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Coarse Proximity, Proximity at Infinity, and Coarse Structures

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I would like to dedicate this work to my parents, Izabela and Andrzej Grzegrzółka, who have given me opportunities that they did not have.

Chciałbym zadedykować tę pracę doktorską moim rodzicom, Izabeli and Andrzejowi Grzegrzółkom, którzy stworzyli mi możliwości, których oni sami nigdy nie mieli.

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Abstract

In this dissertation, we introduce coarse proximities, explore some of their applications (e.g., proximity at infinity), and study the relationships between three different structures capturing large-scale properties of spaces: coarse proximities, asymptotic resemblances, and coarse space structures.

After a short introduction to coarse topology and small-scale proximities, we recall basic definitions and theorems related to coarse spaces, asymptotic resemblance spaces, and bornologies. Then we investigate metric coarse proximities and introduce a general definition of coarse proximities. After exploring a few of their basic properties, we introduce coarse neighborhoods and use them to give an alternative definition of coarse proximities. We then proceed to show that coarse proximities induce weak asymptotic resemblances, and we use this fact to investigate coarse proximity maps to build a category of coarse proximity spaces whose morphisms are closeness classes of coarse proximity maps.

Next we restrict our attention to the metric case and we construct a natural small-scale proximity structure on the set of unbounded subsets of a metric space. We also show how this structure naturally induces a small-scale proximity on the equivalence classes of the weak asymptotic resemblance induced by the metric. We call this space the “proximity space at infinity.” We then proceed to show that the construction is functorial, making up a functor from the category of unbounded metric spaces whose morphisms are closeness classes of coarse proximity maps (equivalently, coarse maps or asymptotic resemblance maps) to the category of proximity spaces whose morphisms are proximity maps.

Finally, we investigate the relationships between coarse proximities, (weak) asymptotic resemblances, and coarse spaces structures. We also explore coarse and asymptotic normality

and we show that under mild conditions, both normal coarse spaces and normal asymptotic resemblance spaces induce coarse proximities.

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List of Symbols

$\lceil x \rceil$	the smallest integer greater than or equal to x
$(\lambda, \mathcal{B})$	asymptotic resemblance with the induced bornology
$(\mathcal{E}, \mathcal{B})$	coarse space structure with the induced bornology
$(f \times f)(E)$	$\{(f(a), f(b)) \mid (a, b) \in E\}$
(X, λ)	asymptotic resemblance space
$(X, \mathcal{B}, \mathbf{b})$	proximity space
$(X, \mathcal{E})$	coarse space
(X, d)	metric space
$(\mathbf{b}, \mathcal{B})$	coarse proximity with the associated bornology
$[f]$	proximal equivalence class of f
$\bar{A}$	topological closure of A
$\boldsymbol{B}X$	Proximity at Infinity of X
$\cap$	intersection
$\cup$	union
δ	small-scale proximity
$\emptyset$	empty set

$\Longrightarrow$	implies or induces
$\in$	in or is an element of
λ	asymptotic resemblance
$\Longleftrightarrow$	if and only if or induces and is induced by
$\mathbb{N}$	natural numbers (i.e., $\{1, 2, 3...\}$)
$\mathbb{R}$	real numbers
$\mathbb{Z}$	integers
$\mathcal{B}$	bornology
$\mathcal{E}$	coarse space structure
$\mathcal{H}_\infty(X)$	Hyperspace at Infinity of X
ϕ	weak asymptotic resemblance
$\prec$	coarse neighborhood relation induced by a coarse space structure or by an asymptotic resemblance structure
$\sim$	is equivalent to
$\subseteq$	is a proper subset of or equal to
$\triangle$	$\{(x,x) \in X \times X \mid x \in X\}$
$A \ll B$	B is a coarse neighborhood of A
$A \times B$	Cartesian product of A and B
$A\bar{\mathbf{b}}B$	the relation $A\mathbf{b}B$ does not hold
$A\bar{\delta}B$	the relation $A\delta B$ does not hold
$A\bar{\lambda}B$	the relation $A\lambda B$ does not hold

$A \setminus B$	relative complement of A in B
$C_{\mathbf{b}}$	the category of coarse proximity spaces
C_d	the category of unbounded metric spaces
d	metric
$d(A,B)$	$inf\{d(x,y) \mid x \in A, y \in B\}$
$d_H(A,B)$	Hausdorff distance between A and B
$E \circ F$	$\{(x,y) \in X \times X \mid \exists\, z \in X$ such that $(x,z) \in E, (z,y) \in F\}$
E^{-1}	$E^{-1} := \{(x,y) \mid (y,x) \in E\}$
$f(A)$	image of A under f
$f : A \to B$	f is a function from A to B
$f^{-1}(A)$	inverse image of A under f
$g \circ f$	composite function of f and g
$U_{x_0}(A,f)$	coarse neighborhood of A of radius f relative to x_0
$\mathbf{b}$	coarse proximity

Chapter 1

Introduction

1.1 General Comments Regarding this Dissertation

The main focus of this dissertation are coarse proximities. Many of the results in this dissertation come from [7] and [8], which are published papers written by the author and his fellow graduate student Jeremy Siegert. This dissertation consists of 6 main chapters (Introduction, Coarse Structures, Coarse Proximity, Proximity at Infinity, Relationships between Coarse Structures, and Concluding Remarks) divided into sections. Chapters 3, 4, and 5 end with the "Questions" section. In the "Questions" sections, the reader can find a selection of chapter-related questions whose answers are not known to the author at the time of the writing of this dissertation. The author hopes to answer these questions in his future research. Also, he hopes that these questions will serve as a starting point for other students/mathematicians for their own research on this fascinating subject.

The author took great care to give credit to other researchers when the results in this dissertation were not due to the author. Thus, when the result can be found in other works, the reader is always referred to the original source. Consequently, whenever the proof is given, this implies that according to the knowledge of the author, the result is due to the author (or the author and the co-author of his papers, Jeremy Siegert).

Finally, the author attempted to make this dissertation as self-contained as possible. Therefore, almost all the results used in this dissertation are included in this document.

The only exceptions are proofs of theorems attributed to other authors, as described in the paragraph above.

1.2 What Is Coarse Topology?

Coarse topology (i.e., large scale geometry) is a branch of mathematics investigating large-scale properties of spaces. However, what does "large-scale" really mean? In classical topology, one is concerned with "small-scale" properties of spaces, i.e., properties that capture what happens as we "zoom in" at the object of interest. A good example of such a property is openness of a set. The definition of a set being open roughly says the following: "for any point in the set, we can find an open set small enough such that that open set contains the point and is itself contained in the original set." In other words, if we have the metric case in mind, we have to "zoom in" at the point in question to see if we can find a small enough open set with the desired properties.

On the other hand, coarse topology is concerned with large-scale properties of spaces. As an example, take $\mathbb{Z}$ and $\mathbb{R}$. When equipped with metric topology, these two spaces are clearly not homeomorphic. However (being as informal as possible), as we "zoom out" and look at $\mathbb{Z}$ from the great distance, $\mathbb{Z}$ will closely resemble $\mathbb{R}$, and the further we "zoom out", the more alike the two spaces will seem. Therefore, $\mathbb{Z}$ and $\mathbb{R}$ are "coarsely equivalent," i.e., they are "large-scale homeomorphic."

Another method to explore large-scale properties of spaces is to study the behavior of spaces "at infinity," i.e., the behavior that stays "consistent" outside of bounded sets. For example, one could investigate slowly oscillating functions: functions from metric spaces to $\mathbb{R}$ that do not change rapidly outside of large enough bounded sets.

Finally, to explore large-scale properties of spaces, coarse topologists study the relationships between the small-scale and the large-scale world. Quite often a link between the two worlds is a useful tool that can be used to investigate large-scale behavior of spaces. Namely, such a connection often allows to investigate large-scale properties of spaces by using an extensive, well-developed, and very well-understood machinery from (small-scale) classical topology.

1.3 What Is Proximity?

Most of this dissertation will be concerned with coarse proximities - the large-scale counterpart of small-scale proximities. To be able to introduce and understand coarse proximities, let us review basic definitions and examples of proximities from [12].

Definition 1.3.1. Let X be a set. A **proximity** on a set X is a relation δ on the power set of X satisfying the following axioms for all $A, B, C \subseteq X$:

1. $A\delta B$ implies $B\delta A$,
2. $A\delta B$ implies $A \neq \emptyset$ and $B \neq \emptyset$,
3. $A \cap B \neq \emptyset$ implies $A\delta B$,
4. $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$,
5. $A\bar{\delta}B$ implies that there exists a subset E such that $A\bar{\delta}E$ and $(X \setminus E)\bar{\delta}B$,

where $A\bar{\delta}B$ means "$A\delta B$ is not true." If $A\delta B$, then we say that A is **close** to (or **near**) B. Axiom 4 is called the **union axiom** and axiom 5 is called the **strong axiom**. A pair (X, δ), where X is a set and δ is a proximity on X, is called a **proximity space.**

In this document, "small-scale" proximity and proximity mean the same thing. The adjective "small-scale" is usually added to remind the reader that the proximity in question is NOT the coarse proximity.

Here are a few examples of proximities:

Example 1.3.2. If (X, d) is a metric space, then the proximity relation defined by

$$A\delta B \quad \text{if and only if} \quad d(A, B) = 0,$$

where $d(A, B) := inf\{d(x, y) \mid x \in A, y \in B\}$, is called the **metric proximity**.

Example 1.3.3. If $(X, \mathcal{T})$ is a topological space, then the proximity relation defined by

$$A\delta B \quad \text{if and only if} \quad \bar{A} \cap \bar{B} \neq \emptyset,$$

where $\bar{A}$ and $\bar{B}$ denote the topological closures of A and B, is called the **topological proximity**.

Example 1.3.4. If (X, d) is a metric space, then the proximity relation defined by

$$A\delta B \quad \text{if and only if} \quad A \cap B \neq \emptyset$$

is called the **discrete proximity**.

Example 1.3.5. If (X, d) is a metric space, then the proximity relation defined by

$$A\delta B \quad \text{if and only if} \quad A, B \neq \emptyset$$

is called the **indiscrete proximity**.

Exercise 1.3.6. Show that the above relations are indeed proximities.

To be able to properly "coarsen" the notion of proximities, we also need to review their basic properties. Recall the following from [12]:

Definition 1.3.7. Given a proximity space (X, δ) and subsets $A, B \subseteq X$, we say that B is a **proximal neighborhood** of A, denoted $A \ll B$, if $A\bar{\delta}(X \setminus B)$.

Definition 1.3.8. Given a proximity space (X, δ), the **induced topology** on X is defined by the closure operator $cl(A) = \{x \in X \mid \{x\}\delta A\}$.

The induced topology is always completely regular. For the proof, see Theorem 3.14 in [12].

Definition 1.3.9. A function $f : (X, \delta_1) \to (Y, \delta_2)$ is called a **proximity map** if for all $A, B \subseteq X$,

$$A\delta_1 B \quad \text{implies} \quad f(A)\delta_2 f(B).$$

Exercise 1.3.10. Show that all proximity maps are continuous with respect to the induced topologies on the domain and codomain.

Definition 1.3.11. Given a set X and two proximities δ_1, δ_2 on X, we say that δ_1 is **finer** than δ_2 (or δ_2 is **coarser** than δ_1), denoted $\delta_1 > \delta_2$, if $A\delta_1 B$ implies $A\delta_2 B$.

In this dissertation, we will utilize a few more complex proximity theorems. The following result is from [12]:

Proposition 1.3.12. *Given a function $f : X \to (Y, \delta_2)$, the coarsest proximity δ_0 on X for which f is a proximity map is defined by*

$$A\bar{\delta}_0 B \text{ if and only if there is a } C \subseteq Y \text{ such that } f(A)\bar{\delta}_2(Y \setminus C) \text{ and } f^{-1}(C) \subseteq (X \setminus B)$$

Proof. See Theorem 4.5 of [12]. □

As the reader will see in Chapter 4, the proximity on the proximity space at infinity will be obtained by inducing a proximity through a surjective function, as in the following definition from [12]:

Definition 1.3.13. Let f be a surjective function from a proximity space (X, δ) onto a set Y. The **quotient proximity** is the finest proximity on Y for which f is a proximity map.

In [6], it is shown that such a proximity always exists. For a detailed description of quotient proximities we refer the reader to [6]. An important property of quotient proximities that will be used in Chapter 4 is the following:

Proposition 1.3.14. *Let (X, δ_1) be a proximity, $f : X \to Y$ a surjective function, and $g : Y \to (Z, \delta_3)$ a function. If δ_2 is the quotient proximity on Y induced by f, then $g \circ f$ is a proximity map if and only if g is a proximity map from (Y, δ_2) to (Z, δ_3).*

$$\begin{tikzcd}
(X, \delta_1) \arrow[d, "f"] \arrow[dr, "g\circ f"] & \\
Y \arrow[r, "g"] & (Z, \delta_3)
\end{tikzcd}$$

Proof. It is clear that if g is a proximity map, then so is $g \circ f$. To prove the converse, assume that $g \circ f$ is a proximity map. Consider the proximity δ_g induced on the set Y by g as defined in Proposition 1.3.12. We will show that $\pi : (X, \delta_1) \to (Y, \delta_g)$ is a proximity mapping. Let

$A, B \subseteq Y$ be such that $A\bar{\delta}_g B$. Then there is a $C \subseteq Z$ such that $g(A)\bar{\delta}_3(Z \setminus C)$. By the strong axiom there is then a $D \subseteq Z$ such that $g(A)\bar{\delta}_3 D$ and $(Z \setminus D)\bar{\delta}_3(Z \setminus C)$. As proven in [12], the set $E = g^{-1}(D)$ is a set such that $A\bar{\delta}_g E$ and $(Y \setminus E)\bar{\delta}_g B$. If $\pi^{-1}(A)\delta_1\pi^{-1}(E)$, then we have

$$(g \circ \pi)(\pi^{-1}(A))\delta_3(g \circ \pi)(\pi^{-1}(E)).$$

However, note that

$$(g \circ \pi)(\pi^{-1}(A)) = g(A) \text{ and } (g \circ \pi)(\pi^{-1}(E)) = g(E) = D,$$

which would imply that $g(A)\delta_3 D$, which is a contradiction. Then, because $\pi^{-1}(B) \subseteq \pi^{-1}(E)$ we have that $\pi^{-1}(A)\bar{\delta}_1\pi^{-1}(B)$, which establishes that π is a proximity map when Y is equipped with the proximity δ_g. By the definition of the quotient proximity we must then have that δ_2 is finer than δ_g. Now assume towards a contradiction that $g : (Y, \delta_2) \to (Z, \delta_3)$ is not a proximity mapping. Then there are subsets $A, B \subseteq Y$ such that $A\delta_2 B$ and $g(A)\bar{\delta}_3 g(B)$. However, because δ_2 is finer than δ_g we have that $A\delta_g B$ and because g is a proximity mapping when Y is equipped with δ_g we have that $g(A)\delta_3 g(B)$, which is a contradiction. Thus, we must have that g is a proximity mapping. □

Even though the above property of quotient proximities seems very natural, the author could not find a proof of it in the literature.

1.4 Motivation for this Dissertation

As mentioned in section 1.2, coarse topology is often interested in the interplay between the large-scale and the small-scale world. In particular, the idea of "translating" a small-scale world to its large-scale counterpart has been extensively explored by coarse topologists. Uniform spaces and surroundings introduced by Weil [15] and Bourbaki [3] were translated to the large-scale world by Roe [13] in terms of coarse structures and controlled sets. Tukey [14] presented a covering definition of a uniform space which inspired Dydak and Hoffland [4] to introduce large scale structures - a covering approach to coarse spaces. Recently there have been other attempts to translate results from the small-scale world to its large-scale counterpart, including “coarsening” the notion of proximity and neighborhoods (see for example [11] or [9]).

In [9], Hartmann defines a binary relation on the power set of a metric space as the negation of asymptotic disjointness. This “closeness” relation is used to construct a uniform space on a set of equivalence classes of certain unbounded subsets of a metric space. However, this relation is defined only for metric spaces. In [11], asymptotic resemblance relations are defined, which generalize the notion of the Hausdorff distance between two subsets of a metric space being finite. As shown in [11], asymptotic resemblances “coarsen” many foundational results of proximity structures; however, there are several significant differences between the two notions. For example, asymptotic equivalences are equivalence relations, whereas small-scale proximity relations are not. In other words, an asymptotic resemblance on a set captures when two subsets are “the same at infinity” instead of capturing their coarse closeness.

Consequently, the author (together with Jeremy Siegert) realized the importance of defining a coarse analog of proximity spaces which stems from a more direct translation of small-scale proximities into the coarse context (see Chapter 3). Additionally, the author wanted to relate this newly-defined coarse proximity to the work done in papers of similar flavor, such as [11] and [2]. It was also important to the author to find the connection between the small-scale and the large-scale proximities that could provide a useful application of coarse proximities (see Chapter 4). Finally, the author wanted to investigate the relationships between coarse proximity structures and other commonly used coarse structures (see Chapter 5).

Chapter 2

Coarse Structures

In this chapter, we will introduce basic coarse structures, i.e., structures that capture large-scale properties of spaces. In Section 2.1 we investigate coarse space structures in the sense of Roe and in Section 2.2 we explore asymptotic resemblance structures. Section 2.1 is based on [13] and section 2.2 is based on [11].

In his monogram (see [13]), John Roe introduced the concept of a coarse structure, which is a large-scale equivalent of a uniform structure (for the introduction to the theory of uniform spaces, the reader is referred to [16] and [10]). Since then, mathematicians introduced other structures capturing large-scale properties of spaces (e.g., asymptotic resemblance, balleans, large-scale spaces). In this dissertation, by a "coarse structure" we mean any structure that captures large-scale properties of spaces. Coarse structures in the sense of Roe ([13]) will be called "coarse space structures."

2.1 Coarse Spaces

In this section, we introduce coarse space structures and several definitions related to coarse spaces.

Definition 2.1.1. A **coarse space structure** on a set X is is a collection $\mathcal{E}$ of subsets of $X \times X$, called **controlled sets** or **entourages**, such that the following are satisfied:

1. $\triangle \in \mathcal{E}$, where $\triangle := \{(x, x) \mid x \in X\}$,

2. if $E \in \mathcal{E}$ and $B \subseteq E$, then $B \in \mathcal{E}$,

3. if $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$, where $E^{-1} := \{(x, y) \mid (y, x) \in E\}$,

4. if $E \in \mathcal{E}$ and $F \in \mathcal{E}$, then $E \cup F \in \mathcal{E}$,

5. if $E \in \mathcal{E}$ and $F \in \mathcal{E}$, then $E \circ F \in \mathcal{E}$, where $E \circ F := \{(x, y) \mid \exists\, z \in X$ such that $(x, z) \in E, (z, y) \in F\}$.

A pair $(X, \mathcal{E})$, where X is a set and $\mathcal{E}$ is a coarse space structure on X, is called a **coarse space**.

The prime example of a coarse space comes from the metric case, where controlled sets are subsets of "metric tubes."

Example 2.1.2. Let (X, d) be a metric space. For each $r \in \mathbb{R}^+$, define

$$E_r = \{(x, y) \in X \times X \mid d(x, y) < r\}.$$

Let $\mathcal{E}$ be the collection of all the subsets of such sets E_r. Then $\mathcal{E}$ is a coarse space structure, called the **metric coarse structure.**

For more examples of coarse space structures, the reader is referred to [13].

Exercise 2.1.3. Show that the metric coarse structure is indeed a coarse space structure.

Definition 2.1.4. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two coarse structures on the same set X. We say that $\mathcal{E}_1$ is **finer** than $\mathcal{E}_2$ (or that $\mathcal{E}_2$ is **coarser** than $\mathcal{E}_1$) if $\mathcal{E}_1 \subseteq \mathcal{E}_2$.

Notice that the above definition uses the opposite inclusion than the small-scale definition for "finer" and "coarser" for topologies.

The following two definitions introduce the concept of connectedness and boundedness in the coarse space.

Definition 2.1.5. A coarse space $(X, \mathcal{E})$ is **coarsely connected** if each point of $X \times X$ belongs to some controlled set.

Exercise 2.1.6. Show that the metric coarse structure is coarsely connected.

Definition 2.1.7. If $(X, \mathcal{E})$ is a coarse space and A a subset of X, we say that A is **coarsely bounded** if $A \times A$ is a controlled set.

Recall that if (X, d) is a metric space, then a set $A \subseteq X$ is called **bounded** if there exists $x \in X$ and $r > 0$ such that for all $a \in A$, we have $d(x, a) < r$.

Exercise 2.1.8. Show that if (X, d) is a metric space endowed with a metric coarse structure, then $A \subseteq X$ is bounded if and only if it is coarsely bounded.

In [13], coarsely connected coarse spaces are simply called "connected" and coarsely bounded sets are simply called "bounded." However, since we are going to encounter several notions of connectedness and boundedness, in this dissertation "connected" coarse spaces in the sense of Roe are called coarsely connected and "bounded" sets in the sense of Roe are called coarsely bounded. Only when the context makes the meaning clear, to simplify notation we are going to use "connected" and "bounded" in the sense of "coarsely connected" and "coarsely bounded."

The following proposition introduces equivalent conditions for coarse boundedness.

Proposition 2.1.9. *Let $(X, \mathcal{E})$ be a coarse space. Let $A \subseteq X$. Then the following are equivalent:*

1. *$A \times A$ is controlled,*

2. *$x \times A$ is controlled for some $x \in X$.*

Proof. See Proposition 2.16 in [13]. □

There are other equivalent definitions for a subset of a coarse space being bounded. The reader is referred to Proposition 2.16 in [13] for such equivalent definitions.

Now we introduce a basic definition and proposition that we are going to use in future sections.

Definition 2.1.10. If $(X, \mathcal{E})$ is a coarse space, A a subset of X, and E a controlled set, then we define

$$E[A] = \{x \in X \mid \exists\, a \in A \text{ such that } (x, a) \in E\}.$$

Proposition 2.1.11. *Let $(X, \mathcal{E})$ be a coarse space, $A \subseteq X$, and $E \in \mathcal{E}$. If A is bounded, then so is $E[A]$.*

Proof. See Proposition 2.19 in [13]. $\square$

Now we will introduce maps between coarse spaces.

Definition 2.1.12. Let $(X, \mathcal{E}_1)$ and $(Y, \mathcal{E}_2)$ be two coarse spaces. Let $f : X \to Y$ be a map. Then

1. f is called **proper** if the inverse image (under f) of any bounded set in Y is bounded in X,

2. f is called **(uniformly) bornologous** if for each $E \in \mathcal{E}_1$, we have that $(f \times f)(E) \in \mathcal{E}_2$, where $(f \times f)(E) := \{(f(a), f(b)) \mid (a, b) \in E\}$,

3. f is called **coarse** if it is proper and bornologous.

If X and Y are metric spaces, then after endowing X and Y with the metric coarse structure, we can translate the above definitions to the following:

Definition 2.1.13. Let (X, d_1) and (Y, d_2) be metric spaces. Let $f : X \to Y$ be a map. Then

1. f is called **proper** if the inverse images (under f) of any bounded set in Y is bounded in X,

2. f is called **(uniformly) bornologous** if uniformly bounded families of sets are sent to uniformly bounded families, i.e., for all $R > 0$ there exists $S > 0$ such that

$$d_1(x_1, x_2) < R \implies d_2(f(x_1), f(x_2)) < S,$$

3. f is called **coarse** if it is proper and bornologous.

Notice that in Definition 2.1.12 "bounded" means "coarsely bounded," whereas "bounded" in Definition 2.1.13 means "bounded" in the metric sense.

Exercise 2.1.14. Show that Definition 2.1.13 is a special case of Definition 2.1.12 when (X, d_1) and (Y, d_2) are endowed with the respective metric coarse structures.

Let us now introduce the concept of "closeness" of coarse maps.

Definition 2.1.15. Let X be a set and $(Y, \mathcal{E})$ a coarse space. Two functions $f, g : X \to Y$ are **coarsely close** if the set $\{(f(x), g(x)) \mid x \in X\}$ is controlled.

Definition 2.1.16. Let X be a set and (Y, d) a metric space. Two functions $f, g : X \to Y$ are **coarsely close** if there exists $C > 0$ such that for all $x \in X$,

$$d(f(x), g(x)) < C.$$

Exercise 2.1.17. Show that Definition 2.1.16 is a special case of Definition 2.1.15 when (Y, d) is endowed with the metric coarse structure.

