8-2017

Localization of Large Scale Structures

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To the Graduate Council:

I am submitting herewith a dissertation written by Ryan James Jensen entitled "Localization of Large Scale Structures." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Jerzy Dydak, Major Professor

We have read this dissertation and recommend its acceptance:

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Localization of Large Scale Structures

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Ryan James Jensen

August 2017

I would like to dedicate this work to several people:

- First to my parents, Heber and Eileen Jensen, without whom this would not have been started.
- Next to my wife, Anna Jensen, without whom this would not have been finished.
- Finally to my children, James, Ashlyn, and Andrew Jensen, without whom this would have been finished much sooner.

Acknowledgments

First and foremost I would like to thank my adviser, Dr. Jerzy Dydak, for his advice, mentorship, and patience during my time as a graduate student. I am truly grateful for all he has done on my behalf. I have learned much from him, not only in my area of concentration but also in teaching philosophy. I hope to emulate some of his characteristics in my own career.

I also wish to thank the other topologists in our department with whom I have had some interaction. Specifically Dr. Brodskiy and Dr. Thistlethwaite for their help and support given during classes and seminars. I also thank them, along with Dr. Berry, for their willingness to serve on my committee.

I should also mention my gratitude to fellow graduate students with whom I have had many good discussions. A surely incomplete list includes Kevin Sinclair, Kyle Austin, Logan Higginbotham, and Thomas Weighill.

Finally I must thank Pam Armentrout for all she does in making graduate school go more smoothly. I am not alone in saying how much she is appreciated.

Abstract

We begin by giving the definition of coarse structures by John Roe, but quickly move to the equivalent concept of large scale geometry given by Jerzy Dydak. Next we present some basic but often used concepts and results in large scale geometry. We then state and prove the equivalence of various definitions of asymptotic dimension for arbitrary large scale spaces. Some of these are generalizations of asymptotic dimension for metric spaces, and many of the proofs are new. Particularly useful in proving the equivalences of the various definitions is the notion of partitions of unity, originally set forth by Jerzy Dydak. We then generalize the concept of bounded geometry, by defining the entropy and capacity of a set with respect to a cover. We show that all covers which are uniform with respect to a gauge form a large scale structure, which has many of the properties that spaces with bounded geometry have. Finally we restrict the uniformly bounded covers in a large scale structure in order to form a new structure called a localization. We seek to determine which large scale properties hold in the new structure.

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Chapter 1

Introduction

This dissertation is in the area of large scale geometry. Piotr Nowak and Guoliang Yu have a book, [NY12] , which gives an introduction to large scale geometry, although it is mostly restricted to the metric case. In this work we deal with what are called large scale spaces by Dydak and Hoffland in [DH08], and will for the most part not restrict ourselves to the metric case.

Before discussing large scale spaces we will briefly mention uniform spaces, which predate large scale spaces and large scale geometry in general. Uniform spaces were first studied in the mid 1930s by André Weil in [Wei38]. However, it was Isbell in the 1960s who brought together most of the theory scattered in various papers into a monograph on the subject, see [Isb64]. Uniform space theory is more general than metric spaces, and concerns itself with what happens at the small scale, such as uniform continuity or uniform convergence. Every uniform structure on a space induces a topology on the space, but the reverse is not true.

Large scale space theory is dual to uniform space theory in the sense that instead of looking at what happens in the small scale, it is concerned with the large scale. Every metric induces a uniform structure, and likewise every metric space induces a large scale structure. Also, just as uniform spaces are more general than metric spaces, large scale spaces are more general than metric spaces. The fact that a metric induces both a uniform structure and a large scale structure hints that spaces with both structures might be of interest. This is indeed the case, and while we don't treat the subject much here, Austin, Dydak, Holloway, and Weighill have proved many interesting results about such spaces, see [ADH16], [Hol16], or [DW16] for example.

Coarse structures, which are a different formulation of large scale structures, were first introduced by John Roe, Nigel Higson, and Erik Pedersen in [HPR97] to, in their words, "describe a general framework in which one can talk about the relationship between analysis (in particular, index theory on open manifolds) and controlled topology." They used their framework, coarse structures, to approach the Novikov Conjecture [Yu98], which is one of the main reasons why large scale geometry is of interest. Other applications of large scale geometry include geometric group theory and the Baum-Connes conjecture [Yu00].

In this dissertation, we only briefly mention the coarse structures, but will use extensively Dydak's formulation of large scale structures. The outline is as follows.

In Chapter 2 we give some preliminary definitions necessary for precisely defining large scale structures. We also present some often used results and comparison theorems. While we have made an effort to reference these results when necessary, we won't guarantee that we always do so. For the most part however this should not be a problem, as the results in this chapter are intuitive and not difficult to prove.

Chapter 3 concerns itself wholly with asymptotic dimension, which is a property of large scale spaces invariant under coarse maps. There are various definitions of asymptotic dimension for large scale spaces, and each has its uses. In [BD05], and for the metric case, Dranishnikov and Bell showed the equivalence of many of these definitions. We show the equivalence of these various definitions of asymptotic dimension for general large scale spaces. Particularly useful in proving these equivalences is the notion of partitions of unity, originally set forth by Dydak in [Dyd02].

Next in Chapter 4 we provide a generalization of spaces with bounded geometry. We use an idea originally given by Roe in [Roe03], however there was an error in one of his results which we rectify. Central to this chapter is the idea that in large scale structures, one can view elements of a cover as points of some larger cover. This idea is also useful in some of the proofs given in Chapter 3.

Finally, Chapter 5 deals with restricting the uniformly bounded covers in a large scale structure in order to form a new structure. We call this process localization. We then seek to determine which large scale properties hold in the new structure. We show by way of a counter example that asymptotic dimension is one property which does not hold after localization.

Chapter 2

Large Scale Structures

2.1 Coarse Structures

Roe gave the definition for coarse structures on a set X by considering subsets of $X \times X$ in [Roe03]. In particular, the **diagonal** of X is defined by $\Delta = \{(x, x) : x \in X\}$. For $E \subset X \times X$ the inverse of E is $E^{-1} = \{(x, y) : (y, x) \in E\}$. The set E is said to be symmetric if $E = E^{-1}$. If F is another subset of $X \times X$, then the product of E with F is $E \circ F = \{(x, y) : \exists z \in X : (x, z) \in E, (z, y) \in F\}$.

Definition 2.1. A **coarse structure** on a set X is a collection $\mathcal{E}$ of subsets of $X \times X$ which satisfy

1. $\Delta \in \mathcal{E}$;
2. If $E \in \mathcal{E}$ and $F \subset E$, then $F \in \mathcal{E}$;
3. If $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$;
4. If $E, F \in \mathcal{E}$ then $E \circ F \in \mathcal{E}$;
5. If $E, F \in \mathcal{E}$ then $E \cup F \in \mathcal{E}$.

A set $E \in \mathcal{E}$ is called a **controlled set** or an **entourage**.

This definition is analytic in nature, and at times is hard to work with. It is especially hard to visualize what is happening. Dydak and Hoffland defined large scale structures which are more topological in nature. They also proved that every large scale structure is a coarse structure and every coarse structure is a large scale structure. In addition, they gave a way of going between coarse structures and large scale structures, which we make use of in Chapter 4. Most of the definitions in Section 2.2 were first given in [DH08], but others appear in [Aus15a] or [Hol16].

2.2 Stars, Refinements, and Coarsenings

Definition 2.2. Let X be a set and $\mathcal{U}$ a collection of subsets of X, and $V \subset X$. The **star** of V with respect to $\mathcal{U}$, $\mathrm{st}(V,\mathcal{U})$, is the union of all elements of $\mathcal{U}$ which intersect V. That is

$$\mathrm{st}(V,\mathcal{U}) = \bigcup_{U \in \mathcal{U},\ U \cap V \neq \emptyset} U.$$

If $\mathcal{V}$ is another collection of subsets of X, then the **star** of $\mathcal{V}$ with respect to $\mathcal{U}$ is

$$\mathrm{st}(\mathcal{V},\mathcal{U}) = \{\mathrm{st}(V,\mathcal{U}) : V \in \mathcal{V}\}.$$

If $\mathcal{V}_1, \ldots \mathcal{V}_n$ are collections of subsets of X, then

$$\mathrm{st}(V, \mathcal{V}_1, \ldots \mathcal{V}_n) = \mathrm{st}(\mathrm{st}(\ldots \mathrm{st}(V, \mathcal{V}_1), \mathcal{V}_2), \ldots \mathcal{V}_n),$$

and

$$\mathrm{st}(\mathcal{V}_1, \ldots \mathcal{V}_n) = \mathrm{st}(\mathrm{st}(\ldots \mathrm{st}(\mathcal{V}_1, \mathcal{V}_2), \mathcal{V}_3), \ldots \mathcal{V}_n).$$

The n-**star** of V with respect to $\mathcal{U}$ is defined as

$$\begin{aligned} \mathrm{st}^0(V,\mathcal{U}) &= V \\ \mathrm{st}^n(V,\mathcal{U}) &= \mathrm{st}\left(\mathrm{st}^{n-1}(V,\mathcal{U}),\mathcal{U}\right) \quad \text{for } n \geq 1. \end{aligned}$$

Likewise

$$\mathrm{st}^n(\mathcal{V},\mathcal{U}) = \{\mathrm{st}^n(V,\mathcal{U}) : V \in \mathcal{V}\}.$$

Notice that we could write the n-star of a set V with respect to a collection of subsets $\mathcal{U}$ of X as

$$\operatorname{st}^n(V,\mathcal{U}) = \operatorname{st}(V,\underset{1}{\mathcal{U}},\ldots,\underset{n}{\mathcal{U}}),$$

where $\underset{i}{\mathcal{U}}$ denotes the index.

Definition 2.3. If X is a set with $\mathcal{U}$ and $\mathcal{V}$ collections of subsets of X, and for each $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ containing U, then we say that $\mathcal{U}$ **refines** $\mathcal{V}$, and we write $\mathcal{U} \prec \mathcal{V}$. In such a case we also say that $\mathcal{V}$ **coarsens** $\mathcal{U}$, and write $\mathcal{V} \succ \mathcal{U}$.

