4 )</td>
<td>3,239,600</td>
<td>767,470</td>
<td>2,009,100</td>
<td>1.43 (43%)</td>
</tr>
<tr>
<td>Approximate Optimal, ( h_{max} = 0.4 )</td>
<td>3,210,500</td>
<td>767,920</td>
<td>1,982,900</td>
<td>1.40 (40%)</td>
</tr>
</tbody>
</table>
Figure 2.9: Landing of the Black Sea anchovy on the southern part with OC case, $h_{max} = 0.4$.

Figure 2.10: Left plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) when the optimal harvesting strategy applied with $h_{max} = 0.4$. Right plot: Biomass of the Jellyfish (red) and Zooplankton (green).
Furthermore, the structure of food web in terms of population biomasses looks sustainable for the long term since the dynamics of the system have almost stable oscillation (See Figure 2.10). The biomass of the anchovy population changes between 70,000 tonnes and 230,000 tonnes in the southern part of the Black Sea. Therefore, anchovy population will be more productive since we leave more anchovy biomass in the system at the end of each fishing season.

When we divide fishery season two pieces, the optimal harvesting strategy starts with harvest rate, $h = 0.4$ and decay to about 2.3 for the first half of the fishing season, and then increase the harvest rate to its maximum value 0.4 for the rest of the fishing season (See Figure 2.11). Having low harvest rates (less than 0.4) in the first half of the fishing season not only will give the anchovy population an opportunity to increase its size in a school of anchovies, but also it will reduce the cost of the fishery in the first half of the fishing season.

To obtain a more feasible policy for the implementation of optimal harvest rates, we approximate the first half of the optimal harvest rates as 0.335 (See Figure 2.12). From Table 2.5, we see that the discounted net profit from the approximate optimal harvest rates is 40%, which is above the constant harvest case, while being close to the discounted net profit of the optimal harvest strategy.

\[ \text{Figure 2.11:} \quad \text{The harvest rates for the Black Sea anchovy in the optimal harvesting strategy for the first three years with } h_{max} = 0.4. \text{ We start the harvesting in first three months and then stop harvesting the system in the rest of the year.} \]
2.6.2 Traditional Fishery Management vs Ecosystem Based Fishery Management

In this subsection, we will discuss about traditional fishery management (TFM) and ecosystem based fishery management (EBFM), and explain why EBFM is more reliable and realistic than TFM in our application. To do that, we will use a single species model, which is a sub-system of our food chain model. We first will estimate the parameter values of the single species model using the landing data, and then compare the results from the single species model with our food chain model. After that we will implement the optimal harvesting strategy of the single species model in our food chain system. Finally, we will discuss the outcomes of the optimal harvesting strategy from the single species model and our food chain model.

Contrary to the traditional fishery management, ecosystem based fishery management increases the chance of having more sustainable and productive ecosystems. Too often traditional fishery management has failed to take a precautionary approach to maintain and protect sustainable fisheries, biodiversity, and marine ecosystem function (Lauck et al. (1998)). To show the weaknesses of using the traditional fishery management in our application, we first introduce the single species model.
Given a control $h$, the corresponding state variables, $A(h)$ satisfies the following state equation:

$$\frac{dA}{dt} = rA\left(1 - \frac{A}{K}\right) - hA$$

(2.13)

with initial condition $A(0) = A_0$. Our optimal control problem becomes

$$J(h) = \int_0^T e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2)dt$$

subject to (2.13) and $h \in A$.

For this single species model, we estimated the parameters in the case of constant harvest rate and found $r = 0.2$, $K = 275,000$ tonnes, which are much smaller than the corresponding parameters $r_1 = 3.5$ and $K_1 = 350,000$ tonnes estimated for our food chain model. We also estimate the initial biomass of the anchovy population as 220,000 tonnes. When we fit the single species model (See Figure 2.13) with constant harvest rate for the Black Sea anchovy, we estimated the constant harvest rate as 0.5, which is close with some rates in literature (Oguz (2017), Gucu et al. (2017), STECF (2017), and Bilgin et al. (2016)).

![Figure 2.13](image)

**Figure 2.13:** Annual landing of the Black Sea anchovy with constant harvest rate, 0.5, intrinsic growth rate, 0.2, and carrying capacity, 275,000 tonnes in the case of parameter estimation for the single species model.
When we apply these two optimal harvesting strategies obtained from the single species model and our food chain model (See Figure 2.14), we see that the trajectory of anchovy biomass in our food chain model stays roughly between 70,000 to 230,000 tonnes, but the trajectory of anchovy biomass in single species model stays between 80,000 to 275,000 tonnes. Thus, we got a better structure in terms of biomass of anchovy population in the single equation model when we compare with the anchovy biomass in our food chain model. This is because of ignoring the food web of the Black Sea anchovy and interactions between species in the single species model.

When we compare the outcomes of the total landing and net discounted value of Black Sea anchovy obtained by the single species model and our food chain model (See Table 2.6), we see that even with smaller carrying capacity and intrinsic growth rate in the single species model, we obtain more landing and profit by using the single species model, with almost the same harvesting strategy (See Figure 2.15). This situation can be explained by the effects of biological relations between species. If we look at the difference in the total landing of anchovy in the single species model and our food chain models (See Table 2.6), we see that yearly approximately 30,000 tonnes landing has been lost in anchovy landing because of biological interactions between species. The comparison shows that the effect of the jellyfish still plays an important role on the Black Sea anchovy population together with zooplankton population.

Figure 2.14: Left plot: Biomass of the Black Sea anchovy for the single species model. Right plot: Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green) in the optimal harvesting strategy.
Even if we got 2,368,400 tonnes discounted net profit by using the single species model, which is much higher than what we got by using our food chain model, it is not realistic. Actually, the discounted net profit would be 1,994,200 tonnes, if we implement the optimal harvesting strategy of the single species model in our food chain model (See Table 2.6). We got 2,368,400 tonnes discounted net profit in the single species model since it does not take account of the predator pressure of jellyfish, and it also assumes the Black Sea ecosystem has always enough zooplankton biomass. But in our real system, ignoring the effect these species results in obtaining incorrect and unsafe information about the system.

If we could consider the outcomes of the single species model for the fishery management of the Black Sea anchovy, we would take out more fish from the system depending on the optimal harvesting strategy in the single species model, which could result in a collapse of the Black Sea anchovy as happened on the northern part of the Black Sea. It is very risky to ignore the effect of these interactions in the anchovy fishery. Therefore, in order to protect the ecosystem and biodiversity of the Black Sea, it is important to observe the structure of the food web in terms of population biomasses by using food chain models before applying any harvesting strategy.
Table 2.6: Comparison of the models under the assumption of having 50% net profit in current harvesting strategy. $h_f$ and $h_s$ denote the optimal harvesting rates of food chain model and single equation, respectively. In the case of “Food Chain with $h_s$”, we implement the optimal harvest rate of the single equation to our food chain model to compare results.

<table>
<thead>
<tr>
<th>Models</th>
<th>Landing (Tonnes)</th>
<th>D. Net Cost</th>
<th>D. Net Profit (Tonnes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food Chain with $h_f$</td>
<td>3,239,600</td>
<td>767,470</td>
<td>2,009,100</td>
</tr>
<tr>
<td>Single Eq. with $h_s$</td>
<td>3,665,400</td>
<td>772,850</td>
<td>2,368,400</td>
</tr>
<tr>
<td>Food Chain with $h_s$</td>
<td>3,234,700</td>
<td>772,850</td>
<td>1,994,200</td>
</tr>
</tbody>
</table>

2.6.3 Estimation of the Numbers of Fishing Fleets

In this section, we estimate the number of fishing fleets needed in management of the anchovy fishery on the southern part of the Black Sea. In order to approximate the number of fishing fleets, we will find a relationship between landing data of the anchovy and effort (the number of fishing fleets). Depending on our numerical results, the average harvest rate for anchovy fishery is about 0.48 in the constant harvesting strategy. By using the harvest rate 0.48, and the corresponding numbers of fishing fleets from the data from 2003 to 2016 (since we fitted our model by using the data from these years), we will build a regression model to estimate the number of fishing fleets that take place in the anchovy fishery for the implementation of the approximate optimal harvesting strategy.

When we look at the data (See the Table B.1 in Appendix B), we have two different regimes in terms of fishing effort. After 2002, the fishing effort almost doubled by 2013, and this high fishing efforts during the these years resulted in low landing of the anchovy in the long term. The average landing of the anchovy population dropped from 270,000 tonnes to 200,000 tonnes during the period of high fishing effort, and even the average landing dropped to 111,000 tonnes in the last five fishing seasons. It is not hard to see that increasing harvest effort not only resulted in low landing, but also resulted in low profit during the period of high fishing effort because of the high cost of the fishery.
By using our food chain model in the case of constant harvest, we found the average (current) harvest rate as 0.48, which corresponds to fishing effort 373 (the average number of fishing fleets from data) depending on the data from 2003 to 2016. On the other hand, our study showed the current harvest rate is high, and need to be reduced. In the approximate optimal harvesting strategy, it is advised to use the harvest rate 0.335 for the first half of fishing season, and then use the harvest rate 0.4 for the second half of the fishing season for ecosystem friendly optimal fishery management.

When we use the information that the average harvest rate, 0.48 corresponds 373 fishing fleets, which is the average number of fishing fleets depending on data from 2003 to 2016, then we can claim that the average number of fishing fleets roughly should lie between 200 and 350 for corresponding harvest rates 0.335 and 0.4 that are obtained in approximate optimal harvesting strategy. Therefore, we will use the data of fishing fleets that lie between 200 and 350 to fit our regression model. We picked the effort in the range of 200 and 350 since we consider the harvest rate between 0.335 and 0.4 in approximate optimal harvesting strategy. Here, the harvest rate 0.335 roughly corresponds 260 fishing fleets and 0.4 roughly corresponds 310 fishing fleets when we use the information that the harvest rate, 0.48 corresponds 373 fishing fleets. To estimate the number of fishing fleets, we consider the data of effort lies between 200 to 350 by adding some variation to these values. The choice of 200 fishing fleets and 350 fishing fleets are just rough estimates depending on available data (See Table B.1 in Appendix B).

Moreover, we consider the data of landing lies between about 200,000 tonnes and 300,000 tonnes to fit our regression model when we estimate the approximate number of fishing fleets. We picked the landing data between 200,000 tonnes and 300,000 tonnes since we obtained the simulated landing between these amounts in our optimal harvesting strategy. Therefore, we will fit our regression model by using the landing data, which are near to the optimal landing, and the effort data, which lies between 200 and 350 fishing fleets.

With these constraints in our data, we first tried to find a direct relationship between the data of landing and number of fishing fleets, but there was not significant relation between
the landing data and the data of number of fishing fleets. But with the landing data (L) and the data of catch per unit effort (CPUE), we found a significant relation between L and CPUE with R-squared 0.9. Therefore, we used this relationship, and built the following nonlinear regression model for the relationship between L and CPUE on the southern part of the Black Sea:

\[ \sqrt{CPUE} = 8.002 + 1.736e^{-9} \times (L^2) - 4.779e^{-20} \times (L^2)^2 + 5.527e^{-31} \times (L^2)^3 - 2.183e^{-42} \times (L^2)^4. \]

In the regression analysis, to fill the requirement of normality for data, we transformed the data to the forms \( \sqrt{CPUE} \) and \((Landing)^2\). Since we have the relationship \( CPUE = \frac{Landing}{Effort} \) between data, we get the approximate \( Effort^* \) by using the equation:

\[ Effort^* = \frac{Landing^*}{CPUE^*} \]  \hspace{1cm} (2.14)

where, \( Landing^* \) is our optimal landing obtained using our food chain model, and \( CPUE^* \) is our approximate \( CPUE \) from our nonlinear regression model. The results of equation (2.14) given in Table 2.7, and the following figure is the visualization of our nonlinear regression model.

![Nonlinear regression model](image)

**Figure 2.16:** Nonlinear regression between CPUE and the landing of the anchovy population depending on the data from 1985 to 2000, and the data from 2013 to 2016 (See Table B.1 in Appendix B).
Table 2.7: The landing of anchovy in the first half and in second half obtained by using our food chain model with the optimal harvesting strategy. Then, by using the non-linear regression model, we obtained the approximated numbers of fishing fleets for the first and second half of the anchovy fishery. The approximated CPUE is about 948 for the first half of the fishery season and 953 for the second half of the fishery season.

<table>
<thead>
<tr>
<th>Years</th>
<th>Landing in 1st half of fishery with h=0.335</th>
<th>Landing in 2nd half of fishery with h=0.4</th>
<th># of fishing fleets in 1st half of fishery</th>
<th># of fishing fleets in 2nd half of fishery</th>
</tr>
</thead>
<tbody>
<tr>
<td>2003</td>
<td>110,530</td>
<td>116,330</td>
<td>233</td>
<td>244</td>
</tr>
<tr>
<td>2004</td>
<td>113,045</td>
<td>116,430</td>
<td>238</td>
<td>244</td>
</tr>
<tr>
<td>2005</td>
<td>113,400</td>
<td>117,145</td>
<td>239</td>
<td>246</td>
</tr>
<tr>
<td>2006</td>
<td>113,755</td>
<td>116,405</td>
<td>240</td>
<td>244</td>
</tr>
<tr>
<td>2007</td>
<td>113,460</td>
<td>117,025</td>
<td>239</td>
<td>246</td>
</tr>
<tr>
<td>2008</td>
<td>113,240</td>
<td>116,710</td>
<td>239</td>
<td>245</td>
</tr>
<tr>
<td>2009</td>
<td>113,145</td>
<td>116,725</td>
<td>239</td>
<td>245</td>
</tr>
<tr>
<td>2010</td>
<td>113,190</td>
<td>116,570</td>
<td>239</td>
<td>245</td>
</tr>
<tr>
<td>2011</td>
<td>112,885</td>
<td>116,755</td>
<td>238</td>
<td>245</td>
</tr>
<tr>
<td>2012</td>
<td>113,475</td>
<td>116,330</td>
<td>239</td>
<td>244</td>
</tr>
<tr>
<td>2013</td>
<td>113,320</td>
<td>116,805</td>
<td>239</td>
<td>245</td>
</tr>
<tr>
<td>2014</td>
<td>113,265</td>
<td>116,385</td>
<td>239</td>
<td>244</td>
</tr>
<tr>
<td>2015</td>
<td>113,990</td>
<td>116,285</td>
<td>240</td>
<td>245</td>
</tr>
<tr>
<td>2016</td>
<td>113,365</td>
<td>116,795</td>
<td>239</td>
<td>245</td>
</tr>
</tbody>
</table>

When we compare the results in Table 2.7 and the data in Table B.1 (See Appendix B), we see that the number of optimal fishing fleets is estimated as 245 in the optimal control case, but the average number of fishing fleets used in the anchovy fishery is 373 in the current harvesting strategy from the data for last 14 years. Therefore, during the last 14 years, the amount of fleets was approximately 53% more than the optimal amount of fishing fleets predicted by our model.

We summarized the results of optimal control strategy, and compared these results with the current harvesting strategy in Table 2.8.
Table 2.8: Summary of the results in the OC strategy. We also compare the advantageous of using OC strategy by comparing with the current harvest strategy

<table>
<thead>
<tr>
<th>Summary of Results in OC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Harvest Rate</td>
</tr>
<tr>
<td>Maximum Annual Landing</td>
</tr>
<tr>
<td>Optimal Annual Landing</td>
</tr>
<tr>
<td>Optimal Annual Net Profit</td>
</tr>
<tr>
<td># of fleets in first half of the fishery</td>
</tr>
<tr>
<td># of fleets in second half of the fishery</td>
</tr>
<tr>
<td>Discounted Net Profit</td>
</tr>
<tr>
<td>Total Landing</td>
</tr>
<tr>
<td>Total Cost</td>
</tr>
</tbody>
</table>

2.7 Conclusions

First of all, our study shows that the Black Sea ecosystem in terms of the anchovy population is not sustainable with the current harvesting strategy on the southern part of the Black Sea (See Figure 2.8). To have a sustainable and productive ecosystem, it is better to reduce the harvest rate and use the optimal harvesting strategy in a long term, or use a constant harvest rate for each fishing season. These two strategies are not only give the Black Sea ecosystem an opportunity to recover and become more sustainable and productive, but also give a chance to the investors of the fishery to have more landing and profit from the Black Sea anchovy fishery on the southern part. The optimal harvesting strategy offers the investors of the fishery to make more than 40% profit than in the current strategy. The optimal harvesting strategy also offers annually about 230,000 tonnes anchovies in the long term. To give the Black Sea ecosystem an opportunity to recover, and become sustainable in the long term, it is really important to impose a quota about 200,000 tonnes in next couple of years, and then increase it about the 230,000 tonnes in the long term since nowadays this Black
Sea ecosystem has sharp increases and decreases in terms of landing and biomass.

Moreover, the optimal harvesting strategy suggests not to harvest more than 250,000 tonnes, which can be considered as the maximum optimal yield for the Black Sea anchovy on the southern part. Since our food chain model suggests to have at least 50,000 tonnes anchovy biomass at the end of each fishing season in the system for sustainable anchovy fishery, we consider the amount as a threshold amount for EBFM in the constant harvesting strategy. (These threshold amount about 70,000 tonnes in the optimal harvesting strategy for EBFM). It is also recommended not to increase the harvest rate above 0.425 in the case of constant harvest rate because it results in having less than 50,000 tonnes anchovy biomass at the end of fishing seasons. Therefore, the harvest rate, 0.425 can be considered as the maximum harvest rate in the case of constant harvesting strategy. These results for the maximum optimal yield and maximum harvest rate in the constant harvesting strategy for a sustainable ecosystem matches one of the current studies presented by Salihoglu et al. (2017).

Traditional fishery management through modeling, taking into account a single species model for the fish stock, gives somewhat unrealistic information about the current harvesting strategy for the Black Sea anchovy. Moreover, it does not show the effect of the interactions between the species on the Black Sea anchovy as we showed using our food chain model. When we compare the results obtained from the single species model and our food chain model, we can see that the jellyfish population still has an important effect on the landing of the Black Sea anchovy together with the zooplankton population. Approximately 30,000 tonnes of anchovy landing have been lost yearly because of biological interactions between species (See Table 2.6).

Our study takes into account the food web of the Black Sea anchovy, resulting in more reliable information. Therefore, using food chain models with optimal control theory for a fishery management is a valuable tool to obtain improved reliable fishery strategies. The food web approach is very useful to keep track of the effects of fisheries on corresponding ecosystems to protect species and biodiversity.
We also estimated the number of fishing fleets needed in management of the anchovy fishing with the help of a nonlinear regression model using the data of landing, effort, and CPUE. Thus, depending on our food chain and regression models, we estimated the number of fishing fleets in the first half of the fishery season as 239, corresponding to the approximate optimal harvest rate $h = 0.335$. Similarly, we found 245 for the corresponding harvest rate, 0.4 for the approximate optimal harvest rate in the second half of the fishery season. Our study shows that the amount of extra effort is approximately 53%, meaning that we were using 53% extra fishing fleets for the anchovy fishery in the current harvesting strategy depending on the data of effort in last 14 years.

In the future, more data would help us to find the growth rates as functions of time to have more realistic growth rates. Another way to protect the biodiversity and populations of the Black Sea ecosystem is the use of reserves, meaning marine protected areas. Later models with spatial features will be considered to investigate the possibility of reserves. In this study, we did not consider the migration of the Black Sea anchovy, but in the future, one should include the migration of the Black Sea anchovy to have more realistic model.
Chapter 3

PDE Model

3.1 Introduction and Background

In current fishery management, overfishing is one of the top problems that ecologists and bio-economists need to investigate and address with ecosystem friendly fishery management strategies. Since fish populations live in complex ecosystems and have natural predator-prey relationships with other species, the effects of interactions between species in a food web should not be ignored (Grishin et al. (2007)). One of the well-known examples of fish populations is the Black Sea anchovy in the Black Sea (See Figure 3.1), which has been facing difficulties competing with $Mnemiopsis leidyi$ for zooplankton populations as a main source of food, and also has been experiencing predation by $Mnemiopsis leidyi$ on its larvae and eggs (Oguz et al. (2008)). Therefore, we can not ignore the effects of predator-prey relations on this anchovy population. To track the effects of the anchovy fishery on its food web, we will use food chain models.

Many studies have been done for the management of fish populations, but most of them represented the fish ecosystem using a single equation (Hilborn and Walters (1992), Skern-Mauritzen et al. (2016)), which ignores the predator-prey relations and the effects of the fishery on the food webs. Using a single equation is not a good choice for modeling sustainable fishery management (Gwaltney et al. (2004) and Lauck et al. (1998)). Ecosystem based fishery management (EBFM) focusing on the whole ecosystem of species becomes an
increasingly useful trend in the commercial fishery to not only conserve and manage natural renewable food resources, but also to have sustainable and reproductive ecosystems (e.g. Pikitch et al. (2004), Fletcher et al. (2010), Fulton et al. (2014)).