Definition 2.1.18. Let $(X, \mathcal{E}_1)$ and $(Y, \mathcal{E}_2)$ be coarse spaces. We call a coarse map $f : X \to Y$ a **coarse equivalence** if there exists a coarse map $g : Y \to X$ such that $g \circ f$ is coarsely close to the identity map id_X and $f \circ g$ is coarsely close to the identity map id_Y. We say that $(X, \mathcal{E}_1)$ and $(Y, \mathcal{E}_2)$ are **coarsely equivalent** if there exists a coarse equivalence $f : X \to Y$.

2.2 Asymptotic Resemblance Spaces

In this section, we introduce asymptotic resemblance structures and several definitions related to asymptotic resemblance spaces.

Definition 2.2.1. Let X be a set. Let λ be an equivalence relation on the power set of X. Then λ is called an **asymptotic resemblance** on X if it satisfies the following properties:

1. $A_1 \lambda B_1$, $A_2 \lambda B_2$ implies $(A_1 \cup A_2)\lambda(B_1 \cup B_2)$,

2. $(B_1 \cup B_2)\lambda A$ and $B_1, B_2 \neq \emptyset$ implies that there are nonempty $A_1, A_2 \subseteq A$ such that $A = A_1 \cup A_2$, $B_1 \lambda A_1$, and $B_2 \lambda A_2$.

A pair (X, λ), where X is a set and λ is an asymptotic resemblance on X, is called an **asymptotic resemblance space**. When $A\lambda B$, we say that A and B are λ **related** or are **asymptotically alike**.

The prime example of an asymptotic resemblance space also comes from the metric case, where two sets are λ related when their Hausdorff distance is finite.

Definition 2.2.2. Let (X, d) be a metric space. Let A and B be subsets of X. Then the **Hausdorff distance** between A and B is defined by

$$d_H(A, B) = \inf\{r \geq 0 \mid A \subseteq B(B, r) \text{ and } B \subseteq B(A, r)\},$$

where $B(A, r)$ is the open ball of radius r about A (i.e., $B(A, r) = \bigcup_{x \in A} B(x, r)$) and $B(B, r)$ is the open ball of radius r about B.

Notice that if (X, d) is a metric space, then by definition $d_H(\emptyset, \emptyset) = 0$ and $d_H(\emptyset, A) = \infty$ for any nonempty $A \subseteq X$.

Example 2.2.3. Let (X, d) be a metric space. Define a relation λ on the power set of X by

$$A\lambda B \quad \text{if and only if} \quad d_H(A, B) < \infty,$$

which is equivalent to

$$A\lambda B \quad \text{if and only if} \quad \text{there exists } r > 0 \text{ such that } A \subseteq B(B, r) \text{ and } B \subseteq B(A, r).$$

Then λ is an asymptotic resemblance, called the **metric asymptotic resemblance** or **asymptotic resemblance induced by the metric d.**

For more examples of asymptotic resemblance relations, the reader is referred to [11].

Exercise 2.2.4. Show that the metric asymptotic resemblance is indeed an asymptotic resemblance.

Definition 2.2.5. Let λ_1 and λ_2 be two asymptotic resemblance structures on the same set X. We say that λ_1 is **finer** than λ_2 (or that λ_2 is **coarser** than λ_1) if $\lambda_1 \subseteq \lambda_2$.

Definition 2.2.6. An asymptotic resemblance space (X, λ) is **asymptotically connected** if $x\lambda y$ for all $x, y \in X$.

To be completely precise, in the above definition we should have written $\{x\}\lambda\{y\}$ instead of $x\lambda y$. However, for simplicity reasons, this abuse of notation will be utilized throughout the paper.

Exercise 2.2.7. Show that the metric asymptotic resemblance is asymptotically connected.

Definition 2.2.8. Let (X, λ) be an asymptotic resemblance space. Then $A \subseteq X$ is called **asymptotically bounded** if A is empty or there exists $x \in X$ such that $A\lambda x$. If A is not asymptotically bounded, then we say that A is **asymptotically unbounded.**

Definition 2.2.9. Two subsets A, C of an asymptotic resemblance space (X, λ) are called **asymptotically disjoint** if for all asymptotically unbounded subsets $A' \subseteq A$ and $C' \subseteq C$, one has $A'\bar{\lambda}C'$.

By Proposition 2.11 of [11], all subsets of asymptotically bounded sets are asymptotically bounded. Therefore, all asymptotically bounded subsets are vacuously asymptotically disjoint from any set.

Definition 2.2.10. An asymptotic resemblance space (X, λ) is **asymptotically normal** if for all asymptotically disjoint subsets $A_1, A_2 \subseteq X$, there are subsets $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$, A_1 is asymptotically disjoint from X_1, and A_2 is asymptotically disjoint from X_2.

Remark 2.2.11. The metric asymptotic resemblance space is asymptotically normal. For the proof, see Proposition 4.5 in [11].

As it was the case with coarse spaces, when the context makes the meaning clear, to simplify notation we are going to use "connected," "bounded," "disjoint," and "normal" in the sense of "asymptotically connected," "asymptotically bounded," "asymptotically disjoint," and "asymptotically normal," respectively.

At this point, the reader may be wondering if there exists a notion of "normality" of coarse spaces. Indeed, we are going to introduce "coarse normality" of coarse spaces in Chapter 5.

Now we will introduce maps between asymptotic resemblance spaces.

Definition 2.2.12. Let (X, λ_1) and (X, λ_2) be two asymptotic resemblance spaces. Let $f : X \to Y$ be a function. Then f is an **asymptotic resemblance map** if

1. $f^{-1}(B)$ is bounded in X for each bounded subset B of Y,

2. $A\lambda_1 B$ implies $f(A)\lambda_2 f(B)$ for all subsets A and B of X.

Theorem 2.2.13. *Let (X, d_1) and (X, d_2) be metric spaces. Let $f : X \to Y$ be a function. Let λ_1 and λ_2 be induced asymptotic resemblance relations, respectively. Then f is a coarse map if and only if it is an asymptotic resemblance map.*

Proof. See Theorem 2.3 in [11]. □

Let us now introduce the concept of "closeness" of asymptotic resemblance maps.

Definition 2.2.14. Let X be a set and (Y, λ) an asymptotic resemblance space. Two functions $f, g : X \to Y$ are **asymptotically close** if for all $A \subseteq X$,

$$f(A)\lambda g(A).$$

Definition 2.2.15. Let X be a set and (Y, d) a metric space. Two functions $f, g : X \to Y$ are **asymptotically close** if for all $A \subseteq X$, there exists $r > 0$ such that

$$A \subseteq B(B, r) \text{ and } B \subseteq B(A, r).$$

Exercise 2.2.16. Show that the Definition 2.2.15 is a special case of Definition 2.2.14 when (Y, d) is endowed with the metric asymptotic resemblance.

The following propositions shows that in the case of metric spaces, the coarse closeness classes coincide with the asymptotic closeness classes.

Proposition 2.2.17. *Let X be a set, (Y, d) a metric space, and $f, g : X \to Y$ two functions. Then f and g are asymptotically close if and only if they are coarsely close.*

Proof. See Proposition 2.16 of [11]. □

Definition 2.2.18. Let (X, λ_1) and (Y, λ_2) be asymptotic resemblance spaces. We call an asymptotic resemblance map $f : X \to Y$ an **asymptotic equivalence** if there exists an asymptotic resemblance map $g : Y \to X$ such that $g \circ f$ is asymptotically close to the identity map id_X and $f \circ g$ is asymptotically close to the identity map id_Y. We say that (X, λ_1) and (Y, λ_2) are **asymptotically equivalent** if there exists a proximal coarse equivalence $f : X \to Y$.

2.3 Bornologies

To be able to talk about large-scale properties of spaces, one needs to have a notion of a set being "bounded," i.e., a set which in the large-scale context is "small" or "insignificant." As we have seen in Section 2.1 and Section 2.2, the notion of a set being "bounded" is defined for both coarse spaces and asymptotic resemblance spaces. In this section, we formalize the notion of "boundedness." Namely, we introduce bornologies and show that, under mild conditions, both coarse spaces and asymptotic resemblance spaces induce natural bornologies. For the introduction to the theory of bornologies, we refer the reader to [1].

Definition 2.3.1. A **bornology** $\mathcal{B}$ on a set X is a family of subsets of X satisfying:

1. $\{x\} \in \mathcal{B}$ for all $x \in X$,
2. $A \in \mathcal{B}$ and $B \subseteq A$ implies $B \in \mathcal{B}$,
3. If $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$.

Elements of $\mathcal{B}$ are called **bounded** and subsets of X not in $\mathcal{B}$ are called **unbounded**. If $X \notin \mathcal{B}$, then we call the bornology **proper**.

Now we state that both coarsely connected coarse spaces and asymptotically connected asymptotic resemblance spaces induce natural bornologies.

Proposition 2.3.2. *Let $(X, \mathcal{E})$ be a connected coarse space. Then the following are true:*

1. *$\{x\}$ is bounded for any $x \in X$,*
2. *if $A \subseteq X$ is bounded and $B \subseteq A$, then B is bounded,*

3. *if $A \subseteq X$ and $B \subseteq X$ are bounded, then $A \cup B$ is bounded.*

Proof. 1 follows from axiom 1 and 2 of a coarse space structure, 2 follows from axiom 2, and 3 follows from Proposition 2.19 of [13]. □

Definition 2.3.3. If $(X, \mathcal{E})$ is a connected coarse space, then the collection of coarsely bounded sets forms a bornology on X, which we call the **bornology induced by $\mathcal{E}$**.

Proposition 2.3.4. *Let (X, λ) be a connected asymptotic resemblance space. Then the following are true:*

1. *$\{x\}$ is bounded for any $x \in X$,*

2. *if $A \subseteq X$ is bounded and $B \subseteq A$, then B is bounded,*

3. *if $A \subseteq X$ and $B \subseteq X$ are bounded, then $A \cup B$ is bounded.*

Proof. Since (X, λ) is connected, $x\lambda x$ for all $x \in X$, i.e., $\{x\}$ is bounded for all $x \in X$. By Proposition 2.11 of [11], all subsets of bounded sets are bounded. Finally, let $A, B \in \mathcal{B}$. If either A or B is empty, then it is clear that $A \cup B$ is bounded, so let us assume that A and B are nonempty. Then there exists $x, y \in X$ such that $A\lambda x$ and $B\lambda y$. By axiom (1) of Definition 2.2.1, $(A \cup B)\lambda\{x, y\}$. Since (X, λ) is asymptotically connected, we also know that $x\lambda y$ and $y\lambda y$. Again by axiom (1) of Definition 2.2.1 this gives us $\{x, y\}\lambda y$. Since λ is an equivalence relation, we get $(A \cup B)\lambda y$, i.e., $(A \cup B)$ is bouded. □

The reader may be wondering if in the above proof we have used axiom 2 of the definition of asymptotic resemblance. Indeed, Proposition 2.11 of [11] used in the above proof utilizes axiom 2.

Definition 2.3.5. Let (X, λ) be a connected asymptotic resemblance space. Then the collection of all asymptotically bounded sets forms a bornology on X, which we call the **bornology induced by λ**.

Proof. See Proposition 2.3.4. □

Chapter 3

Coarse Proximity

In this chapter, we introduce coarse proximities, which stem from a natural translation of small-scale proximities into the coarse context and capture the intuitive notion of two sets being "close at infinity." After introducing metric coarse proximities in Section 3.1, we define general coarse proximities in Section 3.2. Section 3.3 is devoted to coarse neighborhoods, and Section 3.4 introduces an alternative definition of coarse proximities. In Section 3.5 we show that coarse proximities naturally induce an equivalence relation on the power set of a coarse proximity space. Finally, in Section 3.6, Section 3.7, and Section 3.8 we investigate coarse proximity maps and build a category of coarse proximity spaces whose morphisms are closeness classes of coarse proximity maps.

3.1 Metric Coarse Proximity

Before introducing general coarse proximities, we focus our attention on the metric case. In this section, we define a relation on the power set of a metric space. This relation captures the "closeness at infinity" of subsets of a metric space. We also prove several properties of this relation. Recall the following:

Definition 3.1.1. Let (X, d) be a metric space. Let A and B be subsets of X. Then the distance between A and B is defined by

$$d(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\},$$

Recall that by convention the infimum of the empty set is ∞. Thus, if either A or B is the empty set, then $d(A, B) = \infty$.

Definition 3.1.2. Let (X, d) be a metric space. A set $A \subseteq X$ is called **bounded** if it is contained in a ball of finite radius, i.e. there exists $x \in X$ and $r > 0$ such that for all $a \in A$, we have $d(x, a) < r$.

Exercise 3.1.3. Show that the collection of bounded sets of a metric space forms a bornology.

Now we introduce the notion capturing the intuitive notion of two sets being "close at infinity."

Definition 3.1.4. Let (X, d) be a metric space. Let A and B be subsets of X. We say that A and B are **coarsely close**, denoted $A\mathbf{b}B$, if there exists $\epsilon < \infty$ such that for all bounded sets D, $d(A \setminus D, B \setminus D) < \epsilon$.

Exercise 3.1.5. Show that if (X, d) is a metric space and A is a bounded subset of X, then A is not coarsely close to any subset of X. Consequently, if X is bounded, the relation is empty.

Now we introduce equivalent definitions of two subsets of a metric space being coarsely close.

Proposition 3.1.6. *Let (X, d) be a metric space. Let A and B be subsets of X. Then the following are equivalent:*

1. *there exists $\epsilon < \infty$ such that for all bounded sets D, $d(A \setminus D, B \setminus D) < \epsilon$,*
2. *there exists $\epsilon < \infty$ such that for all bounded sets D, there exists $a \in (A \setminus D)$ and $b \in (B \setminus D)$ such that $d(a, b) < \epsilon$,*
3. *there exist unbounded sets $A_1 \subseteq A$, $B_1 \subseteq B$ such that $d_H(A_1, B_1) < \infty$.*

Proof. Exercise. □

For the remainder of this dissertation, all of the equivalent conditions for coarse closeness in a metric space will be used interchangeably without explicit mention. The reader is encouraged to compare the equivalent conditions from the above proposition with the notion of asymptotic disjointness given in Definition 2.2.9 which first appeared in [2].

The following theorem proves crucial properties of the coarse closeness relation in the metric space.

Theorem 3.1.7. *Let (X, d) be a metric space. Let A, B and C be subsets of X. Then the following are true:*

1. *$A\mathbf{b}B$ implies $B\mathbf{b}A$,*
2. *$A\mathbf{b}B$ implies A is unbounded and B is unbounded,*
3. *$A \cap B$ not bounded implies $A\mathbf{b}B$,*
4. *$(A \cup B)\mathbf{b}C$ if and only if $A\mathbf{b}C$ or $B\mathbf{b}C$,*
5. *$A\bar{\mathbf{b}}B$ implies that there exists a set E such that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$.*

where $A\bar{\mathbf{b}}B$ means "$A\mathbf{b}B$" is not true.

Proof. Properties 1,2, and 3 are clear. We will show 4 and 5.

The backward direction of 4 is trivial. To show the forward direction, assume $(A\cup B)\mathbf{b}C$ and for contradiction assume that $A\bar{\mathbf{b}}C$ and $B\bar{\mathbf{b}}C$. Since $(A\cup B)\mathbf{b}C$, there exists $\epsilon < \infty$ such that for all bounded sets D,

$$d((A \cup B) \setminus D, C \setminus D) < \epsilon.$$

Since $A\bar{\mathbf{b}}C$ and $B\bar{\mathbf{b}}C$, there exist bounded sets D_1 and D_2 such that

$$d(A \setminus D_1, C \setminus D_1) > \epsilon \quad \text{and} \quad d(B \setminus D_2, C \setminus D_2) > \epsilon.$$

Let $D := D_1 \cup D_2$. Then notice that D is bounded and

$$d(A \setminus D, C \setminus D) > \epsilon \quad \text{and} \quad d(B \setminus D, C \setminus D) > \epsilon,$$

which implies that

$$d((A \cup B) \setminus D, C \setminus D) > \epsilon,$$

a contradiction.

To prove 5, notice that if A is bounded, then the set $E := (X \setminus A)$ has the desired properties. If B is bounded, then $E := B$ has the desired properties. Thus, assume that both A and B are unbounded and $A\bar{\mathbf{b}}B$. Then for every $n \in \mathbb{N}$ there is a bounded set $D_n \subseteq X$ such that $d(A \setminus D_n, B \setminus D_n) > n^2$. Fix some $x_0 \in X$. Since any bounded set is contained in some large ball centered at x_0, without loss of generality assume that each D_n is a ball centered at x_0 with radius r_n, i.e., $D_n = B(x_0, r_n)$. Additionally, one can assume that the radii are strictly increasing as $n \to \infty$ and that they take integer values. We can even assume that for each n, we have $r_n - r_{n-1} > n + 1$.

For each n, define

$$\begin{aligned} E_0 &:= B, \\ E_n &:= B(B, n) \setminus B(x_0, r_n), \\ E &:= \bigcup_{n \geq 0} E_n. \end{aligned}$$

Notice that this definition implies that $d(X \setminus D_n, D_{n-1}) > n$ for all $n > 1$. Notice that E is unbounded, since $B \subseteq E$. We will show that that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$.

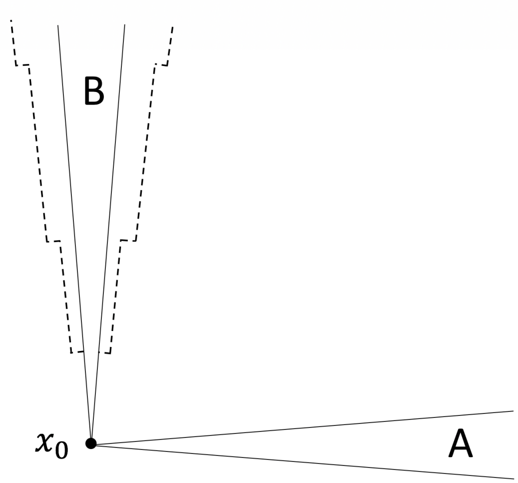

Figure 3.1: Construction of E

First assume that $A\mathbf{b}E$. Then there exists $\epsilon < \infty$ such that for all $n \in \mathbb{N}$, there exists $x_n \in E \setminus D_n$ and $a_n \in A \setminus D_n$ such that $d(x_n, a_n) < \epsilon$. Find n so large that it satisfies the following inequalities:

$$n > 1,$$

$$n > \epsilon,$$

$$(n-1)^2 > n + \epsilon.$$

Notice that the above inequalities are satisfied for any $k \geq n$. Let k be the largest integer such that $x_n \notin D_k$. Clearly $k \geq n$ and $x_n \in D_{k+1}$. Consequently, $x_n \in D_m$ for all $m \geq k+1$. This implies that $x_n \notin E_m$ for all $m \geq k+1$. Therefore, since $x_n \in E$, there exists $b \in B$ such that $d(b, x_n) < k$. Notice that this also implies that $b \notin D_{k-1}$ (because $x_n \in (X \setminus D_k)$ and $d(X \setminus D_k, D_{k-1}) > k$). Similarly, $d(x_n, a_n) < \epsilon$ implies that $a_n \notin D_{k-1}$ (because $d(X \setminus D_k, D_{k-1}) > k > \epsilon$). Thus we have $a_n \in (A \setminus D_{k-1}), b \in (B \setminus D_{k-1})$, and

$$d(b, a_n) \leq d(b, x_n) + d(x_n, a_n) < k + \epsilon < (k-1)^2.$$

contradicting $d(A \setminus D_{k-1}, B \setminus D_{k-1}) > (k-1)^2$. Thus, it has to be that $A\bar{\mathbf{b}}E$.

To show that $(X \setminus E)\bar{\mathbf{b}}B$, for contradiction assume that $(X \setminus E)\mathbf{b}B$. Then there exists $\epsilon < \infty$ such that for all $n \in \mathbb{N}$, there exists $x_n \in (X \setminus E) \setminus B(x_0, r_n)$ and $b_n \in B \setminus B(x_0, r_n)$ such that $d(x_n, b_n) < \epsilon$. Choose n large so that $\epsilon < n$. Then $x_n \notin B(x_0, r_n)$ and $d(x_n, b_n) < \epsilon < n$. In other words, $x_n \in E_n$, contradiciting the fact that $x_n \notin E$. Therefore, it has to be that $(X \setminus E)\bar{\mathbf{b}}B$. □

The specific construction of E in the above proof will be utilized in Chapter 4. However, to prove the strong axiom, one could choose $E := \{x \in X \mid d(x, B) \leq d(x, A)\}$ [1]. We leave the verification of this fact as an exercise for the reader.

Exercise 3.1.8. Show that when B is unbounded, then the set E from property 5 has to be unbounded as well.

[1]The author is grateful to Thomas Weighill for suggesting this alternative construction.

Notice that, unlike asymptotic resemblance, the relation in Definition 3.1.4 is not an equivalence relation. To see that, let $X = \mathbb{R}^2$, A the positive x-axis, B the first quadrant, and C the positive y-axis. Then $A\mathbf{b}B$, $B\mathbf{b}C$, but $A\bar{\mathbf{b}}C$.

3.2 Coarse Proximity

In this section, we generalize the coarse proximity relation from section 3.1 to an arbitrary set X. We also explore several properties of this relation.

Definition 3.2.1. Let X be a set equipped with a bornology $\mathcal{B}$. A **coarse proximity** on a set X is a relation $\mathbf{b}$ on the power set of X satisfying the following axioms for all $A, B, C \subseteq X$:

1. $A\mathbf{b}B$ implies $B\mathbf{b}A$,
2. $A\mathbf{b}B$ implies $A \notin \mathcal{B}$ and $B \notin \mathcal{B}$,
3. $A \cap B \notin \mathcal{B}$ implies $A\mathbf{b}B$,
4. $(A \cup B)\mathbf{b}C$ if and only if $A\mathbf{b}C$ or $B\mathbf{b}C$,
5. $A\bar{\mathbf{b}}B$ implies that there exists a subset E such that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$,

where $A\bar{\mathbf{b}}B$ means "$A\mathbf{b}B$ is not true." If $A\mathbf{b}B$, then we say that A is **coarsely close** to (or **coarsely near**) B. Axiom 4 will be called the **union axiom** and axiom 5 will be called the **strong axiom**. A triple $(X, \mathcal{B}, \mathbf{b})$ where X is a set, $\mathcal{B}$ is a bornology on X, and $\mathbf{b}$ is a coarse proximity relation on X, is called a **coarse proximity space**.

The reader is encouraged to compare the above axioms with the axioms of a (small-scale) proximity given in definition 1.3.1. For the remainder of the paper, the use of axiom 1 is not going to be explicitly mentioned.

Example 3.2.2. Let (X, d) be a metric space, $\mathcal{B}_d$ the collection of all bounded sets of X with respect to the metric d, and $\mathbf{b}_d$ the relation defined in 3.1.4. Then by theorem 3.1.7, this relation is a coarse proximity on X. We call this relation the **metric coarse proximity** and the associated space $(X, \mathcal{B}_d, \mathbf{b}_d)$ the **metric coarse proximity space**.

Example 3.2.3. Let X be a set with any bornology $\mathcal{B}$. For any subsets A and B of X, define

$$A\mathbf{b}B \quad \text{if} \quad A \cap B \notin \mathcal{B}.$$

Then this relation is a coarse proximity on X, called the **discrete coarse proximity**.

Proof. All the axioms are clear besides axiom 5. To show axiom 5, set $E = B$. $\square$

Example 3.2.4. Let X be a set with any bornology $\mathcal{B}$. For any subsets A and B of X, define

$$A\mathbf{b}B \quad \text{if} \quad A, B \notin \mathcal{B}.$$

Then this relation is a coarse proximity on X, called the **indiscrete coarse proximity**.

Proof. All the axioms are clear besides axiom 5. To show axiom 5, assume $A\bar{\mathbf{b}}B$. If $A \in \mathcal{B}$, let $E = X \setminus A$. If $B \in \mathcal{B}$, let $E = B$. $\square$

The reader is encouraged to compare the above examples with similar examples of small-scale proximities, given in Section 1.3.

Notice that if $\mathcal{B}$ is not a proper bornology on a set X (i.e. there are no unbounded sets), then the coarse proximity relation is empty. Also, one can show that the set E from the strong axiom contains B up to some bounded set. Since this fact is used later on, we state it as a proposition.

