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# Asymptotics for Brownian motion in Poisson potential with Riesz kernel and in time-independent Gaussian rough noise

Bo Gao

*University of Tennessee*, bgao3@vols.utk.edu

# Asymptotics for Brownian motion in Poisson potential with Riesz kernel and in time-independent Gaussian rough noise

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Bo Gao

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*This dissertation is dedicated to my beloved parents Liucheng Gao, Weihong Zhang, my dearest wife Xuan Han and my sweet daughter Abigail Gao.*

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# Abstract

In this dissertation, we consider the long time asymptotics for Brownian motion in Poisson random medium or time-independent Gaussian rough medium. We first give answers of the exponential moment asymptotics of Brownian motion in Poisson random medium of a critical case under both quenched and annealed regimes. We then investigate the light tailed case under quenched regime. Finally, we study the quenched large-$t$ asymptotic of Brownian motion in a time-independent Gaussian rough noise.

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# Chapter 1

# Introduction

Random Motions in Random Media (RMRM) are an important subject in Probability theory due to a wide range of applications to real-world problems, including astrophysics, oceanography, chemical reactions, physical theories, statistical mechanics and electrical networks [14, 20, 25]. Because of these and also its connection with stochastic heat equation, they have been studied of much interest and efforts over the last 20 years.

It's well known that RMRM links to the following stochastic heat eqution, called parabolic Anderson model (PAM),

$$\begin{cases} \partial_t u(t,x) = Au(t,x) + V(x)u(t,x), & (t,x) \in [0,\infty) \times \mathbb{R}^d; \\ u(0,x) = 1, & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

where $A$ is the infinitesimal generator of a d-dimensional Markov process and $\{V(x); x \in \mathbb{R}^d\}$ is a homogeneous random potential playing a role of random media that derives the equation.

It's also well known that, under some regularity assumption such as Hölder continuity on $V(x)$, (1.1) has a unique solution with the following Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x \exp\Big\{ \int_0^t V(X_s)ds \Big\}, \tag{1.2}$$

where $\{X_t; t \geq 0\}$ is a Markov process on $\mathbb{R}^d$ with infinitesimal generator $A$ given in (1.1), independent of the random media $\{V(x); x \in \mathbb{R}^d\}$ and $\mathbb{E}_x$ is the expectation with respect to $X_t$ with $X_0 = x$.

From the Feynman-Kac representation (1.2) we can see that, in order to understand the solution of (1.1), we have to understand interaction between the evolution of the random motion $\{X_t; t \geq 0\}$ and the random media $\{V(x); x \in \mathbb{R}^d\}$ where it stays in. This immediately links PAM to RMRM. The studies on $u(t,x)$ are of much concerns due to their strong connections with many areas, such as physics, population dynamics and partial differential equation with random coefficients. Let's take the model of population dynamics as an example. We show that how this model is described by the system (1.1). Imagine that initially or at time $t = 0$, the population is evenly distributed at the space $\mathbb{R}^d$. Let $u(t,x)$ represent the population density at the time $t$ and the site $x$. The individuals will move from regions of high concentration to regions of low concentration with the direction of flux given by $-\frac{1}{2}\nabla u(t,x)$ at time $t$ and site $x$ (Fick's law). Also, at the site $x$, the birth and death rate of the population at time $t$ are given by $V_+(t,x)$ and $V_-(t,x)$, respectively. Put $V(t,x) = V_+(t,x) + V_-(t,x)$. Then given a nice bounded domain $D \subset \mathbb{R}^d$, the change rate of the population in $D$ will be

$$\frac{d}{dt}\int_D u(t,x)dx = \frac{1}{2}\int_{\partial D} (\nabla u(t,x) \cdot \vec{n})dS + \int_D V(t,x)u(t,x)dx$$

where the first part of the right side is the surface integral over the boundary of the domain $D$ and $\vec{n} = \vec{n}(x)$ is the outward pointing unit normal field of the boundary $\partial D$. The above equation illustrates the fact that the population is changing due to death and birth or migration. Notice that by divergence theorem,

$$\int_{\partial D} (\nabla u(t,x) \cdot \vec{n}) dS = \int_D \Delta u(t,x) dx.$$

Therefore, combining the above results, we have

$$\frac{d}{dt} \int_D u(t,x)dx = \frac{1}{2} \int_D \Delta u(t,x)dx + \int_D V(t,x)u(t,x)dx$$

which leads to the Anderson equation

$$\partial_t u(t,x) = \frac{1}{2} \Delta u(t,x) + V(x)u(t,x)$$

with initial condition given by $u(0,x) = 1$. This instantly connects $u(t,x)$, the solution of the system (1.1), to the model of population dynamics.

Motivated by real-world applications given above, one of our major objective in this paper is to investigate the long time behavior of $u(t,x)$ with random motion $X_t$ given as Brownian motion and random media $V(x)$ be defined later in section 2.2 and 2.3. Through out this paper, we denote $\mathbb{P}$ and $\mathbb{E}$ as the law and expectation used for the random media, respectively. Meanwhile we denote $\mathbb{P}_x$ and $\mathbb{E}_x$ as the law and expectation used for the random motion $X_t$ starting at x, respectively.

On the other hand, the model of RMRM is often introduced as the random Gibbs measure which is of great interest in statistical mechanics. The generalized model can be set up as follows. Consider a particle doing a random movement in the space $\mathbb{R}^d$. The trajectory of the particle is described by a d-dimensional Markov process $\{X_t; t \geq 0\}$. Independently, the space $\mathbb{R}^d$ is filled with a random media $\{V(x); x \in \mathbb{R}^d\}$. Due to the two kind of randomness from the random motion and the random media respectively, the model can be considered in two different settings. In the quenched setting, where the random energy is conditioned on random media, the model is formulated in terms of the random Gibbs measure $\mu_{t,\omega}$:

$$\frac{d\mu_{t,\omega}}{d\mathbb{P}_x} = \frac{1}{Z_{t,\omega}} \exp\Big\{ \int_0^t V(X_s)ds \Big\}. \tag{1.3}$$

In the annealed setting, where the energy is obtained by averaging both the random motion and the random media, the Gibbs measure $\mu_t$ is given by

$$\frac{d\mu_t}{d(\mathbb{P}_x \times \mathbb{P})} = \frac{1}{Z_t} \exp\Big\{ \int_0^t V(X_s)ds \Big\}. \tag{1.4}$$

Notice that, in order to make Gibbs measure $\mu_{t,\omega}$ and $\mu_t$ probability measures, it has to be that

$$Z_{t,\omega} = \mathbb{E}_x \exp\Big\{ \int_0^t V(X_s)ds \Big\}$$

and

$$Z_t = \mathbb{E}_x \otimes \mathbb{E} \exp\Big\{ \int_0^t V(X_s)ds \Big\}.$$

Research on large-$t$ asymptotics for the above two kind of exponential moments have become very active in the last two decades. For example, the models of Brownian motion in Poisson random media have been studied a lot in literature. We point out [1, 2, 3, 4, 11, 12, 13, 16, 17, 18, 31, 32] as a partial list of the publications related to this topic. Also, we refer the readers to [24] and [33] for background, motivation and fundamental results on this subject.

In the usual set up, the Poisson random media is defined as follows

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy) \tag{1.5}$$

where $\omega(dx)$ is a Poisson field with Lebesgue measure $dx$ as its intensity measure and $K(x) \geq 0$ is a deterministic function and known as the shape function. Typically, the shape function $K(x)$ is assumed to be bounded and compactly supported or chosen as $\frac{1}{|x|^p} \wedge 1$ where $p > d$, so that the random media function $V(x)$ can be well defined.

We mention some early studies for this case. In 1975, Donsker and Varadhan [13] showed the annealed large-$t$ asymptotics when $K(x) = \frac{1}{|x|^p} \wedge 1$ with $p > d + 2$,

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E} \otimes \mathbb{E}_x \exp\left\{-\int_0^t V(B_s)ds\right\} = -\frac{d+2}{2} w_d^{\frac{2}{d+2}} \left(\frac{2\lambda_d}{d}\right)^{\frac{d}{d+2}} \qquad a.s.\text{-}\mathbb{P} \tag{1.6}$$

where $w_d$ is the volume of the unit ball and $\lambda_d$ denotes the principle eigenvalue of $-\frac{1}{2}\Delta$ on the d-dimensional unit ball with zero boundary values.

Meanwhile, when $d < p < d + 2$, Pastur [28] determined the long time annealed asymptotics,

$$\lim_{t\to\infty} t^{-\frac{d}{p}} \log \mathbb{E} \otimes \mathbb{E}_x \exp\left\{-\int_0^t V(B_s)ds\right\} = -w_d \Gamma\left(\frac{p-d}{p}\right) \qquad a.s.\text{-}\mathbb{P}. \tag{1.7}$$

Later, under the large deviation strategies developed in [13], Ôkura [26] proved the annealed exponential moment asymptotics for the critical case $p = d + 2$,

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E} \otimes \mathbb{E}_x \exp\left\{-\int_0^t V(B_s)ds\right\} = -C \tag{1.8}$$

where $C > 0$ is a constant.

Results of quenched setting appeared much later. In 1993, Sznitman [31] proved that when shape function $K(\cdot)$ is bounded and compactly supported,

$$\lim_{t\to\infty} t^{-1}(\log t)^{\frac{2}{d}} \log \mathbb{E}_x \exp\left\{-\theta \int_0^t V(B_s)ds\right\} = -\lambda_d \left(\frac{\omega_d}{d}\right)^{2/d} \qquad a.s.\text{-}\mathbb{P}. \tag{1.9}$$

Notice that this case can be also seen as the case of $p = \infty$.

Recently, Fukushima [15] gave the quenched long time asymptotics for $K(x) = \frac{1}{|x|^p} \wedge 1$ with $d < p < d+2$,

$$\lim_{t\to\infty} t^{-1}(\log t)^{\frac{p-d}{d}} \log \mathbb{E}_x \exp\left\{-\theta \int_0^t V(B_s)ds\right\}$$
$$= -\frac{d\theta}{p}\left(\frac{p-d}{pd}\right)^{\frac{p-d}{d}} \left(w_d \Gamma\left(\frac{p-d}{d}\right)\right)^{\frac{p}{d}} \qquad a.s.\text{-}\mathbb{P}. \tag{1.10}$$

Under $d/2 < p < d$, Chen [8] proved the quenched large-$t$ asymptotics of renormalized exponential moment,

$$\lim_{t\to\infty} t^{-1}(\log t)^{-\frac{d-p}{d}} \log \mathbb{E}_x \exp\left\{-\theta \int_0^t \overline{V}(B_s)ds\right\}$$
$$= \frac{\theta d^2}{d-p}\left(\frac{w_d}{d}\Gamma\left(\frac{2p-d}{p}\right)\right)^{\frac{p}{d}} \qquad a.s.\text{-}\mathbb{P}. \tag{1.11}$$

where the renormalized Poisson potential:

$$\overline{V}(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^p}[\omega(dy) - dy].$$

It should be mentioned that $V(x)$ will blow up under the choice of $p \leq d$. To make it well defined, we should renormalize it [12]. Another important point is Chen [8] obtained the long time asymptotics result without removing the singularity of the kernel function $K(x)$ at $x = 0$.

Motivated by Chen [8] and the examples given above, we are interested in investigating the above RMRM models without removing the singularity of the kernel function at the original. In particular, we ask the following questions: Whether (1.8) and (1.10) remain true when $K(x) = \frac{1}{|x|^p}$ with $p = d+2$ and $d < p < d+2$, respectively? The answer is "Yes" for the annealed setting, but "No" for the quenched setting. We refer the readers to [1] and Remark 5.8 in [24] for the reasons of "No". Also we are interested in studying the quenched long time asymptotic for the critical case $p = d + 2$. To the best of the author's knowledge, these results have not been provided before.

Another interesting case of RMRM is to take random media $V$ as a Gaussian field. This model has been investigated of great interest by researchers in recent two decades. We give an incomplete list on this topic [4, 6, 9, 16, 17, 18, 21, 22, 30]. We point out the reference [24] for the overview and background of this subject. We mention some early studies related to this topic. In 1995, Carmona and Molchanov [4] discovered a quenched long time asymptotics

$$\lim_{t\to\infty} \frac{1}{t\sqrt{\log t}} \log \mathbb{E}_x \exp\left\{ \int_0^t V(B_s)ds \right\} = \sqrt{2d\gamma(0)}, \qquad a.s. \tag{1.12}$$

where $\{V(x); x \in \mathbb{R}^d\}$ is a mean zero homogeneous Gaussian field with covariance structure given by

$$\gamma(x) = Cov(V(0), V(x)), \quad x \in \mathbb{R}^d$$

and

$$\lim_{|x|\to\infty} \gamma(x) = 0.$$

Later, Chen [9] analyzed quenched exponential moments under the setting of generalized Gaussian field which is rougher than that of [4]. In particular, Chen [9] considered the case when the Gaussian potential is a white noise given by $V(x) = \dot{W}(x)(x \in \mathbb{R})$, where $W(x)$ is a Brownian sheet with Hurst index $H = \frac{1}{2}$. In this case, the covariance function $\gamma(x) = \delta_0(x)$ is the Dirac function and the following almost sure asymptotic is given for any $x \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{t(\log t)^{2/3}} \log \mathbb{E}_x \exp\left\{ \theta \int_0^t V(B_s)ds \right\} = c_\theta, \qquad a.s. \tag{1.13}$$

where $c_\theta > 0$ is an explicit constant.

Motivated by the above works, we want to ask the following question: Do we have the similar asymptotic result as in (1.12) and (1.13) if we make the PAM under much less regularity assumptions, or let the Gaussian environment even rougher? Specifically, we consider a Brownian sheet as in [9], but allow the Hurst index to be less than $\frac{1}{2}$. We will investigate quenched large-$t$ asymptotic of $u(t,x)$ in this case.

The rest of this paper is organized as follows. In Chapter 2, we describe the random medias to be considered in our model in details and present the main results in this paper. In Chapter 3, we provide the proof of quenched asymptotic of the critical case $p = d + 2$. The large deviations for a group of Poisson integrals with respect to the enlarged Poisson field are also given. In Chapter 4, we give the proof of annealed asymptotic of the critical case $p = d+2$. Comparing with the similar result in [26], we include the singularity of the kernel function at the original. In Chapter 5, we investigate the quenched regime of the light tailed case $p > d + 2$ and present the proof of the corresponding long time asymptotic. In Chapter 6, we discuss the quenched large-$t$ asymptotic of Brownian motion in time-independent Gaussian rough noise. Some proof of useful Lemmas are included in Appendix.

# Chapter 2

# Main theorems

In this chapter, main theorems will be presented. We first introduce some notations which will be used throughout this paper. The Possion potential and the Gaussian potential which serve as the random medias in our models will also be introduced. Finally, we state the main results of this paper.

## 2.1 Notation

$\mathbb{P}$ and $\mathbb{E}$ are the law and expectation used for the random media, respectively. Meanwhile $\mathbb{P}_x$ and $\mathbb{E}_x$ are the law and expectation used for Brownian motion starting at x, respectively.

$\mathbb{Z}_+$ is the set of all positive integers. $\mathbb{R}^d$ is the Euclidean space, where $d \in \mathbb{Z}_+$. $\mathcal{B}(\mathbb{R}^d)$ is the Borel algebra on $\mathbb{R}^d$. $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing and infinitely smooth functions. Through out this paper, $D \subset \mathbb{R}^d$ is a fixed and bounded open set. $\mathcal{L}^2(D)$ is $L^2$ space on $D$ with Lebesgue measure. $W^{1,2}(D)$ is the Sobolev space over $D$, defined to be the subset of the functions $f$ in $\mathcal{L}^2(D)$ such that the function $f$ and its weak derivative $|\nabla f|$ are in $\mathcal{L}^2(D)$ under the Sobolev norm

$$\|f\|_{1,2} = (\|f\|^2_{\mathcal{L}^2(D)} + \|\nabla f\|^2_{\mathcal{L}^2(D)})^{1/2}.$$

Denote

$$\mathcal{F}_d(D) = \Big\{ f \in W^{1,2}(\mathbb{R}^d);\ \int_D f^2(x)dx = 1 \Big\},$$

and

$$\mathcal{G}_d(D) = \Big\{ f \in \mathcal{S}(\mathbb{R}^d);\ \int_D f^2(x)dx + \frac{1}{2}\int_D |\nabla f(x)|^2 dx = 1 \Big\}.$$

For simplicity, write $\mathcal{F}_d = \mathcal{F}_d(\mathbb{R}^d)$ and $\mathcal{G}_d = \mathcal{G}_d(\mathbb{R}^d)$. The following two functions

$$\varphi(x) = 1 - e^{-x} \qquad \text{and} \qquad \psi(x) = e^x - 1 \tag{2.1}$$

appear frequently throughout this paper, where $x \geq 0$. It is obvious that they are nonnegative, increasing and convex functions.

For any $R > 0$, $Q_R = (-R, R)^d$ is the open box. $B(x, R)$ is the ball of radius $R$ centering at $x$. $\omega_d$ is the volume of the unit ball of $\mathbb{R}^d$.

## 2.2 The Poisson random media

Let Poisson field $\omega(dx)$ denote the distribution of the random obstacles in the space $\mathbb{R}^d$ with the Lebesgue measure $dx$ as its intensity measure which is specified below.

**Definition 2.2.1.** *(Possion Point Process). A Possion Point Process $\omega(\cdot)$ with d-dimensional Lebesgue measure as its intensity measure can be seen as a Poisson random measure satisfying:*

1. *For any disjoint sets $A_1, A_2, ... \in \mathcal{B}(\mathbb{R}^d)$, $\omega(A_1), \omega(A_2), ...$ are independent.*

2. *For each $A \in \mathcal{B}(\mathbb{R}^d)$, $\omega(A) \sim Possion(|A|)$, that is*

$$\mathbb{P}(\omega(A) = k) = e^{-|A|}\frac{|A|^k}{k!},$$

*where $|\cdot|$ represents the volume.*

The Poisson random media defined by

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy) \tag{2.2}$$

represents the total trapping energy at $x \in \mathbb{R}^d$ generated by Poisson obstacles, where $K(x)$ is known as the shape function. We shall consider two different kinds of shape functions in quenched setting and annealed setting, respectively. In quenched setting, our shape function is defined by

$$K(x) = \frac{1}{|x|^p} \wedge 1 \tag{2.3}$$

which excludes the singularity of $K(x)$ at $x = 0$. In annealed setting, our shape function is defined by

$$K(x) = \frac{1}{|x|^{d+2}} \tag{2.4}$$

which includes the singularity of $K(x)$ at $x = 0$.

**Remark 1.** *1. The Poisson potential $V(x)$ is well defined under $p > d$. We refer the readers to [29] for more details.*

*2. By the space homogeneity of the Poisson field, we have $V(x) \overset{d}{=} V(0)$ for every $x \in \mathbb{R}^d$.*

## 2.3 The Gaussian random media

We assume that V=$\{\dot{W}(x); x \in \mathbb{R}\}$ is a time-independent centered Gaussian noise. More specifically, the Gaussian random media $V$ can be seen as the derivative of a centered Gaussian process $W$ whose covariance is given by

$$\mathbb{E}[W(x)W(y)] = \frac{1}{2}(|x|^{2H} + |y|^{2H} - |x-y|^{2H}), \tag{2.5}$$

where $H < \frac{1}{2}$. Obviously, $W$ is a fractional Brownian motion with Hurst parameter $H < 1/2$. Notice that the Gaussian random media $V$ is a generalized random field which is not defined pointwise. We define it as a linear functional on test function space $\mathcal{S}(\mathbb{R})$, the Schwartz space. Specifically,

$$\langle V, \varphi \rangle = \int_{\mathbb{R}} V(x)\varphi(x)dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}).$$

For more details about generalized functions, see [19].

**Remark 2.** *1. $\langle V, \varphi \rangle$ is homogenous in space, that is*

$$\langle V, \varphi(\cdot - x) \rangle \overset{d}{=} \langle V, \varphi \rangle, \qquad x \in \mathbb{R} \text{ and } \varphi \in \mathcal{S}(\mathbb{R}).$$

*2. $\langle V, \varphi \rangle$ is a mean-zero Gaussian field with*

$$Cov\left(\langle V, \varphi \rangle, \langle V, \phi \rangle\right) = \int_{\mathbb{R}} \mathcal{F}\varphi(\lambda)\overline{\mathcal{F}\phi(\lambda)}\mu(d\lambda), \tag{2.6}$$

*where $\mathcal{F}\varphi(\lambda)$ denotes the Fourier transform of the function $\varphi \in \mathcal{S}(\mathbb{R})$, that is*

$$\mathcal{F}\varphi(\lambda) = \int_{\mathbb{R}} e^{i\lambda x}\varphi(x)dx,$$

*and where we have*

$$\mu(d\lambda) = c_H|\lambda|^{1-2H}d\lambda, \text{ and } \quad c_H = \frac{1}{2\pi}\Gamma(2H+1)sin(\pi H). \tag{2.7}$$

*3. It also should be mentioned that the variance of our noise $V$ has an alternate representation other than (2.6). By (2.8) in [21], we have*

$$\mathbb{E}\left[\langle V, \varphi \rangle\right]^2 = c_H \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\varphi(x+y) - \varphi(x)|^2}{|y|^{2-2H}}dxdy. \tag{2.8}$$

Notice that $\mu(d\lambda)$ in (2.6) is a tempered measure satisfies

$$\int_{\mathbb{R}} \frac{1}{1+|\lambda|^{2(1-\delta)}}\mu(d\lambda) < \infty \tag{2.9}$$

for some $\delta < H$. Consequently, by Lemma A.1 in [9], the $\mathcal{L}^2$-limit

$$\int_0^t V(B_s)ds \overset{def}{=} \lim_{\varepsilon\to 0^+}\int_0^t V_\varepsilon(B_s)ds \tag{2.10}$$

exists for every $t \geq 0$, where the Gaussian field $V_\varepsilon$ is a smooth version of $V$ and it will be specified later. In addition, the integral defined above is continuous in $t$. As a result, the Feynman-Kac representation (1.2) make sense in our setting.

We denote by $\gamma$ the Fourier transform of the measure $\mu(d\lambda)$, that is

$$\gamma(x) = \int_{\mathbb{R}} e^{i\lambda x}|\lambda|^{1-2H}d\lambda. \tag{2.11}$$

Obviously, $\gamma(x)$ is not defined pointwise. We also define it as a linear functional on test function space $\mathcal{S}(\mathbb{R})$. In addition, by (2.6) $c_H\gamma(x)$ is the covariance function of the Gaussian rough noise $V$. Further, $\gamma$ is non-negative definite in the sense that

$$\int_{\mathbb{R}^2}\gamma(x-y)\varphi(x)\varphi(y)dxdy = \int_{\mathbb{R}}|\mathcal{F}\varphi(\lambda)|^2\mu(d\lambda) \geq 0, \qquad \varphi\in\mathcal{S}(\mathbb{R}).$$

## 2.4 Main theorems

In this section, we present main theorems. We first consider the model of Brownian motion in Poisson random potential. The following theorem tells a story of the light tailed case: $p > d+2$.

**Theorem 2.1.** *Under $p > d+2$, for every $\theta > 0$ we have*

$$\lim_{t\to\infty} t^{-1}(\log t)^{\frac{2}{d}}\log\mathbb{E}_0\exp\left\{-\theta\int_0^t V(B_s)ds\right\} = -\lambda_d\left(\frac{\omega_d}{d}\right)^{2/d} \qquad a.s.\text{-}\mathbb{P} \tag{2.12}$$

*where random media $V(x)$ is defined under (2.3) and $\lambda_d$ denotes the principle Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in $B(0,1)$.*

**Remark 3.** *If the random potential $V$ is stationary in space, then also $u(t,\cdot)$, the solution of the system (1.1), is a stationary field for any $t$. As a result, the asymptotic behavior of $u$ will not depend on where does the random motion start. For sake of simplicity, we will thus consider $x = 0$ throughout this paper.*

**Remark 4.** *We refer to $p > d+2$ as the light tailed case. The first observation of our result is that the constant on the right hand side doesn't depend on the choice of $\theta$. It reveals that although values of the quenched exponential moment on the left hand side depend on each configuration $\omega$ of the Possion random media, the result is almost surely not affected by the randomness of the random potential $V$. The second observation is that we get the same asymptotic result as in (1.9) which can be seen as the case of $p = \infty$. In both cases, Brownian motion will try to avoid Poisson obstacles without any tolerance when configuration $\omega$ is fixed.*

**Remark 5.** *We will apply Sznitman's "empty ball" strategy [31] to prove theorem 2.1. That is, up to time $t$ the Brownian particle stays in the box with radius roughly equal to $t$. Inside the box, there are at least one ball with radius $\sim r(\log t)^{1/d}$ which is Poisson obstacles free. Then the exact lower bound will be obtained by forcing the Brownian particle run a roughly "straight" line into the ball where Poisson obstacles are free.*

The next theorem exhibits the long time behavior of exponential moment in the critical case under quenched setting.