The study for the management of natural renewable resources is an increasingly growing field due to its importance for sustainable ecosystems, and for economic value of the renewable resources. There have been some studies investigating different techniques to protect fish populations and conserve marine ecosystems, such as no-take marine reserves (marine protected areas) (FAO (2011), Ozturk et al. (2017)). Besides, this intervention method, another way to find management policies is to use the Optimal Control Theory (OCT) together with food chain models to obtain optimal management policies for the natural renewable food resources. In many cases, no-take marine reserve areas are natural results of the application of OCT in fishery management (Neubert (2003), Joshi et al. (2009), and Moberg et al. (2015)). Thus, using ecosystem based modeling techniques by using food chain models coupled with optimal control theory will be very useful.

![Figure 3.1: The location of the Black Sea (obtained from Gucu et al. (2017))]
Due to overexploitation of marine systems and the lack of ecosystem based management strategies for commercial fisheries, it is essential to the fishery industry to come up with new ecosystem friendly management strategies to protect fish stocks and keep ecosystems sustainable and resilient. That is why ecologists and economists have been investigating new management methods for ecologically sustainable commercial fisheries. In particular, the exploration of spatial strategies for improving regulatory outcomes has received much recent attention (Wilen (2004)). Neubert built a steady state model from a diffusive PDE, which is reduced to a second-order ODE in space at equilibrium and he showed that depending on length of domains, marine reserves are part of optimal harvesting strategy when maximizing yields (Neubert (2003)). The advantage of using a spatial model is the possibility of having marine reserve. Ding and Lenhart extended Neubert’s work to a multidimensional spatial domain and numerically found marine reserves in the center of the domain (Ding and Lenhart (2008)). Furthermore, Herrera and Lenhart showed that when spatial dynamics of a resource are ignored, then management strategies generally produce suboptimal results (Herrera and Lenhart (2009)). These studies have showed us having spatial fishery models will give better ideas and strategies for fishery management.

In our study, we build our model as an extension of the models presented by Kelly et al. (2016) and Joshi et al. (2009). In their studies they used a single PDE for the stock. Those two models are generic, not representing a specific fishery scenario. But in our study we use a system of PDEs, based on three trophic levels to represent the food web of the Black Sea anchovy. This application for the Black Sea anchovy coupled with the optimal control theory is novel. Other such PDE results like Kelly et al. (2016) and Joshi et al. (2009) are for general fish populations using a single equation. Having three trophic levels in a model will give us an opportunity to track any change in the food web due to the effects of commercial fishery. Besides, having the information about the effects of fishing on the food web, we also will able to see the effects of predator-prey relations on the system. This approach and modeling technique are valuable for ecosystem based sustainable fishery management.
In the next section, we formulate our model with biologically feasible boundary and initial conditions, and present our goal in the objective functional for an optimal fishery. In Section 3.3, we prove the positivity, existence and uniqueness of the weak solution for our PDE system by using Banach Fixed Theorem. In section 3.4, we first prove existence of an optimal control and then demonstrate the necessary conditions that an optimal control must satisfy. Moreover, we obtain the optimality system, which consists of the state equations, the adjoint equations, and the characterization of optimal control. In Section 3.5, we first obtain the parameters of our model by adjusting the parameters presented in Chapter 2, then we present numerical solutions to our optimality system for some (ecosystem friendly) scenarios, and then we discuss and compare these scenarios for fishery management of the Black Sea anchovy.

3.2 Model Formulation

In this study, we focus on optimal control of harvesting the Black Sea anchovy on the southern part of the Black Sea by using a food chain model in one space dimension. We consider a parabolic PDE system with Neumann boundary conditions in a bounded domain $Q = (0, L) \times (0, T)$. We defined $A(x, t)$, $P(x, t)$, and $Z(x, t)$ as biomass of the Black Sea anchovy, the Jellyfish ($Mnemiopsis Leidyi$), and the Zooplankton at location $x$, and time $t$, respectively. Each biomass has a logistic growth with an intrinsic growth rate, horizontal movement (Advection), and a simple diffusion (movement from regions of high concentration to regions of low concentration). In this study, we harvest the Black Sea anchovy, $A(x, t)$ and the harvesting term, $h(x, t)A(x, t)$, is proportional to the biomass of the Black Sea anchovy and the effort, $h(x, t)$.\[54\]
Given a control $h$, the corresponding state variables, $A(h)$, $P(h)$, and $Z(h)$ satisfy the state system:

\begin{align*}
L_1A &= f_1(A) + m_0AZ - m_1PA - hA \\
L_2P &= f_2(P) + m_2PA + m_3PZ - m_6P \quad \text{in } (0, L) \times (0, T) \quad (3.1) \\
L_3Z &= f_3(Z) - m_4AZ - m_5PZ
\end{align*}

with homogeneous Neumann boundary conditions:

\begin{align*}
\frac{\partial A}{\partial \eta} &= 0, \quad \frac{\partial P}{\partial \eta} = 0, \quad \frac{\partial Z}{\partial \eta} = 0 \quad \text{on } \partial(0, L) \times (0, T) \quad (3.2)
\end{align*}

and the initial conditions in $L^\infty(0, L)$:
\[ A(x, 0) = A_0(x), \quad P(x, 0) = P_0(x), \quad Z(x, 0) = Z_0(x) \quad \text{for} \quad x \in (0, L) \subset \mathbb{R} \quad (3.3) \]

where \( f_1(A), f_2(P), \) and \( f_3(Z) \) represent the growth term (logistic growth) of \( A, P, \) and \( Z \) respectively with intrinsic growth rates \( r_1, r_2, \) and \( r_3 \) as follow:

\[
\begin{align*}
    f_1(A) &= r_1 A(1 - \frac{A}{K_1}), \\
    f_2(P) &= r_2 P(1 - \frac{P}{K_2}), \\
    f_3(Z) &= r_3 Z(1 - \frac{Z}{K_3}),
\end{align*}
\]

and the terms \( m_0AZ, m_1PA, m_2PA, m_3AZ, m_4AZ, \) and \( m_5PZ \) represent interaction (predation) terms between species. For example, the term \(-m_1PA\) is a decay term for the anchovy population. We use \(+m_2PA\) is a growth term for the Jellyfish population, and we use the following notation for the left hand side of (3.1)

\[
\begin{align*}
    L_1A &= A_t + b_1(x, t)A_x - (D_1(x, t)A_x)_x \\
    L_2P &= P_t + b_2(x, t)P_x - (D_2(x, t)P_x)_x \\
    L_3Z &= Z_t + b_3(x, t)Z_x - (D_3(x, t)Z_x)_x
\end{align*}
\]

where \( b_1, b_2, \) and \( b_3 \) are advection coefficients of Anchovy, Jellyfish, and Zooplankton, respectively, and the advection coefficients are positive since the system moves from left to right during the fishery season. And \( D_1, D_2, \) and \( D_3 \) are the diffusion coefficients of Anchovy, Jellyfish, and Zooplankton, respectively.
Table 3.1: Parameter description and units. The diffusion coefficients are greater than $\theta > 0$ because of the uniform ellipticity on the diffusion coefficients (See assumption 3.6).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>Intrinsic growth rate of $A$</td>
<td>day$^{-1}$</td>
</tr>
<tr>
<td>$r_2$</td>
<td>Intrinsic growth rate of $P$</td>
<td>day$^{-1}$</td>
</tr>
<tr>
<td>$r_3$</td>
<td>Intrinsic growth rate of $Z$</td>
<td>day$^{-1}$</td>
</tr>
<tr>
<td>$K_1$</td>
<td>Carrying Capacity of $A$</td>
<td>Tonnes</td>
</tr>
<tr>
<td>$K_2$</td>
<td>Carrying Capacity of $P$</td>
<td>Tonnes</td>
</tr>
<tr>
<td>$K_3$</td>
<td>Carrying Capacity of $Z$</td>
<td>Tonnes</td>
</tr>
<tr>
<td>$m_0$</td>
<td>Growth rate of $A$ due to predation of $Z$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Consumption rate of $A$ due to its predator $P$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Growth rate of $P$ due to predation of $A$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>Growth rate of $P$ due to predation of $Z$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>Consumption rate of $Z$ due to its predator $A$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_5$</td>
<td>Consumption rate of $Z$ due to its predator $P$</td>
<td>(days x Tonnes)$^{-1}$</td>
</tr>
<tr>
<td>$m_6$</td>
<td>Consumption rate of $P$ due to its predators</td>
<td>days$^{-1}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Harvesting constant of $P$</td>
<td>day$^{-1}$</td>
</tr>
<tr>
<td>$b_i$</td>
<td>Advection Coefficient, $(b_i &gt; 0)$ for $i = 1, 2, 3$</td>
<td>km x day$^{-1}$</td>
</tr>
<tr>
<td>$D_i$</td>
<td>Diffusion Coefficient, $(D_i &gt; \theta &gt; 0)$ for $i = 1, 2, 3$</td>
<td>km$^2$ x day$^{-1}$</td>
</tr>
</tbody>
</table>

We define the class of admissible controls as

$$\mathcal{A} = \{ h \in L^\infty(Q) : 0 \leq h(x, t) \leq M \text{ and } h = 0 \text{ on } (0, L) \times ([0, T] \setminus \Omega) \}$$

where $M \in \mathbb{R}$, and $\Omega = \bigcup_{i=1}^{n} [a_i, b_i]$ with $n$ denoting the number of years of control, and the interval $[a_i, b_i]$ represents the fishery season in the $i$th year, which takes about 3 months (November - January) in commercial fishery of the Black Sea anchovy for the Turkish Coast of the Black Sea.

Our objective functional is

$$J(h) = \int_{Q} e^{-\alpha t}(phA - \mu_1 h - \mu_2 h^2)dxdt = \int_{Q^*} e^{-\alpha t}(phA - \mu_1 h - \mu_2 h^2)dxdt \quad (3.4)$$
subject to the PDE system (3.1), where 

\[ Q^* = (0, L) \times \Omega, \]

\[ e^{-\alpha t} \] represents the discount term with interest rate \( \alpha \), and \( h \) is our control. The term \( e^{-\alpha t}phA \) represents the revenue of the fishery (with price \( p \)), and \( e^{-\alpha t}(\mu_1 h + \mu_2 h^2) \) represents the cost of the fishery. For convenience, we take \( p = 1 \) monetary unit. Our purpose is to maximize the objective functional over the admissible class of controls, such that

\[
J(h^*) = \sup_{h \in A} J(h).
\]

Since our control function is only in \( L^\infty(Q) \), we will consider a weak solution of the system (3.1)-(3.3). We will use the space \( V = L^2(0, T; H^1(0, L)) \) to represent the weak solution, and the space \( V^* = L^2(0, T; (H^1(0, L))^*) \) to represent the time derivative of the weak solution, and then we define the bilinear forms as

\[
B^i(t, u, \phi) = \int_{(0, L)} D_i(x, t)u_x\phi_x dx + \int_{(0, L)} b_i(x, t)u_x\phi dx
\]

for \( u, \phi \in V \), and \( i = 1, 2, 3 \).

**Definition 3.1.** For the system (3.1) with BCs (3.2) and initial conditions (3.3), a weak solution \((A, P, Z) \in V^3 \cap (L^\infty(Q))^3 \) with \( A_t, P_t, Z_t \in V^* \) satisfies

\[
\int_0^T \langle A_t, \phi_1 \rangle dt + \int_0^T B^1(t, A, \phi_1) dt = \int_Q [f_1(A) + A(m_0Z - m_1P - h)]\phi_1 dx dt
\]

\[
\int_0^T \langle P_t, \phi_2 \rangle dt + \int_0^T B^2(t, P, \phi_2) dt = \int_Q [f_2(P) + P(m_2A + m_3Z - m_6)]\phi_2 dx dt \tag{3.5}
\]

\[
\int_0^T \langle Z_t, \phi_3 \rangle dt + \int_0^T B^3(t, Z, \phi_3) dt = \int_Q [f_3(Z) - Z(m_4A + m_5P)]\phi_3 dx dt
\]

for any test functions \( \phi_1, \phi_2, \phi_3 \in V \), where \( \langle \ , \ \rangle \) is the duality between \( H^1(0, L) \) and \((H^1(0, L))^* \), and satisfies initial conditions (3.3).
Remark 3.1.1. When \( A, P, Z \in V \) and \( A, P, Z \in V^* \), then \( A, P, Z \in C([0, T]; L^2(0, L)) \) by using results of Evans (1998). And so, the initial conditions, \( A(x,0), P(x,0), \) and \( Z(x,0) \), make sense in \( L^2(0, L) \).

To prove the existence of a weak solution for our state system, we need to have \( L^\infty(Q) \) bounds on the state variables due to having non-linear terms. Before starting to show existence of a weak solution of the state system, and obtain \( L^\infty(Q) \) bounds on the state variables, let us give our assumptions, and some inequalities (equalities) to use in proofs.

Throughout the part we will consider the following assumptions:

1. The initial conditions satisfy \( A_0(x), P_0(x), Z_0(x) \in L^\infty(0, L) \), and \( 0 \leq A_0(x) < B \), \( 0 \leq P_0(x) < B \), \( 0 \leq Z_0(x) < B \) for some \( B \in \mathbb{R} \).

2. \( r_i, K_i, m_i, m_0 \) are positive constants, and \( h \in L^\infty(Q) \) for \( i = 1, 2, 3 \), and are non-negative.

3. Bounded coefficients:

\[
D_i(x, t), b_i(x, t) \in C^1(\overline{Q}) \quad \text{for } i=1,2,3.
\]

4. Uniform ellipticity on the diffusion coefficient : For one dimension, there exists \( \theta > 0 \) such that

\[
\theta \leq D_i(x, t) \quad \text{for all } (x, t) \in Q. \quad (3.6)
\]

Throughout the part we will use the following inequalities and the inequality:

1. For \( f, g \in V \) and \( \epsilon > 0 \), Cauchy’s Inequality gives

\[
\int_Q fg dxdt \leq \epsilon \int_Q f^2 dxdt + \frac{1}{4\epsilon} \int_Q g^2 dxdt \quad (3.7)
\]
2. For \( f \in V \) and \( \epsilon > 0 \), another version of Cauchy’s Inequality gives

\[
\int_Q |(b, f)_x f| dxdt \leq \frac{\theta}{2} \int_Q (f_x)^2 dxdt + C_{\theta, b_i} \int_Q f^2 dxdt \quad (3.8)
\]

where \( C_{\theta, b_i} \) depends on \( \theta \), and the bounds on the coefficients \( b_i \) (by Assumption 3).

3. For \( f \in V \) and \( s \in [0, T] \), we can obtain the following equality

\[
\int_Q f_t f dxdt = \frac{1}{2} \int_Q (f^2)_x dxdt = \frac{1}{2} \int_{(0, L)} f^2(x, s) - f^2(x, 0) dx \quad (3.9)
\]

4. (Gronwall’s Inequality) Let \( \xi(t) \) be a nonnegative, summable function on \([0, T]\) which satisfies for a.e. \( t \) the integral inequality (Evans (1998))

\[
\xi(t) \leq G_1 \int_0^t \xi(s) ds + G_2 \quad (3.10)
\]

for constants \( G_1, G_2 \geq 0 \). Then

\[
\xi(t) \leq G_2 \left(1 + G_1 te^{G_1 t}\right) \quad (3.11)
\]

for a.e. \( 0 \leq t \leq T \).

### 3.3 Existence and Uniqueness of State Variables

In this part, for given a control \( h \in A \), we prove existence of a solution for our state system, and then we show existence of an optimal control for our system.

**Theorem 3.2.** Given \( h(x, t) \in A \), and sufficiently small \( T \), there exists a unique nonnegative weak solution \( (A, P, Z) \in V^3 \cap (L^\infty(0, L))^3 \) satisfying (3.2), (3.3), and (3.5). Moreover, \( 0 \leq A(x, t) \leq C, 0 \leq P(x, t) \leq C, \) and \( 0 \leq Z(x, t) \leq C \) a.e. \( (x, t) \in Q \) for some constant \( C \).
Proof. We will use Banach’s fixed point theorem to prove there exists a unique non-negative weak solution \((A, P, Z) \in V^3 \cap (L^\infty(0, L))^3\) satisfying (3.2), (3.3), and (3.5). The Banach’s fixed theorem states that for given a Banach space, \(B\), and a nonlinear mapping \(F : B \to B\) such that

\[
\|F(u) - F(\hat{u})\| \leq \gamma \|u - \hat{u}\| \quad \text{for all } u, \hat{u} \in B
\]

for some \(0 < \gamma < 1\), then \(F\) has a unique fixed point.

Evans (1998) states that \(A, P, Z \in C([0, T]; L^2((0, L)))\) for a solution \((A, P, Z) \in V^3\). In addition to this result, we need to get \(L^\infty(Q)\) bounds on \(A, P, Z\) to show existence of a weak solution for (3.2), (3.3), and (3.5). After having these results in hand, we apply Banach’s fixed point theorem in the Banach space, \(B^3\), which is defined as

\[
B = C\left([0, T]; L^2((0, L))\right) \cap \left\{ u \in L^\infty(Q) : 0 \leq u \leq M \quad \text{a.e. } (x, t) \in Q \right\}
\]

together with the norm:

\[
\|(u_1, u_2, u_3)\|_{B^3} = (\|u_1\|_B + \|u_2\|_B + \|u_3\|_B)
\]

where

\[
\|u\|_B = \sup_{0 \leq t \leq T} \|u(t)\|_{L^2((0, L))}
\]

Assume \(h \in \mathcal{A}\) and for \(\lambda \in \mathbb{R}\) to be chosen, let \(A = \hat{a}e^\lambda t\), \(P = \hat{p}e^\lambda t\), \(Z = \hat{z}e^\lambda t\). When we substitute them into (3.1), we will obtain

\[
\begin{align*}
\hat{L}_1\hat{a} &= r_1\hat{a} - \hat{a}^2e^\lambda t \frac{r_1}{K_1} + m_0\hat{a}\hat{z}e^\lambda t - m_1\hat{a}\hat{p}e^\lambda t - \hat{h}\hat{a} \\
\hat{L}_2\hat{p} &= r_2\hat{p} - \hat{p}^2e^\lambda t \frac{r_2}{K_2} + m_2\hat{a}\hat{p}e^\lambda t + m_3\hat{z}\hat{p}e^\lambda t - m_6\hat{p} \\
\hat{L}_3\hat{z} &= r_3\hat{z} - \hat{z}^2e^\lambda t \frac{r_3}{K_3} - m_4\hat{a}\hat{z}e^\lambda t - m_5\hat{p}\hat{z}e^\lambda t
\end{align*}
\]

a.e. \((x, t) \in Q\)
with boundary conditions

\[ \frac{\partial \hat{a}}{\partial \eta}(x,t) = 0, \quad \frac{\partial \hat{p}}{\partial \eta}(x,t) = 0, \quad \frac{\partial \hat{z}}{\partial \eta}(x,t) = 0 \quad \text{for all } x \in \{0, L\}, \quad t \in (0, T) \]  

(3.13)

and initial conditions

\[ \hat{a}(x,0) = A_0(x), \quad \hat{p}(x,0) = P_0(x), \quad \hat{z}(x,0) = Z_0(x) \quad \text{for all } x \in (0, L) \]  

(3.14)

where for \( i = 1, 2, 3 \),

\[ L_i(ue^\lambda t) = e^\lambda t \hat{L}_i u \]

and

\[ \hat{L}_i u = u_t + b_i(x,t)u_x - (D_i(x,t)u_x)_x + u\lambda. \]  

(3.15)

Let \((\hat{v}_1, \hat{v}_2, \hat{v}_3) \in B^3\), and consider the following linear parabolic PDE system:

\[ \hat{L}_1 \hat{a} = r_1 \hat{v}_1 - \hat{a} \hat{v}_1 e^\lambda t \frac{r_1}{K_1} + m_0 \hat{a} \hat{v}_3 e^\lambda t - m_1 \hat{a} \hat{v}_2 e^\lambda t - h \hat{a} \]

\[ \hat{L}_2 \hat{p} = r_2 \hat{v}_2 - \hat{p} \hat{v}_2 e^\lambda t \frac{r_2}{K_2} + m_2 \hat{v}_1 \hat{p} e^\lambda t + m_3 \hat{v}_3 \hat{p} e^\lambda t - m_6 \hat{p} \]

\[ \hat{L}_3 \hat{z} = r_3 \hat{v}_3 - \hat{z} \hat{v}_3 e^\lambda t \frac{r_3}{K_3} - m_4 \hat{v}_1 \hat{z} e^\lambda t - m_5 \hat{v}_2 \hat{z} e^\lambda t \quad \text{a.e. } (x,t) \in Q \]  

(3.16)

with boundary conditions

\[ \frac{\partial \hat{a}}{\partial \eta}(x,t) = 0, \quad \frac{\partial \hat{p}}{\partial \eta}(x,t) = 0, \quad \frac{\partial \hat{z}}{\partial \eta}(x,t) = 0 \quad \text{for all } x \in \{0, L\}, \quad t \in (0, T) \]

and initial conditions

\[ \hat{a}(x,0) = A_0(x), \quad \hat{p}(x,0) = P_0(x), \quad \hat{z}(x,0) = Z_0(x) \quad \text{for all } x \in (0, L). \]

