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Coarse Constructions

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To the Graduate Council:

I am submitting herewith a dissertation written by Logan McKee Higginbotham entitled "Coarse Constructions." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Coarse Constructions

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I dedicate this work to all of those who have led me to the current life I live and have endowed me with the knowledge they've attained. To my parents Angela and John, to my grandfather Quenton Caudill, to my childhood pastor Steve Carney, to my advisor Dr. Jerzy Dydak, and to my dearest Jessica.

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Abstract

Coarse geometry has its roots in an attempt to make progress on the Novikov conjecture. It proved to be useful and resulted in progress on the Coarse Baum-Connes conjecture. This progress in turn led to progress in the Novikov conjecture. This paper investigates various constructions in coarse geometry that make new coarse geometric spaces from old ones.

Chapter one is devoted to introductory material and builds an appropriate framework that we will use throughout the rest of this paper. Chapter two is about the asymptotic filtered colimit, a coarse construction that is intuitively a "coarse version" of the pasting lemma from Topology. We investigate many coarse properties that are (and are not) preserved under this construction.

Chapter three is concerned with asymptotic products, a coarse construction that is an analog of the product topology. We investigate this construction in non-metrizable and metrizable settings. This chapter culminates with a result that, for certain circumstances, coarse embeddings are preserved in the asymptotic product construction.

Chapter four is about coarse quotient mappings similar to the quotient mappings of Topology; we also introduce the coarse category in this chapter. We then go on to talk about coarse quotients by group actions. This leads us to consider a construction called warped spaces. Using warped spaces, we obtain results regarding the preservation of Property A under the warped space construction and (with certain additional assumptions) a "coarse deck transformation theorem".

Table of Contents

Chapter 1

Introduction

Coarse geometry has its roots in an attempt to make progress on the Novikov conjecture (which postulates that the higher signatures of smooth manifolds are homotopy invariants). With this perspective, coarse geometry was only viewed in the lens of a metric space X and certain families of subsets of X that were called uniformly bounded. In the years that followed, Coarse structures were generalized by Higson and Roe as a collection of subsets of $X \times X$ called entourages. It was shown that this rendition of coarse geometry coincided with the metric space version and not every coarse structure was metrizable. Around this time, coarse structures were used in various fields and new properties that coarse structures preserved were discovered. Since then, it was shown by J. Dydak that an alternate definition of coarse structures (which were named large scale structures to avoid confusion) is equivalent to the Higson and Roe's coarse structures in that for every coarse structure there is a corresponding large scale structure and conversely. Roughly, the key difference between the two viewpoints is that a large scale structure is based on families of subsets of a set X as opposed to subsets of $X \times X$. In this dissertation, we will use large scale structures rather than coarse structures. We will make certain constructions on large scale structures from others and show which coarse properties are preserved under these constructions.

1.1 Scales

We first introduce notions associated with families of subsets of a set X:

Definition 1.1.1. Let $\mathcal{U}$ be a family of subsets of a set X and let V be a subset of X. The **star** of V against $\mathcal{U}$, denoted $\mathrm{st}(V,\mathcal{U})$, is the set $\bigcup\limits_{\substack{U\in\mathcal{U}\\ U\cap V\neq\varnothing}} U$. If $\mathcal{V}$ is another family of subsets of X, then the family of subsets of X $\{\mathrm{st}(V,\mathcal{U})|V\in\mathcal{V}\}$ is denoted $\mathrm{st}(\mathcal{V},\mathcal{U})$ for convenience.

Definition 1.1.2. Let $\mathcal{U},\mathcal{V}$ be families of subsets of a set X. We say $\mathcal{U}$ is a **refinement** of $\mathcal{V}$ provided for every $U\in\mathcal{U}$ there is a $V\in\mathcal{V}$ so that $U\subseteq V$. In this same situation, we also say that $\mathcal{V}$ **coarsens** $\mathcal{U}$. Refiniement is denoted as $\mathcal{U}\prec\mathcal{V}$.

It is sometimes needed that we need to consider covers of X instead of collections of subsets of X. To distinguish families of subsets of X from covers of X, we call covers of X scales:

Definition 1.1.3. Given a set X, we say $\mathcal{U}$ is a **scale** of X if $\mathcal{U}$ is a family of subsets of X that covers X. If $\mathcal{U}$ is a collection of subsets of X, we can make $\mathcal{U}$ into a cover via constructing $\mathcal{U}'=\mathcal{U}\cup\{\{x\}\}_{x\in X}$. This extension is often called the **trivial extension** of $\mathcal{U}$.

Stars and refinements have some basic properties:

1. $U\subseteq X$ and $\mathcal{V}$ a scale implies $U\subseteq\mathrm{st}(U,\mathcal{V})$.
2. If $\mathcal{U}$ is a collection of subsets of X and $\mathcal{V}$ is a scale of X, then $\mathcal{U}\prec\mathrm{st}(\mathcal{U},\mathcal{V})$.
3. If $U\subseteq V$ and $\mathcal{W}$ is a collection of subsets of X, then $\mathrm{st}(U,\mathcal{W})\subseteq\mathrm{st}(V,\mathcal{W})$.
4. If $\mathcal{U}\prec\mathcal{V}$ are collections of subsets of X and $\mathcal{W}$ is a collection of subsets of X, then $\mathrm{st}(\mathcal{U},\mathcal{W})\prec\mathrm{st}(\mathcal{V},\mathcal{W})$.
5. If U is a subset of X and $\mathcal{V},\mathcal{W}$ are collections of subsets of X with $\mathcal{V}\prec\mathcal{W}$, then $\mathrm{st}(U,\mathcal{V})\subseteq\mathrm{st}(U,\mathcal{W})$.
6. If $\mathcal{U},\mathcal{V}$, and $\mathcal{W}$ are collections of subsets of X with $\mathcal{V}\prec\mathcal{W}$, then $\mathrm{st}(\mathcal{U},\mathcal{V})\prec\mathrm{st}(\mathcal{U},\mathcal{W})$.
7. If $\mathcal{U}$ is a scale and $\mathcal{V}$ is any collection of subsets of X, then $\mathcal{V}\prec\mathrm{st}(\mathcal{U},\mathcal{V})$.

If any of these properties are used for families of subsets of X and the property requires a scale, it is assumed that the trivial extension is used.

A key example that is used for understanding stars and refinements of scales (which is a special case of families of subsets of a set) are covers of a metric space X by R-balls. Let $B(U, R)$ be an R-ball of a subset $U \subseteq X$ and let $\mathcal{U}_R = \{B(x, R)\}_{x \in X}$. It can be shown that for any other scale $\mathcal{V}$ of X, $\text{st}(\mathcal{V}, \mathcal{U}_R) = \{B(V, 2R)|\ V \in \mathcal{V}\}$. In general, this is to say that any star $\text{st}(\mathcal{U}, \mathcal{V})$ thickens up elements of $\mathcal{U}$ by elements of $\mathcal{V}$. We can think of $\text{st}(\mathcal{U}, \mathcal{V})$ as a $\mathcal{V}$-neighborhood of $\mathcal{U}$. Starring allows us to create "neighborhoods" without the need of a metric. For the sake of completion, we wish to introduce the notion of coarse structures before large scale structures.

1.2 Coarse Structures

Coarse structures were introduced by Higson and Roe for use in index theory and signature theory. Coarse structures were to give an approach for the Novikov and Coarse Baum-Connes conjectures. Much like scales and families of subsets of a set X, a metric space X give a natural example of what is to follow.

Definition 1.2.1. Let X be a set and consider the set $X \times X$:

1. The **diagonal** of X is denoted by Δ and is defined as $\Delta = \{(x, x)|\ x \in X\}$.
2. Let $U \subseteq X \times X$. Define the **inverse** of U, denoted by U^{-1}, to be $U^{-1} = \{(y, x)|\ (x, y) \in U\}$.
3. Let $U, V \subseteq X \times X$ and define the **product** of U and V to be $U \circ V = \{(x, z)|\ (x, y) \in U \textit{ and } (y, z) \in V \text{ forsome } y \in X\}$.

Definition 1.2.2. [9] A **coarse structure** on a set X is a family $\mathcal{X}$ of subsets of $X \times X$ that satisfy:

1. $\Delta \in \mathcal{X}$
2. If $U \in \mathcal{X}$, then $U^{-1} \in \mathcal{X}$.

3. If $U, V \in \mathcal{X}$, then $U \circ V \in \mathcal{X}$.

4. If $U \in \mathcal{X}$ and $V \subseteq U$, then $V \in \mathcal{X}$.

5. If $U, V \in \mathcal{X}$, then $U \cup V \in \mathcal{X}$.

The elements of a large scale structure are called **controlled sets** or **entourages**.

Coarse structures have been useful in establishing coarse geometry, but in recent years the large scale structure viewpoint of coarse geometry (i.e. scales or covers) has grown more popular amongst those involved in coarse geometry. For more on coarse structures, one can find much in [9].

1.3 Large Scale Structures

The definition of large scale structues was given in by Dydak in [4]. This interpretation of coarse structures give coarse geometry a more topological flavor.

Definition 1.3.1. [4] Let X be a set. A **large scale structure** on X is a non-empty set of families of subsets of X $\mathcal{LSS}$ so that the following conditions are satisfied:

1. If $\mathcal{U}, \mathcal{V}$ are families of subsets of X with $\mathcal{V} \in \mathcal{LSS}$ and each element U of $\mathcal{U}$ consisting of more than one point is contained in some V of $\mathcal{V}$, then $\mathcal{U} \in \mathcal{LSS}$.

2. If $\mathcal{U}, \mathcal{V} \in \mathcal{LSS}$, then $\mathrm{st}(\mathcal{U}, \mathcal{V}) \in \mathcal{LSS}$.

Elements $\mathcal{U}$ of $\mathcal{LSS}$ are called **uniformly bounded families** or **uniformly bounded scales**.

We note here closure under refinements implies the first condition above. The advantage of having a weaker first requirement is that a large scale structure as defined ”disregards” one point sets. That is, one point sets do not ”change” the large scale structure. Also, the first item in the definition gives us that the cover $\{\{x\}\}_{x \in X}$ is uniformly bounded for any large scale structure.

In [4], it was shown that coarse structures and large scale structures were equivalent by the following propositions:

Proposition 1.3.2. *Every large scale structure $\mathcal{LSS}$ on X induces a coarse structure $\mathcal{X}$ on X in the following way: A subset $E \subseteq X \times X$ is controlled if and ony if there is a $\mathcal{U} \in \mathcal{LSS}$ so that $E \subseteq \bigcup_{U \in \mathcal{U}} U \times U$.*

Proposition 1.3.3. *Every coarse structure $\mathcal{X}$ on X induces a large scale stucture $\mathcal{LSS}$ on X in the following way: $\mathcal{U}$ is uniformly bounded if and only if there is a controlled set $E \in \mathcal{X}$ so that $\bigcup_{U \in \mathcal{U}} U \times U \subseteq E$.*

Thus, properties of coarse structures also apply to large scale structures too.

1.4 Maps Between Large Scale Structures

It is worth noting that coarse geometry may be thought of as a dualization of "nice" topological structures. I used quotes because there is some subtlety here. Coarse geometry is a dualization of uniform structures, but every completely regular topology induces a uniform structure. Roughly, topology may be seen as "small scale" via the maps between them (called continuous maps). Continuous maps are maps that preserve closeness (think of the $\epsilon - \delta$ definition of continuity). As a consequence, continuous functions had the property that the push-forward of a convergent sequence is convergent. The dualization of this would be that pull-back divergent sequences is divergent. Before we define such maps, we define ∞-pseudo metric spaces.

Definition 1.4.1. An **∞-pseudo metric space** X is a set with a distance function $d : X \to [0, \infty]$ so that:

- $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, y)$ is allowed to assume the value ∞.
- $d(x, y) \leq d(x, z) + d(z, y)$ provided $d(x, z), d(z, y) < \infty$.

Definition 1.4.2. Let X and Y be ∞-psuedo metric spaces and $f : X \to Y$. Then we say f is **large scale continuous** or **bornologous** if for every $R>0$, there is an $S>0$ so that if $x, y \in X$ with $d_X(x, y) < R$, then $d_Y(f(x), f(y)) < S$.

Proposition 1.4.3. *Let (X, ρ_X) and (Y, ρ_Y) be ∞-pseudo-metric spaces with $f : X \to Y$. f is bornologous if and only if for any sequence $\{(a_n, b_n)\} \subseteq X \times X$, $\rho_y(f(a_n), f(b_n))$ diverges (i.e. has an infinite limit) implies $\rho_X(a_n, b_n)$ diverges.*

Proof. $(\Rightarrow)$: Suppose $\rho_Y(f(a_n), f(b_n))$ diverges as $n \to \infty$. Then $\forall M > 0\ \exists N > 0$ such that $n \geq N$ implies $\rho_Y(f(a_n), f(b_n)) > M$. Since f is bornologous, using the contrapositive of the definition of bornologous maps we have that for $M > 0\ \exists K_M$ such that $\rho_Y(f(a_n), f(b_n)) \geq K_M$ implies $\rho_X(a_n, b_n) \geq M$. We now show that $\rho_X(a_n, b_n)$ diverges. Indeed, let $M > 0$. Then as f is bornologous,$\exists K_M$ such that $\rho_Y(f(a_n), f(b_n)) \geq K_M$ implies $\rho_X(a_n, b_n) \geq M$. Since $\rho_Y(f(a_n), f(b_n))$ diverges, $K_M > 0$ implies $\exists N > 0$ such that $n \geq N$ implies $\rho_Y(f(a_n), f(b_n)) \geq K_M$ which implies $\rho_X(a_n, b_n) \geq M$. So for every $M > 0\ \exists N > 0$ such that $n \geq N$ implies $\rho_X(a_n, b_n) \geq M$. Hence $\rho_X(a_n, b_n)$ diverges.
$(\Leftarrow)$: Let $M > 0$ and assume for contradiction that f is not bornologous. Then we have that $\forall n > 0\ \exists\, (a_n, b_n)$ such that $\rho_X(a_n, b_n) < M$ and $\rho_Y(f(a_n), f(b_n)) \geq n$. Then we have that the sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ has the property that $\rho_Y(f(a_n), f(b_n))$ diverges (as $n \to \infty$). So we have that $\rho_X(a_n, b_n)$ diverges by our hypothesis. But $\rho_X(a_n, b_n) < M$ for every n. This is a contradiction which means there is an $N > 0$ such that $\rho_X(a, b) < M$ implies $\rho_Y(f(a), f(b)) \leq N$. □

From the large scale point of view, if $f, g : X \to Y$ for X,Y ∞-pseudo metric spaces are uniformly within some constant, then we wish to regard them as the same. This is because if we "zoom out" far enough (that is, choose a large enough scale), they are the "same" function.

Definition 1.4.4. Let $f, g : X \to Y$ for X, Y ∞-pseudo metric spaces. Then f is **close** to g provided $\exists R > 0$ such that $d_Y(f(x), g(x)) < R\ \forall x \in X$.

With this definition in mind, we can use close functions to define equivalent large scale spaces.

Definition 1.4.5. Let X and Y be ∞-pseudo metric spaces. X is **coarsely equivalent** to Y if $\exists f : X \to Y$ and $g : Y \to X$ so that f and g are bornologous and $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y. We call f a **coarse equivalence**.

Proposition 1.4.6. *Let* (X, ρ_X) *and* (Y, ρ_Y) *be* ∞*-pseudo-metric spaces with* $f : X \to Y$ *a 1-1 correspondence.* f *is a coarse equivalence if and only if for any sequence* $\{(a_n, b_n)\} \subseteq X \times X$*, one has the property* $\rho_Y(f(a_n), f(b_n))$ *diverges if and only if* $\rho_X(a_n, b_n)$ *diverges.*

Proof. $(\Rightarrow)$: Since f is a coarse equivalence, $\exists g : Y \to X$ such that f and g are bornologous and for all $x \in X$, $\rho_X(x, g \circ f(x)) \leq R$. By the previous proposition we have that f bornologous gives us that for any sequence $\{(a_n, b_n)\} \subseteq X \times X$ $\rho_Y(f(a_n), f(b_n))$ diverges implies $\rho_X(a_n, b_n)$ diverges. Further, g bornologous and f is a surjection gives us that $\rho_X(g \circ f(a_n), g \circ f(b_n))$ diverges implies $\rho_Y(f(a_n), f(b_n))$ diverges. Now suppose $\rho_X(a_n, b_n)$ diverges. We will show that $\rho_Y(f(a_n), f(b_n))$ diverges. Notice
$\rho_X(a_n, b_n) \leq \rho_X(a_n, g \circ f(a_n)) + \rho_X(g \circ f(a_n), g \circ f(b_n)) + \rho_X(g \circ f(b_n), b_n) \leq 2R + \rho_X(g \circ f(a_n), g \circ f(b_n))$. Then $\rho_X(a_n, b_n) - 2R \leq \rho_X(g \circ f(a_n), g \circ f(b_n))$ and since $\rho_X(a_n, b_n)$ diverges and R is constant, we have that $\rho_X(a_n, b_n) - 2R$ diverges which implies $\rho_X(g \circ f(a_n), g \circ f(b_n))$ diverges which implies $\rho_Y(f(a_n), f(b_n))$ diverges as desired.
$(\Leftarrow)$: By 1.4.3 we have that f is bornologous. Since f is a 1-1 correspondence, we know there exists a set-theoretic inverse $g : Y \to X$. By construction of g, it's obvious that $g \circ f$ and $f \circ g$ are close to their respective identities. We show that g is bornologous using 1.4.3. By surjectivity of f we may assume that any sequence within $Y \times Y$ is of the form $\{(f(a_n), f(b_n))\}$. Suppose $\rho_X(g \circ f(a_n), g \circ f(b_n))$ diverges. Since g is a set-theoretic inverse, we have that $g \circ f(x) = x$ for all $x \in X$. So $\rho_X(g \circ f(a_n), g \circ f(b_n))$ diverges implies $\rho_X(a_n, b_n)$ diverges which implies (by the hypothesis) $\rho_Y(f(a_n), f(b_n))$ diverges. So g is bornologous. ☐

Definition 1.4.7. Let (X, ρ_X), (Y, ρ_Y) be ∞ pseudo metric spaces. We say that $f : X \to Y$ is a **coarse embedding** provided that f is a coarse equivalence to its image $f(X)$.

Corollary 1.4.8. *Let* (X, ρ_X) *and* (Y, ρ_Y) *be* ∞*-pseudo-metric spaces with* $f : X \to Y$ *an injection.* f *is a coarse embedding if and only if for any sequence* $\{(a_n, b_n)\} \subseteq X \times X$*, one has the property* $\rho_y(f(a_n), f(b_n))$ *diverges if and only if* $\rho_X(a_n, b_n)$ *diverges.*

We can generalize many of the theorems given here. The following is such a generalization:

Definition 1.4.9. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces and let $f : X \to Y$. We say f is **large scale continuous** or **bornologous** if for every $\mathcal{U} \in \mathcal{LSS}_X$, $f(\mathcal{U}) \in \mathcal{LSS}_Y$, where $f(\mathcal{U}) = \{f(U) |\ U \in \mathcal{U}\}$.

At this juncture, it should be questioned whether or not the abstract version of bornologous maps coincides with the ∞-pseudo metric space version of bornologous maps.

Definition 1.4.10. Let (X, d) be an ∞-pseudo metric space. Define $\mathcal{LSS}$ on X via $\mathcal{U} \in \mathcal{LSS}$ if and only if $\sup\limits_{U \in \mathcal{U}} \text{diam}(U) < \infty$, where $\text{diam}(U) = \sup\limits_{x,y \in U} d(x, y)$. This large scale structure is called the **induced large scale structure by the metric d**.

Showing that the induced metric large scale structure is a large scale structure is a simple (and illuminating) exercise for the reader. The same can be said for the following proposition.

Proposition 1.4.11. *Let (X, d_X) and (Y, d_Y) be ∞-pseudo metric spaces and let $f : X \to Y$. Let $\mathcal{LSS}_X$ and $\mathcal{LSS}_Y$ be the large scale structures induced by d_X and d_Y respectively. Then $f : (X, d_X) \to (Y, d_Y)$ is bornologous in the metric space sense if and only if $f : (X, \mathcal{LSS}_X) \to (Y, \mathcal{LSS}_Y)$ is bornologous in the abstract sense.*

Here are some definitions that lead to a general notion of coarse equivalence:

Definition 1.4.12. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces and let $f : X \to Y$. We say f is a **coarse embedding** if for every $\mathcal{V} \in \mathcal{LSS}_Y$, we have $f^{-1}(\mathcal{V}) \in \mathcal{LSS}_X$, where $f^{-1}(\mathcal{V}) = \{f^{-1}(V) | \ V \in \mathcal{V}\}$. We say f is **coarsely surjective** if there exists a $\mathcal{V} \in \mathcal{LSS}_Y$ so that $Y \subseteq \text{st}(f(X), \mathcal{V})$.

Note that if f is not surjective, then $f(\mathcal{U})$ need not be a cover of $f(X)$. However, we may extend $f(\mathcal{U})$ trivially so that it does cover $f(X)$ and $\text{st}(f(X), \mathcal{V})$ makes sense.

Definition 1.4.13. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces and let $f, g : X \to Y$. We say f and g are **close** provided there is a $\mathcal{V} \in \mathcal{LSS}_Y$ so that for any $x \in X$, $f(x), g(x) \in V$ for some $V \in \mathcal{V}$.

Definition 1.4.14. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces and let $f : X \to Y$ be large scale continuous. We say f is a **coarse equivalence** provided f is a coarse embedding and coarsely surjective. If $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ are large scale spaces and a coarse equivalence $f : X \to Y$ exists, we say $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ are **coarsely equivalent**.