**Theorem 2.2.** *Under $p = d+2$, for every $\theta > 0$ we have*

$$\lim_{t\to\infty} t^{-1}(\log t)^{\frac{2}{d}} \log \mathbb{E}_0 \exp\left\{-\theta \int_0^t V(B_s)ds\right\} = -\frac{d}{d+2}\left(\frac{2}{d(d+2)}\right)^{\frac{2}{d}} \rho(d,\theta)^{\frac{d+2}{d}} \qquad a.s.\text{-}\mathbb{P} \tag{2.13}$$

*where random media $V(x)$ is defined under (2.3) and,*

$$\rho(d,\theta) = \inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}. \tag{2.14}$$

*where $\varphi$ is given in (2.1).*

**Remark 6.** *Compared with the light tailed case, the constant we obtained in the critical case is highly influenced by the structure of Poisson random media. In this case, Brownian particle tends to keep away from Poisson obstacles but with some endurance. Due to that, "empty ball" strategy developed by Sznitman [31] can no longer be applied here. Therefore, we need to find alternative strategy to deal with this case.*

*Our proof of the quenched large-$t$ asymptotic for the critical case roughly relies on the following relationship:*

$$E_0\exp\left\{-\theta\int_0^t V(B_s)ds\right\}\approx\exp\{t\lambda_{(-\theta V)}(Q_{R_t})\}, \tag{2.15}$$

*where $\lambda_{(-\theta V)}(Q_{R_t})$ is the principle eigenvalue of the operator $\frac{1}{2}\Delta-\theta V$ with zero boundary condition in $\mathcal{L}^2(Q_{R_t})$, and radius $R_t$ is nearly linear and grows a bit slower than linear in our setting.*

*Another important strategy is that we rescale the Poisson field and reduce the problem to investigate the "enlarge obstacles":*

$$\omega((\log t)dx).$$

*This idea originally comes from Sznitman [31]. Notice that the choice of the rescaling factor $(\log t)^{1/d}$ links to the exponential decay rate of exponential moment in (2.13).*

In [26], Ôkura proved the annealed exponential moment asymptotic for the critical case $p=d+2$. The following theorem tells a slightly different story by including the singularity of the kernel function $K(x)$ at $x=0$.

**Theorem 2.3.** *For any $\theta > 0$,*

$$\lim_{t\to\infty} t^{-\frac{d}{d+2}} \log \mathbb{E}\otimes\mathbb{E}_0 \exp\left\{-\theta\int_0^t V(B_s)ds\right\}$$
$$= -\inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\} \tag{2.16}$$

*where random media $V(x)$ is defined under (2.4).*

**Remark 7.** *Comparing the results of quenched setting (2.13) and annealed setting (2.16), we have the following observation. It has a faster decay rate of $\exp\{-c\frac{t}{(\log t)^{2/d}}\}$ in quenched setting comparing with that of $\exp\{-ct^{d/(d+2)}\}$ in annealed setting. The reason lies in the following inequality*

$$\mathbb{E}\log u(t,x) \leq \log \mathbb{E}u(t,x)$$

*where $u(t,x)$ represents the solution to the system (1.1).*

**Remark 8.** *By comparing the result of Ôkura [26] and ours, we find out that whether removing the singularity of the kernel function or not will not affect either the exponential decay rate of the exponential moment or the constant.*

Next, we consider the model of Brownian motion in Gaussian rough noise. In the next theorem, we take dimension of the space to be 1, i.e. $d = 1$. We also take the random media

$$V = \dot{W}, \tag{2.17}$$

which is defined in the previous section.

**Theorem 2.4.** *Assume (2.17),*

$$\lim_{t\to\infty} t^{-1}(\log t)^{-\frac{1}{1+H}} \log \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\} = 2^{\frac{1}{1+H}} c_H^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}} \qquad a.s.$$

*where*

$$\mathcal{E} = \sup_{g \in \mathcal{G}_1} \left\{ \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x) dx \right|^2 |\lambda|^{1-2H} d\lambda \right\}. \tag{2.18}$$

**Remark 9.** *The first observation is the constant we obtained here is greatly influenced by the structure of the Gaussian noise. Secondly, the growth rate of* $u(t,0)$ *is* $\exp\{ct(\log t)^{1/(1+H)}\}$ *which is related to the Gaussian structure. We can see the rougher the Gaussian environment, the faster the exponential growth rate.*

# Chapter 3

# Quenched Asymptotic: p=d+2

In this chapter, we first establish the large deviations for a group of Poisson integrals with respect to the enlarged Poisson field. Then we give the proof of theorem 2.2. The proof is followed by two steps. First, we reduce the problem to the investigation of the "enlarged obstacles", $\omega((\log)^{1/d}dx)$. Second, we apply the large deviation results developed in section 3.2.

## 3.1 Notation

Through out this chapter, $D \subset \mathbb{R}^d$ is a fixed and bounded open set. Denote

$$\mathcal{F}_d(D) = \Big\{ f \in W^{1,2}(\mathbb{R}^d); \int_D f(x)^2 dx = 1 \Big\}.$$

The following two functions

$$\varphi(x) = 1 - e^{-x} \qquad \text{and} \qquad \psi(x) = e^x - 1 \tag{3.1}$$

appear frequently throughout this chapter, where $x \geq 0$. Notice that $\varphi(x)$ and $\psi(x)$ are nonnegative and increasing function with $\varphi(0) = \psi(0) = 0$. In addition, $\psi(\frac{1}{|x|^p})$

is not integrable on $\mathbb{R}^d$, but for some $c > 0$

$$\int_{|x|\geq c} \psi(\frac{1}{|x|^p})dx < \infty. \tag{3.2}$$

We create a smooth truncation to the shape function $K(x)$. Let the smooth function $\alpha : \mathbb{R}^+ \to [0,1]$ satisfy the following properties: $\alpha(x) = 1$ on $[0,1]$, $\alpha(x) = 0$ for $x \geq 3$ and $-1 \leq \alpha'(x) \leq 0$ for all $x \geq 0$.

For $a > 0$ and $\varepsilon > 0$, define

$$K_{a,\varepsilon}(x) = \big(\frac{1}{|x|^{d+2}} \wedge \varepsilon^{-\frac{d+2}{d}}\big)\alpha(a^{-1}|x|),$$

$$L_{a,\varepsilon}(x) = \big(\frac{1}{|x|^{d+2}} \wedge \varepsilon^{-\frac{d+2}{d}}\big)(1 - \alpha(a^{-1}|x|)),$$

and

$$G_{a,\varepsilon}(g) = \int_{\mathbb{R}^d} \left[\int_D K_{a,\varepsilon}(y-x)g^2(y)dy\right] \omega(\varepsilon^{-1}dx), \qquad g \in \mathcal{F}_d(D),$$

$$F_{a,\varepsilon}(g) = \int_{\mathbb{R}^d} \left[\int_D L_{a,\varepsilon}(y-x)g^2(y)dy\right] \omega(\varepsilon^{-1}dx), \qquad g \in \mathcal{F}_d(D).$$

Write

$$\zeta_\varepsilon(g) = \int_{\mathbb{R}^d} \left[\int_D \left(\frac{1}{|y-x|^{d+2}} \wedge \varepsilon^{-\frac{d+2}{d}}\right) g^2(y)dy\right] \omega(\varepsilon^{-1}dx), \qquad g \in \mathcal{F}_d(D).$$

Write

$$V_{a,\varepsilon}(x) = \int_{\mathbb{R}^d} L_{a,\varepsilon}(y-x)\omega(\varepsilon^{-1}dy), \qquad x \in D.$$

## 3.2 Large deviations of Poisson Stochastic Integrals

In the following section, we are going to present large deviations for the Poisson integrals with respect to the "enlarged obstacles". These results will play a crucial role in proving the lower bound of the theorem 2.2.

**Lemma 3.1.** *For any $a > 0$ and $\theta > 0$,*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \mathbb{E} \exp\{\theta \sup_{x\in D} V_{a,\varepsilon}(x)\} = \int_{\mathbb{R}^d} \psi\left(\theta \frac{1-\alpha(a^{-1}|x|)}{|x|^p}\right) dx. \tag{3.3}$$

*Proof.* By the space homogeneity of the Poisson field and by Poisson integral formula (A.1), for any $a > 0$, $\theta > 0$ and $x \in D$, we have

$$\begin{aligned}
\mathbb{E}\exp\{\theta V_{a,\varepsilon}(x)\} &= \mathbb{E}\exp\{\theta V_{a,\varepsilon}(0)\} \\
&= \exp\left\{\varepsilon^{-1}\int_{\mathbb{R}^d} \psi\left(\theta L_{a,\varepsilon}(x)\right) dx\right\} \\
&= \exp\left\{\varepsilon^{-1}\int_{\mathbb{R}^d} \psi\left(\theta(1-\alpha(a^{-1}|x|))\left(\frac{1}{|x|^{d+2}} \wedge \varepsilon^{-\frac{d+2}{d}}\right)\right) dx\right\}.
\end{aligned}$$

Notice that,

$$\psi\left(\theta(1-\alpha(a^{-1}|x|))\left(\frac{1}{|x|^{d+2}} \wedge \varepsilon^{-\frac{d+2}{d}}\right)\right) \le \psi\left(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^{d+2}}\right).$$

Therefore, by (3.2) and by Dominated convergence theorem,

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \mathbb{E} \exp\{\theta V_{a,\varepsilon}(x)\} = \int_{\mathbb{R}^d} \psi\left(\theta \frac{1-\alpha(a^{-1}|x|)}{|x|^{d+2}}\right) dx. \tag{3.4}$$

Consequently, all we need is to take supremum over $x \in D$ in the exponent on the left-hand sides of (3.4) and push the supremum through the expectation.

By the boundedness of D, we may assume that $D \subseteq (-b, b)^d$ for some $b > 0$. Let $h > 0$ be a constant which will be later specified. Under the substitution $y \to \varepsilon^{h/d} z$, we have that

$$\begin{aligned}
V_{a,\varepsilon}(x) &= \int_{\mathbb{R}^d} L_{a,\varepsilon}(y-x)\omega(\varepsilon^{-1}dy) \\
&= \varepsilon^{-(d+2)h/d}\int_{\mathbb{R}^d} \widetilde{L}_{a,\varepsilon}(z-\varepsilon^{-h/d}x)\omega(\varepsilon^{-1+h}dz) \\
&= \varepsilon^{-(d+2)h/d} H_\varepsilon(\varepsilon^{-h/d}x), \qquad (3.5)
\end{aligned}$$

where

$$\widetilde{L}_{a,\varepsilon}(x) = \left(\frac{1}{|x|^{d+2}} \wedge \varepsilon^{-\frac{(d+2)(1-h)}{d}}\right)\{1-\alpha(a^{-1}\varepsilon^{h/d}|x|)\}$$

and

$$H_\varepsilon(x) = \int_{\mathbb{R}^d} \widetilde{L}_{a,\varepsilon}(z-x)\omega(\varepsilon^{-1+h}dz).$$

By Poisson integral formula (A.1), for any $x, y \in D$ with $x \neq y$, and $\theta > 0$, we have

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{H_\varepsilon(x)-H_\varepsilon(y)}{|x-y|}\right\} \\
&\quad = \exp\left\{\varepsilon^{-1+h}\int_{\mathbb{R}^d}\psi\left(\varepsilon^{-(d+2)h/d}\frac{\theta}{|x-y|}(\widetilde{L}_{a,\varepsilon}(z-x)-\widetilde{L}_{a,\varepsilon}(z-y))\right)dz\right\}.
\end{aligned}$$

Switching x and y, we have a similar result as the above , then we have

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{|H_\varepsilon(x)-H_\varepsilon(y)|}{|x-y|}\right\} \\
&\quad = \mathbb{E}\left[\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{H_\varepsilon(x)-H_\varepsilon(y)}{|x-y|}\right\}, 1_{\{H_\varepsilon(x)>H_\varepsilon(y)\}}\right] \\
&\qquad + \mathbb{E}\left[\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{H_\varepsilon(y)-H_\varepsilon(x)}{|x-y|}\right\}, 1_{\{H_\varepsilon(x)<H_\varepsilon(y)\}}\right] \\
&\quad \leq \mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{H_\varepsilon(x)-H_\varepsilon(y)}{|x-y|}\right\} + \mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{H_\varepsilon(y)-H_\varepsilon(x)}{|x-y|}\right\} \\
&\quad \leq 2\exp\left\{\varepsilon^{-1+h}\int_{\mathbb{R}^d}\psi\left(\varepsilon^{-(d+2)h/d}\frac{\theta}{|x-y|}|\widetilde{L}_{a,\varepsilon}(z-x)-\widetilde{L}_{a,\varepsilon}(z-y)|\right)dz\right\}
\end{aligned}$$

where the last inequality follows from the monotonicity of $\psi$. Substitute $z$ with $\varepsilon^{-h/d}z$, we have that

$$\begin{aligned}\int_{\mathbb{R}^d}\psi\left(\varepsilon^{-(d+2)h/d}\frac{\theta}{|x-y|}|\widetilde{L}_{a,\varepsilon}(z-x)-\widetilde{L}_{a,\varepsilon}(z-y)|\right)dz\\ =\varepsilon^{-h}\int_{\mathbb{R}^d}\psi\left(\frac{\theta}{|x-y|}|L_a(z-\varepsilon^{h/d}x)-L_a(z-\varepsilon^{h/d}y)|\right)dz,\end{aligned}$$

where

$$L_a(z)=\left(\frac{1}{|z|^{d+2}}\wedge\varepsilon^{-\frac{d+2}{d}}\right)(1-\alpha(a^{-1}|z|)).$$

Notice that,

$$\begin{aligned}L_a(z-\varepsilon^{h/d}x)-L_a(z-\varepsilon^{h/d}y)&=(L_a(z-\varepsilon^{h/d}x)-L_a(z-\varepsilon^{h/d}y))1_{\{|z|<C_a^{-1}\}}\\ &\quad+(L_a(z-\varepsilon^{h/d}x)-L_a(z-\varepsilon^{h/d}y))1_{\{|z|\geq C_a^{-1}\}}.\end{aligned}$$

By the definition of $\alpha(\cdot)$, we can make the first part of the right hand side of the above equation 0 if we choose a proper constant $C_a$ and small enough $\varepsilon$. In addition, by basic calculus, when $|z|>\varepsilon^{1/d}$ we have,

$$|\nabla L_a(z)|=\frac{(d+2)(1-\alpha(a^{-1}|z|))}{|z|^{d+1}}+\frac{\alpha'(a^{-1}|z|)}{a|z|^{d+2}}$$

Keep the above in mind, by the mean value theorem, there exists a constant $\xi$ between $z-\varepsilon^{h/d}x$ and $z-\varepsilon^{h/d}y$ such that,

$$\begin{aligned}|L_a(z-\varepsilon^{h/d}x)-L_a(z-\varepsilon^{h/d}y)|&\leq\varepsilon^{h/d}|\nabla L_a(\xi)||x-y|1_{\{|z|\geq C_a^{-1}\}}\\ &\leq C_a\frac{\varepsilon^{h/d}|x-y|}{|z|^{d+2}}1_{\{|z|\geq C_a^{-1}\}}.\end{aligned}$$

for $x, y \in D$. Summarizing what we have,

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{|H_\varepsilon(x)-H_\varepsilon(y)|}{|x-y|}\right\}\\
&\quad\le 2\exp\left\{\varepsilon^{-1}\int_{\{|z|\ge C_a^{-1}\}}\psi\left(\frac{C_a\theta\varepsilon^{\gamma/d}}{|z|^{d+2}}\right)dz\right\}\\
&\quad= 2\exp\left\{\varepsilon^{h/(d+2)-1}\int_{\{|z|\ge C_a^{-1}\varepsilon^{-h/d(d+2)}\}}\psi\left(\frac{C_a\theta}{|z|^{d+2}}\right)dz\right\}.
\end{aligned}$$

Let $h > 0$ satisfy that $h > d+2$. Then for any $\theta > 0$ the quantity

$$\sup_{\substack{x,y\in D\\ x\neq y}}\mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\frac{|H_\varepsilon(x)-H_\varepsilon(y)|}{|x-y|}\right\}$$

is bounded uniformly for small $\varepsilon > 0$. Thus, by Theorem D-6 in [7], for any $\theta > 0$,

$$\lim_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+}\mathbb{E}\exp\left\{\theta\varepsilon^{-(d+2)h/d}\sup_{|x-y|\le\delta}|H_\varepsilon(x)-H_\varepsilon(y)|\right\} = 1. \tag{3.6}$$

On the other hand, for any $x \in \mathbb{R}^d$ and $\theta > 0$, by homogeneity of Poisson field,

$$\begin{aligned}
\mathbb{E}\exp\{\theta\varepsilon^{-(d+2)h/d}H_\varepsilon(x)\} &= \mathbb{E}\exp\{\theta\varepsilon^{-(d+2)h/d}H_\varepsilon(0)\}\\
&= \mathbb{E}\exp\{\theta V_{a,\varepsilon}(0)\}
\end{aligned}$$

where the last step follows from (3.5). By (3.4), for any $x \in D$

$$\lim_{\varepsilon\to 0^+}\varepsilon\log\mathbb{E}\exp\{\theta\varepsilon^{-(d+2)h/d}H_\varepsilon(x)\} = \int_{\mathbb{R}^d}\psi\left(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^{d+2}}\right)dx.$$

Combine them with (3.6). A standard argument of exponential approximation leads to

$$\lim_{\varepsilon\to 0^+}\varepsilon\log\mathbb{E}\exp\{\theta\varepsilon^{-(d+2)h/d}\sup_{x\in D}H_\varepsilon(x)\}$$
$$=\int_{\mathbb{R}^d}\psi\left(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^{d+2}}\right)dx. \tag{3.7}$$

Recall that $D\subseteq(-b,b)^d$. Using (3.5),

$$\begin{aligned}\sup_{x\in D}V_{a,\varepsilon}(x)&=\varepsilon^{-(d+2)h/d}\sup_{x\in\varepsilon^{-h/d}D}H_\varepsilon(x)\\&\le\varepsilon^{-(d+2)h/d}\max_{z\in b\mathbb{Z}^d\cap\varepsilon^{-h/d}D}\left\{\sup_{x\in z+D}H_\varepsilon(x)\right\}.\end{aligned}$$

By the fact that the random variables

$$\sup_{x\in z+D}H_\varepsilon(x);\qquad z\in b\mathbb{Z}^d\cap\varepsilon^{-h/d}D,$$

are identically distributed,

$$\mathbb{E}\exp\{\theta\sup_{x\in D}V_{a,\varepsilon}(x)\}\le\#\{b\mathbb{Z}^d\cap\varepsilon^{-h/d}D\}\mathbb{E}\exp\{\theta\varepsilon^{-(d+2)h/d}\sup_{x\in D}H_\varepsilon(x)\}.$$

Notice that $\#\{b\mathbb{Z}^d\cap\varepsilon^{-h/d}D\}\sim c\varepsilon^{-h}$. Combine this with (3.7), we have

$$\limsup_{\varepsilon\to o^+}\varepsilon\log\mathbb{E}\exp\{\theta\sup_{x\in D}V_{a,\varepsilon}(x)\}\le\int_{\mathbb{R}^d}\psi\left(\theta\frac{1-\alpha(a^{-1}|x|)}{|x|^{d+2}}\right)dx.$$

In view of (3.4), the other side of the above inequality holds obviously. Therefore, we have proved (3.3). □

Next, we are going to prove the main Theorem in this section.

**Theorem 3.1.** *For any $\gamma > 0$ and $\theta > 0$,*

$$\lim_{\varepsilon\to o^+} \varepsilon \log \mathbb{P}\left\{ \inf_{g\in\mathcal{F}_d(D)} \left\{ \theta\zeta_\varepsilon(g) + \frac{1}{2\varepsilon}\int_D |\nabla g(x)|^2 dx \right\} \le \gamma\varepsilon^{-1} \right\} = -I_D(\gamma), \tag{3.8}$$

$$\lim_{a\to\infty} \limsup_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\left\{ \sup_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g) \ge \gamma\varepsilon^{-1} \right\} = -\infty, \tag{3.9}$$

$$\liminf_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}\left\{ \inf_{g\in\mathcal{F}_d(D)} \left\{ \theta G_{a,\varepsilon}(g) + \frac{1}{2\varepsilon}\int_D |\nabla g(x)|^2 dx \right\} \le \gamma\varepsilon^{-1} \right\} \ge -I_D(\gamma), \tag{3.10}$$

*where*

$$\begin{aligned} I_D(\gamma) =& \frac{2}{d+2}\left(\frac{d}{(d+2)\gamma}\right)^{d/2} \\ &\times \inf_{g\in\mathcal{F}_d(D)} \left( \int_{\mathbb{R}^d} \varphi\left(\theta \int_D \frac{g^2(y)}{|y-x|^{d+2}} dy\right) dx + \frac{1}{2}\int_D |\nabla g(x)|^2 dx \right)^{(d+2)/2}. \end{aligned} \tag{3.11}$$

*Proof.* Notice that,

$$\sup_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g) = \sup_{g\in\mathcal{F}_d(D)} \int_D V_{a,\varepsilon}(x) g^2(x) dx \le \sup_{x\in D} V_{a,\varepsilon}(x).$$

Let $\theta > 0$ be fixed but arbitrary. Combine this with (3.3), we have that

$$\limsup_{\varepsilon\to 0^+} \varepsilon \log \mathbb{E} \exp\left\{ \theta \sup_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g) \right\} \le \int_{\mathbb{R}^d} \psi\left(\theta \frac{1-\alpha(a^{-1}|x|)}{|x|^p}\right) dx,$$

Consequently, by Dominate Convergence Theorem

$$\lim_{a\to\infty} \limsup_{\varepsilon\to 0^+} \varepsilon \log \mathbb{E} \exp\left\{ \theta \sup_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g) \right\} = 0.$$

According to Chebyshev's inequality, for any $\varepsilon > 0$ and $\gamma > 0$,

$$\mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g) \geq \gamma\varepsilon^{-1}\right\} \leq \frac{\mathbb{E}\exp\left\{\theta \sup\limits_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g)\right\}}{\exp\left\{\theta\gamma\varepsilon^{-1}\right\}}.$$

Hence,

$$\lim_{a\to\infty}\limsup_{\varepsilon\to 0^+}\varepsilon\log\mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(D)} F_{a,\varepsilon}(g) \geq \gamma\varepsilon^{-1}\right\} \leq -\gamma\theta.$$

Therefore, we obtain (3.9) by letting $\theta \to \infty$.

Next, we turn to prove (3.8). For any $\varrho > 0$ and $g \in \mathcal{F}_d(D)$,

$$\begin{aligned}
&\mathbb{E}\exp\left\{-\varrho\left(\theta\zeta_\varepsilon(g) + \frac{1}{2\varepsilon}\int_D |\nabla g(x)|^2 dx\right)\right\} \\
&= \exp\left\{-\varepsilon^{-1}\int_{\mathbb{R}^d}\varphi\left(\varrho\theta\int_D \frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx - \frac{\varrho}{2\varepsilon}\int_D |\nabla g(x)|^2 dx\right\},
\end{aligned}$$

which leads to

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\varepsilon\log\mathbb{E}\exp\left\{-\varrho\inf_{g\in\mathcal{F}_d(D)}\left\{\theta\zeta_\varepsilon(g) + \frac{1}{2\varepsilon}\int_D |\nabla g(x)|^2 dx\right\}\right\} \\
&= -\inf_{g\in\mathcal{F}_d(D)}\left\{\int_{\mathbb{R}^d}\varphi\left(\varrho\theta\int_D \frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx + \frac{\varrho}{2}\int_D |\nabla g(x)|^2 dx\right\}.
\end{aligned}$$

Notice that, by the substitution

$$g(x) \to \varrho^{-\frac{d}{2(d+2)}} g\left(\varrho^{-\frac{1}{d+2}}x\right),$$

we have

$$\begin{aligned}
&\inf_{g\in\mathcal{F}_d(D)}\left\{\int_{\mathbb{R}^d}\varphi\left(\varrho\theta\int_D \frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx + \frac{\varrho}{2}\int_D |\nabla g(x)|^2 dx\right\} \\
&= \varrho^{d/(d+2)}\inf_{g\in\mathcal{F}_d(D)}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_D \frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx + \frac{1}{2}\int_D |\nabla g(x)|^2 dx\right\}.
\end{aligned}$$

which give us

$$\lim_{\varepsilon\to 0^+}\varepsilon\log\mathbb{E}\exp\left\{-\varrho\inf_{g\in\mathcal{F}_d(D)}\left\{\theta\zeta_\varepsilon(g)+\frac{1}{2\varepsilon}\int_D|\nabla g(x)|^2dx\right\}\right\}$$
$$=-\varrho^{d/(d+2)}\inf_{g\in\mathcal{F}_d(D)}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_D\frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx+\frac{1}{2}\int_D|\nabla g(x)|^2dx\right\}.$$

Following Lemma (B.1), we have that

$$\lim_{\varepsilon\to o^+}\varepsilon\log\mathbb{P}\left\{\inf_{g\in\mathcal{F}_d(D)}\left\{\theta\zeta_\varepsilon(g)+\frac{1}{2\varepsilon}\int_D|\nabla g(x)|^2dx\right\}\le\gamma\varepsilon^{-1}\right\}$$
$$=\sup_{\varrho>0}\left\{-\gamma\varrho+\varrho^{d/(d+2)}\inf_{g\in\mathcal{F}_d(D)}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_D\frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx+\frac{1}{2}\int_D|\nabla g(x)|^2dx\right\}\right\}$$
$$=-I_D(\gamma).$$

It remains to prove (3.10). By comparing $\zeta_\varepsilon(g)$ and $G_{a,\varepsilon}(g)$, we have

$$\mathbb{P}\left\{\inf_{g\in\mathcal{F}_d(D)}\left\{\theta\zeta_\varepsilon(g)+\frac{1}{2\varepsilon}\int_D|\nabla g(x)|^2dx\right\}\le\gamma\varepsilon^{-1}\right\}$$
$$\le\mathbb{P}\left\{\inf_{g\in\mathcal{F}_d(D)}\left\{\theta G_{a,\varepsilon}(g)+\frac{1}{2\varepsilon}\int_D|\nabla g(x)|^2dx\right\}\le\gamma\varepsilon^{-1}\right\}.$$

Applying (3.8) on the left-hand side, we obtain (3.10). □

## 3.3 Upper bound

In this section, we are going to establish the upper bound of Theorem 2.2. That is to obtain an asymptotic upper bound for

$$\mathbb{E}_0\exp\left\{-\theta\int_0^t V(B_s)ds\right\}. \tag{3.12}$$

The rough strategy is as follows. We localize the above exponential moment (3.12) by forcing Bownian motion to stay in a large box of size $t/(\log t)^{1/d}$ up to time $t$, the difference between of the two exponential moments in our setting is negligible.