We will have occasion in Chapter 3 to frequently consider stars of a collection of subsets $\mathcal{U}$ against itself. The following definition is similar to those given above, but this time the order of the elements used in staring is different.

Definition 2.4. Let X be a set and $\mathcal{U}$ a collection of subsets of X. Then **n-star of $\mathcal{U}$** is defined inductively as follows:

$$\operatorname{st}^0(\mathcal{U}) = \mathcal{U}$$
$$\operatorname{st}^{n+1}(\mathcal{U}) = \operatorname{st}(\mathcal{U}, \operatorname{st}^n(\mathcal{U})).$$

Holloway proved several comparison results involving refinements and stars. The results are straightforward to prove, but the interested reader may see [Hol16] for the proofs.

Proposition 2.5.

1. *If $U \subset X$ and $\mathcal{V}$ is a cover of X, then $U \subset \operatorname{st}(U,\mathcal{V})$.*
2. *If U is a collection of subsets of X and $\mathcal{V}$ is a cover of X, then $\mathcal{U} \prec \operatorname{st}(\mathcal{U},\mathcal{V})$.*
3. *If $U \subset V$ and $\mathcal{W}$ is any collection of subsets of X, then $\operatorname{st}(U,\mathcal{W}) \subset \operatorname{st}(V,\mathcal{W})$.*
4. *If $\mathcal{U} \prec \mathcal{V}$ and $\mathcal{W}$ is any collection of subsets of X, then $\operatorname{st}(\mathcal{U},\mathcal{W}) \prec \operatorname{st}(\mathcal{V},\mathcal{W})$.*
5. *If $U \subset X$ and $\mathcal{V}_1, \mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1 \prec \mathcal{V}_2$, then $\operatorname{st}(U,\mathcal{V}_1) \subset \operatorname{st}(U,\mathcal{V}_2)$.*
6. *If $\mathcal{U}$ is any collection of subsets of X and $\mathcal{V}_1, \mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1 \prec \mathcal{V}_2$, then $\operatorname{st}(\mathcal{U},\mathcal{V}_1) \prec \operatorname{st}(\mathcal{U},\mathcal{V}_2)$.*

7. *If $\mathcal{U}$ is a scale (cover, but scale defined below) and $\mathcal{V}$ is any collection of subsets of X, then $\mathcal{V} \prec \operatorname{st}(\mathcal{U}, \mathcal{V})$.*

Some more comparison results are as follows:

Proposition 2.6. *If U_1 and U_2 are subsets of X, and $\mathcal{V}$ is a collection of subsets of X, then*

$$\operatorname{st}(U_1 \cup U_2, \mathcal{V}) = \operatorname{st}(U_1, \mathcal{V}) \cup \operatorname{st}(U_2, \mathcal{V}).$$

Proof. First take $x \in \operatorname{st}(U_1 \cup U_2, \mathcal{V})$. So $x \in V$ for some $V \in \mathcal{V}$ where $V \cap (U_1 \cup U_2) \neq \emptyset$. Thus $V \cap U_1 \neq \emptyset$ or $V \cap U_2 \neq \emptyset$. Hence $x \in \operatorname{st}(U_1, \mathcal{V}) \cup \operatorname{st}(U_2, \mathcal{V})$.

Next take $x \in \operatorname{st}(U_1, \mathcal{V}) \cup \operatorname{st}(U_2, \mathcal{V})$, without loss of generality suppose $x \in \operatorname{st}(U_1, \mathcal{V})$. So there is a $V \in \mathcal{V}$ containing x and intersecting $U_1 \subset U_1 \cup U_2$. Thus $x \in \operatorname{st}(U_1 \cup U_2, \mathcal{V})$. □

In large scale geometry, it is frequently important to see what happens to collections of subsets under maps. Because of this we will introduce some more notation. Suppose X and Y are sets and $f : X \to Y$. If $\mathcal{U}$ is a collection of subsets of X and $\mathcal{V}$ is a collection of subsets of Y then we define

$$f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\},$$

and

$$f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}.$$

In [Hol16], Holloway also proved the following two results.

Proposition 2.7. *If $f : X \to Y$ is a function and $\mathcal{U}$ and $\mathcal{V}$ are collections of subsets of X, then*

$$f(\operatorname{st}(\mathcal{U}, \mathcal{V})) \prec \operatorname{st}(f(\mathcal{U}), f(\mathcal{V})).$$

Proposition 2.8. *Let $f : X \to Y$ be a function with $\mathcal{U}$ and $\mathcal{V}$ collections of subsets of Y. Then*

1. $\operatorname{st}(f^{-1}(\mathcal{U}), f^{-1}(\mathcal{V})) \prec f^{-1}(\operatorname{st}(\mathcal{U}, \mathcal{V}))$*; and*

2. *if f is surjective, then* $\mathrm{st}(f^{-1}(\mathcal{U}), f^{-1}(\mathcal{V})) = f^{-1}(\mathrm{st}(\mathcal{U}, \mathcal{V}))$.

If $\mathcal{U}$ is a cover of X, then we call $\mathcal{U}$ a **scale**. $\mathcal{U}$ is a **smaller scale** than a scale $\mathcal{V}$ if $\mathcal{U} \neq \mathcal{V}$ and $\mathrm{st}(\mathcal{U},\mathcal{U})$ refines $\mathcal{V}$. (See [Aus15a, ADH16]).

2.3 Large Scale Structures

The concluding section in this chapter is dedicated to at last defining large scale structures, as well as functions between those structures.

Definition 2.9. A **large scale structure** $\mathcal{LSS}$ on a set X is a non-empty set of families $\mathcal{B}$ of subsets of X (each family $\mathcal{B}$ is called **uniformly bounded** or **uniformly $\mathcal{LSS}$-bounded** once $\mathcal{LSS}$ is fixed) satisfying the following conditions:

1. $\mathcal{B}_1 \in \mathcal{LSS}$ implies $\mathcal{B}_2 \in \mathcal{LSS}$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.

2. $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}$ implies $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}$.

Now that we have a formal definition for large scale spaces, we talk about functions between such spaces. The following is the dual of the more familiar uniform continuity on metric spaces.

Definition 2.10. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces. A function $f : X \to Y$ is **large scale continuous** (also called **coarse** or by Roe, **bornologous**) if for every $\mathcal{B} \in \mathcal{LSS}_X$, $f(\mathcal{B}) \in \mathcal{LSS}_Y$.

Definition 2.11. Let X and Y be large scale spaces, and $f, g : X \to Y$ two maps. We say f and g are **close** if there is a uniformly bounded cover $\mathcal{B}$ of Y so that for each $x \in X$ there is some $B \in \mathcal{B}$ containing both $f(x)$ and $g(x)$.

Definition 2.12. Let X and Y be large scale spaces and $f : X \to Y$ a large scale continuous map. We say that X and Y are **coarsely equivalent**, and that f is a **coarse equivalence**, if there is a large scale continuous map $g : Y \to X$ so that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X.

Definition 2.13. Let $(X, \mathcal{LSS}_X)$ and $(Y, \mathcal{LSS}_Y)$ be large scale spaces. A function $f : X \to Y$ is called **effectively proper** if for each point $y \in Y$ and each $\mathcal{V} \in \mathcal{LSS}_Y$, there is a $\mathcal{U} \in \mathcal{LSS}_X$ so that

$$f^{-1}(\mathrm{st}(\{y\}, \mathcal{V})) \subset \mathrm{st}(\{x\}, \mathcal{U}).$$

Roe also defines a **rough map** as an effectively proper bornologous map, and notes that f is rough exactly when X is a coarse equivalence with $f(X) \subset Y$, and hence rough maps may be called a **coarse embedding**.

Chapter 3

Asymptotic Dimension

3.1 Introduction

An important topological invariant is the covering dimension, also called the topological dimension or Lebesgue dimension. The notion dual to covering dimension is known as asymptotic dimension, and is invariant under coarse maps. In [BD05] Bell and Dranishnikov give five equivalent definitions of asymptotic dimension for metric spaces. Multiple definitions are valuable since it is at times easier to apply one than another. The purpose of this chapter is to generalize part of the list given by Bell and Dranishnikov to arbitrary large scale spaces.

Asymptotic dimension for large scale spaces is usually defined in terms of the multiplicity of covers. If $\mathcal{U}$ is a cover of a space X then the **multiplicity** of $\mathcal{U}$ is largest n so the n-elements of $\mathcal{U}$ have a point in common, or ∞ if no such n exists. Alternatively, we could define a multiplicity function on points with respect to a cover:

Definition 3.1. Let $\mathcal{U}$ be a cover of a set X. The **multiplicity** $\text{mult}_{\mathcal{U}}(x)$ **of** $\mathcal{U}$ **at** x is the number of elements of $\mathcal{U}$ containing x, or ∞ if there are infinitely many such elements of $\mathcal{U}$.

And then define the multiplicity of $\mathcal{U}$ to be $\sup_{x\in X}\{\text{mult}_{\mathcal{U}}(x)\}$. From this, we get the classic definition of asymptotic dimension for large scale spaces:

Definition 3.2. If X is a space with a large scale structure $\mathcal{LSS}$, then X has **asymptotic dimension** at most n if for every uniformly bounded cover $\mathcal{U}$ of X, there is a uniformly bounded cover $\mathcal{V}$ of X which coarsens $\mathcal{U}$ and has multiplicity at most $n+1$. This is denoted

$$\operatorname{asdim}(X) \leq n.$$

The asymptotic dimension of X is equal to n if $\operatorname{asdim}(X) \leq n$ but $\operatorname{asdim}(X) \not\leq n-1$.

The reason we define asymptotic dimension as less than some number n is we usually only care if we have finite asymptotic dimension, so we only need to show that $\operatorname{asdim}(X) \leq n$ for some n. Also it is often difficult to find the exact asymptotic dimension, but easier to find an upper bound.

3.2 Nerves and Combinatorial Dimension

An equivalent way to define finite asymptotic dimension via covers is using nerves and the combinatorial dimension of a simplicial complex. The following definitions first appeared in [DV16], but we list them here for convenience.