The following is from [Kev]:

Proposition 1.4.15. *Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces and let $f : X \to Y$ be large scale continuous. f is a coarse equivalence if and only if there exists a large scale continuous map $g : Y \to X$ so that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X.*

The above proposition will prove to be useful throughout this text. Note that the closeness conditions are equivalent to saying that the covers $\{\{g \circ f(x), x\}\}_{x \in X} \in \mathcal{LSS}_X$ and $\{\{f \circ g(y), y\}\}_{y \in Y} \in \mathcal{LSS}_Y$.

Chapter 2

Asymptotic Filtered Colimits

We begin this chapter by introducing the notion of an asymptotic filtered colimit.

2.1 Introduction of Asymptotic Filtered Colimit

Definition 2.1.1. Suppose X is a set with $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$ subsets of X and for each $s \in S$, X_s has the large scale structure $\mathcal{LSS}_s$. Further, assume $\bigcup\limits_{s\in S} X_s = X$ and for every $r, s \in S$ we have that the restrictions of the large scale structures $\mathcal{LSS}_r$ and $\mathcal{LSS}_s$ to the set $X_r \cap X_s$ coincide. Also, $\forall r, s \in S\ \exists t \in S$ such that $X_r \cup X_s \subseteq X_t$. Then the **asymptotic filtered colimit** of $\{X_s\}_{s\in S}$ of X is the following large scale structure:
$\mathcal{U}$ is uniformly bounded if and only if $\exists s \in S$ and $\mathcal{V} \in \mathcal{LSS}_s$ so that for any $U \in \mathcal{U}$ with $|U| > 1\ \exists V \in \mathcal{V}$ so that $U \subseteq V$ (and consequently $U \subseteq X_s$).

We note here that another way to think of the uniformly bounded families in the asymptotic filtered colimit $\mathcal{LSS}$ of $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$ is the following: For any $\mathcal{U} \in \mathcal{LSS}$ there is an $s \in S$ so that $\mathcal{U}^* \in \mathcal{LSS}_s$, where $\mathcal{U}^*$ is $\mathcal{U}$ with all one-point sets outside of X_s removed.

Proposition 2.1.2. *The asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$ of X (denoted $\mathcal{LSS}$) is indeed a large scale structure.*

Proof. Let $\mathcal{U} \in \mathcal{LSS}$ and suppose we have a family of subsets of X $\mathcal{W}$ so that $|W| > 1$ implies there exists a $U \in \mathcal{U}$ so that $W \subseteq U$. Since $\mathcal{U} \in \mathcal{LSS}$, $\exists s \in S$ and $\mathcal{V} \in \mathcal{LSS}_s$ so that $|U| > 1$ implies there is a $V \in \mathcal{V}$ such that $U \subseteq V$. If $|W| > 1$ and $W \subseteq U$ along with $U \subseteq V$, then we have that $W \subseteq V$. Then by definition and choice of $s \in S$, we have $\mathcal{W} \in \mathcal{LSS}$.

Now suppose $\mathcal{U}, \mathcal{V} \in \mathcal{LSS}$. Then $\exists r \in S$ and $\mathcal{F} \in \mathcal{LSS}_r$ so that for any $U \in \mathcal{U}$ with $|U| > 1$, we have $\exists F \in \mathcal{F}$ such that $U \subseteq F$. Also, $\exists s \in S$ and $\mathcal{G} \in \mathcal{LSS}_s$ so that for any $V \in \mathcal{V}$ with $|V| > 1$, we have $\exists G \in \mathcal{G}$ such that $V \subseteq G$. Select $t \in S$ such that $X_r \cup X_s \subseteq X_t$. We show that $\operatorname{st}(\mathcal{U}, \mathcal{V}) \in \mathcal{LSS}$. Define $\mathcal{U}^* = \mathcal{U} \setminus \{U \in \mathcal{U} \mid U = \{x\},\ x \in X \setminus X_r\}$. Likewise, define $\mathcal{V}^* = \mathcal{V} \setminus \{V \in \mathcal{V} \mid V = \{x\},\ x \in X \setminus X_s\}$. Notice that we have $\mathcal{U}^* \in \mathcal{LSS}_r$ and that $\mathcal{V}^* \in \mathcal{LSS}_s$. Since $X_r \subseteq X_t$ and $X_s \subseteq X_t$ and the restrictions of the large scale structures of $\mathcal{LSS}_r$ and $\mathcal{LSS}_t$ (respectively $\mathcal{LSS}_s$ and $\mathcal{LSS}_t$) to the intersection $X_r \cap X_t = X_r$ (respectively $X_s \cap X_t = X_s$) coincide, we therefore have that $\mathcal{U}^* \in \mathcal{LSS}_r$ implies $\mathcal{U}^* \in \mathcal{LSS}_t$ along with $\mathcal{V}^* \in \mathcal{LSS}_s$ implies $\mathcal{V}^* \in \mathcal{LSS}_t$. Indeed, $\mathcal{U}^* \in \mathcal{LSS}_r$ implies (by definition of $\{X_s\}$) there is a uniformly bounded family $\mathcal{U}' \in \mathcal{LSS}_t$ so that $\mathcal{U}'|_{X_r} = \mathcal{U}^*$, where $\mathcal{U}'|_{X_r} := \{U' \cap X_r \mid U' \in \mathcal{U}'\}$. But this means that for every $U \in \mathcal{U}$ with $|U| > 1$, there is a $U' \in \mathcal{U}'$ so that $U \subseteq U'$. Thus, $\mathcal{U}^* \in \mathcal{LSS}_t$. Since $\mathcal{U}^*, \mathcal{V}^* \in \mathcal{LSS}_t$, we have that $\operatorname{st}(\mathcal{U}^*, \mathcal{V}^*) \in \mathcal{LSS}_t$. Let $\mathcal{W} = \operatorname{st}(\mathcal{U}^*, \mathcal{V}^*) \cup \{V \in \mathcal{V} \mid V = \{x\},\ x \in X\}$. Then $\mathcal{W} \in \mathcal{LSS}_t$. We show that for any $U \in \mathcal{U}$ with $|\operatorname{st}(U, \mathcal{V})| > 1$, we have that there exists $W \in \mathcal{W}$ so that $\operatorname{st}(U, \mathcal{V}) \in \mathcal{W}$. This would show that $\operatorname{st}(\mathcal{U}, \mathcal{V}) \in \mathcal{LSS}_t$. If $|U| > 1$, then $U \in \mathcal{U}^*$ which implies that $\operatorname{st}(U, \mathcal{V}) \in \mathcal{W}$. If $|U| = 1$, then since $|\operatorname{st}(U, \mathcal{V})| > 1$ we have that there is a $V \in \mathcal{V}$ such that $|V| > 1$ and $U \subseteq V$. This gives us that $\operatorname{st}(U, \mathcal{V}) \in \mathcal{W}$. □

Proposition 2.1.3. *Suppose X is a set and $\mathcal{LSS}_X$ is the asymptotic filtered colimit of subsets $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$ of X and $f : X \to Y$ is a function to a large scale space Y. f is bornologous if and only if $f|_{X_s}$ is bornologous for each s.*

Proof. $(\Rightarrow)$: Let $s \in S$ and $\mathcal{U}_s \in \mathcal{LSS}_s$. Then notice that $\mathcal{U}_s \in \mathcal{LSS}_X$ which implies that $f(\mathcal{U}_s) \in \mathcal{LSS}_Y$. Since $U_s \in \mathcal{U}_s$ gives us $U_s \subseteq X_s$, we have $f(\mathcal{U}_s) = f|_{X_s}(\mathcal{U}_s)$.
$(\Leftarrow)$: Let $\mathcal{U} \in \mathcal{LSS}_X$. Then there is an $s \in S$ and a $\mathcal{V} \in \mathcal{LSS}_s$ such that for any $U \in \mathcal{U}$ with $|U| > 1$, there is a $V \in \mathcal{V}$ such that $U \subseteq V$. Define $\mathcal{U}^* = \mathcal{U} \setminus \{U \in \mathcal{U} \mid U = \{x\},\ x \in X \setminus X_s\}$. Then $\mathcal{U}^* \in \mathcal{LSS}_s$ and $f(\mathcal{U}^*) = f|_{X_s}(\mathcal{U}^*)$. So $f(\mathcal{U}^*) \in \mathcal{LSS}_Y$. We show that if $f(U) \in f(\mathcal{U})$

with $|f(U)| > 1$, then $f(U) \in f(\mathcal{U}^*)$. Indeed, $|f(U)| > 1$ implies $|U| > 1$ and hence $U \in \mathcal{U}^*$ which implies $f(U) \in f(\mathcal{U}^*)$. So $f(\mathcal{U}) \in \mathcal{LSS}_Y$. □

2.2 Coarse Properties Preserved by the Asymptotic Filtered Colimit Construction

2.2.1 Metrizability

For completeness, we remind the reader of the following from [4]. In particular, this statement is a combination of proposition 1.6 and theorem 1.8 in the paper cited:

Theorem 2.2.1. *Let $\mathcal{LSS}$ be a large scale structure on a set X and suppose there exists a set of families of X, $\mathcal{LSS}'$, such that for any $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}'$ there exists $\mathcal{B}_3 \in \mathcal{LSS}'$ such that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \operatorname{st}(\mathcal{B}_1, \mathcal{B}_2)$ refines $\mathcal{B}_3$. Then if the cardinality of $\mathcal{LSS}'$ is countable, then $\mathcal{LSS}$ is metrizable as a coarse space.*

Proof. See [4]. □

Theorem 2.2.2. *Let $(X, \mathcal{LSS})$ be an asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$ and that for every $s \in S$ we have that X_s is metrizable as a coarse space. Then if S is countable, then X is metrizable as a coarse space.*

Proof. By 2.2.1 we have that for every $s \in S$, there is a $\mathcal{LSS'}_s$ such that $|\mathcal{LSS'}_s|$ is countable and $\forall \mathcal{B}^s_1, \mathcal{B}^s_2 \in \mathcal{LSS'}_s$ $\exists \mathcal{B}^s_3$ such that $\mathcal{B}^s_1 \cup \mathcal{B}^s_2 \cup \operatorname{st}(\mathcal{B}^s_1, \mathcal{B}^s_2)$ is a refinement of $\mathcal{B}^s_3$. Let $\mathcal{LSS}' = \bigcup\limits_{s\in S} \mathcal{LSS'}_s$. Then $|\mathcal{LSS}'|$ is countable since the countable union of countable sets is countable. Let $\mathcal{A'}_s, \mathcal{B'}_r \in \mathcal{LSS}'$. Then note that there is a $t \in S$ so that $X_r \cup X_s \subseteq X_t$ and $\mathcal{A'}_s, \mathcal{B'}_s \in \mathcal{LSS'}_t$ which implies there is a $\mathcal{W'}_t \in \mathcal{LSS'}_t$ so that $\mathcal{A'}_s \cup \mathcal{B'}_r \cup \operatorname{st}(\mathcal{A'}_s, \mathcal{B'}_r) \in \mathcal{W'}_t$ Since, $\mathcal{W'}_t \in \mathcal{LSS}'$, we have by 2.2.1 that X is metrizable as a coarse space. □

2.2.2 Finite Asymptotic Dimension

We use the following definition of Asymptotic Dimension from [4]:

Definition 2.2.3. Let $(X, \mathcal{LSS})$ be a large scale structure. We say $(X, \mathcal{LSS})$ has **asymptotic dimension at most n** if for every uniformly bounded family $\mathcal{U}$ in X there is a uniformly bounded coarsening $\mathcal{V}$ such that the multiplicity of $\mathcal{V}$ is at most $n+1$ (i.e. each point $x \in X$ is contained in at most $n+1$ elements of $\mathcal{V}$).

Theorem 2.2.4. *Let $(X, \mathcal{LSS})$ be an asymptotic filtered colimit of the large scale structures $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$ of X. The asymptotic dimension of X is at most n if and only if the asymptotic dimension of each $(X_s, \mathcal{LSS}_s)$ is at most n.*

Proof. $(\Rightarrow)$: Let $\mathcal{U}_s \in \mathcal{LSS}_s$. Then we have that $\mathcal{U}_s \in \mathcal{LSS}$ and hence $\mathcal{U}_s$ has a coarsening $\mathcal{V}$ with multiplicity at most $n+1$. The desired coarsening is $\mathcal{V}' = \{V \cap X_s \mid V \in \mathcal{V}\}$.
$(\Leftarrow)$: Let $\mathcal{U} \in \mathcal{LSS}$. Then there is an $s \in S$ and a $\mathcal{V} \in \mathcal{LSS}_s$ such that for any $U \in \mathcal{U}$ with $|U| > 1$, there is a $V \in \mathcal{V}$ such that $U \subseteq V$. Define $\mathcal{U}^*$ as $\mathcal{U}$ with one point sets removed. Then $\mathcal{U}^* \in \mathcal{LSS}_s$ and hence there is a coarsening $\mathcal{W} \in \mathcal{LSS}_s$ with multiplicity at most $n+1$. Then the family $\mathcal{W} \cup \{U \in \mathcal{U} \mid U = \{x\},\ x \in X \setminus X_s\}$. is the desired coarsening of $\mathcal{U}$ with multiplicity at most $n+1$. □

2.2.3 Exactness

Definition 2.2.5. Let X be a set. We say $(f_i)_{i \in I}$ is a **partition of unity** of X if $f_i : X \to [0, \infty)$ for all i and for all $x \in X$, $\sum\limits_{i \in I} f_i(x) = 1$.

The following definition is adapted from [5]:

Definition 2.2.6. Let $(X, \mathcal{LSS})$ be a large scale structure. $(X, \mathcal{LSS})$ is **exact** if for every $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$ there exists a partition of unity $(f_i)_{i \in I}$ of X so that the cover of X, $\mathcal{V} = \{\text{support}(f_i) \mid i \in I\}$, is uniformly bounded and that if (for $U \in \mathcal{U}$) $x, y \in U$, then $\sum\limits_{i \in I} |f_i(x) - f_i(y)| < \epsilon$.

Theorem 2.2.7. *Suppose X is a set and $\mathcal{LSS}$ is the asymptotic filtered colimit of subsets $\{X_s\}_{s \in S}$ of X. $(X, \mathcal{LSS})$ is exact if and only if for each $s \in S$, $(X_s, \mathcal{LSS}_s)$ is exact.*

Proof. $(\Rightarrow)$: Let $\mathcal{U}_s \in \mathcal{LSS}_s$ and $\epsilon > 0$. Note that for any $s \in S$ $\mathcal{LSS}_s \subseteq \mathcal{LSS}$. Then we have $\mathcal{U}_s \in \mathcal{LSS}$; since $(X, \mathcal{LSS})$ is exact, we can find the desired partition of unity of X. Restrict this partition of unity of X to a partition of unity of X_s. This shows that $(X_s, \mathcal{LSS}_s)$ is exact.

$(\Leftarrow)$: Let $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$. Then there exists $s \in S$ and $\mathcal{V} \in \mathcal{LSS}_s$ such that for every $U \in \mathcal{U}$ with $|U| > 1$ there exists a $V \in \mathcal{V}$ so that $V \subseteq U$. Let $\mathcal{U}^*$ be $\mathcal{U}$ with one point sets removed. Then $\mathcal{U}^* \in \mathcal{LSS}_s$ which means there is a partition of unity of X_s, $(f_i)_{i\in I}$ so that the family $\{\text{support}(f_i) \mid i \in I\}$ is uniformly bounded and if $U \in \mathcal{U}$ and $x, y \in U$, then $\sum\limits_{i\in I} |f_i(x) - f_i(y)| < \epsilon$. For any value $j \in X \setminus X_s$, define $f_j : X \to [0, \infty)$ via $f_j(j) = 1$ and zero elsewhere. Also, for any $i \in I$ extend $f_i : X_s \to [0, \infty)$ to X by defining $f_i(j) = 0$ for any $j \in X \setminus X_s$. Let the set J index the various f_j's and let $K = I \cup J$. We claim that $(f_k)_{k\in K}$ is the desired partition of unity of X. Indeed, notice that aside from a collection of one point sets (i.e. support(f_j) for $j \in J$), we have that the family $\{\text{support}(f_k) \mid k \in K\} = \{\text{support}(f_i) \mid i \in I\} \in \mathcal{LSS}_s \subset \mathcal{LSS}$. Now let $U \in \mathcal{U}$. If $|U| = 1$, then we have that $x, y \in U$ implies that $x = y$ and thus $\sum\limits_{k\in K} |f_k(x) - f_k(y)| = 0 < \epsilon$. If $|U| > 1$, then we have that $U \subseteq X_s$ and since $(f_i)_{i\in I}$ is a partition of unity for X_s and that $f_j(U) \equiv 0$, we have that $x, y \in U$ implies $\sum\limits_{k\in K} |f_k(x) - f_k(y)| = \sum\limits_{i\in I} |f_i(x) - f_i(y)| < \epsilon$. Now we show that for every $x \in X$, $\sum\limits_{k\in K} f_k(x) = 1$. Suppose $x \in X_s$. then for any $i \in I$, $f_i(x) = 0$ and there is a unique $j \in J$ so that $f_j(x) = 1$. So $\sum\limits_{k\in K} f_k(x) = 1$. If $x \in X \setminus X_s$, then we have that for any $j \in J$, $f_j(x) = 0$ and since $(f_i)_{i\in I}$ form a partition of unity for X_s, we have that $\sum\limits_{k\in K} f_k(x) = \sum\limits_{i\in I} f_i(x) = 1$. □

2.2.4 Coarse Embeddability Into a Hilbert Space

Before we present the following theorem, recall that for any two separable Hilbert spaces G and H, there is an isometric isomorphism between the two. We will also use some pinch space theory. The following definition and theorem is adapted from [7]:

Definition 2.2.8. Let $(X, \mathcal{LSS})$ be a large scale space, K a metric space, and $c > 0$. We say $(X, \mathcal{LSS})$ **c-pinch-spaces to K** if for every $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$ there is a $\mathcal{V} \in \mathcal{LSS}$ and a function $f : X \to K$ so that $\sup_{U \in \mathcal{U}} \operatorname{diam}(f(U)) < \epsilon$ and for every $x, y \in X$ so that $\{x, y\} \nsubseteq V$ for every $V \in \mathcal{V}$ we have that $d_K(f(x), f(y)) \geq c$.

Theorem 2.2.9. *If X is a metric space, then X coarsely embedds into a Hilbert space if and only if X c-pinch-spaces to a Hilbert space for some $c > 0$.*

Theorem 2.2.10. *Let S be a countable index set and let H be a fixed separable Hilbert space. Let $(X, \mathcal{LSS})$ be the asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$ with every X_s countable. Then $(X, \mathcal{LSS})$ coarsely embedds into H if and only if $(X_s, \mathcal{LSS}_s)$ coarsely embedds into H for all $s \in S$.*

Proof. $(\Rightarrow)$: This follows via restriction of the embedding function $f : X \to H$ to any X_s. $(\Leftarrow)$: Note that $\bigoplus_{s \in S} H \cong H$ since S is countable. Likewise, $H \oplus H \cong H$. We show $(X, \mathcal{LSS})$ 1-pinch-spaces to H. Let $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$. Define $\mathcal{U}^*$ to be $\mathcal{U}$ with one point sets removed. Then by definition of $\mathcal{LSS}$, $\mathcal{U}^* \in \mathcal{LSS}_s$ for some s. Since $(X_s, \mathcal{LSS}_s)$ 1-pinch-spaces to H, there exists $f^{\mathcal{U}^*}_{\epsilon,s} : X_s \to H$ and $\mathcal{W}_s \in \mathcal{LSS}_s$ such that $\sup_{U \in \mathcal{U}^*} \operatorname{diam}(f^{\mathcal{U}^*}_{\epsilon,s}(U)) < \epsilon$ and for any $x, y \in X_s$ with $\{x, y\} \nsubseteq W$ for every $W \in \mathcal{W}_s$ we have $\|f^{\mathcal{U}^*}_{\epsilon,s}(x) - f^{\mathcal{U}^*}_{\epsilon,s}(y)\| \geq 1$ (the norm is in H). Now, since $X = \bigcup_{s \in S} X_s$ and X_s is countable for every s, we may index an orthronormal basis of H via $\{e_x\}_{x \in X}$. Furthermore, define $f^{\mathcal{U}}_{\epsilon} : X \to H \oplus H$ via $f^{\mathcal{U}}_{\epsilon}(x) = \left(f^{\mathcal{U}^*}_{\epsilon,s}(x), 0\right)$ for any $x \in X_s$ and $f^{\mathcal{U}}_{\epsilon}(x) = (0, e_x)$ for any x not in X_s. $f^{\mathcal{U}}_{\epsilon}$ is well defined since asymptotic filtered colimits agree on their intersection. Define $\mathcal{V} \in \mathcal{LSS}$ as $\mathcal{V} = \mathcal{W}_s \cup \{x\}_{x \in X}$. We will show that $\left(f^{\mathcal{U}}_{\epsilon}, \mathcal{V}\right)$ satisfies the 1-pinch space conditions. Note that $\epsilon > \sup_{U \in \mathcal{U}^*} \operatorname{diam}(f^{\mathcal{U}^*}_{\epsilon,s}(U)) = \sup_{U \in \mathcal{U}} \operatorname{diam}(f^{\mathcal{U}}_{\epsilon}(U))$ since $|U| > 1$ implies $U \subseteq X_s$ and $U \in \mathcal{U}^*$ which implies that $\operatorname{diam}(U) < \epsilon$. If $|U| = 1$, then $\operatorname{diam}(f^{\mathcal{U}}_{\epsilon}(U)) = 0 < \epsilon$. Hence, $\sup_{U \in \mathcal{U}} \operatorname{diam}(f^{\mathcal{U}}_{\epsilon}(U)) < \epsilon$. Now, let $x, y \in X$ so that $\{x, y\} \nsubseteq V$ for every $V \in \mathcal{V}$. We have three cases:

Suppose $\{x, y\} \subseteq X \setminus X_s$. Then $f^{\mathcal{U}}_{\epsilon}(x) = (0, e_x)$ and $f^{\mathcal{U}}_{\epsilon}(y) = (0, e_y)$. Then we have that $\|(0, e_x) - (0, e_y)\|_{H \oplus H} = \sqrt{\|0\|^2_H + \|e_x - e_y\|^2_H} = \sqrt{2} > 1$.