Then we estimate the localized exponential moment from above by investigating the principal eigenvalue of $\frac{1}{2}\Delta - V$ in a large box of size $t$. In the following section, we will give proof of upper bound in details and investigate the long time asymptotics of the principle eigenvalue.

### 3.3.1 Eigenvalue estimates

We first introduce some notations which will be used in this and next sections. For any $R > 0$, let $Q_R = (-R, R)^d$ be an open box. Denote

$$h_t = (\log t)^{-1/d}. \tag{3.13}$$

Given a bounded open set $D \subset \mathbb{R}^d$ and a measurable function $\xi(x)$ on $\mathbb{R}^d$, we denote $\lambda_\xi(D)$ to be the principal eigenvalue of $\frac{1}{2}\Delta + \xi$ in $D$. By (3.5) in [16],

$$\lambda_\xi(D) = \sup_{g \in \mathcal{F}_d(D)} \Big\{ \int_D \xi(x) g^2(x) dx - \frac{1}{2} \int_D |\nabla g(x)|^2 dx \Big\}. \tag{3.14}$$

Clearly, whenever $U \subseteq V$ and $\xi \leq \eta$, we have

$$\lambda_\xi(U) \leq \lambda_\eta(V). \tag{3.15}$$

**Lemma 3.2.** *Under $p = d + 2$, for any $\theta > 0$,*

$$\limsup_{t \to \infty} (\log t)^{2/d} \lambda_{\theta\xi}(Q_t) \leq \Lambda(\theta), \qquad a.s., \tag{3.16}$$

*where $\xi = -V$,*

$$\Lambda(\theta) = -\frac{d}{d+2} \left( \frac{2}{d(d+2)} \right)^{\frac{2}{d}} \rho(d, \theta)^{\frac{d+2}{d}}, \tag{3.17}$$

*and $\rho(d, \theta)$ is defined in (2.14).*

*Proof.* Let $u > 0$ which will be specified later and set $\delta > 0$ to be a fixed but small number. Write

$$\widetilde{h}_t = h_t \sqrt{\frac{u}{1-\delta}}, \tag{3.18}$$

where $h_t$ is given in (3.13). Next, we rescale the Poisson field and define

$$\xi_t(x) = -\theta \widetilde{h}_t^d \int_{\mathbb{R}^d} \left( \frac{1}{|y-x|^{d+2}} \wedge \widetilde{h}_t^{-(d+2)} \right) \omega(\widetilde{h}_t^{-d} dy).$$

Under substitution $g(x) \mapsto \widetilde{h}_t^{d/2} g(\widetilde{h}_t x)$ and then applying Fubini's theorem, we have that

$$\lambda_{\theta\xi}(Q_t) = \widetilde{h}_t^2 \sup_{g \in \mathcal{F}_d(Q_{t\widetilde{h}_t})} \left\{ \int_{Q_{t\widetilde{h}_t}} \xi_t(x) g^2(x) dx - \frac{1}{2} \int_{Q_{t\widetilde{h}_t}} |\nabla g(x)|^2 dx \right\}. \tag{3.19}$$

In the rest of the proof, we are going to give an upper bound to the above variation. According to a nice strategy developed by Gärtner and König [16], the principle eigenvalue in a large $t$-dependent box can be bounded by the maximum of the principle eigenvalues in the sub-boxes of finite length. More precisely, let $r \geq 2$ be large but fixed. By Proposition 1 in [16], also by Lemma 4.6 in [18], there is a nonnegative and continuous function $\phi(x)$ on $\mathbb{R}^d$ whose support is contained in the 1-neighborhood of the grid $2r\mathbb{Z}^d$, such that

$$\lambda_{\xi_t - \phi^y}(Q_{t\widetilde{h}_t}) \leq \max_{z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t + 2r}} \lambda_{\xi_t}(z + Q_{r+1}), \qquad y \in Q_r, \tag{3.20}$$

where $\phi^y(x) = \phi(x+y)$. In addition, $\phi(x)$ is periodic with period $2r$

$$\phi(x + 2rz) = \phi(x); \qquad x \in \mathbb{R}^d, z \in \mathbb{Z}^d,$$

and there is a constant $K > 0$ independent of $r$ and $t$ such that

$$\frac{1}{(2r)^d} \int_{Q_r} \phi(x) dx \leq \frac{K}{r}. \tag{3.21}$$

By periodicity of $\phi$ and by (3.21), we get

$$\begin{aligned}
\lambda_{\xi_t}(Q_{t\widetilde{h}_t}) &= \sup_{g \in \mathcal{F}_d(Q_{t\widetilde{h}_t})} \left\{ \int_{Q_{t\widetilde{h}_t}} \xi_t(x) g^2(x) dx - \frac{1}{2} \int_{Q_{t\widetilde{h}_t}} |\nabla g(x)|^2 dx \right\} \\
&\leq \frac{K}{r} + \sup_{g \in \mathcal{F}_d(Q_{t\widetilde{h}_t})} \left\{ \int_{Q_{t\widetilde{h}_t}} \left( \xi_t(x) - \frac{1}{(2r)^d} \int_{Q_r} \phi^y(x) dy \right) g^2(x) dx - \frac{1}{2} \int_{Q_{t\widetilde{h}_t}} |\nabla g(x)|^2 dx \right\} \\
&= \frac{K}{r} + \sup_{g \in \mathcal{F}_d(Q_{t\widetilde{h}_t})} \left\{ \frac{1}{(2r)^d} \int_{Q_r} \left[ \int_{Q_{t\widetilde{h}_t}} (\xi_t(x) - \phi^y(x)) g^2(x) dx - \frac{1}{2} \int_{Q_{t\widetilde{h}_t}} |\nabla g(x)|^2 dx \right] dy \right\}
\end{aligned}$$

By pushing the supremum through the integral over $Q_r$, we have

$$\begin{aligned}
&\sup_{g \in \mathcal{F}_d(Q_{t\widetilde{h}_t})} \left\{ \frac{1}{(2r)^d} \int_{Q_r} \left[ \int_{Q_{t\widetilde{h}_t}} (\xi_t(x) - \phi^y(x)) g^2(x) dx - \frac{1}{2} \int_{Q_{t\widetilde{h}_t}} |\nabla g(x)|^2 dx \right] dy \right\} \\
&\qquad \leq \frac{1}{(2r)^d} \int_{Q_r} \sup_{g \in \mathcal{F}_d(Q_{t\widetilde{h}_t})} \left\{ \int_{Q_{t\widetilde{h}_t}} (\xi_t(x) - \phi^y(x)) g^2(x) dx - \frac{1}{2} \int_{Q_{t\widetilde{h}_t}} |\nabla g(x)|^2 dx \right\} dy \\
&\qquad = \frac{1}{(2r)^d} \int_{Q_r} \lambda_{\xi_t - \phi^y}(Q_{t\widetilde{h}_t}) dy \\
&\qquad \leq \max_{z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t + 2r}} \lambda_{\xi_t}(z + Q_{r+1}),
\end{aligned}$$

where the last step follows from (3.20). Summarizing our estimates

$$\lambda_{\xi_t}(Q_{t\widetilde{h}_t}) \leq \frac{K}{r} + \max_{z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t + 2r}} \lambda_{\xi_t}(z + Q_{r+1}).$$

Combine this with (3.19)

$$\lambda_{\theta\xi}(Q_t) \leq \frac{uh_t^2}{1-\delta} \left( \frac{K}{r} + \max_{z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t + 2r}} \lambda_{\xi_t}(z + Q_{r+1}) \right).$$

Take $r > 0$ sufficiently large so that the first term on the right-hand side is less than $\frac{\delta u {h_t}^2}{1-\delta}$. As a result,

$$\mathbb{P}\{\lambda_{\theta\xi}(Q_t) \geq -u{h_t}^2\} \leq \mathbb{P}\Big\{ \max_{z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t+2r}} \lambda_{\xi_t}(z + Q_{r+1}) > -1 \Big\}. \tag{3.22}$$

By shifting invariance of the Poisson field, the random variables

$$\lambda_{\xi_t}(z + Q_{r+1}); \qquad\qquad z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t+2r}$$

are identically distributed. Consequently, there is $C > 0$ such that

$$\begin{aligned} &\mathbb{P}\Big\{ \max_{z \in 2r\mathbb{Z}^d \cap Q_{2t\widetilde{h}_t+2r}} \lambda_{\xi_t}(z + Q_{r+1}) > -1 \Big\} \\ &\quad \leq Ct^d \mathbb{P}\{\lambda_{\xi_t}(Q_{r+1}) > -1\}. \end{aligned} \tag{3.23}$$

Notice that,

$$\begin{aligned} \lambda_{\xi_t}(Q_{r+1}) &= \sup_{g \in \mathcal{F}_d(Q_{r+1})} \left\{ \int_{Q_{r+1}} \xi_t(x) g^2(x) dx - \frac{1}{2} \int_{Q_{r+1}} |\nabla g(x)|^2 dx \right\} \\ &= -\widetilde{h}_t^d \inf_{g \in \mathcal{F}_d(Q_{r+1})} \left\{ \theta \int_{\mathbb{R}^d} \left[ \int_{Q_{r+1}} \left( \frac{1}{|y-x|^{d+2}} \wedge \widetilde{h}_t^{-(d+2)} \right) g^2(x) dx \right] \omega(\widetilde{h}_t^{-d} dy) \right. \\ &\qquad \left. + \frac{1}{2\widetilde{h}_t^d} \int_{Q_{r+1}} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

Combining this and taking $\varepsilon = \widetilde{h}_t^d$ and $\gamma = 1$ in (3.8) leads to,

$$\begin{aligned} \lim_{t \to \infty} \frac{1}{\log t} \log \mathbb{P}\{\lambda_{\xi_t}(Q_{r+1}) > -1\} &= -\left( \frac{u}{1-\delta} \right)^{-d/2} I_{Q_{r+1}}(1) \\ &\leq -\left( \frac{u}{1-\delta} \right)^{-d/2} \frac{2}{d+2} \left( \frac{d}{d+2} \right)^{d/2} \rho(d,\theta)^{\frac{d+2}{2}}, \end{aligned} \tag{3.24}$$

where the last step follows from the monotonicity of $I_{Q_r}(1)$ in $r$ and the definition of $I_{Q_r}(\cdot)$ given in (3.11).

Let $u = -(1-2\delta)\Lambda(\theta)$, where $\Lambda(\theta)$ is defined in (3.17). By (3.22), (3.23) and (3.24), there is a $\nu > 0$ such that

$$\mathbb{P}\{\lambda_{\theta\xi}(Q_t) \geq (1-2\delta)\Lambda(\theta)h_t^2\} \leq Ct^d \exp\{-(d+\nu)\log t\} = \frac{C}{t^\nu}$$

for sufficiently large t.

Hence, for any $\gamma > 1$, we have

$$\sum_k \mathbb{P}\{\lambda_{\theta\xi}(Q_{\gamma^k}) \geq (1-2\delta)\Lambda(\theta)h_{\gamma^k}^2\} < \infty$$

Therefore, by the Borel-Cantelli lemma,

$$\limsup_{k\to\infty} h_{\gamma^k}^{-2}\lambda_{\theta\xi}(Q_{\gamma^k}) \leq (1-2\delta)\Lambda(\theta) \qquad a.s.$$

Notice that for sufficiently large $t$, there exists an integer $k > 0$, such that $\gamma^{k-1} < t < \gamma^k$. By definition of $h_t$ in (3.13) and by (3.15),

$$h_t^{-2}\lambda_{\theta\xi}(Q_t) \leq h_{\gamma^k}^{-2}\lambda_{\theta\xi}(Q_{\gamma^k}) \tag{3.25}$$

Taking *limsup* over $t$ and $k$ on both side, we have

$$\limsup_{t\to\infty} h_t^{-2}\lambda_{\theta\xi}(Q_t) \leq (1-2\delta)\Lambda(\theta) \qquad a.s. \tag{3.26}$$

Since $\delta > 0$ can be arbitrarily small, we have proved (3.16). □

The following Lemma gives the other side of the story stated in Lemma (3.2).

**Lemma 3.3.** *Under $p = d+2$, for any $\theta > 0$,*

$$\liminf_{t\to\infty} (\log t)^{2/d}\lambda_{\theta\xi}(Q_t) \geq \Lambda(\theta), \qquad a.s., \tag{3.27}$$

*where $\xi = -V$ and $\Lambda(\theta)$ is defined in* (3.17).

*Proof.* Let $u > 0$ be fixed but arbitrary. Write

$$\hat{h}_t = \sqrt{u} h_t.$$

where $h_t$ is defined in (3.13).

Once again, we rescale the Poisson random field and define

$$\eta_t(x) = -\theta \hat{h}_t^d \int_{\mathbb{R}^d} \left( \frac{1}{|y-x|^{d+2}} \wedge \hat{h}_t^{-(d+2)} \right) \omega(\hat{h}_t^{-d} dy)$$

By the substitution $g(x) \to \hat{h}_t^{d/2} g(\hat{h}_t x)$, we have $\lambda_{\theta\xi}(Q_t) = \hat{h}_t^2 \lambda_{\eta_t}(Q_{t\hat{h}_t})$.

Therefore,

$$\mathbb{P}\{\lambda_{\theta\xi}(Q_t) \le -u {h_t}^2\} = \mathbb{P}\{\lambda_{\eta_t}(Q_{t\hat{h}_t}) \le -1\}.$$

Next, we shall bound the above principle eigenvalue $\lambda_{\eta_t}(Q_{t\hat{h}_t})$ in the rescaled Poisson obstacle case. Let $s > 1$ and $r > 0$ be fixed. Set $\mathcal{N}_t = h_t^{-s}\mathbb{Z}^d \cap Q_{t\hat{h}_t - r}$. Notice that, $z + Q_r \subset Q_{t\hat{h}_t}$ for each $z \in \mathcal{N}_t$ when $t$ is large enough. Therefore, by the monotonicity of $\lambda_{\eta_t}(D)$ in the set $D \subset \mathbb{R}^d$,

$$\lambda_{\eta_t}(Q_{t\hat{h}_t}) \ge \lambda_{\eta_t}(z + Q_r), \qquad z \in \mathcal{N}_t.$$

Consequently,

$$\begin{aligned} \lambda_{\eta_t}(Q_{t\hat{h}_t}) &\ge \max_{z \in \mathcal{N}_t} \lambda_{\eta_t}(z + Q_r) \\ &= \max_{z \in \mathcal{N}_t} \sup_{g \in \mathcal{F}_d(z+Q_r)} \left\{ \int_{\mathbb{R}^d} \eta_t(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}. \end{aligned}$$

Let the smooth function $\alpha(\cdot)$ be given as in Section 3.1. We shall adopt this smooth truncation to the shape function as follows. Given $a > 0$, write

$$K_{t,a}(x) = \left(\frac{1}{|x|^{d+2}} \wedge \hat{h}_t^{-(d+2)}\right) \alpha(a^{-1}|x|)$$

and

$$L_{t,a}(x) = \left(\frac{1}{|x|^{d+2}} \wedge \hat{h}_t^{-(d+2)}\right) (1 - \alpha(a^{-1}|x|)).$$

By the equality

$$\begin{aligned}
&\int_{\mathbb{R}^d} \eta_t(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \\
&= -\theta \hat{h}_t^d \int_{\mathbb{R}^d} \left[\int_{z+Q_r} K_{t,a}(y-x) g^2(y) dy\right] \omega(\hat{h}_t^{-d} dx) - \frac{1}{2} \int_{z+Q_r} |\nabla g(x)|^2 dx \\
&- \theta \hat{h}_t^d \int_{\mathbb{R}^d} \left[\int_{z+Q_r} L_{t,a}(y-x) g^2(y) dy\right] \omega(\hat{h}_t^{-d} dx) \\
&= \hat{h}_t^d (A_z^\theta(g) + B_z^\theta(g))
\end{aligned}$$

where

$$A_z^\theta(g) = -\theta \int_{\mathbb{R}^d} \left[\int_{z+Q_r} K_{t,a}(y-x) g^2(y) dy\right] \omega(\hat{h}_t^{-d} dx) - \frac{1}{2\hat{h}_t^d} \int_{z+Q_r} |\nabla g(x)|^2 dx$$

and

$$B_z^\theta(g) = -\theta \int_{\mathbb{R}^d} \left[\int_{z+Q_r} L_{t,a}(y-x) g^2(y) dy\right] \omega(\hat{h}_t^{-d} dx),$$

Then by triangular inequality, we have

$$\begin{aligned}
&\max_{z \in \mathcal{N}_t} \sup_{g \in \mathcal{F}_d(z+Q_r)} \left\{\int_{\mathbb{R}^d} \eta_t(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx\right\} \\
&\geq \hat{h}_t^d \left\{\max_{z \in \mathcal{N}_t} \sup_{g \in \mathcal{F}_d(z+Q_r)} A_z^\theta(g) - \max_{z \in \mathcal{N}_t} \sup_{g \in \mathcal{F}_d(z+Q_r)} B_z^\theta(g)\right\}.
\end{aligned}$$

By the fact that the random variables

$$\sup_{g\in\mathcal{F}_d(z+Q_r)} B_z^\theta(g); \qquad z\in\mathcal{N}_t$$

are identically distributed. Therefore, for any $\delta>0$,

$$\begin{aligned}&\mathbb{P}\Big\{\max_{z\in h_t^{-s}\mathbb{Z}^d\cap Q_{t\hat{h}_t-r}}\ \sup_{g\in\mathcal{F}_d(z+Q_r)} B_z^\theta(g)\ge\delta\hat{h}_t^{-d}\Big\}\\ &\le \#\{\mathcal{N}_t\}\mathbb{P}\Big\{\sup_{g\in\mathcal{F}_d(Q_r)} B_0^\theta(g)\ge\delta\hat{h}_t^{-d}\Big\}\end{aligned}$$

On the other hand, since $\alpha(\cdot)$ is supported on [0,3] and $s>1$ we have

$$A_z^\theta(g)=-\theta\int_{z+Q_{3^{-1}h_t^{-s}}}\left[\int_{z+Q_r}K_{t,a}(y-x)g^2(y)dy\right]\omega(\hat{h}_t^{-d}dx)+\frac{1}{2\hat{h}_t^d}\int_{z+Q_r}|\nabla g(x)|^2dx$$

as $t$ is sufficiently large. By the fact that $z\in\mathcal{N}_t$ and the sets $z+Q_{3^{-1}h_t^{-s}}$ are disjoint in $z$. Consequently, the sequence

$$\sup_{g\in\mathcal{F}_d(z+Q_r)} A_z^\theta(g); \qquad z\in\mathcal{N}_t$$

is an i.i.d. sequence. Therefore, we have

$$\begin{aligned}&\mathbb{P}\Big\{\max_{z\in\mathcal{N}_t}\ \sup_{g\in\mathcal{F}_d(z+Q_r)} A_z^\theta(g)\le-(1-\delta)\hat{h}_t^{-d}\Big\}\\ &\qquad=\left(\mathbb{P}\Big\{\sup_{g\in\mathcal{F}_d(Q_r)} A_0^\theta(g)\le-(1-\delta)\hat{h}_t^{-d}\Big\}\right)^{\#\{\mathcal{N}_t\}}\\ &\qquad=\left(1-\mathbb{P}\Big\{\sup_{g\in\mathcal{F}_d(Q_r)} A_0^\theta(g)\ge-(1-\delta)\hat{h}_t^{-d}\Big\}\right)^{\#\{\mathcal{N}_t\}}.\end{aligned}$$

Summarizing the above estimates, we have

$$\begin{aligned}
&\mathbb{P}\{\lambda_{\theta\xi}(Q_t) \le -u{h_t}^2\} \\
&\qquad \le \left(1 - \mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(Q_r)} A_0^\theta(g) \ge -(1-\delta)\hat{h}_t^{-d}\right\}\right)^{\#\{\mathcal{N}_t\}} \\
&\qquad\quad + \#\{\mathcal{N}_t\}\mathbb{P}\{\sup_{g\in\mathcal{F}_d(Q_r)} B_0^\theta(g) \ge \delta\hat{h}_t^{-d}\} 
\end{aligned} \tag{3.28}$$

Taking $\varepsilon = \hat{h}_t^d$ and $\gamma = 1-\delta$ in Theorem 3.1, we have

$$\liminf_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(Q_r)} A_0^\theta(g) \ge -(1-\delta)\hat{h}_t^{-d}\right\} \ge -u^{-d/2} I_{Q_r}(1-\delta)$$

$$\lim_{a\to\infty} \limsup_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(Q_r)} B_0^\theta(g) \ge \delta\hat{h}_t^{-d}\right\} = -\infty$$

By monotonicity of $I_{Q_r}(1-\delta)$ in $r$ and by the definition of $I_{Q_r}(\cdot)$ given in (3.11),

$$\begin{aligned}
\lim_{r\to\infty} I_{Q_r}(1-\delta) &= \frac{2}{d+2}\left(\frac{d}{(d+2)(1-\delta)}\right)^{d/2} \\
&\quad\times \inf_{g\in\mathcal{F}_d}\left(\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}\frac{g^2(y)}{|y-x|^{d+2}}dy\right)dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right)^{(d+2)/2}.
\end{aligned}$$

Take $u > -(1-\delta)^{-1}\Lambda(\theta)$ where $\Lambda(\theta)$ is given in (3.17). Then, there is a $\rho > 0$ such that when $a$ and $r$ are sufficiently large,

$$\mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(Q_r)} A_0^\theta(g) \ge -(1-\delta)\hat{h}_t^{-d}\right\} \ge \exp\{-(d-\rho)\log t\} = t^{-(d-\rho)}$$

and

$$\mathbb{P}\left\{\sup_{g\in\mathcal{F}_d(D)} B_0^\theta(g) \ge \delta\hat{h}_t^{-d}\right\} \le \exp\{-2d\log t\} = t^{-2d}$$

for sufficiently large $t$.

Combine these with (3.28) and by the fact that $\#\{\mathcal{N}_t\} \sim ct^d(\log t)^{-(s+1)}$ as $t \to \infty$, we have for any $u > -(1-\delta)^{-1}\Lambda(\theta)$,

$$\begin{aligned}
&\mathbb{P}\{\lambda_{\theta\xi}(Q_t) \le -u{h_t}^2\} \\
&\qquad \le (1 - t^{-(d-\rho)})^{\#\{\mathcal{N}_t\}} + \#\{\mathcal{N}_t\}t^{-2d}\} \\
&\qquad \le \exp\{-ct^{\rho}(\log t)^{-(s+1)}\} + ct^{-d}.
\end{aligned}$$

Hence, picking $\gamma = 2$, for any $u > -(1-\delta)^{-1}\Lambda(\theta)$, we have

$$\sum_k \mathbb{P}\{\lambda_{\theta\xi}(Q_{\gamma^k}) \le -uh_{\gamma^k}^2\} < \infty.$$

Once again, by the Borel-Cantelli lemma,

$$\liminf_{k\to\infty} h_{\gamma^k}^{-2}\lambda_{\theta\xi}(Q_{\gamma^k}) \ge (1-\delta)^{-1}\Lambda(\theta) \qquad a.s.$$

Repeating the similar procedures in (3.25) and (3.26), since $\lambda_{\theta\xi}(Q_t)$ is monotonic in $t$ and $\delta > 0$ can be arbitrarily small, we have proved (3.3). □

### 3.3.2 Upper bound

In this section, we establish the upper bound of theorem 2.2. More precisely, we prove that for any $\theta > 0$

$$\limsup_{t\to\infty} t^{-1}(\log t)^{2/d} \log \mathbb{E}_0 \exp\Big\{ -\theta \int_0^t V(B_s)ds\Big\} \le \Lambda(\theta) \qquad a.s.\text{-}\mathbb{P} \tag{3.29}$$

where $\Lambda(\theta)$ is defined in (3.17).