The **combinatorial dimension** of a simplicial complex K is the supremum over all n so that K has an n-simplex. Let X be a set and $\mathcal{U}$ a family of subsets of X. The **nerve** of $\mathcal{U}$, denoted $\mathcal{N}(\mathcal{U})$, is the simplicial complex which has $\mathcal{U}$ as its vertices, and $[U_0, \ldots U_n]$ is an n-simplex if $\cap_{i=0}^{n} U_i \neq \emptyset$. The **dimension** $\dim(\mathcal{U})$ of $\mathcal{U}$ is the combinatorial dimension of its nerve. Using the definition of dimension just stated, we now have another definition of asymptotic dimension.

Definition 3.3. A large scale structure X is said to have **asymptotic dimension** $\operatorname{asdim}(X)$ at most n if for every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded cover $\mathcal{V}$ of X coarsening $\mathcal{U}$ with dimension at most n.

It is clear that Definition 3.3 and Definition 3.2 are equivalent since if the $\mathcal{U}$-multiplicity is $n+1$, then the nerve of $\mathcal{U}$ has an n-simplex but nothing larger. And if $\mathcal{U}$ has an n-simplex with n maximal, then the $\mathcal{U}$-multiplicity is $n+1$. We prefer definition Definition 3.3 since in this case having asymptotic dimension at most n

corresponds to having a coarsening cover with dimension at most n. The $n+1$ bit is hidden in this case as the indexing starts at 0.

Using Definition 3.3 we can give a definition of covering dimension (see also [BD05]).

Definition 3.4. A topological space X is said to have **covering dimension** at most n if for every open cover $\mathcal{U}$ of X there exists a cover $\mathcal{V}$ of X refining $\mathcal{U}$ with dimension at most n.

Notice that with this definition it is easy to see that asymptotic dimension is dual to covering dimension. Instead of an open cover we take a uniformly bounded cover; and instead of requiring a refining cover we require a coarsening uniformly bounded cover.

We will now deviate a little from discussing asymptotic dimension to explore further the idea of the dimension of a cover $\mathcal{U}$ with respect to another cover $\mathcal{V}$.

Definition 3.5. Let $\mathcal{U}$ and $\mathcal{V}$ be covers of a space X. The **nerve of $\mathcal{U}$ with respect to $\mathcal{V}$**, denoted $\mathcal{N}_{\mathcal{V}}(\mathcal{U})$, is the simplicial complex with $\mathcal{U}$ as vertices, and $[U_0, \ldots, U_n]$ an n-simplex if there is a $V \in \mathcal{V}$ so that

$$V \cap U_i \neq \emptyset$$

for each $0 \leq i \leq n$.

Since we have a simplicial complex, we can talk about the combinatorial dimension of that complex, just as we did before.

Definition 3.6. For $\mathcal{U}$ and $\mathcal{V}$ covers of a space X, the $\mathcal{V}$ **dimension of** $\mathcal{U}$, written $\dim_{\mathcal{V}}(\mathcal{U})$, is the combinatorial dimension of $\mathcal{N}_{\mathcal{V}}(\mathcal{U})$.

For a space X, denote by $\mathcal{D}_X$ the cover of X by singletons. Notice that for a cover $\mathcal{U}$ of X, $\mathcal{N}_{\mathcal{D}_X}(\mathcal{U}) = \mathcal{N}(\mathcal{U})$, and $\dim_{\mathcal{D}_X}(\mathcal{U}) = \dim(\mathcal{U})$. If $\mathcal{V}$ is another cover of X, we can think of $\dim_{\mathcal{V}}(\mathcal{U})$ as one less than the number of elements of $\mathcal{U}$ intersecting a $V \in \mathcal{V}$. We also think of $\dim(\mathcal{U})$ as (one less than) the number of elements of $\mathcal{U}$ with a common intersection. We can also go the other way, and talk about the number of elements of $\mathcal{U}$ contained in an element V of $\mathcal{V}$.

Definition 3.7. Let $\mathcal{U}$ and $\mathcal{V}$ be covers a space X. The **nerve of $\mathcal{U}$ with respect to $-\mathcal{V}$**, denoted $\mathcal{N}_{-\mathcal{V}}(\mathcal{U})$, is the simplicial complex with $\mathcal{U}$ as vertices, and $[U_0, \ldots, U_n]$ an n-simplex if there is a $V \in \mathcal{V}$ containing all U_i. That is

$$\bigcup_{i=0}^{n} U_i \subset V.$$

Definition 3.8. For $\mathcal{U}$ and $\mathcal{V}$ covers of a space X, the $-\mathcal{V}$ **dimension of $\mathcal{U}$**, written $\dim_{-\mathcal{V}}(\mathcal{U})$, is the combinatorial dimension of $\mathcal{N}_{-\mathcal{V}}(\mathcal{U})$.

It is not difficult to get a result comparing these various dimensions.

Proposition 3.9. *Let $\mathcal{U}$ and $\mathcal{V}$ be covers of a space X. Then*

$$\dim_{-\mathcal{V}}(\mathcal{U}) \leq \dim_{\mathcal{V}}(\mathcal{U}) \leq \dim(\mathrm{st}(\mathcal{U}, \mathcal{V})).$$

Proof. Suppose $\dim_{-\mathcal{V}}(\mathcal{U}) = n$. So there are $U_0, \ldots U_n \in \mathcal{U}$ so that each $U_i \subset V$ for some $V \in \mathcal{V}$. Clearly, each U_i intersects V, so $\dim_{\mathcal{V}}(\mathcal{U}) \geq n$. A similar argument works if $\dim_{-\mathcal{V}}(\mathcal{U}) = \infty$. Hence $\dim_{-\mathcal{V}}(\mathcal{U}) \leq \dim_{\mathcal{V}}(\mathcal{U})$.

Next suppose $\dim_{\mathcal{V}}(\mathcal{U}) = n$. This means there is a $V \in \mathcal{V}$ and $U_0, \ldots U_n \in \mathcal{U}$ so that V intersects each U_i. Thus $V \subset \mathrm{st}(U_i, \mathcal{V})$ for each $0 \leq i \leq n$. This in turn means

$$\bigcap_{i=0}^{n} \mathrm{st}(U_i, \mathcal{V}) \neq \emptyset.$$

A similar argument works if $\dim_{\mathcal{V}}(\mathcal{U}) = \infty$. Thus $\dim_{\mathcal{V}}(\mathcal{U}) \leq \dim(\mathrm{st}(\mathcal{U}, \mathcal{V}))$. □

Proposition 3.10. *Let X be a space with a cover $\mathcal{U}$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are also covers of X with $\mathcal{V}_1$ refining $\mathcal{V}_2$, then $\dim_{\mathcal{V}_1}(\mathcal{U}) \leq \dim_{\mathcal{V}_2}(\mathcal{U})$.*

Proof. Suppose $\dim_{\mathcal{V}_1}(\mathcal{U}) = n$. So there is a $V_1 \in \mathcal{V}$ and $U_0, \ldots U_n \in \mathcal{U}$ so that $V_1 \cap U_i$ is non-empty for $0 \leq i \leq n$. Since $\mathcal{V}_1$ refines $\mathcal{V}_2$, there is a $V_2 \in \mathcal{V}_2$ so that $V_1 \subset V_2$.

Thus $V_2 \cap U_i \neq \emptyset$ for $0 \leq i \leq n$, showing that $\dim_{\mathcal{V}_2}(\mathcal{U}) \geq n$. A similar argument holds if $\dim_{\mathcal{V}_1}(\mathcal{U}) = \infty$. □

The $-\mathcal{V}$-dimension of $\mathcal{U}$ will not be of much use to us; however, both the dimension of $\mathcal{U}$ and the $\mathcal{V}$-dimension of $\mathcal{U}$ will be useful in terms of asymptotic dimension.

For the metric case, Dydak and Virk defined in [DV16] the R-dimension of a scale $\mathcal{U}$. Their definition is that if $\mathcal{U}$ is a family of subsets of a metric space X and $R \geq 0$, the R**-dimension** $\dim_R(\mathcal{U})$ of $\mathcal{U}$ is the dimension of the family

$$B(\mathcal{U}, R) = \{B(U, R) : U \in \mathcal{U}\},$$

where $B(U, R)$ is all points $x \in X$ so that there is a $u \in U$ with $d(x, u) < R$. The observant reader will notice that their definition is slightly different than our definition when restricted to the metric case. Indeed, Dydak and Virk's definition is that if $\mathcal{U}$ is a cover of a metric space X, and $R \geq 0$ is a scale, then

$$\dim_R(\mathcal{U}) = \dim(\mathrm{st}\,(\mathcal{U}, \{B(x, R/2)\}_{x \in X})),$$

while ours is

$$\dim_R(\mathcal{U}) = \dim(\mathrm{st}(\mathcal{U}, \{B(x, R)\}_{x \in X})).$$

However this causes no trouble when proving the results to follow in this section.

For the rest of this section, we will use some shorthand notation. If $\mathcal{U}$ is a cover of a space X, and $x \in X$ a point, we will denote by U_x the star of $\{x\}$ with respect to $\mathcal{U}$, that is $U_x = \mathrm{st}(\{x\}, \mathcal{U})$. We will also use $\mathcal{U}_X = \{U_x\}_{x \in X}$.

Proposition 3.11. *Let X be large scale structure, and $\mathcal{U}$ a uniformly bounded cover of X. Then*

$$\mathcal{U} \prec \mathcal{U}_X \prec \mathrm{st}(\mathcal{U}, \mathcal{U}).$$

As a result, $\mathcal{U}_X$ is a uniformly bounded family.