Suppose $\{x, y\} \subseteq X_s$. Then $\{x, y\} \nsubseteq V$ for every $V \in \mathcal{V}$ implies that $\{x, y\} \nsubseteq W$ for every $W \in \mathcal{W}_s$. Then we have that $\|f^{\mathcal{U}}_{\epsilon}(x) - f^{\mathcal{U}}_{\epsilon}(y)\|_{H \oplus H} = \sqrt{\|f^{\mathcal{U}^*}_{\epsilon,s}(x) - f^{\mathcal{U}^*}_{\epsilon,s}(y)\|^2_H + \|0\|^2_H} \geq 1$ by assumption that $(X_s, \mathcal{LSS}_s)$ 1-pinch-spaces to H.

Suppose that $x \in X_s$ and $y \in X \setminus X_s$. Then $f^{\mathcal{U}}_{\epsilon}(x) = (f^{\mathcal{U}^*}_{\epsilon,s}(x), 0)$ and $f^{\mathcal{U}}_{\epsilon}(y) =$

$(0, e_y)$. Then we have that $\|f^{\mathcal{U}}_\epsilon(x) - f^{\mathcal{U}}_\epsilon(y)\|_{H\oplus H} = \|(f^{\mathcal{U}^*}_{\epsilon,s}(x), 0) - (0, e_y)\|_{H\oplus H} = \sqrt{\|f^{\mathcal{U}^*}_{\epsilon,s}(x) - 0\|^2_H + \|0 - e_y\|^2_H} \geq 1$.

So in all cases, $\|f^{\mathcal{U}}_\epsilon(x) - f^{\mathcal{U}}_\epsilon(y)\|_{H\oplus H} \geq 1$. Defining $h : X \to H$ to be the composition of $f^{\mathcal{U}}_\epsilon$ with the isometric isomorphism from $H \oplus H$ to H, we have that $(h, \mathcal{V})$ 1-pinch-spaces to H which means that X coarsely embedds into H. □

2.2.5 Coarse Amenability

Definition 2.2.11. Let X be a set, $A \subseteq X$, and $\mathcal{U}$ a family of subsets of X. Then the **horizon of A against** $\mathcal{U}$, denoted $hor(A,\mathcal{U})$, is the set $\{U \in \mathcal{U} | A \cap U \neq \varnothing\}$.

Here are some useful properties of the horizon that we will use:

Lemma 2.2.12. *Let X be a set, $A, B \subseteq X$, and $\mathcal{U}, \mathcal{V}$ be families of subsets of X. Then:*

1. $A \subseteq B \Rightarrow hor(A,\mathcal{U}) \subseteq hor(B,\mathcal{U})$
2. $\mathcal{U} \prec \mathcal{V} \Rightarrow hor(A,\mathcal{U}) \subseteq hor(A, \mathcal{V})$
3. $A \subseteq B$ *and* $\mathcal{U} \subseteq \mathcal{V} \Rightarrow hor(A,\mathcal{U}) \subseteq hor(B, \mathcal{V})$.

Proof. Let $U \in hor(A,\mathcal{U})$. Then $\varnothing \neq U \cap A \subseteq U \cap B$ which implies that $B \cap U \neq \varnothing$. So $U \in hor(B,\mathcal{U})$.

For the second item, let $U \in hor(A,\mathcal{U})$. Then $\varnothing \neq U \cap A$. But $U \in \mathcal{U} \prec \mathcal{V}$ implies $U \in hor(A, \mathcal{V})$. The last statement is a combination of the first two. □

Definition 2.2.13. Let $(X, \mathcal{LSS})$ be a large scale structure. Then $(X, \mathcal{LSS})$ is **coarsely amenable** if for every $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$, there exists $\mathcal{V} \in \mathcal{LSS}$ so that for any $x \in \bigcup_{U\in\mathcal{U}} U$, $|hor(\mathrm{st}(x,\mathcal{U}), \mathcal{V})| < \infty$ and

$$\frac{|hor(x, \mathcal{V})|}{|hor(\mathrm{st}(x,\mathcal{U}), \mathcal{V})|} > 1 - \epsilon$$

. For simplicity, we denote $hor(\{x\}, \mathcal{V})$ as $hor(x, \mathcal{V})$ and $hor(\mathrm{st}(\{x\},\mathcal{U}), \mathcal{V})$ as $hor(\mathrm{st}(x,\mathcal{U}), \mathcal{V})$.

This definition of coarse amenability is given in [2].

Theorem 2.2.14. *Suppose S is an index set, $(X, \mathcal{LSS})$ the asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$. Then $(X, \mathcal{LSS})$ is coarsely amenable if and only if $(X_s, \mathcal{LSS}_s)$ be coarsely amenable for every $s \in S$.*

Proof. $(\Rightarrow)$: It is shown in [2] that coarse amenability is preserved by taking subspaces.
$(\Leftarrow)$:Let $\mathcal{U} \in \mathcal{LSS}$. Then for some $s \in S$, $\mathcal{U}^* \in \mathcal{LSS}_s$, where $\mathcal{U}^*$ is $\mathcal{U}$ with one point sets outside of X_s removed. As $\mathcal{LSS}_s$ is coarsely amenable, there is a $\mathcal{V}^* \in \mathcal{LSS}_s$ so that for any $x \in \bigcup\limits_{U \in \mathcal{U}^*} U$, $hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*)| < \infty$ and $\frac{|hor(x,\mathcal{V}^*)|}{|hor(\text{st}(x,\mathcal{U}^*),\mathcal{V}^*)|} > 1 - \epsilon$. Define $\mathcal{V} = \mathcal{V}^* \ \cup (\mathcal{U} \setminus \mathcal{U}^*)$. Then $\mathcal{V} \in \mathcal{LSS}$. Note that by construction, $\mathcal{V} \setminus \mathcal{V}^* = \mathcal{U} \setminus \mathcal{U}^*$. We now show that for any $x \in \bigcup\limits_{U \in \mathcal{U}} U$, $hor(\text{st}(x, \mathcal{U}), \mathcal{V}) = hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*) \cup hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*)$. Furthermore, we will show that the $hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*) \cap hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*) = \varnothing$.
$(\subseteq)$: Let $V \in hor(\text{st}(x, \mathcal{U}), \mathcal{V})$. Then $V \in \mathcal{V}^*$ or it isn't. Suppose $V \in \mathcal{V}^*$. Then there is a $U \in \mathcal{U}$ so that $x \in U$ and $U \cap V \neq \varnothing$. We will show that $U \in \mathcal{U}^*$. Suppose not (for contradiction). Then $U \subseteq (X \setminus X_s)$ and $|U| = 1$. Hence $U = \{x\}$ and $U \subseteq V$ as $U \cap V \neq \varnothing$. Thus, $x \in V$ so $V \nsubseteq X_s$ which implies $V \notin \mathcal{V}^*$ which is a contradiction. So we must have that $U \in \mathcal{U}^*$ hence $V \in hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*)$. Now, if $V \notin \mathcal{V}^*$, then there is a $U \in \mathcal{U}$ so that $x \in \mathcal{U}$ and $U \cap V \neq \varnothing$. As $V \notin \mathcal{V}^*$, we have that $|V| = 1$ which means that $V \subseteq U$. As $V \nsubseteq X_s$, we have that $U \nsubseteq X_s$ which implies (by definition of $\mathcal{LSS}$) $|U| = 1$. So $U = V = \{x\}$ and $U \in \mathcal{U} \setminus \mathcal{U}^*$. Therefore, $x \in U$ implies $U \in \text{st}(x, \mathcal{U} \setminus \mathcal{U}^*)$ which implies $V \in hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*)$.
$(\supseteq)$: This follows via two applications of the previous lemma.
We now show that $hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*) \cap hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*) = \varnothing$. Note that $hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*) = \{x\}$ or is the empty set since $hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*)) = \{x\}$ or the empty set. If this set is the singelton $\{x\}$, then $x \notin X_s$ which implies that $\text{st}(x, \mathcal{U}^*) = \varnothing$ which means that $hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*) \cap hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*) = \varnothing$ as desired.
Since $hor(x, \mathcal{V}) = hor(x, \mathcal{V}^*) \cup hor(x, \mathcal{V} \setminus \mathcal{V}^*)$ and (by the previous lemma) $hor(x, \mathcal{V} \setminus \mathcal{V}^*) = hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*)$, we therefore have that:

$$\frac{|hor(x, \mathcal{V})|}{|hor(\text{st}(x, \mathcal{U}), \mathcal{V})|} = \frac{|hor(x, \mathcal{V}^*)| + |hor(x, \mathcal{V} \setminus \mathcal{V}^*)|}{|hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*)| + |hor(\text{st}(x, \mathcal{U} \setminus \mathcal{U}^*), \mathcal{V} \setminus \mathcal{V}^*)|} =$$

$$\frac{|hor(x, \mathcal{V}^*)| + |hor(x, \mathcal{V} \setminus \mathcal{V}^*)|}{|hor(\text{st}(x, \mathcal{U}^*), \mathcal{V}^*)| + |hor(x, \mathcal{V} \setminus \mathcal{V}^*)|}$$

If we can show the fraction on the far right is larger than $1-\epsilon$ for any $x \in \bigcup_{U\in\mathcal{U}} U$, then we're done. Let $x \in \bigcup_{U\in\mathcal{U}} U$. Then $x \in \bigcup_{U\in\mathcal{U}^*} U$ or $x \in \bigcup_{U\in\mathcal{U}\setminus\mathcal{U}^*} U$. If $x \in \bigcup_{U\in\mathcal{U}\setminus\mathcal{U}^*} U$, then $x \in X\setminus X_s$ and for some $U \in \mathcal{U}$, $U = \{x\}$. Thus, $|hor(x, \mathcal{V} \setminus \mathcal{V}^*)| = 1$ and $|hor(\text{st}(x, \mathcal{V}^*))| = 0$ (as $x \notin X_s$) so $|hor(\text{st}(x, \mathcal{U}, \mathcal{V}))| = 1 < \infty$ and for any ϵ between one and zero, $\frac{|hor(x,\mathcal{V})|}{|hor(\text{st}(x,\mathcal{U}),\mathcal{V})|} = 1 > 1-\epsilon$. If $x \in \bigcup_{U\in\mathcal{U}^*} U$, then we have that $x \in X_s$ which implies that $|hor(x, \mathcal{V} \setminus \mathcal{V}^*)| = 0$ and hence $\frac{|hor(x,\mathcal{V})|}{|hor(\text{st}(x,\mathcal{U}),\mathcal{V})|} = \frac{|hor(x,\mathcal{V}^*)|}{|hor(\text{st}(x,\mathcal{U}^*),\mathcal{V}^*)|} > 1-\epsilon$. So $(X, \mathcal{LSS})$ is coarsely amenable. □

2.2.6 Property A

The following definitions are from [Kev].

Definition 2.2.15. $(X, \mathcal{LSS})$ is a **bounded geometry coarse space** if for any $\mathcal{U} \in \mathcal{LSS}$, $\sup_{U\in\mathcal{U}} |U| < \infty$.

Definition 2.2.16. Let $(X, \mathcal{LSS})$ be a bounded geometry coarse space. We say that $(X, \mathcal{LSS})$ has **property A** if for any $\epsilon > 0$ and $\mathcal{U} \in \mathcal{LSS}$ there is a $\mathcal{V} \in \mathcal{LSS}$ and a family of subsets of $X \times \mathbb{N}$, $\{A_x\}_{x\in X}$, so that for each $x \in X$: $|A_x| < \infty$, $(x, 1) \in A_x$, $A_x \subseteq \text{st}(x, \mathcal{V}) \times \mathbb{N}$, and for any $y \in \text{st}(x, \mathcal{U})$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$, where $A_x \Delta A_y$ is the symmetric difference of A_x and A_y.

We will show that property A is preserved by the asymptotic filtered colimit construction.

Proposition 2.2.17. *Let $(X, \mathcal{LSS})$ be an asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$. If $(X_s, \mathcal{LSS}_s)$ is a bounded geometry coarse space with property A for every $s \in S$, then $(X, \mathcal{LSS})$ is a bounded geometry coarse space with property A.*

Proof. Note that $(X, \mathcal{LSS})$ is a bounded geometry coarse space since for any $\mathcal{U} \in \mathcal{LSS}$, we have that $\mathcal{U}^* \in \mathcal{LSS}_s$ for some $s \in S$ and that $(X_s, \mathcal{LSS}_s)$ is a bounded geometry coarse space. We now show that $(X, \mathcal{LSS})$ has property A. Let $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$. Then we have that for some $s \in S$, $\mathcal{U}^* \in \mathcal{LSS}_s$. Since $(X_s, \mathcal{LSS}_s)$ has property A, we have that there is a $\mathcal{V}_s \in \mathcal{LSS}_s$ and a collection of subsets of $X_s \times \mathbb{N}$, $\{A_x\}_{x\in X_s}$, so that the requirements of property A are satisfied in $(X_s, \mathcal{LSS}_s)$. Note that $\mathcal{V}_s \in \mathcal{LSS}$ and define $\mathcal{V} \in \mathcal{LSS}$ via $\mathcal{V} = \mathcal{V}_s \cup \{\{x\} | x \in X \setminus X_s\}$. Define $\{B_x\}_{x\in X}$ via $B_x = A_x$ if $x \in X_s$

and $B_x = \{(x,1)\}$ otherwise. We show that $\mathcal{V}$ and $\{B_x\}_{x\in X}$ satisfy the requirements in the definition of property A. Let $x \in X$. Then $|B_x| < \infty$ and $(x,1) \in B_x$ are obvious. $B_x \subseteq \mathrm{st}(x,\mathcal{V})$ since $x \in X_s$ implies that $B_x = A_x \subseteq \mathrm{st}(x,\mathcal{V}_s) \times \mathbb{N} = \mathrm{st}(x,\mathcal{V}) \times \mathbb{N}$. Otherwise, $B_x = (x,1) \subseteq \mathrm{st}(x,\mathcal{V}) \times \mathbb{N} = \{x\} \times \mathbb{N}$. Lastly, let $y \in \mathrm{st}(x,\mathcal{U})$. If $x \in X_s$, then we have that $\mathrm{st}(x,\mathcal{U}) = \mathrm{st}(x,\mathcal{U}^*)$ hence $y \in \mathrm{st}(x,\mathcal{U}^*)$ (i.e. $y \in X_s$) and $B_x = A_x$ and $B_y = A_y$. So $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$ since $(X_s, \mathcal{LSS}_s)$ has property A. If $x \in X \setminus X_s$, then $y \in \mathrm{st}(x,\mathcal{U})$ implies that $y = x$. Hence $|B_x \Delta B_y| = 0$ and $\frac{|B_x \Delta B_y|}{|B_x \cap B_y|} = 0 < \epsilon$. So $(X, \mathcal{LSS})$ has property A. □

The converse of this theorem is most likely true. One would need to show that Property A is preserved by subspaces. It was shown in [8] that this is true in the case of uniformly discrete metric spaces.

2.2.7 Slowly Oscillating Functions

We remind the reader of the following definitions:

Definition 2.2.18. Let $(X, \mathcal{LSS})$ be given and let $\mathcal{U} \in \mathcal{LSS}$. We say a **$\mathcal{U}$-chain component** of X is an equivalence class of the following equivalence relation. $x \sim y$ if and only if there is a finite sequence $\{U_i\}_{i=1}^n \subseteq \mathcal{U}$ such that $U_i \cap U_{i+1} \neq \varnothing$ for every i and $x \in U_1$ along with $y \in U_n$. A **coarse chain component** of $x \in X$ is the union of its $\mathcal{U}$-chain components, where $\mathcal{U}$ ranges over every uniformly bounded family of $\mathcal{LSS}$. A subset $B \subseteq X$ is called **weakly bounded** if its intersection with each coarse chain component is contained in some U for $U \in \mathcal{U}$ and $\mathcal{U} \in \mathcal{LSS}$.

Definition 2.2.19. Let $f : X \to Y$ where $(X, \mathcal{LSS})$ is a large scale structure and Y is a metric space. f is **slowly oscillating** if $\forall \mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$ $\exists B \subseteq X$ weakly bounded such that for any $U \in \mathcal{U}$ with $U \not\subseteq B$ implies $\mathrm{diam}(f(U)) < \epsilon$.

Theorem 2.2.20. *Let X be a set and let $\mathcal{LSS}$ be the asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s\in S}$. Let Y be a metric space and let $f : X \to Y$. Then f is slowly oscillating if and only if $f|_{X_s}$ is slowly oscillating for all $s \in S$.*

Proof. $(\Rightarrow)$: Let $\mathcal{U}_s \in \mathcal{LSS}_s$ and $\epsilon > 0$. Then there is a $B \subseteq X$ weakly bounded such that for any $\mathcal{U}_s \in \mathcal{LSS}_s$ with $U_s \not\subseteq B$ implies $\mathrm{diam}(f(U_s)) < \epsilon$. But $U_s \subseteq X_s$ implies

$f(U_s) = f|_{X_s}(U_s)$ and we are done with choice of weakly bounded subset $B \cap X_s$. Indeed, suppose $U_s \in \mathcal{U}_s$ and $U \nsubseteq (B \cap X_s)$. Then since $U_s \subseteq X_s$, we have that $U_s \nsubseteq B$ which implies $\operatorname{diam}(f(U_s)) = \operatorname{diam}(f|_{X_s}(U_s)) < \epsilon$.

$(\Leftarrow)$: Let $\mathcal{U} \in \mathcal{LSS}$ and $\epsilon > 0$. Then there is an $s \in S$ and $\mathcal{V} \in \mathcal{LSS}_s$ such that for any $U \in \mathcal{U}$ with $|U| > 1$ implies $U \subseteq V$ for $V \in \mathcal{V}$. Define $\mathcal{U}^*$ to be $\mathcal{U}$ with one point sets removed outside of X_s. Then $\mathcal{U}^* \in \mathcal{LSS}_s$ which implies there is a $B \subseteq X_s \subseteq X$ weakly bounded such that for any $U \in \mathcal{U}^*$ with $U \not\subset B$ we have that $\operatorname{diam}(f(U)) < \epsilon$. Notice that for any $U \in \mathcal{U} \setminus \mathcal{U}^*$ with $U \nsubseteq B$, we have that $\operatorname{diam}(f(U)) = 0 < \epsilon$. Therefore, B is a choice of a weakly bounded set with the property that for any $U \in \mathcal{U}$ with $U \not\subset B$, we have $\operatorname{diam}(f(U)) < \epsilon$. So f is slowly oscillating. □

2.3 Coarse Properties That Might not be Preserved by the Asymptotic Filtered Colimit Construction

We have presented multiple properties that are preserved through asymptotic filtered colimits. It turns out that close functions are not preserved through asymptotic filtered colimits. The following is such an example:

Let $X = (0, 1]$ and let $X_n = \left[\frac{1}{n+1}, 1\right]$ for $n \in \{1, 2, ...\}$. Let X_n have the subspace topology of the usual topology of the real numbers. Then we have that $\bigcup_{n=1}^{\infty} X_n = X$ and that $X_n \subseteq X_{n+1}$ for every n. Let $\mathcal{LSS}$ be the asymptotic filtered colimit of $\{X_n\}$ of X. We have that $\mathcal{LSS}$ is metrizable. Let $f : X \to [0, \infty)$ be definied via $f(x) = \frac{1}{x}$. Also, define $g : X \to \{1\}$ be defined via $g(x) = 1$. For any n, we have that X_n is a compact set. Since the function $|f - g|$ is continuous on X_n, we have that $f|_{X_n}$ is close to $g|_{X_n}$ for all n. However, f is not close to g under the large scale structure $\mathcal{LSS}$. Indeed, suppose f is close to g. Then there is a family of subsets of $[0, \infty)$, $\mathcal{V}$ and an $M > 0$ so that for any $V \in \mathcal{V}$, $\operatorname{diam}(V) < M$; we also have that for any $x \in X$, $\{f(x), g(x)\} \subseteq V$ for some $V \in \mathcal{V} \cup \{\{y\} \mid y \in Y\}$. This implies that for any $x \in X$, $|f(x) - g(x)| < M$ i.e. for any $x \in (0, 1]$, $\frac{1-x}{x} < M$. This is a contradiction. Indeed, choose $x = \frac{1}{M+2}$.

We define a stronger notion of closeness:

Definition 2.3.1. Let $\{f_s\}_{s\in S}$ and $\{g_s\}_{s\in S}$ be families of functions from $\{(X_s, \mathcal{LSS}_s^X)\}_{s\in S}$ to $\{(Y_s, \mathcal{LSS}_s^Y)\}_{s\in S}$ that agree on $X_s \cap X_{s'}$ for all $s, s' \in S$. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be the respective asymptotic filtered colimits. We say that $\{f_s\}_{s\in S}$ is **uniformly close** to $\{g_s\}_{s\in S}$ if there exists a $\mathcal{V} \in \mathcal{LSS}_Y$ such that for any $x \in X$ and $s \in S$ $\exists V_s \in \mathcal{V} \cup \{\{y\} \mid y \in Y\}$ such that $\{f_s(x), g_s(x)\} \subseteq V_s$ or equivalently $\{\{f_s(x), g_s(x)\}\}_{s\in S, x\in X} \prec \mathcal{V}$.