For any open set $D \subset \mathbb{R}^d$, define the stopping time

$$\tau_D = \inf\{s \ge 0; B_s \notin D\}.$$

Write $R_t = Mth_t$ where $h_t$ is given in (3.13), and the constant $M > 0$ is fixed but sufficiently large. Our strategy roughly relies on the following relationship:

$$E_0 \exp\left\{\theta \int_0^t \xi(B_s)ds\right\} \approx \exp\{t\lambda_{\theta\xi}(Q_{R_t})\}, \tag{3.30}$$

where the principle eigenvalue is defined in (3.14), $\xi = -V$, and radius $R_t$ grows a bit slower than linear in our setting. To obtain the upper bound, we consider the decomposition

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \xi(B_s)ds\right\} \\
&\quad = \mathbb{E}_0\left[\exp\left\{\theta \int_0^t \xi(B_s)ds\right\}; \tau_{Q_{R_t}} \geq t\right] \\
&\qquad + \mathbb{E}_0\left[\exp\left\{\theta \int_0^t \xi(B_s)ds\right\}; \tau_{Q_{R_t}} < t\right] \\
&\quad \leq \mathbb{E}_0\left[\exp\left\{\theta \int_0^t \xi(B_s)ds\right\}; \tau_{Q_{R_t}} \geq t\right] + \mathbb{P}_0\{\tau_{Q_{R_t}} < t\},
\end{aligned} \tag{3.31}$$

where the last step follows from the fact $\xi \leq 0$. Notice that the second term on the right hand side of the inequality is negligible in our setting. In fact, by the Gaussian tail estimates,

$$\mathbb{P}_0\{\tau_{Q_{R_t}} < t\} \leq \exp\left\{-cR_t{}^2/t\right\} = \exp\left\{-cM^2th_t^2\right\}.$$

On the other hand, the first term in the above decomposition is the dominating term. Let $\alpha, \beta > 1$ satisfy $\alpha^{-1} + \beta^{-1} = 1$ with $\alpha$ close to 1. According to Lemma 4.1

and Lemma 4.3 in [8], we have

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left\{\theta\int_0^t \xi(B_s)ds\right\};\tau_{Q_{R_t}}\ge t\right]\\
&\le\left(\mathbb{E}_0\exp\left\{\theta\beta\int_0^1\xi(B_s)ds\right\}\right)^{1/\beta}\\
&\quad\times\left\{\frac{1}{(2\pi)^{d/2}}\int_{Q_{R_t}}\mathbb{E}_x\left[\exp\left\{\theta\alpha\int_0^{t-1}\xi(B_s)ds\right\};\tau_{Q_{R_t}}\ge t-1\right]dx\right\}^{1/\alpha}\\
&\le\left(\frac{2R_t^2}{\pi}\right)^{d/(2\alpha)}\left(\mathbb{E}_0\exp\left\{\theta\beta\int_0^1\xi(B_s)ds\right\}\right)^{1/\beta}\exp\left\{(t-1)\lambda_{\alpha\theta\xi}(Q_{R_t})\right\}. \qquad (3.32)
\end{aligned}$$

The idea behind the above step is to localize Brownian motion and rearrange the starting point of Brownian motion uniformly over $R_t$. Then we get an explicit bound by applying Lemma 4.1 in [8]. And the price we paid here is that Brownian motion can reach anywhere else in a finite range of time, which is affordable.

Summarizing our estimates we have

$$\begin{aligned}
\mathbb{E}_0\exp\left\{\theta\int_0^t\xi(B_s)ds\right\}\le\left(\frac{2R_t^2}{\pi}\right)^{d/(2\alpha)}\left(\mathbb{E}_0\exp\left\{\theta\beta\int_0^1\xi(B_s)ds\right\}\right)^{1/\beta}\exp\left\{t\lambda_{\alpha\theta\xi}(Q_{R_t})\right\}\\
+\exp\left\{-cM^2th_t^2\right\}. \qquad (3.33)
\end{aligned}$$

Consequently, (3.29) follows from Lemma (3.2). Indeed, we apply (3.16) to the first term on the right-hand side of (3.33) (with t being replaced by $R_t = Mth_t$ and $\theta$ being replaced by $\alpha\theta$). Notice that $\alpha$ can be arbitrarily close to 1. This term will give us the exact bound in (3.29) by letting $\alpha \to 1^+$. The second term of right hand side in (3.33) is negligible as $M > 0$ can be sufficiently large.

## 3.4 Lower bound

Next we establish the lower bound of Theorem 2.2. More precisely, we prove that

$$\liminf_{t\to\infty} t^{-1}(\log t)^{2/d}\log\mathbb{E}_0\exp\left\{-\theta\int_0^t V(B_s)ds\right\}\ge\Lambda(\theta)\qquad a.s.\text{-}\mathbb{P}. \qquad (3.34)$$

where $\Lambda(\theta)$ is given in (3.17).

On the way of the proof of the lower bound, we need the following Lemma to bound the Poisson potential.

**Lemma 3.4.** *([15], Lemma 5) Under $p > d$,*

$$\sup_{x\in(-t,t)^d} V(x) \le 3d\log t \qquad\qquad a.s.\text{-}\mathbb{P} \tag{3.35}$$

*for sufficiently large $t$.*

Now we turn to prove the lower bound. We write $\xi = -V$. The strategy is as follows. Following the spirit of (3.30), we reduce the problem to investigate the lower bound of the principle eigenvalue and then using the corresponding lower bound given in Lemma (3.3).

Let $0 < h < 1$ be fixed and will be requested to be arbitrarily close to 1. Let $\alpha, \beta > 1$ satisfy $\alpha^{-1} + \beta^{-1} = 1$ with $\alpha$ being close to 1. By choosing $\delta = t^h$ in Lemma 4.3 in [8], we have

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \xi(B_s)ds\right\} \\
&\quad\ge \left(\mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^h} \xi(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\qquad\times\left\{\int_{Q_{t^h}} p_{t^h}(x)\mathbb{E}_x\left[\exp\left\{\alpha^{-1}\theta\int_0^{t-t^h} \xi(B_s)\right\}; \tau_{Q_{t^h}} \ge t - t^h\right]dx\right\}^{\alpha}
\end{aligned} \tag{3.36}$$

where $p_{t^h}(x)$ is the probability density function of $B_{t^d}$, more specifically

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}}\exp\{-\frac{|x|^2}{2t}\}.$$

Combine this with (3.36), we get

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \xi(B_s)ds\right\} \\
&\geq \frac{1}{(2\pi t^h)^{\alpha d/2}} e^{-t^h/2}\left(\mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^h} \xi(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\times \left\{\int_{Q_{t^h}} \mathbb{E}_x\left[\exp\left\{\alpha^{-1}\theta\int_0^{t-t^h} \xi(B_s)\right\}; \tau_{Q_{t^h}} \geq t - t^h\right]dx\right\}^{\alpha}
\end{aligned} \tag{3.37}$$

Again, by taking $\delta = t^h$ in Lemma 4.3 in [8],

$$\begin{aligned}
&\int_{Q_{t^h}} \mathbb{E}_x\left[\exp\left\{\alpha^{-1}\theta\int_0^{t-t^h} \xi(B_s)\right\}; \tau_{Q_{t^h}} \geq t - t^h\right]dx \\
&\geq (2\pi)^{d\alpha/2}(t-t^h)^{dh/2}(t-t^h)^{h\alpha/(2\beta)}(t-t^h)^{-2d\alpha/\beta} \\
&\times \exp\left\{-(\alpha/\beta)t^h\lambda_{\alpha^{-2}\beta\theta\xi}(Q_{t^h}) + \alpha t\lambda_{\alpha^{-2}\theta\xi}(Q_{t^h})\right\}.
\end{aligned} \tag{3.38}$$

Summarizing our estimates and replacing $e^{-t^h/2}$ by $e^{-t^h}$ to absorb all polynomial growth quantities,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\theta \int_0^t \xi(B_s)ds\right\} \\
&\geq e^{-t^h}\left(\mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^h} \xi(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\times \exp\left\{-(\alpha^2/\beta)t^h\lambda_{\alpha^{-2}\beta\theta\xi}(Q_{t^h}) + \alpha^2 t\lambda_{\alpha^{-2}\theta\xi}(Q_{t^h})\right\}
\end{aligned} \tag{3.39}$$

for sufficiently large $t$.

Let me try to explain the strategy applied here. Within a period $[0, t^h]$, we force the Brownian motion quickly running into a box with radius $t^h$ and spend the rest of its life time there. We take $h < 1$ to make sure the price paid by the Brownian motion on the way to the box is insignificant. Living in the box, the Brownian motion is allowed to rearrange its starting point uniformly over $Q_{t^h}$ with the affordable price $e^{-t^h}$.

Next, replacing $V$ by $-\xi$ in (3.35),

$$\log \mathbb{E}_0 \exp\Big\{ -\frac{\theta\beta}{\alpha}\int_0^{t^h} \xi(B_s)ds\Big\} = o(t^{\frac{1+h}{2}}), \qquad a.s.(t\to\infty).$$

Also, by (3.16),

$$\frac{\alpha^2}{\beta} t^h \lambda_{\alpha^{-2}\beta\theta\xi}(Q_{t^h}) = o(t^{\frac{1+h}{2}}) \qquad a.s.$$

for sufficiently large $t$. Therefore, (3.34) follows from Lemma (3.3). Indeed, replacing $\theta$ by $\alpha^{-2}\theta$ in Lemma (3.3) and applying (3.39) gives us,

$$\liminf_{t\to\infty} \frac{(\log t^h)^{2/d}}{t} \log \mathbb{E}_0 \exp\left\{\theta \int_0^t \xi(B_s)ds\right\} \geq \alpha^2 \Lambda(\alpha^{-2}\theta) \qquad a.s.$$

Notice that $(\log t^h)^{2/d} = h^{2/d}(\log t)^{2/d}$. By letting $\alpha \to 1^+$ and $h \to 1^-$, we obtain (3.34).

# Chapter 4

# Long Time Annealed Asymptotic: p=d+2

In this Chapter, we give the proof of Theorem 2.3. The Poisson random potential investigated in this chapter is given by

$$V(x) = \int_{\mathbb{R}^d} K(y-x)\omega(dy)$$

where $K(x) = \frac{1}{|x|^{d+2}}$ which includes singularity of kernel function $K(x)$ at the original. Ôkura [26] investigated this annealed case with the kernel function $K(x)$ bounded at $x = 0$ which is different from ours. Here, we will present a different strategy.

## 4.1 Two analytic lemmas

In the following section, we will discuss the proof of two analytic lemmas which are useful for the proof of the upper bound of Theorem 2.3.

**Lemma 4.1.** *The following inequality holds:*

$$\liminf_{M\to\infty}\inf_{g\in\mathcal{F}_d}\left\{\int_{[-M,M]^d}\varphi\left(\int_{\mathbb{R}^d}K_R(x-y)\hat{g}^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$
$$\geq\inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}K_R(x-y)g^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\} \tag{4.1}$$

*where* $\varphi$ *is defined in* (2.1), $K_R$ *is defined in* (4.24) *and*

$$\hat{g}(x)=\left(\sum_{z\in\mathbb{Z}^d}g^2(2Mz+x)\right)^{1/2}.$$

*Proof.* Let $g\in\mathcal{F}_d$ be fixed and since

$$\hat{g}^2(x)=\sum_{z\in\mathbb{Z}^d}g^2(2Mz+x) \qquad x\in\mathbb{R}^d.$$

Then $\hat{g}$ is absolutely continuous and we have

$$\int_{[-M,M]^d}\hat{g}^2(x)dx=\int_{\mathbb{R}^d}g^2(x)dx=1. \tag{4.2}$$

After computation of $\nabla\hat{g}(x)$ and then using the Cauchy-Schwarz inequality we have

$$|\nabla\hat{g}(x)|^2\leq\sum_{z\in\mathbb{Z}^d}|\nabla g(2Mz+x)|^2. \tag{4.3}$$

For large enough $M$ we set $m=M-\sqrt{M}$ and write

$$E=\bigcup_{i=1}^{d}(\{-M\leq x_i\leq -m+\sqrt{m}\}\bigcup\{m-\sqrt{m}\leq x_i\leq M\}).$$

Taking $\lambda=\sqrt{M}+\sqrt{m}$ in Lemma 3.4 in [13], there is an $a\in\mathbb{R}^d$ such that

$$\int_E\hat{g}^2(x-a)dx\leq\frac{d(\sqrt{M}+\sqrt{m})}{2^{d-1}M}\leq\frac{d}{2^{d-2}\sqrt{M}}$$

where the last inequality follows from the simple fact $m \leq M$.

Without loss of generality, we may assume $a = 0$, that is,

$$\int_E \hat{g}^2(x)dx \leq \frac{d}{2^{d-2}\sqrt{M}}. \tag{4.4}$$

for otherwise we can replace $\hat{g}(\cdot)$ by $\hat{g}(\cdot + a)$. Define the function $\kappa$ on $\mathbb{R}$ by

$$\kappa(x) = \begin{cases} xm^{-1/2} & \text{if } -m \leq x \leq -m + m^{1/2} \\ 1 & \text{if } -m + m^{-1/2} \leq x \leq m - m^{1/2} \\ m^{1/2} - xm^{-1/2} & \text{if } m - m^{1/2} \leq x \leq m \\ 0 & \text{otherwise} \end{cases}$$

and write

$$\hat{\kappa}(x) = \kappa(x_1)\kappa(x_2)\cdots\kappa(x_d), \qquad x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d.$$

It is not hard to verify that $|\hat{\kappa}| \leq 1$ and $|\nabla\hat{\kappa}| \leq \sqrt{d/m}$. Next, we define

$$f(x) = \hat{g}(x)\hat{\kappa}(x)\left(\int_{\mathbb{R}^d} \hat{g}^2(x)\hat{\kappa}^2(x)dx\right)^{-1/2} \tag{4.5}$$

which obviously belongs to $\mathcal{F}_d$. Denote $A = \int_{\mathbb{R}^d} \hat{g}^2(x)\hat{\kappa}^2(x)dx$. Clearly, $0 \le A \le 1$. By (4.2), (4.3) and Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla f|^2 dx &= \frac{1}{A}\Big\{ \int_{\mathbb{R}^d} |\nabla \hat{g}|^2 |\hat{\kappa}|^2 dx + \int_{\mathbb{R}^d} |\hat{g}|^2 |\nabla \hat{\kappa}|^2 dx + 2\int_{\mathbb{R}^d} \hat{g}\hat{\kappa}\langle \nabla \hat{g}, \nabla \hat{\kappa}\rangle dx \Big\} \\
&\le \frac{1}{A}\Big\{ \int_{[-m,m]^d} |\nabla \hat{g}|^2 dx + \frac{d}{m}\int_{[-m,m]^d} |\hat{g}|^2 dx + 2\Big( \int_{[-m,m]^d} |\nabla \hat{g}|^2 |\nabla \hat{\kappa}|^2 dx \Big)^{1/2} \Big\} \\
&\le \frac{1}{A}\Big\{ \int_{[-m,m]^d} |\nabla \hat{g}|^2 dx + \frac{d}{m} + 2\sqrt{\frac{d}{m}}\Big( \int_{[-m,m]^d} |\nabla \hat{g}|^2 |dx \Big)^{1/2} \Big\} \\
&\le \frac{1}{A}\Big\{ \Big(1+\sqrt{\frac{d}{m}}\Big) \int_{[-m,m]^d} |\nabla \hat{g}|^2 dx + \frac{d}{m} + \sqrt{\frac{d}{m}} \Big\} \\
&\le \frac{1}{A}\Big\{ \Big(1+\sqrt{\frac{d}{m}}\Big) \int_{[-M,M]^d} |\nabla \hat{g}|^2 dx + 2\sqrt{\frac{d}{m}} \Big\} \\
&\le \frac{1}{A}\Big\{ \Big(1+\sqrt{\frac{d}{m}}\Big) \int_{\mathbb{R}^d} |\nabla g|^2 dx + 2\sqrt{\frac{d}{m}} \Big\} \qquad (4.6)
\end{aligned}$$

where the fourth step follows from the inequality $2ab \le a^2 + b^2$, more specifically,

$$2\sqrt{\frac{d}{m}}\Big( \int_{[-m,m]^d} |\nabla \hat{g}|^2 |dx \Big)^{1/2} \le \sqrt{\frac{d}{m}} + \sqrt{\frac{d}{m}} \int_{[-m,m]^d} |\nabla \hat{g}|^2 |dx.$$

On the other hand, let $M \ge 9R^2$, by (4.5), definition of the functional $K_R$ and monotonicity of the functional $\psi$,

$$\begin{aligned}
\int_{[-M,M]^d} \varphi\left( \int_{\mathbb{R}^d} K_R(x-y)\hat{g}^2(y)dy \right) dx &\ge \int_{\mathbb{R}^d} \varphi\left( A\int_{\mathbb{R}^d} K_R(x-y) f^2(y)dy \right) dx \\
&\ge A\int_{\mathbb{R}^d} \varphi\left( \int_{\mathbb{R}^d} K_R(x-y) f^2(y)dy \right) dx
\end{aligned} \qquad (4.7)$$

where the last step follows from the inequality $\varphi(\theta x) \ge \theta\varphi(x)$ for $0 < \theta < 1$ and $x > 0$.

Next we estimate $A$, by (4.4)

$$\begin{aligned}A &= \int_{\mathbb{R}^d} \hat{g}^2(x)\hat{\kappa}^2(x)dx = \int_{[-M,M]^d} \hat{g}^2(x)\hat{\kappa}^2(x)dx \\ &= \int_{[-M,M]^d\setminus E} \hat{g}^2(x)dx + \int_E \hat{g}^2(x)\hat{\kappa}^2(x)dx \\ &\geq \int_{[-M,M]^d} \hat{g}^2(x)dx - \int_E \hat{g}^2(x)dx \geq 1 - \frac{d}{2^{d-2}\sqrt{M}}. \end{aligned} \tag{4.8}$$

Combining this with (4.6) and (4.7), we have

$$\begin{aligned}&\int_{[-M,M]^d} \varphi\left(\int_{\mathbb{R}^d} K_R(x-y)\hat{g}^2(y)dy\right)dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \\ &\geq A\left(1+\sqrt{\frac{d}{m}}\right)^{-1}\left\{\left(1+\sqrt{\frac{d}{m}}\right)\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d} K_R(x-y)f^2(y)dy\right)dx \right. \\ &\quad \left. +\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2 dx\right\} - 2\left(1+\sqrt{\frac{d}{m}}\right)^{-1}\sqrt{\frac{d}{m}} \\ &\geq \left(1-\frac{d}{2^{d-2}\sqrt{M}}\right)\left(1+\sqrt{\frac{d}{m}}\right)^{-1}\inf_{f\in\mathcal{F}_d}\left\{\left(1+\sqrt{\frac{d}{m}}\right)\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d} K_R(x-y)f^2(y)dy\right)dx \right. \\ &\quad \left. +\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2 dx\right\} - 2\left(1+\sqrt{\frac{d}{m}}\right)^{-1}\sqrt{\frac{d}{m}}.\end{aligned}$$

Since $m = M - \sqrt{M}$, $m \to \infty$ as $M \to \infty$. By taking the infimum on the left-hand side over $g \in \mathcal{F}_d$ and letting $M \to \infty$ on the both sides, we obain (4.1). □

**Lemma 4.2.** *The following inequality holds*

$$\begin{aligned}&\liminf_{R\to\infty}\inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d} K_R(x-y)g^2(y)dy\right)dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2 dx\right\} \\ &\geq \inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx + \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2 dx\right\}\end{aligned} \tag{4.9}$$

*where $K_R$ is defined in (4.24).*

*Proof.* For any $\varepsilon > 0$, there exists a $g_\varepsilon \in \mathcal{F}_d$, such that,

$$\inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}K_R(x-y)g^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$
$$\geq\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}K_R(x-y)g_\varepsilon^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g_\varepsilon(x)|^2dx-\varepsilon.$$

By the continuity of the functional $\varphi$ and fatou's lemma,

$$\liminf_{R\to\infty}\left\{\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}K_R(x-y)g_\varepsilon^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g_\varepsilon(x)|^2dx-\varepsilon\right\}$$
$$\geq\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}\frac{g_\varepsilon^2(y)}{|x-y|^{d+2}}dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g_\varepsilon(x)|^2dx-\varepsilon$$
$$\geq\inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}-\varepsilon.$$

(4.9) is obtained by combining the above results and letting $\varepsilon\to 0^+$. □

## 4.2 Annealed asymptotic of p=d+2

In this section we will give the proof of Theorem 2.3 in details.

**Theorem 4.1.** *For any $\theta > 0$,*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{-t\int_{\mathbb{R}^d}\varphi\left(\frac{\theta}{t}\int_0^t K(x-B_s)ds\right)dx\right\}$$
$$=-\inf_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_{\mathbb{R}^d}\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\tag{4.10}$$

***Proof of theorem 2.3 based on Theorem 4.1.*** Notice that, by Fubini theorem

$$\int_0^t V(B_s)ds=\int_{\mathbb{R}^d}\int_0^t K(x-B_s)ds\omega(dx).$$

Again by Fubini theorem and by Poisson integral formula,

$$\mathbb{E}\otimes\mathbb{E}_0\exp\left\{-\theta\int_0^t V(B_s)ds\right\}=\mathbb{E}_0\exp\left\{-\int_{\mathbb{R}^d}\varphi\left(\theta\int_0^t K(x-B_s)ds\right)dx\right\}\tag{4.11}$$

Replacing $s$ with $t^{2/(d+2)}s$ and then by scaling property of the Brownian motion, one can easily get

$$\int_{\mathbb{R}^d} \varphi\left(\theta \int_0^t K(x-B_s)ds\right) dx \overset{d}{=} t^{d/(d+2)} \int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t^{d/(d+2)}} \int_0^{t^{d/(d+2)}} K(x-B_s)ds\right) dx.$$

Combine this with (4.11), we have

$$\begin{aligned}
&\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{-\theta \int_0^t V(B_s)ds\right\} \\
&\qquad = \mathbb{E}_0 \exp\left\{-t^{d/(d+2)} \int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t^{d/(d+2)}} \int_0^{t^{d/(d+2)}} K(x-B_s)ds\right) dx\right\}.
\end{aligned}$$

By the above fact and by replacing $t^{d/(d+2)}$ with $t$, we reduced the proof of (2.16) to the proof of (4.10).

### 4.2.1 Lower bound

First we establish the lower bound of (4.10). For any bounded open domain $D \subseteq \mathbb{R}^d$ containing 0, let $D^\delta$ be the $\delta$-neighborhood of $D$, that is, $D^\delta = \{x \in \mathbb{R}^d; d(x, D) < \delta\}$ where $\delta > 0$ is fixed but small. We shall avoid the singularity of the kernel function $K$ by restricting Brownian particle to stay inside the domain $D$ and forcing Poisson obstacles to live outside of $D^\delta$ up to time $t$,

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{-t \int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t} \int_0^t K(x-B_s)ds\right) dx\right\} \\
&\geq \mathbb{E}_0 \left[\exp\left\{-t \int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t} \int_0^t \frac{ds}{|x-B_s|^{d+2}}\right) dx\right\}; \tau_D \geq t\right] \\
&\geq \exp\{-t|D^\delta|\}\mathbb{E}_0 \left[\exp\left\{-t \int_{\mathbb{R}^d \setminus D^\delta} \varphi\left(\frac{\theta}{t} \int_0^t \frac{ds}{|x-B_s|^{d+2}}\right) dx\right\}; \tau_D \geq t\right] \\
&= \exp\{-t|D^\delta|\}\mathbb{E}_0 \left[\exp\left\{-t \int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t} \int_0^t \frac{1_{\mathbb{R}^d \setminus D^\delta}(x)}{|x-B_s|^{d+2}} ds\right) dx\right\}; \tau_D \geq t\right],
\end{aligned} \tag{4.12}$$

where the second step follows from the simple fact $\varphi(x) \leq 1$ for all $x \in \mathbb{R}$.

Define

$$\xi(g) = -\int_{\mathbb{R}^d} \varphi(g(x))dx; \qquad g \in \mathcal{L}(\mathbb{R}^d), \tag{4.13}$$

which is a convex function on $\mathcal{L}(\mathbb{R}^d)$.

Since $\xi$ is finite and continuous, it is subdifferetiable on $\mathcal{L}(\mathbb{R}^d)$. For any $g_0 \in \mathcal{L}(\mathbb{R}^d)$, let $f \in \mathcal{L}^\infty(\mathbb{R}^d)$ be a sub-derivative of $\xi(g)$ at $g = g_0$, i.e.,

$$\xi(g) - \xi(g_0) \geq \langle f, g - g_0 \rangle; \qquad g \in \mathcal{L}(\mathbb{R}^d),$$

or

$$\xi(g) \geq \eta(g_0) + \langle f, g \rangle; \qquad g \in \mathcal{L}(\mathbb{R}^d), \tag{4.14}$$

where $\eta(g_0) = \xi(g_0) - \langle f, g_0 \rangle$.