Given \((v_1, v_2, v_3) \in B^3\), the linear system (3.12) - (3.14) has a unique weak solution \((\hat{a}, \hat{p}, \hat{z}) \in V^3 \) (Evans (1998)). We will also show that the unique weak solution \((\hat{a}, \hat{p}, \hat{z}) \in B^3\).
It will be enough to show that \(0 \leq \hat{a} \leq M\), \(0 \leq \hat{p} \leq M\), and \(0 \leq \hat{z} \leq M\) since know that \((\hat{a}, \hat{p}, \hat{z}) \in C([0, T]; L^2(0, L))\) from Evans (1998). Now, let rewrite (3.16) as

\[
\begin{align*}
\hat{L}_1\hat{a} + [e^{\lambda t}(m_1\hat{v}_2 - m_0\hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}]\hat{a} &= r_1v_1 \geq 0 \\
\hat{L}_2\hat{p} + [-e^{\lambda t}(m_2\hat{v}_1 + m_3\hat{v}_3) + m_6 + \hat{v}_2 e^{\lambda t} \frac{r_2}{K_2}]\hat{p} &= r_2v_2 \geq 0 \\
\hat{L}_3\hat{z} + [e^{\lambda t}(m_4\hat{v}_1 + m_5\hat{v}_2) + \hat{v}_3 e^{\lambda t} \frac{r_3}{K_3}]\hat{z} &= r_3v_3 \geq 0
\end{align*}
\]

(3.17) since \(r_k\) and \(0 \leq v_i \leq M\) for \(k = 1, 2, 3\). Moreover, we have \(m_i\), which are non-negative constants for \(i = 0, 1, \ldots, 6\), and also \(h \in L^\infty(Q)\). Thus, we can choose \(\lambda\) sufficiently large and choose \(T\) small enough to have the following inequalities:

\[
\begin{align*}
\lambda + e^{\lambda t}(m_1\hat{v}_2 - m_0\hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1} \geq 0 \\
\lambda - e^{\lambda t}(m_2\hat{v}_1 + m_3\hat{v}_3) + m_6 + \hat{v}_2 e^{\lambda t} \frac{r_2}{K_2} \geq 0 \\
\lambda + e^{\lambda t}(m_4\hat{v}_1 + m_5\hat{v}_2) + \hat{v}_3 e^{\lambda t} \frac{r_3}{K_3} \geq 0 \hspace{1cm} \text{a.e. } (x, t) \in Q
\end{align*}
\]

(3.18)

Hence, the extension of the parabolic maximum principle to weak solutions (Krylov (1987)) gives us

\[
0 \leq \hat{a}(x, t) \\
0 \leq \hat{p}(x, t) \\
0 \leq \hat{z}(x, t) \hspace{1cm} \text{a.e. } (x, t) \in Q.
\]

Let us now show \(\hat{a}(x, t) \leq M\), \(\hat{p}(x, t) \leq M\), and \(\hat{z}(x, t) \leq M\) a.e. \((x, t) \in Q\). Since it is not hard to see from (3.17) that we can find \(C\) such that \(C = C_1M\), where \(C_1\) is a positive constant, we can obtain
\[
\hat{L}_1 \hat{a} + [e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] \hat{a} \leq C
\]
\[
\hat{L}_2 \hat{p} + [-e^{\lambda}(m_2 \hat{v}_1 + m_3 \hat{v}_3) + m_6 + \hat{v}_2 e^{\lambda t} \frac{r_2}{K_2}] \hat{p} \leq C
\]
(3.19)
\[
\hat{L}_3 \hat{z} + [e^{\lambda}(m_4 \hat{v}_1 + m_5 \hat{v}_2) + \hat{v}_3 e^{\lambda t} \frac{r_3}{K_3}] \hat{z} \leq C
\] a.e. \((x, t) \in Q\)

Now, consider the functions
\[
a(x, t) = \hat{a}(x, t) - Ct
\]
\[
p(x, t) = \hat{p}(x, t) - Ct
\]
(3.20)
\[
z(x, t) = \hat{z}(x, t) - Ct
\]

By using (3.18), (3.19), and (3.20), we can get the following
\[
\hat{L}_1 a + [e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] a
\]
\[
= \hat{L}_1 \hat{a} + [e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] \hat{a} - \hat{L}_1(Ct)
\]
\[
- [e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] Ct
\]
\[
= \hat{L}_1 \hat{a} + [e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] \hat{a} - C
\]
\[
- [\lambda + e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] Ct
\]
\[
\leq -[\lambda + e^{\lambda}(m_1 \hat{v}_2 - m_0 \hat{v}_3) + h + \hat{v}_1 e^{\lambda t} \frac{r_1}{K_1}] Ct \leq 0.
\]

since \(\hat{L}_1(Ct) = C + \lambda Ct\). Similarly, we can get the other inequalities. Hence, choosing \(\lambda\) sufficiently large, and \(T\) sufficiently small, we can get
\[
\hat{L}_2 p + [-e^{\lambda}(m_2 \hat{v}_1 + m_3 \hat{v}_3) + m_6 + \hat{v}_2 e^{\lambda t} \frac{r_2}{K_2}] p \leq 0
\]
\[
\hat{L}_3 z + [e^{\lambda}(m_4 \hat{v}_1 + m_5 \hat{v}_2) + \hat{v}_3 e^{\lambda t} \frac{r_3}{K_3}] z \leq 0
\] a.e. \((x, t) \in Q\)
together with boundary conditions
\[
\frac{\partial a}{\partial \eta}(x, t) = 0, \quad \frac{\partial p}{\partial \eta}(x, t) = 0, \quad \frac{\partial z}{\partial \eta}(x, t) = 0 \quad \text{for all } x \in \{0, L\}, \quad t \in (0, T)
\]
and initial conditions
\[
a(x, 0) = A_0(x), \quad p(x, 0) = P_0(x), \quad z(x, 0) = Z_0(x) \quad \text{for all } x \in (0, L)
\]
Having this in hand, applying the extension of the Maximum Principle to weak solutions (Krylov (1987)), we can result the following:
\[
a(x, t) \leq ||A_0(x)||_{L^\infty((0, L))}
\]
\[
p(x, t) \leq ||P_0(x)||_{L^\infty((0, L))}
\]
\[
z(x, t) \leq ||Z_0(x)||_{L^\infty((0, L))} \quad \text{a.e. } (x, t) \in Q
\]
Equivalently,
\[
\hat{a}(x, t) \leq ||A_0(x)||_{L^\infty((0, L))} + Ct
\]
\[
\hat{p}(x, t) \leq ||P_0(x)||_{L^\infty((0, L))} + Ct
\]
\[
\hat{z}(x, t) \leq ||Z_0(x)||_{L^\infty((0, L))} + Ct \quad \text{a.e. } (x, t) \in Q
\]
Assuming \( T \leq \frac{M}{2C} \), and \( \max\{||A_0(x)||_{L^\infty((0, L))} + ||P_0(x)||_{L^\infty((0, L))} + ||Z_0(x)||_{L^\infty((0, L))}\} \leq \frac{M}{2} \), we obtain
\[
\hat{a}(x, t) \leq ||A_0(x)||_{L^\infty((0, L))} + \frac{M}{2}
\]
\[
\leq \max\left\{||A_0(x)||_{L^\infty((0, L))} + ||P_0(x)||_{L^\infty((0, L))} + ||Z_0(x)||_{L^\infty((0, L))}\right\} + \frac{M}{2}
\]
\[
\leq M \quad \text{a.e. } (x, t) \in Q
\]
Similarly, we can get
\[
\hat{p}(x,t) \leq M
\]
\[
\hat{z}(x,t) \leq M \quad \text{a.e.}(x,t) \in Q
\]

Thus, so far we have shown for \((\hat{v}_1, \hat{v}_2, \hat{v}_3) \in \mathcal{B}^3\), the solution \((\hat{a}, \hat{p}, \hat{z}) \in \mathcal{V}^3\) to (3.1)-(3.3) is also in \(\mathcal{B}^3\).

Now, define the map \(F : \mathcal{B}^3 \rightarrow \mathcal{B}^3\) such that \(F(\hat{v}_1, \hat{v}_2, \hat{v}_3) = (\hat{a}, \hat{p}, \hat{z})\) for \((\hat{v}_1, \hat{v}_2, \hat{v}_3) \in \mathcal{B}^3\), and show \(F\) is a strict contraction, which is the main part of this proof.

To show that \(F\) is a strict contraction, consider \((\hat{v}_1, \hat{v}_2, \hat{v}_3), (v_1, v_2, v_3) \in \mathcal{B}^3\), and define
\[
(\hat{a}, \hat{p}, \hat{z}) = F(\hat{v}_1, \hat{v}_2, \hat{v}_3)
\]
\[
(a, p, z) = F(v_1, v_2, v_3).
\]

We need to show the following
\[
||F(v_1, v_2, v_3) - F(\hat{v}_1, \hat{v}_2, \hat{v}_3)||_{\mathcal{B}^3} \leq \gamma ||(v_1, v_2, v_3) - (\hat{v}_1, \hat{v}_2, \hat{v}_3)||_{\mathcal{B}^3}
\]
where \(0 < \gamma < 1\). Equivalently, we have
\[
||a - \hat{a}||_{\mathcal{B}} + ||p - \hat{p}||_{\mathcal{B}} + ||z - \hat{z}||_{\mathcal{B}} \leq \gamma \left(||v_1 - \hat{v}_1||_{\mathcal{B}} + ||v_2 - \hat{v}_2||_{\mathcal{B}} + ||v_3 - \hat{v}_3||_{\mathcal{B}}\right)
\]

Now, consider the differences \((a - \hat{a}), (p - \hat{p})\), and \((z - \hat{z})\), and substitute it in the PDE system, (11) to get the following
\[
\hat{L}_1(a - \hat{a}) = \hat{f}_1(a) - \hat{f}_1(\hat{a}) + m_0e^{\lambda t}(v_3a - \hat{v}_3\hat{a}) - m_1e^{\lambda t}(v_2a - \hat{v}_2\hat{a}) - h(a - \hat{a})
\]
\[
\hat{L}_2(p - \hat{p}) = \hat{f}_2(p) - \hat{f}_2(\hat{p}) + m_2e^{\lambda t}(v_1p - \hat{v}_1\hat{p}) + m_3e^{\lambda t}(v_3p - \hat{v}_3\hat{p}) - m_6(p - \hat{p})
\]
\[
\hat{L}_3(z - \hat{z}) = \hat{f}_3(z) - \hat{f}_3(\hat{z}) - m_4e^{\lambda t}(v_1z - \hat{v}_1\hat{z}) - m_5e^{\lambda t}(v_2z - \hat{v}_2\hat{z}) \quad \text{a.e. } (x,t) \in Q
\]
with boundary conditions

\[ \frac{\partial (a - \hat{a})}{\partial \eta}(x, t) = 0, \quad \frac{\partial (p - \hat{p})}{\partial \eta}(x, t) = 0, \quad \frac{\partial (z - \hat{z})}{\partial \eta}(x, t) = 0 \quad \text{for all } x \in \{0, L\}, \quad t \in (0, T) \]

and initial conditions

\[ (a - \hat{a})(x, 0) = 0, \quad (p - \hat{p})(x, 0) = 0, \quad (z - \hat{z})(x, 0) = 0 \quad \text{for all } x \in (0, L) \]

where

\[ \hat{L}_i(u - \hat{u}) = (u - \hat{u})_t + \lambda(u - \hat{u}) + b_i(x, t)(u - \hat{u})_x - (D_i(x, t)(u - \hat{u})_x)_x \]

for \( i = 1, 2, 3 \), and any function \( u \in V \).

Let us multiply the PDE system (3.21) by \((a - \hat{a})\), \((p - \hat{p})\), and \((z - \hat{z})\) respectively, which are test functions, and integrate over \( Q_s = ((0, L)) \times (0, s) \) for \( s \in (0, T) \) as

\[
\int_{Q_s} \hat{L}_1(a - \hat{a})(a - \hat{a})dxdt = \int_{Q_s} (\hat{f}_1(a) - \hat{f}_1(\hat{a}))(a - \hat{a})dxdt + \int_{Q_s} m_0 e^{\lambda t}(v_3 a - \hat{v}_3 \hat{a})(a - \hat{a})dxdt \\
- \int_{Q_s} m_1 e^{\lambda t}(v_2 a - \hat{v}_2 \hat{a})(a - \hat{a})dxdt - \int_{Q_s} h(a - \hat{a})(a - \hat{a})dxdt
\]

\[
\int_{Q_s} \hat{L}_2(p - \hat{p})(p - \hat{p})dxdt = \int_{Q_s} (\hat{f}_2(p) - \hat{f}_2(\hat{p}))(p - \hat{p})dxdt + \int_{Q_s} m_2 e^{\lambda t}(v_1 p - \hat{v}_1 \hat{p})(p - \hat{p})dxdt \\
+ \int_{Q_s} m_3 e^{\lambda t}(v_3 p - \hat{v}_3 \hat{p})(p - \hat{p})dxdt - \int_{Q_s} m_6(p - \hat{p})(p - \hat{p})dxdt
\]

\[
\int_{Q_s} \hat{L}_3(z - \hat{z})(z - \hat{z})dxdt = \int_{Q_s} (\hat{f}_3(z) - \hat{f}_3(\hat{z}))(z - \hat{z})dxdt - \int_{Q_s} m_4 e^{\lambda t}(v_1 z - \hat{v}_1 \hat{z})(z - \hat{z})dxdt \\
- \int_{Q_s} m_5 e^{\lambda t}(v_2 z - \hat{v}_2 \hat{z})(z - \hat{z})dxdt
\]
where

\[ \int_{Q_s} \hat{L}_i(u - \hat{u})(u - \hat{u}) dx dt = \int_{Q_s} (u - \hat{u})_t (u - \hat{u}) dx dt + \int_{Q_s} \lambda (u - \hat{u})(u - \hat{u}) dx dt \\
+ \int_{Q_s} b_i(x,t)(u - \hat{u})_x (u - \hat{u}) dx dt - \int_{Q_s} (D_i(x,t)(u - \hat{u})_x)(u - \hat{u}) dx dt \]

After taking integration by parts for the diffusion term, and making some simplification, we obtain for \( i = 1, 2, 3 \).

\[ \int_{Q_s} \hat{L}_i(u - \hat{u})(u - \hat{u}) dx dt = \int_{Q_s} (u - \hat{u})_t (u - \hat{u}) dx dt + \lambda \int_{Q_s} (u - \hat{u})^2 dx dt \\
+ \int_{Q_s} D_i(u - \hat{u})_x^2 dx dt + \int_{Q_s} b_i(u - \hat{u})_x (u - \hat{u}) dx dt \]

Now, let us firstly apply the equality (3.8) to the time derivative term. Since initial conditions are zero, we get

\[ \int_{Q_s} \hat{L}_i(u - \hat{u})(u - \hat{u}) dx dt = \frac{1}{2} \int_{(0,L)} (u - \hat{u})^2(x,s) dx + \lambda \int_{Q_s} (u - \hat{u})^2 dx dt \\
+ \int_{Q_s} D_i(u - \hat{u})_x^2 dx dt + \int_{Q_s} b_i(u - \hat{u})_x (u - \hat{u}) dx dt \]

(3.23)

Next we substitute the equation (3.23) to the PDE system, (3.22) to get the following

\[ \frac{1}{2} \int_{(0,L)} (a - \hat{a})^2(x,s) dx + \lambda \int_{Q_s} (a - \hat{a})^2 dx dt + \int_{Q_s} D_i(a - \hat{a})_x^2 dx dt + \int_{Q_s} b_i(a - \hat{a})_x (a - \hat{a}) dx dt \\
= \int_{Q_s} (\hat{f}_1(a) - \hat{f}_1(\hat{a}))(a - \hat{a}) dx dt + \int_{Q_s} m_0 e^{\lambda t} (v_3a - \hat{v}_3\hat{a})(a - \hat{a}) dx dt \\
- \int_{Q_s} m_1 e^{\lambda t} (v_2a - \hat{v}_2\hat{a})(a - \hat{a}) dx dt - \int_{Q_s} h(a - \hat{a})^2 dx dt \]
\[
\frac{1}{2} \int_{(0,L)} (p - \hat{p})^2(x, s) dx + \lambda \int_{Q_s} (p - \hat{p})^2 dx dt + \int_{Q_s} D_1(p - \hat{p})_x^2 dx dt + \int_{Q_s} b_1(p - \hat{p})(p - \hat{p}) dx dt \\
= \int_{Q_s} (\hat{f}_2(p) - \hat{f}_2(\hat{p}))(p - \hat{p}) dx dt + \int_{Q_s} m_2 e^{\lambda t}(v_1 p - \hat{v}_1 \hat{p})(p - \hat{p}) dx dt \\
+ \int_{Q_s} m_3 e^{\lambda t}(v_3 p - \hat{v}_3 \hat{p})(p - \hat{p}) dx dt - \int_{Q_s} m_6(p - \hat{p})^2 dx dt
\]

\[
\frac{1}{2} \int_{(0,L)} (z - \hat{z})^2(x, s) dx + \lambda \int_{Q_s} (z - \hat{z})^2 dx dt + \int_{Q_s} D_1(z - \hat{z})_x^2 dx dt + \int_{Q_s} b_1(z - \hat{z})_x(z - \hat{z}) dx dt \\
= \int_{Q_s} (\hat{f}_1(z) - \hat{f}_3(\hat{z}))(z - \hat{z}) dx dt - \int_{Q_s} m_4 e^{\lambda t}(v_1 z - \hat{v}_1 \hat{z})(z - \hat{z}) dx dt \\
- \int_{Q_s} m_5 e^{\lambda t}(v_2 z - \hat{v}_2 \hat{z})(z - \hat{z}) dx dt
\]

Let us now make some estimates by using ellipticity condition (3.5) for diffusion term, and Cauchy’s inequality (3.7) for advection term as

\[
\frac{1}{2} \int_{(0,L)} (a - \hat{a})^2(x, s) dx + \lambda \int_{Q_s} (a - \hat{a})^2 dx dt + \theta \int_{Q_s} |(a - \hat{a})_x|^2 dx dt \leq \frac{\theta}{2} \int_{Q_s} |(a - \hat{a})_x|^2 dx dt \\
+ \int_{Q_s} m_0 e^{\lambda t}(v_3 a - \hat{v}_3 \hat{a})(a - \hat{a}) dx dt - \int_{Q_s} m_1 e^{\lambda t}(v_2 a - \hat{v}_2 \hat{a})(a - \hat{a}) dx dt \\
+ C_{\theta, b_1} \int_{Q_s} |a - \hat{a}|^2 dx dt + \int_{Q_s} (\hat{f}_1(a) - \hat{f}_1(\hat{a}))(a - \hat{a}) dx dt
\]

\[
\frac{1}{2} \int_{(0,L)} (p - \hat{p})^2(x, s) dx + \lambda \int_{Q_s} (p - \hat{p})^2 dx dt + \theta \int_{Q_s} |(p - \hat{p})_x|^2 dx dt \leq \frac{\theta}{2} \int_{Q_s} |(p - \hat{p})_x|^2 dx dt \\
+ \int_{Q_s} m_2 e^{\lambda t}(v_1 p - \hat{v}_1 \hat{p})(p - \hat{p}) dx dt + \int_{Q_s} m_3 e^{\lambda t}(v_3 p - \hat{v}_3 \hat{p})(p - \hat{p}) dx dt \\
+ C_{\theta, b_2} \int_{Q_s} |p - \hat{p}|^2 dx dt + \int_{Q_s} (\hat{f}_2(p) - \hat{f}_2(\hat{p}))(p - \hat{p}) dx dt
\]
\[
\frac{1}{2} \int_{(0,L)} (z - \hat{z})^2(x,s)dx + \lambda \int_{Q_s} (z - \hat{z})^2dxdt + \theta \int_{Q_s} |(z - \hat{z})_x|^2dxdt \leq \frac{\theta}{2} \int_{Q_s} |(z - \hat{z})_x|^2dxdt
\]
\[
- \int_{Q_s} m_4 e^{\lambda t}(v_1z - \hat{v}_1\hat{z})(z - \hat{z})dxdt - \int_{Q_s} m_5 e^{\lambda t}(v_2z - \hat{v}_2\hat{z})(z - \hat{z})dxdt
\]
\[
+ C_{\theta,b_3} \int_{Q_s} |z - \hat{z}|^2dxdt + \int_{Q_s} (\hat{f}_3(z) - \hat{f}_3(\hat{z}))(z - \hat{z})dxdt
\]

(3.24)

We will estimate the terms in the right hand side of the inequality by using Cauchy’s Inequality (3.7) and \( L^\infty(Q) \) bounds on the parameters as follow:

Firstly, we estimate the term, \( \int_{Q_s} (\hat{f}_1(a) - \hat{f}_1(\hat{a}))(a - \hat{a})dxdt \). Using

\[
\left| \hat{f}_1(a) - \hat{f}_1(\hat{a}) \right| = \left| r_1 a (1 - \frac{ae^\lambda}{K_1}) - r_1 \hat{a} (1 - \frac{\hat{a}e^\lambda}{K_1}) \right| \leq r_1 |a - \hat{a}| + \frac{e^\lambda}{K_1} |a + \hat{a}| |a - \hat{a}|
\]

\[
\leq \left( r_1 + 2M \frac{e^\lambda}{K_1} \right) |a - \hat{a}|
\]

and that all the non-negative parameters \( r_i \), and uniformly boundedness of state variables, we can get

\[
\int_{Q_s} (\hat{f}_1(a) - \hat{f}_1(\hat{a}))(a - \hat{a})dxdt \leq (C_1 + M_1 e^{\lambda T}) \int_{Q_s} (a - \hat{a})^2dxdt
\]

Similarly, we can obtain

\[
\int_{Q_s} (\hat{f}_2(p) - \hat{f}_2(\hat{p}))(p - \hat{p})dxdt \leq (C_2 + M_2 e^{\lambda T}) \int_{Q_s} (p - \hat{p})^2dxdt
\]

\[
\int_{Q_s} (\hat{f}_3(z) - \hat{f}_3(\hat{z}))(z - \hat{z})dxdt \leq (C_3 + M_3 e^{\lambda T}) \int_{Q_s} (z - \hat{z})^2dxdt
\]

where \( C_i \) depends on \( r_i \), and \( M_i \) depends on \( M \) and \( K_i \) for \( i = 1, 2, 3 .. \). Secondly, we estimate the term, \( \int_{Q_s} m_0 e^{\lambda t}(v_3a - \hat{v}_3\hat{a})(a - \hat{a})dxdt \) by using the inequality (3.6) as
Similarly, we can get

\[ \int_{Q_s} m_0 e^{\lambda t} (v_3a - \hat{v}_3 \hat{a})(a - \hat{a})dxdt \]

\[ = \int_{Q_s} m_0 e^{\lambda t} (v_3a - a\hat{v}_3 + a\hat{v}_3 - \hat{v}_3 \hat{a})(a - \hat{a})dxdt \]

\[ = \int_{Q_s} m_0 e^{\lambda t} [a(v_3 - \hat{v}_3) + \hat{v}_3(a - \hat{a})](a - \hat{a})dxdt \]

\[ = \int_{Q_s} m_0 e^{\lambda t} a(v_3 - \hat{v}_3)(a - \hat{a}) + \int_{Q_s} \hat{v}_3(a - \hat{a})^2dxdt \]

\[ \leq \int_{Q_s} m_0 e^{\lambda t} a(v_3 - \hat{v}_3)(a - \hat{a})dxdt + \int_{Q_s} m_0 e^{\lambda t} \hat{v}_3(a - \hat{a})^2dxdt \]

\[ \leq \frac{1}{2} \int_{Q_s} m_0^2 e^{2\lambda t} a^2(a - \hat{a})^2dxdt + \frac{1}{2} \int_{Q_s} (v_3 - \hat{v}_3)^2dxdt + \int_{Q_s} m_0 e^{\lambda t} \hat{v}_3(a - \hat{a})^2dxdt \]

\[ \leq \frac{1}{2} m_0^2 e^{2\lambda T} M^2 \int_{Q_s} (a - \hat{a})^2dxdt + \frac{1}{2} \int_{Q_s} (v_3 - \hat{v}_3)^2dxdt + m_0 e^{\lambda T} M \int_{Q_s} (a - \hat{a})^2dxdt \]

\[ \leq C_{m_0} e^{2\lambda T} (M^2 + M) \int_{Q_s} (a - \hat{a})^2dxdt + \frac{1}{2} \int_{Q_s} (v_3 - \hat{v}_3)^2dxdt \]

Similarly, we can get

\[ \int_{Q_s} m_1 e^{\lambda t} (v_2a - \hat{v}_2 \hat{a})(a - \hat{a})dxdt \leq C_{m_1} e^{2\lambda T} (M^2 + M) \int_{Q_s} (a - \hat{a})^2dxdt + \frac{1}{2} \int_{Q_s} (v_2 - \hat{v}_2)^2dxdt \]

\[ \int_{Q_s} m_2 e^{\lambda t} (v_1p - \hat{v}_1 \hat{p})(p - \hat{p})dxdt \leq C_{m_2} e^{2\lambda T} (M^2 + M) \int_{Q_s} (p - \hat{p})^2dxdt + \frac{1}{2} \int_{Q_s} (v_1 - \hat{v}_1)^2dxdt \]

\[ \int_{Q_s} m_3 e^{\lambda t} (v_3p - \hat{v}_3 \hat{p})(p - \hat{p})dxdt \leq C_{m_3} e^{2\lambda T} (M^2 + M) \int_{Q_s} (p - \hat{p})^2dxdt + \frac{1}{2} \int_{Q_s} (v_3 - \hat{v}_3)^2dxdt \]

\[ \int_{Q_s} m_4 e^{\lambda t} (v_1z - \hat{v}_1 \hat{z})(z - \hat{z})dxdt \leq C_{m_4} e^{2\lambda T} (M^2 + M) \int_{Q_s} (z - \hat{z})^2dxdt + \frac{1}{2} \int_{Q_s} (v_1 - \hat{v}_1)^2dxdt \]

\[ \int_{Q_s} m_5 e^{\lambda t} (v_2z - \hat{v}_2 \hat{z})(z - \hat{z})dxdt \leq C_{m_5} e^{2\lambda T} (M^2 + M) \int_{Q_s} (z - \hat{z})^2dxdt + \frac{1}{2} \int_{Q_s} (v_2 - \hat{v}_2)^2dxdt \]
where $C_{m_i} > 0$ for $i = 0, 1,..5$. Now, let us substitute these estimates in the previous PDE system, (3.22), and obtain the following PDE system:

$$
\frac{1}{2} \int_{(0,L)} (a - \hat{a})^2(x,s)dx + \lambda \int_{Q_s} (a - \hat{a})^2 dxdt + \frac{\theta}{2} \int_{Q_s} |(a - \hat{a})_x|^2 dxdt \\
\leq C_{b_1} \int_{Q_s} (a - \hat{a})^2 dxdt + (C_1 + M_1 e^{\lambda T}) \int_{Q_s} (a - \hat{a})^2 dxdt + \frac{1}{2} \int_{Q_s} (v_3 - \hat{v}_3)^2 dxdt \\
+ C_{m_6} e^{2\lambda T} (M^2 + M) \int_{Q_s} (a - \hat{a})^2 dxdt + C_{m_2} e^{2\lambda T} (M^2 + M) \int_{Q_s} (a - \hat{a})^2 dxdt \\
+ \frac{1}{2} \int_{Q_s} (v_2 - \hat{v}_2)^2 dxdt
$$

$$
\frac{1}{2} \int_{(0,L)} (p - \hat{p})^2(x,s)dx + \lambda \int_{Q_s} (p - \hat{p})^2 dxdt + \frac{\theta}{2} \int_{Q_s} |(p - \hat{p})_x|^2 dxdt \\
\leq C_{b_2} \int_{Q_s} (p - \hat{p})^2 dxdt + (C_2 + M_2 e^{\lambda T}) \int_{Q_s} (p - \hat{p})^2 dxdt + \frac{1}{2} \int_{Q_s} (v_1 - \hat{v}_1)^2 dxdt \\
+ C_{m_2} e^{2\lambda T} (M^2 + M) \int_{Q_s} (p - \hat{p})^2 dxdt + C_{m_3} e^{2\lambda T} (M^2 + M) \int_{Q_s} (p - \hat{p})^2 dxdt \\
+ \frac{1}{2} \int_{Q_s} (v_3 - \hat{v}_3)^2 dxdt
$$

$$
\frac{1}{2} \int_{(0,L)} (z - \hat{z})^2(x,s)dx + \lambda \int_{Q_s} (z - \hat{z})^2 dxdt + \frac{\theta}{2} \int_{Q_s} |(z - \hat{z})_x|^2 dxdt \\
\leq C_{b_3} \int_{Q_s} (z - \hat{z})^2 dxdt + (C_3 + M_3 e^{\lambda T}) \int_{Q_s} (z - \hat{z})^2 dxdt + \frac{1}{2} \int_{Q_s} (v_1 - \hat{v}_1)^2 dxdt \\
+ C_{m_4} e^{2\lambda T} (M^2 + M) \int_{Q_s} (z - \hat{z})^2 dxdt + C_{m_3} e^{2\lambda T} (M^2 + M) \int_{Q_s} (z - \hat{z})^2 dxdt \\
+ \frac{1}{2} \int_{Q_s} (v_2 - \hat{v}_2)^2 dxdt.
$$

When we sum the PDEs, we can obtain the following:
\[ \frac{1}{2} \int_{(0,L)} [(a - \hat{a})^2(x,s) + (p - \hat{p})^2(x,s) + (z - \hat{z})^2(x,s)] dx \\
+ \lambda \int_{Q^s} [(a - \hat{a})^2 + (p - \hat{p})^2 + (z - \hat{z})^2] dx dt \\
+ \frac{\theta}{2} \int_{Q^s} [(a - \hat{a})_x^2 + (p - \hat{p})_x^2 + (z - \hat{z})_x^2] dx dt \\
\leq C_1^* \int_{Q^s} [(a - \hat{a})^2 + (p - \hat{p})^2 + (z - \hat{z})^2] dx dt \\
+ M_1^* e^{\lambda T} \int_{Q^s} [(a - \hat{a})^2 + (p - \hat{p})^2 + (z - \hat{z})^2] dx dt \\
+ \frac{1}{2} \int_{Q^s} [(v_1 - \hat{v}_1)^2 + (v_2 - \hat{v}_2)^2 + (v_3 - \hat{v}_3)^2] dx dt \\
+ C_{m_i} e^{2\lambda T} (M^2 + M) \int_{Q^s} [(a - \hat{a})^2 + (p - \hat{p})^2 + (z - \hat{z})^2] dx dt \\
+ \frac{1}{2} \int_{Q^s} [(v_1 - \hat{v}_1)^2] dx dt \]

where \( C_1^* \) depends on \( \theta \), and \( b_i \), \( M_1^* \) depends on \( L^\infty(Q) \) bounds, and \( C_{m_i} \) depends on \( m_i \) for \( i = 0, 1, \ldots, 5 \). Equivalently, we can write

\[ \frac{1}{2} \int_{(0,L)} [(a - \hat{a})^2(x,s) + (p - \hat{p})^2(x,s) + (z - \hat{z})^2(x,s)] dx \\
+ \lambda^* \int_{Q^s} [(a - \hat{a})^2 + (p - \hat{p})^2 + (z - \hat{z})^2] dx dt \\
+ \frac{\theta}{2} \int_{Q^s} [(a - \hat{a})_x^2 + (p - \hat{p})_x^2 + (z - \hat{z})_x^2] dx dt \\
\leq \int_{Q^s} [(v_1 - \hat{v}_1)^2 + (v_2 - \hat{v}_2)^2 + (v_3 - \hat{v}_3)^2] dx dt \]
where $\lambda^* = \lambda - C_1^* - M_1^* e^{\lambda T} - C_{m_1}^* e^{2\lambda T} (M^2 + M)$. Now, we choose $\lambda$ sufficiently large, and $T$ sufficiently small to obtain $\lambda^* > 0$. Therefore, we rewrite the previous inequality as follow:

$$
\int_{(0,L)} [(a - \hat{a})^2(x, s) + (p - \hat{p})^2(x, s) + (z - \hat{z})^2(x, s)]dx \\
\leq 2 \int_{Q_s} [(v_1 - \hat{v}_1)^2 + (v_2 - \hat{v}_2)^2 + (v_3 - \hat{v}_3)^2]dxdt
$$

(3.25)

Since $||u||_{B} = \sup_{0 \leq t \leq T} ||u||_{L^2((0,L))}$, when we take supremum of right hand side of the inequality over $0 \leq t \leq T$ we will obtain the following

$$
\int_{(0,L)} [(a - \hat{a})^2(x, s) + (p - \hat{p})^2(x, s) + (z - \hat{z})^2(x, s)]dx \\
\leq 2 \int_{Q_s} [(v_1 - \hat{v}_1)^2 + (v_2 - \hat{v}_2)^2 + (v_3 - \hat{v}_3)^2]dxdt \\
\leq 2T \left( \sup_{0 \leq t \leq T} \int_{(0,L)} (v_1 - \hat{v}_1)^2 dx + \sup_{0 \leq t \leq T} \int_{(0,L)} (v_2 - \hat{v}_2)^2 dx + \sup_{0 \leq t \leq T} \int_{(0,L)} (v_3 - \hat{v}_3)^2 dx \right)
$$

(3.26)

To reach our goal of having norm of the terms as we wish, we first need to take square root, and then take supremum over $0 \leq s \leq T$ to obtain the following

$$
||a - \hat{a}||_{B} \leq (2T)^{1/2} \left( ||v_1 - \hat{v}_1||_{B} + ||v_2 - \hat{v}_2||_{B} + ||v_3 - \hat{v}_3||_{B} \right)
$$

$$
||p - \hat{p}||_{B} \leq (2T)^{1/2} \left( ||v_1 - \hat{v}_1||_{B} + ||v_2 - \hat{v}_2||_{B} + ||v_3 - \hat{v}_3||_{B} \right)
$$

$$
||z - \hat{z}||_{B} \leq (2T)^{1/2} \left( ||v_1 - \hat{v}_1||_{B} + ||v_2 - \hat{v}_2||_{B} + ||v_3 - \hat{v}_3||_{B} \right)
$$

It follows that

$$
||a - \hat{a}||_{B} + ||p - \hat{p}||_{B} + ||z - \hat{z}||_{B} \leq 3(2T)^{1/2} \left( ||v_1 - \hat{v}_1||_{B} + ||v_2 - \hat{v}_2||_{B} + ||v_3 - \hat{v}_3||_{B} \right)
$$

By assuming $T$ is sufficiently small as we chose in previous assumptions, we can have $3(2T)^{1/2} < 1$ for sufficiently small $T$. 

74
Finally, we can say that the map $F$ is a strict contradiction. Thus, by using Banach’s fixed Theorem, we can say that there exists a unique solution $(\hat{a}, \hat{p}, \hat{z}) \in V^3 \cap B^3$ to the PDE system (3.11)–(3.13).

Therefore, we can say for sufficiently small $T$, we have an unique solution $(A, P, Z) \in V^3$ satisfying (3.2)–(3.4). Thus, for the system (3.1)–(3.3), we obtain a weak solution $(A, P, Z) \in V^3 \cap L^\infty(Q)$ with $A_t, P_t, Z_t \in L^2(0, T; (H^1(0, L))^*)$ by assuming $T$ is sufficiently small. Moreover, we obtain

\[
0 \leq A(x, t) \leq e^{\lambda t} M,
0 \leq P(x, t) \leq e^{\lambda t} M,
0 \leq Z(x, t) \leq e^{\lambda t} M, \quad \text{a.e.} \ (x, t) \in Q
\]

\[\square\]

### 3.4 Existence of Optimal Control

We have showed in section 3.3 that for a given control $h$ in $A$ there exists a unique weak state solution $(A, P, Z) = (A, P, Z)(h)$. Now, we will show that there exist an optimal control $h^*$ and the corresponding state solutions $(A^*, P^*, Z^*) = (A, P, Z)(h^*)$ to our optimal control problem.

**Theorem 3.3.** There exists an optimal control $h^*$ in $A$, which maximizes the objective functional $J(h)$ subject to our PDE system (3.1)-(3.3).

**Proof.** The state variables, $A, P, Z$, and the control variable, $h$ are uniformly $L^\infty(Q)$ bounded in $Q$. Thus, we can obtain

\[
\sup_{h \in A} J(h) = \int_{Q^*} e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2)dxdt < \infty
\]
and thus there exists a maximizing sequence \( h^n \in \mathcal{A} \) such that

\[
\lim_{n \to \infty} J(h^n) = \sup_{h \in \mathcal{A}} J(h).
\]

By using the results of Theorem 3.2, there exists a unique solution to the state system (3.1)-(3.3), and we define

\[
(A^n, P^n, Z^n) = (A, P, Z)(h^n) \quad \text{for all } n \in \mathbb{N}
\]

Now, we use the weak formulation of the state system (3.1) satisfied by \( A^n, P^n, \) and \( Z^n \). And also consider \( A^n, P^n, \) and \( Z^n \) as the test functions, then we integrate over \( Q_s = (0, L) \times (0, s) \), where \( s \in (0, T) \). Thus, for any \( n \), we can get

\[
\int_{Q_s} \langle A^n_t, A^n \rangle dt + \int_0^s B^1(t, A^n, A^n) dt = \int_{Q_s} [f_1(A^n) + A^n(m_0 Z^n - m_1 P^n - h)] A^n dx dt
\]

\[
\int_{Q_s} \langle P^n_t, P^n \rangle dt + \int_0^s B^2(t, P^n, P^n) dt = \int_{Q_s} [f_2(P^n) + P^n(m_2 A^n + m_3 Z^n - m_6)] P^n dx dt
\]

\[
\int_{Q_s} \langle Z^n_t, Z^n \rangle dt + \int_0^s B^3(t, Z^n, Z^n) dt = \int_{Q_s} [f_3(Z^n) - Z^n(m_4 A^n + m_5 P^n)] Z^n dx dt
\]

(3.27)

Before estimating some terms here by using Cauchy’s inequality (3.6), the ellipticity condition, and uniformly boundedness of state variables, we will rewrite the time derivative term as

\[
\int_{Q_s} \langle A^n_t, A^n \rangle dt = \frac{1}{2} \int_{Q_s} \frac{d}{dt}(A^n)^2 dx dt = \frac{1}{2} \int_{(0, L) \times \{s\}} (A^n)^2(x, s) dx - \frac{1}{2} \int_{(0, L) \times \{0\}} (A^n)^2(x, 0) dx
\]

by Evans (1998). Similarly, we can get the same equality for other time derivative terms. Therefore, by using the equalities together with uniform ellipticity condition on the diffusion coefficients, we can get
Using Cauchy’s inequality (3.6) on the advection term, we can estimate it as

\[
\frac{1}{2} \int_{Q_s} b_1(x,t)(A^n)_x A^n dx \leq C_{\theta,b_1} \int_{Q_s} (A^n)_x^2 dx + \frac{\theta}{2} \int_{Q_s} |A^n_x|^2 dx,
\]

where \(C_{\theta,b_1}\) depends on \(\theta\) and the advection coefficient \(b_1\). Similarly, we can estimate the other advection terms, and obtain the inequality (3.28) as

\[
\frac{1}{2} \int_{Q_s} b_1(x,t)(A^n)_x A^n dx \leq C_{\theta,b_1} \int_{Q_s} (A^n)_x^2 dx + \frac{\theta}{2} \int_{Q_s} |A^n_x|^2 dx,
\]

\[
\frac{1}{2} \int_{Q_s} b_1(x,t)(P^n)_x P^n dx \leq C_{\theta,b_1} \int_{Q_s} (P^n)_x^2 dx + \frac{\theta}{2} \int_{Q_s} |P^n_x|^2 dx,
\]

\[
\frac{1}{2} \int_{Q_s} b_1(x,t)(Z^n)_x Z^n dx \leq C_{\theta,b_1} \int_{Q_s} (Z^n)_x^2 dx + \frac{\theta}{2} \int_{Q_s} |Z^n_x|^2 dx.
\]
\[
\frac{1}{2} \int_{(0,L)} (Z^n)^2(x,s)dx + \frac{\theta}{2} \int_{Q_s} |Z^n_x|^2dx dt \leq C_{\theta,b_3} \int_{Q_s} (Z^n)^2dx dt + \frac{1}{2} \int_{(0,L)} (Z^n_0)^2(x)dx \\
+ \int_{Q_s} [f_3(Z^n) - Z^n(m_4A^n + m_5P^n)]Z^n dx dt 
\] (3.29)

Since we define \( f_i = u g_i(u) \) for \( i = 1, 2, 3 \), we can obtain that

\[
\frac{1}{2} \int_{(0,L)} (A^n)^2(x,s)dx + \frac{\theta}{2} \int_{Q_s} |A^n_x|^2dx dt \leq C_{\theta,b_1} \int_{Q_s} (A^n)^2dx dt + \frac{1}{2} \int_{(0,L)} (A^n_0)^2(x)dx \\
+ \int_{Q_s} [g_1(A^n) + m_0Z^n - m_1P^n - h](A^n)^2 dx dt 
\]

\[
\frac{1}{2} \int_{(0,L)} (P^n)^2(x,s)dx + \frac{\theta}{2} \int_{Q_s} |P^n_x|^2dx dt \leq C_{\theta,b_2} \int_{Q_s} (P^n)^2dx dt + \frac{1}{2} \int_{(0,L)} (P^n_0)^2(x)dx \\
+ \int_{Q_s} [g_2(P^n) + m_2A^n + m_3Z^n - m_6](P^n)^2 dx dt 
\]

\[
\frac{1}{2} \int_{(0,L)} (Z^n)^2(x,s)dx + \frac{\theta}{2} \int_{Q_s} |Z^n_x|^2dx dt \leq C_{\theta,b_3} \int_{Q_s} (Z^n)^2dx dt + \frac{1}{2} \int_{(0,L)} (Z^n_0)^2(x)dx \\
+ \int_{Q_s} [g_3(Z^n) - m_4A^n - m_5P^n](Z^n)^2 dx dt 
\] (3.30)