Theorem 2.3.2. *Let $(X, \mathcal{LSS}_X)$ be the asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_{s,X})\}_{s\in S}$ and let $(Y, \mathcal{LSS}_Y)$ be a large scale space. Let $\{f_s\}_{s\in S}$ and $\{g_s\}_{s\in S}$ be families of functions so that $f_s, g_s : X_s \to Y$ for every s and suppose that the families of functions are uniformly close. Further, suppose that for every $s, t \in S$ with $X_s \cap X_t \neq \varnothing$ we have that $f_s|_{X_s\cap X_t} \equiv f_t|_{X_s\cap X_t}$ and $g_s|_{X_s\cap X_t} \equiv g_t|_{X_s\cap X_t}$. Then the induced functions $f, g : X \to Y$ are close.*

Proof. Let $\mathcal{V} \in \mathcal{LSS}_Y$ be the uniformly bounded family given by uniform closeness and let $x \in X$. Then select a $s \in S$ so that $f_s(x)$ and $g_s(X)$ have x in their domains. By uniform closeness, there is a $V \in \mathcal{V}$ so that $f_s(x), g_s(x) \in V$. Hence $f \sim g$ as desired. □

Given how nicely asymptotic filtered colimits preserve finite asymptotic dimension, one might wonder if the asymptotic filtered colimit construction preserves asymptotic property C. It is not currently known if this is the case. However, you would need a more abstract notion of Asymptotic property C if you would like to keep the full generality of the construction. Such a generalized notion of asymptotic property C is given in the following paragraphs. Before we begin, we define asymptotic property C for metric spaces; this is the commonly used definition. For sake of clarity, we will call it asymptotic property C I. The more general definition will be called (temporarily) asymptotic property C II:

Definition 2.3.3. Let X be a metric space. We say X has **asymptotic property C I** or **APCI** if for any increasing sequence $0 < R_1 \leq R_2 \leq ...$ of real numbers there is a natural number n and uniformly bounded families $\mathcal{U}_1, ..., \mathcal{U}_n$ so that $\bigcup_{i=1}^{n} \mathcal{U}_i$ covers X and for all j, $1 \leq j \leq n$, $U, U' \in \mathcal{U}_j$ with $U \neq U'$, $d(U, U') \geq R_j$.

Definition 2.3.4. Let $(X, \mathcal{LSS})$ be a large scale structure. We say that $(X, \mathcal{LSS})$ has **asymptotic property C II** or APC II if for any sequence of uniformly bounded families $\mathcal{U}_1 \prec \mathcal{U}_2 \prec ...$ there is a natural number n and $\mathcal{V}_1, ..., \mathcal{V}_n \in \mathcal{LSS}$ so that $\bigcup_{i=1}^{n} \mathcal{V}_i$ covers X and for all j, $1 \leq j \leq n$, $V, V' \in \mathcal{V}_j$ with $V \neq V'$, $\mathrm{st}(V, \mathcal{U}_j) \cap V' = \varnothing$.

Proposition 2.3.5. *Let X be a metric space and let $\mathcal{LSS}$ be the large scale structure on X via $\mathcal{U} \in \mathcal{LSS}$ if and only if $\sup_{U \in \mathcal{U}} \mathrm{diam}(U) < \infty$. Then X has APCI if and only if $(X, \mathcal{LSS})$ has APCII.*

Proof. $(\Rightarrow)$: Let $\mathcal{U}_1 \prec \mathcal{U}_2 \prec ...$ be given where for all i, $\mathcal{U}_i \in \mathcal{LSS}$. Define $R_i = 1 + \sup_{U \in \mathcal{U}_i} \mathrm{diam}(U)$. Since for all i, $\mathcal{U}_i \prec \mathcal{U}_{i+1}$, we have a sequence $0 < R_1 \leq R_2 \leq ....$ Therefore, there is a natural number n and $\mathcal{V}_1, ..., \mathcal{V}_n \in \mathcal{LSS}$ so that $\bigcup_{i=1}^{n} \mathcal{V}_i$ covers X and for all j, $1 \leq j \leq n$, $V, V' \in \mathcal{V}_j$ with $V \neq V'$, $d(V, V') \geq R_j$. We show that for $1 \leq j \leq n$ and $V, V' \in \mathcal{V}_j$ with $V \neq V'$, $\mathrm{st}(V, \mathcal{U}_j) \cap V' = \varnothing$. For a contradiction, suppose not. Then there is a $y \in V'$ and a $U \in \mathcal{U}_j$ so that $y \in U$ and $U \cap V \neq \varnothing$. Let $x \in U \cap V$. Then $x, y \in U$ and $d(V, V') \leq d(x, y) \leq \mathrm{diam}(U) \leq \sup_{U \in \mathcal{U}_i} \mathrm{diam}(U) < R_j$ which is a contradiction. So $\mathrm{st}(V, \mathcal{U}_j) \cap V' = \varnothing$.

$(\Leftarrow)$: Let $0 < R_1 < R_2 < ...$ be given. Define $\mathcal{U}_i = \{B(x, R_i) | x\} \in X$, where $B(x, R_i)$ is the ball centered at x with radius R_i. Then we have a sequence of uniformly bounded families $\mathcal{U}_{R_1} \prec \mathcal{U}_{R_2} \prec ...$ which implies there are uniformly bounded families $\mathcal{V}_1, ..., \mathcal{V}_n$ so that $\bigcup_{i=1}^{n} \mathcal{V}_i$ covers X and for all j, $1 \leq j \leq n$, $V, V' \in \mathcal{V}_j$ with $V \neq V'$, $\mathrm{st}(V, \mathcal{U}_j) \cap V' = \varnothing$. We show that for $V \neq V' \in \mathcal{V}_j$, $d(V, V') \geq R_j$ for $1 \leq j \leq n$. Indeed, suppose (for contradiction) that $d(V, V') < R_j$. Then there are $x \in V$ and $y \in V'$ so that $d(x, y) < R_j$. Then there is a $U \in \mathcal{U}_j$ so that $x, y \in U$, namely $U = B(x, R_j)$. Hence $y \in V'$ and $y \in st(V, \mathcal{U}_j)$ which is a contradiction. So $d(V, V') \geq R_j$ as desired. □

From now on, we will call APC II just APC as their definitions coincide with the appropriate large scale structure. We now show that APC is preserved under coarse embeddings and that APC passes through subspaces (with the appropriate large scale structure on the subspace).

Proposition 2.3.6. *Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale structures so that $(X, \mathcal{LSS}_X)$ coarsely embedds into $(Y, \mathcal{LSS}_Y)$ via the function $f : X \to Y$. If $(Y, \mathcal{LSS}_Y)$ has APC, then $(X, \mathcal{LSS}_X)$ has APC.*

Proof. Let $\mathcal{U}_1 \prec \mathcal{U}_2 \prec ...$ and for all i, $\mathcal{U}_i \in \mathcal{LSS}_X$ be given. Then for all i, $f(\mathcal{U}_i) \in \mathcal{LSS}_Y$ and $f(\mathcal{U}_1) \prec f(\mathcal{U}_2) \prec ....$ Then there is a natural number n and $\mathcal{V}_1, ..., \mathcal{V}_n \in \mathcal{LSS}_Y$ so that $\bigcup_{i=1}^{n} \mathcal{V}_i$ covers Y and for all j, $1 \leq j \leq n$, $V, V' \in \mathcal{V}_j$ with $V \neq V'$, $\mathrm{st}(V, f(\mathcal{U}_j)) \cap V' = \varnothing$. Since f is a coarse embedding, we have that $f^{-1}(\mathcal{V}_i) \in \mathcal{LSS}_X$ for all $1 \leq i \leq n$ and clearly, $\bigcup_{i=1}^{n} f^{-1}(\mathcal{V}_i)$ covers X. We show that for $1 \leq j \leq n$ and $f^{-1}(V), f^{-1}(V') \in f^{-1}(\mathcal{V}_j)$ with $f^{-1}(V) \neq f^{-1}(V')$, $\mathrm{st}(f^{-1}(V), \mathcal{U}_j) \cap f^{-1}(V') = \varnothing$. For a contradiction, suppose not. Then there is a $x \in \mathrm{st}(f^{-1}(V), \mathcal{U}_j) \cap f^{-1}(V')$. Then $f(x) \in V'$ and there is a $U \in \mathcal{U}_j$ so that $U \cap f^{-1}(V) \neq \varnothing$ and $x \in U$. Let $y \in U \cap f^{-1}(V)$. Then $f(y) \in V$ and $f(x), f(y) \in f(U)$. But $f(U) \in f(\mathcal{U}_j)$ and we thus have that $f(x) \in \mathrm{st}(V, f(\mathcal{U}_j)) \cap V'$ which is a contradiction. So $(X, \mathcal{LSS}_X)$ has APC. □

Corollary 2.3.7. *Let $(X, \mathcal{LSS}_X)$ be a large scale structure and let $Y \subseteq X$. Let $\mathcal{LSS}_Y$ be so that $(Y, \mathcal{LSS}_Y)$ is a subspace large scale structure of $(X, \mathcal{LSS}_X)$. If $(X, \mathcal{LSS}_X)$ has APC, then $(Y, \mathcal{LSS}_Y)$ has APC.*

Proof. This follows from [**?**]. □

Corollary 2.3.8. *Let X be a set and let $\mathcal{LSS}$ be the asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$. If $\mathcal{LSS}$ has APC, then $(X_s, \mathcal{LSS}_s)$ has APC for any $s \in S$.*

Question 2.3.9. *Let X be a set and let $\mathcal{LSS}$ be the asymptotic filtered colimit of $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$. If for all $s \in S$, $(X_s, \mathcal{LSS}_s)$ has APC, then does $(X, \mathcal{LSS})$ has APC?*

Chapter 3

Asymptotic Products

The next construction that is considered in this text is the asymptotic product. Unlike the asymptotic filtered colimit, we create a new large scale structure from the product of large scale structures as opposed to a union of large scale structures.

3.1 Biproducts and the Asymptotic Product of Large Scale Spaces

Definition 3.1.1. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces; let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps. Define the **biproduct large scale structure** $\mathcal{LSS}_{X\times Y}$ over $X \times Y$ via $\mathcal{W} \in \mathcal{LSS}_{X\times Y}$ if and only if $\pi_1(\mathcal{W}) \in \mathcal{LSS}_X$ and $\pi_2(\mathcal{W}) \in \mathcal{LSS}_Y$.

Proposition 3.1.2. *The biproduct large scale structure* $(X \times Y, \mathcal{LSS}_{X\times Y})$ *on* $(X, \mathcal{LSS}_X)$ *and* $(Y, \mathcal{LSS}_Y)$ *is a large scale structure.*

Proof. Let $\mathcal{U} \in \mathcal{LSS}_{X\times Y}$ and let $\mathcal{V}$ be a family of subsets of $X \times Y$ so that for every $V \in \mathcal{V}$ with more than one point, there is a $U \in \mathcal{U}$ so that $V \subseteq U$. We show $\mathcal{V} \in \mathcal{LSS}_{X\times Y}$. Indeed, $V \subseteq U$ implies (for i=1 or 2) $\pi_i(V) \subseteq \pi_i(U)$. So $\pi_i(\mathcal{V}) \prec \pi_i(\mathcal{U}) \cup \{\pi_i(V) | \ |V| = 1\}$ and as $\mathcal{LSS}_X$ and $\mathcal{LSS}_Y$ are large scale structures, we have that $\pi_1(\mathcal{V}) \in \mathcal{LSS}_X$ and $\pi_2(\mathcal{V}) \in \mathcal{LSS}_Y$ hence $\mathcal{V} \in \mathcal{LSS}_{X\times Y}$.

Now suppose $\mathcal{U}, \mathcal{V} \in \mathcal{LSS}_{X\times Y}$ and consider $\mathrm{st}(\mathcal{U}, \mathcal{V})$ with $\mathrm{st}(U, \mathcal{V}) \in \mathrm{st}(\mathcal{U}, \mathcal{V})$. Then (for i=1 or 2) $\pi_i(\mathrm{st}(U,\mathcal{V})) = \pi_i\left(\bigcup_{\substack{V\in\mathcal{V}\\ V\cap U\neq\varnothing}} V\right) = \bigcup_{\substack{V\in\mathcal{V}\\ V\cap U\neq\varnothing}} \pi_i(V) = \bigcup_{\substack{\pi_i(V)\in\pi_i(\mathcal{V})\\ V\cap U\neq\varnothing}} \pi_i(V) \subseteq \bigcup_{\substack{\pi_i(V)\in\pi_i(\mathcal{V})\\ \pi_i(V)\cap\pi_i(U)\neq\varnothing}} \pi_i(V)$ since $U\cap V\neq\varnothing \Rightarrow \pi_i(U\cap V)\neq\varnothing$ and $\pi_i(U\cap V)\subseteq\pi_i(U)\cap\pi_i(V)$. This set is equal to $\mathrm{st}(\pi_i(U),\pi_i(\mathcal{V}))$ and as U as arbitrary, we have that $\pi_i(\mathrm{st}(\mathcal{U},\mathcal{V}))$ is a refinement of $\mathrm{st}(\pi_i(\mathcal{U}),\pi_i(\mathcal{V}))\in\mathcal{LSS}_{X\times Y}$ hence $\pi_i(\mathrm{st}(\mathcal{U},\mathcal{V}))\in\mathcal{LSS}_{X\times Y}$. □

We now show that the biproduct construction "commutes" with the asymptotic filtered colimit construction with the condition that there aren't one point sets.

Proposition 3.1.3. *Let X and Y be sets and S be an index set. Let $\mathcal{LSS}_*^X$ be the asymptotic filtered colimit with respect to $\left\{(X_s,\mathcal{LSS}_s^X)\right\}_{s\in S}$ and let $\mathcal{LSS}_*^Y$ be the asymptotic filtered colimit with respect to $\left\{(Y_s,\mathcal{LSS}_s^Y)\right\}_{s\in S}$. Suppose that any $\mathcal{U}\in\mathcal{LSS}_*^X\cup\mathcal{LSS}_*^Y$ does not contain a one point set. Let $\mathcal{LSS}_*$ be the asymptotic filtered colimit of $\left\{(X_s\times Y_s,\mathcal{LSS}_s^{X\times Y})\right\}_{s\in S}$ where $\mathcal{LSS}_s^{X\times Y}$ is the biproduct of $\mathcal{LSS}_s^X$ and $\mathcal{LSS}_s^Y$ for a fixed $s\in S$. Let $\mathcal{LSS}^{X\times Y}$ be the biproduct of $\mathcal{LSS}_*^X$ and $\mathcal{LSS}_*^Y$. Then $\mathcal{LSS}^{X\times Y}=\mathcal{LSS}_*$*

Proof. $(\subseteq)$:Let $\mathcal{U}\in\mathcal{LSS}^{X\times Y}$. Then $\pi_1(\mathcal{U})\in\mathcal{LSS}_*^X$ and $\pi_2(\mathcal{U})\in\mathcal{LSS}_*^Y$. Then there are $s_1,s_2\in S$ so that $\pi_1(\mathcal{U})\in\mathcal{LSS}_{s_1}^X$ and $\pi_2(\mathcal{U})\in\mathcal{LSS}_{s_2}^Y$. Choose $s\in S$ so that $(X_{s_1}\times Y_{s_1})\cup(X_{s_2}\times Y_{s_2})\subseteq X_s\times Y_s$. Then $\pi_1(\mathcal{U})\in\mathcal{LSS}_s^X$ and $\pi_2(\mathcal{U})\in\mathcal{LSS}_s^Y$ hence $\mathcal{U}\in\mathcal{LSS}_s^{X\times Y}$. Since $\mathcal{LSS}_s^{X\times Y}\subseteq\mathcal{LSS}_*$ for any s, we have that $\mathcal{U}\in\mathcal{LSS}_*$.
$(\supseteq)$: Let $\mathcal{U}\in\mathcal{LSS}_*$. Then we have that for some s, $\mathcal{U}\in\mathcal{LSS}_s^{X\times Y}$. We do not have to remove one point sets since both $\mathcal{LSS}_*^X$ and $\mathcal{LSS}_*^Y$ lack families that have one point sets in them. So for some $s\in S$, $\mathcal{U}\in\mathcal{LSS}_s^{X\times Y}$ which implies $\pi_1(\mathcal{U})\in\mathcal{LSS}_s^X$ and $\pi_2(\mathcal{U})\in\mathcal{LSS}_s^Y$. Since $\mathcal{LSS}_s^X\subseteq\mathcal{LSS}_*^X$ and $\mathcal{LSS}_s^Y\subseteq\mathcal{LSS}_*^Y$, we have that $\pi_1(\mathcal{U})\in\mathcal{LSS}_*^X$ and $\pi_2(\mathcal{U})\in\mathcal{LSS}_*^Y$. But by definition this means that $\mathcal{U}\in\mathcal{LSS}^{X\times Y}$. □

The previous proposition implies that if you have large scale structures with one point sets and wish to "commute" the asymptotic filtered colimit construction with the biproduct construction, you can remove one point sets, commute the construction, and then add back in one point sets.

Definition 3.1.4. Let $\{(X_s,\mathcal{LSS}_s)\}_{s\in S}$ be a set of large scale spaces. Define a large scale structure $\mathcal{LSS}$ on $\prod_{s\in S} X_s$ via $\mathcal{U}\in\mathcal{LSS}$ provided $\pi_s(\mathcal{U})\in\mathcal{LSS}_s$ for every $s\in S$ and $\exists E\subseteq S$

so that E is finite in cardinality and $\forall U \in \mathcal{U}$ and $s \in S \setminus E$, $|\pi_s(U)| = 1$. We call $\mathcal{LSS}$ the **asymptotic product** of the large scale spaces $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$.

Proposition 3.1.5. *Let $\mathcal{LSS}$ be an asymptotic product of $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$. Then $\mathcal{LSS}$ is a large scale structure for $\prod\limits_{s \in S} X_s$.*

Proof. Let $\mathcal{U} \in \mathcal{LSS}$ and let $\mathcal{V}$ be a family of subsets of $\prod\limits_{s \in S} X_s$ such that $V \in \mathcal{V}$ with $|V| > 1$ implies there is a $U \in \mathcal{U}$ with $V \subseteq U$. Since $\mathcal{U} \in \mathcal{LSS}$, there is a finite subset E of S so that for any $U \in \mathcal{U}$ and $s \in S \setminus E$ we have $|\pi_s(U)| = 1$. Suppose $V \in \mathcal{V}$ and $|V| = 1$. Then clearly $|\pi_s(V)| = 1 \; \forall s \in S \setminus E$. If $|V| > 1$, then there is a $U \in \mathcal{U}$ such that $V \subseteq U$ and hence (for $s \in S$) $\pi_s(V) \subseteq \pi_s(U)$ which implies $|\pi_s(V)| = 1$. So $\mathcal{V} \in \mathcal{LSS}$.
Let $\mathcal{U}, \mathcal{V} \in \mathcal{LSS}$. Then there are finite subsets of S, E and F, so that for any $U \in \mathcal{U}$ and $s \in S \setminus E$, $|\pi_s(U)| = 1$; also, for any $V \in \mathcal{V}$ and $s \in S \setminus F$, $|\pi_s(V)| = 1$. We show $\operatorname{st}(\mathcal{U}, \mathcal{V}) \in \mathcal{LSS}$. Let $\operatorname{st}(U, \mathcal{V}) \in \operatorname{st}(\mathcal{U}, \mathcal{V})$. Then for any $s \in S$, we have

$$\pi_s(\operatorname{st}(U), \mathcal{V}) = \pi_s \left(\bigcup_{\substack{V \in \mathcal{V} \\ V \cap U \neq \varnothing}} V \right) = \bigcup_{\substack{V \in \mathcal{V} \\ V \cap U \neq \varnothing}} \pi_s(V) \subseteq \bigcup_{\substack{\pi_s(V) \in \pi_s(\mathcal{V}) \\ \pi_s(V) \cap \pi_s(U) \neq \varnothing}} \pi_s(V) = \operatorname{st}(\pi_s(U), \pi_s(\mathcal{V})) \in \mathcal{LSS}_s$$

The last subset above holds as $V \in \mathcal{V}$ imples $\pi_s(V) \in \pi_s(\mathcal{V})$ and that $U \cap V \neq \varnothing$ implies $\pi_s(U) \cap \pi_s(V) \neq \varnothing$. Now let $s \in S \setminus (E \cup F)$. Then $|\pi_s(U)| = |\pi_s(V)| = 1$ for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since we've shown $\operatorname{st}(U, \mathcal{V}) \subseteq \operatorname{st}(\pi_s(U), \pi_s(\mathcal{V}))$ and $|\operatorname{st}(\pi_s(U), \pi_s(\mathcal{V}))| \leq 1$, then $|\operatorname{st}(U, \mathcal{V})| \leq 1$. So $\operatorname{st}(\mathcal{U}, \mathcal{V}) \in \mathcal{LSS}$ and $\mathcal{LSS}$ is a large scale structure. □

Definition 3.1.6. Let $(X, \mathcal{LSS})$ be given. A **coarse component** of X is an equivalence class of the following equivalence relation: $x \sim y$ if and only if there is a $\mathcal{U} \in \mathcal{LSS}$ and a $U \in \mathcal{U}$ so that $x, y \in U$.