Next, we denote

$$h(x) = \frac{\theta}{t} \int_0^t \frac{1_{\mathbb{R}^d \setminus D^\delta}(x)}{|x - B_s|^{d+2}} ds. \tag{4.15}$$

Obviously, $h(x)$ belongs to $\mathcal{L}(\mathbb{R}^d)$ on $\{\tau_D > t\}$. Thus, by (4.14), for any $g_0 \in \mathcal{L}(\mathbb{R}^d)$,

$$\begin{aligned}
&\mathbb{E}_0 \left[ \exp\left\{ -t \int_{\mathbb{R}^d} \varphi\left( \frac{\theta}{t} \int_0^t \frac{1_{\mathbb{R}^d \setminus D^\delta}(x)}{|x - B_s|^{d+2}} ds \right) dx \right\}; \tau_D \geq t \right] \\
&\geq \exp\{t\eta(g_0)\} \mathbb{E}_0 \left[ \exp\left\{ \theta \int_{\mathbb{R}^d} f(x) \int_0^t \frac{1_{\mathbb{R}^d \setminus D^\delta}(x)}{|x - B_s|^{d+2}} ds dx \right\}; \tau_D \geq t \right] \\
&= \exp\{t\eta(g_0)\} \mathbb{E}_0 \left[ \exp\left\{ \theta \int_0^t \hat{f}(B_s) ds \right\}; \tau_D \geq t \right]
\end{aligned} \tag{4.16}$$

where $f \in \mathcal{L}^\infty(\mathbb{R}^d)$ is sub-derivative of $\xi(g)$ at $g = g_0$ and

$$\hat{f}(y) = \int_{\mathbb{R}^d} \frac{1_{\mathbb{R}^d \setminus D^\delta}(x)}{|x - y|^{d+2}} f(x) dx; \qquad y \in D.$$

Notice that $\hat{f}$ is bounded and continuous on $D$. By Theorem 4.1.6 in [8],

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\left[\exp\left\{\theta\int_0^t\hat{f}(B_s)ds\right\};\tau_D\geq t\right]\\&=\sup_{g\in\mathcal{F}_d(D)}\left\{\theta\int_D\hat{f}(x)g^2(x)dx-\frac{1}{2}\int_D|\nabla g(x)|^2dx\right\}.\end{aligned}\tag{4.17}$$

Summarizing our estimates above, by (4.12), (4.16) and (4.17) we have

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{-t\int_{\mathbb{R}^d}\varphi\left(\frac{\theta}{t}\int_0^tK(x-B_s)ds\right)dx\right\}\\&\geq-|D^\delta|+\sup_{g\in\mathcal{F}_d(D)}\left\{\eta(g_0)+\int_{\mathbb{R}^d}f(x)\left[\theta 1_{\mathbb{R}^d\setminus D^\delta}(x)\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy\right]dx-\frac{1}{2}\int_D|\nabla g(x)|^2dx\right\}.\end{aligned}\tag{4.18}$$

where $\eta$ is defined in (4.14). Denote

$$g^*(x)=\theta 1_{\mathbb{R}^d\setminus D^\delta}(x)\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy;\qquad x\in\mathbb{R}^d.$$

where $g\in\mathcal{F}_d(D)$. Clearly, $g^*\in\mathcal{L}(\mathbb{R}^d)$. Then let $f^*\in\mathcal{L}^\infty(\mathbb{R}^d)$ be a sub-derivative of $\xi(g)$ at $g=g^*$. Notice that

$$\begin{aligned}&\sup_{g_0\in\mathcal{L}(\mathbb{R}^d)}\left(\eta(g_0)+\int_{\mathbb{R}^d}f(x)\left[\theta 1_{\mathbb{R}^d\setminus D^\delta}(x)\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy\right]dx\right)\\&\qquad\geq\eta(g^*)+\int_{\mathbb{R}^d}f^*(x)\left[\theta 1_{\mathbb{R}^d\setminus D^\delta}(x)\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy\right]dx\\&\qquad=\xi(g^*)=-\int_{\mathbb{R}^d\setminus D^\delta}\varphi\left(\theta\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx.\end{aligned}\tag{4.19}$$

where the last step follows from (4.13). Following (4.18), (4.19) and taking supremum over $g_0\in\mathcal{L}(\mathbb{R}^d)$ on the right hand side of (4.18), we have

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{-t\int_{\mathbb{R}^d}\varphi\left(\frac{\theta}{t}\int_0^tK(x-B_s)ds\right)dx\right\}\\&\geq-|D^\delta|-\inf_{g\in\mathcal{F}_d(D)}\left\{\int_{\mathbb{R}^d\setminus D^\delta}\varphi\left(\theta\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx+\frac{1}{2}\int_D|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Letting $\delta \to 0^+$ on the right-hand side leads to

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{-t\int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t}\int_0^t K(x-B_s)ds\right)dx\right\}$$
$$\geq - \inf_{g\in\mathcal{F}_d(D)} \left\{|D| + \int_{\mathbb{R}^d\setminus D} \varphi\left(\theta\int_D \frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx + \frac{1}{2}\int_D |\nabla g(x)|^2 dx\right\}. \quad (4.20)$$

Let $\widetilde{\mathcal{F}}_d(D)$ be the sub-class of $\mathcal{F}_d(D)$ consisting of the functions that are almost non-zero on $D$. Then by (4.20), we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{-t\int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t}\int_0^t \frac{ds}{|x-B_s|^{d+2}}\right)dx\right\}$$
$$\geq - \inf_{g\in\widetilde{\mathcal{F}}_d(D)} \left\{|D| + \int_{\mathbb{R}^d\setminus D} \varphi\left(\theta\int_D \frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx + \frac{1}{2}\int_D |\nabla g(x)|^2 dx\right\}. \quad (4.21)$$

Let $r > 0$ be fixed but small, notice that for any $x \in D$ and $g \in \widetilde{\mathcal{F}}_d(D)$,

$$\int_D \frac{g^2(y)}{|x-y|^{d+2}}dy \geq r^{-(d+2)} \int_{|y|<r} g^2(x+y)dy. \quad (4.22)$$

According to Lebesgue differentiation theorem, as $r \to 0$,

$$r^{-d}\int_{|y|<r} g^2(x+y)dy \to g^2(x) \qquad a.e.$$

Bringing this to (4.22), for almost every $x \in D$, we have

$$\int_D \frac{g^2(y)}{|x-y|^{d+2}}dy = \infty.$$

Consequently,

$$\int_D \varphi\left(\theta\int_D \frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx = |D|.$$

Combine this result with (4.21),

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{-t\int_{\mathbb{R}^d}\varphi\left(\frac{\theta}{t}\int_0^t K(x-B_s)ds\right)dx\right\}$$
$$\geq -\inf_{g\in\widetilde{\mathcal{F}}_d(D)}\left\{\int_{\mathbb{R}^d}\varphi\left(\theta\int_D\frac{g^2(y)}{|x-y|^{d+2}}dy\right)dx+\frac{1}{2}\int_D|\nabla g(x)|^2dx\right\}. \tag{4.23}$$

The right-hand side of (4.23) can be extended to all $g\in\mathcal{F}_d$ for the following reasons: First, $\widetilde{\mathcal{F}}_d(D)$ is dense in $\mathcal{F}_d(D)$ under the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d)}$. Second, the infinitely smooth, rapidly decreasing and locally supported functions are dense in the Sobolev space $W^{1,2}(\mathbb{R}^d)$ under the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d)}$. Taking supremum over $g\in\mathcal{F}_d$ on the right-hand side of (4.23) we obtain the lower bound of (4.10).

### 4.2.2 Upper bound

Next, we turn to prove the upper bound of (4.10). To remove the singularity of the shape function $K$ we create a smooth truncation. Let the smooth function $\alpha:\mathbb{R}^+\to[0,1]$ satisfy the following properties: $\alpha(u)=1$ on $[0,1]$, $\alpha(u)=0$ for $u\geq 3$ and $-1\leq\alpha'(u)\leq 0$ for all $u\geq 0$. Let $R>0$ be fixed but large. Define

$$K_R(x)=\frac{1}{|x|^{d+2}}\alpha(R^{-1}|x|)(1-\alpha(R|x|));\qquad x\in\mathbb{R}^d. \tag{4.24}$$

Let $M > 3R$ be fixed but arbitrary. Recall $K(x) = \frac{1}{|x|^{d+2}}$. By the fact that $K_R(x) \le K(x)$ for all $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t}\int_0^t K(x - B_s)ds\right)dx \\
&\quad \ge \int_{\mathbb{R}^d} \varphi\left(\frac{\theta}{t}\int_0^t K_R(x - B_s)ds\right)dx \\
&\quad = \sum_{z\in\mathbb{Z}^d}\int_{[-M,M]^d} \varphi\left(\frac{\theta}{t}\int_0^t K_R(2Mz + x - B_s)ds\right)dx \\
&\quad \ge \int_{[-M,M]^d} \varphi\left(\frac{\theta}{t}\int_0^t \sum_{z\in\mathbb{Z}^d} K_R(2Mz + x - B_s)ds\right)dx \\
&\quad = \int_{[-M,M]^d} \varphi\left(\frac{\theta}{t}\int_0^t \hat{K}_R(x - B_s)ds\right)dx \qquad (4.25)
\end{aligned}$$

where we denote

$$\hat{K}_R(x) = \sum_{z\in\mathbb{Z}^d} K_R(2Mz + x); \qquad x \in \mathbb{R}^d$$

and the second inequality follows from the fact that

$$\varphi(u_1 + u_2) \le \varphi(u_1) + \varphi(u_2); \qquad u_1, u_2 \ge 0.$$

An critical observation to the proof is that $\hat{K}_R$ is a periodic extension of $K_R$ by the locality of $K_R$ and the fact $M > 3R$. In particular, $\hat{K}_R$ is uniformly continuous and bounded with the same supremum bounds as $K_R$. Consequently, for any integer $k \ge 1$ there is a $\delta_k > 0$ such that

$$\max\left\{|\hat{K}_R(x) - \hat{K}_R(y)|; \quad x, y \in [-M,M]^d \text{ and } |x - y| < \delta_k\right\} \le \frac{1}{k}. \qquad (4.26)$$

Denote

$$A \equiv \sup_{x\in\mathbb{R}^d} K_R(x) < \infty,$$

and define the subset $\mathcal{C} \subseteq \mathcal{L}([-M,M]^d)$ as

$$\mathcal{C} = \bigcap_{k=1}^{\infty} \Big\{ h \in \mathcal{L}([-M,M]^d); \quad 0 \le h(x) \le A \text{ and } \sup_{|x-y|\le \delta_k} |h(x)-h(y)| \le \frac{1}{k} \Big\}.$$

Clearly, the class $\mathcal{C}$ is uniformly bounded and equicontinuous. According to Arzelá-Ascoli theorem, $\mathcal{C}$ is relatively compact in $\mathcal{L}([-M,M]^d)$,

For any fixed $t > 0$, the following process

$$Z_t(x) = \frac{1}{t}\int_0^t \hat{K}_R(x - B_s)ds; \qquad x \in [-M,M]^d,$$

can be considered as a stochastic process with values in the Hilbert space $\mathcal{L}([-M,M]^d)$. By (4.26), it is not hard to see that $Z_t \in \mathcal{C}$. Let $\mathcal{K}$ be the closure of $\mathcal{C}$ in $\mathcal{L}([-M,M]^d)$.

Consider the convex function

$$\xi_M(g) := -\int_{[-M,M]^d} \varphi(g(x))dx, \qquad g \in \mathcal{L}([-M,M]^d).$$

Let $D\xi_M(g) \in \mathcal{L}^\infty([-M,M]^d)$ be a sub-derivative of $\xi_M(g)$. By convexity of $\xi_M(g)$, for any $g \in \mathcal{L}([-M,M]^d)$,

$$\xi_M(h) \ge \xi_M(g) + \langle D\xi_M(g), h - g\rangle \qquad h \in \mathcal{L}([-M,M]^d),$$

or

$$\xi_M(h) \ge \eta_M(g) + \langle D\xi_M(g), h\rangle \qquad h \in \mathcal{L}([-M,M]^d), \tag{4.27}$$

where $\eta_M(g) = \xi_M(g) - \langle D\xi_M(g), g\rangle$.

Let $\varepsilon > 0$ be fixed. By continuity of $\xi_M$ and by the fact that the equality in (4.27) holds as $h = g$, the family of the open sets

$$\mathcal{O}_g = \big\{ h \in \mathcal{L}([-M,M]^d); \qquad \xi_M(h) < \varepsilon + \eta_M(g) + \langle D\xi_M(g), h\rangle \big\}; \qquad g \in \theta\mathcal{K}$$

forms an open cover of the compact set $\theta\mathcal{K}$. By the finite-cover theorem, one can pick $\mathcal{O}_{g_1}, ..., \mathcal{O}_{g_m}$ as a finite sub-cover such that

$$\xi_M(h) \le \varepsilon + \max_{1\le i\le m}\Big\{\eta_M(g_i) + \langle D\xi_M(g_i), h\rangle\Big\}$$

for any $h \in \theta\mathcal{K}$. In particular,

$$\xi_M(\theta Z_t) \le \varepsilon + \max_{1\le i\le m}\Big\{\eta_M(g_i) + \theta\langle D\xi_M(g_i), Z_t\rangle\Big\}.$$

Consequently,

$$\begin{aligned}\mathbb{E}_0 \exp&\left\{-t\int_{[-M,M]^d}\varphi\left(\frac{\theta}{t}\int_0^t \hat{K}_R(x-B_s)ds\right)dx\right\}\\ &\le \exp\Big\{\varepsilon t\Big\}\sum_{i=1}^m \exp\Big\{t\eta_M(g_i)\Big\}\mathbb{E}_0\exp\Big\{\theta t\langle D\xi_M(g_i), Z_t\rangle\Big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp&\left\{-t\int_{[-M,M]^d}\varphi\left(\frac{\theta}{t}\int_0^t \hat{K}_R(x-B_s)ds\right)dx\right\}\\ &\le \varepsilon + \max_{1\le i\le m}\Big\{\eta_M(g_i) + \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\Big\{\theta t\langle D\xi_M(g_i), Z_t\rangle\Big\}\end{aligned} \tag{4.28}$$

Let $1 \le i \le m$ be fixed. Notice that

$$t\langle D\xi_M(g_i), Z_t\rangle = \int_0^t \hat{f}_i(B_s)ds,$$

where

$$\hat{f}_i(y) = \int_{[-M,M]^d}\hat{K}_R(x-y)D\xi_M(g_i)(x)dx$$

is bounded and continuous. By Theorem 4.1.6 in [8], we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{\theta\int_0^t \hat{f}_i(B_s)ds\right\}$$
$$=\sup_{g\in\mathcal{F}_d}\left\{\theta\int_{\mathbb{R}^d}\hat{f}_i(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\} \tag{4.29}$$

Combining this with (4.28), we get

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{-t\int_{[-M,M]^d}\varphi\left(\frac{\theta}{t}\int_0^t\hat{K}_R(x-B_s)ds\right)dx\right\}$$
$$\le \varepsilon+\max_{1\le i\le m}\left\{\sup_{g\in\mathcal{F}_d}\left\{\eta_M(g_i)+\theta\int_{\mathbb{R}^d}\hat{f}_i(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\right\}$$
$$\le \varepsilon-\inf_{g\in\mathcal{F}_d}\left\{\int_{[-M,M]^d}\varphi\left(\theta\int_{\mathbb{R}^d}\hat{K}_R(x-y)g^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

where the last step follows from (4.27).

Letting $\varepsilon\to 0^+$ on the right hand side and noticing that

$$\int_{\mathbb{R}^d}\hat{K}_R(x-y)g^2(y)dy=\int_{\mathbb{R}^d}K_R(x-y)\hat{g}^2(y)dy,$$

where

$$\hat{g}(y)=\Big(\sum_{z\in\mathbb{Z}^d}g^2(2Mz+y)\Big)^{1/2}.$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{-t\int_{[-M,M]^d}\varphi\left(\frac{\theta}{t}\int_0^t\hat{K}_R(x-B_s)ds\right)dx\right\}$$
$$\le -\inf_{g\in\mathcal{F}_d}\left\{\int_{[-M,M]^d}\varphi\left(\int_{\mathbb{R}^d}K_R(x-y)\hat{g}^2(y)dy\right)dx+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.$$

Combining this with (4.25) and letting $M\to\infty$ and $R\to\infty$, the upper bound of (4.10) follows from Lemma 4.1 and Lemma 4.2. □

# Chapter 5

# Quenched Asymptotic: p>d+2

In this chapter, we give the proof of Theorem 2.1. On the way of the proof of the upper bound of the Theorem 2.1, some estimates on the integrated density of states of the random Schrödinger operator $-\frac{1}{2}\Delta + \theta V$ are obtained. For the proof of the lower bound, we employ the "empty-ball" strategy developed by Sznitman in [31].

## 5.1 Estimate on the integrated density of states

In this section, Some estimates on the integrated density of states of the random Schrödinger operator $-\frac{1}{2}\Delta + \theta V$ are given. The proof is very close to that in [15].

The integrated density of states measures the number of energy levels per unit volume below a given energy. It involves counting the eigenvalues of an operator below a certain threshold. Consider the random Schrödinger operator $-\frac{1}{2}\Delta + \theta V$. Then the integrated density of states is defined by

$$N(\lambda) = \lim_{R\to\infty} \frac{1}{|Q_R|}\mathbb{E}[\#\{n; \bar{\lambda}_n(Q_R) \le \lambda\}] \tag{5.1}$$

where $Q_R = (-R, R)^d$ is an open box and $\bar{\lambda}_n(Q_R)$ is the $n$-th smallest eigenvalue of $-\frac{1}{2}\Delta + \theta V$ in $Q_R$ with the Dirichlet boundary condition. We refer the readers to [23] and [27] for more details about the integrated density of states function. Denote

$\bar{\lambda}_{\theta V}(Q_R) = \bar{\lambda}_1(Q_R)$. Let $D \subseteq \mathbb{R}^d$ be an open set, by the Rayleigh-Ritz formula,

$$\bar{\lambda}_{\theta V}(D) = \inf_{g \in \mathcal{F}_d(D)} \Big\{ \theta \int_D V(x) g^2(x) dx + \frac{1}{2} \int_D |\nabla g(x)|^2 dx \Big\}. \tag{5.2}$$

Recall that $\lambda_\xi(D)$ defined in (3.14). Clearly, we have

$$\lambda_{\theta\xi}(D) = -\bar{\lambda}_{\theta V}(D) \tag{5.3}$$

where $\xi = -V$.

Now we give the estimate on the integrated density of states.

**Theorem 5.1.** *Assume that $p > d+2$. Then*

$$\lim_{\lambda \to 0^+} \lambda^{d/2} \log N(\lambda) = -\omega_d \lambda_d^{d/2} \tag{5.4}$$

*where $w_d$ is the volume of the unit ball and $\lambda_d$ denotes the principle Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in $B(0,1)$.*

*Proof.* The proof here is very similar to that of Theorem 3 in [15], details are left to the reader for the sake of practice. □

**Remark 10.** *The connection between the integrated density of states function $N$ and our problem is the fact [23] that*

$$N(\lambda) = \sup_{R>0} \frac{1}{|Q_R|} \mathbb{E}[\#\{n; \bar{\lambda}_n(Q_R) \le \lambda\}.$$

*Key observations are*

$$\mathbb{E}[\#\{n; \bar{\lambda}_n(Q_R) \le \lambda\} \ge \mathbb{P}\{\bar{\lambda}_{\theta V}(Q_R) \le \lambda\}$$

*and*

$$N(\lambda) \approx \exp\{-c\lambda^{-d/2}\}$$

*for sufficiently small $\lambda$, which give us*

$$\mathbb{P}\{\bar{\lambda}_{\theta V}(Q_R) \le \lambda\} \preceq (2R)^d \exp\{-c\lambda^{-d/2}\}. \tag{5.5}$$

*Consequently, by using Borel Cantelli lemma, we can obtain a sharp almost-sure asymptotic bound of $\bar{\lambda}_{\theta V}(Q_R)$ by carefully picking $\lambda$ which tends to $0$ in a proper way.*

## 5.2 Upper bound

In this section, we will prove the upper bound of Theorem 2.1. More precisely, we prove that under $p > d+2$, for any $\theta > 0$

$$\limsup_{t\to\infty} t^{-1}(\log t)^{2/d} \log \mathbb{E}_0 \exp\Big\{ -\theta \int_0^t V(B_s)ds\Big\} \le -\lambda_d \Big(\frac{\omega_d}{d}\Big)^{2/d} \qquad a.s.\text{-}\mathbb{P} \tag{5.6}$$

Repeating the similar procedures in (3.31), (3.32), (3.33) and by (5.3), to prove (5.6), all we need is to show

$$\liminf_{t\to\infty} (\log t)^{2/d} \bar{\lambda}_{\theta V}(Q_t) \ge \lambda_d \Big(\frac{\omega_d}{d}\Big)^{2/d} \qquad a.s.\text{-}\mathbb{P}. \tag{5.7}$$

where $\bar{\lambda}_{\theta V}(Q_t)$ is defined in (5.2).

First notice that, taking $R = t$ in (5.1), for any $\lambda > 0$ we have

$$N(\lambda) \ge (2t)^{-d}\mathbb{P}\{\bar{\lambda}_{\theta V}(Q_t) \le \lambda\}. \tag{5.8}$$

Let $u > 0$ be a constant which will be specified later. By (5.8) and by taking $\lambda = u(\log t)^{-2/d}$ in (5.4), we have

$$\mathbb{P}\Big\{\bar{\lambda}_{\theta V}(Q_t) \le u(\log t)^{-2/d}\Big\} \le (2t)^d \exp\Big\{ -u^{-d/2}\omega_d \lambda_d^{d/2} \log t(1+o(1))\Big\}$$

for sufficiently large $t$.

Let $\delta > 0$ be fixed but arbitrary. Take $u = (1-\delta)\lambda_d \left(\frac{\omega_d}{d}\right)^{2/d}$. Then there is a $\nu > 0$ such that

$$\mathbb{P}\{\bar{\lambda}_{\theta V}(Q_t) \leq (1-\delta)\lambda_d \left(\frac{\omega_d}{d}\right)^{2/d} (\log t)^{-2/d}\} \leq Ct^d \exp\{-(d+\nu)\log t\} = \frac{C}{t^\nu}$$

for sufficiently large t.

Hence, for any $\gamma > 1$, we have

$$\sum_{k=1}^{\infty} \mathbb{P}\{\bar{\lambda}_{\theta V}(Q_{\gamma^k}) \leq (1-\delta)\lambda_d \left(\frac{\omega_d}{d}\right)^{2/d} (\log \gamma^k)^{-2/d}\} < \infty$$

By the Borel-Cantelli lemma,

$$\liminf_{k\to\infty} (\log \gamma^k)^{2/d} \bar{\lambda}_{\theta V}(Q_{\gamma^k}) \geq (1-\delta)\lambda_d \left(\frac{\omega_d}{d}\right)^{2/d} \qquad a.s.$$

Since $\bar{\lambda}_{\theta V}(Q_t)$ is monotonic in $t$ and $\delta > 0$ can be arbitrarily small, by repeating the similar procedures in (3.25) and (3.26), we have proved (5.7).

## 5.3 Lower bound

In this section, we turn to prove the lower bound of Theorem 2.1. Let us first introduce some notations. Write $R_t = t/(\log t)^{3/d}$, $b_t = (\log t)^{3/d}$ and $r_t = r(\log t)^{1/d}$ where $r > 0$. Denote $\mathcal{N}_t = b_t\mathbb{Z}^d \cap B(0, R_t)$. Set

$$D_z = B(z, r_t); \qquad z \in \mathcal{N}_t. \tag{5.9}$$

For large enough $t$, $D_z$ are disjoint evenly located identical micro-balls. Set $t_k = 2^k$ $(k = 1, 2, ...)$. The following lemma will describe the behavior of Poisson obstacles, that is, at least one of the micro-balls $\{D_z;\ z \in \mathcal{N}_{t_k}\}$ will be Poisson obstacles free for large enough $k$.

**Lemma 5.1.** *For $\mathbb{P}$ almost every $\omega$, there exists $k_0(\omega) < \infty$, such that when $k > k_0(\omega)$ and $r < (\frac{d}{\omega_d})^{1/d}$, we have*

$$\omega(D_{z_k}) = 0,$$

*for some $z_k \in \mathcal{N}_{t_k}$.*

*Proof.* By shifting invarance of the Poisson field, the random variables

$$\omega(D_z); \qquad z \in \mathcal{N}_t$$

are independent and identically distributed. Consequently,

$$\begin{aligned}\mathbb{P}\{\min_{z\in\mathcal{N}_t} \omega(D_z) \geq 1\} &= (1 - \mathbb{P}\{\omega(D_0) = 0\})^{\#(\mathcal{N}_t)} \\ &= (1 - \exp\{-\omega_d r^d \log t\})^{\#(\mathcal{N}_t)}\end{aligned}$$

By the fact that $\#(\mathcal{N}_t) \sim ct^d/(\log t)^6$ as $t \to \infty$, the above expression is equivalent to

$$\exp\{-ct^{d-\omega_d r^d}(\log t)^6\},$$

for sufficiently large $t$. Notice that, for $r < \left(\frac{\omega_d}{d}\right)^{1/d}$,

$$\sum_{k=1}^{\infty} \mathbb{P}\{\min_{z\in\mathcal{N}_{t_k}} \omega(D_z) \geq 1\} < \infty.$$

Hence, by Borel-Cantelli lemma, we obtain our claim. □

The proof of the lower bound also relies on the following lemma.