Nothing that our state variables, control variable are uniformly bounded in \( Q \), and coefficients are \( L^\infty(Q) \) bounded in \( Q \), when we sum the integral equations, we get

\[
\frac{1}{2} \int_{(0,L)} (A^n)^2(x,s) + (P^n)^2(x,s) + (Z^n)^2(x,s)dx + \frac{\theta}{2} \int_{Q_s} |A^n_x|^2 + |P^n_x|^2 + |Z^n_x|^2dx dt \\
\leq (C_{\theta,b_1} + C_3) \int_{Q_s} (A^n)^2 + (P^n)^2 + (Z^n)^2 dx dt + \frac{1}{2} \int_{(0,L)} (A^n_0)^2(x) + (P^n_0)^2(x) + (Z^n_0)^2(x)dx 
\] (3.31)
It follows that

\[
\int_{(0,L)} (A^n)^2(x,s) + (P^n)^2(x,s) + (Z^n)^2(x,s) \, dx \leq C_4 \int_{Q_s} (A^n)^2 + (P^n)^2 + (Z^n)^2 \, dxdt + \int_{(0,L)} (A^0_n)^2(x) + (P^0_n)^2(x) + (Z^0_n)^2(x) \, dx
\]

Now, let us apply the Gronwall’s Inequality to obtain

\[
\int_{(0,L)} (A^n)^2(x,s) + (P^n)^2(x,s) + (Z^n)^2(x,s) \, dx \leq (1 + C_4 e^{C_4 s}) \int_{(0,L)} (A^n_0)^2(x) + (P^n_0)^2(x) + (Z^n_0)^2(x) \, dx.
\]

When we take integral over \(0 \leq t \leq T\), we obtain the following

\[
\int_Q |A^n|^2 + |P^n|^2 + |Z^n|^2 \, dxdt \leq T \left(1 + C_4 T e^{C_4 T}\right) \int_{(0,L)} (A^n_0)^2(x) + (P^n_0)^2(x) + (Z^n_0)^2(x) \, dx
\]

(3.32)

By using inequalities (3.29), and (3.30), we can conclude that

\[
\sup_{s \in (0,T)} \left(\int_{(0,L)} (A^n)^2(x,s) + (P^n)^2(x,s) + (Z^n)^2(x,s) \, dx\right) + \theta \int_{Q_s} |A^n_s|^2 + |P^n_s|^2 + |Z^n_s|^2 \, dxdt \\
\leq C_5 \int_{(0,L)} (A^n_0)^2(x) + (P^n_0)^2(x) + (Z^n_0)^2(x) \, dx
\]

where \(C_5\) depends on \(T, \theta, \) and \(L^\infty\) bounds on the coefficients, the states, and the control.

Finally, we obtain that \(|A^n|_V, |P^n|_V, |Z^n|_V\) are uniformly bounded for any \(n \in \mathbb{N}\). These bounds imply together with the PDE system (3.1) that \(|A^n_t|_{V^*}, |P^n_t|_{V^*}, |Z^n_t|_{V^*}\) are uniformly bounded in \(V^* = L^2(0,T; H^1(0,L)^*)\) for any \(n \in \mathbb{N}\).
Since our state variables, and its time derivatives are uniformly bounded, and \( h^n \) is uniformly \( L^\infty(Q) \) bounded, we can obtain that there exists subsequences \( A^n, P^n, Z^n, A^n_t, P^n_t, Z^n_t, v^n, \) and \((A^*, P^*, Z^*) \in V^3 \cap L^\infty(Q) \) with \((A^*_t, P^*_t, Z^*_t) \in (V^*)^3 \) such that

\[
A^n \rightharpoonup A^*, \quad P^n \rightharpoonup P^*, \quad Z^n \rightharpoonup Z^* \quad \text{weakly in } V
\]

(3.33)

\[
A^n_t \rightharpoonup A^*_t, \quad P^n_t \rightharpoonup P^*_t, \quad Z^n_t \rightharpoonup Z^*_t \quad \text{weakly in } V^*
\]

(3.34)

and

\[
h^n \rightharpoonup h^* \quad \text{weakly in } L^2(Q).
\]

(3.35)

Also, by using the compactness result from Simon (1986), we can obtain the following:

\[
A^n \to A^*, \quad P^n \to P^*, \quad Z^n \to Z^* \quad \text{in } L^2(Q).
\]

(3.36)

Now, we need to show that \((A^*, P^*, Z^*) = (A, P, Z)(h^*) \) in weak sense. The weak formulation of the system (3.1) satisfied by \( A^n, P^n, \) and \( Z^n \) with test functions \( \phi_1, \phi_2, \phi_3 \in V \) is

\[
\int_0^T \langle A^n_t, \phi_1 \rangle dt + \int_0^T B^1(t, A^n, \phi_1) dt = \int_Q [f_1(A^n) + A^n(m_0Z^n - m_1P^n - h^n)]\phi_1 dx dt
\]

\[
\int_0^T \langle P^n_t, \phi_2 \rangle dt + \int_0^T B^2(t, P^n, \phi_2) dt = \int_Q [f_2(P^n) + P^n(m_2A^n + m_3Z^n - m_6)]\phi_2 dx dt
\]

\[
\int_0^T \langle Z^n_t, \phi_3 \rangle dt + \int_0^T B^3(t, Z^n, \phi_3) dt = \int_Q [f_3(Z^n) - Z^n(m_4A^n + m_5P^n)]\phi_3 dx dt
\]

(3.37)

Now, we will show each term in the PDEs for \( A^n, P^n, Z^n, \) and \( h^n \) converges to the corresponding term with \( A^*, P^*, Z^*, \) and \( h^* \), respectively. By the weak convergence (3.30),
and (3.31), we obtain
\[
\int_0^T (A^n_t - A^*_t)\phi_1 dt \to 0 \\
\int_0^T (P^n_t - P^*_t)\phi_2 dt \to 0 \\
\int_0^T (Z^n_t - Z^*_t)\phi_3 dt \to 0
\]
for \(\phi_1, \phi_2, \phi_3 \in V \subset L^2(Q)\).

Since \(A^n \to A^*, P^n \to P^*,\) and \(Z^n \to Z^*\) strongly in \(L^2(Q)\) (pointwise a.e.), and \(f_i\) is continuous for \(i = 1, 2, 3\), we can obtain \(f_1(A^n) \to f_1(A^*), f_2(P^n) \to f_2(P^*),\) and \(f_3(Z^n) \to f_1(Z^*)\) pointwise so that
\[
\int_Q (f_1(A^n) - f_1(A^*))\phi_1 dxdt \to 0 \\
\int_Q (f_2(P^n) - f_2(P^*))\phi_2 dxdt \to 0 \\
\int_Q (f_3(Z^n) - f_3(Z^*))\phi_3 dxdt \to 0
\]
Now, let us observe the other nonlinear terms. Since \(h^n\) is bounded in \(L^\infty(Q)\), and \(A^*, P^*,\) and \(Z^*\) are uniformly bounded in \(V\), when we adding and subtracting terms, we obtain
\[
\left| \int_Q (h^n A^n \phi_1 - h^n A^* \phi_1) dxdt \right| \leq \left| \int_Q (h^n A^n \phi_1 - h^n A^* \phi_1) dxdt \right| \\
= \left| \int_Q (h^n A^n - h^n A^* + h^n A^* - h^n A^*)\phi_1 dxdt \right| \\
\leq \left| \int_Q h^n (A^n - A^*)\phi_1 dxdt \right| + \left| \int_Q (h^n - h^n A^*)\phi_1 dxdt \right| \to 0.
\]
(3.38)

Since \(A^n \to A^*\) strongly in \(L^2(Q)\), \(\phi_1 \in V \subset L^2(Q)\), and \(h^n \leq M\) for each \(n \in \mathbb{N}\), we can get the following
\[
\left| \int_{Q} h^{n}(A^{n} - A^{*}) \phi_{1} \, dx \, dt \right| \leq \int_{Q} |h^{n}|(A^{n} - A^{*})|\phi_{1}| \, dx \, dt \leq M \int_{Q} |(A^{n} - A^{*})|\phi_{1}| \, dx \, dt \\
\leq M \left( \int_{Q} (A^{n} - A^{*})^{2} \, dx \, dt \right)^{1/2} \left( \int_{Q} (\phi_{1})^{2} \, dx \, dt \right)^{1/2} \rightarrow 0.
\]

For the second part, since \( \phi_{1} \in V \subset L^{2}(Q) \), and \( A^{*} \) is uniformly bounded in \( V \subset L^{2}(Q) \), we can obtain \( A^{*}\phi_{1} \in L^{2}(Q) \). Thus, by weak convergence of \( h^{n} \) in \( L^{2} \), we have

\[
\left| \int_{Q} (h^{n} - h^{*})A^{*}\phi_{1} \, dx \, dt \right| \rightarrow 0
\]

Since \( A^{n} \rightarrow A^{*} \), \( P^{n} \rightarrow P^{*} \), and \( Z^{n} \rightarrow Z^{*} \) strongly in \( L^{2}(Q) \) and \( \phi_{1} \in V \subset L^{2}(Q) \), we can get

\[
\int_{Q} (P^{n}A^{n}\phi_{1} - P^{*}A^{*}\phi_{1}) \, dx \, dt = \int_{Q} (P^{n}A^{n} - P^{n}A^{*} + P^{n}A^{*} - P^{*}A^{*})\phi_{1} \, dx \, dt \\
\leq \int_{Q} P^{n}(A^{n} - A^{*})\phi_{1} \, dx \, dt + \int_{Q} (P^{n} - P^{*})A^{*}\phi_{1} \, dx \, dt \\
\leq M \int_{Q} |A^{n} - A^{*}|\phi_{1}| \, dx \, dt + M \int_{Q} |P^{n} - P^{*}|\phi_{1}| \, dx \, dt \rightarrow 0
\]

(3.39)

Similarly, we can show the convergence of other nonlinear terms. Note that the diffusion coefficients are bounded. Since \( \phi \in V = L^{2}(0, T; H^{1}(0, L)) \), it follows that not only \( \phi_{x} \) exists and is in \( L^{2}(Q) \). We also have \( A^{n}_{x} \rightarrow A^{*}_{x} \) weakly in \( L^{2}(Q) \). We can do the same for other terms. Hence, we can obtain the following

\[
\int_{Q} D_{1}(x, t)(A^{n}_{x} - A^{*}_{x})(\phi_{1})_{x} \, dx \, dt \rightarrow 0 \\
\int_{Q} D_{2}(x, t)(P^{n}_{x} - P^{*}_{x})(\phi_{2})_{x} \, dx \, dt \rightarrow 0 \\
\int_{Q} D_{3}(x, t)(Z^{n}_{x} - Z^{*}_{x})(\phi_{3})_{x} \, dx \, dt \rightarrow 0
\]

82
Similarly, together with assumptions on the boundedness of the advection coefficients, we can get

\[ \int_Q b_1(x,t)(A^n_x - A^*_x)(\phi_1)dxdt \to 0 \]
\[ \int_Q b_2(x,t)(P^n_x - P^*_x)(\phi_2)dxdt \to 0 \]
\[ \int_Q b_3(x,t)(Z^n_x - Z^*_x)(\phi_3)dxdt \to 0 \]

Therefore, we can conclude that

\[(A^*, P^*, Z^*) = (A, P, Z)(h^*)\]

To complete the proof, we will use the lower semi-continuity argument for our objective functional, which says every lower semi-continuous convex function of a real vector space remains lower semi-continuous when supplied with the weak topology (Ekeland (1999)). Since we have

\[ h^n \rightharpoonup h^* \text{ weakly in } L^2(Q) \]

we can obtain the following from Ekeland (1999)

\[ \int_{Q^*} (h^*)^2 dxdt \leq \liminf_{n \to \infty} \int_{Q^n} (h^n)^2 dxdt \]

This gives us

\[ \sup_{h \in A} J(h) = \lim_{n \to \infty} J(h^n) = \liminf_{n \to \infty} J(h^n) \]
\[ = \liminf_{n \to \infty} \int_{Q^*} e^{-\alpha t}(h^n A^n - \mu h^n - \mu (h^n)^2)dxdt \]
\[ \leq \int_{Q^*} e^{-\alpha t}(h^* A^* - \mu h^* - \mu (h^*)^2)dxdt = J(h^*) \]
Therefore, $h^*$ is the optimal control which maximizes the objective functional such that

$$J(h^*) = \max_{h \in A} J(h).$$

\[\square\]

### 3.5 Derivation of the Optimality System

We now derive the optimality system which consists of the state system coupled with the adjoint system and the optimal control characterization. We differentiate the objective functional with respect to the control to obtain necessary conditions for the optimality system. Since the objective functional contains the anchovy population, $A$, we also have to differentiate $h \rightarrow (A,P,Z)(h)$ map with respect to the control, $h$. These derivatives are called the sensitivity functions.

**Theorem 3.4 (Sensitivity PDEs).** Let $h$ be an optimal control for the optimal control problem (3.4) with the corresponding state solution $(A,P,Z) = (A,P,Z)(h)$, and let $h' \in A$ be another control with the corresponding state solutions $(A',P',Z') = (A(h + \epsilon l), P(h + \epsilon l), Z(h + \epsilon l))$ such that $h' = h + \epsilon l$, where $h + \epsilon l \in A$ for all sufficiently $\epsilon > 0$ with $l \in L^\infty(0,L)$.

Then, for any $h \in A$ the mapping, $h \rightarrow (A(h),P(h),Z(h))$ is weakly differentiable in the directional derivative sense and there exists $\Psi_1, \Psi_2, \Psi_3 \in L^2((0,T),H^1(0,L))$ with $(\Psi_1)_t, (\Psi_2)_t, (\Psi_3)_t \in L^2((0,T),H^1(0,L)^*)$ such that

\[
\begin{align*}
\lim_{\epsilon \to 0} \frac{A(h + \epsilon l) - A(h)}{\epsilon} &= \Psi_1 	ext{ weakly in } L^2(Q) \\
\lim_{\epsilon \to 0} \frac{P(h + \epsilon l) - P(h)}{\epsilon} &= \Psi_2 	ext{ weakly in } L^2(Q) \\
\lim_{\epsilon \to 0} \frac{Z(h + \epsilon l) - Z(h)}{\epsilon} &= \Psi_3 	ext{ weakly in } L^2(Q)
\end{align*}
\]

Furthermore, the sensitivity functions $\Psi_1$, $\Psi_2$, and $\Psi_3$ satisfy the linearized system (Sensitivity Equations):
\[ L_1 \Psi_1 - f_1'(A) \Psi_1 - m_0(A \Psi_3 + Z \Psi_1) + m_1(P \Psi_1 + A \Psi_2) + h \Psi_1 = -lA \]

\[ L_2 \Psi_2 - f_2'(P) \Psi_2 - m_2(P \Psi_1 + A \Psi_2) - m_3(P \Psi_3 + Z \Psi_2) + m_6 \Psi_2 = 0 \quad \text{in } (0,L) \times (0,T) \]

\[ L_3 \Psi_3 - f_3'(Z) \Psi_3 + m_4(A \Psi_3 + Z \Psi_1) + m_5(P \Psi_3 + Z \Psi_2) = 0 \quad (3.40) \]

with boundary conditions:

\[ \frac{\partial \Psi_1}{\partial \eta} = 0, \quad \frac{\partial \Psi_2}{\partial \eta} = 0, \quad \frac{\partial \Psi_3}{\partial \eta} = 0 \quad \text{on } \partial(0,L) \times (0,T) \quad (3.41) \]

and initial conditions:

\[ \Psi_1(x,0) = 0, \quad \Psi_2(x,0) = 0, \quad \Psi_3(x,0) = 0 \quad \text{for } x \in (0,L) \subset \mathbb{R} \quad (3.42) \]

Proof. Let us choose \( l \in L^\infty(Q) \) and \( h \in \mathcal{A} \) such that \((h + \epsilon l) \in \mathcal{A}\) for small \( \epsilon > 0 \). Then, the following equations are satisfied by the quotients \( \frac{A^\epsilon - A}{\epsilon} \), \( \frac{P^\epsilon - P}{\epsilon} \), and \( \frac{Z^\epsilon - Z}{\epsilon} \).

\[
\begin{align*}
\left( \frac{A^\epsilon - A}{\epsilon} \right)_t &= r_1 \left( \frac{A^\epsilon - A}{\epsilon} \right) - r_1 \left( \frac{(A^\epsilon)^2 - A^2}{\epsilon} \right) - b_1 \left( \frac{A^\epsilon - A}{\epsilon} \right)_x + D_1 \left( \frac{A^\epsilon - A}{\epsilon} \right)_x \\
&+ m_0 \left( \frac{A^\epsilon Z^\epsilon - AZ}{\epsilon} \right) - m_1 \left( \frac{P^\epsilon A^\epsilon - PA}{\epsilon} \right) - h \left( \frac{A^\epsilon - A}{\epsilon} \right) - lA^\epsilon \\
\left( \frac{P^\epsilon - P}{\epsilon} \right)_t &= r_2 \left( \frac{P^\epsilon - P}{\epsilon} \right) - r_2 \left( \frac{(P^\epsilon)^2 - P^2}{\epsilon} \right) - b_2 \left( \frac{P^\epsilon - P}{\epsilon} \right)_x + D_2 \left( \frac{P^\epsilon - P}{\epsilon} \right)_x \\
&+ m_2 \left( \frac{P^\epsilon A^\epsilon - PA}{\epsilon} \right) + m_3 \left( \frac{P^\epsilon Z^\epsilon - PZ}{\epsilon} \right) - m_6 \left( \frac{P^\epsilon - P}{\epsilon} \right) \\
\left( \frac{Z^\epsilon - Z}{\epsilon} \right)_t &= r_3 \left( \frac{Z^\epsilon - Z}{\epsilon} \right) - r_3 \left( \frac{(Z^\epsilon)^2 - Z^2}{\epsilon} \right) - b_3 \left( \frac{Z^\epsilon - Z}{\epsilon} \right)_x + D_3 \left( \frac{Z^\epsilon - Z}{\epsilon} \right)_x \\
&- m_4 \left( \frac{A^\epsilon Z^\epsilon - AZ}{\epsilon} \right) - m_5 \left( \frac{P^\epsilon Z^\epsilon - PZ}{\epsilon} \right)
\end{align*}
\]
with boundary conditions:
\[ \frac{\partial (A^\epsilon - A)}{\partial \eta} = 0, \quad \frac{\partial (P^\epsilon - P)}{\partial \eta} = 0, \quad \frac{\partial (Z^\epsilon - Z)}{\partial \eta} = 0 \quad \text{on} \quad \partial(0, L) \times (0, T) \]

and initial conditions:
\[ \left( \frac{A^\epsilon - A}{\epsilon} \right)(x, 0) = 0, \quad \left( \frac{P^\epsilon - P}{\epsilon} \right)(x, 0) = 0, \quad \left( \frac{Z^\epsilon - Z}{\epsilon} \right)(x, 0) = 0 \quad \text{for} \quad x \in (0, L) \subset \mathbb{R} \]

Since the state and control variables are bounded in \( L^\infty(Q) \), using estimation techniques as in Theorem 3.3, we find for any \( s \in (0, s] \) that
\[
\begin{align*}
&\sup_{s \in (0, T]} \left( \int_{(0,L)} \left( \frac{A^\epsilon - A}{\epsilon} \right)^2(x, s) + \left( \frac{P^\epsilon - P}{\epsilon} \right)^2(x, s) + \left( \frac{Z^\epsilon - Z}{\epsilon} \right)^2(x, s)dx \right) \\
&\quad + \theta \int_{Q_0} |(\frac{A^\epsilon - A}{\epsilon})_x|^2 + |(\frac{P^\epsilon - P}{\epsilon})_x|^2 + |(\frac{Z^\epsilon - Z}{\epsilon})_x|^2dxdt \\
&\leq C_5 \int_{(0,L)} \left( \frac{A^\epsilon - A}{\epsilon} \right)^2(x, 0) + \left( \frac{P^\epsilon - P}{\epsilon} \right)^2(x, 0) + \left( \frac{Z^\epsilon - Z}{\epsilon} \right)^2(x, 0)dx
\end{align*}
\]

where \( C_5 \) depends on \( T, \theta, \) and \( L^\infty \) bounds on the coefficients, the states, and the control.