Proposition 3.2.5. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and let $A, B \subseteq X$. If there exists $E \subseteq X$ such that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$, then E contains B up to some bounded set, i.e. $B \setminus E \in \mathcal{B}$. In particular, if $B \notin \mathcal{B}$, then E has to be unbounded.*

Proof. If the conclusion of the proposition is false, then $B \cap (X \setminus E) \notin \mathcal{B}$, which by axiom 2 implies that $(X \setminus E)\mathbf{b}B$, a contradiction to the definition of E. $\square$

Notice that in the above proof we have not used the fact that $A\bar{\mathbf{b}}E$. Thus, the above proof actually shows the following:

Proposition 3.2.6. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and let $B \subseteq X$. If there exists $E \subseteq X$ such that $(X \setminus E)\bar{\mathbf{b}}B$, then E contains B up to some bounded set, i.e. $B \setminus E \in \mathcal{B}$. In particular, if $B \notin \mathcal{B}$, then E has to be unbounded.* $\square$

Now we are going to show several basic properties of coarse proximities.

Proposition 3.2.7. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let A, B, C, and D be subsets of X. If $A \subseteq C, B \subseteq D$, and $A\mathbf{b}B$, then $C\boldsymbol{b}D$. In particular, X is coarsely near every unbounded subset.*

Proof. Notice that $A \cup C = C$. Thus, by axiom 4, $C\mathbf{b}B$. Since $B \cup D = D$, axiom 4 implies that $C\mathbf{b}D$. $\square$

Remark 3.2.8. The above lemma implies that if $A \subseteq C, B \subseteq D$, and $C\bar{\boldsymbol{b}}D$, then $A\bar{\boldsymbol{b}}B$.

Proposition 3.2.9. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let A and B be subsets of X. Then $A\mathbf{b}B$ if and only if for all $D_1, D_2 \in \mathcal{B}$, $(A \setminus D_1)\mathbf{b}(B \setminus D_2)$.*

Proof. The converse direction follows from Proposition 3.2.7. To prove the forward direction, assume $A\mathbf{b}B$ and let $D_1, D_2 \in \mathcal{B}$ be arbitrary. For contradiction, assume $(A \setminus D_1)\bar{\mathbf{b}}(B \setminus D_2)$, Then notice that since D_1 is bounded, $D_1\bar{\mathbf{b}}(B \setminus D_2)$, so by axiom 4, $A\bar{\mathbf{b}}(B \setminus D_2)$. Similarly, $A\bar{\mathbf{b}}D_2$, which again by axiom 4 gives us $A\bar{\mathbf{b}}B$, a contradiction. Thus, $(A \setminus D_1)\bar{\mathbf{b}}(B \setminus D_2)$. $\square$

Notice that the property from proposition 3.2.9 is a large scale equivalent of the trivial property of a small scale proximity, namely

$$A\delta B \text{ if and only if } (B \setminus \emptyset)\delta(A \setminus \emptyset).$$

Proposition 3.2.10. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let A and B be subsets of X. Then the converse of the strong axiom holds, i.e., if there exists $E \subseteq X$ such that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$, then $A\bar{\mathbf{b}}B$.*

Proof. Assume that there exists $E \subseteq X$ such that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$. By Proposition 3.2.5, there exists a bounded set D such that $(B \setminus D) \subseteq E$. By Remark 3.2.8, this implies that $A\bar{\mathbf{b}}(B \setminus D)$. Also, by axiom 2 we have that $A\bar{\mathbf{b}}D$, and thus by axiom 4, $A\bar{\mathbf{b}}B$. $\square$

3.3 Coarse Neighborhoods

In this section, we introduce the definition of a coarse neighborhood and explore several of its basic properties. We show that if X is a metric space, then coarse neighborhoods coincide with asymptotic neighborhoods defined in [2] and that coarse maps copreserve coarse neighborhoods.

Definition 3.3.1. Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Given subsets $A, B \subseteq X$, we say that B is a **b-coarse neighborhood** (or just **coarse neighborhood** if the proximity relation is clear) of A, denoted $A \ll B$, if $A\bar{\mathbf{b}}(X \setminus B)$.

Theorem 3.3.2. *Given a coarse proximity space $(X, \mathcal{B}, \mathbf{b})$, the relation $\ll$ satisfies the following properties:*

1. *$X \ll (X \setminus D)$ for all $D \in \mathcal{B}$,*
2. *$A \ll B$ implies that $A \subseteq B$ up to some bounded set D, i.e., there exists $D \in \mathcal{B}$ such that $A \setminus D \subseteq B$,*
3. *$A \subseteq B \ll C \subseteq D$ implies $A \ll D$,*
4. *$A \ll B_1$ and $A \ll B_2$ if and only if $A \ll (B_1 \cap B_2)$,*
5. *$A \ll B$ if and only if $(X \setminus B) \ll (X \setminus A)$,*
6. *$A \ll B$ implies that there exists $C \subseteq X$ such that $A \ll C \ll B$.*

Proof. Axiom 2 of a proximity space implies that bounded sets are not related to any sets. Thus, $X\bar{\mathbf{b}}D$ for any $D \in \mathcal{B}$. This is the same as saying $X\bar{\mathbf{b}}(X \setminus (X \setminus D))$ for any $D \in \mathcal{B}$, or equivalently $X \ll (X \setminus D)$ for any $D \in \mathcal{B}$, which is the statement of 1. To show 2, notice that if $A \cap (X \setminus B) \notin \mathcal{B}$, then $A\mathbf{b}(X \setminus B)$, a contradiction to $A \ll B$. To show 3, for contradiction assume that $A \not\ll D$, i.e., $A\mathbf{b}(X \setminus D)$. The union axiom implies then that $B\mathbf{b}(X \setminus D)$. Since $(X \setminus D) \subseteq (X \setminus C)$, again by the union axiom we get $B\mathbf{b}(X \setminus C)$, a contradiction to $B \ll C$. To show 4, notice that by the union axiom

$$\begin{aligned} A \ll B_1 \text{ and } A \ll B_2 &\iff A\bar{\mathbf{b}}(X \setminus B_1) \text{ and } A\bar{\mathbf{b}}(X \setminus B_2) \\ &\iff A\bar{\mathbf{b}}((X \setminus B_1) \cup (X \setminus B_2)) \\ &\iff A\bar{\mathbf{b}}(X \setminus (B_1 \cap B_2)) \\ &\iff A \ll (B_1 \cap B_2). \end{aligned}$$

To show 5, notice that

$$\begin{aligned} A \ll B &\iff A\bar{\mathbf{b}}(X \setminus B) \\ &\iff (X \setminus B)\bar{\mathbf{b}}A \\ &\iff (X \setminus B)\bar{\mathbf{b}}(X \setminus (X \setminus A)) \\ &\iff (X \setminus B) \ll (X \setminus A). \end{aligned}$$

To show 6, assume $A \ll B$, i.e., $A\bar{\mathbf{b}}(X \setminus B)$. Strong axiom implies that there exists $E \subseteq X$ such that $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}(X \setminus B)$. In other words, we have that $A\bar{\mathbf{b}}(X \setminus (X \setminus E))$ and $(X \setminus E)\bar{\mathbf{b}}(X \setminus B)$, i.e., $A \ll (X \setminus E) \ll B$. Setting $C = (X \setminus E)$ gives the desired result. $\square$

Notice that property 4 can be extended to finite intersections in an obvious way. The reader is encouraged to compare the above theorem with the similar theorem for the small-scale proximity (see for example Theorem 3.9 in [12]). In fact, due to the similarity of definitions of coarse proximities and proximities, the proof of the above theorem closely resembles the proof of its small-scale counterpart. In particular, the proofs of properties 3 through 6 of Theorem 3.3.2 only use axioms 1,4, and 5 of coarse proximities. Since these axioms are exactly the same for small-scale proximities, small-scale proximities satisfy the same properties.

The above theorem demonstrates the usefulness of the resemblance of definitions of coarse proximity and proximity. Despite capturing completely different notions, the similarity of definitions allowed us to adjust the proof from the small-scale world to its large-scale counterpart.

Let us now explore a few more basic properties of coarse neighborhoods. The reason why we split these properties in 2 theorems is quite simple: the upcoming properties can be derived from properties in Theorem 3.3.2. This will become more clear in Section 3.4.

Proposition 3.3.3. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let A, B, and C be subsets of X. Then the following are true:*

1. *if $A \in \mathcal{B}$, then $A \ll E$ for any $E \subseteq X$,*

2. *if $(A \setminus B) \in \mathcal{B}$ and $B \ll C$, then $A \ll C$,*

3. *if* $A \ll B$ *and* $B \ll C$*, then* $A \ll C$*,*

4. *if* $A \ll B$ *and* $A \ll (X \setminus B)$*, then* $A \in \mathcal{B}$*,*

Proof. Property 1 is trivial. Property 2 follows from Propostion 3.2.9. To see 3, notice that $A \ll B$ and property 2 of Theorem 3.3.2 imply $(A \setminus B) \in \mathcal{B}$. Thus, property 2 shows $A \ll C$. To see 4, notice that $A \ll B$ and $A \ll (X \setminus B)$ imply $(A \setminus B) \in \mathcal{B}$, and $(A \setminus (X \setminus B)) \in \mathcal{B}$, respectively. Thus, $A = (A \setminus B) \cup (A \setminus (X \setminus B)) \in \mathcal{B}$. $\square$

Notice that 2 in Proposition 3.3.2 implies that if A is unbounded, then so is its coarse neighborhood. Also, property 2 implies that if $A \ll B$ and D is bounded, then $(A \cup D) \ll B$. Finally, notice that by using coarse neighborhoods, the strong axiom can be translated to (assuming axiom 1): $A\bar{\mathbf{b}}B$ implies that there exists a subset E such that $A \ll (X \setminus E)$ and $B \ll E$.

The following proposition characterizes the strong axiom in terms of coarse neighborhoods.

Proposition 3.3.4. *Let* $(X, \mathcal{B})$ *be a set with a bornology, and let* $\mathbf{b}$ *be a relation on* 2^X *satisfying axioms 1 through 4 of a coarse proximity. Define, for subsets* $A, B \subseteq X$*,* $A \ll B \iff A\bar{\mathbf{b}}(X \setminus B)$*. Then for subsets* $A, B \subseteq X$*, the following are equivalent:*

1. $A\bar{\mathbf{b}}B$ *implies that there exists a subset* E *such that* $A\bar{\mathbf{b}}E$ *and* $(X \setminus E)\bar{\mathbf{b}}B$*,*

2. $A \ll B$ *implies that there exists* $C \subseteq X$ *such that* $A \ll C \ll B$*.*

Proof. ($1 \implies 2$) It is clear that $\mathbf{b}$ is a coarse proximity on X. Thus, the result follows from property 6 of Theorem 3.3.2.

($2 \implies 1$) Assume $A\bar{\mathbf{b}}B$. This can be written as $A\bar{\mathbf{b}}(X \setminus (X \setminus B))$, i.e., $A \ll (X \setminus B)$. Therefore, there exists $C \subseteq X$ such that $A \ll C \ll (X \setminus B)$, i.e. $A\bar{\mathbf{b}}(X \setminus C)$ and $C\bar{\mathbf{b}}(X \setminus (X \setminus B))$. Let $E = X \setminus C$. Then $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$. $\square$

Now we will investigate the relationship between coarse neighborhoods, asymptotic neighborhoods, and asymptotic disjointness. In particular, we will show that in the case of metric spaces, coarse neighborhoods and asymptotic neighborhoods coincide. Recall the following definitions from [2]:

Definition 3.3.5. In a metric space (X, d), a subset $B \subseteq X$ is an **asymptotic neighborhood** of a set $A \subseteq X$ if there exists $x_0 \in X$ such that

$$\lim_{r \to \infty} d(A \setminus B(x_0, r), X \setminus B) = \infty.$$

Definition 3.3.6. In a metric space (X, d), two subsets $A, B \subseteq X$ are said to be **asymptotically disjoint** if for some (and hence every) point $x_0 \in X$ one has

$$\lim_{r \to \infty} d(A \setminus B(x_0, r), B \setminus B(x_0, r)) = \infty.$$

Notice that the above definition of asymptotic disjointness seems to differ from the general definition of asymptotic disjointness given in 2.2.9. However, it is shown in Proposition 4.4 of [11] that the two definitions agree.

The following result follows directly from definitions.

Proposition 3.3.7. *Let (X, d) be a metric space with the corresponding metric coarse proximity $\mathbf{b}_d$. Then A and B are asymptotically disjoint in the sense of Definition 3.3.6 if and only if $A \bar{\mathbf{b}} B$.*

To compare asymptotic neighborhoods and coarse neighborhoods, we need the following lemma:

Lemma 3.3.8. *Let (X, d) be a metric space and let $A, B \subseteq X$. Then A and B are asymptotically disjoint if and only if for every $n \in \mathbb{N}$ there is a bounded set C such that $d(A \setminus C, B) > n$.*

Proof. If either A or B is bounded, then the result is trivial. Thus, assume that A and B are unbounded. The reverse direction is trivial. Assume that A and B are asymptotically disjoint and assume towards a contradiction that $n \in \mathbb{N}$ is such that for all bounded $C \subseteq X$, $d(A \setminus C, B) \leq n$. Thus, for every such bounded set C there is a pair $(x_C, y_C) \in A \times B$ such that $x_C \notin C$ and $d(x_C, y_C) \leq n$. Since A and B are asymptotically disjoint, there is a bounded set D such that $d(A \setminus D, B \setminus D) > n$. Without loss of generality we can assume that $D = B(x_0, r)$ for some $x_0 \in X$ and some radius r. Thus, for any $r' > r$, if $C = B(x_0, r')$, then

$x_C \notin C$ and $y_C \in D$. In particular, if $r' > r + n$, then we have $x_C \notin B(x_0, r'), y_C \in B(x_0, r)$ and $d(x_C, y_C) \leq n$, a contradiction. $\square$

Proposition 3.3.9. *Given a metric space (X, d) and subsets $A, B \subseteq X$, B is an asymptotic neighborhood of A if and only if B is a coarse neighborhood of A with respect to the metric coarse proximity $\mathbf{b}_d$.*

Proof. Assume that B is an asymptotic neighborhood of A. Then there exists $x_0 \in X$ such that $\lim_{r\to\infty} d(A \setminus B(x_0, r), X \setminus B) = \infty$. For contradiction, assume that $A\mathbf{b}(X \setminus B)$. Then there exists $\epsilon < \infty$ such that for any r we can find $x \in A \setminus B(x_0, r)$ and $y \in (X \setminus B) \setminus B(x_0, r)$ with the property that $d(x, y) < \epsilon$. In particular, we can find $x \in A \setminus B(x_0, r)$ and $y \in (X \setminus B)$ such that $d(x, y) < \epsilon$, contradicting the fact that $\lim_{r\to\infty} d(A \setminus B(x_0, r), X \setminus B) = \infty$. The converse follows from Proposition 3.3.7 and Lemma 3.3.8. $\square$

The following proposition shows that in the case of metric spaces, coarse maps copreserve asymptotic neighborhoods.

Proposition 3.3.10. *Let $(X, d_1), (Y, d_2)$ be metric spaces and $h : X \to Y$ a coarse map. If $A, B \subseteq Y$ such that $A \ll B$ with respect to the metric coarse proximity structure induced by d_2, then $h^{-1}(A) \ll h^{-1}(B)$ with respect to the metric coarse proximity structure on X induced by d_1.*

Proof. If A is bounded, then since h is a coarse map, $h^{-1}(A)$ is bounded. By Proposition 3.3.2, this implies that any set is a coarse neighborhood of A. In particular, $h^{-1}(A) \ll h^{-1}(B)$. So let us assume that A is unbounded. Let $x_0 \in X$. If $h^{-1}(A) \not\ll h^{-1}(B)$ then there is an $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $x_n \in h^{-1}(A) \setminus B(x_0, n)$ and $y_n \in (X \setminus h^{-1}(B)) \setminus B(x_0, n)$ such that $d(x_n, y_n) < \epsilon$. The sets $A' := \{x_n\}_{n\in\mathbb{N}}$ and $B' := \{y_n\}_{n\in\mathbb{N}}$ are unbounded sets such that $A'\mathbf{b}_{d_1}B'$, which by the coarseness of h implies that $h(A')\mathbf{b}_{d_2}h(B')$. Therefore, by Proposition 3.2.7, $A\mathbf{b}_{d_2}(Y \setminus B)$, a contradiction. Thus, $h^{-1}(A) \ll h^{-1}(B)$. $\square$

Notice that if $A, B \subseteq X$, then $A \ll B$, does not imply $h(A) \ll h(B)$. To see that, let $X = \mathbb{R}, Y = \mathbb{R}^2, A = B = X$, and let $f : X \to Y$ be defined by $f(x) = (x, 0)$. Then f is a coarse map, $A \ll B$, but it is not true that $h(A) \ll h(B)$.

3.4 Alternative Definition of Coarse Proximities

In this section, we introduce a definition of a coarse proximity in terms of coarse neighborhoods. This alternative definition will be extensively used in Chapter 5.

Theorem 3.4.1. *Let X be a set with bornology $\mathcal{B}$. If $\ll$ is a binary relation on the power set of X satisfying 1 through 6 of Theorem 3.3.2 and $\mathbf{b}$ is a relation on the power set of X defined by*

$$A\bar{\mathbf{b}}B \quad \textit{if and only if} \quad A \ll (X \setminus B),$$

then $\mathbf{b}$ is a coarse proximity on X. Also, B is a $\mathbf{b}$-coarse neighborhood of A if and only if $A \ll B$.

Proof. To show axiom 1 of a proximity, assume $A\bar{\mathbf{b}}B$. Then $A \ll (X \setminus B)$, which by 5 of Theorem 3.3.2 implies that $B \ll (X \setminus A)$, i.e., $B\bar{\mathbf{b}}A$. To show axiom 2, notice that 1 and 3 of Theorem 3.3.2 imply that $A \ll (X \setminus B)$ for all $B \in \mathcal{B}$, i.e., $A\bar{\mathbf{b}}B$ for all $B \in \mathcal{B}$. By symmetry proven in axiom 1, this implies axiom 2. To show axiom 3, assume $A\bar{\mathbf{b}}B$, i.e., $A \ll (X \setminus B)$. By 2 of Theorem 3.3.2, this means that there exists $D \in \mathcal{B}$ such that $(A \setminus D) \subseteq (X \setminus B)$, which is the the same as saying that $(A \setminus D) \cap B = \emptyset$. Thus, $A \cap B \subseteq D$, showing that $A \cap B \in \mathcal{B}$. To show axiom 4, first assume $(A \cup B)\bar{\mathbf{b}}C$, i.e., $(A \cup B) \ll (X \setminus C)$. Property 3 of Theorem 3.3.2 implies that $A \ll (X \setminus C)$ and $B \ll (X \setminus C)$, i.e., $A\bar{\mathbf{b}}C$ and $B\bar{\mathbf{b}}C$. To prove the forward direction, assume $(A \cup B)\mathbf{b}C$, which by symmetry gives us $C\mathbf{b}(A \cup B)$, i.e., $C \not\ll X \setminus (A \cup B)$. This is the same as saying $C \not\ll ((X \setminus A) \cap (X \setminus B))$ which by 4 of Theorem 3.3.2 implies that $C \not\ll (X \setminus A)$ or $C \not\ll (X \setminus B)$, i.e. $C\mathbf{b}A$ or $C\mathbf{b}B$. This again by symmetry implies that $A\mathbf{b}C$ or $B\mathbf{b}C$, proving axiom 4. To show the strong axiom, assume $A\bar{\mathbf{b}}B$, i.e., $A \ll (X \setminus B)$. Therefore, by 6 of Theorem 3.3.2, there exists $C \subseteq X$ such that $A \ll C \ll (X \setminus B)$, or equivalently $A \ll (X \setminus (X \setminus C)) \ll (X \setminus B)$. This implies that $A\bar{\mathbf{b}}(X \setminus C)$ and $C\bar{\mathbf{b}}B$. Let $E = X \setminus C$. Then $A\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}B$. Finally, notice that

$$\begin{aligned} B \text{ is a } \mathbf{b}\text{-coarse neighborhood of } A &\iff A\bar{\mathbf{b}}(X \setminus B) \\ &\iff A \ll (X \setminus (X \setminus B)) \\ &\iff A \ll B. \end{aligned}$$

$\square$

The reader is encouraged to compare the above theorem with the similar theorem for the small-scale proximity (see for example Theorem 3.11 in [12]). In fact, due to the similarity of definitions of coarse proximities and proximities, the proof of the above theorem closely resembles the proof of its small-scale counterpart.

Definition 3.4.2. In the setting of the above theorem, we say that the relation $\ll$ **induces a coarse proximity on the pair** $(X, \mathcal{B})$.

3.5 Equivalence Relation Induced by Coarse Proximities

Our goal for the remainder of this chapter is to construct an appropriate category of coarse proximity spaces. Thus, we have to introduce maps between coarse proximity spaces that preserve the coarse proximity structure. We are going to do that in Section 3.6, where we call such maps "coarse proximity maps." However, as it is usual for coarse topology, the morphisms in the category of coarse proximity spaces should not simply be coarse proximity maps, but instead certain equivalence classes (i.e., "closeness classes") of such maps. Otherwise, the isomorphisms in this category would require the existence of two coarse proximity maps f and g such that $f \circ g = id$. However, "large-scale isomorphism" should not require the two spaces to be in the bijective correspondence (take for example $\mathbb{N}$ and $\mathbb{R}$; they are not in bijective correspondence, but they are coarsely equivalent).

How should one define the "closeness classes" of coarse proximity maps? The natural thing to do would be to imitate the asymptotic closeness definition of two asymptotic resemblance maps. Namely, one could define two coarse proximity maps $f, g : X \to (Y, \mathcal{B}, \mathbf{b})$ to be "close" if and only if for all $A \subseteq X$, $f(A)\mathbf{b}g(A)$. However, this immediately raises a question: is this an equivalence relation? This is not clear. Thus, to properly define "closeness classes" of coarse proximity maps, we need to first find an equivalence relation on the power set of a coarse proximity space. Such an equivalence relation needs to be naturally induced by coarse proximities and hopefully have a nice description in the metric case.

In this section, we show that every coarse proximity space induces a weak asymptotic resemblance. We also show that in the case of metric coarse proximity spaces, the weak asymptotic resemblance coincides with the metric asymptotic resemblance when one considers nonempty sets.

Definition 3.5.1. Let X be a set and ϕ an equivalence relation on 2^X satisfying the following property:

$$A\phi B,\ C\phi D \text{ implies } (A \cup C)\phi(B \cup D).$$

Then we call ϕ a **weak asymptotic resemblance**. If $A\phi B$, then we say that A and B are ϕ **related**. By $A\bar{\phi}B$ we mean "$A\phi B$ is not true."

In other words, a weak asymptotic resemblance does not need to satisfy condition 2 of the definition of asymptotic resemblance (for the definition, see 2.2.1). Also, all the definitions related to asymptotic resemblances (e.g., asymptotic connectedness, asymptotic boundedness) can be easily extended to weak asymptotic resemblances.

Now we are going to show that every coarse proximity space induces a weak asymptotic resemblance.

Theorem 3.5.2. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let ϕ be the relation on the power set of X defined in the following way: $A\phi B$ if and only if the following hold:*

(i) for every unbounded $B' \subseteq B$ we have $A\mathbf{b}B'$,

(ii) for every unbounded $A' \subseteq A$ we have $A'\mathbf{b}B$.

Then ϕ is a weak asymptotic resemblance that we call the ***weak asymptotic resemblance (on X) induced by the coarse proximity $\mathbf{b}$****. If the coarse proximity is induced by a metric d, then we call ϕ the* ***weak asymptotic resemblance (on X) induced by d****.*

To prove the above theorem, we need the following remarks and lemmas.

Remark 3.5.3. If ϕ is the relation defined in Theorem 3.5.2 and A and B are bounded, then they are always ϕ related. If A is bounded and B unbounded, then they are not ϕ related.

Remark 3.5.4. If ϕ is the relation defined in Theorem 3.5.2, then notice that $A\phi A$ for all subsets A of X. Also, for all $A, B \subseteq X$ we have $A\phi B$ if and only if $B\phi A$.

Notice that Remark 3.5.3 implies that if ϕ is the relation defined in Theorem 3.5.2, then ϕ is not an asymptotic resemblance. It is because in asymptotic resemblance spaces the empty set is only related to itself, whereas the induced ϕ relation forces the empty set to be ϕ related to other bounded sets.