**Lemma 5.2.** *For $p > d + 2$ and $r > 0$, we have*

$$\lim_{t\to\infty} (\log t)^{2/d} \int_{\{|x-z|\geq r(\log t)^{1/d}\}} \frac{1}{|x - z|^p} \omega(dx) = 0 \qquad a.s.\text{-}\mathbb{P}. \tag{5.10}$$

*Proof.* Let $\theta > 0$ be fixed but arbitrary, by Poisson integral formula,

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\int_{\{|x|\geq r(\log t)^{1/d}\}}\frac{(\log t)^{\frac{d+2}{d}}}{|x|^p}\omega(dx)\right\} \\
&= \exp\left\{\int_{\{|x|\geq r(\log t)^{1/d}\}}\psi\left(\frac{\theta(\log t)^{\frac{d+2}{d}}}{|x|^p}\right)dx\right\} \\
&= \exp\left\{\log t\left\{\int_{\{|x|\geq r\}}\psi\left(\frac{\theta}{(\log t)^{\frac{p-d-2}{d}}|x|^p}\right)dx\right\}\right\} \qquad (5.11)
\end{aligned}$$

where $\psi(x) = e^x - 1$. Notice that $p > d+2$. Consequently, by (5.11) and by Dominated Convergence Theorem,

$$\lim_{t\to\infty}\frac{1}{\log t}\log\mathbb{E}\exp\left\{\theta\int_{\{|x|\geq r(\log t)^{1/d}\}}\frac{(\log t)^{\frac{d+2}{d}}}{|x|^p}\omega(dx)\right\} = 0.$$

Therefore, by a standard application of Chebyshev's inequality, for any $\delta > 0$ we have

$$\lim_{t\to\infty}\frac{1}{\log t}\log\mathbb{P}\left\{\int_{\{|x|\geq r(\log t)^{1/d}\}}\frac{1}{|x-z|^p}\omega(dx) \geq \delta\,(\log t)^{-2/d}\right\} = -\infty.$$

Hence, for sufficiently large t, we have

$$\mathbb{P}\left\{(\log t)^{2/d}\int_{\{|x-z|\geq r(\log t)^{1/d}\}}\frac{1}{|x-z|^p}\omega(dx) \geq \delta\right\} \leq t^{-2d}. \qquad (5.12)$$

For any $\gamma > 1$, set $t_k = \gamma^k$ $(k = 1, 2, ...)$. By (5.12),

$$\sum_{k=1}^{\infty}\mathbb{P}\left\{(\log t_k)^{2/d}\int_{\{|x-z|\geq r(\log t_k)^{1/d}\}}\frac{1}{|x-z|^p}\omega(dx) \geq \delta\right\} < \infty.$$

By Borel-Cantelli lemma,

$$\lim_{t\to\infty}(\log t_k)^{2/d}\int_{\{|x-z|\geq r(\log t_k)^{1/d}\}}\frac{1}{|x-z|^p}\omega(dx) = 0 \qquad a.s.\text{-}\mathbb{P}.$$

By monotonicity we have proved (5.10). $\square$

We are now ready to prove the lower bound. Repeat the similar procedures in (3.36), (3.37), (3.38) and (3.39), and by the relationship in (5.3), all we need is to show

$$\limsup_{t\to\infty}(\log t)^{2/d}\bar{\lambda}_{\theta V}(Q_t) \le \lambda_d \left(\frac{\omega_d}{d}\right)^{2/d} \qquad a.s.\text{-}\mathbb{P}. \tag{5.13}$$

We first fix $\omega$ in the setting of Lemma 5.1 and Lemma 5.2. Set $t_k = 2^k$ ($k = 1, 2, ...$). For large enough $k$ which depends on $\omega$, we denote by $z_k(\omega)$, the center of the micro-ball $D_{z_k}$ defined in (5.9) where no Poisson obstacle falls. For any large enough $t > 0$, there is a $k$ such that

$$t_{k-1} < t < t_k. \tag{5.14}$$

Notice that $D_{z_k} \subset Q_t$ for sufficiently large $k$. By monotonicity of $\bar{\lambda}_{\theta V}(D)$ in $D$, we have

$$\bar{\lambda}_{\theta V}(Q_t) \le \bar{\lambda}_{\theta V}(D_{z_k}) \tag{5.15}$$

where $D_{z_k} = B(z_k, r(\log t_k)^{1/d})$.

Let $r < s < (\frac{d}{\omega_d})^{1/d}$ be fixed but arbitrarily close to $(\frac{d}{\omega_d})^{1/d}$. By (5.15) and by the fact that $\omega(B(z_k, s(\log t_k)^{1/d})) = 0$, we see that for large enough $t$,

$$\begin{aligned}\bar{\lambda}_{\theta V}(Q_t) &\le \inf_{g\in\mathcal{F}_d(D_{z_k})}\left\{\theta\int_{D_{z_k}}\int_{\mathbb{R}^d}\frac{g^2(x)}{|y-x|^p}\omega(dy)dx + \frac{1}{2}\int_{D_{z_k}}|\nabla g(x)|^2dx\right\}\\ &= \inf_{g\in\mathcal{F}_d(D_{z_k})}\left\{\theta\int_{D_{z_k}}\int_{|y-z_k|\ge s(\log t_k)^{1/d}}\frac{g^2(x)}{|y-x|^p}\omega(dy)dx + \frac{1}{2}\int_{D_{z_k}}|\nabla g(x)|^2dx\right\}.\end{aligned}$$

Combine this with the inequality

$$|y-x| \ge |y-z_k| - |x-z_k| \ge (1-\frac{r}{s})|y-z_k|.$$

We have

$$\bar{\lambda}_{\theta V}(Q_t) \le \inf_{g\in\mathcal{F}_d(D_{z_k})}\left\{\theta(1-\frac{r}{s})^{-p}\int_{D_{z_k}}\int_{|y-z_k|\ge s(\log t_k)^{1/d}}\frac{g^2(x)}{|y-z_k|^p}\omega(dy)dx+\frac{1}{2}\int_{D_{z_k}}|\nabla g(x)|^2dx\right\}$$
$$=\theta(1-\frac{r}{s})^{-p}\int_{|y-z_k|\ge s(\log t_k)^{1/d}}\frac{1}{|y-z_k|^p}\omega(dy)+\inf_{g\in\mathcal{F}_d(D_0)}\left\{\frac{1}{2}\int_{D_0}|\nabla g(x)|^2dx\right\} \tag{5.16}$$

where $D_0 = B(0, r(\log t_k)^{1/d})$.

The first term on the right hand side of (5.16) is negligible in our setting according to Lemma 5.2. For the second term, by Theorem 4.1.6 in [8] and by a classic result of small ball estimate, we have

$$\inf_{g\in\mathcal{F}_d(D_0)}\left\{\frac{1}{2}\int_{D_0}|\nabla g(x)|^2dx\right\} = (\lambda_d r^{-2}+o(1))(\log t_k)^{-2/d} \tag{5.17}$$

for large enough $t_k$. Then by (5.16), (5.17) and Lemma 5.2,

$$\limsup_{k\to\infty}(\log t_k)^{2/d}\bar{\lambda}_{\theta V}(Q_t) \le \lambda_d r^{-2}. \tag{5.18}$$

By (5.14),

$$\begin{aligned}(\log t)^{2/d}\bar{\lambda}_{\theta V}(Q_t) &\le (\log t_{k-1})^{2/d}\bar{\lambda}_{\theta V}(Q_t)\\ &= \left(\frac{t_{k-1}}{t_k}\right)^{2/d}(\log t_k)^{2/d}\bar{\lambda}_{\theta V}(Q_t)\\ &= \left(\frac{k-1}{k}\right)^{2/d}(\log t_k)^{2/d}\bar{\lambda}_{\theta V}(Q_t).\end{aligned}$$

Combining this with (5.18) and letting $r \to (\frac{d}{\omega_d})^{1/d}$, we obtain (5.13).

# Chapter 6

# Quenched asymptotic: a case of time-independent Gaussian rough noise

In this chapter, we will give the proof of theorem 2.4. We first give a bound of exponential moment which is critical for the solvability of the PAM (6.1) in our case. Then we give the proof of theorem 2.4 which consists of two main steps. First, we associate the quenched exponential moment to the principal eigenvalue of the operator $\frac{1}{2}\Delta + V$ with the zero boundary on a open box of radius nearly linear. Then we give an estimate on the principal eigenvalue.

## 6.1 Preliminary

In this chapter we consider the following stochastic heat equation:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(x)u(t,x), & (t,x) \in [0,\infty) \times \mathbb{R}; \\ u(0,x) = 1, & x \in \mathbb{R}. \end{cases} \tag{6.1}$$

where $V = \dot{W}$ is a time-independent Gaussian rough noise (see Section 2.3 for details).

**Definition 6.1.1.** *Let $u = \{u(t,x); (t,x) \in [0,\infty) \times \mathbb{R}\}$ be a real-valued adapted random field. Assume that the process $\{p_{t-s}(x-y)u(s,y)1_{[0,t]}(s)\}$ is Skorokhod integrable with respect to $W(dy)$. Then we say that $u$ is a mild solution of (6.1) if it satisfies the following integral equation*

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)u(s,y)W(dy)ds. \tag{6.2}$$

*where $p_t(x)$ is the heat kernel.*

We shall establish a Feynman-Kac representation of the mild solution of (6.1),

$$u(t,0) = \mathbb{E}_0 \exp\Big\{\int_0^t V(B_s)ds\Big\}. \tag{6.3}$$

Notice that the random media $V$ is not even pointwise defined. Let's first make sense of the exponential moment in (6.3). Denote the mollifier $h \in \mathcal{S}(\mathbb{R})$ by

$$h(x) = \begin{cases} c\exp(\frac{1}{x^2-1}), & x \in (-1,1), \\ 0, & x \notin (-1,1), \end{cases} \tag{6.4}$$

where $c$ is a normalizing constant such that $\int_{\mathbb{R}} h(x)dx = 1$. Write

$$h_\varepsilon(x) = \varepsilon^{-1}h(\varepsilon^{-1}x). \tag{6.5}$$

For every $\varepsilon > 0$, we denote the Gaussian random field $V_\varepsilon$ as

$$V_\varepsilon(x) = \langle V, h_\varepsilon(\cdot - x)\rangle, \qquad x \in \mathbb{R}. \tag{6.6}$$

which is pointwise defined. As discussed in section 2.3, the $\mathcal{L}^2$-limit

$$\int_0^t V(B_s)ds \overset{def}{=} \lim_{\varepsilon \to 0^+} \int_0^t V_\varepsilon(B_s)ds \tag{6.7}$$

exists for every $t \geq 0$. In addition, the integral defined above is continuous in $t$. As a result, the Feynman-Kac representation (6.3) make sense in our setting. Also, by Lemma A.1 in [9], the process

$$\int_0^t V(B_s)ds, \qquad\qquad t \geq 0$$

is a mean-zero Gaussian with the variance given by

$$\mathbb{E}\Big\{\int_0^t V(B_s)ds\Big\}^2 = \int_{\mathbb{R}} \Big|\int_0^t e^{i\lambda B_s} ds\Big|^2 \mu(d\lambda).$$

where the tempered measure $\mu(d\lambda)$ is given in (2.7).

Consequently,

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\} = \mathbb{E}_0 \exp\left\{\frac{1}{2}\int_0^t \int_0^t \gamma(B_s - B_r)dsdr\right\}. \tag{6.8}$$

where $\gamma$ is defined in (2.11). In fact, the solvability of the system (6.1) relies on the exponential integrability of the integral

$$\int_0^t \int_0^t \gamma(B_s - B_r)dsdr.$$

We refer the readers to Section 2 in [10] or Section 4 in [5] for details about this fact. As a result, we can justify the Feynman-Kac representation (6.3) by showing the result below.

**Proposition 1.** *For $t > 0$, we have*

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\} < \infty. \tag{6.9}$$

*Proof.* By (6.8) and (2.11), we can easily see that

$$\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\} = \mathbb{E}_0 \exp\Big\{\frac{1}{2}\int_{\mathbb{R}} \Big|\int_0^t e^{i\lambda B_s} ds\Big|^2 \mu(d\lambda)\Big\}.$$

Now consider a process $Z$ defined as

$$Z_T = \max_{t\le T} \int_{\mathbb{R}} \Big| \int_0^t e^{i\lambda B_s} ds \Big|^2 \mu(d\lambda).$$

According to Theorem 1.3.5 in [7], to prove (6.9) all we need is to show that the stochastic process $Z_t$ is sub-additive. That is, for any $s,t \ge 0$, $Z_{s+t} \le Z_s + \hat{Z}_t$ for a random variable $\hat{Z}_t$ independent of $\{Z_u; 0 \le u \le s\}$ with $\hat{Z}_t \overset{d}{=} Z_t$.

Set

$$\hat{Z}_t = \max_{s<u\le t+s} \int_{\mathbb{R}} \Big| \int_s^u e^{i\lambda B_r} dr \Big|^2 \mu(d\lambda).$$

Obviously, we have $Z_{s+t} \le Z_s + \hat{Z}_t$. Also, because of

$$|e^{-i\lambda B_s}|^2 = 1,$$

$\hat{Z}_t$, under some proper substitution, can be written as

$$\hat{Z}_t = \max_{u\le t} \int_{\mathbb{R}} \Big| \int_0^u e^{i\lambda (B_{r+s} - B_s)} dr \Big|^2 \mu(d\lambda)$$

which is independent of $\{B_u; u \le s\}$. As a result, $\hat{Z}_t$ is independent of $\{Z_u; u \le s\}$. What's more, we have $\hat{Z}_t \overset{d}{=} Z_t$. Therefore, by Theorem 1.3.5 in [7], for all $\theta > 0$ and $t > 0$

$$\mathbb{E}_0 \exp\Big\{\theta Z_t\Big\} < \infty.$$

Consequently, we have proved (6.9). □

## 6.2 Gaussian supremum

In this section, Our main goal is to show that Gaussian supremum in (6.10) is finite when $D$ is bounded, which is important to link the supremum with the principal eigenvalue of the linear operator $\frac{1}{2}\Delta + V$. A nice strategy in [8] is employed here.

Let $D \subset \mathbb{R}$ be open and bounded. Set

$$\mathcal{K}_1(D) = \{g \in \mathcal{S}(D); \|g\|_2^2 = 1\}$$

and

$$\mathcal{G}_1(D) = \{g \in \mathcal{S}(D); \|g\|_2^2 + \frac{1}{2}\|g'\|_2^2 = 1\}.$$

where $\mathcal{S}(D)$ is the space of rapidly decreasing and infinitely smooth functions that vanish at the boundary of an open domain $D$. Our approach largely relies on the estimate of the supremum

$$\sup_{g \in \mathcal{K}_1(D)} \left\{ \langle V, g^2 \rangle - \frac{1}{2} \int_{\mathbb{R}} |g'(x)|^2 dx \right\}. \tag{6.10}$$

Notice that, $\mathcal{K}_1(D)$ is not a compact set. It is not sure if the above supremum is finite. When it is finite, we will have

$$\lambda_V(D) = \sup_{g \in \mathcal{K}_1(D)} \left\{ \langle V, g^2 \rangle - \frac{1}{2} \int_{\mathbb{R}} |g'(x)|^2 dx \right\}, \tag{6.11}$$

where $\lambda_V(D)$ is the principal eigenvalue of the linear operator $\frac{1}{2}\Delta + V$ with the zero boundary condition over D. We shall use strategy in [8] to prove that the supremum is finite when D is bounded.

Consider a pseudometric space $(E, \rho)$ with the pseudometric $\rho(\cdot, \cdot)$. Let $N(E, \rho, \varepsilon)$ be the minimal number of the open balls of the diameter no greater than $\varepsilon$, which are necessary for covering E. We take $E = \mathcal{G}_1(D)$ and

$$\rho(f, g) = \{\mathbb{E}[\langle V, f^2 \rangle - \langle V, g^2 \rangle]^2\}^{1/2}, \qquad f, g \in \mathcal{G}_1(D).$$

We have that

$$\rho(f,g) = \left\{ \int_{\mathbb{R}^2} \gamma(x-y)(f(x)^2 - g(x)^2)(f(y)^2 - g(y)^2)dxdy \right\}^{1/2}, \qquad f,g \in \mathcal{G}_1(D). \tag{6.12}$$

We first give an entropy type bound below. The proof is very similar to Proposition 2.1 in [8].

**Lemma 6.1.** *For $\beta > 1$, we have*

$$\lim_{\varepsilon \to 0^+} \varepsilon^\beta \log N(\mathcal{G}_1(D), \rho, \varepsilon) = 0. \tag{6.13}$$

*Noticing that we can choose some $\beta < 2$, then we have*

$$\int_0^1 \sqrt{\log N(\mathcal{G}_1(D), \rho, \varepsilon)} d\varepsilon < \infty. \tag{6.14}$$

*Proof.* Consider the $\varepsilon$-mollifier $h_\varepsilon(x)$ given in (6.5). For every $\varepsilon > 0$, we define the operator $\mathcal{S}_\varepsilon$ on $\mathcal{S}(\mathbb{R})$ as

$$\mathcal{S}_\varepsilon g(x) = \left\{ \int_{\mathbb{R}} g^2(x-y) h_\varepsilon(y) dy \right\}^{1/2}, \qquad x \in \mathbb{R}.$$

By Fourier transform, for $g \in \mathcal{G}_1(D)$

$$\mathbb{E}[\langle V, g^2 \rangle - \langle V, \mathcal{S}_\varepsilon(g)^2 \rangle]^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |1 - \mathcal{F}(h)(\varepsilon\lambda)|^2 |\mathcal{F}(g^2)(\lambda)|^2 \mu(d\lambda),$$

where $h$ is given in (6.5) and $\mathcal{F}(\varphi)(\lambda)$ denotes the Fourier transform of function $\varphi \in \mathcal{S}(\mathbb{R})$.

Notice that, we have $|1 - \mathcal{F}(h)(\varepsilon\lambda)| \leq 1$, $\mathcal{F}(h)(0) = 1$ and $|\mathcal{F}(h)'| < c$ for some constant $c > 0$. By mean-value theorem,

$$|1 - \mathcal{F}(h)(\varepsilon\lambda)| \leq |1 - \mathcal{F}(h)(\varepsilon\lambda)|^\delta \leq c^\delta |\varepsilon\lambda|^\delta, \qquad \lambda \in \mathbb{R}, \varepsilon > 0,$$

where $\delta$ is chosen by (2.9).

On the other hand, for any $g \in \mathcal{G}_1(D)$, we have $|\mathcal{F}(g^2)(\lambda)| \leq \|g\|_2^2 \leq 1$. In addition, for any $\lambda \in \mathbb{R} \setminus \{0\}$, integration by parts give us

$$\mathcal{F}(g^2)(\lambda) = \frac{2i}{\lambda} \int_{\mathbb{R}} g(x) g'(x) e^{i\lambda x} dx.$$

Hence, for $\lambda \in \mathbb{R} \setminus \{0\}$

$$|\mathcal{F}(g^2)(\lambda)| \leq \frac{2}{|\lambda|} \|g\|_2 \|g'\|_2 \leq \frac{2}{|\lambda|}.$$

Consequently, for any $g \in \mathcal{G}_1(D)$ and $\lambda \in \mathbb{R}$

$$|\mathcal{F}(g^2)(\lambda)|^2 \leq 4 \left(1 \wedge \frac{1}{|\lambda|^2}\right).$$

Summarizing our estimates, there is a constant $C_\delta > 0$ such that

$$\begin{aligned} \rho(g, \mathcal{S}_\varepsilon g) &= \mathbb{E}[\langle V, g^2 \rangle - \langle V, \mathcal{S}_\varepsilon(g)^2 \rangle]^2 \\ &\leq C_\delta \varepsilon^\delta \int_{\mathbb{R}} |\lambda|^\delta \left(1 \wedge \frac{1}{|\lambda|^2}\right) \mu(d\lambda). \end{aligned}$$

By (2.9) and by the choice of $C_\delta$ independent of $g \in \mathcal{G}_1(D)$, there is a $C > 0$ such that

$$\sup_{g \in \mathcal{G}_1(D)} \rho(g, \mathcal{S}_\varepsilon g) \leq C \varepsilon^\delta.$$

Write $\phi(\varepsilon) = \varepsilon^{\delta^{-1}}$. We have that

$$\sup_{g \in \mathcal{G}_1(D)} \rho(g, \mathcal{S}_{\phi(\varepsilon)} g) \leq C \varepsilon \tag{6.15}$$

Denote the pseudometric $\rho_\varepsilon$ by $\rho_\varepsilon(f,g) = \rho(\mathcal{S}_{\phi(\varepsilon)}f, \mathcal{S}_{\phi(\varepsilon)}g)$. Notice that by triangle inequality and by (6.15), for any $f, g \in \mathcal{G}_1(D)$,

$$\begin{aligned}\rho(f,g) &\leq \rho(f, \mathcal{S}_{\phi(\varepsilon)}f) + \rho(g, \mathcal{S}_{\phi(\varepsilon)}g) + \rho(\mathcal{S}_{\phi(\varepsilon)}f, \mathcal{S}_{\phi(\varepsilon)}g) \\ &\leq 2C\varepsilon + \rho_\varepsilon(f,g).\end{aligned}$$

Hence, for small enough $\varepsilon > 0$, we have $\rho(f,g) \leq \rho_\varepsilon(f,g) + o(1)$. Therefore, to prove (6.13), all we need is to show for $\beta > 1$,

$$\lim_{\varepsilon \to 0^+} \varepsilon^\beta \log N(\mathcal{G}_1(D), \rho_\varepsilon, \varepsilon) = 0. \tag{6.16}$$

Notice that by (6.12)

$$\begin{aligned}\rho_\varepsilon(f,g) \leq &\left(\int_{\mathbb{R}} |(\mathcal{S}_{\phi(\varepsilon)}f)^2(x) - (\mathcal{S}_{\phi(\varepsilon)}g)^2(x)| dx\right)^{1/2} \\ &\times \left(\sup_{x \in D_1} \left|\int_{\mathbb{R}} \gamma(x-y)\{(\mathcal{S}_{\phi(\varepsilon)}f)^2(x) - (\mathcal{S}_{\phi(\varepsilon)}g)^2(x)\} dx\right|\right)^{1/2},\end{aligned}$$

where $D_1$ is the 1-neighborhood of $D$. For $x \in D_1$, take

$$A_\varepsilon(f)(x) = (\mathcal{S}_{\phi(\varepsilon)}f)^2(x)$$

and

$$B_\varepsilon(f)(x) = \int_{\mathbb{R}} \gamma(x-y)(\mathcal{S}_{\phi(\varepsilon)}f)^2(x) dx$$

in Lemma $A.3$ in [9]. All we need is to exam that there are $p > 1$ satisfying

$$\beta > \frac{2p}{2p-1} > 1$$

and $C > 0$, $m > 0$ independent of $\varepsilon > 0$ such that

$$|(\mathcal{S}_{\phi(\varepsilon)}f)^2(x) - (\mathcal{S}_{\phi(\varepsilon)}f)^2(y)| \leq C\varepsilon^{-m}|x-y|, \tag{6.17}$$

$$\left|\int_{\mathbb{R}}\{\gamma(x-z)-\gamma(y-z)\}(\mathcal{S}_{\phi(\varepsilon)}f)^2(z)dz\right| \le C\varepsilon^{-m}|x-y|, \tag{6.18}$$

$$\int_{\mathbb{R}}|(\mathcal{S}_{\phi(\varepsilon)}f)^2(x)|^{2p}dx \le C \tag{6.19}$$

and

$$\left|\int_{\mathbb{R}}\gamma(x-z)(\mathcal{S}_{\phi(\varepsilon)}f)^2(z)dz\right| \le C. \tag{6.20}$$

for all $x, y \in D_1$ and $f \in \mathcal{G}_1(D)$.

By the mean value theorem

$$\begin{aligned}
|(\mathcal{S}_{\phi(\varepsilon)}f)^2(x)-(\mathcal{S}_{\phi(\varepsilon)}f)^2(y)| &\le \int_{\mathbb{R}}|h_{\phi(\varepsilon)}(x+z)-h_{\phi(\varepsilon)}(y+z)|f^2(z)dz \\
&\le C\phi^{-2}(\varepsilon)\|f\|_2^2|x-y| \\
&\le C\phi^{-2}(\varepsilon)|x-y|.
\end{aligned}$$

Thus, (6.17) follows with $m = 2\delta^{-1}$.