Definition 3.1.7. Let $(X, \mathcal{LSS})$ be given and let $\{C_i\}_{i \in I}$ be the coarse components of X. Let $\{Y_s\}_{s \in S}$ range over all possible finite unions of coarse components of X. The **coarse component large scale structure of $\mathcal{LSS}$**, denoted $\mathcal{LSS}_c$, is defined via $\mathcal{U} \in \mathcal{LSS}_c$ if and only if for any $s \in S$, $\mathcal{U}|_{Y_s} = \{U \cap Y_s | U \in \mathcal{U}\} \in \mathcal{LSS}$ and there exists a $s_0 \in S$ so that for any $U \in \mathcal{U}$ with $|U| > 1$, $U \subseteq Y_{s_0}$.

To show that this is indeed a large scale structure, we will show that $\mathcal{LSS}_c$ is a special type of asymptotic filtered colimit.

Proposition 3.1.8. *Let $(X, \mathcal{LSS})$ be a large scale structure and let $\{C_i\}_{i\in I}$, $\{Y_s\}_{s\in S}$, and $\mathcal{LSS}_c$ be as in the above definition. For any $s \in S$, let $\mathcal{LSS}_s$ be a large scale structure on Y_s via $\mathcal{U} \in \mathcal{LSS}_s$ if and only if for every $U \in \mathcal{U}$, $U \subseteq Y_s$ and $\mathcal{U} \in \mathcal{LSS}$. Let $\mathcal{LSS}_{afc}$ be the asymptotic filtered colimit of $\{(Y_s, \mathcal{LSS}_s)\}_{s\in S}$. Then $\mathcal{LSS}_c = \mathcal{LSS}_{afc}$.*

Proof. We first show that $\{Y_s\}_{s\in S}$ behave in the appropriate way per 2.1.1. Note that by construction, $X = \bigcup_{s\in S} Y_s$ since $X = \bigcup_{i\in I} C_i$. Let $r, s \in S$. Since $Y_r \cup Y_s$ is a finite union of coarse components, there is a $t \in S$ so that $Y_r \cup Y_s \subseteq Y_t$ (in fact, they're equal). Also, $\mathcal{LSS}_r|_{Y_s} = \mathcal{LSS}_s|_{Y_r}$ since either the intersection is empty (so $\mathcal{LSS}_r|_{Y_s} = \mathcal{LSS}_s|_{Y_r}$ trivially) or the intersection is another finite union of coarse components, call it Y_l. Then we have that $\mathcal{LSS}_r|_{Y_s} = \mathcal{LSS}_l = \mathcal{LSS}_s|_{Y_r}$. So $\mathcal{LSS}_{afc}$ is well-defined. We now show $\mathcal{LSS}_c = \mathcal{LSS}_{afc}$.
$(\subseteq)$: Let $\mathcal{U} \in \mathcal{LSS}_c$. Then there is a $s_0 \in S$ so that for any $U \in \mathcal{U}$ with $|U| > 1$, we have $U \subseteq Y_{s_0}$ and for any $s \in S$, $\mathcal{U}|_{Y_s} \in \mathcal{LSS}$. So $\mathcal{U}|_{Y_{s_0}} \in \mathcal{LSS}$ and hence $\mathcal{U} \in \mathcal{LSS}_{afc}$ as $\mathcal{U}^* \in \mathcal{LSS}_{s_0}$.
$(\supseteq)$: Let $\mathcal{U} \in \mathcal{LSS}_{afc}$. Then there is an $s_0 \in S$ so that $\mathcal{U}^* \in \mathcal{LSS}_{s_0}$. Note $\mathcal{U}^* = \mathcal{U}|_{Y_{s_0}}$ and that since $\mathcal{LSS}_{s_0} \subseteq \mathcal{LSS}$, we have that $\mathcal{U}|_{Y_{s_0}} \in \mathcal{LSS}$. Let $Y_{s_0} = \bigcup_{i=1}^{n} C_i$ and let $s \in S$. We will show that $\mathcal{U}|_{Y_s} \in \mathcal{LSS}$. Indeed, if $Y_s \cap Y_{s_0} = \varnothing$, then $\mathcal{U}|_{Y_s}$ is comprised of one point sets or is empty. In either case, $\mathcal{U}|_{Y_s} \in \mathcal{LSS}$. If $Y_s \cap Y_{s_0} \neq \varnothing$, then there is a natural number $m \leq n$ so that $Y_s \cap Y_{s_0} = \bigcup_{j=0}^{m} C_{i_j}$, where $C_{i_j} \in \{C_1, ..., C_n\}$ for $1 \leq j \leq m$. Note $\mathcal{U}$ consists of one point sets outside of $Y_s \cap Y_{s_0}$. Then we have that $\mathcal{U}^*|_{Y_s} = \mathcal{U}^*|_{Y_s \cap Y_{s_0}} \prec \mathcal{U}|_{Y_{s_0}} \in \mathcal{LSS}$ hence $\mathcal{U}^*|_{Y_s} \in \mathcal{LSS}$ and since $\mathcal{U}$ consists of one point sets outside of $Y_s \cap Y_{s_0}$, we have that $\mathcal{U}|_{Y_s} \in \mathcal{LSS}$. So $\mathcal{U} \in \mathcal{LSS}_c$. □

From the above proposition, we find:

Corollary 3.1.9. *Let $(X, \mathcal{LSS})$ be a large scale structure and let $\{C_i\}_{i\in I}$, $\{Y_s\}_{s\in S}$, and $\mathcal{LSS}_c$ be as in the above definition. Then $\mathcal{LSS}_c$ is a large scale structure on X. Furthermore, let $(Y, \mathcal{LSS}_Y)$ be a large scale structure and $f : (X, \mathcal{LSS}_c) \to (Y, \mathcal{LSS}_Y)$. Then f is bornologous if and only if $f|_C$ is bornologous for every coarse component C of X.*

Question 3.1.10. *Is every coarse component of an asymptotic product an asymptotic filtered colimit?*

3.2 Asymptotic Products of Metric Spaces

Definition 3.2.1. Suppose D is an infinite countable set. A function $\alpha : D \to [0, \infty)$ **has limit infinity at infinity** if for each $M > 0$ there is a finite subset E of D such that $\alpha(d) > M$ for all $d \in D \setminus E$.

Proposition 3.2.2. $\alpha : D \to [0, \infty)$ *has limit infinity at infinity if and only if its extension* $\bar{\alpha} : \omega(D) \to \omega((0, \infty))$ *defined by* $\bar{\alpha}(\infty) = \infty$ *is continuous. Here* $\omega(D)$ *is the one-point compactification of* D *equipped with the discrete topology.*

Proof. First note that as D is a countably infinite space with the discrete topology, D is locally compact hence $\omega(D)$ exists. Also, α is continuous as D has the discrete topology.
$(\Rightarrow)$: Since α is continuous it suffices to show that $\overline{\alpha}$ is continuous at ∞. Let U be a closed set of $\omega((0, \infty))$ which contains ∞. Then the complement of U, U^c is open and doesn't contain ∞ hence $\overline{\alpha}^{-1}(U^c) = \alpha^{-1}(U^c)$ is open and doesn't contain ∞ by continuity of α and $\overline{\alpha}(\infty) = \infty$. Since $\alpha^{-1}(U^c)$ is open in D, $\alpha^{-1}(U^c)$ is open in $\omega(D)$ which means that $\left(\overline{\alpha}^{-1}(U^c)\right)^c$ is closed in $\omega(D)$. We show that $\left(\overline{\alpha}^{-1}(U^c)\right)^c = \overline{\alpha}^{-1}(U)$ which proves the left implication. This amounts to showing $\overline{\alpha}^{-1}(U^c) = \left(\overline{\alpha}^{-1}(U)\right)^c$. Let $x \in \overline{\alpha}^{-1}(U^c)$. Then $\overline{\alpha}(x) \in U^c$ iff $\overline{\alpha}(x) \notin U$ iff $x \notin \overline{\alpha}^{-1}(U)$ iff $x \in \left(\overline{\alpha}^{-1}(U)\right)^c$. So we have equality which means that $\overline{\alpha}^{-1}(U)$ is closed. Therefore $\overline{\alpha}$ is continuous.
$(\Leftarrow)$: Suppose $\overline{\alpha}$ is continuous and choose $M > 0$. Let $U = (M, \infty] \subseteq \omega((0, \infty))$. Then $\overline{\alpha}^{-1}(U)$ is open in $\omega(D)$ and contains ∞ hence its complement is compact. Note that $\left(\overline{\alpha}^{-1}(U)\right)^c = \{d \in D | \alpha(d) \leq M\} \subseteq D$. Since this set is compact and D has the discrete topology, the set is finite. Hence α has limit ∞ at ∞. □

As an aside, suppose D is a set and X_d is a set for each $d \in D$. One may view the cartesian product $\prod_{d \in D} X_d$ as the set of functions $f : D \to \bigcup_{d \in D} X_d$ such that $f(d) \in X_d$ for each $d \in D$.

Definition 3.2.3. Suppose D is an infinite countable set and a function $\alpha : D \to (0, \infty)$ has limit infinity at infinity. Given ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$, the **asymptotic product** $(\prod\limits_{d \in D} X_d, \rho_d)$ is the cartesian product $\prod\limits_{d \in D} X_d$ equipped with the ∞-metric ρ_α defined as follows:
Given $u, v \in \prod\limits_{d \in D} X_d$, $\rho_\alpha(u, v)$ is the sum $\sum\limits_{d \in D} r_d$, where r_d is equal to $\alpha(d)$ if $0 \leq \rho_d(u(d), v(d)) \leq \alpha(d)$ provided $u(d) \neq v(d)$, $r_d = \rho_d(u(d), v(d))$ if $\rho_d(u(d), v(d)) > \alpha(d)$, and $r_d = 0$ if and only if $u(d) = v(d)$. We note here that r_d depends on the choice of u and v.

Proposition 3.2.4. *ρ_α is an ∞-metric.*

Proof. It is clear that $\rho_\alpha(u, v) \geq 0$ and $\rho_\alpha(u, u) = 0$. Suppose $\rho_\alpha(u, v) = 0$. Then we have that $r_d = 0 \ \forall d \in D$. Since α can't take on the value of zero, we have that $r_d = 0 \Rightarrow u(d) = v(d)$. Therefore, $u(d) = v(d) \forall d \in D$ so $u = v$. We now show the triangle inequality: For any $d \in D$ and $u, v, w \in \prod\limits_{d \in D} X_d, \rho_d$ we have $\rho_d(u(d), v(d)) \leq \rho_d(u(d), w(d)) + \rho_d(w(d), v(d))$. Let s_d be $\alpha(d)$ for $0 < \rho_d(u(d), w(d)) \leq \alpha(d)$, $s_d = \rho_d(u(d), w(d))$ if $\rho_d(u(d), w(d)) > \alpha(d)$, and $s_d = 0$ if $\rho_d(u(d), w(d)) = 0$.
Also, let t_d be $\alpha(d)$ for $0 < \rho_d(v(d), w(d)) \leq \alpha(d)$,$t_d = \rho_d(v(d), w(d))$ if $\rho_d(v(d), w(d)) > \alpha(d)$, and $t_d = 0$ if $\rho_d(v(d), w(d)) = 0$. By definition, for any $d \in D$ we have $0 \leq r_d \leq s_d + t_d$. Now, we have
$\rho_\alpha(u, v) = \sum\limits_{d \in D} r_d \leq \sum\limits_{d \in D} (s_d + t_d) \leq \sum\limits_{d \in D} s_d + \sum\limits_{d \in D} t_d = \rho_\alpha(u, w) + \rho_\alpha(w, v).$ □

Definition 3.2.5. Suppose $(\prod\limits_{d \in D} X_d, \rho_\alpha)$ is an asymptotic product of ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$. Given $c \in D$ by $e_c : (\prod\limits_{d \in D} X_d, \rho_\alpha) \to X_c$ we denote the **evaluation function** defined by

$$e_c(f) = f(c).$$

Traditionally, such functions are called **projections**.

We note here that D will be assumed to have the discrete topology from now on.

Proposition 3.2.6. *Suppose $(\prod\limits_{d \in D} X_d, \rho_\alpha)$ is an asymptotic product of ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$ and $f : X \to (\prod\limits_{d \in D} X_d, \rho_\alpha)$ is a function from an ∞-pseudo-metric space X. f is bornologous if and only if $e_d \circ f : X \to X_d$ is bornologous for each $d \in D$*

and whenever $x, y \in X$ have finite distance, then there is a finite subset E of D such that $e_d(f(x)) = e_d(f(y))$ for all $d \in D \setminus E$.

Proof. Let the metric on X be called ρ_X.
$(\Rightarrow)$: Let $M > 0$. Then $\exists N > 0$ such that $\rho_X(x, y) < M \Rightarrow \rho_\alpha(f(x), f(y)) < N$. Then for $d \in D$, $\rho_d(e_d(f(x)), e_d(f(y))) \leq r_d \leq \sum_{d \in D} r_d \leq \rho_\alpha(f(x), f(y)) < N$. So $e_d \circ f$ is bornologous. Now, note that D discrete $\Rightarrow \alpha$ has limit ∞ at ∞ hence $\exists E \subseteq D$, E finite, such that $\alpha(d) > N \; \forall d \in D \setminus E$. Since $\rho_\alpha(f(x), f(y)) < N$, we have $r_d \nleq \alpha(d)$ lest $\rho_\alpha(f(x), f(y)) \geq N$. Then by definition of r_d, we must have $r_d = 0$ which is to say $\forall d \in D \setminus E,\; e_d(f(x)) = e_d(f(y))$. $(\Leftarrow)$: Let $M > 0$. Then $\rho_X(x, y) < M < \infty \Rightarrow \exists E \subseteq D$, E finite, such that $\forall d \in D \setminus E,\; \rho_\alpha(e_d(f(x)), e_d(f(y))) = 0$. This is since $e_d(f(x)) = e_d(f(y)) \; \forall d \in D \setminus E$. Since $e_d \circ f$ is bornologous $\forall d \in D$, $\forall a \in E \; \exists N_a > 0$ such that
$\rho_\alpha(e_a(f(x)), e_a(f(y))) < N_a$. Choose each N_a so that $N_a > \alpha(a)$. Then we have $\forall a \in E,\; r_a < N_a$. Define $N = 1 + \sum_{a \in E} N_a$. Then $\rho_\alpha(f(x), f(y)) = \sum_{d \in D} r_d = \sum_{a \in E} r_a < N$. Note that $N < \infty$ since E is finite in cardinality. □

Corollary 3.2.7. *Given two asymptotic products $(\prod_{d \in D} X_d, \rho_\alpha)$ and $(\prod_{d \in D} X_d, \rho_\beta)$ of ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$, the identity function between them is a coarse equivalence.*

Proof. Let f be the identity function. Since f is close to itself, we need only to show that f and f^{-1} are boronologous. Let $M > 0$. Then $e_d \circ f$ and $e_d \circ f^{-1}$ are clearly bornologous with choice $N = M$. We now show that if $\rho_\alpha(x, y) < \infty$, $\exists E \subseteq D$, E finite such that
$\forall d \in D \setminus E,\; e_d(f(x)) = e_d(f(y))$ (which is equivalent to saying that $e_d(x) = e_d(y)$ or $e_d(f^{-1}(x)) = e_d(f^{-1}(y))$). Indeed,we show this statement for the function f^{-1} whose image has the metric ρ_α. For the case of f, the proof is similar; replace every α with β. Let $E = \{d \in D | r_d > 0\}$. Then $\rho_\alpha(x, y) < \infty \Rightarrow \sum_{d \in D} r_d < \infty$ which implies that E is finite or E is infinite with limit zero at ∞. Assume (for contradiction) the latter case. Then note that E is discrete hence α has limit ∞ at ∞ by 3.2.2 and that $\forall d \in E,\; \alpha(d) \leq r_d$.
Since r_d has limit zero at ∞, we have α has limit zero at ∞. This contradicts 3.2.2 and we thus have that E is finite. In other words, $\forall d \in D \setminus E$, $r_d = 0$ which implies that $\forall d \in D \setminus E$, $e_d(x) = e_d(y)$. By 3.2.6, we have that f and f^{-1} are bornologous and hence f is a coarse equivalence. □

Theorem 3.2.8. *Given two asymptotic products* $(\prod_{d\in D} X_d, \rho_\alpha)$ *and* $(\prod_{d\in D} Y_d, \rho_\beta)$ *of* ∞*-pseudo-metric spaces* (X_d, ρ_d) *and* (Y_d, ρ_d), $d \in D$, *the function between them,* f, *induced by injective coarse embeddings* $f_d : X_d \to Y_d$ *is a coarse embedding.*

Proof. We first note that f being induced by the f_d's means that for any $d \in D$ $f|_{X_d} \equiv f_d$. It follows that if f_d is injective $\forall d \in D$, then f is injective. Furthermore, by construction the function f induced by the f_d's is unique. One can show this by checking the coordinates $e_d(u)$ for all $d \in D$. Another important fact we will make use of is that for any $d \in D$ and $u \in \prod_{d\in D} X_d$, $e_d \circ f(u) = f_d \circ e_d(u)$. This is because each f_d is a coordinate function of f by construction.

We begin the proof by showing the result for when $f_d \equiv id_{X_d}$ for every d. By uniqueness, we have that the induced function f is the identity on the asymptotic product. To prove the result for this case, we use 3.2.6. By hypothesis, each f_d is bornologous. Now suppose $u, v \in \prod_{d\in D} X_d$ and $\rho_\alpha(u, v) < \infty$. Then by the logic used in 3.2.7, $\exists E \subseteq D$ finite such that $\forall d \in D\backslash E$, $e_d(u) = e_d(v)$ which implies (as f is the identity function) $\forall d \in D\backslash E$, $e_d \circ f(u) = e_d \circ f(v)$. So f is bornologous. Clearly, a function g exists so that $f \circ g$ and $g \circ f$ are close to their respective identities (namely, f). So we have that f is a coarse equivalence.

Now, let f be as in the statement of the theorem. By use of 3.2.7, we may assume the metric on $\prod_{d\in D} X_d$ to be ρ_α as opposed to ρ_β. So $f : \prod_{d\in D} X_d \to \prod_{d\in D} Y_d$. Define a new metric on X_d, $\psi_d : X_d \times X_d \to [0, \infty)$ via $\psi_d(a, b) = \rho_d(f_d(a), f_d(b))$. Using the ψ_d's, one may define a new metric on $\prod_{d\in D} X_d$ called ψ_α as follows: $\psi_\alpha(u, v) = \sum_{d\in D} s_d$, where

$$s_d = \begin{cases} 0 & e_d(u) = e_d(v) \\ \alpha(d) & e_d(u) \neq e_d(v),\ \psi_d(e_d(u), e_d(v)) \leq \alpha(d) \\ \psi_d(e_d(u), e_d(v)) & \psi_d(e_d(u), e_d(v)) > \alpha(d) \end{cases}$$

By use of the special case, we have the identity map $id : \left(\prod_{d\in D} X_d, \rho_\alpha\right) \to \left(\prod_{d\in D} X_d, \psi_\alpha\right)$ is a coarse embedding. Consider $f : \left(\prod_{d\in D} X_d, \psi_\alpha\right) \to \left(\prod_{d\in D} Y_d, \rho_\alpha\right)$. We claim that f under these metrics is an isometric embedding (and hence a coarse embedding by 1.4.3). Since the composition of coarse embeddings is a coarse embeddings, we would be done. It remains to be shown that the claim is indeed true. Note here that the inherited metric of the image of f is of the form: $\rho_\alpha(f(u), f(v)) = \sum\limits_{d\in D} r_d$, where

$$r_d = \begin{cases} 0 & e_d \circ f(u) = e_d \circ f(v) \\ \alpha(d) & e_d \circ f(u) \neq e_d \circ f(v),\ \rho_d(e_d \circ f(u), e_d \circ f(v)) \leq \alpha(d) \\ \rho_d(e_d \circ f(u), e_d \circ f(v)) & \rho_d(e_d \circ f(u), e_d \circ f(v)) > \alpha(d) \end{cases}$$

To show that f is an isometric embedding, it suffices to show that for any $d \in D,\ r_d = s_d$. We note here that $e_d \circ f(u) = e_d \circ f(v)$ if and only if $f_d \circ e_d(u) = f_d \circ e_d(u)$ if and only if (by injectivity of each f_d) $e_d(u) = e_d(v)$. We will make use of this shortly. Now,

$$\begin{aligned} r_d &= \begin{cases} 0 & e_d \circ f(u) = e_d \circ f(v) \\ \alpha(d) & e_d \circ f(u) \neq e_d \circ f(v),\ \rho_d(e_d \circ f(u), e_d \circ f(v)) \leq \alpha(d) \\ \rho_d(e_d \circ f(u), e_d \circ f(v)) & \rho_d(e_d \circ f(u), e_d \circ f(v)) > \alpha(d) \end{cases} \\ &= \begin{cases} 0 & e_d(u) = e_d(v) \\ \alpha(d) & e_d(u) \neq e_d(v),\ \rho_d(f_d \circ e_d(u), f_d \circ e_d(v)) \leq \alpha(d) \\ \rho_d(f_d \circ e_d(u), f_d \circ e_d(v)) & \rho_d(f_d \circ e_d(u), f_d \circ e_d(v)) > \alpha(d) \end{cases} \\ &= \begin{cases} 0 & e_d(u) = e_d(v) \\ \alpha(d) & e_d(u) \neq e_d(v),\ \psi_d(e_d(u), e_d(v)) \leq \alpha(d) \\ \psi_d(e_d(u), e_d(v)) & \psi_d(e_d(u), e_d(v)) > \alpha(d) \end{cases} \\ &= s_d \end{aligned}$$

□

Proposition 3.2.9. *Suppose D is an infinite countable set and a function $\alpha : D \to (0, \infty)$ has limit infinity at infinity. Given the ∞-pseudo-metric space (X_d, ρ_d), $d \in D$, the asymptotic product $(\prod\limits_{d\in D} X_d, \rho_\alpha)$ is coarsely equivalent to the cartesian product $\prod\limits_{d\in D} X_d$ equipped with the ∞-metric ϕ defined as follows:*
Given $u, v \in \prod\limits_{d\in D} X_d,\ \phi(u, v) = \sup\limits_{d\in D} r_d,$ where
1. r_d is equal to $\alpha(d)$ if $0 \leq \rho_d(u(d), v(d)) \leq \alpha(d)$ and $u(d) \neq v(d)$,

2. $r_d = \rho_d(u(d), v(d))$ *if* $\rho_d(u(d), v(d)) > \alpha(d)$,

3. $r_d = 0$ *if* $u(d) = v(d)$.