Finally, we obtain that \( \|(\frac{A^\epsilon - A}{\epsilon})\|_V, \|(\frac{P^\epsilon - P}{\epsilon})\|_V, \|(\frac{Z^\epsilon - Z}{\epsilon})\|_V \) are uniformly bounded for any \( n \in \mathbb{N} \). These bounds imply together with the sensitivity PDE system that \( \|(\frac{A^\epsilon - A}{\epsilon})_t\|_{V^*}, \|(\frac{P^\epsilon - P}{\epsilon})_t\|_{V^*}, \|(\frac{Z^\epsilon - Z}{\epsilon})_t\|_{V^*} \) are uniformly bounded in \( V^* = L^2(0, T; H^1(0, L)^*) \) for any \( n \in \mathbb{N} \). These bounds justify the existence of \( \Psi_1, \Psi_2, \Psi_3 \in V \) and \( (\Psi_1)_t, (\Psi_2)_t, (\Psi_3)_t \in V \), such that

\[ \frac{A^\epsilon - A}{\epsilon} \rightharpoonup \Psi_1, \quad \frac{P^\epsilon - P}{\epsilon} \rightharpoonup \Psi_2, \quad \frac{Z^\epsilon - Z}{\epsilon} \rightharpoonup \Psi_3, \quad \text{weakly in} \ V, \]
\[ \left( \frac{A^\epsilon - A}{\epsilon} \right)_t \rightharpoonup \Psi_1, \quad \left( \frac{P^\epsilon - P}{\epsilon} \right)_t \rightharpoonup \Psi_2, \quad \left( \frac{Z^\epsilon - Z}{\epsilon} \right)_t \rightharpoonup \Psi_3, \quad \text{weakly in} \ V^* \]
\[
\frac{A^\epsilon - A}{\epsilon} \rightarrow \Psi_1, \quad \frac{P^\epsilon - P}{\epsilon} \rightarrow \Psi_2, \quad \frac{Z^\epsilon - Z}{\epsilon} \rightarrow \Psi_3, \quad \text{in } L^2(Q)
\]

By using the strong convergences in $L^2(Q)$, we also can obtain the following convergences in weak sense

\[
\frac{(A^\epsilon)^2 - (A)^2}{\epsilon} \rightarrow 2A\Psi_1,
\frac{(P^\epsilon)^2 - (P)^2}{\epsilon} \rightarrow 2P\Psi_2,
\frac{(Z^\epsilon)^2 - (Z)^2}{\epsilon} \rightarrow 2Z\Psi_3,
\frac{P^\epsilon A^\epsilon - PA}{\epsilon} \rightarrow P\Psi_1 + A\Psi_2,
\frac{P^\epsilon Z^\epsilon - PZ}{\epsilon} \rightarrow P\Psi_3 + Z\Psi_2,
\frac{A^\epsilon Z^\epsilon - AZ}{\epsilon} \rightarrow A\Psi_3 + Z\Psi_1,
\]

Thus, as $\epsilon \rightarrow 0$, we obtain the resulting PDEs for $\Psi_1$, $\Psi_2$, and $\Psi_3$ as follows:

\[
\begin{align*}
(\Psi_1)_t &= r_1(\Psi_1) - 2A \frac{r_1}{K_1} (\Psi_1) - b_1(x,t)(\Psi_1)_x + (D_1(x,t)(\Psi_1)_x)_x + m_0(A\Psi_3 + Z\Psi_1) \\
&\quad - m_1(P\Psi_1 + A\Psi_2) - h\Psi_1 - lA
\end{align*}
\]

\[
(\Psi_2)_t = r_2(\Psi_2) - 2P \frac{r_2}{K_2} (\Psi_2) - b_2(x,t)(\Psi_2)_x + (D_2(x,t)(\Psi_2)_x)_x + m_2(P\Psi_1 + A\Psi_2) \\
&\quad + m_3(P\Psi_3 + Z\Psi_2) - m_6\Psi_2
\]

\[
(\Psi_3)_t = r_3(\Psi_3) - 2Z \frac{r_3}{K_3} (\Psi_3) - b_3(x,t)(\Psi_3)_x + (D_3(x,t)(\Psi_3)_x)_x - m_4(A\Psi_3 + Z\Psi_1) \\
&\quad - m_5(P\Psi_3 + Z\Psi_2).
\]

When we arrange the previous PDE System, we obtain the following linearized System, and $\Psi_1$, $\Psi_2$, and $\Psi_3$ satisfy the Sensitivity Equations:

\[
\begin{align*}
L_1\Psi_1 - f_1'(A)\Psi_1 - m_0(A\Psi_3 + Z\Psi_1) &+ m_1(P\Psi_1 + A\Psi_2) + h\Psi_1 = -lA \\
L_2\Psi_2 - f_2'(P)\Psi_2 - m_2(P\Psi_1 + A\Psi_2) &- m_3(P\Psi_3 + Z\Psi_2) + m_6\Psi_2 = 0 \quad \text{in } (0,L) \times (0,T) \\
L_3\Psi_3 - f_3'(Z)\Psi_3 + m_4(A\Psi_3 + Z\Psi_1) &+ m_5(P\Psi_3 + Z\Psi_2) = 0
\end{align*}
\]

with boundary conditions: 87
\[
\frac{\partial \Psi_1}{\partial \eta} = 0, \quad \frac{\partial \Psi_2}{\partial \eta} = 0, \quad \frac{\partial \Psi_3}{\partial \eta} = 0 \quad \text{on} \quad \partial(0, L) \times (0, T)
\]

and initial conditions:

\[
\Psi_1(x, 0) = 0, \quad \Psi_2(x, 0) = 0, \quad \Psi_3(x, 0) = 0 \quad \text{for} \quad x \in (0, L) \subset \mathbb{R}
\]

Therefore, these convergences justify that \((\Psi_1, \Psi_2, \Psi_3)\) solve system (4.38) with initial conditions (3.39) and boundary conditions (4.40).

The system (3.38) can be written in terms of the linear operator \(L\) as follow:

\[
L \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} -lA \\ 0 \\ 0 \end{pmatrix}
\]

(3.44)

where

\[
L \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} L_1 \Psi_1 \\ L_2 \Psi_2 \\ L_3 \Psi_3 \end{pmatrix} + M \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}
\]

Here,

\[
M = \begin{pmatrix} -f_1'(A) - m_0 Z + m_1 P + h & m_1 A & -m_0 A \\ -m_2 P & -f_2'(P) - m_2 A - m_3 Z + m_6 & -m_3 P \\ m_4 Z & m_5 Z & -f_3'(Z) + m_4 A + m_5 P \end{pmatrix}
\]

and

\[
\begin{pmatrix} L_1 \Psi_1 \\ L_2 \Psi_2 \\ L_3 \Psi_3 \end{pmatrix} = \begin{pmatrix} (\Psi_1)_t + b_1(x, t)(\Psi_1)_x - (D_1(x, t)(\Psi_1)_x)_x \\ (\Psi_2)_t + b_2(x, t)(\Psi_1)_x - (D_2(x, t)(\Psi_2)_x)_x \\ (\Psi_3)_t + b_3(x, t)(\Psi_3)_x - (D_3(x, t)(\Psi_3)_x)_x \end{pmatrix}
\]
Now let’s derive the adjoint equations, denoted by \( \lambda_i, \ i = 1, 2, 3 \). Let \( \mathcal{L}^* \) be our adjoint operator of the \( \mathcal{L} \) associated with the Sensitivity PDEs (3.42). We need to find the adjoint operator \( \mathcal{L}^* \) such that

\[
\int_Q e^{-\alpha t}(\Psi_1, \Psi_2, \Psi_3)\mathcal{L}^* \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix} = \int_Q e^{-\alpha t}(\lambda_1, \lambda_2, \lambda_3)\mathcal{L} \begin{pmatrix}
\Psi_1 \\
\Psi_2 \\
\Psi_3
\end{pmatrix}.
\] (3.45)

We write the adjoint system as

\[
\mathcal{L}^* \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix} = \begin{pmatrix}
h \\
0 \\
0
\end{pmatrix}
\] on \( Q \). (3.46)

The LHS of above equation can be written as follow:

\[
\mathcal{L}^* \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix} = \begin{pmatrix}
L_1^*\lambda_1 \\
L_2^*\lambda_2 \\
L_3^*\lambda_3
\end{pmatrix} + M^T \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix} + \alpha \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix},
\]

where \( \alpha \) is the interest rate,

\[
\begin{pmatrix}
L_1^*\lambda_1 \\
L_2^*\lambda_2 \\
L_3^*\lambda_3
\end{pmatrix} = \begin{pmatrix}
-(\lambda_1)_t - (b_1(x, t)\lambda_1)_x - (D_1(x, t)(\lambda_1)_x)_x \\
-(\lambda_2)_t - (b_2(x, t)\lambda_2)_x - (D_2(x, t)(\lambda_2)_x)_x \\
-(\lambda_3)_t - (b_3(x, t)\lambda_3)_x - (D_3(x, t)(\lambda_3)_x)_x
\end{pmatrix},
\]

and

\[
M^T = \begin{pmatrix}
-f_1'(A) - m_0 Z + m_1 P + h & -m_2 P & m_4 Z \\
m_1 A & -f_2'(P) - m_2 A - m_3 Z + m_6 & m_5 Z \\
-m_0 A & -m_3 P & -f_3'(Z) + m_4 A + m_5 P
\end{pmatrix}.
\]
The RHS of the adjoint system (3.46) is obtained by the derivative of the integrand of the objective functional (without $e^{-\alpha t}$) with respect to the states. To justify the term,

$$\alpha \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix},$$

we use the integration by parts as follows

$$\int_Q e^{-\alpha t}(\Psi_1)_t \lambda_1 dx dt = -\int_Q (e^{-\alpha t}(\lambda_1)_t - \alpha e^{-\alpha t} \lambda_1) \Psi_1 dx dt + \int_{(0,L) \times \{T\}} e^{-\alpha t} \lambda_1 \Psi_1 \tilde{n} ds$$

$$+ \int_{(0,L) \times \{0\}} e^{-\alpha t} \lambda_1 \Psi_1 \tilde{n} ds.$$

Assuming $\lambda_1(x,T) = 0$, and by $\Psi_1(x,0) = 0$, we can get

$$\int_Q e^{-\alpha t}(\Psi_1)_t \lambda_1 dx dt = \int_Q e^{-\alpha t}[-(\lambda_1)_t + \alpha \lambda_1] \Psi_1 dx dt. \quad (3.47)$$

**Theorem 3.5.** Given an optimal control $h^*$ and corresponding state solution $(A^*, P^*, Z^*)$, there exists a weak solution $(\lambda_1, \lambda_2, \lambda_3)$ satisfying the adjoint system (3.44), boundary conditions and transversality conditions

$$L_1^* \lambda_1 = h + \left[ f_1'(A) + m_0 Z - m_1 P - h - \alpha \right] \lambda_1 + m_2 P \lambda_2 - m_4 Z \lambda_3$$

$$L_2^* \lambda_2 = \left[ f_2'(P) + m_2 A + m_3 Z - m_6 - \alpha \right] \lambda_2 - m_1 A \lambda_1 - m_5 A \lambda_3 \quad \text{in } (0,L) \times (0,T)$$

$$L_3^* \lambda_3 = \left[ f_3'(Z) - m_3 A - m_5 P - \alpha \right] \lambda_3 + m_0 A \lambda_1 + m_3 P \lambda_2 \quad (3.48)$$
with boundary conditions:

\[ D_1(x,t) \frac{\partial \lambda_1}{\partial \eta} + (b_1(x,t) \cdot \vec{\eta}) \lambda_1 = 0 \]
\[ D_2(x,t) \frac{\partial \lambda_2}{\partial \eta} + (b_2(x,t) \cdot \vec{\eta}) \lambda_2 = 0 \quad \text{on } \partial(0,L) \times (0,T) \]
\[ D_3(x,t) \frac{\partial \lambda_3}{\partial \eta} + (b_3(x,t) \cdot \vec{\eta}) \lambda_3 = 0 \]

and transversality conditions:

\[ \lambda_1(x,T) = 0, \quad \lambda_2(x,T) = 0, \quad \lambda_3(x,T) = 0 \quad \text{for } x \in (0,L) \subset \mathbb{R} \] (3.50)

Furthermore, the optimal control is characterized by

\[ h^* = \min \left( \left( \frac{(1 - \lambda_1)A - \mu_1}{2\mu_2} \right)^+, M \right) \quad \text{on } Q \] (3.51)

where

\[ n^+ = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases} \]

Proof. Since the adjoint PDE system is linear, there exist \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) satisfying the adjoint PDE system in a weak sense. The system (3.48) can be written as

\[ \mathcal{L}^* \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix} \quad \text{on } Q^* \] (3.52)

where

\[ \mathcal{L}^* \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} L_1^* \lambda_1 \\ L_2^* \lambda_2 \\ L_3^* \lambda_3 \end{pmatrix} + M^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \alpha \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}. \]
Now, let’s get the characterization of optimal control, $h$. Assume $h$ is an optimal control, and $A(h), P(h),$ and $Z(h)$ are its corresponding solutions. Consider $h + \epsilon l \in A$ with associated solutions $A^\epsilon, P^\epsilon,$ and $Z^\epsilon$. Since the maximum of the objective functional is attained at $h$, we obtain the following:

\[
0 \geq \lim_{\epsilon \to 0^+} \frac{J(h + \epsilon l) - J(h)}{\epsilon} = \lim_{\epsilon \to 0^+} \int_{Q^*} e^{-\alpha t} \left[ \left( \frac{(h + \epsilon l)A^\epsilon - \mu_1(h + \epsilon l) - \mu(h + \epsilon l)^2}{\epsilon} - \frac{hA - \mu_1 h - \mu_2 h^2}{\epsilon} \right) \right]
\]

\[
= \lim_{\epsilon \to 0^+} \int_{Q^*} e^{-\alpha t} \left[ \left( \frac{(h + \epsilon l)A^\epsilon - hA}{\epsilon} + \frac{\mu_1 h - \mu_1(h + \epsilon l)}{\epsilon} + \frac{\mu_2 h^2 - \mu_2(h + \epsilon l)^2}{\epsilon} \right) \right]
\]

\[
= \lim_{\epsilon \to 0^+} \int_{Q^*} e^{-\alpha t} \left[ h \left( \frac{A^\epsilon - A}{\epsilon} \right) + lA^\epsilon \right] + \lim_{\epsilon \to 0^+} \int_{Q^*} \left[ \mu_2 e^{-\alpha t} \left( -2hl - \epsilon l^2 \right) - e^{-\alpha t} \mu_1 l \right]
\]

\[
= \int_{Q^*} e^{-\alpha t} h\Psi_1 + e^{-\alpha t} lA - \int_{Q^*} e^{-\alpha t} \left( 2\mu_2 hl + \mu_1 l \right)
\]

\[
= \int_{Q^*} e^{-\alpha t} (\Psi_1, \Psi_2, \Psi_3) \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix} + e^{-\alpha t} lA - \int_{Q^*} e^{-\alpha t} \left( 2\mu_2 hl + \mu_1 l \right)
\]

\[
= \int_{Q^*} e^{-\alpha t} (\lambda_1, \lambda_2, \lambda_3) \begin{pmatrix} -lA \\ 0 \\ 0 \end{pmatrix} + \int_{Q^*} e^{-\alpha t} \left( -\lambda_1 lA + A - 2\mu_2 h - \mu_1 \right)
\]

\[
= \int_{Q^*} e^{-\alpha t} \left( -\lambda_1 lA + A - 2\mu_2 h - \mu_1 \right)
\]

\[
= \int_{Q^*} e^{-\alpha t} \left( (1 - \lambda_1) A - 2\mu_2 h - \mu_1 \right)
\]
Thus, we obtain

$$0 \geq \int_{Q^*} e^{-\alpha t} \left[ (1 - \lambda_1)A - 2\mu h - \mu_1 \right].$$  \hspace{1cm} (3.53)

To determine $h^*$, we use standard optimality techniques. Consider the following three cases:

**Case 1:** On the set \{$(x, t) \in Q^* : h^*(x, t) = 0$\}, take the variation, $l$ to have support on this set, and $l \geq 0$. It follows from the inequality (3.53) that

$$0 \geq A(1 - \lambda_1) - 2\mu_2 h^* - \mu_1 \quad \text{or} \quad h^* = 0 \geq \frac{A(1 - \lambda_1) - \mu_1}{2\mu_2}$$

**Case 2:** On the set $(x, t) \in Q^* : h^*(x, t) = h_{max} = M$, take the variation, $l$ to have support on this set, and $l \leq 0$. Thus the inequality (3.53) implies that

$$0 \leq A(1 - \lambda_1) - 2\mu_2 h^* - \mu_1 \quad \text{or} \quad h^* = M \leq \frac{A(1 - \lambda_1) - \mu_1}{2\mu_2}$$

**Case 3:** On the set $(x, t) \in Q^* : 0 < h^*(x, t) < h_{max} = M$, take the variation, $l$ with arbitrary sign, and with support on the set. Thus, it follows from the inequality (3.53) that

$$0 = A(1 - \lambda_1) - 2\mu_2 h^* - \mu_1 \quad \text{or} \quad h^* = \frac{A(1 - \lambda_1) - \mu_1}{2\mu_2}$$

Therefore, considering the three cases, we obtain an characterization of $h^*$ as

$$h^* = \min \left( \left( \frac{(1 - \lambda_1)A - \mu_1}{2\mu_2} \right)^+, M \right) \quad \text{on } Q^*$$

where

$$n^+ = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$
Thus, our optimality system has the form:

\[ L_1 A = f_1(A) + m_0 AZ - m_1 PA - h^* A \]
\[ L_2 P = f_2(P) + m_2 PA + m_3 PZ - m_6 P \]
\[ L_3 Z = f_3(Z) - m_4 AZ - m_5 PZ \]
\[ L_1^* \lambda_1 = h^* + \left[ f_1'(A) + m_0 Z - m_1 P - h^* - \alpha \right] \lambda_1 + m_2 P \lambda_2 - m_4 Z \lambda_3 \]
\[ L_2^* \lambda_2 = \left[ f_2'(P) + m_2 A + m_3 Z - m_6 - \alpha \right] \lambda_2 - m_1 A \lambda_1 - m_5 A \lambda_3 \]
\[ L_3^* \lambda_3 = \left[ f_3'(Z) - m_3 A - m_5 P - \alpha \right] \lambda_3 + m_0 A \lambda_1 + m_3 P \lambda_2 \]

in \((0, L) \times (0, T)\)

with boundary conditions for \(i = 1, 2, 3\)

\[
\begin{cases}
\frac{\partial A}{\partial \eta} = 0, \quad \frac{\partial P}{\partial \eta} = 0, \quad \frac{\partial Z}{\partial \eta} = 0 \\
D_i(x, t) \frac{\partial \lambda_i}{\partial \eta} + (b_i(x, t) \cdot \vec{n}) \lambda_i = 0
\end{cases}
\]
on \(\partial(0, L) \times (0, T)\)

initial conditions,

\[ A(x, 0) = A_0(x), \quad P(x, 0) = P_0(x), \quad Z(x, 0) = Z_0(x) \]

for \(x \in (0, L) \subset \mathbb{R}\)

transversality conditions,

\[ \lambda_1(x, T) = 0, \quad \lambda_2(x, T) = 0, \quad \lambda_3(x, T) = 0 \]

for \(x \in (0, L) \subset \mathbb{R}\)

and the characterization of the control, \(h^*\)

\[ h^* = \min \left( \left( \frac{(1 - \lambda_1) A - \mu_1}{2 \mu_2} \right)^+, M \right) \]
on \(Q^*\)

where

\[ n^+ = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases} \]
Theorem 3.6. When $T$ is small enough, then the solution of the optimality system is unique.

Proof. Suppose $(A, P, Z, \lambda_1, \lambda_2, \lambda_3)$ and $(\hat{A}, \hat{P}, \hat{Z}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$ are two solutions of the optimality system (3.54) with initial conditions, boundary conditions, and transversality conditions of the optimality system. Then, the related control characterizations are

$$h = \min \left( \left( \frac{(1 - \lambda_1)A - \mu_1}{2\mu_2} \right)^+, M \right) \text{ on } Q^*$$

and

$$\hat{h} = \min \left( \left( \frac{(1 - \hat{\lambda}_1)\hat{A} - \mu_1}{2\mu_2} \right)^+, M \right) \text{ on } Q^*.$$

By using similar tools as in Theorem 3.2 and Theorem 3.3, we obtain that for sufficiently small $T$, $A = \hat{A}$, $P = \hat{P}$, $Z = \hat{Z}$, $\lambda_1 = \hat{\lambda}_1$, $\lambda_2 = \hat{\lambda}_2$, and $\lambda_3 = \hat{\lambda}_3$. See the work by Miller Neilan (2009) for details of the proof. Finally, since the optimal control is characterized by the state and adjoint variables, we obtain $h = \hat{h}$. 