Notice the relation

$$\begin{aligned}
&\int_{\mathbb{R}}\{\gamma(x-z)-\gamma(y-z)\}(\mathcal{S}_{\phi(\varepsilon)}f)^2(z)dz \\
&\quad = \int_{\mathbb{R}}\gamma(z)\{(\mathcal{S}_{\phi(\varepsilon)}f)^2(z-x)-(\mathcal{S}_{\phi(\varepsilon)}f)^2(z-y)\}dz,
\end{aligned}$$

and the fact that

$$\int_{D'}|\gamma(z)|dz < \infty$$

where $D' = \{z_1 + z_2 \in \mathbb{R}; z_1, z_2 \in D_1\}$. Combine these with (6.17), we have (6.18).

We now turn to (6.19). For any $p > 1$, by Jensen's inequality,

$$\int_{\mathbb{R}}|(\mathcal{S}_{\phi(\varepsilon)}f)^2(x)|^{2p}dx \le \int_{\mathbb{R}}|f(z)|^{2p}dz.$$

Plus, by Gagliardo-Nirenberg inequality,

$$\int_{\mathbb{R}} |f(z)|^{2p} dz \le C \|f\|_2^{p-1} \|f'\|_2^{p+1} \le C.$$

Thus, we have proved (6.19).

It remains to prove (6.20). First notice that integration by parts give us

$$\int_{\mathbb{R}} e^{i\lambda x} (\mathcal{S}_{\phi(\varepsilon)} f)^2(x) dx = \frac{2i}{\lambda} \int_{\mathbb{R}} e^{i\lambda x} (\mathcal{S}_{\phi(\varepsilon)} f)(x) (\mathcal{S}_{\phi(\varepsilon)} f)'(x) dx.$$

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \gamma(x-z) (\mathcal{S}_{\phi(\varepsilon)} f)^2(z) dz \\
&= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} e^{i\lambda(x-z)} (\mathcal{S}_{\phi(\varepsilon)} f)^2(z) dz \right] |\lambda|^{1-2H} d\lambda \\
&= 2i \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} e^{i\lambda(x-z)} \frac{|\lambda|^{1-2H}}{\lambda} d\lambda \right] (\mathcal{S}_{\phi(\varepsilon)} f)(z) (\mathcal{S}_{\phi(\varepsilon)} f)'(z) dz.
\end{aligned}$$

Next we want to show that there exists a $M > 0$ such that for all $x \in \mathbb{R}$

$$\left| \int_{\mathbb{R}} e^{i\lambda x} \frac{|\lambda|^{1-2H}}{\lambda} d\lambda \right| < M.$$

In fact,

$$\begin{aligned}
&\int_{\mathbb{R}} e^{i\lambda x} \frac{|\lambda|^{1-2H}}{\lambda} d\lambda \\
&= \int_{\mathbb{R}} \frac{cos(\lambda x) |\lambda|^{1-2H}}{\lambda} d\lambda + i \int_{\mathbb{R}} \frac{sin(\lambda x) |\lambda|^{1-2H}}{\lambda} d\lambda \\
&= 2i \int_0^\infty \frac{sin(\lambda x)}{\lambda^{2H}} d\lambda.
\end{aligned}$$

Then by Dirichlet's test of improper integrals, there exists a $M > 0$ such that for all $x \in \mathbb{R}$

$$\left| \int_0^\infty \frac{sin(\lambda x)}{\lambda^{2H}} d\lambda \right| \le M.$$

Summarizing our estimates

$$\begin{aligned}&\left|\int_{\mathbb{R}}\gamma(x-z)(\mathcal{S}_{\phi(\varepsilon)}f)^2(z)dz\right|\\&\le 2\int_{\mathbb{R}}\left|\int_{\mathbb{R}}e^{i\lambda(x-z)}\frac{|\lambda|^{1-2H}}{\lambda}d\lambda\right||(\mathcal{S}_{\phi(\varepsilon)}f)(z)(\mathcal{S}_{\phi(\varepsilon)}f)'(z)|dz\\&\le 2M\int_{\mathbb{R}}|(\mathcal{S}_{\phi(\varepsilon)}f)(z)(\mathcal{S}_{\phi(\varepsilon)}f)'(z)|dz\\&\le 2M\|\mathcal{S}_{\phi(\varepsilon)}f\|_2^2\|\mathcal{S}_{\phi(\varepsilon)}f'\|_2^2\le C.\end{aligned}$$

where the last inequality is from $\|\mathcal{S}_{\phi(\varepsilon)}f\|_2\le\|f\|_2\le 1$, $\|\mathcal{S}_{\phi(\varepsilon)}f'\|_2\le\|f'\|_2\le 1$ and setting $C=2M$. □

By (6.14), the supremum in (6.10) is finite, integrable and $\{\langle V,g^2\rangle; g\in\mathcal{G}_1(D)\}$ has continuous sample paths with respect to the pseudometric induced by its covariance. Plus, such sample continuity is extended to $\mathcal{S}(\mathbb{R})$. We refer the reader to Section 2.2 in [9] and Appendix D in [7] for more detailed explanation.

Keep the above results in mind, we can get the following bound of Gaussian supremum.

**Lemma 6.2.** *Under the assumption of Theorem 2.4, we have*

$$\lim_{\varepsilon\to 0^+}\mathbb{E}\sup_{g\in\mathcal{G}_1(-\varepsilon,\varepsilon)}\langle V,g^2\rangle=0 \tag{6.21}$$

*Proof.* We refer the readers to Lemma 5.3 in [5] for a detailed proof. □

Then, we can apply this Lemma to get an explicit almost-sure asymptotic bound of the principle eigenvalue given in (6.11) as $D$ expends to the whole real line. More precisely, we will try to prove the following inequality:

$$\limsup_{t\to\infty}(\log t)^{-\frac{1}{1+H}}\lambda_V(Q_t)\le 2^{\frac{1}{1+H}}c_H{}^{\frac{1}{1+H}}\mathcal{E}^{\frac{1}{1+H}}\qquad a.s.$$

Our approach relies on the idea that the principal eigenvalue over a large domain can be dominated by the maximum of eigenvalues on some sub-domains, which is a strategy we applied several times in this paper. See Proposition 1 in [16] or Lemma 4.6 in [18] for details. A careful reader may find that the result holds only for pointwise defined potentials which is not our case. However, the same result is stated to be true for generalized functions in [9]. We first present this result in a lemma below.

**Lemma 6.3.** *Let $r \geq 2$ be large but fixed. There is a nonnegative and continuous function $\phi(x)$ on $\mathbb{R}$ whose support is contained in the 1-neighborhood of the grid $2r\mathbb{Z}$, such that for any generalized function $\zeta$,*

$$\lambda_{\zeta-\phi^y}(Q_{th_t}) \leq \max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} \lambda_\zeta(z+Q_{r+1}), \qquad\qquad y \in Q_r,$$

*where $\phi^y(x) = \phi(x+y)$. In addition, $\phi(x)$ is periodic with period $2r$*

$$\phi(x+2rz) = \phi(x); \qquad\qquad x \in \mathbb{R}, z \in \mathbb{Z},$$

*and there is a constant $K > 0$ independent of $r$ and $t$ such that*

$$\frac{1}{2r}\int_{Q_r} \phi(x)dx \leq \frac{K}{r}. \tag{6.22}$$

Now we are ready to give a sharp bound of our principle eigenvalue.

**Lemma 6.4.** *Under assumption of the Theorem 2.4,*

$$\limsup_{t\to\infty}(\log t)^{-\frac{1}{1+H}}\lambda_V(Q_t) \leq 2^{\frac{1}{1+H}} c_H^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}} \qquad a.s. \tag{6.23}$$

*where $\mathcal{E}$ is given in (2.18).*

*Proof.* Let $u > 0$ be fixed, and write $h_t = \sqrt{u}(\log t)^{\frac{1}{2(1+H)}}$. Write

$$g_t(x) = h_t^{1/2} g(h_t x), \qquad x \in \mathbb{R}$$

Under the substitution $g \to g_t$, we have that

$$\lambda_V(Q_t) = {h_t}^2 \sup_{g\in\mathcal{K}_1(Q_{th_t})} \left\{ h_t^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\}. \tag{6.24}$$

Let $\{\langle V_t, \psi\rangle; \psi \in \mathcal{S}(\mathbb{R})\}$ be the generalized Gaussian field defined as $\langle V_t, \psi\rangle = \langle V, \psi_t\rangle$, where $\psi_t(x) = h_t\psi(h_t x)$. Then we have $\langle V, g_t^2\rangle = \langle V_t, g^2\rangle$. Thus, by (6.24),

$$\begin{aligned}
\frac{1}{{h_t}^2}\lambda_V(Q_t) &= \sup_{g\in\mathcal{K}_1(Q_{th_t})} \left\{ h_t^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\} \\
&= \sup_{g\in\mathcal{K}_1(Q_{th_t})} \left\{ h_t^{-2}\langle V_t, g^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\} \\
&= \sup_{g\in\mathcal{K}_1(Q_{th_t})} \left\{ \langle h_t^{-2}V_t - \frac{1}{2r}\int_{Q_r} \phi^y(x)dy, g^2\rangle + \langle \frac{1}{2r}\int_{Q_r} \phi^y(x)dy, g^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\}
\end{aligned}$$

where $\phi$ is a periodic function given in Lemma 6.3. Then by (6.22), we have

$$\begin{aligned}
\frac{1}{{h_t}^2}\lambda_V(Q_t) \le& \frac{K}{r} + \sup_{g\in\mathcal{K}_1(Q_{th_t})} \left\{ \langle h_t^{-2}V_t - \frac{1}{2r}\int_{Q_r} \phi^y(x)dy, g^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\} \\
\le& \frac{K}{r} + \frac{1}{2r}\int_{Q_r} \sup_{g\in\mathcal{K}_1(Q_{th_t})} \left\{ \langle h_t^{-2}V_t - \phi^y, g^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\} dy \\
=& \frac{K}{r} + \frac{1}{2r}\int_{Q_r} \lambda_{h_t^{-2}V_t - \phi^y}(Q_{th_t}) dy
\end{aligned}$$

Consequently, by Lemma 6.3 and by the fact $|Q_r| = 2r$,

$$\begin{aligned}
\frac{1}{{h_t}^2}\lambda_V(Q_t) \le& \frac{K}{r} + \max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} \lambda_{h_t^{-2}V_t}(z + Q_{r+1}) \\
\le& \frac{K}{r} + \max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t).
\end{aligned}$$

where the last step follows from the fact $\langle V, g_t^2\rangle = \langle V_t, g^2\rangle$ and

$$X_z(t) = \sup_{g\in\mathcal{K}_1(z+Q_{r+1})} \left\{ h_t^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \right\}, \quad z \in 2r\mathbb{Z}\cap Q_{th_t}$$

Therefore, if we take $r > 0$ sufficiently large, we have

$$\mathbb{P}\{\lambda_V(Q_t) \geq {h_t}^2\} \leq \mathbb{P}\{\max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t) > 1\}. \tag{6.25}$$

By homogeneity of the Gaussian field $\{\langle V, \varphi\rangle; \varphi \in \mathcal{S}(\mathbb{R})\}$, the random variables

$$X_z(t), \quad z \in 2r\mathbb{Z} \cap Q_{th_t}$$

are identically distributed. Consequently, there exists a constant $C > 0$ such that

$$\begin{aligned}\mathbb{P}\{\max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t) > 1\} &\leq Cth_t\mathbb{P}\{X_0(t) > 1\} \\ &= Cth_t\mathbb{P}\{\sup_{g\in\mathcal{G}_1(Q_{r+1})} \langle V, g_t^2\rangle > h_t^2\}. \end{aligned} \tag{6.26}$$

where the last step follows from Lemma A.2 in [8].

Notice that for each $g \in \mathcal{G}_1(Q_{r+1})$, $(1 + \|g'\|^2h_t^2)^{-1/2}g_t(\cdot) \in \mathcal{G}_1(Q_{(r+1)h_t^{-1}})$. Then by linearity, we have

$$\begin{aligned}\mathbb{E}\sup_{g\in\mathcal{G}_1(Q_{r+1})}\langle V, g_t^2\rangle &= (1 + \|g'\|^2h_t^2)\mathbb{E}\sup_{g\in\mathcal{G}_1(Q_{r+1})}\langle V, \frac{g_t^2}{1 + \|g'\|^2h_t^2}\rangle \\ &\leq (1 + h_t^2)\mathbb{E}\sup_{f\in\mathcal{G}_1(Q_{(r+1)h_t^{-1}})}\langle V, f^2\rangle \\ &= o(h_t^2) \qquad\qquad (t \to \infty)\end{aligned}$$

where the last step is by Lemma 6.2. Then by the concentration inequality for Gaussian field,

$$\begin{aligned}\mathbb{P}\{\sup_{g\in\mathcal{G}_1(Q_{r+1})}\langle V, g_t^2\rangle > h_t^2\} &= \mathbb{P}\{\sup_{g\in\mathcal{G}_1(Q_{r+1})}\langle V, g_t^2\rangle - \mathbb{E}\sup_{g\in\mathcal{G}_1(Q_{r+1})}\langle V, g_t^2\rangle > (1 + o(1))h_t^2\} \\ &\leq \exp\left\{-(1 + o(1))\frac{h_t^4}{2\sigma_t^2}\right\},\end{aligned}$$

where

$$\sigma_t^2 = \sup_{g\in\mathcal{G}_1(Q_{r+1})} Var(\langle V, g_t^2\rangle).$$

In fact, by (2.6) we have

$$\begin{aligned}\sigma_t^2 &= c_H \sup_{g\in\mathcal{G}_1(Q_{r+1})} \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{i\lambda x} g_t^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda \\ &\le c_H h_t^{2-2H} \sup_{g\in\mathcal{G}_1} \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda\end{aligned}$$

Therefore, by the definition of $h_t$, we have

$$\mathbb{P}\{\sup_{g\in\mathcal{G}_1(Q_{r+1})} \langle V, g_t^2\rangle > h_t^2\} \le \exp\{-(1+\nu)\log t\} = \frac{1}{t^{1+\nu}} \tag{6.27}$$

for some $\nu > 0$, whenever $t$ is large and the constant $u$ satisfies $u > 2^{\frac{1}{1+H}} c_H{}^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}}$.

Summarising our estimates, by (6.25), (6.26) and (6.27) for any $\gamma > 1$ and $u > 2^{\frac{1}{1+H}} c_H{}^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}}$,

$$\sum_k \mathbb{P}\{\lambda_V(Q_{\gamma^k}) \ge h_{\gamma^k}^2\} < \infty.$$

By the Borel-Cantelli lemma,

$$\limsup_{k\to\infty} (\log\gamma^k)^{-\frac{1}{1+H}} \lambda_V(Q_{\gamma^k}) \le 2^{\frac{1}{1+H}} c_H{}^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}} \qquad a.s.$$

Since $\lambda_V(Q_t)$ is monotonic in $t$, by repeating the similar procedures in (3.25) and (3.26), we have proved (6.23). □

Finally, we give an estimation on the variation $\mathcal{E}$ which will be needed in our future computations.

**Lemma 6.5.** *We have the variational quantity*

$$\mathcal{E} = \sup_{g\in\mathcal{G}_1} \left\{ \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda \right\} < \infty. \tag{6.28}$$

*Proof.* First notice that for any $g \in \mathcal{G}_1$,

$$\left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x) dx \right| \leq \int_{\mathbb{R}} |g^2(x)| dx \leq 1.$$

In addition, an elementary integration by parts argument shows that

$$\int_{\mathbb{R}} e^{i\lambda x} g^2(x) dx = \frac{2i}{\lambda} \int_{\mathbb{R}} e^{i\lambda x} g(x) g'(x) dx.$$

Hence, for any $g \in \mathcal{G}_1$ we get

$$\left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x) dx \right| \leq 2|\lambda|^{-1} \int_{\mathbb{R}} |g(x)||g'(x)| dx \leq 2|\lambda|^{-1},$$

where the last inequality follows from Cauchy-Schwarz inequality and the fact that $\|g\|_2 \leq 1$ and $\|g'\|_2 \leq 1$ for $g \in \mathcal{G}_1$. Gathering the above two bounds we have obtained, we end up with

$$\begin{aligned} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x) dx \right|^2 |\lambda|^{1-2H} d\lambda &\leq \int_{-1}^{1} |\lambda|^{1-2H} d\lambda + 4 \int_{|\lambda| \geq 1} |\lambda|^{-(1+2H)} d\lambda \\ &= \frac{1}{1-H} + \frac{4}{H}. \end{aligned}$$

□

## 6.3 Upper bound

In this section we prove the upper bound of Theorem 2.4. More precisely, we prove that,

$$\begin{aligned} \limsup_{t \to \infty} t^{-1} (\log t)^{-\frac{1}{1+H}} \log \mathbb{E}_0 \exp \left\{ \int_0^t V(B_s) ds \right\} \\ \leq 2^{\frac{1}{1+H}} c_H^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}}, \qquad a.s. \end{aligned} \tag{6.29}$$

where $\mathcal{E}$ is defined in (2.18).

Write $R_k = R_k(t) = (Mt(\log t)^{1/2(1+H)})^k (k = 1, 2, ...)$ where the constant $M > 0$ is fixed but sufficiently large. Once again, we will connect the exponential moment in (6.3) to the principal eigenvalue of the operator $\frac{1}{2}\Delta + V$ with the zero boundary on a open box of radius nearly linear by using Lemmas 4.1 and 4.3 in [8]. However, the Lemmas were hold only for the pointwise defined potentials. Consequently, we should start with $V_\varepsilon$, a smooth version of the potential $V$. And then pass the inequality from $V_\varepsilon$ to $V$ by letting $\varepsilon \to 0^+$. Therefore, the following lemma is needed. We refer the readers to Lemma 2.2 in [9] for a proof.

**Lemma 6.6.** *Let $V_\varepsilon$ be given in (6.6). For any bounded open domain $D \subset \mathbb{R}$, let $D_\varepsilon$ be $\varepsilon$-neiborhood of $D$ and define*

$$\lambda_V^+(D) \equiv \lim_{\varepsilon\to 0^+} \lambda_V(D_\varepsilon).$$

*Then we have*

$$\lambda_V(D) \le \liminf_{\varepsilon\to 0^+} \lambda_{V_\varepsilon}(D) \le \limsup_{\varepsilon\to 0^+} \lambda_{V_\varepsilon}(D) \le \lambda_V^+(D), \qquad a.s. \tag{6.30}$$

The next Lemma will be also needed in our future computation.

**Lemma 6.7.** *Under assumption of the Theorem 2.4,*

$$\liminf_{\varepsilon\to 0^+} \mathbb{E}_0 \exp\left\{\int_0^t V_\varepsilon(B_s)ds\right\} = \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\}, \qquad a.s. \tag{6.31}$$

*Proof.* By letting $q = 1$ in Proposition 4.4 (ii) in [5], we have

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\otimes\mathbb{E}_0 \left|\exp\left\{\int_0^t V_\varepsilon(B_s)ds\right\} - \exp\left\{\int_0^t V(B_s)ds\right\}\right| = 0.$$

Therefore, applying Fatou's lemma, one have

$$\liminf_{\varepsilon\to 0^+} \mathbb{E}_0 \left|\exp\left\{\int_0^t V_\varepsilon(B_s)ds\right\} - \exp\left\{\int_0^t V(B_s)ds\right\}\right| = 0, \qquad a.s.$$

which gives us (6.31). ☐

Now we can turn to prove (6.29). Consider the decomposition

$$
\begin{aligned}
&\mathbb{E}_0 \exp\{\int_0^t V_\varepsilon(B_s)ds\} \\
&= \mathbb{E}_0 \left[\exp\{\int_0^t V_\varepsilon(B_s)ds\}; \tau_{Q_{R_1}} \geq t\right] \\
&\quad + \sum_{k=1}^{\infty} \mathbb{E}_0 \left[\exp\{\int_0^t V_\varepsilon(B_s)ds\}; \tau_{Q_{R_k}} < t \leq \tau_{Q_{R_k+1}}\right] \\
&\leq \mathbb{E}_0 \left[\exp\{\int_0^t V_\varepsilon(B_s)ds\}; \tau_{Q_{R_1}} \geq t\right] \\
&\quad + \sum_{k=1}^{\infty} (\mathbb{P}_0\{\tau_{Q_{R_k}} < t\})^{1/2} \left\{\mathbb{E}_0 \left[\exp\{2\int_0^t V_\varepsilon(B_s)ds\}; \tau_{Q_{R_k+1}} \geq t\right]\right\}^{1/2} \qquad (6.32)
\end{aligned}
$$

The well-known result on the Gaussian tail gives that

$$
(\mathbb{P}\{\tau_{Q_{R_k}} < t\})^{1/2} \leq \exp\{-c{R_k}^2/t\} = \exp\{-cM^2 t^{2k-1} (\log t)^{k/(1+H)}\}.
$$

Let $\alpha, \beta > 1$ satisfy $\alpha^{-1} + \beta^{-1} = 1$ with $\alpha$ close to 1. By Lemma 4.3 and Lemma 4.1 in [8], we have

$$
\begin{aligned}
&\mathbb{E}_0 \left[\exp\{\int_0^t V_\varepsilon(B_s)ds\}; \tau_{Q_{R_1}} \geq t\right] \\
&\quad \leq \frac{1}{(2\pi)^{d/\alpha}} \left(\mathbb{E}_0 \exp\{\beta \int_0^1 V_\varepsilon(B_s)ds\}\right)^{1/\beta} \\
&\qquad \times \left\{\int_{Q_{R_1}} dx \mathbb{E}_x \left[\exp\{\alpha \int_0^{t-1} V_\varepsilon(B_s)ds\}; \tau_{Q_{R_1}} \geq t-1\right]\right\}^{1/\alpha} \\
&\quad \leq \left(\frac{R_1}{\pi}\right)^{d/\alpha} \left(\mathbb{E}_0 \exp\{\beta \int_0^1 V_\varepsilon(B_s)ds\}\right)^{1/\beta} \exp\{(t-1)\lambda_{\alpha V_\varepsilon}(Q_{R_1})\}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\{2\int_0^t V_\varepsilon(B_s)ds\}; \tau_{Q_{R_k+1}} \geq t\right] \\
&\quad \leq \left(\frac{R_k+1}{\pi}\right)^{d/\alpha}\left(\mathbb{E}_0 \exp\{2\beta\int_0^1 V_\varepsilon(B_s)ds\}\right)^{1/\beta}\exp\{(t-1)\lambda_{2\alpha V_\varepsilon}(Q_{R_k+1})\}.
\end{aligned}$$

Summarizing our estimates we have

$$\begin{aligned}
&\mathbb{E}_0 \exp\{\int_0^t V_\varepsilon(B_s)ds\} \\
&\quad \leq \left(\frac{R_1}{\pi}\right)^{d/\alpha}\left(\mathbb{E}_0 \exp\{\beta\int_0^1 V_\varepsilon(B_s)ds\}\right)^{1/\beta}\exp\{t\lambda_{\alpha V_\varepsilon}(Q_{R_1})\} \\
&\qquad + \left(\mathbb{E}_0 \exp\{2\beta\int_0^1 V_\varepsilon(B_s)ds\}\right)^{1/2\beta} \\
&\qquad\quad \times \sum_{k=1}^\infty \left(\frac{R_k+1}{\pi}\right)^{d/2\alpha}\exp\{-cM^2t^{2k-1}(\log t)^{k/(1+H)}\}\exp\left\{\frac{t}{2}\lambda_{2\alpha V_\varepsilon}(Q_{R_{k+1}})\right\}.
\end{aligned}$$

To pass the above inequality from $V_\varepsilon$ to $V$, we let $\varepsilon \to 0^+$. By Lemma 6.7 and by (6.30),

$$\begin{aligned}
&\mathbb{E}_0 \exp\{\int_0^t V(B_s)ds\} \\
&\quad \leq \left(\frac{R_1}{\pi}\right)^{d/\alpha}\left(\mathbb{E}_0 \exp\{\beta\int_0^1 V(B_s)ds\}\right)^{1/\beta}\exp\{t\lambda^+_{\alpha V}(Q_{R_1})\} \\
&\qquad + \left(\mathbb{E}_0 \exp\{2\beta\int_0^1 V(B_s)ds\}\right)^{1/2\beta} \\
&\qquad\quad \times \sum_{k=1}^\infty \left(\frac{R_k+1}{\pi}\right)^{d/2\alpha}\exp\{-cM^2t^{2k-1}(\log t)^{k/(1+H)}\}\exp\left\{\frac{t}{2}\lambda^+_{2\alpha V}(Q_{R_{k+1}})\right\}.
\end{aligned} \tag{6.33}$$

Consequently, (6.29) follow from Lemma 6.4. Indeed, we apply (6.23) to the first term on the right-hand side of (6.33)(with t being replaced by $R_1$). Notice that $\alpha$ can be arbitrarily close to 1. This term contributes essentially up to the bound given in (6.29) if we let $\alpha \to 1^+$. To control the infinite series on the right-hand side of (6.33), we apply (6.23) to each term with t being replaced by $R_{k+1} = (Mt(\log t)^{1/2(1+H)})^{k+1}$. In this way, the series is dominated by

$$\sum_{k=1}^{\infty} \left(\frac{R_k + 1}{\pi}\right)^{d/2\alpha} \exp\{-c' t^{2k-2} {h_t}^{2k}\} = O(1) \qquad a.s.\text{-}\mathbb{P} \quad (t \to \infty),$$

where $c' > 0$ is a constant. Here we point out that to control the first term of the series in (6.33), $M > 0$ is required to be sufficiently large.