Lemma 3.5.5. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Then the relation ϕ defined in Theorem 3.5.2 is transitive.*

Proof. Let A, B, and C be subsets of X such that $A\phi B$ and $B\phi C$. Then either all of them are bounded or all of them are unbounded. If all of them are bounded, then by remark 3.5.3 we have $A\phi C$. So let us assume that all of them are unbounded. For contradiction, assume $A\bar{\phi}C$. Then, without loss of generality there exists an unbounded set $A' \subseteq A$ such that $A'\bar{\mathbf{b}}C$ (the other case will follow similarly by symmetry). Thus, there exists an unbounded set E such that $A'\bar{\mathbf{b}}E$ and $(X \setminus E)\bar{\mathbf{b}}C$. If there exists an unbounded $B' \subseteq B$ such that $B' \subseteq (X \setminus E)$, then $(X \setminus E)\bar{\mathbf{b}}C$ and remark 3.2.8 imply that $B'\bar{\mathbf{b}}C$, a contradiction to $B\phi C$. Thus, it has to be that $B \subseteq E$ up to some bounded set D, i.e., $(B \setminus D) \subseteq E$. Thus, since $A'\bar{\mathbf{b}}E$, by remark 3.2.8 we must have that $A'\bar{\mathbf{b}}(B \setminus D)$, which by proposition 3.2.9 implies that $A'\bar{\mathbf{b}}B$, a contradiction to $A\phi B$. Therefore, it has to be that $A\phi C$. □

Lemma 3.5.6. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and let ϕ be the relation on the power set of X as defined in Theorem 3.5.2. If $A\phi B$, then for any bounded sets D_1 and D_2, we have $(A \cup D_1)\phi(B \cup D_2)$.*

Proof. If A and B are both bounded, then the result also follows from Remark 3.5.3. So let us assume that A and B are unbounded. Let $E \subseteq A \cup D_1$ be unbounded. Then there exists unbounded $E' \subseteq E$ such that $E' \subseteq A$. Thus, since $A\phi B$, we have $E'\phi B$, which by Proposition 3.2.7 implies that $E\phi(B \cup D_2.)$ The other condition follows similarly. □

Finally we are ready to prove Theorem 3.5.2.

Proof of Theorem 3.5.2. The fact that ϕ is an equivalence relation follows from remark 3.5.4 and Lemma 3.5.5. To see that ϕ satisfies the property from Definition 3.5.1, let $A, B, C, D \subseteq X$ be such that $A\phi B$ and $C\phi D$. If either pair (A and B or C and D) is bounded, then the result follows from lemma 3.5.6. Therefore, we will assume that all of them are unbounded.

Now let $E \subseteq A \cup C$ be an unbounded set. Then either $E' \cap A$ or $E' \cap C$ is unbounded. Let us call that unbounded set E'. Then we have either $E'\mathbf{b}B$ or $E'\mathbf{b}D$, which by Proposition 3.2.7 implies that $E\mathbf{b}(B \cup D)$. Similarly in the reverse direction. Thus $(A \cup C)\phi(B \cup D)$. $\square$

The following proposition shows that the ϕ relation preserves boundedness.

Proposition 3.5.7. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and let ϕ be the relation on the power set of X as defined in Theorem 3.5.2. Then $A \subseteq X$ is bounded if and only if A is asymptotically bounded.*

Proof. The statement is clear when A is empty, so assume that $A \neq \emptyset$. Assume that $A \in \mathcal{B}$. Let $a \in A$. Since $\mathcal{B}$ is a bornology, $a \in \mathcal{B}$. Thus, by Remark 3.5.3 we have that $A\phi a$. i.e., A is asymptotically bounded. Conversely, let A be asymptotically bounded, i.e., there exists $x \in X$ such that $A\phi x$. Since x is bounded, again by Remark 3.5.3, we have that $A \in \mathcal{B}$. $\square$

Notice that the above proposition also shows that despite not having condition 2 of an asymptotic resemblance space, bounded sets of an induced weak asymptotic resemblance space form a bornology.

Proposition 3.5.8. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and let ϕ be the relation on the power set of X as defined in Theorem 3.5.2. Then the induced weak asymptotic resemblance space is connected.*

Proof. This follows immediately from the fact that singletons are bounded and that bounded sets are always ϕ related. $\square$

Next proposition implies that in the case of metric spaces, the induced weak asymptotic resemblance and the induced asymptotic resemblance coincide when one considers nonempty subsets.

Proposition 3.5.9. *Let (X, d) be a metric space and let ϕ be the weak asymptotic resemblance induced by the metric d. Then given nonempty $A, B \subseteq X$, we have that $A\phi B$ if and only if A and B have finite Hausdorff distance.*

Proof. To prove the forward direction, assume that $A\phi B$ and assume towards a contradiction that $d_H(A,B)=\infty$. Then for each $n\in\mathbb{N}$ there exists $x_n\in A$ such that $d(x_n,B)>n$ or there exists $y_n\in B$ such that $d(y_n,A)>n$. Define A' to be the collection of all such x_n and B' to be the collection of all such y_n. Without loss of generality we may assume that A' is not finite. Notice that A' has to be unbounded (if A' is bounded, then $d(a_i,a_j)<M$ for all $a_i,a_j\in A'$. Let $a_k\in A'$. Then $d(a_k,B)\leq N$ for some N, and consequently $d(a_i,B)\leq M+N$ for all $a_i\in A'$, a contradiction to the construction of A'). Because $A\phi B$ we have that $A'\mathbf{b}B$, which implies that there are unbounded subsets $A''\subseteq A'$ and $B''\subseteq B$ such that $d_H(A'',B'')\leq n$ for some $n<\infty$. Therefore, for all $a\in A''$ there exists $b\in B''$ such that $d(a,b)<n$, a contradiction to the construction of A'.

To prove the converse direction, assume that $d_H(A,B)=m<\infty$. If A and B are bounded, then $A\phi B$ trivially. If A and B are unbounded and $A'\subseteq A$ is an unbounded set, then we know that $A'\subseteq B(B,m)$. Therefore, for all $a\in A'$ we can find $b_a\in B$ such that $d(a,b_a)<m$. Let $B'=\{b_a\}_{a\in A'}$. Then by construction of B' we have that $d_H(A',B')\leq m<\infty$. which implies that $A'\mathbf{b}B$. Running through the same argument replacing A's with B's yields $A\phi B$. $\square$

The above proposition also implies that in the case of a metric space (X,d), the underlying coarse proximity relation induces the relation ϕ that agrees with the asymptotic resemblance λ induced by d on nonempty sets. The only difference between these two relations is the fact that $\emptyset\phi A$ for all bounded set sets $A\subseteq X$, whereas $\emptyset\bar{\lambda}A$ for all such A's (besides the empty set, since $\emptyset\lambda\emptyset$ for any asymptotic resemblance relation). Since "at infinity" the empty set resembles bounded sets, ϕ seems to better capture the nature of the empty set. For the remainder of this paper, we are often going to use the fact that in the case of metric spaces, these two relations coincide for nonempty sets. The fact that they do not agree on the empty set is usually not going to interfere with utilizing the full strength of Hausdorff distance. For the discussion regarding this difference between asymptotic resemblances and the relation ϕ, the reader is referred to Section 6.1.

Now we are going to show a few basic properties related to weak asymptotic resemblance relations. The following proposition shows that in any coarse proximity space two subsets are ϕ related if and only if they share all coarse neighborhoods.

Proposition 3.5.10. *Let* $(X, \mathcal{B}, \mathbf{b})$ *be a coarse proximity space and* ϕ *the weak asymptotic resemblance induced by the coarse proximity* $\mathbf{b}$. *Then for all* $A, B \subseteq X$, *the following are equivalent:*

(i) For all $C \subseteq X$ $A \ll C$ *if and only if* $B \ll C$,

(ii) $A\phi B$.

Proof. ($(ii) \implies (i)$) Assume $A\phi B$ and let C be such that $A \ll C$. Theorem 3.3.2 implies the existence of E such that $A \ll E \ll C$. Notice that $B \subseteq E$ up to a bounded set D, i.e., $(B \setminus D) \subseteq E$. For if that is not the case, then D is an unbounded subset of $X \setminus E$ such that $D\mathbf{b}A$ (because $D \subseteq B$ and $A\phi B$), and therefore implying that $(X \setminus E)\mathbf{b}A$, a contradiction to $A \ll E$. Thus, we know that $(B \setminus D) \subseteq E$ and since $E \ll C$ (*i.e.*, $E\bar{\mathbf{b}}(X \setminus C)$), we have that $(B \setminus D)\bar{\mathbf{b}}(X \setminus C)$, which by Proposition 3.2.9 shows that $B\bar{\mathbf{b}}(X \setminus C)$, i.e. $B \ll C$. The other implication follows by symmetry.

($(i) \implies (ii)$) Let $B' \subseteq B$ be an unbounded subset and assume towards a contradiction that $A\bar{\mathbf{b}}B'$. Then by the strong axiom there is an $E \subseteq X$ such that $A \ll (X \setminus E)$ and $B' \ll E$. However, by assumption we have that $B \ll (X \setminus E)$. In particular, this implies that $B' \ll (X \setminus E)$. So we have that $B' \ll E$ and $B' \ll (X \setminus E)$, which by Theorem 3.3.2 implies that B' is bounded, a contradiction. Therefore $A\mathbf{b}B'$ for every unbounded $B' \subseteq B$. Similarly one can show that $A'\mathbf{b}B$ for every unbounded $A' \subseteq A$. Thus $A\phi B$. □

One could expect that $A\phi B$ implies that for all $C \subseteq X$ $C \ll A$ if and only if $C \ll B$. However, that is not the case.

Example 3.5.11. Consider $\mathbb{R}^2$. Let $A = \{(x, y) \mid y = |x|\}$, $B = \{(x + 1, y) \mid (x, y) \in A\}$, and $C = \{(x, y) \mid (x, y) \in A \text{ and } x \leq 0\}$. Let $X = A \cup B$ with the metric inherited from $\mathbb{R}^2$. Then $A\phi B$ and $C \ll A$, but it is not true that $C \ll B$ (in fact, C is unbounded and disjoint from B).

The following corollary is a direct consequence of Proposition 3.5.10 and will be used in the next section.

Corollary 3.5.12. *Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space and ϕ the corresponding equivalence relation on 2^X. Let A, B, C, and D be subsets of X such that $A\phi C$ and $B\phi D$. Then $A\mathbf{b}B$ if and only if $C\mathbf{b}D$.*

Proof. Assume $A\bar{\mathbf{b}}B$. Then there exists $E \subseteq X$ such that $E\bar{\mathbf{b}}A$ and $(X \setminus E)\bar{\mathbf{b}}B$. This can be translated to $A \ll (X \setminus E)$ and $B \ll E$. By Proposition 3.5.10, this implies that $C \ll (X \setminus E)$ and $D \ll E$, i.e. $E\bar{\mathbf{b}}C$ and $(X \setminus E)\bar{\mathbf{b}}D$. By the converse of the strong axiom, this implies that $C\bar{\mathbf{b}}D$. The converse direction follows by symmetry. □

3.6 Coarse Proximity Maps

In this section, we introduce functions preserving coarse proximity relations, called coarse proximity maps, and we investigate their basic properties. We show that in the case of metric spaces, these maps coincide with coarse maps and asymptotic resemblance maps.

Definition 3.6.1. Let $(X, \mathcal{B}_1, \mathbf{b}_1)$ and $(Y, \mathcal{B}_2, \mathbf{b}_2)$ be coarse proximity spaces. Let $f : X \to Y$ be a function and A and B subsets of X. Then f is a **coarse proximity map** provided that the following are satisfied:

(i) $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$,

(ii) $A\mathbf{b}_1B$ implies $f(A)\mathbf{b}_2f(B)$.

Proposition 3.6.2. *Coarse proximity maps send unbounded sets to unbounded sets. Consequently, preimages of bounded sets are bounded.*

Proof. Assume $B \notin \mathcal{B}_1$. Then $B\mathbf{b}_1B$. Thus, $f(B)\mathbf{b}_2f(B)$, implying that $f(B) \notin \mathcal{B}_2$. □

Exercise 3.6.3. Show the composition of two coarse proximity maps is a coarse proximity map.

The following proposition shows that in the case of metric spaces, coarse maps and coarse proximity maps coincide.

Proposition 3.6.4. *Let (X, d_1) and (Y, d_2) be metric spaces and $(X, \mathcal{B}_{d_1}, \mathbf{b}_1)$ and $(Y, \mathcal{B}_{d_2}, \mathbf{b}_2)$ be induced metric coarse proximity spaces. Let $f : X \to Y$ be a function. Then f is a coarse map if and only if f is a coarse proximity map.*

Proof. To prove the forward direction, assume that f is a coarse map. Since f is (uniformly) bornologous, it sends bounded sets to bounded sets. Now assume $A, B \subseteq X$ are such that $A\mathbf{b}_1 B$, and for contradiction assume that $f(A)\bar{\mathbf{b}}_2 f(B)$. Then there exists a set $E \subseteq Y$ such that $f(A)\bar{\mathbf{b}}_2 E$ and $(Y \setminus E)\bar{\mathbf{b}}_2 f(B)$, i.e.,

$$f(A) \ll (Y \setminus E) \quad \text{and} \quad f(B) \ll E.$$

Since coarse maps copreserve coarse neighborhoods (see Proposition 3.3.10), this implies that

$$A \subseteq f^{-1}(f(A)) \ll f^{-1}(Y \setminus E) = (X \setminus f^{-1}(E)) \quad \text{and} \quad B \subseteq f^{-1}(f(B)) \ll f^{-1}(E),$$

i.e., $A\bar{\mathbf{b}} f^{-1}(E)$ and $B\bar{\mathbf{b}}(X \setminus f^{-1}(E))$. By Proposition 3.2.10, this shows that $A\bar{\mathbf{b}}B$, a contradiction. Thus, it has to be that $f(A)\mathbf{b}f(B)$, completing the proof that f is a coarse proximity map.

To prove the converse, let f be a coarse proximity map and λ_1 and λ_2 be asymptotic resemblance relations induced by the metrics d_1 and d_2 respectively. By Proposition 3.5.9 we have that for nonempty sets these relations are precisely the ϕ_1 and ϕ_2 relations constructed from the respective coarse proximity structures as in Proposition 3.5.2 (i.e., they are weak asymptotic resemblances induced by d_1 and d_2, respectively). We will show that f is an asymptotic resemblance map. Let $A, B \subseteq X$ be such that $A\lambda_1 B$. It is trivial to show that $f(A)\lambda_2 f(B)$ (the implication $A\phi B \implies f(A)\phi f(B)$ is actually true for any coarse proximity map. For the proof, see Proposition 3.8.3). Thus, f is an asymptotic resemblance map. Since Proposition 3.6.2 implies that f is also proper, by Theorem 2.2.13, f must also be a coarse mapping between the metric spaces (X, d_1) and (Y, d_2). □

Corollary 3.6.5. *Let X and Y be metric spaces and let $f : X \to Y$ be a function. Then f is a coarse map if and only if f is a coarse proximity map if and only if f is an asymptotic resemblance map.*

Proof. This follows from Proposition 3.6.4 and Theorem 2.2.13. □

The following corollary shows that if X is a metric space, then any coarse proximity map (equivalently, asymptotic resemblance map) copreserves coarse neighborhoods.

Corollary 3.6.6. *Let $(X, d_1), (Y, d_2)$ be metric spaces and $h : X \to Y$ a coarse proximity map (equivalently, asymptotic resemblance map). If $A, B \subseteq Y$ such that $A \ll B$ with respect to the metric coarse proximity structure induced by d_2, then $h^{-1}(A) \ll h^{-1}(B)$ with respect to the metric coarse proximity structure on X induced by d_1.*

Proof. This is an immediate consequence of Proposition 3.6.4 and Proposition 3.3.10. □

3.7 Closeness Relation of Coarse Proximity Maps

As is usual for coarse topology, the morphisms in the category of coarse proximity spaces will not simply be coarse proximity maps, but instead equivalence classes of such maps. In this section, we define the proximal closeness of two maps whose codomain is a coarse proximity space. We also show that in the case of metric spaces, this closeness relation coincides with coarse closeness (see Definition 2.1.16) and asymptotic closeness (see Definition 2.2.14). We take our definition of proximal closeness to be aesthetically similar to the definition of asymptotic closeness for maps whose codomain in an asymptotic resemblance space, as in Definition 2.2.14.

Definition 3.7.1. Let X be a set and $(Y, \mathcal{B}, \mathbf{b})$ a coarse proximity space. Two functions $f, g : X \to Y$ are **proximally close**, denoted $f \sim g$, if for all $A \subseteq X$

$$f(A)\phi g(A),$$

where ϕ is the weak asymptotic resemblance relation on Y induced by the coarse proximity structure $\mathbf{b}$.

Notice that since ϕ is an equivalence relation, the closeness relation from Definition 3.7.1 is an equivalence relation. We will denote the equivalence class of a function f by $[f]$.

As we mentioned before, there are at least 3 ways to define closeness relation on maps from X to Y. If Y is a coarse proximity space, we can define the closeness relation with respect to that relation, as in Definition 3.7.1. If Y is a coarse space, then we can define the closeness relation with respect to that relation, as in Definition 2.1.15. Finally, if Y is an asymptotic resemblance space, we can define the closeness relation with respect to that relation, as in Definition 2.2.14. The following proposition shows that in the case of metric spaces, all of these definitions of closeness coincide.

Proposition 3.7.2. *Let X be a set, (Y, d) a metric space, and $f, g : X \to Y$ two functions. Then the following are equivalent:*

1. *f and g are are proximally close,*
2. *f and g are are asymptotically close,*
3. *f and g are coarsely close.*

Proof. Since in the case of metric spaces asymptotic resemblance induced by the metric and the ϕ relation coincide for nonempty sets (see Proposition 3.5.9), the closeness relation from Definition 3.7.1 (i.e., the definition of proximally close) coincides with the closeness relation defined in 2.2.14 (i.e., the definition of asymptotically close). This shows the equivalence of 1 and 2. The equivalence of 2 and 3 is the statement of Proposition 2.2.17. □

Thanks to the above proposition, whenever we deal with metric spaces, the sentence "closeness class of a function f" is unambiguous.

Corollary 3.7.3. *Let $f, g : (X, d_1) \to (Y, d_2)$ be maps between metric spaces. Then the following are equivalent:*

1. *f and g are coarse proximity maps and are proximally close,*
2. *f and g are asymptotic resemblance maps maps and are asymptotically close,*
3. *f and g are coarse maps maps and are coarsely close.*

Proof. This follows immediately from Proposition 3.7.2 and Proposition 3.6.5. □

3.8 Category of Coarse Proximity Spaces

In this section, we show that the collections of coarse proximity spaces and proximal closeness classes of coarse proximity maps make up a category.

Definition 3.8.1. Let $(X, \mathcal{B}_1, \mathbf{b}_1)$ and $(Y, \mathcal{B}_2, \mathbf{b}_2)$ be coarse proximity spaces. We call a coarse proximity map $f : X \to Y$ a **proximal coarse equivalence** if there exists a coarse proximity map $g : Y \to X$ such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$. We say that $(X, \mathcal{B}_1, \mathbf{b}_1)$ and $(Y, \mathcal{B}_2, \mathbf{b}_2)$ are **proximally coarse equivalent** if there exists a proximal coarse equivalence $f : X \to Y$.

Proposition 3.8.2. *Let $f(X, d_1) \to (Y, d_2)$ be a map between metric spaces. Then the following are equivalent:*

1. *f is a proximal coarse equivalence,*
2. *f is an asymptotic equivalence,*
3. *f is a coarse equivalence.*

Proof. This follows immediately from Proposition 3.6.5 and Proposition 3.7.2. □

To define a reasonable definition of composition of two closeness classes of coarse proximity maps, we need to know that coarse proximity functions preserve the ϕ relation.

Proposition 3.8.3. *Let $(X, \mathcal{B}_1, \mathbf{b}_1)$ and $(Y, \mathcal{B}_2, \mathbf{b}_2)$ be coarse proximity spaces and let $f : X \to Y$ be a coarse proximity map. Let ϕ_1 and ϕ_2 be weak asymptotic resemblance relations induced by $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Then for any $A, B \subseteq X$, we have*

$$A \phi_1 B \implies f(A) \phi_2 f(B).$$

Proof. Let $A, B,$ and f be as in the statement of the proposition. If A and B are bounded, then the result is trivial. So assume that A and B are unbounded. For contradiction assume that $f(A) \bar{\phi}_2 f(B)$. Then there exists $A' \subseteq f(A)$ such that A' is unbounded and $A' \bar{\mathbf{b}}_2 f(B)$. Then $A'' := f^{-1}(A') \cap A$ is unbounded, $A'' \subseteq A$ and $A'' \bar{\mathbf{b}}_1 B$ (because otherwise $f(A'') \mathbf{b}_2 f(B)$, and since $f(A'') \subseteq f(A)$, $f(A) \mathbf{b}_2 f(B)$), a contradiction to $A \phi_1 B$. □

The following proposition implies that if $f \sim g$, then f is a coarse proximity map/equivalence if and only if g is.

Proposition 3.8.4. *Let* $(X, \mathcal{B}_1, \mathbf{b}_1)$ *and* $(Y, \mathcal{B}_2, \mathbf{b}_2)$ *be coarse proximity spaces. Let* $f : X \to Y$ *and* $g : X \to Y$ *be two proximally close functions. If* f *is a coarse proximity map, then so is* g*. If* f *is a proximal coarse equivalence, then so is* g*.*

Proof. Let ϕ_1 and ϕ_2 be weak asymptotic resemblance relations induced by $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Let us first assume that f is a coarse proximity map. Let $B \subseteq X$ be bounded. Since f is a coarse proximity map, $f(B)$ is bounded. Since $f(B)\phi_2 g(B)$, Remark 3.5.3 implies that $g(B)$ is bounded. Now let $A, C \subseteq X$ and assume $A\mathbf{b}_1C$. Since f is a coarse proximity map, $f(A)\mathbf{b}_2f(C)$. Since $f \sim g$, we have that $f(A)\phi_2 g(A)$ and $f(C)\phi_2 g(C)$. Then Corollary 3.5.12 implies that $g(A)\mathbf{b}_2 g(C)$. Thus, g is a coarse proximity map.

Now assume that f is a proximal coarse equivalence, i.e., there exists a coarse proximity map $f_1 : Y \to X$ such that $f_1 \circ f \sim id_X$ and $f \circ f_1 \sim id_Y$. We will show that $f_1 \circ g \sim id_X$ and $g \circ f_1 \sim id_Y$. To see that $f_1 \circ g \sim id_X$, let $A \subseteq X$. Then since $g \sim f$, we have that $g(A)\phi_2 f(A)$. Since f_1 is a coarse proximity map, Proposition 3.8.3 implies that $(f_1(g(A))\phi_1(f_1(f(A))$. Since A was arbitrary, this implies that

$$(f_1 \circ g) \sim (f_1 \circ f) \sim id_X.$$

To see that $g \circ f_1 \sim id_Y$, let $C \subseteq Y$. Since g is close to f, we have $g(f_1(C))\phi_2 f(f_1(C))$. Since C was arbitrary, this implies that

$$(g \circ f_1) \sim (f_1 \circ f_1) \sim id_Y. \qquad \square$$

Notice that the proof of the above proposition actually shows that if $f \sim g$ and f_1 is an "inverse" of f (as in the definition of a proximal coarse equivalence), then f_1 is also an "inverse" of g.

Now we will define the composition of two proximal classes of coarse proximity maps.

Proposition 3.8.5. *Let* $(X, \mathcal{B}_1, \mathbf{b}_1), (Y, \mathcal{B}_2, \mathbf{b}_2),$ *and* $(Z, \mathcal{B}_3, \mathbf{b}_3)$ *be coarse proximity spaces. Let* $f : X \to Y$ *and* $g : Y \to Z$ *be coarse proximity functions and let* $[f]$ *and* $[g]$ *be respective proximal closeness classes. Then the operation* $[f] \circ [g] := [f \circ g]$ *is well-defined.*

Proof. Let f and g be as in the statement of the proposition. Let ϕ_2 and ϕ_3 be weak asymptotic resemblance relations induced by $\mathbf{b}_2$ and $\mathbf{b}_3$, respectively. Let $f' \in [f]$ and $g' \in [g]$. By Remark 3.6.3, $g \circ f$ and $g' \circ f'$ are coarse proximity maps from X to Z. Let us show that $g \circ f$ and $g' \circ f'$ are close, which will show that $[g \circ f] = [g' \circ f']$. Let A be a set. Since $f \sim f'$, we have that $f(A)\phi_2 f'(A)$. Therefore, we have

$$\Big(g(f(A))\Big)\phi_3\Big(g(f'(A))\Big)\phi_3\Big(g'(f'(A))\Big),$$

where the first equivalence follows from Proposition 3.8.3 and the second equivalence follows from $g \sim g'$. Since ϕ_3 is an equivalence relation, this completes the proof that $[g \circ f] = [g' \circ f']$. $\square$

Definition 3.8.6. The collection of coarse proximity spaces and closeness classes of coarse proximity maps (with the composition of morphisms defined as in Proposition 3.8.5) makes up the category $C_{\mathbf{b}}$ of coarse proximity spaces.