Therefore, the solutions to the optimality system are unique, which implies a unique optimal control.

3.6 Numerical Simulations and Parameters

We used the model parameters given in Table 3.2 to obtain all the outcomes in this chapter. These parameters are obtained from Chapter 2 and justified for our spatial food chain model in the study. The state and adjoint system (3.54) are solved using an explicit finite difference method. See Appendix A for details of the finite difference scheme. We solve the optimality system using the forward-backward sweep method. In our numerical results, we picked the length of our space domain as 4, and it corresponds $1700km$. Also we picked the time length, $T$ as 12, and it corresponds 12 months.
Table 3.2: Parameter descriptions and values obtained from Chapter 2, and justified for our spatial food chain model. Here $e$ is a scientific notation in MATLAB and it is a shorthand for 10

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Descriptions</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>Initial biomass of anchovy, $A$</td>
<td>$3e^5$</td>
</tr>
<tr>
<td>$P_0$</td>
<td>Initial biomass of jellyfish, $A$</td>
<td>$3e^4$</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>Initial biomass of zooplankton, $A$</td>
<td>$1e^5$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>Intrinsic growth rate of anchovy, $A$</td>
<td>0.35</td>
</tr>
<tr>
<td>$r_2$</td>
<td>Intrinsic growth rate of jelly fish, $P$</td>
<td>0.4</td>
</tr>
<tr>
<td>$r_3$</td>
<td>Intrinsic growth rate of zoo-plankton, $Z$</td>
<td>0.8</td>
</tr>
<tr>
<td>$K_1$</td>
<td>Carrying capacity of anchovy, $A$</td>
<td>$6e^5$</td>
</tr>
<tr>
<td>$K_2$</td>
<td>Carrying capacity of jelly fish, $P$</td>
<td>$9e^3$</td>
</tr>
<tr>
<td>$K_3$</td>
<td>Carrying capacity of zoo-plankton, $Z$</td>
<td>$1.5e^5$</td>
</tr>
<tr>
<td>$m_0$</td>
<td>Growth rate of $A$ due to predation of $Z$</td>
<td>$1e^{-6}$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Consumption rate of $A$ by its predator, $P$</td>
<td>$2.2e^{-5}$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Growth rate of $P$ due to predation of $A$</td>
<td>$1.95e^{-6}$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>Growth rate of $P$ due to predation of $Z$</td>
<td>$5.7e^{-6}$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>Consumption rate of $Z$ due to its predator $A$</td>
<td>$5e^{-6}$</td>
</tr>
<tr>
<td>$m_5$</td>
<td>Consumption rate of $Z$ due to its predator $P$</td>
<td>$1e^{-5}$</td>
</tr>
<tr>
<td>$m_6$</td>
<td>Consumption rate of $P$ due to its predators</td>
<td>0.2</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>Coefficient of linear part of the cost function</td>
<td>20000</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>Coefficient of quadratic part of the cost function</td>
<td>0.1</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>The interest rate of the discount rate</td>
<td>0.01</td>
</tr>
<tr>
<td>$b_i$</td>
<td>Advection Coefficient, ($b_i &gt; 0$ for $i = 1, 2, 3$)</td>
<td>varied</td>
</tr>
<tr>
<td>$D_i$</td>
<td>Diffusion Coefficient, $(D_i &gt; \theta &gt; 0$ for $i = 1, 2, 3$)</td>
<td>varied</td>
</tr>
</tbody>
</table>
3.6.1 Numerical Results

We will investigate different scenarios in the anchovy fishery and then discuss their corresponding effects. We first will apply optimal control strategies when the populations are concentrated in center of the domain (See Figure 3.3), and then we will compare with the current harvesting strategy in the anchovy fishery. The following figure (See Figure 3.4) gives the spatial distribution of anchovy population under different diffusion rates when the control is not applied (no harvesting). As the diffusion rate increases from 0.01 to 0.1, the biomass of anchovy population is getting lower near the final time (See Figure 3.5).

When diffusion rates are high, the anchovy population hits the boundaries faster. We do not know which diffusion rate is more realistic for the anchovy population. That is why we will apply optimal control rates for different diffusion rates to get some ideas about the true diffusion rate for the Black Sea anchovy system (See Figures (3.6)-(3.8)). In Figures (3.6)-(3.8), even if the maximum harvest rate is 0.4, we do not harvest at maximum harvest level since the cost of the fishery is high. See Figure 3.10, we almost harvest at maximum harvest level since we reduced the cost of the fishery.

Figure 3.3: Initial Biomass of anchovy (blue), jellyfish (red), and zooplankton (green).
Figure 3.4: Spatial dynamics of anchovy population without harvesting and with different diffusion rates

In Figures (3.6)-(3.8), optimal control harvest rates with the maximum harvest rate, $h = 0.4$ are applied to the entire year with different diffusion rates. This means that we consider the time domain $[0, T]$ in our control set. When we look at the optimal harvest rates in these figures, we see that the optimal harvest rates are almost zero in the first half of the year, and the harvesting starts right after the 8th month depending on the diffusion rates. Thus, we obtain a seasonal fishery when we apply optimal harvest rates depending on dynamics of the system. This totally matches with the current management strategy that has applied on the southern part of the Black Sea. These results show the power of the optimal control theory since it advises a seasonal fishery.

When we look at the dynamics of the populations, especially the zooplankton population for the different diffusion rates, we will see different dynamics in each case. In the case of diffusion rate, 0.01, the anchovy population almost wipes out the zooplankton around the center, and that does not seem realistic in the long term. Therefore, we believe that the true diffusion rate for the anchovy population should be between 0.05 and 0.1 (See Figures 3.6 - 3.8).
Figure 3.5: Dynamics of anchovy population without harvesting, which is integrated over space at each time step with different diffusion rates (for 12 months).

\[ D_1 = D_2 = D_3 = 0.1 \quad D_1 = D_2 = D_3 = 0.05 \quad D_1 = D_2 = D_3 = 0.01 \]

Figure 3.6: Optimal harvesting rate applied for the entire year with diffusion rates, \( D_i = 0.01 \) and \( b_i = 0 \) for \( i = 1, 2, 3 \). The total landing is 376,170 tonnes, and the discounted net profit is \( J = 284,590 \) tonnes.
Figure 3.7: Optimal harvesting rate applied for the entire year with diffusion rates, $D_i = 0.05$ and $b_i = 0$ for $i = 1, 2, 3$. The total landing is 330,060 tonnes, and the discounted net profit is $J = 213,840$ tonnes.

Figure 3.8: Optimal harvesting rate applied for the entire year with diffusion rates, $D_i = 0.1$ and $b_i = 0$ for $i = 1, 2, 3$. The total landing is 278,080 tonnes, and the discounted net profit is $J = 175,620$ tonnes.
Figure 3.9: Dynamics of populations without harvesting in the case of diffusion rates, $D_i = 0.05$ for $i = 1, 2, 3$.

From the results for different diffusion rates, we will consider the diffusion rate for the anchovy population as $0.05$ for this study. In the case of no harvesting with $D_i = 0.05$ (See Figure 3.9), the anchovy population almost wipes out the center of the zooplankton population since the biomass concentration of anchovy is very high on the center of the zooplankton population. Even if the biomass of zooplankton decreases, there is no dramatic change in jellyfish biomass since there is enough anchovy population in the system. When we compare Figures 3.7 and 3.9, we see that the zooplankton biomass recovers and jellyfish population decreases in the end of the year in Figure 3.7 since we apply harvesting in last 3 months.

When we applied optimal harvesting rates for a whole year, we do not harvest the system in first months of the year and harvest the system in last months. When we reduce the cost of the fishery by reducing value of $\mu_1$ from 20000 to 500, then the fishery season enlarges from 4th month to 6th month in optimal control harvesting strategy (See the left plot in Figure 3.10). We ask why we get seasonal fishery when we apply the optimal harvest rates without forcing seasonal harvesting. It may be because of the small initial biomass of the anchovy population (See Figure 3.10). We increased the initial biomass of anchovy population from 300,000 tonnes to 500,000 tonnes and we obtained a different version of seasonal harvesting (See the right plot in Figure 3.10).
Figure 3.10: Optimal harvesting rate applied for the entire year with different initial conditions, diffusion rates, $D_i = 0.1$ and advection rates $b_i = 0$ for $i = 1, 2, 3$. We also reduced the cost by reducing $\mu_1$ from 20000 to 500. In the left plot, we used true initial biomass, and we increased the initial to $5e^5$ in the right plot to show the effects of having different initial conditions.

These results show that seasonal fishery is the best option for the anchovy fishery on the southern part of the Black Sea depending on the dynamics of food web when we apply optimal control tools.

3.6.2 Imposed Seasonal Optimal Harvesting

In the case, we will impose a seasonal harvest for the Black Sea anchovy on the southern part of the Black Sea (See Figure 3.11). We apply harvesting for 3 months on the southern part, and we now consider the set $\Omega$ in our optimal control strategy with the maximum harvest rate, 0.4. In simulations, we will take the diffusion rate of anchovy population as 0.05. When we apply imposed seasonal fishery with optimal control rates for one year, we obtain the net profit as 202,600 with diffusion rate 0.05 (See Figure 3.11). This is less than
what we obtained in optimal control cases with the diffusion rate 0.05 (See Figure 3.7) and harvest for three months rather than about 4 months in the case of imposed seasonal fishery. Also, the dynamics of populations are similar with those in Figure 3.7 since we almost have the same strategy.

We also applied a seasonal fishery with optimal control rates with no advection and low diffusion \((D_i = 10^{-100} \text{ for } i = 1, 2, 3)\) for five years, and our results match with the results we obtained in Chapter 2. We almost get similar predator prey relations between species (See Figure 3.11), and get similar optimal harvest rates (See Figure 3.12). In the optimal harvest rates, we reduce the harvest rates in the middle of the fishing seasons to let the anchovy population increase its size (See Figure 3.13).

Figure 3.11: Imposed seasonal optimal harvesting rates applied for 3 months with diffusion rate, \(D_i = 0.05\) and advection rate, \(b_i = 0\) for \(i = 1, 2, 3\). The total landing is 306,550 tonnes, and the discounted net profit is \(J = 202,600\) tonnes.
**Figure 3.12:** Population biomasses integrated over space at each time step. Biomass of the Black Sea anchovy (blue), Jellyfish (red), and Zooplankton (green).

**Figure 3.13:** Optimal harvest rates for five years.
Figure 3.14: Stock dynamics of the anchovy population and seasonal optimal harvesting strategy for five consecutive years with almost zero diffusion and $b_i = 0$ for $i = 1, 2, 3$.

3.6.3 Effect of Movement

In the southern part of the Black Sea, we have movement from the western part to the eastern part of the Black Sea during the fishery season. Therefore, we would like to include a positive advection coefficient. We apply different advection rates to see the effects of movement (See Figures (3.15) and (3.16)). When we increase the constant advection rate from 0.05 to 0.1, the area of the region of harvest decreases and shifts to the right side of the region in the imposed seasonal harvesting since the anchovy biomass also shifts to the right side of the Black Sea. If any change happens in the advection rate of anchovy population, then it will directly affect the harvest regions. When the speed of the advection is high, it decreases the total landing and the discounted net profit of the anchovy fishery. (See Figures 3.14 and 3.15). When the advection rates are 0.05, the landing of the anchovy is 299,900 tonnes and the net profit is 197,340. On the other hand, when we increase the advection rate from 0.05 to 0.1, the total landing decreases to 281,000 tonnes and the net profit decreases to 183,120 tonnes.
Figure 3.15: Imposed seasonal optimal harvesting rate applied for 3 months with diffusion rates, $D_i = 0.05$ and $b_i = 0.05$ for $i = 1, 2, 3$. The total landing is 299,900 tonnes, and the discounted net profit is $J = 197,340$ tonnes.

Figure 3.16: Imposed seasonal optimal harvesting rate applied for 3 months with diffusion rates, $D_i = 0.05$ and $b_i = 0.1$ for $i = 1, 2, 3$. The total landing is 281,900 tonnes, and the discounted net profit is $J = 183,120$ tonnes.
3.6.4 Comparison of Strategies

We will compare two strategies, the imposed seasonal optimal harvest strategy and the constant harvest strategy. We will use 0.4 as a constant harvest rate since we used the same harvest rate in the ODE model. With this way, we can compare our results from our ODE and PDE models. We will place the initial biomasses in the left size of the region to obtain more realistic results since we have advection from the western coast to the eastern coast of the Black Sea right before the fishery seasons. We use the diffusion rate $D_i = 0.08$ for $i = 1, 2, 3$ and the advection rates $b_1 = 0.2$, $b_2 = 0.15$, and $b_3 = 0.1$ when we compare these strategies.

![Optimal Harvest Rates vs Constant Harvest Rate](image)

**Figure 3.17:** Comparison of imposed seasonal optimal harvest rates and the constant harvest rate applied for three months with $D_i = 0.08$ and $b_1 = 0.2$, $b_2 = 0.15$, and $b_3 = 0.1$. The left plots show the dynamics of populations when we apply the optimal harvesting strategy with the maximum harvest rate 0.4, and the right plots show the dynamics of populations when we apply the constant harvesting strategy with $h = 0.4$. 

Landing = 224,260

J = 142,080

Landing = 244,030

J = 132,790
We catch 224,260 tonnes anchovy with the net profit as \( J = 142,080 \) tonnes in optimal harvest strategy, and catch 244,030 tonnes anchovy with the net profit as \( J = 132,790 \) tonnes in the constant harvest strategy (See Figure 3.17). These results are very close to what we obtained in ODE model. In the ODE model, we catch yearly approximately 230,000 tonnes anchovy with the yearly net profit as \( J = 143,500 \) tonnes.

In the PDE model, we obtain 7.5% more net profit in the optimal control strategy than in the constant harvest strategy. In the optimal control strategy, we harvest mostly in the eastern part of the region because of the direction of the advection (See Figure 3.17), and this result of harvested areas in the optimal control strategy matches with literature (Gucu et al. (2017) and STECF (2017)).

### 3.7 Conclusions

In the study, we modeled the spatial features of the Black Sea anchovy. We showed the advantages of using food chain models coupled with optimal control theory as a new approach. The benefits of using this approach include investigating the marine protected areas and finding spatial harvesting strategies depending on the food web dynamics of the system. For example, in this study, we showed that seasonal harvesting is the best option for the Black Sea anchovy by using optimal control theory.

Including advection and diffusion to our dynamical system, we get extra information about the Black Sea in terms of spatial features. For example, when the speed of movement is high, then it decreases the net profit of the anchovy fishery on the southern part of the Black Sea. It is better to see the effects of advection and diffusion on the food webs since advection and diffusion rates have an important role in spatial management strategies. Look at Figure 3.6 to see the effect of having small diffusion rate on the zooplankton biomass. In this case, the anchovy population almost wipes out the zooplankton biomass in the center of the spatial domain.
In the case of implementing the initial biomasses in the left side of the region with imposed seasonal harvesting, we obtained very close results as in the ODE Model. This case is very realistic in terms of the harvested areas (Gucu et al. (2017)), the landing, and the discounted net profit (See Chapter 2). In this case, we harvest the stocks mostly on the eastern part of the Turkish coast. It is because of the direction of the advection (movement of the anchovy population from western coast to eastern coast in the southern part of the Black Sea).

We also got similar results with ODE model in the case of no advection and low diffusion (See Figure 3.12 and 3.13). As you see in Figure 3.13, we get almost a similar harvesting strategy in the long term management as we got in Chapter 2. We start with high harvest rates in the beginning of the fishery seasons, reduce the harvest rates in the middle of the fishing seasons, and then we again increase the harvest rates. This strategy gives the system an opportunity to increase its size.
Chapter 4

Summary and Future Directions

In summary, this dissertation contributes ecosystem-based fishery management approaches using a novel method of coupling food chain models with optimal control tools. The results in Chapter 2 show the importance of food chain models in the investigation of ecosystem-based fishery management strategies. Using food chain models, we are able to see the effects of predator-prey relations and harvesting on the related food web by observing key species in food webs.

However, using single equations in fishery management does not offer reliable information in our study. For example, using landing data in our food chain model and in the single-species anchovy model in chapter 2, we showed that the single equation overestimates the biomass in the single-species anchovy model and contributes to overfishing instead of reducing the risk of overfishing. Using single-species models in fishery management can be very risky since it ignores the predator-prey relations in ecosystems. Therefore, our study recommends the usage of food chain models in fishery management strategies to be able to obtain more reliable policies about the food web of target fish populations.

We compared three different strategies in Chapter 2, and showed the benefits of the optimal harvesting strategy in the management of the Black Sea anchovy in the long term. We showed that the optimal harvesting strategy offers 43% more net profit and 12% more landing on the southern part of the Black Sea. This strategy not only offers the best food
web structure in terms of population biomass, but also offers a more productive ecosystem by offering more landing in the long term.

We also accomplished finding a link between the number of fishing fleets used in harvesting and the harvest rate implemented in fishery models with the help of a nonlinear regression model in chapter 2. Thus, depending on our food chain model and the nonlinear regression model, we estimated the number of fishing fleets in the case of approximate optimal harvest rates as 245, corresponding to the optimal harvest rate, \( h = 0.4 \) in the second half of fishery seasons. Similarly, we found 239 for the corresponding optimal harvest rate, 0.335 in the first half of the fishery seasons. Our study showed that the amount of extra effort in the recent harvest strategies was approximately 53%. Therefore, during the last 14 years, the amount of fleets was approximately 53% more than the optimal amount of fishing fleets predicted by our food chain and regression model.

In this dissertation, we also included spatial features of population dynamics in Chapter 3, and the results without forcing seasonal harvesting showed the power of optimal control theory. It offers us a seasonal harvesting just depending on dynamics of populations. This result matches with the harvesting of the Black Sea anchovy on the southern part since we have seasonal harvesting for anchovy fishing. With low diffusion and no advection, and with initial biomasses in the left side of our spatial domain in Chapter 3, we obtained very close results to that in Chapter 2. In this case, we also obtained very realistic results in terms of the harvested areas (Gucu et al. (2017)), the landing (See the average landing in STECF (2017) and Chapter 2), and the discounted net profit from Chapter 2. In chapter 3, we also investigated the effect of advection on spatial fishery management and showed that we obtain less landing and net profit when advection is high.

Our first future work will be to include migration to our models to able to obtain more realistic anchovy dynamics. The anchovy population first move to northwest of the Black Sea right after the fishery season, and move to southern part when temperature drops, and finally move from the west coast to the east coast during the fishery season. Therefore,
including migration will give us more realistic information about the dynamical system and also help us to investigate our spatial dynamics of populations in the long term. Moreover, we would like to build age-structured models for the anchovy population since we have available age-structured data about the Black Sea anchovy. Also, we can extend the PDE model to 2 spatial dimensions and try to find some spatial data.

Another important future work would be adding more trophic levels in the models. With this way, we can control any population in food webs. For example, we can reduce the pressure of a predator on a prey by harvesting or removing the predator, and in the way both population may reach equilibrium. Lastly, we would like to apply the food web approach with optimal control tools to other ecosystems.
Bibliography


FAO (2011). *Fisheries management, Marine protected areas and fisheries*. Food and Agriculture Organization (FAO) of the United Nations, Rome. 3, 14, 52


Miller Neilan, R. L. (2009). Optimal control applied to population and disease models, Dissertation, University of Tennessee, Knoxville. 95


Appendices
Appendix A

Finite Difference Method

We used an explicit finite difference method as a numerical method to obtain the numerical solutions of the PDE system. In the method, we approximate the derivatives in the model with difference equations, which is based on Taylor Series.