Proof. We first show that $\mathcal{U}$ refines $\mathcal{U}_X$. Let $U \in \mathcal{U}$, and choose $x \in U$. Then $U_x = \mathrm{st}(\{x\}, \mathcal{U}) \in \mathcal{U}$, and as U intersects the point x, $U \subset U_x$. Next we show that

$\mathcal{U}_X$ refines st($\mathcal{U}$,$\mathcal{U}$). Take $U_x \in \mathcal{U}_X$, and take U to be any set in $\mathcal{U}$ which intersects the point x, where x is the point corresponding to U_x. Then notice that st(U,$\mathcal{U}$) contains $U_x = \text{st}(\{x\},\mathcal{U})$. This shows that $\mathcal{U}_X \prec \text{st}(\mathcal{U},\mathcal{U})$. Finally as X is a large scale structure and $\mathcal{U}$ is a uniformly bounded cover with st($\mathcal{U}$,$\mathcal{U}$) uniformly bounded as well, it follows that $\mathcal{U}_x$ is a uniformly bounded cover since it refines st($\mathcal{U}$,$\mathcal{U}$). □

Proposition 3.12. *If X is a space, with $\mathcal{U}$ and $\mathcal{V}$ covers of X, then*

$$\dim(\text{st}(\mathcal{U},\mathcal{V})) = \dim_{\mathcal{V}_X}(\mathcal{U}).$$

Proof. First suppose that $\dim(\text{st}(\mathcal{U},\mathcal{V})) = n$. This means there are $\text{st}(U_0,\mathcal{V}),\ldots,\text{st}(U_n,\mathcal{V}) \in \text{st}(\mathcal{U},\mathcal{V})$ (for some $U_0,\ldots U_n \in \mathcal{U}$) with non-empty intersection. So take

$$x \in \bigcap_{i=0}^{n} \text{st}(U_i,\mathcal{V}).$$

Then take $V_x = \text{st}(\{x\},\mathcal{V}) \in \mathcal{V}_X$. Notice that $U_i \cap V_x$ is non-empty for each $0 \leq i \leq n$. This shows that

$$\dim(\text{st}(\mathcal{U},\mathcal{V})) \leq \dim_{\mathcal{V}_X}(\mathcal{U}).$$

Next suppose $\dim_{\mathcal{V}_X}(\mathcal{U}) = n$. So there exists $U_0,\ldots U_n \in \mathcal{U}$ and a $V_x \in \mathcal{V}_X$ so that $V_x \cap U_i = \text{st}(\{x\},\mathcal{V}) \cap U_i$ is non-empty for $0 \leq i \leq n$. So for each i, there is a V_i so that $x \in V_i$ and $U_i \cap V_i$ is non-empty. Thus $x \in \text{st}(U_i,\mathcal{V})$, showing that $\dim_{\mathcal{V}_X}(\mathcal{U}) \leq \dim(\text{st}(\mathcal{U},\mathcal{V})$. This together with the first half of the proof shows

$$\dim(\text{st}(\mathcal{U},\mathcal{V})) = \dim_{\mathcal{V}_X}(\mathcal{U}).$$

□

The main reason for defining the nerve of a cover with respect to another cover is that it gives us another definition of asymptotic dimension. The equivalence of Definition 3.3 and Definition 3.13 is proved in Theorem 3.27.

Definition 3.13. A large scale structure X has **asymptotic dimension** at most n if for every uniformly bounded cover $\mathcal{U}$ of X, there is a uniformly bounded cover $\mathcal{V}$ of X so that $\dim_{\mathcal{U}}(\mathcal{V}) \leq n$.

3.3 Partitions of Unity

We are interested in partitions of unity because they give us another definition of asymptotic dimension. The definitions given in this section are a conglomeration pulled from [Dyd02], [CDV15], [Hol16], and [CDV14].

The classical definition of a partition of unity on a space X is a collection of functions $\{\phi_s : X \to [0,1]\}_{s\in S}$ so that for each point $x \in X$ one has that all but finitely many $\phi_s(x) = 0$, and

$$\sum_{s\in S} \phi_s(x) = 1.$$

We will use a different definition, originally given by Dydak in [Dyd02].

For a space X and collection of functions $\Phi = \{\phi_s : X \to [0,\infty)\}_{s\in S}$, the notation $\phi = \sum_{s\in S} \phi_s$ means that for each $x \in X$

$$\phi(x) = \sup\left\{\sum_{t\in T} \phi_t(x) : T \subset S, |T| < \infty\right\}.$$

In such a case, the family Φ is called a **partition of** $\phi : X \to [0,\infty]$. In the case that $\phi = \sum_{s\in S} \phi_s = 1$, then the collection is called a **partition of unity**. Φ is a **finite partition of** ϕ if ϕ_s is the zero function for all but finitely many $s \in S$, and Φ is a **point finite partition of** ϕ if $\Phi|\{x\}$ is a finite partition of $\phi|\{x\}$ for all $x \in X$.

For a set S, and $1 \leq p < \infty$ define the ℓ_p space on S to be

$$\ell_P(S) = \left\{f : S \to \mathbb{R} : \sum_{s\in S} |f(s)|^p < \infty\right\}.$$

Let us also define

$$\Delta_p(S) = \left\{ f : X \to [0,1] : \sum_{s\in S} f(s)^p = 1, f \text{ has finite support} \right\}.$$

For each p, it is the case that $\ell_p(S)$ is a metric space with norm

$$\|f\|_p = \left(\sum_{s\in S} |f(s)|^p \right)^{1/p}.$$

Notice that $\Delta_p(S) \subset \ell_p(S)$ for each p and inherits the ℓ_p metric. We are interested in the case when $p = 1$, and will use the notation $\Delta(S) = \Delta_1(S)$.

A simplicial complex can also be viewed in terms of functions. Given a set S, we take the elements of S to be vertices, and we call $\Delta(S)$ the **full complex** over S. A **simplicial complex** K is then a subcomplex of $\Delta(S)$. This is useful for us since we have a simplicial complex $\mathcal{N}(\mathcal{U})$ for some cover $\mathcal{U}$ of a space X, and we wish to relate it to partitions of unity. The **star** of a vertex $v \in K^{(0)}$, denoted $\mathrm{st}(v)$ is all the functions $f \in K$ so that $f(v) > 0$.

It will be useful for us to go between point finite partitions of unity on X, each index by the set S, and functions of the form $\psi : X \to \Delta(S)$. In fact there is a one-to-one correspondence between the two. Suppose $\{\phi_s : X \to [0,1]\}_{s\in S}$ is a point finite partition of unity on X. Then

$$\{\phi_s\}_{s\in S} \mapsto \psi : X \to \Delta(S), \tag{3.1}$$

defined by $\psi(x)(s) = \phi_s(x)$. Notice that $\psi(x)$ is in fact in $\Delta(S)$ since

$$\sum_{s\in S} \psi(x)(s) = \sum_{s\in S} \phi_s(x) = 1,$$

as $\{\phi_s\}_{s\in S}$ was a point finite partition of unity on X.

We are working our way towards building a partition of unity on a space X based on two covers $\mathcal{U}$ and $\mathcal{V}$ of X. We first need a few more definitions.

Definition 3.14. Let X be a space with a cover $\mathcal{U}$. A **$\mathcal{U}$-chain** is a sequence of points $x_0, \dots x_n$ so that for each $i = 0, \dots n-1$, there is a $U_i \in \mathcal{U}$ containing both x_i, and x_{i+1}.

Definition 3.15. Let X be a space, $\mathcal{U}$ a cover of X, and $V \subset X$. For each $x \in X$, the **index of x in V with respect to $\mathcal{U}$**, denoted $\operatorname{ind}_{\mathcal{U}}(x, V)$, is the smallest $k \geq 0$ so that there is a $\mathcal{U}$-chain $x = x_0, \dots, x_k$ where $x_k \notin V$. If no such chain exists, we put $\operatorname{ind}_{\mathcal{U}}(x, V) = \infty$. Notice that $\operatorname{ind}_{\mathcal{U}}(x, V) = 0$ if and only if $x \notin V$.

Now let us take a space X with a cover $\mathcal{U}$, and $\mathcal{V} = \{V_s\}_{s \in S}$ another cover so that $\operatorname{mult}_{\mathcal{V}}(x) < \infty$ for all $x \in X$. Then for each $s \in S$ we can define a function $(\phi_{\mathcal{U}}^{\mathcal{V}})_s : X \to [0, 1]$ by

$$(\phi_{\mathcal{U}}^{\mathcal{V}})_s(x) = \begin{cases} 0 & \text{if } \operatorname{ind}_{\mathcal{U}}(x, V_s) < \infty; \operatorname{ind}_{\mathcal{U}}(x, V_t) = \infty \text{ for } k \text{ distinct } t \in S \\ \dfrac{1}{k} & \text{if } \operatorname{ind}_{\mathcal{U}}(x, V_s) = \infty; \operatorname{ind}_{\mathcal{U}}(x, V_t) = \infty \text{ for } k \text{ distinct } t \in S \\ \dfrac{\operatorname{ind}_{\mathcal{U}}(x, V_s)}{\sum_{t \in S} \operatorname{ind}_{\mathcal{U}}(x, V_t)} & \text{otherwise.} \end{cases} \tag{3.2}$$

Proposition 3.16. *If X is a space with cover $\mathcal{U}$ and $\mathcal{V} = \{V_s\}_{s \in S}$ another cover so that $\operatorname{mult}_{\mathcal{V}}(x) < \infty$ for all $x \in X$, then the*

$$\phi_{\mathcal{U}}^{\mathcal{V}} = \sum_{s \in S} (\phi_{\mathcal{U}}^{\mathcal{V}})_s$$

defined in Equation (3.2) is a point finite partition of unity.

Proof. First notice that since $\operatorname{mult}_{\mathcal{V}}(x) < \infty$ for each $x \in X$ that $\phi_{\mathcal{U}}^{\mathcal{V}}$ is point finite. Also all sums involved have all but finitely many terms as 0, and hence are also finite (this also shows that a finite k exists when applicable in Equation (3.2)). We wish to show $1 = \sum_{s \in S} (\phi_{\mathcal{U}}^{\mathcal{V}})_s(x)$ for each $x \in X$. So fix $x \in X$, then there are two cases.