Associativity of morphisms in the above definition follows from the associativity of composition of functions. The identity morphism is the equivalence class of the identity map. Also, notice that if $(X, \mathcal{B}_1, \mathbf{b}_1)$ and $(Y, \mathcal{B}_2, \mathbf{b}_2)$ are coarse proximity spaces and $f : X \to Y$ is a proximal coarse equivalence, then $[f]$ is an equivalence in the category of coarse proximity spaces.

As we will see in Section 4, sometimes it is useful to restrict one's attention to the metric case. Thus, we introduce the following definition.

Definition 3.8.7. The collection of unbounded metric spaces and closeness classes of coarse proximity maps (with the composition of morphisms defined as in Proposition 3.8.5) makes up the category C_d of unbounded metric spaces.

Notice that thanks to Corollary 3.7.3, the words "coarse proximity maps" in the above definition could be replaced with "coarse maps" or "asymptotic resemblance maps," without any change in meaning.

3.9 Questions

Question 3.9.1. Let $(X, \mathcal{B}, \mathbf{b})$ be a coarse proximity space. Let ϕ be the weak asymptotic resemblance induced by the coarse proximity $\mathbf{b}$, as in Theorem 3.5.2. If one restricts oneself to nonempty sets, is ϕ an asymptotic resemblance relation? Conjecture: false.

Question 3.9.2. If the answer to the above question in no, what are the conditions under which coarse proximity spaces naturally induce asymptotic resemblance relations?

Question 3.9.3. Besides the examples provided in this dissertation, what are other examples of coarse proximity spaces?

Question 3.9.4. When are coarse proximity spaces metrizable? In other words, given a coarse proximity space, when can we find a metric on that space that induces the given coarse proximity structure?

Question 3.9.5. In Chapter 4, we will introduce one application of coarse proximities, called the proximity space at infinity. What are other applications of coarse proximities?

Question 3.9.6. Let X be a set, $(Y, \mathcal{B}, \mathbf{b})$ a coarse proximity space, and $f, g : X \to Y$ two functions. Define

$$f \sim g \quad \text{if and only if} \quad f(A)\mathbf{b}g(A) \text{ for all } A \subseteq X.$$

Is this an equivalence relation? Conjecture: false.

Question 3.9.7. One of the greatests strengths of coarse proximities is that despite capturing entirely different notions than small-scale proximities, the definitions of coarse proximities and proximities are strikingly similar. Thus, what other small-scale proximity theorems can be translated to the large-scale context in terms of coarse proximities?

Chapter 4

Proximity at Infinity

In this chapter, we construct a natural small-scale proximity structure on the set of unbounded subsets of a metric space. We also show how this structure naturally induces a small-scale proximity on the equivalence classes of the weak asymptotic resemblance induced by the metric. We call this space the "proximity space at infinity." We then proceed to show that the construction is functorial, making up a functor from the category of unbounded metric spaces whose morphisms are closeness classes of coarse proximity maps (equivalently, coarse maps or asymptotic resemblance maps) to the category of proximity spaces whose morphisms are proximity maps.

The idea of defining topological structures on equivalence classes of unbounded sets has been utilized previously. In [9], a functor from metric spaces to totally bounded metric spaces, called "spaces of ends", is constructed. For a variety of unbounded metric spaces the space of ends is empty. As we will see, the proximity space at infinity for every unbounded metric space is always nonempty. Our construction was inspired by considering the Vietoris topology on the hyperspace of the Higson corona of a proper metric space.

4.1 Coarse Neighborhoods of Radius f

To be able to construct small-scale proximity on the set of unbounded subsets of a metric space, we have to have to find a way to construct coarse neighborhoods in a controlled way. In this section, we introduce such construction and we show several of its properties.

Definition 4.1.1. A sequence $f : \mathbb{N} \to \mathbb{R}$ is called **adequate** if it is positive and

$$f(n) - f(n-1) > n + 1 \quad \text{for all} \quad n > 1.$$

Notice that if f and g are adequate sequences, then so is $h := \max\{f, g\}$.

Definition 4.1.2. Let (X, d) be a metric space and let x_0 be a point in X. If $A \subseteq X$ is a set and $f : \mathbb{N} \to \mathbb{R}$ is an adequate sequence, then we define the **coarse neighborhood of** A **of radius** f **relative to** x_0, denoted $U_{x_0}(A, f) \subseteq X$, in the following way:

$$\begin{aligned} A_0^f &:= A, \\ A_n^f &:= B(A, n) \setminus B(x_0, f(n)), \\ U_{x_0}(A, f) &:= \bigcup_{n \geq 0} A_n^f. \end{aligned}$$

To simplify notation, when the base point is clear from the context we will denote $B(x_0, f(n))$ by $B_{f(n)}$ and $U_{x_0}(A, f)$ by $U(A, f)$. In this notation, the definition of $U(A, f)$ becomes

$$\begin{aligned} A_0^f &:= A, \\ A_n^f &:= B(A, n) \setminus B_{f(n)}, \\ U(A, f) &:= \bigcup_{n \geq 0} A_n^f. \end{aligned}$$

The reader is encouraged to compare the above definition to the construction of the coarse neighborhood in section 3.1. As expected, we will show that a coarse neighborhood of A of radius f relative to x_0 is really a coarse neighborhood.

Proposition 4.1.3. *Given a metric space (X, d), a point $x_0 \in X$, and a set $A \subseteq X$, we have $A \ll U(A, f)$ for every adequate sequence f.*

Proof. For contradiction assume that $A\mathbf{b}(X \setminus U(A, f))$. Then there exists $\epsilon < \infty$ such that for all $n \in \mathbb{N}$, there exists $a_n \in A \setminus B_{f(n)}$ and $x_n \in (X \setminus U(A, f)) \setminus B_{f(n)}$ such that $d(a_n, x_n) < \epsilon$. Choose n large such that $\epsilon < n$. Then $x_n \notin B_{f(n)}$ and $d(a_n, x_n) < \epsilon < n$. Thus, $x_n \in A_n^f$, contradicting the fact that $x_n \notin U(A, f)$. Therefore, by contradiction, $A\bar{\mathbf{b}}(X \setminus U(A, f))$, i.e., $A \ll U(A, f)$. $\square$

The following definition and proposition justify why it is reasonable to restrict ourselves to considering only coarse neighborhoods of the form $U(A, f)$.

Definition 4.1.4. Given a coarse proximity space $(X, \mathcal{B}, \mathbf{b})$ and a set $A \subseteq X$, we say that a collection $\mathcal{A} \subseteq 2^X$ of coarse neighborhoods of A is a **coarse neighborhood base** at A if for every coarse neighborhood $D \subseteq X$ of A there is $E \in \mathcal{A}$ such that

$$A \ll E \ll D.$$

The following proposition shows that for any set B, all $U(B, f)$ form a coarse neighborhood base at B.

Proposition 4.1.5. *Let (X, d) be a metric space and $x_0 \in X$ a point. For each set $B \subseteq X$, define $\mathcal{C}_{x_0}(B)$ to be the set of all coarse neighborhoods of the form $U(B, f)$, where f is an adequate sequence. Then $\mathcal{C}_{x_0}(B)$ is a coarse neighborhood base at B.*

Proof. The statement is trivial if B is bounded, so assume that B is unbounded. Let $B \subseteq X$ be an unbounded set and $D \subseteq X$ a coarse neighborhood of B. Then $B\bar{\mathbf{b}}(X \setminus D)$. Set $A = (X \setminus D)$. Then the set E from Theorem 3.1.7 is the desired coarse neighborhood such that $E \in \mathcal{C}_{x_0}(B)$ and $B \ll E \ll D$. $\square$

Let us explore a few basic properties of coarse neighborhoods of the form $U(A, f)$.

Proposition 4.1.6. *Let (X, d) be a metric space, $x_0 \in X$ a point, f and g adequate sequences, and A and B unbounded subsets of X. Then the following are true:*

1. *$A \subseteq U(A, f)$,*

2. *if A is bounded, then so is $U(A, f)$,*

3. *if $B \subseteq A$, then $U(B, f) \subseteq U(A, f)$,*

4. *if $f \leq g$, then $U(A, g) \subseteq U(A, f)$,*

5. $U(A, f) \cup U(B, f) = U(A \cup B, f)$,

6. *if* $A \ll B$, *then there exists a bounded set* D *such that* $U(A, f) \setminus D \subseteq U(B, f)$.

Proof. The first four properties are direct consequences of definitions. Property 5 follows from the fact that $B(A, n) \cup B(B, n) = B(A \cup B, n)$. To show property 6, let $A \ll B$. Then there exists a bounded set D' such that $(A \setminus D') \subseteq B$. Thus, by property 3, we have $U((A \setminus D'), f) \subseteq U(B, f)$. Thus, by property 5, $U(A, f) \subseteq U(B, f) \cup U(D', f)$. Let $D = U(D', f)$. Then by property 2, D is bounded, and we get that $U(A, f) \setminus D \subseteq U(B, f)$. $\square$

One could expect that if A, B are subsets of a metric space such that $A \ll B$, then for every adequate sequence f one has $U(A, f) \ll U(B, f)$. However, this is not the case.

Example 4.1.7. Consider $\mathbb{R}^2$. Let $A = B = \{(0, y) \mid y > 0\}$. Let x_0 be the origin and let f and g be two adequate sequences such that $f(n) < g(n)$ for all $n \in \mathbb{N}$. Let $X = (\mathbb{R}^2 \setminus U(A, g)) \cup A$. Then $A \ll B$ (Since $(X \setminus B) = (X \setminus U(A, g))$), but it is not true that $U(A, f) \ll U(B, f)$ (since $U(A, f) = U(B, f)$ and $U(A, f)$ is unbounded in $X \setminus U(A, g)$).

One could also expect that if A, B are subsets of a metric space such that $A \phi B$, then for every adequate sequence f one has:

1. $U(A, f) \phi U(B, f)$,

2. $C \ll U(A, f)$ if and only if $C \ll U(B, f)$ for any $C \subseteq X$.

However, the following example shows that neither of these statements is true.

Example 4.1.8. Let $1 > \epsilon > 0$, $A = \{(0, t) \in \mathbb{R}^2 \mid t \geq 0\}$, $B = \{(-1, t) \in \mathbb{R}^2 \mid t \geq 0\}$ and $x_0 = (0, 0)$. Notice that A and B have finite Hausdorff distance. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = n^3$. Then f is an adequate sequence. For each $n \in \mathbb{N}$, define $x_n = ((n - \epsilon), f(n) + \epsilon)$. Let $C = \{x_n\}_{n \in \mathbb{N}}$. Let $X = A \cup B \cup C$ with the subspace metric inherited from $\mathbb{R}^2$. Then $C \subseteq U(A, f)$ and $C \cap U(B, f) = \emptyset$. Notice that by this specific construction there are bounded sets D_1 and D_2 such that $X = U(A, g) \cup D_1$ and $B \cup A = U(B, g) \cup D_2$. Consequently, since X and $A \cup B$ do not have finite Hausdorff distance, neither do $U(A, g)$ and $U(B, g)$. Also, notice that $C \ll U(A, f)$ (since $(X \setminus U(A, f)$ is a bounded set), but it is not true that $C \ll U(B, f)$ (since $C \cap (X \setminus U(B, f)) = C$, which is unbounded).

To be able to prove the "star-refinement" property of coarse neighborhoods (the meaning of "star-refinement" will be made clear in Definition 4.1.13), we need the following lemmas:

Lemma 4.1.9. *Let (X, d) be a metric space, A an unbounded subset of X, x_0 a point in X, $f : \mathbb{N} \to \mathbb{R}$ an adequate sequence, and $n \in \mathbb{N}$ such that $n > 1$. Then*

$$d\Big((X \setminus U(A, f)) \setminus B_{f(n)}, A \setminus B_{f(n)}\Big) > n - 1.$$

Proof. For contradiction, assume that there exists $x \in (X \setminus U(A, f)) \setminus B_{f(n)}$ and $a \in A \setminus B_{f(n)}$ such that $d(x, a) < n$. Then we have that $x \notin B_{f(n)}$ and $x \in B(A, n)$. Thus, $x \in A_n^f$, a contradiction to $x \notin U(A, f)$. □

For the remainder of this dissertation, we will use the following notation: for each n, define

$$C_n := B_{f(n)} \setminus B_{f(n-1)}.$$

Lemma 4.1.10. *Let (X, d) be a metric space, A an unbounded subset of X, x_0 a point in X, $f : \mathbb{N} \to \mathbb{R}$ an adequate sequence, and $n \in \mathbb{N}$ such that $n > 1$. If $x \in C_n$ and $y \in X$ such that $d(x, y) < n$, then y can only belong to C_{n-1}, C_n, or C_{n+1}. In particular, $y \in B_{f(n+1)}$ and $y \notin B_{f(n-2)}$.*

Proof. The fact that $y \notin C_k$ for $k \leq n - 2$ follows from the fact that $B_{f(k)} \subseteq B_{f(n-2)}$ for all $k \leq n - 2$ and the fact that for $n > 1$ the difference in radii between $B_{f(n-2)}$ and $B_{f(n-1)}$ is bigger than n. The fact that $y \notin C_k$ for $k \geq n + 2$ follows from the fact that $B_{f(n+2)} \subseteq B_{f(k)}$ for all $k \geq n + 2$ and the fact that the difference in radii between $B_{f(n+1)}$ and $B_{f(n)}$ is bigger than $n + 2$ for $n > 1$. □

Lemma 4.1.11. *Let (X, d) be a metric space, A an unbounded subset of X, x_0 a point in X, $f : \mathbb{N} \to \mathbb{R}$ an adequate sequence, and $n \in \mathbb{N}$ such that $n > 1$. If $x \in U(A, f)$ and $x \in B_{f(n)}$, then there exists $a \in A$ such that $d(x, a) < n$.*

Proof. Since $x \in U(A, f)$, we know that $x \in A_m^f$ for some m. Thus, there exists $a \in A$ such that $d(x, a) < m$. Also, since $x \in A_m^f$, $x \notin B_{f(m)}$. Since $x \in B_{f(n)}$, it has to be that $m < n$. Thus $d(x, a) < m < n$. □

Proposition 4.1.12. *Let (X, d) be a metric space and $x_0 \in X$. Then given an adequate sequence $f : \mathbb{N} \to \mathbb{R}$, there is another adequate sequence g such that for all unbounded $A \subseteq X$ we have that*

$$U(U(A, g), g) \ll U(A, f).$$

Proof. Let f be as in the statement of the proposition. Define

$$g(n) = f(n^2).$$

Since f is adequate, so is g. Let $A \subseteq X$ be an arbitrary unbounded subset. To simplify notation, we will define the following:

$$D := U(U(A, g), g),$$
$$E := U(A, g),$$
$$F := U(A, f).$$

We wish to show that $D\bar{\mathbf{b}}(X \setminus F)$, where $\mathbf{b}$ is the coarse proximity relation induced by the metric d. For contradiction assume that $D\mathbf{b}(X \setminus F)$. Then there is an $\epsilon < \infty$ such that for every natural number n there exists $x_n \in ((X \setminus F) \setminus B_{f(n)})$ and $y_n \in (D \setminus B_{f(n)})$ such that

$$d(x_n, y_n) < \epsilon.$$

Since F is a coarse neighborhood of A of radius f, by Lemma 4.1.9 we have that for any $n > 4$

$$d((X \setminus F) \setminus B_{f(n-3)}, A \setminus B_{f(n-3)}) > n - 4.$$

Find n so large that it satisfies the following inequalities:

$$n > 4,$$
$$n - 4 > \epsilon + 2\sqrt{n+2} + 3,$$
$$n > \lceil \sqrt{n+2} \rceil,$$
$$n - 1 > \lceil \sqrt{n+2} \rceil + 1.$$

Notice that the above inequalities are satisfied for any $k \geq n$. Let k be the largest number such that $x_n \notin B_{f(k)}$. Then $x_n \in B_{f(k+1)}$. Clearly $k \geq n$. Since $d(x_n, y_n) < \epsilon < n \leq k$, by Lemma 4.1.10, y_n can be in C_{k+2}, C_{k+1}, or C_k. In particular, $y_n \in B_{f(k+2)}$. Therefore y_n has to be in $B_{g(\lceil\sqrt{k+2}\rceil)}$. Because if it is not, then

$$y_n \notin B_{g(\lceil\sqrt{k+2}\rceil)} = B_{f(\lceil\sqrt{k+2}\rceil^2)} \supseteq B_{f(k+2)},$$

a contradiction. Thus, since $y_n \in D$ and $y_n \in B_{g(\lceil\sqrt{k+2}\rceil)}$, by Lemma 4.1.11 there exists $z \in E$, such that

$$d(y_n, z) < \lceil\sqrt{k+2}\rceil.$$

Since $y_n \in B_{g(\lceil\sqrt{k+2}\rceil)}$ and $d(y_n, z) < \lceil\sqrt{k+2}\rceil$, by the proof of the Lemma 4.1.10 we have that $z \in B_{g(\lceil\sqrt{k+2}\rceil+1)}$. Thus, since $z \in E$, and $z \in B_{g(\lceil\sqrt{k+2}\rceil+1)}$, again by Lemma 4.1.11 there exists $a \in A$ such that

$$d(z, a) < \lceil\sqrt{k+2}\rceil + 1.$$

Let us now examine how close a is to x_0. We do it step by step. We know that y_n can be in C_{k+2}, C_{k+1}, or C_k. In particular, $y_n \notin B_{f(k-1)}$. Since $d(y_n, z) < \lceil\sqrt{k+2}\rceil < k$ we have that $z \notin B_{f(k-2)}$. Since $d(z, a) < \lceil\sqrt{k+2}\rceil + 1 < k - 1$, we have that $a \notin B_{f(k-3)}$. So, we have $x_n \in ((X \setminus F) \setminus B_{f(k-3)}), a \in (A \setminus B_{f(k-3)})$, and

$$\begin{aligned} d(x_n, a) &\leq d(x_n, y_n) + d(y_n, z) + d(z, a) \\ &\leq \epsilon + \lceil\sqrt{k+2}\rceil + \lceil\sqrt{k+2}\rceil + 1 \\ &\leq \epsilon + \sqrt{k+2} + 1 + \sqrt{k+2} + 1 + 1 \\ &= \epsilon + 2\sqrt{k+2} + 3 \\ &< k - 4, \end{aligned}$$

a contradiction to

$$d((X \setminus F) \setminus B_{f(k-3)}, A \setminus B_{f(k-3)}) > k - 4. \qquad \square$$

Definition 4.1.13. Given a metric space (X, d), a point $x_0 \in X$, and two adequate sequences f and g such that g satisfies the relation in Proposition 4.1.12, the sequence g is said to be a **coarse star refinement of f with respect to x_0**.

Corollary 4.1.14. *Let (X, d) be a metric space and $x_0 \in X$. Then given two adequate sequences $f, g : \mathbb{N} \to \mathbb{R}$, there is another such sequence h such that for all unbounded $A \subseteq X$, we have that*

$$U(A, h) \ll U(A, f) \quad \text{and} \quad U(A, h) \ll U(A, g).$$

Proof. By Proposition 4.1.12, there exist adequate sequences f_1 and g_1 such that

$$U(U(A, f_1), f_1) \ll U(A, f) \quad \text{and} \quad U(U(A, g_1), g_1) \ll U(A, g).$$

In particular, we have that

$$U(A, f_1) \ll U(A, f) \quad \text{and} \quad U(A, g_1) \ll U(A, g).$$

Define $h(n) = \max\{f(n), g(n)\}$. Notice that h is an adequate sequence and by Proposition 4.1.6,

$$U(A, h) \ll U(A, f) \quad \text{and} \quad U(A, h) \ll U(A, g). \qquad \square$$

4.2 Hyperspace at Infinity

In this section, we will construct the small-scale proximity on the set of unbounded subsets of a metric space.

Definition 4.2.1. Given a coarse proximity space $(X, \mathcal{B}, \mathbf{b})$, the **hyperspace at infinity** of X, denoted $\mathcal{H}_\infty(X)$, is the set $\{A \subseteq X \mid A \notin \mathcal{B}\}$.

Notice that unbounded subsets of a metric space become "points" in the hyperspace at infinity. The following theorem defines a proximity on the hyperspace at infinity of a metric space.

Theorem 4.2.2. *Given a metric space (X, d), a point $x_0 \in X$, and its corresponding hyperspace at infinity $\mathcal{H}_\infty(X)$, define a relation δ on the powerset of $\mathcal{H}_\infty(X)$ in the following way:*

$\mathcal{A}\delta\mathcal{C}$ if and only if for every adequate sequence f there exist $A \in \mathcal{A}$ and $C \in \mathcal{C}$ such that $A \ll U(C, f)$ and $C \ll U(A, f)$.

Then δ is a proximity on $\mathcal{H}_\infty(X)$.

Remark 4.2.3. Notice that the statement $\mathcal{A}\bar{\delta}\mathcal{C}$ is equivalent to the existence of an adequate sequence f such that for all $A \in \mathcal{A}$ and $C \in \mathcal{C}$, either $A \not\ll U(C, f)$ or $C \not\ll U(A, f)$. Such a sequence f will be called a **witnessing (adequate) sequence** for $\mathcal{A}\bar{\delta}\mathcal{C}$.

Proof of theorem 4.2.2. The only axioms of a proximity that are not immediate from the definition of δ are the union and strong axioms. We will show these here:

Union axiom: Assume that $\mathcal{A}, \mathcal{C}, \mathcal{D} \subseteq \mathcal{H}_\infty(X)$ and $(\mathcal{C} \cup \mathcal{D})\delta\mathcal{A}$. Assume towards a contradiction that $\mathcal{C}\bar{\delta}\mathcal{A}$ and $\mathcal{D}\bar{\delta}\mathcal{A}$. Then there are witnessing adequate sequences f_1 and f_2, respectively. By Corollary 4.1.14 there exists an adequate sequence g such that for all unbounded sets $A \subseteq X$ we have

$$U(A, g) \ll U(A, f_1) \text{ and } U(A, g) \ll U(A, f_2).$$

Because $(\mathcal{C} \cup \mathcal{D})\delta\mathcal{A}$, there is some $C \in (\mathcal{C} \cup \mathcal{D})$ and some $A \in \mathcal{A}$ such that $C \ll U(A, g)$ and $A \ll U(C, g)$. If $C \in \mathcal{C}$, then we have

$$C \ll U(A, g) \ll U(A, f_1) \quad \text{and} \quad A \ll U(C, g) \ll U(C, f_1),$$

a contradiction to f_1 being a witnessing cover for $\mathcal{A}\bar{\delta}\mathcal{C}$. If $C \in \mathcal{D}$, then we have

$$C \ll U(A, g) \ll U(A, f_2) \quad \text{and} \quad A \ll U(C, g) \ll U(C, f_2),$$

a contradiction to f_2 being a witnessing cover for $\mathcal{A}\bar{\delta}\mathcal{D}$. The converse direction of the union axiom is trivial.

Strong Axiom: Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{H}_\infty(X)$ be such that $\mathcal{A}\bar{\delta}\mathcal{C}$. Then there exists a witnessing adequate sequence f such that for all $A \in \mathcal{A}$ and all $C \in \mathcal{C}$ one either has $A \not\ll U(C, f)$ or $C \not\ll U(A, f)$. Let g be an adequate sequence such that for all $A \in \mathcal{H}_\infty(X)$ we have

$$U(U(A, g), g) \ll U(A, f).$$

Define

$$\mathcal{E} = \{K \in \mathcal{H}_\infty(X) \mid \exists C \in \mathcal{C},\, C \ll U(K, g) \ll U(C, f)\}.$$

We claim that $\mathcal{A}\bar{\delta}\mathcal{E}$ and $(\mathcal{H}_\infty(X) \setminus \mathcal{E})\bar{\delta}\mathcal{C}$. If $\mathcal{A}\delta\mathcal{E}$, then there is some $A \in \mathcal{A}$ and some $K \in \mathcal{E}$ such that $A \ll U(K, g)$ and $K \ll U(A, g)$. Let C be a member of $\mathcal{C}$ that witnesses K being a member of $\mathcal{E}$. Then $U(K, g) \ll U(C, f)$, which implies $A \ll U(C, f)$. Also, by 6 of Proposition 4.1.6, we have that $K \ll U(A, g)$ implies that there is a bounded set D such that $U(K, g) \setminus D \subseteq U(U(A, g), g) \ll U(A, f)$. Thus, $U(K, g) \ll U(A, f)$, and hence $C \ll U(A, f)$. Therefore, we have $A \ll U(C, f)$ and $C \ll U(A, f)$, which is a contradiction to f being a witnessing sequence. Therefore $\mathcal{A}\bar{\delta}\mathcal{E}$.