Proof. Let $\{(u_n, v_n)\}_{n=1}^{\infty}$ be a sequence of points in $\prod_{d \in D} X_d \times \prod_{d \in D} X_d$. By 1.4.6 it suffices to show that $\rho_\alpha(u_n, v_n)$ diverges if and only if $\phi(u_d, v_d)$ diverges.

Define r_d^n as follows: $r_d^n = \alpha(d)$ if $0 \leq \rho_d(u_n(d), v_n(d)) \leq \alpha(d)$ and $u_n(d) \neq v_n(d)$,

$r_d^n = \rho_d(u_n(d), v_n(d))$ if $\rho_d(u_n(d), v_n(d)) > \alpha(d)$,

$r_d^n = 0$ if $u_n(d) = v_n(d)$.

$(\Rightarrow)$: Let n_k be a convergent subsequence of $\{\phi(u_n, v_n)\}_{n=1}^{\infty}$ (by convergent, we mean that the value the subsequence converges to is an element of $[0, \infty]$) and suppose for contradiction that $\lim_{k \to \infty} \phi(u_{n_k}, v_{n_k}) \neq \infty$. Then there is an $L > 0$ so that $\phi(u_{n_k}, v_{n_k}) < L$ for $n_k \geq N_1$ for some number N_1. By definition of $\phi(u_n, v_n)$, we have $\forall n_k \geq N_1$, $r_d^{n_k} < L$. Since α has limit ∞ at ∞, $\exists E \subseteq D$ such that E is finite in cardinality and $\alpha(d) > L$ $\forall d \in D \setminus E$. Using this, we have that $\forall n_k \geq N_1$, $d \in D \setminus E$, $r_d^{n_k} = 0$. This is because $r_d^{n_k} \not\geq \alpha(d)$ lest $r_d^{n_k} > L$. Since $\rho_\alpha(u_n, v_n)$ diverges, $\rho_\alpha(u_{n_k}, v_{n_k})$ diverges which implies $\exists N$ so that $n_k \geq N$ implies $\rho_\alpha(u_{n_k}, v_{n_k}) = \sum_{d \in D} r_d^{n_k} > L \cdot |E|$. Choose $N > N_1$ if need be. Now, $n_k \geq N$ implies $L \cdot |E| < \sum_{d \in D} r_d^{n_k} = \sum_{d \in E} r_d^{n_k} \leq |E| \cdot \sup_{d \in E} r_d^{n_k} = |E| \cdot \sup_{d \in D} r_d^{n_k}$ and hence $|E| \cdot L < |E| \cdot \sup_{d \in D} r_d^{n_k}$ which implies that $L < \sup_{d \in D} r_d^{n_k} = \phi(u_{n_k}, v_{n_k})$ which is a contradiction. So we must have that $\lim_{k \to \infty} \phi(u_{n_k}, v_{n_k}) = \infty$. In particular, the limit infimum of the sequence $\{\phi(u_n, v_n)\}_{n=1}^{\infty}$ must be ∞ hence $\{\phi(u_n, v_n)\}_{n=1}^{\infty}$ must diverge.

$(\Leftarrow)$: Let $M > 0$ be given. Then $\exists N$ so that $n \geq N$ implies $\phi(u_n, v_n) = \sup_{d \in D} r_d^n > M$ which implies (for $n \geq N$) $\sum_{d \in D} r_d^n > M$. Hence, $\lim_{n \to \infty} \rho_\alpha(u_n, v_n) = \infty$. Indeed, the contrapositive of the previous implication is true. Suppose $\sum_{d \in D} r_d^n \leq M$. Then $r_d^n \leq \sum_{d \in D} r_d^n \leq M$ which implies $\sup_{d \in D} r_d^n \leq M$. □

As an ending aside, one may wonder whether the metric space rendition of the asymptotic product is the "correct" generalization of the large scale rendition of the asymptotic product. Below is a proposition that answers this question in the affirmative.

Proposition 3.2.10. *Let S be a countable set and for all $s \in S$ let (X_s, ϕ_s) be an ∞-psuedometric space. Let $\alpha : S \to (0, \infty)$ be so that α has limit ∞ at ∞. Let ϕ_α be the asymptotic product metric on $X = \prod_{s \in S} X_s$ as in the previous proposition. Let $\mathcal{LSS}_m$ be* *a large scale structure on X defined via $\mathcal{U} \in \mathcal{LSS}_m$ if and only if* $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$. *Likewise, define $\mathcal{LSS}_s$ on X_s via $\mathcal{U}_s \in \mathcal{LSS}_s$ if and only if* $\sup_{U_s \in \mathcal{U}_s} \operatorname{diam}(U_s) < \infty$. *Let $\mathcal{LSS}$ be the asymptotic product large scale structure of $\{(X_s, \mathcal{LSS}_s)\}_{s \in S}$. Then $\mathcal{LSS}_m = \mathcal{LSS}$.*

Proof. $(\subseteq)$: Let $\mathcal{U} \in \mathcal{LSS}$ and choose $F \subseteq S$ so that for any $s \in S \setminus F$ and $\mathcal{U} \in \mathcal{LSS}$ with $U \in \mathcal{U}$, $|\pi_s(U)| = 1$. So we know that for any $s \in S \setminus F$, $r_s = 0$ (for r_s as defined in the previous proposition). Furthermore, $\pi_s(\mathcal{U}) \in \mathcal{LSS}_s$ for all $s \in S$ implies that for all s, $\sup_{\pi_s(U) \in \pi_s(\mathcal{U})} \operatorname{diam}(\pi_s(U)) = R_s < \infty$. Now let $U \in \mathcal{U}$. Then: $\operatorname{diam}(U) = \sup_{x,y \in U} \sup_{s \in S} r_s(x,y) = \sup_{x,y \in U} \sup_{s \in F} r_s(x,y) = \sup_{s \in F} \sup_{x,y \in U} r_s(x,y) \leq \sup_{s \in F} \{\operatorname{diam}(\pi_s(U)),\ \alpha(s)\} \leq \sup_{s \in F} \{R_s,\ \alpha(s)\} < \infty$.

Hence $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$ and $\mathcal{U} \in \mathcal{LSS}_m$. Note here that we used $r_s(x,y)$ as opposed to r_s since r_s depends on the x and y chosen.

$(\supseteq)$: Let $\mathcal{U} \in \mathcal{LSS}_m$ and let $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) = R - 1 < \infty$. Choose $F \subseteq S$ so that for any $s \in S \setminus F$, $\alpha(s) > R$. Let $U \in \mathcal{U}$ and suppose (for contradiction) that there is a $s \in S \setminus F$ so that $|\pi_s(U)| > 1$. Then there is a $x, y \in U$ so that $\pi_s(x) \neq \pi_s(y)$ so $r_s(x,y) \neq 0$. So $r_s(x,y) = \alpha(s)$ or $r_s(x,y) = \phi_s(\pi_s(x), \pi_s(y))$. But $s \in S \setminus F$ which means that $\alpha(s) > R > \phi_\alpha(x,y)$ since $x, y \in U$ and $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) = R - 1$. This implies that $r_s(x,y) \neq \alpha(s)$ lest $\phi_\alpha(x,y) > r_s(x,y) > R > R - 1 \geq \operatorname{diam}(U)$. So we have that $r_s = \phi_s(\pi_s(x), \pi_s(y))$. But this would mean that $r_s(x,y) > \alpha(s) > R > R - 1 \geq \operatorname{diam}(U)$. This means that $x, y \notin U$ which is a contradiction. Thus, for any $U \in \mathcal{U}$ and $s \in S \setminus F$, $|\pi_s(U)| = 1$. Lastly, we have that for any $s \in S$, $\pi_s(\mathcal{U}) \in \mathcal{LSS}_s$ since for any $U \in \mathcal{U}$, $\operatorname{diam}(\pi_s(U)) \leq \operatorname{diam}(U)$. So $\mathcal{U} \in \mathcal{LSS}$ as desired. □

3.3 Reduced Products

Definition 3.3.1. Suppose D is an infinite countable set and a function $\alpha : D \to (0, \infty)$ has limit infinity at infinity. Given ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$, the **reduced product** $(\times_{d \in D} X_d, \psi_\alpha)$ is the cartesian product $\prod_{d \in D} X_d$ equipped with the ∞-pseudo-metric ψ_α defined as the sum $\sum_{d \in D} \alpha(d) \cdot \rho_d(u(d), v(d))$.

Proposition 3.3.2. *Suppose $(\prod_{d \in D} X_d, \rho_\alpha)$ is an asymptotic product of ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$. If there is $c > 0$ such that each X_d is c-discrete (i.e. $\rho_d(x,y) \geq c$ if*

$x \neq y \in X_d$), then the identity function from the asymptotic product to the reduced product $(\times_{d\in D} X_d, \psi_\alpha)$ is a coarse equivalence.

Proof. Let f be the identity function. Since f is close to itself, we need only to show that f and f^{-1} are boronologous. We first show that f is bornologous. Let $M > 0$ be given and suppose $\rho_\alpha(x, y) < M$. Then by the logic in 3.2.7, $\exists E \subseteq D$, E finite, so that $\rho_d(e_d(x), e_d(y)) = 0\ \forall d \in D \setminus E$. Define $N = 1 + \sum_{d\in E} \alpha(d) \cdot \rho_d(e_d(x), e_d(y))$. Then
$\chi_\alpha(f(x), f(y)) = \psi_\alpha(x, y) = \sum_{d\in D} \alpha(d) \cdot \rho_d(e_d(x), e_d(y)) = \sum_{d\in E} \alpha(d) \cdot \rho_d(e_d(x), e_d(y)) < N$. So f is bornologous.

To show f^{-1} is bornologous, we use 3.2.6. Note that $e_d \circ f^{-1} = e_d$ is bornologous since $\rho_d(e_d(x), e_d(y)) \leq r_d \leq \rho_\alpha(x, y)$ if $\rho_\alpha(x, y)$ is finite. Now suppose $\psi_\alpha(x, y) < \infty$. Define $E = \{d \in D | \rho_d(e_d(x), e_d(y)) \neq 0\}$. Since each X_d is c-discrete, we have that $\forall d \in D \setminus E,\ \rho_d(e_d(x), e_d(y)) \geq c$. Then
$\infty > \sum_{d\in D} \alpha(d) \cdot \rho_d(e_d(x), e_d(y)) = \sum_{d\in E} \alpha(d) \cdot \rho_d(e_d(x), e_d(y)) \geq c \cdot \sum_{d\in E} \alpha(d)$ which implies E is finite in cardinality lest α have limit zero at ∞. By 3.2.6, f^{-1} is bornologous and hence f is a coarse equivalence. □

Corollary 3.3.3. *Given two reduced products $(\times_{d\in D} X_d, \rho_\alpha)$ and $(\times_{d\in D} X_d, \rho_\beta)$ of ∞-pseudo-metric spaces (X_d, ρ_d), $d \in D$, the identity function between them is a coarse equivalence provided there is $c > 0$ such that each X_d is c-discrete.*

Proof. This is a direct consequence of 3.2.8 and 3.3.2. □

Chapter 4

Coarse Quotients

In an introductory Topology course, one is introduced to the notion of quotient maps and the quotient topology. Recall that for a surjective map $f : X \to Y$ and X a topological space, one can endue Y with a topology via the declaration $V \subseteq Y$ is open if and only if $f^{-1}(V)$ is open in X. Not only is this a topology on Y, but this topology on Y also renders f to be a continuous map. This topology on Y is called the quotient topology. We remind the reader with a couple facts concerning this topology:

- Let $f : X \to Y$ surjective and X a topological space. The quotient topology on Y is the largest topology on Y that makes f continuous.

- Let X be a topological space and $\sim$ an equivalence relation on X. Let $f : X \to X/\sim$ be the map $f(x) = \bar{x}$. Then $X/\sim$ with the quotient topology and f as a pair have the following universal property: If $g : X \to Z$ is a continuous map so that $a \sim b \Rightarrow g(a) \sim g(b)$ (i.e. g is constant on the fibers of f), then there is a unique continuous map $h : X/\sim \to Z$ so that $g = h \circ f$.

We seek to dualize these ideas into a coarse geometric sense. It would make sense to check to see if, given $f : X \to Y$ surjective and $(X, \mathcal{LSS}_X)$ a large scale structure, the space $\mathcal{LSS}_Y = f(\mathcal{LSS}_X)$ is a large scale space. Alas, $\mathcal{LSS}_Y$ need not be closed under the star operation in general. Then how does one proceed?

4.1 Types of coarse quotient maps

4.1.1 Quotient large scale structures

Definition 4.1.1. Given a family of subsets of X, $\mathcal{A}$, we can consider the smallest large scale structure on X containing $\mathcal{LSS}$, which we call the large scale structure **generated by** $\mathcal{A}$, denoted by $\overline{\mathcal{A}}$.

It has been shown in [9] that there are at least two possible constructions for $\overline{\mathcal{LSS}}$ given $\mathcal{LSS}$ of X. One can either:

- take the intersection of all large scale structures containing $\mathcal{LSS}$ (this is based on Proposition 2.12 in [9]),
- add the cover by singletons to $\mathcal{LSS}$ if necessary, close $\mathcal{LSS}$ under starring and then close the resulting collection under refinement.

The following is a useful lemma whose proof is left to the reader.

Lemma 4.1.2. *Let $f : X \to Y$ be a map between sets and let $\mathcal{LSS}_X$ be a collection of families of subsets of X. Then*

$$f(\overline{\mathcal{LSS}_X}) \subseteq \overline{f(\mathcal{LSS}_X)}$$

Definition 4.1.3. Let $(X, \mathcal{LSS})$ be a large scale space and let $f : X \to Y$ be a surjective map. Then the **quotient large scale structure on** Y is defined to be $\overline{f(\mathcal{LSS})}$.

Clearly the quotient large scale structure is the smallest large scale structure which makes the map f large scale continuous (it is after all the smallest large scale structure generated by the image of every uniformly bounded family). The quotient large scale structure also satisfies the appropriate dual of the universal property stated above. Note that the existence of the quotient large scale structure and the universal property are stated in [3].

Proposition 4.1.4. *Let $f : X \to Y$ be a surjective large scale continuous map. Then Y has the quotient large scale structure with respect to f if and only f satisfies the following universal property: For any large scale continuous map $g : X \to Z$ which is constant on the fibers of f, there is a unique large scale continuous map $h : Y \to Z$ such that $h \circ f = g$.*

4.1.2 The Coarse Category

Some decisions made in future definitions will make more sense under the light of the coarse category. The coarse category has different definitions across the literature. Among all of the definitions, there is a common thread in that all close maps are identified within the category. We shall use the following definition:

Definition 4.1.5. The category **Coarse**/$\sim$, called the **coarse category**, is the category whose objects are large scale spaces and whose morphisms are equivalence classes of large scale continuous maps under the closeness relation $\sim$.

One can check that this is all well defined. Also, composition is defined in terms of the representatives. That is, $[f] \circ [g] = [f \circ g]$. Given the definition of a coarse embedding and proposition 1.4.15, we have the following statement.

Proposition 4.1.6. *Let f represent a morphism $[f]$ in* ***Coarse***/$\sim$. *Then:*

1. *$[f]$ is an epimorphism if and only if f is coarsely surjective.*

2. *$[f]$ is a monomorphism if and only if f is a coarse embedding.*

3. *$[f]$ is an isomorphism if and only if f is a coarse equivalence.*

4.1.3 Coarse quotient maps

It may seem apt to define a coarse quotient map as a surjective large scale continuous map $f : X \to Y$ where Y is equipped with the quotient large scale structure. However, from the categorical perspective this would not be a very good idea. This is because that definition would not behave well with close maps (which are identified in the coarse category). Since the epimorphisms of **Coarse**/$\sim$ are coarse surjections, it would make sense to have the definition of a coarse quotient map to agree with coarse surjections.

Definition 4.1.7. Let $f : X \to Y$ be a large scale continuous map. Then f is a **coarse quotient map** if it is coarsely surjective and there exists uniformly bounded cover $\mathcal{V}$ of Y such that the large scale structure on Y is generated by the quotient large scale structure on $f(X)$ and $\mathcal{V}$. A cover $\mathcal{V}$ satisfying this property is called a **quotient scale** of f.

Note that if $\mathcal{V}$ is a quotient scale for the coarse quotient map $f : X \to Y$, then so is any uniformly bounded coarsening of $\mathcal{V}$. In particular, we can always pick a quotient scale $\mathcal{V}$ such that $Y \subseteq \mathrm{st}(f(X), \mathcal{V})$. We now motivate our definition. Firstly, we note that a coarse quotient map satisfies a universal property.

Proposition 4.1.8. *Let $f : X \to Y$ be a large scale continuous map and let $\mathcal{V}$ be a uniformly bounded cover of Y. Then the following are equivalent.*

- *f is a coarse quotient map with quotient scale $\mathcal{V}$.*
- *for any large scale continuous map $g : X \to Z$ such that $g(f^{-1}(\mathcal{V}))$ is uniformly bounded, there exists a unique-up-to-closeness map $h : Y \to Z$ such that $h \circ f$ is close to g.*

Proof. $(\Rightarrow)$: Without loss of generality, choose the coarse quotient scale $\mathcal{V}$ so that $Y \subseteq \mathrm{st}(f(X), \mathcal{V})$.Suppose there is a large scale continuous $g : X \to Z$ so that $g \circ f^{-1}(\mathcal{V}) \in \mathcal{LSS}_Z$. Let $y \in Y$. Then there is a $V \in \mathcal{V}$ so that $y \in V$ and a $f(x) \in f(X)$ so that $f(x) \in f(X) \cap V$. Choose a $z \in g \circ f^{-1}(V)$. Define $h : Y \to Z$ via $h(y) = z$. We show that $g \circ f^{-1}(\mathcal{V})$ witnesses the closeness of $h \circ f$ and g. Indeed, let $x \in X$ and let $y = f(x)$. Choose $V \in \mathcal{V}$ so that $y \in V \cap \mathrm{st}(f(X), \mathcal{V})$. Then we have that $z = h \circ f(x) \in g \circ f^{-1}(V)$. Also, $f(x) \in V$ implies $x \in f^{-1}(V)$ which implies $g(x) \in g \circ f^{-1}(V)$. So $g \circ f^{-1}(\mathcal{V})$ witnesses the closeness of $h \circ f$ and g. h is large scale continuous as $h(\overline{f(\mathcal{LSS}_X)}) \subseteq \overline{h \circ f(\mathcal{LSS}_X)} =$ (by closeness) $\overline{g(\mathcal{LSS}_X)} \subseteq \mathcal{LSS}_Z$. The uniqueness of h is up to closeness i.e. up to choice of a different z made during the construction of h.
$(\Leftarrow)$: Let $Y' \subseteq Y$ be the underlying subset of Y so that $f' : X \to Y'$ is a coarse quotient map with scale $\mathcal{V}$ and $f(x) = f'(x)$ for all $x \in X$. Then as $\mathcal{V} \in \overline{f'(\mathcal{LSS}_X)}$ and $\overline{f'(\mathcal{LSS}_X)} \subseteq \mathcal{LSS}_Y$, we have that $\mathcal{V} \in \mathcal{LSS}_Y$. Furthermore, $f' \circ f^{-1}(\mathcal{V}) = \mathcal{V} \in \overline{f'(\mathcal{LSS}_X)}$ which implies that there is a unique map up to closeness $h : Y \to Y'$ so that $h \circ f \sim f'$. This implies that the set identity map $Y \to Y'$ is large scale continuous which gives us the result. □

We now show that this definition of coarse quotient maps behaves well with close maps. Before we begin, we need a lemma with regard to refinements. These are all easy to check.

Lemma 4.1.9. *1. Let $f : X \to Y$ be a map of sets and $\mathcal{U}, \mathcal{V}$ be families of subsets of X so that $\mathcal{U} \prec \mathcal{V}$. Then $f(\mathcal{U}) \prec f(\mathcal{V})$.*

2. Let $f : X \to Y$ be a map of sets and $\mathcal{A}, \mathcal{B}$ be families of subsets of Y so that $\mathcal{A} \prec \mathcal{B}$. Then $f^{-1}(\mathcal{U}) \prec f^{-1}(\mathcal{V})$.

3. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale structures and $f : X \to Y$ a map of sets. Suppose $\mathcal{U}, \mathcal{V} \in \mathcal{LSS}_X$ so that $\mathcal{U} \prec \mathcal{V}$, $f(\mathcal{V}) \in \mathcal{LSS}_Y$, and there is another map of sets $g : X \to Y$ so that f is close to g. Then $g(\mathcal{U}) \in \mathcal{LSS}_Y$.