## 6.4 Lower bound

In this section, we will give a lower bound of the exponential moment in (6.3). We first link it to the principle eigenvalue given in (6.11). Then we give a bound of the principle eigenvalue from below.

### 6.4.1 Eigenvalue estimate

To prove the lower bound of the principle eigenvalue, we need the following lemma to bound the maximum entry of a Gaussian vector.

**Lemma 6.8.** *([9], Lemma 4.2) Let $(\xi_1, ..., \xi_n)$ be a mean-zero Gaussian vector with identically distributed components. Write*

$$R = \max_{i \neq j} |Cov(\xi_i, \xi_j)|$$

*and assume that $Var(\xi_1) \geq 2R$. Then for any $A, B > 0$,*

$$\mathbb{P}\Big\{\max_{k \leq n} \xi_k \leq A\Big\} \leq \left(\mathbb{P}\left\{\xi_1 \leq \sqrt{\frac{2R + Var(\xi_1)}{Var(\xi_1)}}(A+B)\right\}\right)^n + \mathbb{P}\{U \geq B/\sqrt{2R}\},$$

*where $U$ is a standard normal random variable.*

**Lemma 6.9.** *Under assumption of the Theorem 2.4,*

$$\liminf_{t\to\infty}(\log t)^{-\frac{1}{1+H}}\lambda_V(Q_t) \geq 2^{\frac{1}{1+H}} c_H^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}} \qquad a.s.$$

*where $\mathcal{E}$ is defined in (2.18).*

*Proof.* Notice that, by (6.24) we have

$$\begin{aligned}\mathbb{P}\{\lambda_V(Q_t) \leq h_t^2\} &= \mathbb{P}\left\{\sup_{g\in\mathcal{K}_1(Q_{th_t})}\left\{h_t^{-2}\langle V, g_t^2\rangle - \frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\} \leq 1\right\}\\ &= \mathbb{P}\left\{\sup_{g\in\mathcal{G}_1(Q_{th_t})}\langle V, g_t^2\rangle \leq h_t^2\right\}\end{aligned}$$

where the last step follows from Lemma A.2 in [8].

Let the constant $r > 0$ be fixed but arbitrary and set $\mathcal{N}_t = 2r\mathbb{Z} \cap Q_{t-r}$. When $t$ is large, we have $h_t z + Q_r \subset Q_{th_t}$ for each $z \in \mathcal{N}_t$. Hence,

$$\sup_{g\in\mathcal{G}_1(Q_{th_t})}\langle V, g_t^2\rangle \geq \max_{z\in\mathcal{N}_t}\sup_{g\in\mathcal{G}_1(Q_{h_t z+r})}\langle V, g_t^2\rangle.$$

For any $g \in \mathcal{G}_1(Q_r)$ and $z \in \mathcal{N}_t$, notice that $g^z(\cdot) \equiv g(\cdot - h_t z) \in \mathcal{G}_1(h_t z + Q_r)$. Hence,

$$\sup_{g\in\mathcal{G}_1(Q_{th_t})}\langle V, g_t^2\rangle \geq \max_{z\in\mathcal{N}_t}\langle V, (g^z)_t^2\rangle.$$

Summarizing our estimates,

$$\mathbb{P}\{\lambda_V(Q_t) \leq h_t^2\} \leq \mathbb{P}\{\max_{z\in\mathcal{N}_t}\langle V, (g^z)_t^2\rangle \leq h_t^2\}$$

for any $g \in \mathcal{G}_1(Q_r)$ and $z \in \mathcal{N}_t$.

Notice that the random variables

$$\langle V, (g^z)_t^2\rangle, \qquad z \in \mathcal{N}_t$$

are not independent. Hence, we need to control the covariance in order to show that the assumptions of Lemma 6.8 are met. Write $\xi_z(t) = \langle V, (g^z)_t^2 \rangle$. Notice that $(g^z)_t(x) = h_t^{1/2} g(h_t x - h_t z)$. For each $z, z' \in \mathcal{N}_t$,

$$\begin{aligned} Cov(\xi_z, \xi_{z'}) &= \int_{\mathbb{R}\times\mathbb{R}} \gamma(x-y)(g^z)_t^2(x)(g^{z'})_t^2(y)dxdy \\ &= \int_{\mathbb{R}\times\mathbb{R}} \gamma(x-y+(z-z'))g_t^2(x)g_t^2(y)dxdy \\ &= \int_{\mathbb{R}\times\mathbb{R}} \gamma(h_t^{-1}(x-y)+(z-z'))g^2(x)g^2(y)dxdy \\ &= c_H h_t^{2-2H} \int_{\mathbb{R}} e^{ih_t\lambda(z-z')} \left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H} d\lambda. \end{aligned} \tag{6.34}$$

In particular, taking $z = z'$, we have

$$Var(\xi_0(t)) = c_H \sigma^2(g) h_t^{2-2H} \tag{6.35}$$

where

$$\sigma^2(g) = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H} d\lambda \tag{6.36}$$

is finite for $g \in \mathcal{G}(Q_r)$ according to (6.28).

Define

$$G(\lambda) = \left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H}.$$

By (6.28), $G(\lambda)$ belongs to $L^1(\mathbb{R})$. Hence, by Riemann-Lebesgue lemma, for any $z \neq z'$,

$$\lim_{t\to\infty} \int_{\mathbb{R}} e^{ih_t\lambda(z-z')} \left| \int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H} d\lambda = 0.$$

Combine this with (6.34), we get

$$R_t \equiv \max_{\substack{z,z' \in \mathcal{N}_t \\ z \neq z'}} |Cov(\xi_z, \xi_{z'})| = o(h_t^{2-2H}) \qquad (t \to \infty). \tag{6.37}$$

Given a small but fixed $\nu > 0$, taking $A = h_t^2$ and $B = \nu h_t^2$ in Lemma 6.8, we have

$$\mathbb{P}\{\max_{z \in \mathcal{N}_t} \xi_z(t) \leq h_t^2\} \leq \left(\mathbb{P}\Big\{\xi_0(t) \leq (1+\nu)h_t^2\sqrt{\frac{2R_t + Var(\xi_0(t))}{Var(\xi_0(t))}}\Big\}\right)^{\#(\mathcal{N}_t)} + \mathbb{P}\{U \geq \nu h_t^2/\sqrt{2R_t}\},$$

where $U$ is a standard normal random variable.

First, for the second term on the right hand side, a standard way of using Gaussian tail estimates give us,

$$\mathbb{P}\{U \geq \nu h_t^2/\sqrt{2R_t}\} = \exp\Big\{-(1+o(1))\frac{\nu^2 h_t^4}{4R_t}\Big\} \leq \exp\{-2\log t\}$$

for large $t$, where the last step follows from (6.37).

On the other side, we want to bound the first term. Notice that $\xi_0(t) \sim N(0, c_H h_t^{2-2H}\sigma^2(g))$ where $\sigma^2(g)$ is defined in (6.36) and

$$\left(\mathbb{P}\Big\{\xi_0(t) \leq (1+\nu)h_t^2\sqrt{\frac{2R_t + Var(\xi_0(t))}{Var(\xi_0(t))}}\Big\}\right)^{\#(\mathcal{N}_t)}$$
$$= \left(1 - \mathbb{P}\Big\{\xi_0(t) \geq (1+\nu)h_t^2\sqrt{\frac{2R_t + Var(\xi_0(t))}{Var(\xi_0(t))}}\Big\}\right)^{\#(\mathcal{N}_t)}$$

Notice that, by (6.35) and (6.36),

$$\frac{2R_t + Var(\xi_0(t))}{Var(\xi_0(t))} \sim 1 \qquad (t \to \infty).$$

Therefore, by Gaussian tail estimate,

$$\mathbb{P}\Big\{\xi_0(t) \geq (1+\nu)h_t^2\sqrt{\frac{2R_t + Var(\xi_0(t))}{Var(\xi_0(t))}}\Big\} = \exp\left\{ -(1+o(1))\frac{(1+\nu)^2 h_t^{2+2H}}{2c_H\sigma^2(g)}\right\}$$
$$= \exp\left\{ -(1+o(1))\frac{(1+\nu)^2 u^{1+H}\log t}{2c_H\sigma^2(g)}\right\}.$$

By the fact that $\#(\mathcal{N}_t) \sim (2r)^{-1}t$ as $t \to \infty$, we have

$$\left(\mathbb{P}\Big\{\xi_0(t) \leq (1+\nu)h_t^2\sqrt{\frac{2R_t + Var(\xi_0(t))}{Var(\xi_0(t))}}\Big\}\right)^{\#(\mathcal{N}_t)} \leq \exp\{-t^{\beta}\}$$

for some $\beta > 0$, whenever $\nu$ is small and $u$ satisfies

$$u < 2^{\frac{1}{1+H}} c_H{}^{\frac{1}{1+H}} \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda\right)^{\frac{1}{1+H}}.$$

Summarizing our estimates, for any $\gamma > 1$, $g \in \mathcal{G}_1(Q_r)$ and $u$ satisfies the above condition, we have

$$\sum_k \mathbb{P}\{\lambda_V(Q_{\gamma^k}) \leq h_{\gamma^k}^2\} < \infty.$$

By the Borel-Cantelli lemma,

$$\liminf_{k\to\infty}(\log\gamma^k)^{-\frac{1}{1+H}}\lambda_V(Q_{\gamma^k}) \geq 2^{\frac{1}{1+H}} c_H{}^{\frac{1}{1+H}} \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda\right)^{\frac{1}{1+H}} \quad a.s.$$

Since $\lambda_V(Q_t)$ is monotonic in $t$, the liminf along the sub-sequence $\gamma^k$ above can be extended into the liminf along the continuous time $t$. Hence, we get

$$\liminf_{t\to\infty}(\log t)^{-\frac{1}{1+H}}\lambda_V(Q_t) \geq 2^{\frac{1}{1+H}} c_H{}^{\frac{1}{1+H}} \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{i\lambda x} g^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda\right)^{\frac{1}{1+H}} \quad a.s.$$

Taking supremum over $g \in \mathcal{G}_1(Q_r)$ and letting $r \to \infty$ give us the lower bound. $\square$

### 6.4.2 Lower bound

In this section we prove the lower bound of Theorem 2.4. More precisely, we prove that,

$$\liminf_{t\to\infty} t^{-1}(\log t)^{-\frac{1}{1+H}} \log \mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\} \geq 2^{\frac{1}{1+H}} c_H^{\frac{1}{1+H}} \mathcal{E}^{\frac{1}{1+H}}, \qquad a.s. \tag{6.38}$$

Let $0 < q < 1$ be fixed but close to 1. Let $\alpha, \beta > 1$ satisfy $\alpha^{-1} + \beta^{-1} = 1$ with $\alpha$ being close to 1. According to Lemma 4.3 in [8], we have

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\int_0^t V_\varepsilon(B_s)ds\right\} \\
&\geq \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^{t^q} V_\varepsilon(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\quad \times \left\{\int_{Q_{t^q}} p_{t^q}(x)\mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t-t^q} V_\varepsilon(B_s)\right\}; \tau_{Q_{t^q}} \geq t - t^q\right] dx\right\}^\alpha \\
&\geq \frac{1}{(2\pi t^q)^{\alpha d/2}} e^{-ct^q} \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^{t^q} V_\varepsilon(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\quad \times \left\{\int_{Q_{t^q}} \mathbb{E}_x\left[\exp\left\{\alpha^{-1}\int_0^{t-t^q} V_\varepsilon(B_s)\right\}; \tau_{Q_{t^q}} \geq t - t^q\right] dx\right\}^\alpha \\
&\geq e^{-ct^q} \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^{t^q} V_\varepsilon(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\quad \times \exp\{-(\alpha^2/\beta)t^q\lambda_{(\beta/\alpha^2)V_\varepsilon}(Q_{t^q}) + \alpha^2 t\lambda_{\alpha^{-2}V_\varepsilon}(Q_{t^q})\}
\end{aligned}$$

for large $t$, where the last step follows from Lemma 4.2 in [8]. Once again, by Lemma 6.7 and by (6.30),

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{\int_0^t V(B_s)ds\right\} \\
&\quad \geq e^{-ct^q} \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^{t^q} V(B_s)ds\right\}\right)^{-\alpha/\beta} \\
&\qquad \times \exp\{-(\alpha^2/\beta)t^q\lambda^+_{(\beta/\alpha^2)V}(Q_{t^q}) + \alpha^2 t\lambda_{\alpha^{-2}V}(Q_{t^q})\}.
\end{aligned} \tag{6.39}$$

Notice that $V \stackrel{d}{=} -V$, then replacing $V$ by $-V$ and $t$ by $t^q$ in (6.29),

$$\log \mathbb{E}_0 \exp\left\{-\frac{\theta\beta}{\alpha}\int_0^{t^q} V(B_s)ds\right\} = o(t) \qquad a.s.$$

for sufficiently large $t$. In addition, by Lemma 6.4,

$$t^q \lambda^+_{(\beta/\alpha^2)V}(Q_{t^q}) = o(t) \qquad a.s.$$

as $t \to \infty$. Therefore, by (6.39) and by the fact that $\alpha$ and $q$ can be arbitrary close to 1, (6.38) follows from Lemma 6.9.

# Bibliography

[1] M. Biskup and W. König. Screening effect due to heavy lower tails in one-dimensional parabolic Anderson model. *Journal of Statistical Physics*, 102(5-6): 1253–1270, 2001. 4, 6

[2] A. Cadel, S. Tindel, and F. Viens. Sharp asymptotics for the partition function of some continuous-time directed polymers. *Potential Analysis*, 29(2):139–166, 2008. 4

[3] R. A. Carmona and S. A. Molchanov. *Parabolic Anderson problem and intermittency*, volume 518. American Mathematical Soc., 1994. 4

[4] R. A. Carmona and S. A. Molchanov. Stationary parabolic Anderson model and intermittency. *Probability theory and related fields*, 102(4):433–453, 1995. 4, 7

[5] P. Chakraborty, X. Chen, B. Gao, and S. Tindel. Quenched asymptotics for a 1-d stochastic heat equation driven by a rough spatial noise. *ArXiv e-prints*, October 2018. 68, 76, 82

[6] Hu Y. Z. Nualart D. Chen, X. and S. Tindel. Spatial asymptotics for the parabolic Anderson model driven by a Gaussian rough noise. *Electronic Journal of Probability*, 22, 2017. 7

[7] X. Chen. *Random walk intersections: large deviations and related topics*. Number 157. American Mathematical Soc., 2010. 23, 69, 76, 101

[8] X. Chen. Quenched asymptotics for Brownian motion of renormalized Poisson potential and for the related parabolic Anderson models. *The Annals of*

*Probability*, 40(4):1436–1482, 2012. 6, 39, 40, 41, 51, 57, 65, 69, 70, 71, 79, 82, 83, 86, 90

[9] X. Chen. Quenched asymptotics for Brownian motion in generalized Gaussian potential. *The Annals of Probability*, 42(2):576–622, 2014. 7, 8, 13, 68, 73, 76, 77, 82, 85

[10] X. Chen. Parabolic Anderson model with rough or critical Gaussian noise. 2018. 68

[11] X. Chen and A. M. Kulik. Asymptotics of negative exponential moments for annealed Brownian motion in a renormalized Poisson potential. *International Journal of Stochastic Analysis*, 2011, 2011. 4

[12] X. Chen and A. M. Kulik. Brownian motion and parabolic Anderson model in a renormalized Poisson potential. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 48, pages 631–660. Institut Henri Poincaré, 2012. 4, 6

[13] Donsker M. D. and Varadhan S. R. S. Asymptotics for the Wiener sausage. *Communications on Pure and Applied Mathematics*, 28(4):525–565, 1975. 4, 5, 44

[14] F. Den Hollander. *Random Polymers*. Springer, 2009. 1

[15] R. Fukushima. Second order asymptotics for Brownian motion in a heavy tailed Poissonian potential. *Markov Processes and Related Fields*, 17(3):447–482, 2011. 6, 40, 58, 59

[16] J. Gärtner and W. König. Moment asymptotics for the continuous parabolic Anderson model. *Annals of Applied Probability*, pages 192–217, 2000. 4, 7, 28, 29, 77

[17] J. Gärtner and S. Molchanov. Parabolic problems for the Anderson model. *Communications in mathematical physics*, 132(3):613–655, 1990. 4, 7

[18] W. Gärtner, J. König and S. A. Molchanov. Almost sure asymptotics for the continuous parabolic Anderson model. *Probability theory and related fields*, 118 (4):547–573, 2000. 4, 7, 29, 77

[19] IM Guelfand, N Ya Vilenkin, and MI Graev. *Generalized Functions*. Academic Press, 1964. 12

[20] S. Havlin and D. Ben-Avraham. Diffusion in disordered media. *Advances in Physics*, 36(6):695–798, 1987. 1

[21] Huang J. Y. Lê K. Nualart D Hu, Y. Z. and S. Tindel. Stochastic heat equation with rough dependence in space. *The Annals of Probability*, 45(6b):4561–4616, 2017. 7, 12

[22] Huang J. Y. Nualart D Hu, Y. Z. and S. Tindel. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electronic Journal of Probability*, 20, 2015. 7

[23] W. Kirsch and F. Martinelli. On the density of states of Schrödinger operators with a random potential. *Journal of Physics A: Mathematical and General*, 15 (7):2139, 1982. 58, 59

[24] W. König. *The Parabolic Anderson Model: Random Walk in Random Potential*. Birkhäuser, 2016. 4, 6, 7

[25] S. Molchanov. Lectures on random media. In *Lectures on probability theory*, pages 242–411. Springer, 1994. 1

[26] H. Ôkura. An asymptotic property of a certain Brownian motion expectation for large time. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 57(3):155–159, 1981. 5, 8, 15, 16, 43

[27] L. A. Pastur. Spectra of random self adjoint operators. *Russian mathematical surveys*, 28(1):1, 1973. 58

[28] L. A. Pastur. Behavior of some wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential. *Teoreticheskaya i Matematicheskaya Fizika*, 32(1):88–95, 1977. 5

[29] B. S. Rajput and J. Rosiński. Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields*, 82(3):451–487, 1989. 11, 98

[30] D. Slepian. The one-sided barrier problem for Gaussian noise. *Bell System Technical Journal*, 41(2):463–501, 1962. 7

[31] A. S. Sznitman. Brownian asymptotics in a Poissonian environment. *Probability theory and related fields*, 95(2):155–174, 1993. 4, 5, 14, 15, 58

[32] A. S. Sznitman. Brownian survival among Gibbsian traps. *The Annals of Probability*, pages 490–508, 1993. 4

[33] A. S. Sznitman. *Brownian motion, obstacles and random media*. Springer Science & Business Media, 1998. 4

# Appendix

# Appendix A

# Possion integrals

Recall that $\omega(dx)$ denote the Poisson random field defined in Definition 2.2.1. By Theorem 2.7 in [29], a Borel-measurable function $K(x)$ is integrable on $\mathbb{R}^d$ with respect to $\omega(dx)$ if and only if

$$\int_{\mathbb{R}^d} \varphi(|K(x)|)dx < \infty$$

where $\varphi(x) = 1 - e^{-x}$. With this in mind, the follow lemma tells us the Poisson random media $V$ is well defined in our setting.

**Lemma A.1.** *Under $p > d$,*

$$\int_{\mathbb{R}^d} [1 - \exp\{-\frac{1}{|x|^p}\}]dx = w_d \Gamma\left(\frac{p-d}{p}\right)$$

*where $w_d$ is the volume of the d-dimensional unit ball.*

*Proof.* By the sphere substitution,

$$\begin{aligned}\int_{\mathbb{R}^d} [1 - \exp\{-\frac{1}{|x|^p}\}]dx &= dw_d \int_o^\infty \rho^{d-1}(1 - \exp\{-\frac{1}{\rho^p}\})d\rho \\ &= \frac{dw_d}{p} \int_0^\infty \gamma^{-\frac{d+p}{p}}(1 - e^{-\gamma})d\gamma\end{aligned}$$

where the second step follows from the substitution $\rho = \gamma^{-\frac{1}{p}}$.

Applying the integration by parts (under the assumption $p > d$),

$$\int_0^\infty \gamma^{-\frac{d+p}{p}}(1-e^{-\gamma})d\gamma = \frac{p}{d}\int_0^\infty \gamma^{-\frac{d}{p}}e^{-\gamma}d\gamma$$
$$= \frac{p}{d}\Gamma\left(\frac{p-d}{p}\right)$$

We have proved identity. □

Further more, if $K(x)$ is integrable on $\mathbb{R}^d$ with respect to $\omega(dx)$, we have

$$\mathbb{E}\exp\Big\{\int_{\mathbb{R}^d} K(x)\omega(dx)\Big\} = \exp\Big\{\int_{\mathbb{R}^d}\psi(K(x))dx\Big\}. \tag{A.1}$$

where $\psi(x) = e^x - 1$.

# Appendix B

# A basic result for large deviation

We assume $Y_n$ be a sequence of real random variables and take non-negative values, and let $b_n$ be a positive sequence such that $b_n \to \infty$.

**Lemma B.1.** *Assume that for all $\theta > 0$, the limit*

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\{-\theta b_n Y_n\} \tag{B.1}$$

*exists as an extended real number, and that the function $\Lambda(\theta)$ is essentially smooth on $\mathbb{R}^+$. Then for every $\lambda > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \leq \lambda\} = -I(\lambda). \tag{B.2}$$

*where the function*

$$I(\lambda) = \sup_{\theta>0}\{-\theta\lambda - \Lambda(\theta)\}$$

*is strictly decreasing and continuous on $\mathbb{R}^+$.*

*Proof.* First, we proof the upper bound of (B.2). By Chebyshev inequality, we have

$$\mathbb{P}\{Y_n \leq \lambda\} \leq \frac{\mathbb{E}\exp\{-\theta b_n Y_n\}}{\exp\{-\theta b_n \lambda\}}$$

which by (B.1) gives us

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\{Y_n\le\lambda\}\le\theta\lambda+\Lambda(\theta).$$

Consequently, we get the upper bound of (B.2) by taking infimum over $\theta>0$.

We now come to the proof of the lower bound of (B.2). By essential smoothness of $\Lambda(\cdot)$ there is a $\theta_0$ such that $-\lambda\theta_0-\Lambda(\theta_0)=I(\lambda)$. Then for any $\delta>0$

$$\begin{aligned}\mathbb{P}\{Y_n\le\lambda\}&\ge\mathbb{P}\{Y_n\in(\lambda-\delta,\lambda)\}\\&\ge\exp\{-\theta_0 b_n(\delta-\lambda)\}\mathbb{E}\left[\exp\{-\theta b_n Y_n\}1_{\{Y_n\in(\lambda-\delta,\lambda)\}}\right].\end{aligned}$$

By the continuity of $I(\cdot)$,

$$\Lambda(\theta_0)>\sup_{\lambda\notin(\lambda-\delta,\lambda)}\{-\lambda\theta_0-I(\lambda)\}.$$

Consider the following decomposition,

$$\begin{aligned}\mathbb{E}\left[\exp\{-\theta b_n Y_n\}\right]&=\mathbb{E}\left[\exp\{-\theta b_n Y_n\}1_{\{Y_n\in(\lambda-\delta,\lambda)\}}\right]\\&\quad+\mathbb{E}\left[\exp\{-\theta b_n Y_n\}1_{\{Y_n\notin(\lambda-\delta,\lambda)\}}\right].\end{aligned}$$

By (B.1) and (1.1.6) in [7], we get

$$\begin{aligned}\Lambda(\theta_0)\le\max\Bigg\{&\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\{-\theta b_n Y_n\}1_{\{Y_n\in(\lambda-\delta,\lambda)\}}\right],\\&\sup_{\lambda\notin(\lambda-\delta,\lambda)}\{-\lambda\theta_0-I(\lambda)\}\Bigg\}.\end{aligned}$$

Consequently,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\{-\theta b_n Y_n\}1_{\{Y_n\in(\lambda-\delta,\lambda)\}}\right]\ge\Lambda(\theta_0).$$

Thus,

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\{Y_n \le \lambda\} \ge -\theta_0\delta + \theta_0\lambda + \Lambda(\theta_0) = -I(\lambda) - \theta_0\delta.$$

Finally, by letting $\delta \to 0^+$ on the right hand side, we complete the proof. $\square$

# Vita

Bo Gao was born in Shi Jiazhuang, China in 1986. He attended Central China Normal University to study mathematics in 2005 and received a Bachelor of Science in 2009. He then attended the graduate school of Donghua University in September 2010 and graduated with a Master of Science in Mathematics three years later.

In August 2013, he joined the Department of Mathematics at the University of Tennessee, Knoxville as a graduate student, concentrating on probability theory and stochastic processes.