Now assume towards a contradiction that $(\mathcal{H}_\infty(X) \setminus \mathcal{E})\delta\mathcal{C}$. Then let $K \in (\mathcal{H}_\infty(X)) \setminus \mathcal{E}$ and $C \in \mathcal{C}$ be such that $K \ll U(C, g)$ and $C \ll U(K, g)$. The first of these implies that there is a bounded set D such that $U(K, g) \setminus D \subseteq U(U(C, g), g) \ll U(C, f)$, which in turn implies that $U(K, g) \ll U(C, f)$. However this implies that $K \in \mathcal{E}$, which is a contradiction. Therefore $(\mathcal{H}_\infty(X) \setminus \mathcal{E})\bar{\delta}\mathcal{C}$ which established the strong axiom for δ. □

Note that the coarse neighborhoods $U(A, f)$ and hence the proximity δ are defined with respect to a particular point x_0 within our metric space X. Proposition 4.1.5 showed that regardless of the choice of point x_0, the resulting coarse neighborhoods of the form $U(A, f)$ will make up a coarse neighborhood base at any subset A of X. Now we will show that the proximity on the hyperspace also does not depend on the choice of the base point. For the sake of clarity, we will return to our previous notation involving the basepoint, i.e., $U_{x_0}(A, f)$.

Lemma 4.2.4. *Let (X, d) be a metric space and x_0, x_1 distinct points of X. Then for any adequate sequence f, there exists an adequate sequence g such that for all subsets $C \subseteq X$,*

$$U_{x_0}(C, g) \ll U_{x_1}(C, f).$$

Proof. Let f be an adequate sequence. Without loss of generality we can assume that f takes integer values. We define an adequate sequence g in the following way: for each $n \in \mathbb{N}$, there is a least natural number $T(n)$ such that $B(x_1, f(n)) \subseteq B(x_0, T(n))$. Define $g : \mathbb{N} \to \mathbb{R}$ by setting $g(1) = T(1)$ and then inductively by

$$g(n) := \max\{T(n^2), g(n-1) + n + 2\}.$$

Notice that the second condition implies that g is an adequate sequence. We then claim that for all subsets $C \subseteq X$ we have

$$U_{x_0}(C, g) \ll U_{x_1}(C, f).$$

Denote the set on the left hand side by D and the set on the right hand side by E. Assume towards a contradiction that $D\mathbf{b}(X \setminus E)$, where $\mathbf{b}$ is the coarse proximity induced by the metric. Then there is an $\epsilon < \infty$ such that for every $n \in \mathbb{N}$ there is

$$x_n \in (X \setminus E) \setminus B(x_0, g(n)) \quad \text{and } y_n \quad \in D \setminus B(x_0, g(n))$$

such that $d(x_n, y_n) < \epsilon$. Let k_n be the greatest natural number such that $x_n \notin B(x_0, g(k_n))$. Then $x_n \in B(x_0, g(k_n+1))$. By Lemma 4.1.10 we have that for any $n > \epsilon$, $y_n \notin B(x_0, g(k_n - 1))$ and $y_n \in B(x_0, g(k_n+2))$. Then, by Lemma 4.1.11 we have that there must be a $c_n \in C$ such that $d(y_n, c_n) < k_n+2$. Notice that since $y_n \notin B(x_0, g(k_n-1))$ and $d(y_n, c_n) < k_n+2$, g being is an adequate sequence implies $c_n \notin B(x_0, g(k_n-3))$. Also, by the triangle inequality we have that for all $n > \epsilon$,

$$d(x_n, c_n) < \epsilon + k_n + 2.$$

However, for all $n > \epsilon$ we also have that

$$x_n, c_n \notin B(x_0, g(k_n - 3)) \supseteq B(x_0, T((k_n - 3)^2)) \supseteq B(x_1, f((k_n - 3)^2)).$$

Thus, by Lemma 4.1.9 we have that $d(x_n, c_n) > (k_n - 3)^2$ for all $n > \epsilon$. But for large enough n, this contradicts $d(x_n, c_n) < \epsilon + k_n + 2$. Therefore, it has to be that $U_{x_0}(C, g) \ll U_{x_1}(C, f)$. $\square$

Theorem 4.2.5. *Let (X, d) be a metric space and x_0, x_1 distinct points of X. If δ_0 and δ_1 are the respective proximities on $\mathcal{H}_\infty(X)$ constructed using x_0 and x_1 as in Theorem 4.2.2, then the proximity relations δ_0 and δ_1 are equal.*

Proof. Assume $\mathcal{A}, \mathcal{C} \subseteq \mathcal{H}_\infty(X)$ be such that $\mathcal{A}\delta_0\mathcal{C}$. Let f be an arbitrary adequate sequence. Then by Lemma 4.2.4, there exists an adequate sequence g such that for all subsets $C \subseteq X$,

$$U_{x_0}(C, g) \ll U_{x_1}(C, f).$$

Since $\mathcal{A}\delta_0\mathcal{C}$, there exists $A \in \mathcal{A}$ and a $C \in \mathcal{C}$ such that

$$A \ll U_{x_0}(C, g) \quad \text{and} \quad C \ll U_{x_0}(A, g),$$

which by the property of g gives us

$$A \ll U_{x_0}(C, g) \ll U_{x_1}(C, f) \quad \text{and} \quad C \ll U_{x_0}(A, g) \ll U_{x_1}(A, f).$$

Thus, we have $\mathcal{A}\delta_1\mathcal{C}$. Similarly one can show that $\mathcal{A}\delta_1\mathcal{C}$ implies $\mathcal{A}\delta_0\mathcal{C}$. Therefore $\delta_0 = \delta_1$. $\square$

4.3 Proximity at Infinity

The goal of this section is to construct a functor from the category C_d, i.e., the category of unbounded metric spaces whose morphisms are closeness classes of coarse proximity maps (equivalently, coarse maps or asymptotic resemblance maps), to the category of proximity spaces whose morphisms are proximity maps. The natural thing to try would be to assign to each unbounded metric space its hyperspace at infinity with the induced small-scale proximity, and to each closeness class of coarse proximity maps the induced map on the hyperspace, as in the following diagram:

$$\begin{tikzcd}
(X, d_0) \arrow[r, "{[h]}"] \arrow[d, dashed] & (Y, d_1) \arrow[d, dashed] \\
(\mathcal{H}_\infty(X), \delta_0) \arrow[r, "h"] & (\mathcal{H}_\infty(Y), \delta_1)
\end{tikzcd}$$

where the dashed arrow means the assignment of a hyperspace at infinity to the given metric space. However, the question immediately arises: is that functor well-defined? In other words, if we choose some other $g \in [h]$, is it true that $h(A) = g(A)$ for any $A \in \mathcal{H}_\infty(X)$? One can immediately see that this does not have to be the case (since h and g are not the same, but only proximally close). Thus, to create a well-defined functor, when given two proximally close functions h and g, we have to make sure that $h(A)$ and $g(A)$ are identified for all unbounded A. How can one do it? Since h and g are proximally close, we know that $h(A)\phi g(A)$ for any $A \in \mathcal{H}_\infty(X)$. Since in the metric case ϕ relation is equivalent to the relation of having finite Hausdorff distance for nonempty sets, we are going to identify all sets that have finite Hausdorff distance. We are going to denote that space by $\boldsymbol{B}X$. We are also going to induce a proximity $\boldsymbol{\delta}$ on that new space. The resulting space will be called the "proximity space at infinity." Consequently, we are going to get the following diagram

$$\begin{tikzcd}
(X, d_0) \arrow[r, "{[h]}"] \arrow[d, dashed] & (Y, d_1) \arrow[d, dashed] \\
(\mathcal{H}_\infty(X), \delta_0) \arrow[d, "\pi_X"] & (\mathcal{H}_\infty(Y), \delta_1) \arrow[d, "\pi_Y"] \\
(\boldsymbol{B}X, \boldsymbol{\delta}_0) \arrow[r, "\boldsymbol{B}h"] & (\boldsymbol{B}Y, \boldsymbol{\delta}_1)
\end{tikzcd}$$

where π is a surjective projection. This diagram is going to help us build the desired functor.

Definition 4.3.1. Let (X, d) be a metric space and $(\mathcal{H}_\infty(X), \delta_0)$ the corresponding proximity space (constructed with respect to some point $x_0 \in X$). Define the set $\mathbf{B}X$ to be the set of all ϕ equivalence classes of unbounded sets in X, where ϕ is the weak asymptotic resemblance induced by the coarse proximity induced by d. By Proposition 3.5.9 this relation is equivalent to the relation of having finite Hausdorff distance. Endow $\mathbf{B}X$ with quotient proximity $\boldsymbol{\delta}$ induced by the projection $\pi : (\mathcal{H}_\infty(X), \delta_0) \to \mathbf{B}X$, as in Definition 1.3.13. The quotient proximity space $(\mathbf{B}X, \boldsymbol{\delta})$ is called the **proximity space at infinity** of X.

If A is a subset of X, then ϕ equivalence class of A (i.e. a point in $\mathbf{B}X$) will be denoted by $[A]$. Also notice that since proximities induce completely regular topologies, the proximity space at infinity is a completely regular topological space.

Theorem 4.3.2. *Let $(X, d_0), (Y, d_1)$ be unbounded metric spaces, $h : X \to Y$ a coarse proximity map, and $(\boldsymbol{B}X, \boldsymbol{\delta}_0), (\boldsymbol{B}Y, \boldsymbol{\delta}_1)$ the corresponding proximity spaces at infinity. Then the map $\boldsymbol{B}h : \boldsymbol{B}X \to \boldsymbol{B}Y$ defined by*

$$\boldsymbol{B}h([A]) = [h(A)]$$

is a well-defined proximity map. Moreover, if $l : X \to Y$ is a coarse proximity map that is close to h, then $\boldsymbol{B}h = \boldsymbol{B}l$.

Proof. The well-definedness of $\boldsymbol{B}h$ follows from Proposition 3.8.3. The equality of $\boldsymbol{B}h$ and $\boldsymbol{B}l$ for close coarse proximity maps h and l follows from the definition of closeness of coarse proximity maps. Let us show that $\mathbf{B}h$ is a proximity map. Let $x_0 \in X$ and $y_0 = h(x_0)$. Let δ_0 be the proximity on $\mathcal{H}_\infty(X)$ constructed using the basepoint x_0 and let δ_1 be proximity on $\mathcal{H}_\infty(Y)$ constructed using the point $y_0 = h(x_0)$. We then consider the following commutative diagram:

$$\begin{tikzcd}
(\mathcal{H}_\infty(X), \delta_0) \arrow[r, "h"] \arrow[d, "\pi_X"] \arrow[dr, "\mathbf{B}h \circ \pi"] & (\mathcal{H}_\infty(Y), \delta_1) \arrow[d, "\pi_Y"] \\
(\mathbf{B}X, \boldsymbol{\delta}_0) \arrow[r, "\mathbf{B}h"] & (\mathbf{B}Y, \boldsymbol{\delta}_1)
\end{tikzcd}$$

where h is the obvious induced map on the hyperspaces. Notice that since the diagram is commutative, the map $\mathbf{B}h \circ \pi$ is well-defined. By Proposition 1.3.14 the function $\mathbf{B}h$ is a proximity map if and only if the function $\mathbf{B}h \circ \pi$ is a proximity map. To show that

$\mathbf{B}h \circ \pi$ is a proximity map, it is enough to show that h is a proximity map ($\mathbf{B}h \circ \pi$ is then a composition of two proximity maps, and therefore a proximity map). Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{H}_\infty(X)$ be such that $\mathcal{A}\delta_0\mathcal{C}$. We will show that $h(\mathcal{A})\delta_1 h(\mathcal{C})$, which will complete our proof. Let f_1 be an adequate sequence and f_2 an adequate sequence that coarse star refines f_1. Since h is a coarse proximity map, it is proper. Thus, for every $n \in \mathbb{N}$ there is a least $k \in \mathbb{N}$ such that

$$h^{-1}(B(y_0, f_2(n))) \subseteq B(x_0, k).$$

We will denote this natural number by $T(n)$. Likewise, h is bornologous, so for every $n \in \mathbb{N}$ there is a greatest natural number m (possibly also ∞ for the first few n's) such that

$$d(x,y) \leq m \implies d(h(x), h(y)) < n.$$

We will denote this number by $\rho(n)$ (if $\rho(n) = \infty$ for some n, then set $\rho(n) = 1$ instead). Since X and Y are unbounded anf f is a coarse proximity map, the functions ρ and T as sequences must be nondecreasing and divergent. We can choose a sequence (n_k) of natural numbers such that for any $k \in \mathbb{N}$, the following conditions hold:

(i) $k < T(n_k)$,

(ii) $k + 1 < \rho(n_k)$,

(iii) $\max\{T(n_k), \rho(n_k)\} - \max\{T(n_{k-1}), \rho(n_{k-1})\} > k + 1$.

We then define an adequate sequence g by

$$g(k) = \max\{T(n_k), \rho(n_k)\}.$$

Because $\mathcal{A}\delta_0\mathcal{C}$ we have that there is an $A \in \mathcal{A}$ and a $C \in \mathcal{C}$ such that $A \ll U_{x_0}(C, g)$ and $C \ll U_{x_0}(A, g)$. We then claim the following:

$$h(A) \ll U_{y_0}(h(C), f_1) \text{ and } h(C) \ll U_{y_0}(h(A), f_1)]$$

We will show the first of these. The second is shown similarly. Let $x \in A \ll U_{x_0}(C, g)$. Then there is a greatest integer k such that $x \notin B(x_0, g(k))$. Then $x \in B(x_0, g(k+1))$. This implies that there is a $c \in C$ such that $d(x, c) < k + 1$. Since $x \notin B(x_0, g(k))$, we have that $x \notin B(x_0, T(n_k)) \cup B(x_0, \rho(n_k))$. This implies that $h(x) \notin B(y_0, f_2(n_k))$. Likewise, because $k + 1 < \rho(n_k)$ we have that $d(h(x), h(c)) < n_k$. Therefore we have that $h(x) \in h(C)^{f_2}_{n_k}$ and hence, up to a bounded set, $h(A) \subseteq U_{y_0}(h(C), f_2)$. Then, because f_2 coarse star refines f_1 we have $h(A) \ll U_{y_0}(h(C), f_1)$. Similarly $h(C) \ll U_{y_0}(h(A), f_1)$. Thus, $h(\mathcal{A})\delta_1 h(\mathcal{C})$, which establishes that $h : (\mathcal{H}_\infty(X), \delta_0) \to (\mathcal{H}_\infty(Y), \delta_1)$ is a proximity map, which consequently implies that $\mathbf{B}h$ is a proximity map. $\square$

Corollary 4.3.3. *The assignment of the proximity space $(\boldsymbol{B}X, \boldsymbol{\delta})$ to an unbounded metric space (X, d) and the assignment of $\boldsymbol{B}f : \boldsymbol{B}X \to \boldsymbol{B}Y$ to a closeness equivalence class of coarse proximity maps $[f] : X \to Y$ between unbounded metric spaces makes up a functor $\boldsymbol{B}$ from the category of unbounded metric spaces whose morphisms are close equivalence classes of coarse proximity maps to the category of proximity spaces whose morphisms are proximity maps.* $\square$

The existence of the functor $\boldsymbol{B}$ shows that the proximity at infinity is a coarse invariant, as in the following corollary.

Corollary 4.3.4. *If (X, d_1) and (Y, d_2) are unbounded proximally coarse equivalent metric spaces (or to say the same thing, coarse equivalent or asymptotically equivalent), then their corresponding proximity spaces at infinity are proximally isomorphic. In particular, they are homeomorphic.* $\square$

Let us now consider a few examples of proximity spaces at infinity.

Example 4.3.5. Let $X = \{n^2 \mid n \in \mathbb{N}\} \cup \{0\}$ be equipped with its usual metric. Then if $A, B \subseteq X$ are unbounded subsets we have that the Hausdorff distance between A and B is finite if and only if A and B differ by a bounded set. Likewise, there is an adequate sequence g such that for all unbounded sets A one has that $U_0(A, g) \setminus A$ is bounded (one could take $g(n) = n^3$ for example). Then if $\mathcal{H}_\infty(X)$ is given the proximity δ constructed using the basepoint 0, we have that two subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{H}_\infty(X)$ are close if any only if there is an

$A \in \mathcal{A}$ and a $B \in \mathcal{B}$ such that the Hausdorff distance between A and B is finite. Thus the proximity $\hat{\boldsymbol{\delta}}$ on $\mathbf{B}X$ defined by $\pi_X(\mathcal{A})\hat{\boldsymbol{\delta}}\pi_X(\mathcal{B})$ if and only if $\pi_X(\mathcal{A}) \cap \pi_X(\mathcal{B}) \neq \emptyset$ is a proximity on $\mathbf{B}X$ for which the projection $\pi_X : X \to \mathbf{B}X$ is a proximity map. The proximity $\hat{\boldsymbol{\delta}}$ is the finest possible proximity on $\mathbf{B}X$, and hence the finest proximity on $\mathbf{B}X$ for which π_X is a proximity map. Thus, $\hat{\boldsymbol{\delta}}$ is the quotient proximity and $(\mathbf{B}X, \hat{\boldsymbol{\delta}})$ is the proximity at infinity of X. The topology on $\mathbf{B}X$ is discrete.

Proposition 4.3.6. *Let $\mathbb{Z}$ have its natural metric structure and corresponding coarse proximity structure. Then $\mathbf{B}\mathbb{Z}$ is not connected and has at least 3 connected components.*

Proof. Consider the following subsets of $\mathcal{H}_\infty(\mathbb{Z})$

$$\begin{aligned}
\mathcal{A} &:= \{A \in \mathcal{H}_\infty(\mathbb{Z}) \mid \exists z \in \mathbb{Z} \forall x \in A\, z \leq x\}, \\
\mathcal{C} &:= \{C \in \mathcal{H}_\infty(\mathbb{Z}) \mid \exists z \in \mathbb{Z} \forall x \in C\, z \geq x\}, \\
\mathcal{D} &:= \mathcal{H}_\infty(\mathbb{Z}) \setminus (\mathcal{A} \cup \mathcal{C}),
\end{aligned}$$

i.e., $\mathcal{A}$ is the set of unbounded subsets of $\mathbb{Z}$ that have a lower bound, $\mathcal{C}$ is the set of unbounded subsets of $\mathbb{Z}$ that have an upper bound, and $\mathcal{D}$ is the set of unbounded subsets of $\mathbb{Z}$ that have neither a lower bound nor an upper bound. Clearly $\mathcal{A}, \mathcal{C}$, and $\mathcal{D}$ are mutually disjoint. These three sets are trivially closed under the relation of having finite Hausdorff distance, i.e., if $A, B \in \mathcal{H}_\infty(\mathbb{Z})$ and $d_H(A, B) < \infty$, then both A and B are in $\mathcal{A}$, both A and B are in $\mathcal{C}$, or both A and B are in $\mathcal{D}$. If f is any adequate sequence, then given $A \in \mathcal{A}$, there is no $C \in \mathcal{C}$ or $D \in \mathcal{D}$ such that $D \ll U(A, f)$ or $C \ll U(A, f)$, regardless of the choice of a basepoint. Similarly, if f is any adequate sequence, then given $C \in \mathcal{C}$, there is no $A \in \mathcal{A}$ or $D \in \mathcal{D}$ such that $D \ll U(C, f)$ or $A \ll U(C, f)$, regardless of the choice of a basepoint. Consequently, no two of $\mathcal{A}, \mathcal{C}$, or $\mathcal{D}$ are close in the hyperspace at infinity. We then let $\mathbf{3} = \{a, c, d\}$ be the discrete proximity space on 3 elements. The function $h : (\mathcal{H}_\infty(\mathbb{Z}), \delta) \to \mathbf{3}$ defined by $h(\mathcal{A}) = a$, $h(\mathcal{C}) = c$, and $h(\mathcal{D}) = d$ is a proximity mapping that is constant on the fibers of the projection $\pi : \mathcal{H}_\infty(\mathbb{Z}) \to \mathbf{B}\mathbb{Z}$. Thus there is a unique proximity map $g : \mathbf{B}\mathbb{Z} \to \mathbf{3}$ such that $g \circ \pi = h$. Thus $\mathbf{B}\mathbb{Z}$ is not connected and has at least 3 connected components. □

Proposition 4.3.7. *Let* $\mathbb{N}$ *have its natural metric structure and corresponding coarse proximity structure. Then the proximity on* $\mathbf{B}\mathbb{N}$ *is not discrete.*

Proof. Let $\mathcal{A} \subseteq \mathcal{H}_\infty(\mathbb{N})$ be the set of all unbounded subsets of $\mathbb{N}$ that have asymptotic dimension 0. Let $\mathcal{C} \subseteq \mathcal{H}_\infty(\mathbb{N})$ be the set of all unbounded subsets of $\mathbb{N}$ that have asymptotic dimension 1. It is clear that $\mathcal{A}$ and $\mathcal{C}$ are closed under the relation of having finite Hausdorff distance. It is also clear that $\mathcal{A}$ and $\mathcal{C}$ are disjoint. We will show that $\mathcal{A}\delta\mathcal{C}$, which will imply that $\pi(\mathcal{A})\boldsymbol{\delta}\pi(\mathcal{C})$, showing that the proximity on $\mathbf{B}\mathbb{N}$ is not discrete.

We will use 1 as a basepoint. Let f be an adequate sequence. Let us first construct an unbounded set A of asymptotic dimension 0 in the following way: define

$$g(n) = \lceil f(100n) \rceil,$$

and define A_1 to be the integral interval $[1, g(1)]$. Then, for every natural number $n > 1$ let $\eta_n = \max(A_{n-1})$, and define

$$A_n = \{m \in \mathbb{N} \mid \exists k \in \mathbb{N} \cup \{0\},\, m = \eta_n + kn,\, m \leq g(n)\}$$

Finally define $A = \bigcup A_n$. This set is clearly unbounded and has asymptotic dimension 0 because for each real number $r \geq 0$ the set of r-components is uniformly bounded. Also, by construction of A we have that $U_1(A, f) = \mathbb{N}$, and consequently $U_1(A, f)$ is a coarse neighborhood of every subset of $\mathbb{N}$. Thus, setting $C = \mathbb{N}$, we have that $A \in \mathcal{A}, C \in \mathcal{C}, A \ll U_1(C, f)$, and $C \ll U_1(A, f)$. Since f was arbitrary, this shows that $\mathcal{A}\delta\mathcal{C}$, which consequently implies that $\pi(\mathcal{A})\boldsymbol{\delta}\pi(\mathcal{C})$, as desired. □

Remark 4.3.8. The above proof shows that as an element of the hyperspace at infinity of $\mathbb{N}$, the singleton $\mathbb{N}$ is close to the set consisting of asymptotic dimension 0 sets.

Corollary 4.3.9. *If* (X, d) *is an unbounded metric space into which* $\mathbb{N}$ *coarsely embeds, then* $\mathbf{B}X$ *is not discrete.*

4.4 Questions

Question 4.4.1. Can the construction of the proximity space at infinity be generalized to arbitrary coarse proximity spaces?

Question 4.4.2. What is the relationship between the asymptotic dimension of the unbounded metric space and the topological dimension of its proximity space at infinity?

Question 4.4.3. The proximity on $\mathcal{H}_\infty(X)$ is constructed, whereas the proximity on $\boldsymbol{B}X$ is induced. Can one obtain the proximity on $\boldsymbol{B}X$ in a more constructive way?

Question 4.4.4. What are other examples of proximity spaces at infinity?

Question 4.4.5. Is the proximity (and therefore topology) on the proximity space at infinity Hausdorff?

Question 4.4.6. Besides being a coarse invariant of unbounded metric spaces, what are other applications of proximity spaces at infinity?