Proposition 4.1.10. *If $f, g : X \to Y$ are large scale continuous maps which are close, then f is a coarse quotient map if and only if g is.*

Proof. Suppose f is a coarse quotient map and $\mathcal{V}$ is a quotient scale for f and $\mathcal{W}$ be so that for every $x \in X$ we have $f(x), g(x) \in W$ for some $W \in \mathcal{W}$. We show that g is a coarse quotient map with quotient scale $\mathcal{V} \cup \mathcal{W}$. We will use the universal property. Let $(Z, \mathcal{LSS}_Z)$ be given and $h : X \to Z$ be so that $h(g^{-1}(\mathcal{V} \cup \mathcal{W})) \in \mathcal{LSS}_Z$. Note that $h(g^{-1}(\mathcal{V} \cup \mathcal{W})) = h \circ g^{-1}(\mathcal{V}) \cup h \circ g^{-1}(\mathcal{W})$ which implies that (since large scale structures are closed under refinements) $h \circ g^{-1}(\mathcal{V}), h \circ g^{-1}(\mathcal{W}) \in \mathcal{LSS}_Z$ and hence $\mathrm{st}(h \circ g^{-1}(\mathcal{V}), h \circ g^{-1}(\mathcal{W})) \in \mathcal{LSS}_Z$. We show $f^{-1}(\mathcal{V}) \prec \mathrm{st}(g^{-1}(\mathcal{V}), g^{-1}(\mathcal{W}))$. Indeed, let $f^{-1}(V) \in f^{-1}(\mathcal{V})$ be given and $x \in f^{-1}(V)$. Then $f(x) \in V$. Choose $W \in \mathcal{W}$ so that $f(x), g(x) \in W$. Then $V \cap W \neq \varnothing$ which means $g^{-1}(V) \cap g^{-1}(W) = g^{-1}(V \cap W) \neq \varnothing$. Further, $g(x) \in W$ implies $x \in g^{-1}(W)$. Thus, $f^{-1}(V) \subseteq \mathrm{st}(g^{-1}(V), g^{-1}(\mathcal{W}))$ as desired. Hence, $f^{-1}(\mathcal{V}) \prec \mathrm{st}(g^{-1}(\mathcal{V}), g^{-1}(\mathcal{W}))$ which implies $h \circ f^{-1}(\mathcal{V}) \prec h(\mathrm{st}(g^{-1}(\mathcal{V}), g^{-1}(\mathcal{W}))) \prec \mathrm{st}(h \circ g^{-1}(\mathcal{V}), h \circ g^{-1}(\mathcal{W})) \in \mathcal{LSS}_Z$. So $h \circ f^{-1}(\mathcal{V}) \in \mathcal{LSS}_Z$ which implies (as f is a coarse quotient map) there is a $k : Y \to Z$ unique up to closeness so that $f \circ k = h$ which means (up to closeness) $g \circ k = h$ since f is close to g. By the universal property, g is a coarse quotient map. □

Proposition 4.1.11. *Any coarse equivalence is a coarse quotient map and coarse quotient maps are closed under composition.*

Proof. Let $f : (X, \mathcal{LSS}_X) \to (Y, \mathcal{LSS}_Y)$ be a coarse equivalence. Let $k : Y \to X$ be large scale continuous so that $k \circ f \sim id_X$. Let $\mathcal{V} \in \mathcal{LSS}_Y$ and suppose there is a large scale continuous function $g : (X, \mathcal{LSS}_X) \to (Z, \mathcal{LSS}_Z)$ so that $g \circ f^{-1}(\mathcal{V}) \in \mathcal{LSS}_Z$. Define

$h : Y \to Z$ via $h = g \circ k$. Then h is large scale continuous and $h \circ f = g \circ (k \circ f) = g \circ id_X = g$. By the universal property, we have that f is a coarse quotient map.

Let $f : (X, \mathcal{LSS}_X) \to (Y, \mathcal{LSS}_Y)$ and $k : (Y, \mathcal{LSS}_Y) \to (Z, \mathcal{LSS}_Z)$ be coarse quotient maps with coarse quotient scales $\mathcal{V}_f$ and $\mathcal{V}_k$ respectively. Without loss of generality, assume $Y \subseteq \mathrm{st}(f(X), \mathcal{V}_f)$ and $Z \subseteq \mathrm{st}(k(Y), \mathcal{V}_k)$. Choose $\mathcal{V} \in \mathcal{LSS}_Z$ so that $k(\mathcal{V}_f), \mathcal{V}_k \prec \mathcal{V}$. Suppose there is a $g : (X, \mathcal{LSS}_X) \to (W, \mathcal{LSS}_W)$ so that g is large scale continuous and that $g \circ f^{-1} \circ k^{-1}(\mathcal{V}) \in \mathcal{LSS}_W$. Since $k(\mathcal{V}_f) \prec \mathcal{V}$, we have by the lemma above that $k^{-1} \circ k(\mathcal{V}_f) \prec k^{-1}(\mathcal{V})$. Since $\mathcal{V}_f \prec k^{-1} \circ k(\mathcal{V}_f)$, we see $\mathcal{V}_f \prec k^{-1}(\mathcal{V})$ which by the lemma gives us $g \circ f^{-1}(\mathcal{V}_f) \prec g \circ f^{-1} \circ k^{-1}(\mathcal{V}) \in \mathcal{LSS}_W$. So $g \circ f^{-1}(\mathcal{V}_f) \in \mathcal{LSS}_W$ and by the universal property, there is a $h' : Y \to W$ so that $h' \circ f \sim g$. We will now show $h' \circ k^{-1}(\mathcal{V}_k) \in \mathcal{LSS}_W$. Indeed, $\mathcal{V}_k \prec \mathcal{V}$ implies (by the lemma above) $g \circ f^{-1} \circ k^{-1}(\mathcal{V}_k) \prec g \circ f^{-1} \circ k^{-1}(\mathcal{V})$. But $g \circ f^{-1} \circ k^{-1} \sim h' \circ f \circ f^{-1} \circ k^{-1} = h' \circ k^{-1}$. By the third part of the lemma above, we have $h' \circ k^{-1}(\mathcal{V}_k) \in \mathcal{LSS}_W$. By the universal property, there is a unique map (up to closeness) $h'' : Z \to W$ so that $h'' \circ k \sim h'$. Using this with $h' \circ f \sim g$, we have that $h'' \circ k \circ f \sim h' \circ f \sim g$. By the universal property, we have that $f \circ k$ is a coarse quotient map. □

4.1.4 Strong coarse quotients

Let $f : X \to Y$ be a map of large scale structures.Rather than requiring that the collection $f(\mathcal{LSS}_X)$ generate the large scale structure $\mathcal{LSS}_Y$ on Y, one might want to actually have the equality $\mathcal{LSS}_Y = f(\mathcal{LSS}_X)$. A "coarsening" of this notion leads to the following definition whose metric version was introduced by [11], which we call a strong quotient to differentiate it from the previous case.

Definition 4.1.12. Let $f : X \to Y$ be a map between large scale spaces. Then f is a **strong coarse quotient map** if there exists a uniformly bounded cover $\mathcal{V}$ (which we call a **strong quotient scale** of f) such that for every uniformly bounded cover $\mathcal{W}$ in Y there exists a uniformly bounded cover $\mathcal{U}$ in X with $\mathcal{W} \prec \mathrm{st}(f(\mathcal{U}), \mathcal{V})$.

It's easy to see that every strong coarse quotient map is coarsely surjective and that every strong coarse quotient map is a coarse quotient map. Also note that for a map $f : X \to Y$ and the large scale structure $\mathcal{LSS}$ on X, f was called **weakly soft** in [3] if $f(\mathcal{LSS})$ was a large scale structure (the original terminology is stated in terms of coarse structures). Thus every weakly soft coarse quotient map is a strong coarse quotient map. A canonical example of a strong coarse quotient map is the quotient of a large scale space by an appropriate group action which is seen below.

Proposition 4.1.13. *Let $(X, \mathcal{LSS}_X)$ be a large scale space and let G be a group which acts on the underlying set of X such that for every uniformly bounded family $\mathcal{U}$ of subsets of X, the family*

$$G(\mathcal{U}) = \{\bigcup_{g\in G} g(U)|U \in \mathcal{U}\}$$

is uniformly bounded, where $g(U) = \{g \cdot x|x \in U\}$. Let X/G be the set of orbits under this action, with $q : X \to X/G$ the orbit map. Endow X/G with the quotient large scale structure with respect to q. Then q is a strong coarse quotient with the strong quotient scale being the trivial cover.

Proof. It suffices to show that $q(\mathcal{LSS}_X)$ is closed under refinement and stars. Since q is surjective, $q(\mathcal{LSS}_X)$ is closed by refinements. To show that $q(\mathcal{LSS}_X)$ is closed under stars, we will show that for elements $q(\mathcal{U}), q(\mathcal{V}) \in q(\mathcal{LSS}_X)$ we have $\mathrm{st}(q(\mathcal{U}), q(\mathcal{V})) \prec q(\mathrm{st}(G(\mathcal{U}), G(\mathcal{V})))$. Let $U \in \mathcal{U}$ and let $G \cdot U = \bigcup_{g\in G} g \cdot U \in G(\mathcal{U})$. Let $\bar{x} \in \mathrm{st}(q(U), q(\mathcal{V}))$. Then there is a $V \in \mathcal{V}$ so that $\bar{x} \in q(V)$ and $q(U) \cap q(V) \neq \varnothing$. Since $\bar{x} \in q(V)$, there is a $h \in G$ so that $x \in h \cdot V$ which means that $x \in G \cdot V$. Because $q(U) \cap q(V) \neq \varnothing$, there is a $k, l \in G$ so that $k \cdot U \cap l \cdot V \neq \varnothing$ which implies $G \cdot U \cap G \cdot V \neq \varnothing$. We therefore have that $x \in \mathrm{st}(G \cdot U, G(\mathcal{V}))$ which means that $q(x) = \bar{x} \in q(\mathrm{st}(G(\mathcal{U}), G(\mathcal{V})))$ as desired. □

As an aside, the condition $G(\mathcal{U}) \in \mathcal{LSS}_X$ for every $\mathcal{U} \in \mathcal{LSS}_X$ may be strong for infinite groups G, but if G is finite, this condition is equivalent to requiring that every g acts on X as a bornologous function. Indeed, $G(\mathcal{U})$ is a finite union of uniformly bounded sets and hence is uniformly bounded.

4.2 More on Quotients by Group Actions

Given the previous chapters of this text, there should come a point where we discuss coarse properties that are preserved by coarse quotients. However, (as this section shows) even in a

”nice” case like quotients by group actions, it becomes difficult to find coarse properties that are preserved. By the above proposition, group quotients of finitely generated groups may be viewed as coarse quotients (give these groups the induced large scale structure from the word metric). Consider the following counter-examples:

Let F_2 be the free group on two generators. From [8], the asymptotic dimension of F_2 is one. Also from [8] is that the wreath product of the integers, denoted $\mathbb{Z}\wr\mathbb{Z}$, has two generators and does not have finite asymptotic dimension. From group theory, every finitely generated group is the quotient of a free group and hence there is a quotient map $q : F_2 \to \mathbb{Z} \wr \mathbb{Z}$. This is an example of a coarse quotient map that does not preserve finite asymptotic dimension.

For another counter-example, recall that F_2 is not amenable and that $\mathbb{Z}^2$ is amenable. But there is a quotient $q : F_2 \to \mathbb{Z}^2$ where we quotient out by the commutator subgroup of F_2.

The outlook of coarse property permanence may look dour, but it’s not hard to show that both property A and finite asymptotic dimension are preserved through the orbit map $q : X \to X/G$ if X has bounded geometry and G is finite. We will give a generalization of the former. Before we do that, we need justify a slight shift in focus.

Let $(X, \mathcal{LSS})$ be a large scale structure and let G act on X by coarse equivalences. Let X/G be the space of orbits and $q : X \to X/G$ the orbit map. A disadvantage of considering this map is that X/G ”forgets” the large scale structure on G. So we will create a large scale structure that ”encodes” $\mathcal{LSS}$ with the large scale structure of the group. The following name is inspired by [10]:

Definition 4.2.1. Let $(X, \mathcal{LSS})$ be a large scale structure and let G be a group that acts on X by coarse equivalences. Define the large scale structure $\mathcal{LSS}_G$ on X to be the closure of $\mathcal{LSS}$ with the addition of families of the form $\{(x, g \cdot x)\}_{x\in X}$ for each $g \in G$. We call this large scale space the **warped space** of $\mathcal{LSS}$.

We will note here that while acting by coarse equivalences may sound like a strong condition, acting by coarse equivalences is the same as requiring that every g act on X as a bornologous function. This is because every action function h_g which sends x to $g \cdot x$ has a bornologous inverse function, namely $h_{g^{-1}}$. Hence by 1.4.15 every h_g is a coarse equivalence.

Proposition 4.2.2. *Let $(X, \mathcal{LSS})$ be a large scale structure, let G act on X by coarse equivalences, and let $q : X \to X/G$ be the orbit map. If the order of G is finite, then $q : (X, \mathcal{LSS}_G) \to (X/G, q(\mathcal{LSS}))$ is a coarse equivalence.*

Proof. q is clearly surjective (and is hence coarsely surjective) and since the image of every subset of the form $\{x, g\cdot x\}_{x\in X}$ is cover by singletons, we have that q is large scale continuous. It remains to show that q is a coarse embedding. Let $\mathcal{V} \in q(\mathcal{LSS})$ and choose $\mathcal{U} \in \mathcal{LSS}$ so that $q(\mathcal{U}) = \mathcal{V}$. Let $\mathcal{G} = \bigcup_{g\in G} \{\{x, g \cdot x\}\}_{x\in X}$. Then $\mathcal{G} \in \mathcal{LSS}_G$ since G is finite and hence $\operatorname{st}(\mathcal{U}, \mathcal{G}) \in \mathcal{LSS}_G$. We show that $q^{-1}(\mathcal{V}) \prec \operatorname{st}(\mathcal{U}, \mathcal{G})$ which completes the proof. Indeed, let $q^{-1}(V) \in q^{-1}(\mathcal{V})$ and let $x \in q^{-1}(V)$; select $U \in \mathcal{U}$ so that $q(U) = V$. Then there is a $g \in G$ so that $x \cdot g \in U$ which means $U \cap \{x, g \cdot x\} \neq \varnothing$ and $x \in \operatorname{st}(U, \mathcal{G})$. So $q^{-1}(V) \subseteq \operatorname{st}(U, \mathcal{G})$ as desired. □

So we have that if $(X, \mathcal{LSS})$ is a large scale structure, G a finite group acting on X, and q the orbit map then $(X/G, q(\mathcal{LSS}))$ and $(X, \mathcal{LSS}_G)$ are coarsely equivalent. The next lemma gives us a generating set for $\mathcal{LSS}_G$.

Lemma 4.2.3. *Let $(X, \mathcal{LSS})$ be a large scale space and let G act on X by coarse equivalences. Then $\mathcal{LSS}_G$ is the collection of refinements of families of the form $\operatorname{st}(\mathcal{U}, \mathcal{F})$ where $\mathcal{U} \in \mathcal{LSS}$ and $\mathcal{F} = \{\{F \cdot x\}\}_{x\in X}$ for $F \subseteq G$ finite.*

Proof. Let $\mathcal{X}$ be the collection of refinements of families of the form $\operatorname{st}(\mathcal{U}, \mathcal{F})$ where $\mathcal{U} \in \mathcal{LSS}$ and $\mathcal{F} = \{\{F \cdot x\}\}_{x\in X}$ for $F \subseteq G$ finite. We will show that $\mathcal{LSS}_G = \mathcal{X}$. It is clear that $\mathcal{X} \subseteq \mathcal{LSS}_G$ since for any $\mathcal{U} \in \mathcal{LSS}$ and $\mathcal{F}$ as described above, $\mathcal{U}, \mathcal{F} \in \mathcal{LSS}_G$ which implies $\operatorname{st}(\mathcal{U}, \mathcal{F}) \in \mathcal{LSS}_G$. The other inclusion will follow if we show that $\mathcal{X}$ is star closed i.e. $\mathcal{X}$ is a large scale structure containing $\mathcal{LSS}$ and $\{g \cdot x\}_{x\in X}$.

We first note that for $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{LSS}$, $\mathcal{F}_1 = \{F_1 \cdot x\}_{x\in X}$, and $\mathcal{F}_2 = \{F_2 \cdot x\}_{x\in X}$ (for $F_1, F_2 \subseteq G$ finite), $\operatorname{st}(\operatorname{st}(\mathcal{U}_1, \mathcal{F}_1), \operatorname{st}(\mathcal{U}_2, \mathcal{F}_2)) \prec \operatorname{st}(\operatorname{st}(\operatorname{st}(\operatorname{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2), \mathcal{U}_2), \mathcal{F}_2)$. If we show that $\operatorname{st}(\operatorname{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2) \in \mathcal{X}$ and $\operatorname{st}(\operatorname{st}(\mathcal{U}_1, \mathcal{F}_1)\mathcal{U}_2) \in \mathcal{X}$ for arbitrary $\mathcal{U}_1, \mathcal{U}_2, \mathcal{F}_1, \mathcal{F}_2$ as described above, then we will have that $\operatorname{st}(\operatorname{st}(\mathcal{U}_1, \mathcal{F}_1), \operatorname{st}(\mathcal{U}_2, \mathcal{F}_2)) \prec \operatorname{st}(\operatorname{st}(\operatorname{st}(\operatorname{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2), \mathcal{U}_2), \mathcal{F}_2) \in \mathcal{X}$ which will complete the proof since $\mathcal{X}$ is closed under refinements by assumption.

We first show that for $\mathcal{U}_1 \in \mathcal{LSS}$, $\mathcal{F}_1 = \{F_1 \cdot x\}_{x\in X}$, and $\mathcal{F}_2 = \{F_2 \cdot x\}_{x\in X}$ for $F_i \subseteq G$ finite and with the property $g \in F_i$ implies $g^{-1} \in F_i$ (this may be done without loss of generality due to closure of refinements), $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2) \in \mathcal{X}$. Let e be the identity element of G and define $F_3 = F_1 \cdot F_2 \cup F_2 \cup \{e\}$ and $\mathcal{F}_3 = \{F_3 \cdot x\}_{x \in X}$. Then $\text{st}(\mathcal{U}_1, \mathcal{F}_3) \in \mathcal{X}$ and $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2)$. Indeed, let $U_1 \in \mathcal{U}_1$ and $y \in \text{st}(\text{st}(U_1, \mathcal{F}_1), \mathcal{F}_2)$. Then there is a $g_2 \in F_2 \cup \{e\}$ so that $g_2 \cdot y \in \text{st}(U_1, \mathcal{F}_1)$. If $g_2 \cdot y \in U_1$, then $y \in \text{st}(U_1, \mathcal{F}_3)$ as $g_2 \in F_3$. If $g_2 \cdot y \in \text{st}(U_1, \mathcal{F}_1) \setminus U_1$, then there is a $g_1 \in F_1$ so that $g_1 g_2 \cdot y \in U_1$ which implies $y \in \text{st}(U_1, \mathcal{F}_3)$ since $g_1 g_2 \in F_3$. So we've shown $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2) \prec \text{st}(\mathcal{U}_1, \mathcal{F}_3) \in \mathcal{X}$ which means that $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2) \in \mathcal{X}$ as desired.

We will now show that $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1)\mathcal{U}_2) \in \mathcal{X}$. Assume (without loss of generality) that $e \in F_1$ and $g \in F_1$ implies $g^{-1} \in F_1$. Define $\mathcal{V}' = \bigcup_{\substack{U_2 \in \mathcal{U}_2 \\ g \in F_1}} g \cdot U_2$ and $\mathcal{V}$ to be the trivial extension of $\mathcal{V}'$. Then $\mathcal{V} \in \mathcal{LSS}$ which implies that $\text{st}(\mathcal{U}_1, \mathcal{V}) \in \mathcal{LSS}$. We will show that $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1)\mathcal{U}_2) \prec \text{st}(\text{st}(\mathcal{U}_1, \mathcal{V}), \mathcal{F}_1)$. Indeed, let $U_1 \in \mathcal{U}_1$ and $y \in \text{st}(\text{st}(U_1, \mathcal{F}_1), \mathcal{U}_2)$. If $y \in U_1$, then $y \in \text{st}(U_1, \mathcal{V})$ which implies that (as $e \in F_1$) $y \in \text{st}(\text{st}(\mathcal{U}_1, \mathcal{V}), \mathcal{F}_1)$. If $y \in \text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1)\mathcal{U}_2) \setminus U_1$, then $y \in U_2$ where $U_2 \in \mathcal{U}_2$ and $U_2 \cap \text{st}(\mathcal{U}_1, \mathcal{F}_1) \neq \varnothing$. Let $z \in U_2 \cap \text{st}(\mathcal{U}_1, \mathcal{F}_1)$. Then there is a $g \in F_1$ so that $g \cdot z \in U_1$. But since y and z are in U_2, $g \cdot y, g \cdot z \in g \cdot U_2$. Further, $g \cdot U_2 \in \mathcal{V}$. By definition, $g \cdot y \in \text{st}(U_1, \mathcal{V})$. Again by definition, $y \in \text{st}(\text{st}(U_1, \mathcal{V}), \mathcal{F}_1)$. Thus, $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1)\mathcal{U}_2) \prec \text{st}(\text{st}(\mathcal{U}_1, \mathcal{V}), \mathcal{F}_1)$ and $\text{st}(\text{st}(\mathcal{U}_1, \mathcal{F}_1)\mathcal{U}_2) \in \mathcal{X}$. □

This lemma in turn shows that $\mathcal{LSS}_G$ is indeed a large scale structure for more general types of groups (i.e. infinite groups). We now prove an interesting theorem regarding the preservation of property A. Recall that in a bounded geometry metric space, the definition of exactness from [5] is equivalent to Property A.