The first case is when $\operatorname{ind}_{\mathcal{U}}(x, V_t) = \infty$ for k distinct $t \in S$. In this case we have

$$\sum_{s \in S} (\phi_{\mathcal{U}}^{\mathcal{V}})_s(x) = k \cdot (1/k) = 1.$$

The next case is when $\mathrm{ind}_{\mathcal{U}}(x, V_t) < \infty$ for all $t \in S$. In this case

$$\sum_{s\in S}(\phi_{\mathcal{U}}^{\mathcal{V}})_s(x) = \sum_{s\in S}\frac{\mathrm{ind}_{\mathcal{U}}(x, V_s)}{\sum_{t\in S}\mathrm{ind}_{\mathcal{U}}(x, V_t)} = \frac{1}{\sum_{t\in S}\mathrm{ind}_{\mathcal{U}}(x, V_t)} \cdot \sum_{s\in S}\mathrm{ind}_{\mathcal{U}}(x, V_s) = 1,$$

as for each $t \in S$, $\mathrm{ind}_{\mathcal{U}}(x, V_t)$ is finite and the sums are finite as stated above. Thus $\phi_{\mathcal{U}}^{\mathcal{V}}$ is a point finite partition of unity. □

Due to the correspondence given in Equation (3.1), we may think of the partition of unity $\phi_{\mathcal{U}}^{\mathcal{V}}$ in Proposition 3.16 as a function $\psi_{\mathcal{U}}^{\mathcal{V}} : X \to \Delta(S)$, where S is the index set for $\mathcal{V}$. In fact we can do even better than this, since $\mathrm{im}(\psi_{\mathcal{U}}^{\mathcal{V}}) \subset \mathcal{N}(\mathcal{V})$. To see this notice that from the correspondence given above that for each $s \in S$ $\psi_{\mathcal{U}}^{\mathcal{V}}(x)(s) = (\phi_{\mathcal{U}}^{\mathcal{V}})_s(x)$. Then take $\psi_{\mathcal{U}}^{\mathcal{V}}(x) \in \mathrm{im}(\psi_{\mathcal{U}}^{\mathcal{V}}) \subset \Delta(S)$. We have that

$$1 = \sum_{s\in S}\psi_{\mathcal{U}}^{\mathcal{V}}(x)(s) = \sum_{t\in T}\psi_{\mathcal{U}}^{\mathcal{V}}(x)(t),$$

where $T = \{t_0, \dots t_n\}$ is a finite subset of S, and $t \in T$ if and only if $\psi_{\mathcal{U}}^{\mathcal{V}}(x)(t) > 0$. But for each $t \in T$,

$$0 < \psi_{\mathcal{U}}^{\mathcal{V}}(x)(t) = (\phi_{\mathcal{U}}^{\mathcal{V}})_t(x)$$

means that $x \in V_t$. Hence $[t_0, \dots t_n]$ is an n-simplex in $N(\mathcal{V})$, showing that $\mathrm{im}(\psi_{\mathcal{U}}^{\mathcal{V}}) \subset \mathcal{N}(\mathcal{V})$, and we may write $\psi_{\mathcal{U}}^{\mathcal{V}} : X \to \mathcal{N}(\mathcal{V})$.

It should be noted that because of the correspondence given in Equation (3.1), that we use $\psi_{\mathcal{U}}^{\mathcal{V}} : X \to \mathcal{N}(\mathcal{V})$ and $\phi_{\mathcal{U}}^{\mathcal{V}} : X \to [0, 1]$ interchangeably. In particular, we may refer to the partition of unity $\phi_{\mathcal{U}}^{\mathcal{V}}$ defined by Equation (3.2) and think of its image lying in $\mathcal{N}(\mathcal{V})$.

The next few lemmas can be found in [CDV15], we include their proofs for completeness.

Lemma 3.17. *Let X be a space with $\mathcal{U}$ and $\mathcal{V} = \{V_s\}_{s\in S}$ covers of X with $\mathcal{U} \prec \mathcal{V}$ and $\mathrm{mult}_{\mathcal{V}}(x) < \infty$ for all $x \in X$. Let $\phi_{\mathcal{U}}^{\mathcal{V}} : X \to [0, 1]$ be the point finite partition of* *unity defined in Equation (3.2), with corresponding $\psi_{\mathcal{U}}^{\mathcal{V}} : X \to \mathcal{N}(\mathcal{V})$. Then the cover*

$$\{(\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_S))\}_{V_s \in \mathcal{V}}$$

coarsens $\mathcal{U}$.

Proof. Take $U \in \mathcal{U}$ and choose $x \in U$. There are two cases.

The first case is when $\mathrm{ind}_{\mathcal{U}}(x, V_s) < \infty$ for all $V_s \in \mathcal{V}$. In this case choose $V_t \in \mathcal{V}$ containing U. Notice that $\mathrm{ind}_{\mathcal{U}}(x, V_t) > 0$, so $\psi_{\mathcal{U}}^{\mathcal{V}}(x)(t) = (\phi_{\mathcal{U}}^{\mathcal{V}})_t(x) > 0$. Thus $x \in (\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_t))$. Now let $y \in U$. If we show that $(\phi_{\mathcal{U}}^{\mathcal{V}})_t(y) > 0$ then, as with x, $y \in (\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_t))$ and since $y \in U$ was arbitrary, $U \subset (\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_t))$. So by way of contradiction, suppose $(\phi_{\mathcal{U}}^{\mathcal{V}})_t(y) = 0$. It is not the case that $y \notin V_t$ as $V_t \supset U$. So it must be that $\mathrm{ind}_{\mathcal{U}}(y, V_t) < \infty$ and $\mathrm{ind}_{\mathcal{U}}(y, V_r) = \infty$ for some $r \in S$. Since $x, y \in U$, it must be the case that $x \in V_r$ as otherwise $\mathrm{ind}_{\mathcal{U}}(y, V_r) = 1$. But if $x \in V_r$ then $\mathrm{ind}_{\mathcal{U}}(x, V_r) < \infty$, implying that $\mathrm{ind}_{\mathcal{U}}(y, V_r) < \infty$ a contradiction. Hence $(\phi_{\mathcal{U}}^{\mathcal{V}})_t(y) = 0$ as desired.

The next case is when $\mathrm{ind}_{\mathcal{U}}(x, V_t) = \infty$ for some $V_t \in \mathcal{V}$. Then if there is an $x_1 \in U$ which is not in V_t we would have $\mathrm{ind}_{\mathcal{U}}(x, V_t) = 1$ which is a contradiction, so $U \subset V_t$. Now take $y \in U$. If $\mathrm{ind}_{\mathcal{U}}(y, V_t) < \infty$, then as before we would have $\mathrm{ind}_{\mathcal{U}}(x, V_t) < \infty$. Thus $y \in (\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_t))$ and $U \subset (\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_t))$. Hence $\mathcal{U} \prec \{(\psi_{\mathcal{U}}^{\mathcal{V}})^{-1}(\mathrm{st}(V_s))\}_{V_s \in \mathcal{V}}$. □

Definition 3.18. For $\mathcal{U}$ a cover of a space X, M a metric space, and $f : X \to M$, the **$\mathcal{U}$-variation** of f, denoted $\mathrm{var}_{\mathcal{U}}(f)$ is defined as

$$\mathrm{var}_{\mathcal{U}}(f) = \sup\{d(f(x), f(y)) : \text{for some } U \in \mathcal{U}, U \supset \{x, y\}\}.$$

Lemma 3.19. *Let $\mathcal{U}$ be a cover of a space X, $p : X \to [0, \infty)$, and $q : X \to [m, \infty)$ for some $m > 0$. If $p \leq q$, $\mathrm{var}_{\mathcal{U}}(p) \leq 1$, and $\mathrm{var}_{\mathcal{U}}(q) \leq n$ then $\mathrm{var}_{\mathcal{U}}(p/q) \leq \frac{n+1}{m}$.*

Proof. Take $U \in \mathcal{U}$ and choose $x, y \in U$. Notice that $|p(x) - p(y)| \leq 1$ and $|q(x) - q(y)| \leq n$. Next

$$\begin{aligned}\left|\frac{p(x)}{q(x)} - \frac{p(y)}{q(y)}\right| &= \left|\frac{p(x)q(y) - p(y)q(x)}{q(x)q(y)}\right| = \left|\frac{p(x)(q(y) - q(x)) + (p(x) - p(y))q(x)}{q(x)q(y)}\right| \\ &\leq \frac{p(x)|q(y) - q(x)|}{q(x)q(y)} + \frac{|p(x) - p(y)|q(x)}{q(x)q(y)} \leq \frac{p(x)n}{q(x)q(y)} + \frac{1}{q(y)} \\ &\leq \frac{q(x)n}{q(x)m} + \frac{1}{m} = \frac{n+1}{m}.\end{aligned}$$

This holds for all $x, y \in U$ as well as all $U \in \mathcal{U}$, so $\operatorname{var}(p/q) \leq \frac{n+1}{m}$. □

Definition 3.20. Let $\mathcal{U}$ be a cover of a space X with a large scale structure $\mathcal{LSS}$. If $\epsilon > 0$ and $\phi : X \to K$ is a partition of unity, then ϕ is called a $(\mathcal{U}, \epsilon)$**-partition of unity** if

1. $\operatorname{var}_{\mathcal{U}}(\phi) < \epsilon$;

2. for each $U \in \mathcal{U}$ there is an $s \in K^{(0)}$ so that $\phi_s(y) > 0$ for all $y \in U$, that is $\{\phi^{-1}(\operatorname{st}(s))\}_{s \in K^{(0)}}$ coarsens $\mathcal{U}$.

3. $\{\phi^{-1}(\operatorname{st}(s))\}_{s \in K^{(0)}}$ forms a uniformly bounded cover of X.