Chapter 5

Relationships between Coarse Structures

In this dissertation, we have talked about 3 different coarse structures: coarse space structures, asymptotic resemblance structures, and coarse proximity structures. We have also showed (see Section 3.5) that coarse proximity structures induce weak asymptotic resemblances (and in the metric case, asymptotic resemblances for nonempty sets). In this chapter, we explore other relationships between coarse structures.

5.1 Coarse Spaces $\implies$ Asymptotic Resemblance Spaces

In this section, we show that coarse space structures induce asymptotic resemblance structures. The following definition comes from [11].

Definition 5.1.1 (coarse space structure $\implies$ asymptotic resemblance)**.** Let $(X, \mathcal{E})$ be a coarse structure. For any two subsets A and B of X, define

$$A\lambda_{\mathcal{E}}B \quad \text{if and only if} \quad A \subset E[B] \text{ and } B \subset E[A] \text{ for some } E \in \mathcal{E}.$$

Then the relation $\lambda_{\mathcal{E}}$ is an asymptotic resemblance on X. We call $\lambda_{\mathcal{E}}$ the **asymptotic resemblance induced by the coarse structure** $\mathcal{E}$.

Exercise 5.1.2. Show that that asymptotic resemblance induced by the coarse structure is indeed an asymptotic resemblance.

Without loss of generality we can always assume that the set E from Definition 5.1.1 is symmetric and contains the diagonal. Also, when it is clear that the asymptotic resemblance was induced by the coarse structure $\mathcal{E}$, for the simplicity of notation we will denote $\lambda_{\mathcal{E}}$ by λ. Finally, whenever we say "coarse structure induces asymptotic resemblance structure," we always mean "induces" in the sense of Definition 5.1.1.

Now we are going to show that the induced asymptotic resemblance preserves connectedness and boundedness.

Proposition 5.1.3. *Let $(X, \mathcal{E})$ be a coarse space and λ the asymptotic resemblance induced by the coarse structure $\mathcal{E}$. Then $(X, \mathcal{E})$ is coarsely connected if and only if (X, λ) is asymptotically connected.*

Proof. First assume that $(X, \mathcal{E})$ is coarsely connected. Let $x, y \in X$ be arbitrary. Since $(X, \mathcal{E})$ is coarsely connected, $\{x\} \times \{y\}$ is controlled. Let $E := (\{x\} \times \{y\}) \cup (\{y\} \times \{x\})$. Then $E \in \mathcal{E}$ and we have that $\{x\} \subset E[y]$ and $\{y\} \subset E[x]$. Thus, $x\lambda y$. Since x and y were arbitrary, this shows that if (X, λ) is asymptotically connected. Conversely, assume that (X, λ) is asymptotically connected. Let $x, y \in X$ be arbitrary. Then $x\lambda y$ implies the existence of $E \in \mathcal{E}$ such that $\{x\} \subset E[y]$ and $\{y\} \subset E[x]$. Thus, $(x, y) \in E$, showing that $(X, \mathcal{E})$ is coarsely connected. $\square$

Proposition 5.1.4. *Let $(X, \mathcal{E})$ be a coarse space, λ the asymptotic resemblance induced by the coarse structure $\mathcal{E}$, and $A \subseteq X$. Then A is coarsely bounded if and only if it is asymptotically bounded.*

Proof. The statement is true when A is empty, so let us assume $A \neq \emptyset$. Assume that A is coarsely bounded, i.e., $E := A \times A$ is controlled. Since A is nonempty, let $x \in A$ be arbitrary. Then $A \subset E[x]$ and $\{x\} \subset E[A]$, i.e., $A\lambda x$. Thus, A is asymptotically disjoint. Conversely, assume that A is asymptotically bounded, i.e., there exists $x \in X$ such that $A\lambda x$. Thus, there exists $E \in \mathcal{E}$ such that $A \subset E[x]$ and $\{x\} \subset E[A]$. Since $A \subset E[x]$, we have that $(\{x\} \times A) \subseteq E$. Thus, $(\{x\} \times A) \in \mathcal{E}$, which by Proposition 2.1.9 shows that A is coarsely bounded. $\square$

Proposition 5.1.5. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two coarse space structures on the same set X. Let λ_1 and λ_2 be the respective induced asymptotic resemblance structures. Then $\mathcal{E}_1$ being coarser than $\mathcal{E}_2$ implies that λ_1 is coarser that λ_2.*

Proof. Assume that $\mathcal{E}_1$ is coarser than $\mathcal{E}_2$, i.e., $\mathcal{E}_2 \subseteq \mathcal{E}_1$. Let $A, B \subseteq X$ be such that $A\lambda_2 B$. Then there exists $E \in \mathcal{E}_2$ such that $A \subset E[B]$ and $B \subset E[A]$. Since $\mathcal{E}_2 \subseteq \mathcal{E}_1$, this also shows that $A\lambda_1 B$. Thus, $\lambda_2 \subseteq \lambda_1$. $\square$

At this point, the reader may be wondering if the assignment of asymptotic resemblance spaces to coarse spaces is injective. As it was shown in [11], such assignment is usually not one-to-one, i.e., two different coarse spaces may induce the same asymptotic resemblance. The reader is referred to Example 3.1 of [11] for an explicit example.

5.2 Asymptotic Resemblance Spaces $\implies$ Coarse Spaces

Now we focus on the opposite induction than in the previous section, i.e., we investigate how asymptotic resemblances induce coarse space structures. The following definition comes from [7] (whose proof was based on Proposition 3.2 of [11]).

Definition 5.2.1 ((weak) asymptotic resemblance $\implies$ coarse space structure)**.** Let X be a set and λ a (weak) asymptotic resemblance on X. Then the collection $\mathcal{E}_\lambda$ of all subsets $E \subseteq X \times X$ such that $\pi_1(F)\lambda\pi_2(F)$ for all $F \subseteq E$ (where π_1 and π_2 denote projection maps onto the first and second factor, respectively) is a coarse structure on X, called the **coarse space structure induced by the (weak) asymptotic resemblance** λ.

From now on, whenever we say "(weak) asymptotic resemblance structure induces coarse space structure," we always mean "induces" in the sense of Definition 5.2.1.

Proposition 5.2.2. *Let (X, λ) be a (weak) asymptotic resemblance space and $\mathcal{E}$ the coarse space structure induced by the (weak) asymptotic resemblance λ. Then (X, λ) is asymptotically connected if and only if $(X, \mathcal{E})$ is coarsely connected.*

Proof. First assume that (X, λ) is asymptotically connected. Let $x, y \in X$ be arbitrary. Then $x\lambda y$. Let E consist of a singleton (x, y). Then for any $F \subseteq E$, we have that $\pi_1(F)\lambda\pi_2(F)$.

Thus, $E \in \mathcal{E}$ and consequently $(X, \mathcal{E})$ is coarsely connected. Conversely, let $x, y \in X$ be arbitrary. Since $(X, \mathcal{E})$ is coarsely connected, the singleton $E := \{(x, y)\} \in \mathcal{E}$. Thus, $x = \pi_1(E)\lambda\pi_2(E) = y$, showing that (X, λ) is asymptotically connected. $\square$

Proposition 5.2.3. *Let (X, λ) be an asymptotic resemblance space, $\mathcal{E}$ the coarse space structure induced by the asymptotic resemblance λ, and $A \subseteq X$. Then A is asymptotically bounded if and only if A is coarsely bounded.*

Proof. The statement is true when A is empty, so let us assume $A \neq \emptyset$. To prove the forward direction, assume that A is asymptotically bounded, i.e., there exists $x \in X$ such that $A\lambda x$. Consider $A \times \{x\}$. We know that $B\lambda x$ for all $B \subseteq A$ (see Proposition 2.11 of [11]). Thus, for all $F \subseteq E$ we have that $\pi_1(F)\lambda\pi_2(F)$, i.e., $E \in \mathcal{E}$. Thus, by Proposition 2.1.9, A is coarsely bounded. Conversely, let A be coarsely bounded. Thus, there exists $x \in X$ such that $E =: A \times \{x\} \in \mathcal{E}$. Thus, $A = \pi_1(E)\lambda\pi_2(E) = x$. Consequently, A is asymptotically bounded. $\square$

Notice that in the above proof the condition that (X, λ) was an asymptotic resemblance space and not only a weak asymptotic resemblance space was crucial. Namely, we have used the fact that subsets of asymptotically bounded sets are asymptotically bounded. The proof of that fact utilizes condition 2 of asymptotic resemblance.

Proposition 5.2.4. *Let λ_1 and λ_2 be two (weak) asymptotic resemblance structures on the same set X. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the respective induced coarse structures. Then λ_1 being coarser than λ_2 implies that $\mathcal{E}_1$ is coarser that $\mathcal{E}_2$.*

Proof. Assume that λ_1 is coarser than λ_2 i.e., $\lambda_2 \subseteq \lambda_1$. Let $E \in \mathcal{E}_2$. Then for all $F \subseteq E$, we have that $\pi_1(F)\lambda_2\pi_2(F)$. Since $\lambda_2 \subseteq \lambda_1$, we have that $\pi_1(F)\lambda_1\pi_2(F)$ for all $F \subseteq E$, showing $E \in \mathcal{E}_1$. Thus, $\mathcal{E}_2 \subseteq \mathcal{E}_1$, i.e., $\mathcal{E}_1$ is coarser that $\mathcal{E}_2$. $\square$

At this point, the reader may be wondering if the assignment of coarse spaces to asymptotic resemblance spaces is injective. At the time of this dissertation, this is not known.

5.3 Asymptotic Resemblance Spaces $\Longleftrightarrow$ Coarse Spaces

In [11], it is actually shown that if the given asymptotic resemblance structure was induced by a coarse space structure, then the coarse space structure induced by that asymptotic resemblance is the coarsest coarse space structure inducing that asymptotic resemblance. We are going to state this fact as a proposition:

Proposition 5.3.1. *Let X be a set. Let $\mathcal{E}_1$ be a coarse structure on X. Induce the following coarse structures as in the following diagram (where $\Longrightarrow$ means "induces"):*

$$\begin{array}{c} \textit{coarse structure } \mathcal{E}_1 \\ \Downarrow \\ \textit{asymptotic resemblance } \lambda_1 \\ \Downarrow \\ \textit{coarse structure } \mathcal{E}_2 \\ \Downarrow \\ \textit{asymptotic resemblance } \lambda_2. \end{array}$$

Then $\lambda_1 = \lambda_2$ and $\mathcal{E}_1 \subseteq \mathcal{E}_2$. □

Now we focus on the case when we start with the (weak) asymptotic resemblance. We know that every (weak) asymptotic resemblance induces a coarse structure, and every coarse structure induces an asymptotic resemblance. The following result show shows that composition of these two operations does not enlarge the collection of related sets.

Proposition 5.3.2. *Let X be a set. Let λ_1 be a (weak) asymptotic resemblance on X. Induce the following coarse structures as in the following diagram (where $\Longrightarrow$ means "induces"):*

$$\begin{array}{c} \textit{(weak) asymptotic resemblance } \lambda_1 \\ \Downarrow \\ \textit{coarse structure } \mathcal{E}_1 \\ \Downarrow \\ \textit{asymptotic resemblance } \lambda_2 \\ \Downarrow \\ \textit{coarse structure } \mathcal{E}_2. \end{array}$$

Then $\mathcal{E}_1 = \mathcal{E}_2$ and $\lambda_2 \subseteq \lambda_1$.

Proof. Let us first show that $\lambda_2 \subseteq \lambda_1$. Let $A, B \subseteq X$ such that $A\lambda_2 B$. Then there exists a symmetric $E \in \mathcal{E}_1$ such that $A \subseteq E[B]$, $B \subseteq E[A]$, i.e., the following are satisfied:

1. for all $a \in A$, there exists $b \in B$ such that $(a, b) \in E$,

2. for all $b \in B$, there exists $a \in A$ such that $(b, a) \in E$.

Since E is symmetric, these are equivalent to the following:

1. for all $a \in A$, there exists $b \in B$ such that $(a, b) \in E$,

2. for all $b \in B$, there exists $a \in A$ such that $(a, b) \in E$.

Let F be a subset of E that consists of the union of the points (a, b) described in conditions 1 and 2. Then clearly $\pi_1(F) = A$ and $\pi_2(F) = B$, which by the definition of the coarse structure induced by λ_1 implies that $A\lambda_1 B$. Thus, $\lambda_2 \subseteq \lambda_1$.

To show $\mathcal{E}_1 = \mathcal{E}_2$, first notice that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ by Proposition 5.3.1. Since λ_1 is coarser than λ_2, it follows from Proposition 5.2.4 than $\mathcal{E}_1$ is coarser than $\mathcal{E}_2$, i.e., $\mathcal{E}_2 \subseteq \mathcal{E}_1$. Thus, $\mathcal{E}_1 = \mathcal{E}_2$. □

The reader is encouraged to compare Proposition 5.3.1 with Proposition 5.3.2, especially the difference in inclusions in the conclusions of both propositions.

Corollary 5.3.3. *Let X be a set. Let $\mathcal{E}_1$ be a coarse structure on X. Induce the following coarse structures as in the following diagram:*

$$\mathcal{E}_1 \implies \lambda_1 \implies \mathcal{E}_2 \implies \lambda_2 \implies \mathcal{E}_3 \implies \lambda_3 \implies \cdots.$$

Then the following hold:

1. *$\lambda_1 = \lambda_i$ for all $i \geq 1$,*

2. *$\mathcal{E}_1 \subseteq \mathcal{E}_2 = \mathcal{E}_i$ for all $i \geq 2$,*

3. *if one of the given coarse structures is connected, then they are all connected,*

4. *if $A \subseteq X$ is bounded with respect to any of the given coarse structures, then A is bounded with respect to all of the given coarse structures.*

Proof. Properties 1 and 2 are immediate consequences of Proposition 5.3.1 and Proposition 5.3.2. Property 3 is an immediate consequence of Proposition 5.1.3 and Proposition 5.2.2. Property 4 is an immediate consequence of Proposition 5.1.4 and Proposition 5.2.3 □

Corollary 5.3.4. *Let X be a set. Let λ_1 be a (weak) asymptotic resemblance structure on X. Induce the following coarse structures as in the following diagram (where λ_i denotes an asymptotic resemblance):*

$$\lambda_1 \Longrightarrow \mathcal{E}_1 \Longrightarrow \lambda_2 \Longrightarrow \mathcal{E}_2 \Longrightarrow \lambda_3 \Longrightarrow \mathcal{E}_3 \Longrightarrow \cdots .$$

Then the following hold:

1. $\mathcal{E}_1 = \mathcal{E}_i$ *for all* $i \geq 1$,

2. $\lambda_1 \supseteq \lambda_2 = \lambda_i$ *for all* $i \geq 2$,

3. *if one of the given coarse structures is connected, then they are all connected,*

4. *if $A \subseteq X$ is bounded with respect to any of the given coarse structures, then A is bounded with respect to all of the given coarse structures.*

Proof. Properties 1 and 2 are immediate consequences of Proposition 5.3.1 and Proposition 5.3.2. Property 3 is an immediate consequence of Proposition 5.1.3 and Proposition 5.2.2. Property 4 is an immediate consequence of Proposition 5.1.4 and Proposition 5.2.3 □

5.4 Coarse Normality of Coarse Spaces

As we have seen in Section 2.2, we have a notion of an asymptotic resemblance space being "normal." In this section, we introduce a similar concept for coarse spaces. Such normality condition for coarse spaces is quite important, since (as we are going to see in the next section) every "normal" coarse space induces a coarse proximity structure.

According to the knowledge of the author, the first attempt to define large-scale normality was made in [11], where asymptotically normal asymptotic resemblance spaces are defined. Next, Dydak and Weighill introduced a normality condition for large-scale spaces, simply called large scale normality (see [5]). In this section, we present "coarse normality" - a normality condition for coarse spaces. The definition of coarse normality was obtained after translating and slightly modifying large scale normality from large scale structures to coarse space structures (for introduction to large scale structures, see [4]). As we will see in this section, in some sense all these normality conditions (asymptotic normality, large scale normality, coarse normality) coincide.

Definition 5.4.1. Let $(X, \mathcal{E})$ be a coarse space and let $A, B \subseteq X$ be any two subsets. Define $A \prec B$, if for every entourage $E \in \mathcal{E}$, we have that $E[A] \subseteq B \cup K$ for some bounded set $K \subseteq X$.

Remark 5.4.2. The above definition implies that $A \subseteq B$ up to some bounded set K, i.e., $(A \setminus K) \subseteq B$.

The following proposition introduces equivalent definitions of $\prec$.

Proposition 5.4.3. *Let $(X, \mathcal{E})$ be a coarse space, λ the asymptotic resemblance induced by $\mathcal{E}$, $\mathcal{B}$ the collection of coarsely bounded sets, and $A, B \subseteq X$ any two subsets. Then the following are equivalent:*

1. $A \prec B$,

2. *A and $X \setminus B$ are asymptotically disjoint,*

3. *For all $E \in \mathcal{E}$, there exists $D \in \mathcal{B}$ such that* $\Big((A \setminus D) \times ((X \setminus B) \setminus D)\Big) \cap E = \emptyset$.

Proof. (1 $\implies$ 2). Assume $A \prec B$. For contradiction, assume that $A' \subseteq A$ and $C' \subseteq (X \setminus B)$ are unbounded subsets such that $A' \lambda C'$, i.e., there exists $E \subseteq \mathcal{E}$ such that $A' \subseteq E[C']$ and $C' \subseteq E[A']$. Since $A \prec B$, we have that $E'[A] \subseteq B \cup K$ for some bounded set $K \subseteq X$. Since $A' \subseteq A$, we have that

$$C' \subseteq E[A'] \subseteq E[A] \subseteq B \cup K.$$

Thus, $(C' \setminus K) \subseteq B$. Since C' is unbounded and K is bounded, $(C' \setminus K)$ is nonempty. But this is a contradiction, since $(C' \setminus K) \subseteq (X \setminus B)$, by the definition of C'.

(2 $\implies$ 1). For contradiction, assume that $A \not\prec B$, i.e., there exists $E \in \mathcal{E}$ such that $E[A] \not\subseteq B \cup K$ for any bounded $K \subseteq X$. In other words, $E[A] \cap (X \setminus B)$ is unbounded. Without loss of generality we can assume that E is symmetric. Set $C' = E[A] \cap (X \setminus B)$. For each $c \in C'$ there exists $a \in A$ such that $(c, a) \in E$. Let A' be the collection of all such a's. Notice that A' is unbounded, since if it is bounded, then so is $E[A']$. But $E[A']$ contains C', so it has to be unbounded. So we have an unbounded $A' \subseteq A$, an unbounded $C' \subseteq (X \setminus B)$, and $E \in \mathcal{E}$ such that

$$C' \subseteq E[A'] \quad \text{and} \quad A' \subseteq E[C'],$$

a contradiction to $A' \bar{\lambda} C'$.

(1 $\implies$ 3) Let $E \in \mathcal{E}$ be arbitrary. Without loss of generality we can assume that E contains the diagonal. Since $A \prec B$, there exists $K \in \mathcal{B}$ such that $(E[A] \setminus K) \subseteq B$. Let D be all those elements of A such that $E[D] \subseteq K$. Since K is bounded, so is $E[D]$. Since E contains the diagonal, D is bounded as well. Thus, by the construction of D we have that $E[A \setminus D] \subseteq B$. In other words, if there exists $x \in X$ and $a \in (A \setminus D)$ such that $(x, a) \in E$, then x cannot be in $(X \setminus B)$. In particular, it cannot be in $((X \setminus B) \setminus D)$, which shows (3).

(3 $\implies$ 1) For contradiction, assume that $A \not\prec B$, i.e., there exists $E \in \mathcal{E}$ such that $E[A] \cap (X \setminus B)$ is unbounded. Let $D \in \mathcal{B}$ be arbitrary. Then $C := (E[A] \cap (X \setminus B)) \setminus (E \cup \triangle)[D]$ is nonempty. Let $c \in C$. Then there exists $a \in A$ such that $(c, a) \in E$. What is more, $a \notin D$. For if $a \in D$, then $c \in E[D]$, a contradiction. So we have $c \in ((X \setminus B) \setminus D)$, $a \in (A \setminus D)$ and $(a, c) \in E$. Since D was an arbitrary unbounded subset, this contradicts (3). $\square$

Notice that if $(X, \mathcal{E})$ is a connected coarse space, then the collection of bounded sets forms a bornology.

Now we are ready to introduce coarse normality.

Definition 5.4.4. A coarse space $(X, \mathcal{E})$ is called **coarsely normal** if for every pair of subsets $A, B \subseteq X$ such that $A \prec B$, there is a subset $C \subseteq X$ satisfying $A \prec C \prec B$.

The reader familiar with [5] will spot an immediate resemblance to large scale normality defined for large scale structures. Indeed, after translating from large scale structures to coarse structures, the two notions coincide for coarse spaces, as the following lemma and proposition show:

Lemma 5.4.5. *Let $(X, \mathcal{E})$ be a connected coarse structure, $\mathcal{B}$ the bornology induced by $\mathcal{E}$, and $D_1, D_2 \in \mathcal{B}$. If A and B are two subsets of X such that $A \prec B$, then the following hold:*

1. $A \cup D_1 \prec B \setminus D_2$,

2. $A \setminus D_1 \prec B \cup D_2$.

Proof. Exercise. Use the fact that in a connected coarse structure the union of two bounded sets is bounded. $\square$

For the reader unfamiliar with large scale structures, we present a definition of large scale normality given in [5] translated from large scale structures to coarse space structures.

Definition 5.4.6. Let $(X, \mathcal{E})$ be a coarse structure. For any $A, B \subseteq X$, define $A \prec^* B$ if $A \subseteq B$ and $A \prec B$. If $A \prec^* B$ implies that there exists a set $C \subseteq X$ satisfying $A \prec^* C \prec^* B$, then $(X, \mathcal{E})$ is called **large scale normal**.

Proposition 5.4.7. *Let $(X, \mathcal{E})$ be a connected coarse structure. Then the following are equivalent:*

1. *$(X, \mathcal{E})$ is coarsely normal,*

2. *$(X, \mathcal{E})$ is large scale normal*

Proof. To show (1) $\implies$ (2), assume $A \prec^* B$, i.e., $A \subseteq B$ and $A \prec B$. By coarse normality, this implies the existence of $C' \subseteq X$ such that $A \prec C' \prec B$. In particular, this shows that there exist bounded sets D_1 and D_2 such that $A \subseteq C' \cup D_1$ and $C' \subseteq B \cup D_2$. We can assume that $D_1 \subseteq A$ and $D_2 \subseteq (X \setminus B)$. Set $C = (C' \cup D_1) \setminus D_2$. By repeated application of Lemma 5.4.5, we have that $A \prec C \prec B$. Also, $A \subseteq C \subseteq B$, which follows from the fact that $D_2 \cap A = \emptyset$ (which in particular shows that $D_2 \cap D_1 = \emptyset$). To show (2) $\implies$ (1), assume $A \prec B$. In particular, this means that $\triangle[A] = A \subseteq B \cup D$ for some bounded set D. By Lemma 5.4.5, this means that $A \setminus D \prec^* B$, and thus there exists $C \subseteq X$ such that $A \setminus D \prec^* C \prec^* B$. In particular, this means that $A \setminus D \prec C \prec B$, and by Lemma 5.4.5, we have $A \prec C \prec B$. $\square$

Now we will show that in the case of coarse spaces, coarse normality is also equivalent to asymptotic normality (for the definition, see Definition 2.2.10).

Proposition 5.4.8. *Let $(X, \mathcal{E})$ be a coarse space and λ the asymptotic resemblance induced by $\mathcal{E}$. Then the following are equivalent:*

1. *$(X, \mathcal{E})$ is coarsely normal,*

2. *(X, λ) is asymptotically normal.*

Proof. ($1 \implies 2$) Assume $A_1, A_2 \subseteq X$ such that A_1 and A_2 are asymptotically disjoint, i.e., $A_1 \prec (X \setminus A_2)$. Thus, there exists C such that $A_1 \prec C \prec (X \setminus A_2)$. Set $X_1 = (X \setminus C)$ and $X_2 = C$. Then clearly $X = X_1 \cup X_2$, A_1 is asymptotically disjoint from X_1, and A_2 is asymptotically disjoint from X_2.