Theorem 4.2.4. *Let G act on a discrete bounded geometry metric space X by coarse equivalences. If G is amenable and X has Property A, then X_G has Property A.*

Proof. We will prove that X_G is exact. Let $\mathcal{LSS}$ be the induced large scale structure on X by its metric and let $\text{st}(\mathcal{U}, \mathcal{F})$ be a uniformly bounded family in X_G, with $\mathcal{F} = \{F \cdot x \mid x \in X\}$ and $\mathcal{U} \in \mathcal{LSS}$, and let $\epsilon > 0$. By the amenability of G, we have that there is a finite $E \subseteq G$ so that for all $g \in F \cdot F^{-1}$,

$$\frac{|E \Delta E \cdot g|}{|E|} < \varepsilon/3.$$

Since G acts by coarse equivalences, we have that $g \cdot \mathcal{U} = \{g \cdot U\}_{U \in \mathcal{U}}$ is in $\mathcal{LSS}$ for all g and hence

$$\mathcal{U}^E = \bigcup_{k \in E} k \cdot \mathcal{U}$$

is also in $\mathcal{LSS}$.

Since X is exact, we have that there is a partition of unity $(\phi_i)_{i \in I}$ such that the family $\mathcal{V} = (V_i)_{i \in I}$ is in $\mathcal{LSS}$, where $V_i = \{x \in X \mid \phi_i(x) \neq 0\}$, and for every $x, y \in U$ with $U \in \mathcal{U}^E$, we have

$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)| < \frac{\epsilon}{3}.$$

Define a new partition of unity $(\psi_i)_{i \in I}$ on X via

$$\psi_i(x) = \frac{1}{|E|} \sum_{k \in E} \phi_i(k \cdot x),$$

and let $\mathcal{W} = (W_i)_{i \in I}$ be the cover of X given by $W_i = \{x \in X \mid \psi_i(x) \neq 0\}$. We claim that $\mathcal{W}$ is uniformly bounded in $\mathcal{LSS}_G$. Indeed, $x \in W_i$ implies $\psi_i(x) \neq 0$ so there is a $k \in E$ so that $k \cdot x \in V_i$. It follows that $\mathcal{W}$ is a refinement of the cover $\bigcup_{k \in E} k^{-1} \cdot \mathcal{V}$. Hence, $\mathcal{W} \in \mathcal{LSS}_G$.

It remains to show that for any $x, y \in \mathrm{st}(U, \mathcal{F})$ with $U \in \mathcal{U}$, we have $\sum_{i \in I} |\psi_i(x) - \psi_i(y)| < \epsilon$. It is enough to show that (1) for any $x, y \in U$ we have $\sum_{i \in I} |\psi_i(x) - \psi_i(y)| < \epsilon/3$ and (2) for $x \in X$ and $g, h \in F$ we have $\sum_{i \in I} |\psi_i(g \cdot x) - \psi_i(h \cdot x)| < \epsilon/3$.

We first show inequality (1). Let $x, y \in U$ for some $U \in \bigcup_{k \in E} k \cdot \mathcal{U}$. Then

$$\sum_{i \in I} |\psi_i(x) - \psi_i(y)| \leq \frac{1}{|E|} \sum_{k \in E} \sum_{i \in I} |\phi_i(k \cdot x) - \phi_i(k \cdot y)|.$$

For any $k \in E$, $x, y \in U$ implies $k \cdot x, k \cdot y \in k \cdot U \in \mathcal{U}^E$, so by the construction of the ϕ_i,

$$\sum_{i \in I} |\psi_i(x) - \psi_i(y)| \leq \frac{1}{|E|} \cdot |E| \cdot \frac{\epsilon}{3} = \frac{\epsilon}{3}$$

We now show (2). Let $x \in X$ and $g, h \in F$. Then

$$\sum_{i \in I} |\psi_i(g \cdot x) - \psi_i(h \cdot x)| = \frac{1}{|E|} \sum_{i \in I} |\sum_{k \in E} \phi_i(k \cdot g \cdot x) - \phi_i(k \cdot h \cdot x)|$$

Notice that if $k \cdot g \in E \cdot g \cap E \cdot h$, then the term $\phi_i(k \cdot g \cdot x)$ is canceled out. So from the above we get

$$\begin{aligned}\sum_{i\in I} |\psi_i(g\cdot x) - \psi_i(h\cdot x)| &= \frac{1}{|E|}\sum_{i\in I} |\sum_{l\in E\cdot g\setminus E\cdot h} \phi_i(l\cdot x) \quad - \sum_{m\in E\cdot h\setminus E\cdot g} \phi_i(m\cdot x)| \\ &\le \frac{1}{|E|}\sum_{l\in E\cdot g\Delta E\cdot h}\ \sum_{i\in I} |\phi_i(l\cdot x)| \\ &= \frac{|E\cdot g\Delta E\cdot h|}{|E|}\end{aligned}$$

since each ϕ_i is a partition of unity. But $|E \cdot g\Delta E \cdot h| = |E\Delta E \cdot hg^{-1}|$ (it's a group translation over a finite subset of G), so by the condition on E, we have

$$\frac{|E\cdot g\Delta E\cdot h|}{|E|} \le \epsilon/3$$

□

The next result can be thought of as a coarse analog of proposition 1.40 parts a and b from [6]. The theorem says the following:

Theorem 4.2.5. *Let Y be a [topological] space and suppose a group G acts on Y properly discontinuously. Then:*

- *The quotient map $p : Y \to Y/G$ via $p(y) = G \cdot y$ is a normal covering space.*
- *G is the group of deck transformations of this covering space $Y \to Y/G$ if Y is path connected and locally path connected.*

From this theorem, it follows that if we want such an analog, then we need a coarse version of properly discontinuous actions and some sort of connectedness condition.

Definition 4.2.6. Let X be a metric space and let $\mathcal{LSS}_d$ be the induced large scale structure by the metric. Let a group G act on X by coarse equivalences. We say G acts on X **coarsely discontinuously** if for every pair $(\mathcal{U}, g)$ for $\mathcal{U} \in \mathcal{LSS}_d$ and $g \in G \setminus e$ there is a bounded set $K \subseteq X$ so that for any $U \in \mathcal{U}$ that does not meet K, we have that $U \cap g \cdot U = \varnothing$.

An example of such an action would be the group $G = \mathbb{R}/2\pi\mathbb{R}$ (under addition) on the set $X = \mathbb{R}^2$ with the euclidean metric via (for $\alpha \in [0, 2\pi)$) $\alpha \cdot (r\cos(\theta), r\sin(\theta)) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$. Indeed, let $\mathcal{U} \in \mathcal{LSS}_d$ and if necessary, coarsen $\mathcal{U}$ to $\mathcal{V}$ where $\mathcal{V}$ is the cover of X by R-balls. Let $\alpha \in (0, 2\pi)$ be given and define K to be the ball of radius $\frac{2R}{\sqrt{2-2\cos(\alpha)}}$ centered at zero. Then outside of K we have that for any point $z \in X$, the distance between z and $\alpha \cdot z$ is greater than $2R$. Hence for any $V \in \mathcal{V}$ that doesn't meet K, we have that $V \cap \alpha \cdot V = \varnothing$. Since $\mathcal{V}$ is a coarsening of $\mathcal{U}$, the same holds for $\mathcal{U}$ as well. So the action is coarsely discontinuous.

Definition 4.2.7. Let $(X, \mathcal{LSS}_d)$ be a metric space with the induced large scale structure by the metric. We say that $(X, \mathcal{LSS}_d)$ is **coarsely one-ended** if there is a scale $\mathcal{U} \in \mathcal{LSS}_d$ so that for every $K \subseteq X$ bounded there is a $L \subseteq X$ bounded so that $K \subseteq L$ and $X \setminus L$ has one $\mathcal{U}$ chain component.

It is easy to see that the example space given for a coarsely discontinuous action is coarsely one ended. Here is an example of a space that is not coarsely one-ended. Let X be the axes of $\mathbb{R}^2$ with the metric given from the euclidean metric on $\mathbb{R}^2$. Let the mesh of $\mathcal{U}$ be R. Then for the bounded set K, the ball of radius $2\sqrt{2}R + 1$ centered at zero, we have that for any $L \subseteq X$ bounded, there is no $\mathcal{U}$ chain from any two points that belong to any two distinct topologically connected components of $X \setminus L$.

An inspiration for the connectnedness condition being called coarsely one-ended comes from the following:

Proposition 4.2.8. *Let X be a geodesic metric space and $\mathcal{LSS}_d$ be the large scale structure induced by the metric d. Then $(X, \mathcal{LSS}_d)$ is coarsely one-ended if and only if for every bounded set $K \subseteq X$ there is a bounded set $K' \subseteq X$ so that $K \subseteq K'$ and $X \setminus K'$ is topologically connected.*

Proof. $(\Rightarrow)$: Let $\mathcal{U}$ be the scale as per the definition of coarse one-endedness. Without loss of generality, let $\mathcal{U}$ be a cover by R-balls. Let K be given and consider $N = \mathrm{st}(K, \mathcal{U})$. Then there is a L containing N so that $N \subseteq L$ and $X \setminus L$ is $\mathcal{U}$ connected. If $X \setminus L$ is topologically connected, then we're done. Otherwise, let $\{C_i\}_{i \in I}$ be the connected components of $X \setminus L$. Since $|I| > 1$ and $X \setminus L$ has one $\mathcal{U}$ chain component, for each pair of distinct connected components C_i and C_j there are representatives $x_i \in C_i$ and $x_j \in C_j$ so that for some $U \in \mathcal{U}$ we have $x_i, x_j \in U$. That is, $d(x_i, x_j) < 2R$. Let $\gamma_{i,j}$ be the geodesic from x_i to x_j. Then $\gamma_{i,j}$ must intersect L (lest C_i and C_j not be distinct connected components) but $\gamma_{i,j}$ does not intersect K since N is an 2R-ball about K and the length of $\gamma_{i,j}$ is at most 2R. The new set $\gamma_{i,j} \cup X \setminus K$ makes C_i and C_j become one connected component. We can do this process for as many pairs of connected components as we like; the set of all $\bigcup_{\gamma_{i,j}} \gamma_{i,j} \cup X \setminus L$ for various collections of geodesics forms a poset under inclusion. One can find a maximal such element via Zorn's lemma (the upper bound of every chain is the union of all chain elements). Let $M \cup X \setminus L$ be a maximal element in this poset. By construction, all of the geodesics unioned in M do not meet K. Furthermore, $M \cup X \setminus L$ is topologically connected by maximality. Define $K' = L \setminus M$. Then K' is bounded since L is bounded and since $X \setminus K' = M \cup X \setminus L$, we're done.

$(\Leftarrow)$: Suppose that for every bounded set $K \subseteq X$ there is a bounded set $K' \subseteq X$ so that $K \subseteq K'$ and $X \setminus K'$ is topologically connected. For every K' associated to some K, choose an open cover $\mathcal{O}_{K'}$ so that $X \setminus K'$ is $\mathcal{O}$-connected. Define $\mathcal{U}$ to be the trivial cover unioned with $\mathcal{O}_{K'}$ over all possible K'. Then $\mathcal{U}$ is a cover of X and $X \setminus K'$ is $\mathcal{U}$-connected by construction. □

The condition described in the above proposition may be interpreted as "the metric topology on X is one-ended", hence the name.

Theorem 4.2.9. *Let $(X, \mathcal{LSS}_d)$ be an unbounded metric space with the large scale structure induced by the metric d. Let G be a group that acts on X coarsely discontinuously by coarse equivalences. Define $\mathsf{Aut}(X/X_G)$ to be the set of all f coarse equivalences on X so that f is close to the identity map via the large scale structure $\mathcal{LSS}_G$. Define $\sim$ to be the relation of closeness via $\mathcal{LSS}_d$. If $(X, \mathcal{LSS}_d)$ is coarsely one-ended with scale $\mathcal{U}$, then $G \cong \mathsf{Aut}(X/X_G)/\sim$ as groups.*

Proof. Define $h_g : X \to X$ via $h_g(x) = g \cdot x$. It is clear that $h_g \in \mathsf{Aut}(X/X_G)$. We will first show that $\mathsf{Aut}(X/X_G) = \{[h_g] | g \in G\}$. Let $f \in \mathsf{Aut}(X/X_G)$ and suppose (by use of 4.2.3) that f is close to the identity on X via $\mathrm{st}(\mathcal{V}, \mathcal{F})$ with $\mathcal{V} \in \mathcal{LSS}_d$ and $\mathcal{F} = \{\{F \cdot x\}\}_{x \in X}$ for $F \subseteq G$ finite. Then for every $x \in X$ there is a $g \in F$ and $V \in \mathcal{V}$ so that $x, g \cdot f(x) \in V$.

Then we have that for every $x \in X$ there is a $g \in F^{-1}$ so that $g \cdot x, f(x) \in g \cdot V$. Define $\mathcal{W} = \bigcup_{g \in F^{-1} \cup \{e\}} g \cdot \mathcal{V}$.; note that $\mathcal{W} \in \mathcal{LSS}_d$ since F is finite. For each element $k \in F \cdot F^{-1}$, use coarse discontinuity so that there is a bounded set $J_k \subseteq X$ so that for any $\mathrm{st}(W, \mathcal{W}) \in \mathrm{st}(\mathcal{W}, \mathcal{W})$ that does not meet J_k, $\mathrm{st}(W, \mathcal{W}) \cap k \cdot \mathrm{st}(W, \mathcal{W}) = \varnothing$. Define $J = \bigcup_{k \in F \cdot F^{-1}} J_k$; then J is bounded.

By construction of J, for every $\mathrm{st}(W, \mathcal{W}) \in \mathrm{st}(\mathcal{W}, \mathcal{W})$ that does not meet J we have that $\mathrm{st}(W, \mathcal{W}) \cap k \cdot \mathrm{st}(W, \mathcal{W}) = \varnothing$ for every $k \in F \cdot F^{-1}$. We will show that for any $x \in X \setminus \mathrm{st}(J, \mathrm{st}(\mathcal{W}, \mathcal{W}))$ there is a unique $g_x \in F^{-1}$ so that $g_x \cdot x, f(x) \in g_x \cdot V$ for some $V \in \mathcal{V}$ that does not meet J. Indeed, suppose not (for contradiction) and choose $x \in X \setminus \mathrm{st}(J, \mathcal{W})$.Then there are two distinct elements $g, h \in F^{-1}$ so that $g \cdot x, f(x) \in g \cdot V$ and $h \cdot x, f(x) \in h \cdot V'$ for some V and V' that do not meet J. Then we have that $g \cdot x, h \cdot x \in \mathrm{st}(g \cdot V, \mathcal{W})$ which implies $g \cdot x \in \mathrm{st}(g \cdot V, \mathcal{W}) \cap gh^{-1} \cdot \mathrm{st}(g \cdot V, \mathcal{W})$ which contradicts coarse discontinuity. So we have uniqueness of such an element in F^{-1}.

Using this uniqueness, we will now that there is a bounded set K so that $J \subseteq K$ and that for any two points $x, y \in U$ with $U \in \mathcal{U}$ not meeting K we have that $g_x = g_y$. Indeed, define $\mathcal{E}$ to be the trivial cover and define $\mathcal{B} = \bigcup_{g \in F^{-1}} g \cdot \mathcal{U}$. Then these are scales in $\mathcal{LSS}_d$ and so is the scale $\mathcal{R} = \mathrm{st}(\mathcal{E}, \mathrm{st}(\mathcal{W}, \mathrm{st}(f(\mathcal{U}), \mathrm{st}(\mathcal{W}, \mathcal{B}))))$. Using coarse discontinuity of the action for each $g \in F^{-1}$ with the cover $\mathcal{A} = \bigcup\limits_{g \in F^{-1}} g \cdot \mathcal{R}$, we have that for each element $g \in F \cdot F^{-1}$ we have that there is a bounded set K_g so that for any $A \in \mathcal{A}$ that does not meet K_g, we have $A \cap g \cdot A = \varnothing$. Define $K = \bigcup_{g \in F \cdot F^{-1}} K_g \ \cup J$. Then K is bounded and $J \subseteq K$. Suppose (for contradiction) that $g_x \neq g_y$ where $x, y \in U$ for some $U \in \mathcal{U}$ and $x, y \in X \setminus \mathrm{st}(K, \mathcal{A})$. Then there is an $A \in \mathcal{A}$ so that $g_x \cdot x, g_y \cdot x \in A$ and A does not meet K. Indeed, this follows from the following: $g_x \cdot x, f(x) \in W$ for $W \in \mathcal{W}$, $f(x), f(y) \in f(U)$ for $f(U) \in f(\mathcal{U})$, $g_y \cdot y, f(y) \in W'$ for $W' \in \mathcal{W}$, and $g_y \cdot y, g_y \cdot x \in g_y \cdot U$, for $g_y \cdot U \in \mathcal{B}$. Hence $g_x \cdot x, g_y \cdot x \in A$; furthermore, this is independent of the choice of x and y.We know that A can't intersect K because x is outside of $\mathrm{st}(K, \mathcal{A})$ and $x \in g_x^{-1} \cdot A$. But $g_x \cdot x, g_y \cdot x \in A$ implies $g_y \cdot x \in A \cap g_y g_x^{-1} A$ which is a contradiction. So $g_y = g_x$.

By use of coarse one-endedness, we have that there is a $L \subseteq X$ bounded so that $K \subseteq L$ and every pair of points outside of $X \setminus L$ is $\mathcal{U}$ connected. By the uniqueness of g_x for elements in the same $U \in \mathcal{U}$, we have that for any point $x \in X \setminus \text{st}(L, \mathcal{W})$, there is a unique $g \in F^{-1}$ so that $g \cdot x, f(x) \in W$ for some $W \in \mathcal{W}$ that does not meet L. Therefore, outside of $\text{st}(L, \mathcal{W})$ we have that $f \sim h_g$ via $\mathcal{W}$. Define $\mathcal{C}$ to be the cover ov X by $\text{diam}(\text{st}(L, \mathcal{W})) + 1$ balls. Then $f \sim h_g$ over all of X via $\mathcal{C}$ and hence (as elements of $\mathsf{Aut}(X/X_G)/\sim$), $[f] = [h_g]$. We therefore have that $\mathsf{Aut}(X/X_G) = \{[h_g] | g \in G\}$ as desired.

We now show the isomorphism as in the statement of the theorem. Define $\Phi : G \to \mathsf{Aut}(X/X_G)/\sim$ via $\Phi(g) = [h_g]$. This map is clearly a surjective homorphism. We will show injectivity. Suppose that there is a non-identity element $g \in G$ so that $g \in \text{Ker}(\Phi)$. Then we have that $h_g \sim h_e$. Let $\mathcal{D}$ witness the closeness. By coarse discontinuity there is a $P \subseteq X$ bounded so that for every $D \in \mathcal{D}$ that does not meet K, $D \cap g^{-1} \cdot D = \varnothing$. Let $x \in X$. By closeness, there is a $D_x \in \mathcal{D}$ so that $x, g \cdot x \in D_x$. If D_x does not meet K, then we would have that $x \in D_x \cap g^{-1} \cdot D_x$. Hence D_x meets K and thus $X \subseteq \text{st}(K, \mathcal{D})$. But $\mathcal{D} \in \mathcal{LSS}_d$ and K is bounded which implies that $\text{st}(K, \mathcal{D})$ is bounded which would mean that X is bounded, a contradiction. We must have that $g = e$ which implies Φ is injective and hence is an isomorphism. □

Given the theorem above, it is a natural question to ask if we can replace the coarsely one-ended condition with something weaker (like scale connectnedness for some $\mathcal{U} \in \mathcal{LSS}_d$). It turns out that scale conectnedness is not a strong enough conectnedness condition! Let X be the axes of $\mathbb{R}^2$ with the metric given from the euclidean metric on $\mathbb{R}^2$. We showed above that this space is not coarsely one-ended. However, X is scale connected (choose any covering by R-balls). Let $G = \mathbb{Z}/4\mathbb{Z}$ and let 1 act on X via a $\frac{\pi}{4}$ radian rotation of the axes counter-clockwise; call this action function h_1. This action is coarsely discontinuous via the methods shown in the example we gave of coarsely discontinuous actions. Recall that by [**?**] we have that $(X, \mathcal{LSS}_G)$ is coarsely equivalent to $(X/G, q(\mathcal{LSS}))$ which is coarsely equivalent to the positive real numbers with the large scale structure induced from the euclidean metric via the map $f : X/G \to \mathbb{R}$ where $f([(0, y)]) = (0, y)$. Note that the map $\phi : X \to X$ which swaps the negative x axis with the positive x axis and fixes the entire y axis is a coarse equivalence on X that is close to the identity on X in $\mathcal{LSS}_G$. This is because ϕ fixes the positive y axis which is coarsely equivalent to $\mathcal{LSS}_G$. However, there is no power of h_1 that is close to ϕ in $\mathcal{LSS}_d$. This is because for any $(x, 0) \in X$, that point must be moved a distance of 2|x|. There is no cover of finite mesh $\mathcal{U} \in \mathcal{LSS}_d$ with the property that for all $(x, y) \in X$ and a fixed $g \in G$, there is a $U \in \mathcal{U}$ so that $(x, y), h_g((x, y)) \in U$. Hence $[\phi] \neq [h_g]$ for all $g \in G$ $[f], [h_g] \in H(X/X_G)/\sim$, where $\sim$ is closeness by $\mathcal{LSS}_d$.

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Vita

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