Proposition 3.21. *Let X be a space with a large scale structure $\mathcal{LSS}$, and let $\mathcal{V} = \{V_s\}_{s \in S} \in \mathcal{LSS}$ be a cover so that $\operatorname{mult}_{\mathcal{V}}(x) \leq n + 1$ for all $x \in X$. If $\mathcal{U}$ is another cover and $\operatorname{st}^k(\mathcal{U})$ refines $\mathcal{V}$ for some $k \geq 1$, then the partition $\phi^{\mathcal{V}}_{\mathcal{U}} : X \to \mathcal{N}(\mathcal{V})$ defined in Proposition 3.16 is a $\left(\mathcal{U}, \frac{(2n+3)^2}{k+1}\right)$-partition of unity.*

Proof. Since $\mathcal{U} \prec \operatorname{st}^k(\mathcal{U}) \prec \mathcal{V}$, by Lemma 3.17 $\{(\phi^{\mathcal{V}}_{\mathcal{U}})^{-1}(\operatorname{st}(V_s))\}_{s \in S}$ coarsens $\mathcal{U}$. Also notice that this collection refines $\mathcal{V}$ and hence is a uniformly bounded cover. To see this, take $x \in (\phi^{\mathcal{V}}_{\mathcal{U}})^{-1}(\operatorname{st}(V_s))$ for some $s \in S$. This means that $(\phi^{\mathcal{V}}_{\mathcal{U}})_s(x) > 0$. Thus $\operatorname{ind}\mathcal{U}(x, V_s) > 0$ as well and hence $x \in V_s$. This shows that $(\phi^{\mathcal{V}}_{\mathcal{U}})^{-1}(\operatorname{st}(V_s)) \subset V_s$ and hence

$$\{(\phi^{\mathcal{V}}_{\mathcal{U}})^{-1}(\operatorname{st}(V_s))\}_{s \in S} \prec \mathcal{V}.$$

It now remains to show that $\operatorname{var}(\phi^{\mathcal{V}}_{\mathcal{U}}) < \frac{(2n+3)^2}{k}$. Take $x \in X$. There are two cases. First suppose x is a point where $\operatorname{ind}_{\mathcal{U}}(x, V_s) = \infty$ for some $s \in S$. Then for any U containing x and another point y, if $\mathrm{ind}_{\mathcal{U}}(x, V_t) = \infty$ for some $V_t \in \mathcal{V}$, then $\mathrm{ind}_{\mathcal{U}}(y, V_t) = \infty$, as in the second part of the proof for Lemma 3.17. On the other hand, if $\mathrm{ind}_{\mathcal{U}}(x, V_t) < \infty$ for some $V_t \in \mathcal{V}$, then $\mathrm{ind}_{\mathcal{U}}(y, V_t) < \infty$. Thus $\phi^{\mathcal{V}}_{\mathcal{U}}(x) = \phi^{\mathcal{V}}_{\mathcal{U}}(y)$.

The next case is when $\mathrm{ind}_{\mathcal{U}}(x, V_s) < \infty$ for all $s \in S$. Suppose y is another point, and $x, y \in U$ for some $U \in \mathcal{U}$. Define $T_x = \{t \in S : \mathrm{ind}_{\mathcal{U}}(x, V_t) > 0\}$ and likewise define T_y. Notice that both T_x and T_y have at most $n + 1$ elements, as $\mathrm{mult}_{\mathcal{V}}(x) \leq n + 1$. Then $T = T_x \cup T_y$ has at most $2n + 2$ elements, and for $s \in S \setminus T$, $\mathrm{ind}_{\mathcal{U}}(x, V_s) = 0 = \mathrm{ind}_{\mathcal{U}}(y, V_s)$.

Fix $s \in S$ and take $p(x) = \mathrm{ind}_{\mathcal{U}}(x, V_s)$ and $q(x) = \sum_{t \in S} \mathrm{ind}_{\mathcal{U}}(x, V_t)$. Clearly $p(x) \leq q(x)$. Notice that since $x, y \in U$, then $|\mathrm{ind}_{\mathcal{U}}(x, V_s) - \mathrm{ind}_{\mathcal{U}}(y, V_s)| \leq 1$, so $\mathrm{var}_{\mathcal{U}}(p) \leq 1$. Also from this $\mathrm{var}_{\mathcal{U}}(q) \leq 2n + 2$. Next since $\mathrm{st}^k(\mathcal{U})$ covers X, $x \in W$ for some $W \in \mathrm{st}^k(\mathcal{U})$, and $W \subset V_r$ for some $V_r \in \mathcal{V}$ since $\mathrm{st}^k(U) \prec \mathcal{V}$. Therefore $\mathrm{ind}_{\mathcal{U}}(x, V_r) \geq k + 1$, and so $q(x) = \sum_{t \in S} \mathrm{ind}_{\mathcal{U}}(x, V_t) \geq k + 1$. By using Lemma 3.19,

$$\mathrm{var}_{\mathcal{U}}\left((\phi^{\mathcal{V}}_{\mathcal{U}})_s\right) = \mathrm{var}_{\mathcal{U}}\left(\frac{\mathrm{ind}_{\mathcal{U}}(x, V_s)}{\sum_{t \in S} \mathrm{ind}_{\mathcal{U}}(x, V_t)}\right) = \mathrm{var}_{\mathcal{U}}\left(\frac{p}{q}\right) \leq \frac{2n + 3}{k + 1}.$$

There are only $2n + 2$ functions where $(\phi^{\mathcal{V}}_{\mathcal{U}})_s(x)$ and $(\phi^{\mathcal{V}}_{\mathcal{U}})_s(y)$ are non-zero, and so

$$|\phi^{\mathcal{V}}_{\mathcal{U}}(x) - \phi^{\mathcal{V}}_{\mathcal{U}}(y)| \leq \left(\frac{2n + 3}{k + 1}\right)(2n + 2) < \frac{(2n + 3)^2}{k + 1}.$$

Finally this shows $\mathrm{var}_{\mathcal{U}}\left(\phi^{\mathcal{V}}_{\mathcal{U}}\right) < \frac{(2n+3)^2}{k+1}$, and $\phi^{\mathcal{V}}_{\mathcal{U}}$ is a $\left(\mathcal{U}, \frac{(2n+3)^2}{k+1}\right)$-partition of unity. □

Using partitions of unity, we can give two more definitions for asymptotic dimension, again their equivalence with previous definitions is proved in Theorem 3.27.

Definition 3.22. A large scale structure X has **asymptotic dimension** at most n if for every $\epsilon > 0$ and every uniformly bounded cover $\mathcal{U}$ of X there is a $(\mathcal{U}, \epsilon)$-partition of unity $\phi : X \to K^{(n)}$.

Definition 3.23. A large scale structure X has **asymptotic dimension** at most n if for every uniformly bounded cover $\mathcal{U}$ of X, there is a $(\mathcal{U}, \infty)$-partition of unity $\phi : X \to K^{(n)}$.

3.4 $\mathcal{U}$-Disjoint Families

Definition 3.24. If $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of X, then we say $\mathcal{V}$ **is $\mathcal{U}$-disjoint** if the collection $\mathrm{st}(\mathcal{V},\mathcal{U})$ is disjoint. That is if $V_1 \neq V_2$ are elements of $\mathcal{V}$, then $\mathrm{st}(V_1,\mathcal{U})$ does not intersect $\mathrm{st}(V_2,\mathcal{U})$.

The following is a generalization of a result in [DV16], where it is proved for the metric case only.

Proposition 3.25. *Suppose X is a large scale space, $n \geq 1$, and $\mathcal{U}$ a uniformly bounded cover. If $\mathcal{W} = \{W_s\}_{s\in S}$ is a cover of X with $\dim_{\mathrm{st}^{2n+3}(\mathcal{U})}(\mathcal{W}) \leq n$ then there is a cover $\mathcal{V} = \bigcup_{i=0}^{n} \mathcal{V}_i$ of X such that each $\mathcal{V}_i$ is $\mathcal{U}$-disjoint. Furthermore, every element of $\mathcal{V}$ is contained in an intersection of at most $(n+1)$-many elements of $\mathrm{st}(\mathcal{W}, \mathrm{st}^{2n+2}(\mathcal{U}))$.*

Proof. For convenience in writing, define $f_s(x) = \mathrm{ind}_{\mathcal{U}}(x, \mathrm{st}(W_s, \mathrm{st}^{2n+2}(\mathcal{U})))$. Then for each finite subset T of S define

$$U_T = \{x \in X : f_t(x) > f_s(x)\ \forall t \in T, \forall s \in S \setminus T\}.$$

First we have Fact 1: $U_T = \emptyset$ if T contains more than $n+1$ elements. To prove this, suppose that T has $n+2$ or more elements. Choose $x \in X$. If $f_t(x) = 0$ for some $t \in T$, then $x \notin U_T$. So suppose $f_t(x) > 0$ for all $t \in T$. This implies that

$$x \in \bigcap_{i=1}^{n+2} \mathrm{st}(W_{t_i}, \mathrm{st}^{2n+2}(\mathcal{U})).$$

But then $\mathrm{st}(x, \mathrm{st}^{2n+2}(\mathcal{U}))$ intersects $n+2$ elements of $\mathcal{W}$. However, this is a contradiction as $\mathrm{st}(x, \mathrm{st}^{2n+2}(\mathcal{U})) \subset \mathrm{st}(U, \mathrm{st}^{2n+2}(\mathcal{U})) \in \mathrm{st}^{2n+3}(\mathcal{U})$ for any U containing x. But $\mathrm{st}(U, \mathrm{st}^{2n+2}(\mathcal{U}))$ can only intersect $n+1$ elements of $\mathcal{W}$ since $\dim_{\mathrm{st}^{2n+3}(\mathcal{U})}(\mathcal{W}) \leq n$. Thus $U_T = \emptyset$.

Also, notice Fact 2: $U_T \cap U_F = \emptyset$ if both T and F are different but contain the same number of elements. To see this take $t \in T \setminus F$ and $\tau \in F \setminus T$. If $x \in U_T \cap U_F$, then since $t \in T$, $f_t(x) > f_s(x)$ for all $s \in S \setminus T$. In particular for $s = \tau$, we have

$f_t(s) > f_\tau(x)$. Likewise $f_\tau(x) > f_t(x)$, since $\tau \in F$. This is a contradiction, so $U_T \cap U_F = \emptyset$.

Fix x, and arrange the non-zero values of $f_s(x)$ in increasing order, and add 0 to the left hand side. Look at the gaps between these values. The largest value is at least $2n+2$, by virtue of how $f_s(x)$ is defined. Also there are at most $n+1$ gaps. So one gap must be of size 2 or greater. Let T_x consists all t so that $f_t(x)$ is to the right of this gap. Then $\mathrm{st}(x,\mathcal{U}) \subset U_{T_x}$. This follows from the fact that if $y \in \mathrm{st}(x,\mathcal{U})$, then $f_s(x)$ and $f_s(y)$ are within distance one of each other for all $s \in S$ (or are both infinite).