($2 \implies 1$) Assume $A, B \subseteq X$ such that $A \prec B$, i.e., A and $(X \setminus B)$ are asymptotically disjoint. Thus, there exists $X_1, X_2 \subseteq X$ such that $X = X_1 \cup X_2$, A is asymptotically disjoint from X_1, and $(X \setminus B)$ is asymptotically disjoint from X_2. Let $C = X_2$. Then the following hold:

1. A is asymptotically disjoint from $X_1 = (X \setminus X_2) = (X \setminus C)$,

2. $(X \setminus B)$ is asymptotically disjoint from $X_2 = C$,

which is the same as saying $A \prec C \prec B$. ∎

Thanks to the above proposition, it follows from [11] that the class of coarsely normal coarse spaces in nonempty. In particular, all metric spaces (with the metric coarse structure) are coarsely normal. Also, notice that the above proposition shows that the assignment of asymptotic resemblance spaces to coarse spaces preserves normality. Finally, notice that in the above proof we used the definition of the $\prec$ relation that involved asymptotic resemblance. In particular, the fact that λ was induced by a coarse structure was not used. Therefore, the same proof will show the following proposition:

Proposition 5.4.9. *Let (X, λ) be an asymptotic resemblance space. For any $A, B \subseteq X$, define $A \prec_\lambda B$ if and only if A and $X \setminus B$ are asymptotically disjoint. Then the following are equivalent:*

(i) $A \prec_\lambda B$ *implies there exists* $C \subseteq X$ *such that* $A \prec_\lambda C \prec_\lambda B$,

(ii) (X, λ) *is asymptotically normal.* □

Proposition 5.4.10. *Let* $(X, \mathcal{E})$ *be a connected coarse structure and* λ *the asymptotic resemblance induced by* $\mathcal{E}$*. Then the following are equivalent:*

1. $(X, \mathcal{E})$ *is coarsely normal,*

2. $(X, \mathcal{E})$ *is large scale normal,*

3. (X, λ) *is asymptotically normal*

Proof. This follows immediately from Proposition 5.4.7 and Proposition 5.4.8. □

Thanks to the above proposition, the expression "$(X, \mathcal{E})$ is a connected normal coarse structure" is unambiguous.

At this point the reader may be wondering if there exist coarse spaces that are not coarsely normal. Indeed, in [5] it is shown that there exist such coarse spaces. The following example is taken from Corollary 11.4 in that paper (translated to the setting of coarse spaces):

Example 5.4.11. Let $X = \mathbb{R}^+$ and let $\mathcal{E}'$ be the collection of subsets of $\mathbb{R}^+ \times \mathbb{R}^+$ that consist of finitely many half-lines starting at the y or x axis and parallel to the diagonal. Let $\mathcal{E}$ be the collection of all the subsets of elements of $\mathcal{E}'$. Then it is easy to see that $\mathcal{E}$ is a coarse structure whose bounded sets are the subsets of $\mathbb{R}^+$ of finite cardinality. Let $A = (0, 1)$ and let $B = \mathbb{R}^+ \setminus \mathbb{N}$. It is clear that $A \prec B$. Also, notice that any $C \subseteq X$ such that $A \prec C$ needs to contain a set of the form $\mathbb{R}^+ \setminus D$, where D is a sequence of points diverging to infinity (it is because for any $x \in \mathbb{R}^+$ we can always find $E \in \mathcal{E}$ such that $(0, x) \subseteq E[A]$). However, since we can always draw a half-line parallel to the diagonal that misses countably many points (more precisely, misses all the points in $D \times D$), there exists $E \in \mathcal{E}$ such that $E[C] \supseteq E[\mathbb{R}^+ \setminus D] = \mathbb{R}^+$, i.e., $E[C] = \mathbb{R}^+$. But this means that $C \not\prec B$ for any C such that $A \prec C$, i.e., $(X, \mathcal{E})$ is not coarsely normal.

5.5 Connected Normal Coarse Spaces $\implies$ Coarse Proximity Spaces

Finally, we are ready to prove that $\prec$ relation on a connected normal coarse space induces a coarse proximity.

Theorem 5.5.1. *Let $(X, \mathcal{E})$ be a connected coarse space and $\mathcal{B}$ the bornology induced by $\mathcal{E}$. The relation $\prec$ induces a coarse proximity on the pair $(X, \mathcal{B})$ if and only if $(X, \mathcal{E})$ is coarsely normal.*

Proof. If $\prec$ induces a coarse proximity on the pair $(X, \mathcal{B})$, then $(X, \mathcal{E})$ is coarsely normal by 6 of Theorem 3.3.2. To prove the converse, assume that $(X, \mathcal{E})$ is coarsely normal. To show that $\prec$ induces a coarse proximity, it is enough to show that the relation $\prec$ satisfies 1 through 6 of Theorem 3.3.2. First, notice that for any $E \in \mathcal{E}$, we have that $E[X] \subseteq X$. In particular, for any $D \in \mathcal{B}$, we have that for any $E \in \mathcal{E}$, it is true that $E[X] \subseteq (X \setminus D) \cup D$, i.e., $X \prec (X \setminus D)$, which shows 1. To show 2, assume $A \prec B$. Thus, there has to exist a bounded set D such that $\triangle[A] \subseteq B \cup D$. However, $\triangle[A] = A$, and therefore we have that $(A \setminus D) \subseteq B$. To show 3, assume that $A \subseteq B \prec C \subseteq D$. Then for any $E \in \mathcal{E}$, there exists $K \in \mathcal{B}$ such that

$$E[A] \subseteq E[B] \subseteq C \cup K \subseteq D \cup K,$$

i.e., $A \prec D$. To show 4, Let $E \in \mathcal{E}$ be arbitrary. Notice that $A \prec B_1$ and $A \prec B_2$ implies that there exists $K_1, K_2 \in \mathcal{B}$ such that

$$E[A] \subseteq B_1 \cup K_1 \quad \text{and} \quad E[A] \subseteq B_2 \cup K_2,$$

which in turn implies that

$$E[A] \subseteq (B_1 \cup K_1) \cap (B_2 \cup K_2) = (B_1 \cap B_2) \cup (B_1 \cap K_2) \cup (K_1 \cap B_2) \cup (K_1 \cap K_2).$$

Since the last three unions are bounded sets, we have that $A \prec (B_1 \cap B_2)$. Conversely, let $A \prec (B_1 \cap B_2)$. Let $E \in \mathcal{E}$ be arbitrary. Then there exists $K \in \mathcal{B}$ such that for any $i \in \{1, 2\}$,

$$E[A] \subseteq (B_1 \cap B_2) \cup K \subseteq B_i \cup K,$$

showing that $A \prec B_1$ and $A \prec B_2$. To show 5, assume $A \prec B$ and for contradiction assume that $(X \setminus B) \not\prec (X \setminus A)$. Thus, there exists $E \in \mathcal{E}$ such that $E[X \setminus B] \not\subseteq (X \setminus A) \cup K$ for any bounded K, i.e., $A' := E[X \setminus B] \cap A$ is unbounded. Without loss of generality we can assume that E is symmetric. For each $a \in A'$ there exists $c \in (X \setminus B)$ such that $(a, c) \in E$. Let C' be the collection of all such c's. Notice that C' is unbounded, since if it is bounded, then so is $E[C']$. But $E[C']$ contains A', so it has to be unbounded. So we have an unbounded $A' \subseteq A$, an unbounded $C' \subseteq (X \setminus B)$, and $E \in \mathcal{E}$ such that

$$A' \subseteq E[C'] \quad \text{and} \quad C' \subseteq E[A'],$$

a contradiction to $A' \bar{\lambda} C'$ (which should hold by Proposition 5.4.3). The converse direction of 5 follows by symmetry. Finally, 6 is the coarse normality. $\square$

Corollary 5.5.2. *Let $(X, \mathcal{E})$ be a connected coarsely normal coarse space, λ the asymptotic resemblance induced by $\mathcal{E}$, $\mathcal{B}$ the bornology induced by $\mathcal{E}$, and $A, B \subseteq X$ any two subsets. Define the relation* $\mathbf{b}$ *on the power set of X by any of the following equivalent conditions:*

1. *$A\mathbf{b}B$ if and only if there exists $E \in \mathcal{E}$ such that $E[A] \cap B$ is unbounded,*

2. *$A\mathbf{b}B$ if and only if there exists an unbounded $A' \subseteq A$ and an unbounded $B' \subseteq B$ such that $A'\lambda B'$,*

3. *$A\mathbf{b}B$ if and only if there exists $E \in \mathcal{E}$ such that for all $D \in \mathcal{B}$,*

$$\Big((A \setminus D) \times (B \setminus D)\Big) \cap E \neq \emptyset.$$

Then $\mathbf{b}$ *is a coarse proximity.*

Proof. This is a direct consequence of Theorem 5.5.1, Proposition 5.4.3, and Theorem 3.4.1. □

Notice that 3 of the above corollary is in line with the definition of the metric coarse proximity given in [7] (or in Proposition 3.1.6), where two subsets A and B of a metric space (X, d) are coarsely close if and only if there exists $\epsilon < \infty$ such that for all bounded sets D, there exists $a \in (A \setminus D)$ and $b \in (B \setminus D)$ such that $d(a, b) < \epsilon$.

5.6 Connected Normal Asymptotic Resemblance Spaces $\implies$ Coarse Proximity Spaces

The following relation will allow us to induce a coarse proximity on an asymptotic resemblance space.

Definition 5.6.1. For any $A, B \subseteq X$, define $A \prec_\lambda B$ if and only if A and $X \setminus B$ are asymptotically disjoint.

When the meaning is clear, to simplify notation we will use $\prec$ instead of $\prec_\lambda$. Notice that we have already used the notation $\prec$ in the previous section. However, this should not cause much confusion, especially that the relation $\prec$ on a coarse space equals $\prec_\lambda$, where λ is the asymptotic resemblance induced by the given coarse structure.

Proposition 5.6.2. *Let (X, λ) be an asymptotic resemblance space. Then the following are equivalent:*

(i) $A \prec B$ implies there exists $C \subseteq X$ such that $A \prec C \prec B$,

(ii) (X, λ) is asymptotically normal.

Proof. This is the exact statement of Proposition 5.4.9. □

Theorem 5.6.3. *Let (X, λ) be a connected asymptotic resemblance space and $\mathcal{B}$ the bornology induced by λ. Then the relation $\prec$ from Definition 5.6.1 induces a coarse proximity if and only if (X, λ) is asymptotically normal.*

Proof. If $\prec$ induces a coarse proximity on the pair $(X, \mathcal{B})$, then $(X, \mathcal{E})$ is asymptotically normal by 6 of Theorem 3.3.2. To prove the converse, assume that $(X, \mathcal{E})$ is asymptotically normal. To show that $\prec$ induces a coarse proximity, it is enough to show that the relation $\prec$ satisfies 1 through 6 of Theorem 3.3.2. To show 1, let $D \in \mathcal{B}$ be arbitrary. Since subsets of bounded sets are bounded, there is not such $D' \subseteq D$ such that D' is unbounded. Therefore, $X\bar{\lambda}D$ is satisfied vacuously, i.e., $X \prec (X \setminus D)$. To show 2, assume $A \prec B$. For contradiction, assume that $C := A \cap (X \setminus B)$ is unbounded. Then $C \subseteq A$, $C \subseteq (X \setminus B)$, and C is unbounded. By Proposition 2.22 of [11], we have that $C\lambda C$, which contradicts the fact that A is asymptotically disjoint from $(X \setminus B)$. Thus, it has to be that $A \cap (X \setminus B)$ is bounded, i.e., A is contained in B up to some bounded set. To show 3, assume that $A \subseteq B \prec C \subseteq D$. If $A \not\prec D$, then there exist unbounded $A' \subseteq A \subseteq B$ and unbounded $D' \subseteq (X \setminus D) \subseteq (X \setminus C)$ such that $A'\lambda D'$, a contradiction to $B \prec C$. So it has to be that $A \prec D$. To show 4, assume $A \prec B_1$ and $A \prec B_2$, i.e., A is asymptotically disjoint from $(X \setminus B_1)$ and $(X \setminus B_2)$. For contradiction, assume that $A \not\prec (B_1 \cap B_2)$, i.e., there exists unbounded $A' \subseteq A$ and unbounded $C' \subseteq X \setminus (B_1 \cap B_2)$, such that $A'\lambda C'$. However, notice that $X \setminus (B_1 \cap B_2) = (X \setminus B_1) \cup (X \setminus B_2)$. Thus, there has to exist unbounded $C'' \subseteq C'$ such that $C'' \subseteq (X \setminus B_1)$ or $C'' \subseteq (X \setminus B_2)$ (otherwise C' would be bounded, being the union of two bounded sets). Without loss of generality assume that $C'' \subseteq (X \setminus B_1)$. Notice that since $A'\lambda C'$, by Proposition 2.6 of [11], there exists $A'' \subseteq A'$ such that $A''\lambda C''$. Notice that A'' has to be unbounded (for if it is bounded, then there exists $x \in X$ such that $x\lambda A''\lambda C''$, contradicting the fact that C'' is unbounded). So we have unbounded $A'' \subseteq A$, and unbounded $C'' \subseteq (X \setminus B_1)$ such that $A''\lambda C''$, a contradiction to $A \prec B_1$. So it has to be the case that $A \prec (B_1 \cap B_2)$. To show the converse, assume $A \prec (B_1 \cap B_2)$. If without loss of generality $A \not\prec B_1$, then there exist unbounded $A' \subseteq A$ and unbounded $C' \subseteq (X \setminus (B_1)) \subseteq (X \setminus (B_1 \cap B_2))$ such that $A'\lambda C'$, a contradiction to $A \prec (B_1 \cap B_2)$. To show 5, assume $A \prec B$ and for contradiction assume that $(X \setminus B) \not\prec (X \setminus A)$. Then there exist unbounded $B' \subseteq (X \setminus B)$ and unbounded $A' \subseteq (X \setminus (X \setminus A)) = A$ such that $B'\lambda A'$, which contradicts $A \prec B$. The converse is shown similarly. Finally, 6 is the asymptotic normality. $\square$

Corollary 5.6.4. *Let (X, λ) be a connected normal asymptotic resemblance space, $\mathcal{B}$ the bornology induced by λ, and $A, B \subseteq X$ any two subsets. Define the relation $\mathbf{b}$ on the power set of X by*

$A\mathbf{b}B$ if and only if there exists an unbounded $A' \subseteq A$ and an unbounded $B' \subseteq B$ such that $A'\lambda B'$,

i.e., A and B are not asymptotically disjoint. Then $\mathbf{b}$ is a coarse proximity.

Proof. This is a direct consequence of Theorem 5.6.3, Definition 5.6.1 and Theorem 3.4.1. □

Recall that when given a coarse space $(X, \mathcal{E})$, the relation $\prec$ is equivalent to the relation $\prec_\lambda$ when X is equipped with the asymptotic resemblance λ induced by $\mathcal{E}$. Consequently, the proof of Theorem 5.6.3 could be used to show Theorem 5.5.1.

5.7 Summary of Relationships between Coarse Structures

In this section, we summarize all the results regarding the relationships between coarse structures in the form of the diagrams. In all of the diagrams below, we use the following notation:

$\Longrightarrow$	induces
$\Longleftrightarrow$	induces and is induced by
$\mathcal{E}$	coarse space structure
λ	asymptotic resemblance
ϕ	weak asymptotic resemblance
$\mathbf{b}$	coarse proximity
$\mathcal{B}$	bornology
$(\mathcal{E}, \mathcal{B})$	coarse space structure with the induced bornology
$(\lambda, \mathcal{B})$	asymptotic resemblance with the induced bornology
$(\mathbf{b}, \mathcal{B})$	coarse proximity and the associated bornology

Claims neither about the injectivity or surjectivity of such inductions nor about the commutativity of the following diagrams are made, unless otherwise stated. In fact, in this dissertation we have shown that certain inductions are not injective. Each diagram is preceeded by a short description/discussion. We will focus on general inductions, as well as on the preservability of connectedness, boundedness, and normality.

The following diagrams show the relationships between asymptotic resemblances and coarse spaces. It is known that $\mathcal{E} \implies \lambda$ is not injective. The diagram on the right implies that the boundedness is preserved.

$$\mathcal{E} \Longleftrightarrow \lambda \qquad\qquad (\mathcal{E}, \mathcal{B}) \Longleftrightarrow (\lambda, \mathcal{B})$$

The following diagrams show that connectedness is also preserved. It is not known if $\lambda \implies \mathcal{E}$ preserves normality.

$$\text{connected } \mathcal{E} \Longleftrightarrow \text{ connected } \lambda \qquad\qquad \text{normal } \mathcal{E} \Longrightarrow \text{ normal } \lambda$$

The following diagrams show the relationships between connected normal coarse spaces, connected normal asymptotic resemblance spaces, and coarse proximity spaces. It also shows that boundedness is preserved. The following diagram is commutative.

$$\begin{tikzcd}
\text{connected normal } (\mathcal{E}, \mathcal{B}) \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \text{connected normal } (\lambda, \mathcal{B}) \arrow[dl, Rightarrow] \\
& (\mathbf{b}, \mathcal{B}) &
\end{tikzcd}$$

Since coarse proximities do not induce asymptotic resemblance spaces, but instead they induce weak asymptotic resemblance spaces, for the rest of this section we are going to restrict ourselves to weak asymptotic resemblances. The following diagram shows that coarse proximities induce coarse spaces.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, Leftrightarrow] & \phi \\
\mathbf{b} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Since weak asymptotic resemblance spaces do not induce a bornology (more precisely, it does not have to be true that subsets of bounded sets are bounded), the question of preserving bornology is not very meaningful. However, it is worth noting that when coarse proximities induce weak asymptotic resemblances, the boundedness is preserved. In particular, asymptotically bounded sets form a bornology.

The following diagram shows that connectedness is preserved.

$$\begin{tikzcd}
\text{connected } \mathcal{E} \arrow[r, Leftrightarrow] & \text{connected } \phi \\
\mathbf{b} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

It is not known if coarse proximities induce normal weak asymptotic resemblance spaces.

5.8 Questions

Question 5.8.1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two coarse space structures on the same set X. Let λ_1 and λ_2 be the respective induced asymptotic resemblance structures. Thanks to Proposition 5.1.5 we know that $\mathcal{E}_1$ being coarser than $\mathcal{E}_2$ implies that λ_1 is coarser that λ_2. Is the converse true? In other words, does λ_1 being coarser that λ_2 implies that $\mathcal{E}_1$ is coarser than $\mathcal{E}_2$? Conjecture: false.

Question 5.8.2. Let λ_1 and λ_2 be two (weak) asymptotic resemblance structures on the same set X. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the respective induced coarse space structures. Thanks to Proposition 5.2.4 we know that λ_1 being coarser than λ_2 implies that $\mathcal{E}_1$ is coarser that $\mathcal{E}_2$. Is the converse true? In other words, does $\mathcal{E}_1$ being coarser that $\mathcal{E}_2$ implies that λ_1 is coarser than λ_2? Conjecture: false.

Question 5.8.3. Is the assignment of coarse spaces to asymptotic resemblance spaces injective? In other words, do there exist 2 different asymptotic resemblance spaces that induce the same coarse structure?

Question 5.8.4. Do normal asymptotic resemblance spaces induce normal coarse spaces? In other words, does $\lambda \implies \mathcal{E}$ preserve normality?

Question 5.8.5. We know that coarse proximities induce connected weak asymptotic resemblance spaces. However, is the induced weak asymptotic resemblance space normal?

Question 5.8.6. We know that coarse proximities induce weak asymptotic resemblance spaces, and weak asymptotic resemblance spaces induce coarse space structures. Consequently, coarse proximities induce coarse space structures. Can one find an easy description of such induction? Also, is there an other way to directly induce coarse structures from coarse proximities?

Question 5.8.7. Can one claim that any of the inductions from Section 5.7 are either injective or surjective? If not, what are the conditions under which such inductions are injective or surjective?

Chapter 6

Concluding Remarks

6.1 Discussion on Asymptotic Resemblance

Recall the definition of an asymptotic resemblance.

Definition 6.1.1. Let X be a set. Let λ be an equivalence relation on the power set of X. Then λ is called an **asymptotic resemblance** on X if it satisfies the following properties:

1. $A_1\lambda B_1$, $A_2\lambda B_2$ implies $(A_1 \cup A_2)\lambda(B_1 \cup B_2)$,

2. $(B_1 \cup B_2)\lambda A$ and $B_1, B_2 \neq \emptyset$ implies that there are nonempty $A_1, A_2 \subseteq A$ such that $A = A_1 \cup A_2$, $B_1\lambda A_1$, and $B_2\lambda A_2$.

A pair (X, λ), where X is a set and λ is an asymptotic resemblance on X, is called an **asymptotic resemblance space**. When $A\lambda B$, we say that A and B are λ **related** or are **asymptotically alike**.

Notice that condition 2 in the above definition does not require A to be nonempty. Consequently, in any asymptotic resemblance space, the empty set is related only to itself. However, such requirement is necessary if we want certain obvious candidates for asymptotic resemblance spaces to work. For example, one could define two sets to be related when their symmetric difference is finite (Example 2.9 in [11]). According to this definition, the empty set is related to any finite set. Thus, condition 2 without the requirement of A being nonempty is not satisfied. Additionally, intuitively the empty set and bounded sets should be "the same at infinity" or "asymptotically alike," since they do not intersect the Higson corona. Consequently, it seems that one should consider allowing the empty set to be related to some sets besides itself.

To allow the empty set to be related to other sets besides itself, we could change condition 2 to the following:

"$(B_1 \cup B_2)\lambda A$ and $A, B_1, B_2 \neq \emptyset$ implies that there are nonempty $A_1, A_2 \subseteq A$ such that $A = A_1 \cup A_2$, $B_1 \lambda A_1$, and $B_2 \lambda A_2$"

Doing so allows the empty set to be related to other sets. It also makes the Example 2.9 from [11] an asymptotic resemblance. Since the new condition 2 is only slightly more restrictive than the original one (it only affects the relations of the empty set with other sets), all of the theorems and results from [11] regarding asymptotic resemblance should still hold.

However, such change affects a few results in this dissertation. If we replace the original condition 2 with the proposed one, the following hold true:

1. The relation ϕ induced by a coarse proximity $\mathbf{b}$ as in Theorem 3.5.2 can still possibly be an asymptotic resemblance (with the original condition 2 it is known that the induced ϕ relation is not an asymptotic resemblance, since under the induced ϕ relation the empty set is ϕ related to all bounded sets)

2. The assignment of coarse spaces to asymptotic resemblance spaces is not injective, as the following example shows:

   Example 6.1.2. Let X be a set consisting of a single point x. Define λ_1 by

   $$x\lambda_1 x, \quad \emptyset\lambda_1\emptyset,$$

   and define λ_2 by

   $$x\lambda_2 x, \quad \emptyset\lambda_2 x, \quad x\lambda_2\emptyset, \quad \emptyset\lambda_2\emptyset.$$

   Clearly both λ_1 and λ_2 are asymptotic resemblance relations. It is also clear that both of them induce a coarse structure $\mathcal{E} = \{(x, x), \emptyset\}$.

However, the above example is not that satisfactory, since we "added" relations involving the empty set. It is not known if one can find an example that does not use this technique.

In other words, the proposed condition 2 allows one to capture the intuitive notion of bounded sets and the empty set being similar "at infinity," and it gives hope for showing that coarse proximities induce asymptotic resemblances. On the other hand, the proposed condition 2 allows one to "add" relations involving the empty set. This can give rise to trivial counterexamples, such as Example 6.1.2. Consequently, the need for technical statements, such as "up to the empty set relations..." may arise. The author leaves it to the reader to decide which condition 2 seems more appropriate.

6.2 Note to an Interested Reader

The author is very interested in answering the questions listed at the end of the chapters of this dissertation. If the reader knows the answer to any of the questions or would like to discuss topics included in this dissertation, the reader is encouraged to contact the author via pgrzegrz@vols.utk.edu.

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Vita

Pawel Grzegrzolka was born in Otwock, Poland, on March 1, 1991. He is the son of Andrzej and Izabela Grzegrzolka and has a younger brother and a younger sister. Pawel grew up in Warsaw, Poland, where he attended Liceum Ogolnokształcace imienia Tadeusza Czackiego (high school). At 19 years old, he moved to Kansas City, Missouri, to start his studies at Rockhurst University, where he competed for the Varsity Men's Tennis Team. After obtaining a Bachelor of Science in Mathematics and Psychology in 2014, he moved to Knoxville, Tennessee, to pursue a doctorate degree in mathematics at the University of Tennessee. He achieved a Master in Mathematics in 2016 and a Doctor of Mathematics under the supervision of Dr. Jerzy Dydak in 2019.