A Forward Finite Difference Method for the State Variables with Explicit Scheme

First, we obtain the forward difference equations for corresponding time derivatives as follows. The space variable, $x$ and the time variable, $t$ are represented, by grid points, denoted by $i$ and $j$ respectively.

\[
A_t(i, j) = \frac{A(i, j + 1) - A(i, j)}{\Delta t}
\]
\[
P_t(i, j) = \frac{P(i, j + 1) - P(i, j)}{\Delta t}
\]
\[
Z_t(i, j) = \frac{Z(i, j + 1) - Z(i, j)}{\Delta t}
\]

Then, we get the difference equations for corresponding space derivatives as
\[ A_x(i, j) = \frac{A(i+1, j) - A(i, j)}{\Delta x} \]
\[ P_x(i, j) = \frac{P(i+1, j) - P(i, j)}{\Delta x} \]
\[ Z_x(i, j) = \frac{Z(i+1, j) - Z(i, j)}{\Delta x} \]

and

\[ A_{xx}(i, j) = \frac{A(i+1, j) - 2A(i, j) + A(i-1, j)}{(\Delta x)^2} \]
\[ P_{xx}(i, j) = \frac{P(i+1, j) - 2P(i, j) + P(i-1, j)}{(\Delta x)^2} \]
\[ Z_{xx}(i, j) = \frac{Z(i+1, j) - 2Z(i, j) + Z(i-1, j)}{(\Delta x)^2} \]

Hence, having the time derivatives and space derivatives in hand, the following difference equations with source terms will represent the discretization of the PDE model as

\[ \frac{A(i, j+1) - A(i, j)}{\Delta t} = D_1 \frac{A(i+1, j) - 2A(i, j) + A(i-1, j)}{(\Delta x)^2} - b_1 \frac{A(i+1, j) - A(i, j)}{\Delta x} \]
\[ + r_1 A(i, j) - \frac{r_1}{K_1} A(i, j)^2 + m_0 A(i, j) Z(i, j) - m_1 P(i, j) A(i, j) - h A(i, j), \]

\[ \frac{P(i, j+1) - P(i, j)}{\Delta t} = D_2 \frac{P(i+1, j) - 2P(i, j) + P(i-1, j)}{(\Delta x)^2} - b_2 \frac{P(i+1, j) - P(i, j)}{\Delta x} \]
\[ + r_2 P(i, j) - \frac{r_2}{K_2} P(i, j)^2 + m_2 A(i, j) P(i, j) + m_3 P(i, j) Z(i, j) - m_6 P(i, j), \]

\[ \frac{Z(i, j+1) - Z(i, j)}{\Delta t} = D_3 \frac{Z(i+1, j) - 2Z(i, j) + Z(i-1, j)}{(\Delta x)^2} - b_3 \frac{Z(i+1, j) - Z(i, j)}{\Delta x} \]
\[ + r_3 Z(i, j) - \frac{r_3}{K_3} Z(i, j)^2 - m_4 A(i, j) Z(i, j) - m_5 P(i, j) Z(i, j). \]

When we arrange the discretized version of the PDE model, we will obtain following one, which will help us to get the approximated solution of the PDE system together with BCs
and Initial Conditions:

\[ A(i, j + 1) = A(i, j) + D_1 \frac{\Delta t}{(\Delta x)^2} [A(i + 1, j) - 2A(i, j) + A(i - 1, j)] - b_1 \frac{\Delta t}{\Delta x} [A(i + 1, j) - A(i, j)] + [r_1 A(i, j) - \frac{r_1}{K_1} A(i, j)^2 + m_0 A(i, j) Z(i, j) - m_1 P(i, j) A(i, j) - h A(i, j)] \Delta t, \]

\[ P(i, j + 1) = P(i, j) + D_2 \frac{\Delta t}{(\Delta x)^2} [P(i + 1, j) - 2P(i, j) + P(i - 1, j)] - b_2 \frac{\Delta t}{\Delta x} [P(i + 1, j) - P(i, j)] + [r_2 P(i, j) - \frac{r_2}{K_2} P(i, j)^2 + m_2 A(i, j) P(i, j) + m_3 P(i, j) Z(i, j) - m_6 P(i, j)] \Delta t, \]

\[ Z(i, j + 1) = Z(i, j) + D_3 \frac{\Delta t}{(\Delta x)^2} [Z(i + 1, j) - 2Z(i, j) + Z(i - 1, j)] - b_3 \frac{\Delta t}{\Delta x} [Z(i + 1, j) - Z(i, j)] + [r_3 Z(i, j) - \frac{r_3}{K_3} Z(i, j)^2 - m_4 A(i, j) Z(i, j) - m_5 P(i, j) Z(i, j)] \Delta t. \]

**Approximation of the BCs:**

Now, we need to arrange the BCs in terms of difference equations as we did in previous part. we have homogeneous Neumann boundary conditions as

\[ \frac{\partial A}{\partial \eta} = 0, \quad \frac{\partial P}{\partial \eta} = 0, \quad \frac{\partial Z}{\partial \eta} = 0 \quad \text{on} \quad \partial(0, L) \times (0, T) \]

we can arrange the previous BCs as

\[ \frac{\partial A}{\partial \eta} = A_x \cdot \eta = 0, \quad \frac{\partial P}{\partial \eta} = P_x \cdot \eta = 0, \quad \frac{\partial Z}{\partial \eta} = Z_x \cdot \eta = 0 \quad \text{on} \quad \partial(0, L) \times (0, T) \]

where \( \eta \) is normal to the boundary \( \partial(0, L) \). At the boundary point \((0,t)\), the normal vector will get the value \( \eta = -1 \), and at the boundary point \((L,t)\), it takes, \( \eta = +1 \) so that
\[
\frac{\partial A}{\partial v} = A_x \cdot (-1) = 0, \quad \frac{\partial P}{\partial v} = P_x \cdot (-1) = 0, \quad \frac{\partial Z}{\partial v} = Z_x \cdot (-1) = 0 \quad \text{at} \ (0, t)
\]
\[
\frac{\partial A}{\partial \eta} = A_x = 0, \quad \frac{\partial P}{\partial \eta} = P_x = 0, \quad \frac{\partial Z}{\partial \eta} = Z_x = 0 \quad \text{at} \ (L, t)
\]

By using difference equations for \(i = 1\) and \(i = N\), which represent the boundary points \((0,t)\), and \((L,t)\), we get (by using forward difference equations)

\[
A_x(1,j) = \frac{A(1,j) - A(2,j)}{\Delta x} = 0
\]
\[
P_x(1,j) = \frac{P(1,j) - P(2,j)}{\Delta x} = 0 \quad \text{at} \ (0,t)
\]
\[
Z_x(1,j) = \frac{Z(1,j) - Z(2,j)}{\Delta x} = 0
\]

and using backward difference equations

\[
A_x(N,j) = \frac{A(N,j) - A(N-1,j)}{\Delta x} = 0
\]
\[
P_x(N,j) = \frac{P(N,j) - P(N-1,j)}{\Delta x} = 0 \quad \text{at} \ (L,t)
\]
\[
Z_x(N,j) = \frac{Z(N,j) - Z(N-1,j)}{\Delta x} = 0
\]

After arranging the previous equations, we obtain the following for BCs:

\[
A(1,j) = A(2,j)
\]
\[
P(1,j) = P(2,j) \quad \text{at} \ (0,t)
\]
\[
Z(1,j) = Z(2,j)
\]

and
\[ A(N, j) = A(N - 1, j) \]
\[ P(N, j) = P(N - 1, j) \] at \((L, t)\)
\[ Z(N, j) = Z(N - 1, j). \]

### B Backward Finite Difference Method for the Adjoint Variables with Explicit Scheme

First, we obtain the backward difference equations for corresponding time derivatives as follows. The space variable, \(x\) and the time variable, \(t\) are represented, by grid points, denoted by \(i\) and \(j\) respectively.

\[
(\lambda_1)_t(i, j) = \frac{\lambda_1(i, j) - \lambda_1(i, j - 1)}{\Delta t}
\]
\[
(\lambda_2)_t(i, j) = \frac{\lambda_2(i, j) - \lambda_2(i, j - 1)}{\Delta t}
\]
\[
(\lambda_3)_t(i, j) = \frac{\lambda_3(i, j) - \lambda_3(i, j - 1)}{\Delta t}
\]

Then, we get the difference equations for corresponding space derivatives as

\[
(\lambda_1)_x(i, j) = \frac{\lambda_1(i, j) - \lambda_1(i - 1, j)}{\Delta x}
\]
\[
(\lambda_2)_x(i, j) = \frac{\lambda_2(i, j) - \lambda_2(i - 1, j)}{\Delta x}
\]
\[
(\lambda_3)_x(i, j) = \frac{\lambda_3(i, j) - \lambda_3(i - 1, j)}{\Delta x}
\]

and

\[
(\lambda_1)_{xx}(i, j) = \frac{\lambda_1(i - 1, j) - 2\lambda_1(i, j) + \lambda_1(i + 1, j)}{(\Delta x)^2}
\]
\[
(\lambda_2)_{xx}(i, j) = \frac{\lambda_2(i - 1, j) - 2\lambda_2(i, j) + \lambda_2(i + 1, j)}{(\Delta x)^2}
\]
\[
(\lambda_3)_{xx}(i, j) = \frac{\lambda_3(i - 1, j) - 2\lambda_3(i, j) + \lambda_3(i + 1, j)}{(\Delta x)^2}
\]
Hence, having the time derivatives and space derivatives of the adjoint variable in hand, the following difference equations with source terms will represent the discretization of the adjoint equations as

$$\frac{\lambda_1(i, j) - \lambda_1(i, j - 1)}{\Delta t} = -D_1 \frac{\lambda_1(i - 1, j) - 2\lambda_1(i, j) + \lambda_1(i + 1, j)}{(\Delta x)^2} - b_1 \frac{\lambda_1(i, j) - \lambda_1(i - 1, j)}{\Delta x}
- h - \left[r_1 - 2A(i, j) \frac{r_1}{K_1} + m_0Z(i, j) - m_1P(i, j) - h - \alpha\right] \lambda_1 - m_2P(i, j)\lambda_2(i, j) + m_4Z(i, j)\lambda_3$$

$$\frac{\lambda_2(i, j) - \lambda_2(i, j - 1)}{\Delta t} = -D_1 \frac{\lambda_2(i - 1, j) - 2\lambda_2(i, j) + \lambda_2(i + 1, j)}{(\Delta x)^2} - b_1 \frac{\lambda_2(i, j) - \lambda_2(i - 1, j)}{\Delta x}
- \left[r_2 - 2P(i, j) \frac{r_2}{K_2} + m_2A(i, j) + m_3Z(i, j) - m_6 - \alpha\right] \lambda_2 + m_1A(i, j)\lambda_1(i, j) + m_5A(i, j)\lambda_3$$

$$\frac{\lambda_3(i, j) - \lambda_3(i, j - 1)}{\Delta t} = -D_1 \frac{\lambda_3(i - 1, j) - 2\lambda_3(i, j) + \lambda_3(i + 1, j)}{(\Delta x)^2} - b_1 \frac{\lambda_3(i, j) - \lambda_3(i - 1, j)}{\Delta x}
- \left[r_3 - 2Z(i, j) \frac{r_3}{K_3} + m_3A(i, j) - m_5P(i, j) - \alpha\right] \lambda_3 - m_0A(i, j)\lambda_1(i, j) - m_3P(i, j)\lambda_2$$

When we arrange the discretized version of the adjoint system, we will obtain following one, which will help us to get the approximated solution of the adjoint system together with BCs and Final Conditions:

$$\lambda_1(i, j - 1) = \lambda_1(i, j) + D_1 \frac{\Delta t}{(\Delta x)^2} \left[\lambda_1(i - 1, j) - 2\lambda_1(i, j) + \lambda_1(i + 1, j)\right] + b_1 \frac{\Delta t}{\Delta x} \left[\lambda_1(i, j) - \lambda_1(i - 1, j)\right]
+ \left[h + \left[r_1 - 2A(i, j) \frac{r_1}{K_1} + m_0Z(i, j) - m_1P(i, j) - h - \alpha\right] \lambda_1\right] \Delta t
+ \left[m_2P(i, j)\lambda_2(i, j) - m_4Z(i, j)\lambda_3\right] \Delta t$$

127
\[\lambda_2(i, j - 1) = \lambda_2(i, j) + D_1 \frac{\Delta t}{(\Delta x)^2} \left[ \lambda_2(i - 1, j) - 2\lambda_2(i, j) + \lambda_2(i + 1, j) \right] + b_1 \frac{\Delta t}{\Delta x} \left[ \lambda_2(i, j) - \lambda_2(i - 1, j) \right]
+ \left[ r_2 - 2P(i, j) \frac{r_2}{K_2} + m_2A(i, j) + m_3Z(i, j) - m_6 - \alpha \right] \lambda_2 \Delta t
+ \left[ m_1A(i, j)\lambda_1(i, j) + m_5A(i, j)\lambda_3 \right] \Delta t\]

\[\lambda_3(i, j - 1) = \lambda_3(i, j) + D_1 \frac{\Delta t}{(\Delta x)^2} \left[ \lambda_3(i - 1, j) - 2\lambda_3(i, j) + \lambda_3(i + 1, j) \right]
+ b_1 \frac{\Delta t}{\Delta x} \left[ \lambda_3(i, j) - \lambda_3(i - 1, j) \right]
+ \left[ r_3 - 2Z(i, j) \frac{r_3}{K_3} + m_3A(i, j) - m_5P(i, j) - \alpha \right] \lambda_3 \Delta t
\]

Approximation of the BCs for Adjoint Equations:

Now, we need to arrange the BCs in terms of difference equations as we did in previous part. Now, we have robin boundary conditions as

\[D_i(x, t) \frac{\partial \lambda_i}{\partial \eta} + (b_i(x, t) \cdot \eta) \lambda_i = 0, \quad \text{for } i = 1, 2, 3 \quad \text{on } \partial (0, L) \times (0, T)\]

The outward normal \( \eta = -1 \) at the boundary \((0, t)\), and \( \eta = +1 \) at the boundary \((L, t)\). By using backward difference method at the boundary point \((L, t)\), we will obtain the following form:
\[
D_p \left( \frac{\lambda_p(i, j) - \lambda_p(i - 1, j)}{\Delta x} \right) (1) + b_p (1) \lambda_p(i, j) = 0
\]

After arranging the equation, and taking \( i = N \), we obtain

\[
\lambda_p(N, j) = \left( \frac{D_p}{D_p + b_p dx} \right) \lambda_p(N - 1, j), \quad \text{for} \ p = 1, 2, 3 \quad \text{at} \ (L, t)
\]

Similarly, by using forward difference method at the boundary point \((0, t)\), we will get the following discretized form for \( i = 1 \):

\[
\lambda_p(1, j) = \left( \frac{D_p}{D_p - b_p dx} \right) \lambda_p(2, j), \quad \text{for} \ p = 1, 2, 3 \quad \text{at} \ (0, t)
\]

Thus, for \( i = 1 \), we obtain the following boundary conditions at \((0, t)\)

\[
\lambda_1(1, j) = \left( \frac{D_1}{D_1 - b_1 dx} \right) \lambda_1(2, j),
\]

\[
\lambda_2(1, j) = \left( \frac{D_2}{D_2 - b_2 dx} \right) \lambda_2(2, j), \quad \text{at} \ (0, t)
\]

\[
\lambda_3(1, j) = \left( \frac{D_3}{D_3 - b_3 dx} \right) \lambda_3(2, j),
\]

Similarly, for \( i = N \), we obtain the following boundary conditions at \((L, t)\)

\[
\lambda_1(N, j) = \left( \frac{D_1}{D_1 + b_1 dx} \right) \lambda_1(N - 1, j),
\]

\[
\lambda_2(N, j) = \left( \frac{D_1}{D_2 + b_2 dx} \right) \lambda_2(N - 1, j), \quad \text{at} \ (L, t)
\]

\[
\lambda_3(N, j) = \left( \frac{D_1}{D_3 + b_3 dx} \right) \lambda_3(N - 1, j).
\]
Appendix B

Data of the Black Sea Anchovy

In the following Table B.1, the annual landing data (total tonnes) and the effort (the total number of fleets used in a fishing season) of the Black Sea anchovy fishing on the southern part of the Black Sea is given. We also give the catch per unit effort (CPUE) and the numbers of fishing fleets (efforts) in the Table B.1 on the the southern part. The fishing fleets are used daily during the each fishing season. The CPUE denotes the average amount of anchovy landing taken from a single fleet, which was calculated by $\frac{\text{Landing}}{\text{Effort}}$. These data are obtained from Scientific, Technical and Economic Committee for Fishery (STECF (2017)).
Table B.1: The effort (number of fishing fleets), catch per unit effort (CPUE), and landing of the Black Sea anchovy in Turkish coast of the Black Sea (STECF, 2017)

<table>
<thead>
<tr>
<th>Years</th>
<th>Effort (# of fleets)</th>
<th>CPUE</th>
<th>Landing Effort</th>
<th>Landing (Tonnes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>104</td>
<td>2301</td>
<td></td>
<td>239,289</td>
</tr>
<tr>
<td>1981</td>
<td>121</td>
<td>2143</td>
<td></td>
<td>259,767</td>
</tr>
<tr>
<td>1982</td>
<td>145</td>
<td>1838</td>
<td></td>
<td>266,523</td>
</tr>
<tr>
<td>1983</td>
<td>162</td>
<td>1789</td>
<td></td>
<td>289,860</td>
</tr>
<tr>
<td>1984</td>
<td>171</td>
<td>1865</td>
<td></td>
<td>318,917</td>
</tr>
<tr>
<td>1985</td>
<td>195</td>
<td>1401</td>
<td></td>
<td>273,274</td>
</tr>
<tr>
<td>1986</td>
<td>210</td>
<td>1308</td>
<td></td>
<td>274,740</td>
</tr>
<tr>
<td>1987</td>
<td>229</td>
<td>1292</td>
<td></td>
<td>295,902</td>
</tr>
<tr>
<td>1988</td>
<td>247</td>
<td>1194</td>
<td></td>
<td>295,000</td>
</tr>
<tr>
<td>1989</td>
<td>262</td>
<td>369</td>
<td></td>
<td>96,806</td>
</tr>
<tr>
<td>1990</td>
<td>280</td>
<td>237</td>
<td></td>
<td>66,409</td>
</tr>
<tr>
<td>1991</td>
<td>284</td>
<td>279</td>
<td></td>
<td>79,225</td>
</tr>
<tr>
<td>1992</td>
<td>163</td>
<td>953</td>
<td></td>
<td>155,417</td>
</tr>
<tr>
<td>1993</td>
<td>287</td>
<td>764</td>
<td></td>
<td>218,866</td>
</tr>
<tr>
<td>1994</td>
<td>243</td>
<td>1147</td>
<td></td>
<td>278,667</td>
</tr>
<tr>
<td>1995</td>
<td>262</td>
<td>1427</td>
<td></td>
<td>373,782</td>
</tr>
<tr>
<td>1996</td>
<td>278</td>
<td>983</td>
<td></td>
<td>273,239</td>
</tr>
<tr>
<td>1997</td>
<td>248</td>
<td>862</td>
<td></td>
<td>213,780</td>
</tr>
<tr>
<td>1998</td>
<td>209</td>
<td>938</td>
<td></td>
<td>195,996</td>
</tr>
<tr>
<td>1999</td>
<td>199</td>
<td>1562</td>
<td></td>
<td>310,601</td>
</tr>
<tr>
<td>2000</td>
<td>262</td>
<td>995</td>
<td></td>
<td>260,069</td>
</tr>
<tr>
<td>2001</td>
<td>299</td>
<td>965</td>
<td></td>
<td>288,616</td>
</tr>
<tr>
<td>2002</td>
<td>419</td>
<td>803</td>
<td></td>
<td>336,419</td>
</tr>
<tr>
<td>2003</td>
<td>473</td>
<td>563</td>
<td></td>
<td>266,069</td>
</tr>
<tr>
<td>2004</td>
<td>388</td>
<td>790</td>
<td></td>
<td>306,656</td>
</tr>
<tr>
<td>2005</td>
<td>407</td>
<td>240</td>
<td></td>
<td>119,255</td>
</tr>
<tr>
<td>2006</td>
<td>428</td>
<td>496</td>
<td></td>
<td>212,081</td>
</tr>
<tr>
<td>2007</td>
<td>473</td>
<td>755</td>
<td></td>
<td>357,089</td>
</tr>
<tr>
<td>2008</td>
<td>566</td>
<td>398</td>
<td></td>
<td>225,344</td>
</tr>
<tr>
<td>2009</td>
<td>483</td>
<td>384</td>
<td></td>
<td>185,606</td>
</tr>
<tr>
<td>2010</td>
<td>409</td>
<td>496</td>
<td></td>
<td>203,026</td>
</tr>
<tr>
<td>2011</td>
<td>384</td>
<td>642</td>
<td></td>
<td>246,390</td>
</tr>
<tr>
<td>2012</td>
<td>339</td>
<td>322</td>
<td></td>
<td>109,187</td>
</tr>
<tr>
<td>2013</td>
<td>197</td>
<td>1296</td>
<td></td>
<td>255,309</td>
</tr>
<tr>
<td>2014</td>
<td>195</td>
<td>367</td>
<td></td>
<td>71,530</td>
</tr>
<tr>
<td>2015</td>
<td>186</td>
<td>1050</td>
<td></td>
<td>195,350</td>
</tr>
<tr>
<td>2016</td>
<td>198</td>
<td>568</td>
<td></td>
<td>112,500</td>
</tr>
</tbody>
</table>
Vita

Mahir Demir was born in Erzurum, Turkey, on April 1, 1985. After graduating from Şentepe High School in 2003, he went on to Inonu University in Turkey, where he studied mathematics. He received his Bachelor of Arts in May 2009. In August 2009, Mahir began his graduate career at Adiyaman University, in the Department of Mathematics Education. After graduating with Master Degree from Adiyaman University, he went to Gaziantep University, where he got his second master’s degree in the Department of Mathematics. In 2014, he began his Ph.D. work in the University of Tennessee, Knoxville, and was supported by a graduate teaching assistantship from the Department of Mathematics. Mahir received his third master’s degree in the Department of Mathematics in 2017, and then graduated in May 2019 with his Ph.D. in Mathematics with a concentration in Mathematical Ecology as well as having two Ph.D. minors in Computational Science and in Statistics.