The above paragraph showed that $\mathrm{st}(x,\mathcal{U})$ is contained in some U_T. Hence we can define

$$\mathcal{V}_i = \big\{ V_T = \{x \in U_T : \mathrm{st}(x,\mathcal{U}) \subset U_T\} : T \subset S, |T| = i \big\}.$$

By Fact 2 above each $\mathcal{V}_i$ is a family which is $\mathcal{U}$-disjoint.

Also if $T \subset S$ is finite, then since $f_t > 0$ on U_T for each $t \in T$, we have that

$$V_T = \{x \in U_T : \mathrm{st}(x,\mathcal{U}) \subset U_T\} \subset \bigcap_{t \in T} \mathrm{st}(W_t, \mathrm{st}^{2n+2}(\mathcal{U})),$$

which completes the proof as $|T| \leq n+1$ by Fact 1. □

With this we are now ready to give one more definition of asymptotic dimension. The proof that this definition is equivalent is given in the next section.

Definition 3.26. A large scale structure X has **asymptotic dimension** at most n if for every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover $\mathcal{V} = \bigcup_{i=0}^n \mathcal{V}_i$ so that $st(\mathcal{V}_i, \mathcal{U})$ is a disjoint collection for $i = 0, 1, \ldots, n$.

3.5 Equivalent Definitions of Asymptotic Dimension

We are now finally in a position to show the equivalence of the various definitions of asymptotic dimension given throughout this chapter.

Theorem 3.27. *Let X be a space with a ls-structure $\mathcal{LSS}$. The following are equivalent definitions of X having asymptotic dimension at most n:*

(a) For every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover $\mathcal{V}$ coarsening $\mathcal{U}$ so that $\dim(\mathcal{V}) \leq n$.

(b) For every $\epsilon > 0$ and every uniformly bounded cover $\mathcal{U}$ of X there is a $(\mathcal{U}, \epsilon)$-partition of unity $\phi : X \to K^{(n)}$.

(c) For every uniformly bounded cover $\mathcal{U}$ of X, there is a $(\mathcal{U}, \infty)$-partition of unity $\phi : X \to K^{(n)}$.

(d) For every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover $\mathcal{V}$ so that $\dim_{\mathcal{U}}(\mathcal{V}) \leq n$.

(e) For every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover $\mathcal{V} = \bigcup_{i=0}^{n} \mathcal{V}_i$ so that $st(\mathcal{V}_i, \mathcal{U})$ is a disjoint collection for $i = 0, 1, \ldots, n$.

Proof. We will show $(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (a)$.

$(a) \implies (b)$ Let $\mathcal{U}$ be a uniformly bounded cover of X and $\epsilon > 0$. Choose $k \geq 0$ so that $\frac{(2n+3)^2}{k+1} < \epsilon$. Let $\mathcal{V}$ be a uniformly bounded cover coarsening the uniformly bounded cover $\mathrm{st}^k(\mathcal{U})$ with $\dim(\mathcal{V}) \leq n$, as guaranteed by the hypothesis. So for each $x \in X$, $\mathrm{mult}_{\mathcal{V}}(x) \leq n$. Then $\phi_{\mathcal{U}}^{\mathcal{V}} : X \to \mathcal{N}(\mathcal{V})$ defined as in Proposition 3.16 is a $\left(\mathcal{U}, \frac{(2n+3)^2}{k+1}\right)$-partition of unity by Proposition 3.21.

$(b) \implies (c)$ This is easy since if ϕ is a $(\mathcal{U}, \epsilon)$-partition of unity it is a $(\mathcal{U}, \infty)$-partition of unity.

$(c) \implies (d)$ Let $\mathcal{U}$ be a uniformly bounded cover, and notice that $\mathrm{st}^2(\mathcal{U})$ is also a uniformly bounded cover. By hypothesis, there is a $(\mathrm{st}^2(\mathcal{U}), \infty)$-partition of unity $\phi : X \to K^{(n)}$. Hence

$$\mathcal{W} = \{W_s = \phi^{-1}(\mathrm{st}(s))\}_{s \in K^{(0)}}$$

is a uniformly bounded cover which coarsens $\mathrm{st}^2(\mathcal{U})$. For each $s \in K^{(0)}$, define V_s to be W_S take away all $U \in \mathcal{U}$ not completely contained in W_s. Then take $\mathcal{V} = \{V_s\}_{s \in K^{(0)}}$. We first show that $\mathcal{V}$ coarsens $\mathcal{U}$, which is not strictly necessary but does show that

$\mathcal{V}$ is a cover (it is uniformly bounded as $\mathcal{W}$ is and $\mathcal{V}$ refines $\mathcal{W}$). Let $U \in \mathcal{U}$ and notice that $\mathrm{st}(U,\mathcal{U}) \subset W_s$ for some W_s. Then as $U \subset \mathrm{st}(U,\mathcal{U}) \subset W_s$, U is completely contained in W_s, and hence $U \subset V_s$. Finally we show that $U \in \mathcal{U}$ can intersect at most $n+1$ elements of $\mathcal{V}$. By way of contradiction, suppose some U intersects $n+2$ elements of $\mathcal{V}$. If U intersects V_s, then U is completely contained in W_s, but then U is completely contained in $n+2$ elements of $\mathcal{W}$, a contradiction to the multiplicity of $\mathcal{W}$. Thus $\dim_{\mathcal{U}}(\mathcal{V}) \leq n$.

$(d) \implies (e)$ Let $\mathcal{U}$ be a uniformly bounded cover, and notice that $\mathrm{st}^{2n+3}(\mathcal{U})$ is also uniformly bounded. By hypothesis, there is a $\mathcal{W}$ so that $\dim_{\mathrm{st}^{2n+3}(\mathcal{U})}(\mathcal{W}) \leq n$. Then by Proposition 3.25, there is a uniformly bounded cover $\mathcal{V} = \cup_{i=0}^{n} \mathcal{V}_i$ so that $\mathrm{st}(\mathcal{V}_i,\mathcal{U})$ is a disjoint collection for $i = 0, 1, \ldots, n$ (uniformly bounded comes from the furthermore statement in Proposition 3.25).

$(e) \implies (a)$ Let $\mathcal{U}$ be a uniformly bounded cover of X. By hypothesis, there is a uniformly bounded cover $\mathcal{W} = \cup_{i=0}^{n} \mathcal{W}_i$ so that $\mathrm{st}(\mathcal{W}_i,\mathcal{U})$ is a disjoint family for $i = 0, \ldots, n$. Take $\mathcal{V} = \mathrm{st}(\mathcal{W},\mathcal{U})$ and notice that by Proposition 2.5, $\mathcal{U} \prec \mathcal{V}$. For each $x \in X$, x is in only one element of $\mathrm{st}(\mathcal{W}_i,\mathcal{U})$ for each i. Hence x is in at most $n+1$ elements of

$$\mathrm{st}(\cup_{i=0}^{n} \mathcal{W}_i,\mathcal{U}) = \mathrm{st}(\mathcal{W},\mathcal{U}) = \mathcal{V},$$

showing that $\dim(\mathcal{V}) \leq n$. □

It should be noted that some of the equivalences above have been shown before, see [Aus15b] and [CDV15] for example. Of course most of these were proved for the metric case in [BD05]. However, the results presented here are for any large scale space, not limited to metric spaces, and many of the proofs are new. There are also some other definitions, given in [CDV12] which are in appearance similar to the related concept of Property A.

Chapter 4

Capacity and Entropy

4.1 Introduction

In his book on Coarse Geometry, Roe gives a method of creating coarse structures on a set. The main result is Proposition 3.7 of [Roe03]. Unfortunately, his statement of the proposition lacks an important hypothesis, and hence his proof is also incorrect. While he fixes the statement of the Proposition on his website, the proof remains incorrect and is more complicated than reading the book would lead one to believe. The purpose of this chapter is then twofold, first to convert the construction given by Roe to one suitable for larges scale structures; and second, to provide a correct proof Proposition 3.7 of [Roe03], given here in its large scale version as Theorem 4.18.

In his dissertation, [Bun11], Bunn used Roe's Proposition 3.7. However it appears he was aware of the mistake (at least in the statement of the Proposition) as he uses the correct version.

It should also be noted that the main idea of the chapter (that of generalizing bounded geometry based on a gauge) is very related to that of large scale doubling spaces, see [CDV14] or [Aus15a] for example, and [Hei01] for the original concept from analysis.

4.2 Capacity and Entropy

Let us start this chapter with a motivating example:

Example 4.1. Let X be a set. Define a large scale structure $\mathcal{LSS}$ on X by declaring $\mathcal{B} \in \mathcal{LSS}$ if there is a number $N(\mathcal{B})$ so that each $B \in \mathcal{B}$ has at most $N(\mathcal{B})$ elements, and each $x \in X$ intersects at most $N(\mathcal{B})$ elements of $\mathcal{B}$.

It is not difficult to show that Example 4.1 is a way of defining a large scale structure on the set X. Indeed we will give a sketch of the proof here. Let $\mathcal{LSS}$ be the purported large scale structure as defined in Example 4.1. Suppose $\mathcal{B}_1 \in \mathcal{LSS}$ and for some other family, $\mathcal{B}_2$, every non-singleton element of $\mathcal{B}_2$ is contained in some element of $\mathcal{B}_1$. Since $\mathcal{B}_1 \in \mathcal{LSS}$, there is a number $N(\mathcal{B}_1)$ so each element of $\mathcal{B}_1$ has at most $N(\mathcal{B}_1)$ elements, and each $x \in X$ intersects at most $N(\mathcal{B}_1)$ elements of $\mathcal{B}_1$. So if $B_2 \in \mathcal{B}_2$ then B_2 has at most $N(\mathcal{B}_1)$ elements since it is contained in some element of $\mathcal{B}_1$. Next each $B_1 \in \mathcal{B}_1$ has at most $N(\mathcal{B}_1)$ elements and so has at most $2^{N(\mathcal{B}_1)}$ subsets. Hence for any $x \in X$, x is in at most $N(\mathcal{B}_1)$ elements of $\mathcal{B}_1$, and thus in at most $N(\mathcal{B}_1) \cdot 2^{N(\mathcal{B}_1)}$ subsets of $\mathcal{B}_2$. Take $N(\mathcal{B}_2)$ to be $N(\mathcal{B}_1) \cdot 2^{N(\mathcal{B}_1)}$ and then $\mathcal{B}_2 \in \mathcal{LSS}$.

Next we will show that if $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}$ then $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}$. For any $\mathrm{st}(B_1, \mathcal{B}_2) \in \mathrm{st}(\mathcal{B}_1, \mathcal{B}_2)$ we know that B_1 contains at most $N(\mathcal{B}_1)$ points, and each of those points is in at most $N(\mathcal{B}_2)$ elements of $\mathcal{B}_2$. Each $B_2 \in \mathcal{B}_2$ contains at most $N(\mathcal{B}_2)$ points. Thus $\mathrm{st}(B_1, \mathcal{B}_2)$ contains at most $N(\mathcal{B}_1) \cdot N(\mathcal{B}_2)^2$ points. Finally take $x \in X$, we wish to count the elements of $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2)$ which contain the point x. Suppose $x \in \mathrm{st}(B_1, \mathcal{B}_2) \in \mathrm{st}(\mathcal{B}_1, \mathcal{B}_2)$. There are two cases, first $x \in B_1$. There are at most $N(\mathcal{B}_1)$ times this can occur since each $B_1 \in \mathcal{B}_1$ corresponds to only one element of $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2)$. The second case is that $x \in B_2$ for some $B_2 \in \mathcal{B}_2$. x could be in $N(\mathcal{B}_2)$ such sets, but each B_2 could be in $N(\mathcal{B}_1) \cdot N(\mathcal{B}_2)$ elements of $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2)$ (a maximum of $N(B_2)$ points in each B_2 and each point could intersect $N(\mathcal{B}_1)$ elements of $\mathcal{B}_1$). Hence take

$$N(\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2)) = N(\mathcal{B}_1)^2 \cdot N(\mathcal{B}_2)^2$$

and we have that $\mathrm{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}$, and $\mathcal{LSS}$ is in fact a large scale structure.

We wish to expand on how the large scale structure in Example 4.1 was created. What follows gives a general way of defining a large scale structure on a set X, given a suitable cover $\mathcal{U}$.

Throughout the remainder of this chapter, because of its frequent use, we will denote by $\mathcal{D}_X$ the cover of X by singletons. That is for a set X,

$$\mathcal{D}_X = \{\{x\}\}_{x \in X}.$$

Since we are going to define which subsets of a space are bounded, we need some way to measure the "size" of sets. We shall use capacity and entropy as such a measure. The original definitions were given by Kolmogorov for metric spaces (see [Kol93]), and then Roe in [Roe03] gave definitions for his coarse spaces involving subsets of $X \times X$. The generalizations of the Kolmororov definitions to non-metric spaces are as follows:

Definition 4.2. Let X be a space and $\mathcal{U}$ a cover of X, and $S \subset X$.

1. The **$\mathcal{U}$-capacity of** S, denoted $\mathrm{cap}_{\mathcal{U}}(S)$ is the largest m such that there are m points $x_1, \ldots, x_m \in S$ where no two distinct $x_i, x_j \in U$ for any $U \in \mathcal{U}$. If there are infinitely many such points, then the capacity is ∞, and $\mathrm{cap}_{\mathcal{U}}(\emptyset) = 0$.

2. If $\mathcal{V}$ is a collection of subsets of X, then the **$\mathcal{U}$-capacity of** $\mathcal{V}$ is

$$\mathrm{cap}_{\mathcal{U}}(\mathcal{V}) = \sup_{V \in \mathcal{V}} \{\mathrm{cap}_{\mathcal{U}}(V)\}.$$

If $\mathcal{U}$ is simply a collection of subsets (not necessarily a cover) of X then it is still possible to define the $\mathcal{U}$-capacity of a subset S of X. However, doing this creates some difficulties: there are some nuances in the definition, many of the theorems proved below only hold when $\mathcal{U}$ is a cover, and we are only interested in the case when $\mathcal{U}$ is a cover anyway. Hence we will use Definition 4.2 as our definition of capacity, we lose some generality, but gain more as theorems are stated in a simpler manner.

Definition 4.3. Let X be a space, $\mathcal{U}$ a cover of X, and $S \subset X$.

1. The **$\mathcal{U}$-entropy of** S, denoted $\mathrm{ent}_{\mathcal{U}}(S)$ is the smallest n so that there are points $x_1, \dots x_n \in X$ so that $\{\mathrm{st}(\{x_i\},\mathcal{U})\}_{i=1}^{n}$ forms a cover of S or ∞ if no such n exists. The $\mathcal{U}$-entropy of the empty set is 0.

2. If $\mathcal{V}$ is a collection of subsets of X, then the **$\mathcal{U}$-entropy of $\mathcal{V}$** is

$$\mathrm{ent}_{\mathcal{U}}(\mathcal{V}) = \sup_{V \in \mathcal{V}} \{\mathrm{ent}_{\mathcal{U}}(V)\}.$$

The $\mathcal{U}$-entropy of a set S is more familiar than the $\mathcal{U}$-capacity of S, and is reminiscent of compactness. Indeed one may say it is a measure of "the star $\mathcal{U}$ compactness" of S. It turns out that the $\mathcal{U}$-capacity of S is also related to compactness, as Remark 4.4 shows.

Remark 4.4. An equivalent definition of the **$\mathcal{U}$-capacity of** S is the largest m such that there are m points $x_1, \dots x_m \in S$ where

$$x_j \notin \mathrm{st}\left(\{x_i\},\mathcal{U}\right)$$

for any distinct $i, j \in \{1, \dots m\}$, or infinity if no such m exists, and again the $\mathcal{U}$-capacity of the empty set is zero.. Also notice that if $\mathrm{cap}_{\mathcal{U}}(S) = m$, and $\{x_1, \dots x_m\} \subset S$ are the points as in Definition 4.2, then

$$\bigcup_{i=1}^{m} \mathrm{st}\left(\{x_i\},\mathcal{U}\right)$$

covers S.

Proof. First notice that distinct points x_i and x_j are in the same subset U of X if and only if one is in the star of the other.

For the second part, suppose that $\mathrm{cap}_{\mathcal{U}}(S) = m$, so there are points $x_1, \dots x_m$ where no two distinct x_i, x_j are in the same set U for any $U \in \mathcal{U}$. Take $s \in S$. If $x_i \notin \mathrm{st}(\{s\},\mathcal{U})$ for any i, then $\{x_1, \dots x_m\}$ is not maximal. So there is a $U \in \mathcal{U}$

containing both s and x_i for some i. Hence

$$\bigcup_{i=1}^{m} \text{st}\left(\{x_i\}, \mathcal{U}\right)$$

covers S. □

4.3 Relation between Capacity and Entropy

It was mentioned above that both capacity and entropy are related to compactness, so it should be no surprise that capacity and entropy are related to each other. The following proposition follows immediately from Definition 4.2, Definition 4.3, and Remark 4.4.

Proposition 4.5. *For a set X with subset S and cover $\mathcal{U}$, $\text{ent}_{\mathcal{U}}(S) \leq \text{cap}_{\mathcal{U}}(S)$. Furthermore, if $\mathcal{V}$ is a collection of subsets of X then $\text{ent}_{\mathcal{U}}(\mathcal{V}) \leq \text{cap}_{\mathcal{U}}(\mathcal{V})$.*

While capacity and entropy are related, they are not the same. Consider for example the set S equal to the interval from 0 to 1, viewed as a subset of $\mathbb{R}$. Take $\mathcal{U}$ to be the collection of all closed balls of radius 1/2 centered at points in $\mathbb{Z}$, that is $\mathcal{U} = \{\overline{B(z, 1/2)} : z \in \mathbb{Z}\}$. Notice that $\mathcal{U}$ is a cover of $\mathbb{R}$. Then $\text{ent}_{\mathcal{U}}(S) = 1$ since

$$\text{st}(\{1/2\}, \mathcal{U}) = [-1/2, 3/2]$$

covers $S = [0, 1]$. But $\text{cap}_{\mathcal{U}}(S) = 2$, simply take $x_1 = 0$ and $x_2 = 1$ and notice that

$$0 \notin \text{st}(\{1\}, \mathcal{U}) = [1/2, 3/2],$$

and

$$1 \notin \text{st}(\{0\}, \mathcal{U}) = [-1/2, 1/2].$$

So capacity and entropy can be different. In fact it is even possible for a set to have finite entropy but infinite capacity, as the large scale Hawaiian Earring in Example 4.6

shows. Of course there are instances where capacity and entropy are equal; continuing our above example for instance, it is easy to see that $\mathrm{ent}_{\mathcal{U}}(\{1\}) = \mathrm{cap}_{\mathcal{U}}(\{1\}) = 1$.

Example 4.6. Define the large scale Hawaiian Earring $\mathcal{H}$ to be the union of circles centered at $(0, n)$ of radius n for $n \in \{1, 2, \ldots\}$. Define $\mathcal{U}$ to be the collection of circles which make up the Hawaiian Earring, that is

$$\mathcal{U} = \{C((0, n), n) : n \in \{1, 2, \ldots\}\},$$

where $C((x, y), r)$ is the circle centered at (x, y) of radius r. Take

$$S = \{(0, n) : n \in \{1, 2, \ldots\}\}.$$

Notice that for $s = (0, n) \in S$, $\mathrm{st}(\{s\}, \mathcal{U}) = C((0, n), n)$. Also for $s' = (0, n') \in S, s' \neq s$,

$$s' \notin \mathrm{st}(\{s\}, \mathcal{U}) = C((0, n), n).$$

Hence $\mathrm{cap}_{\mathcal{U}}(S) = \infty$. Another way to see this is that finitely many $C((0, n), n)$ cannot cover all of S.

On the other hand, $\mathrm{ent}_{\mathcal{U}}(S) = 1$ since

$$\mathrm{st}((0, 0), \mathcal{U}) = \mathcal{H},$$

as $(0, 0)$ intersects each $C((0, n), n) = U \in \mathcal{U}$.

4.4 Properties of Capacity and Entropy

Next we will prove some useful properties of capacity and entropy. For the most part, the properties one thinks should hold do hold. In particular, both capacity and entropy work well with subsets, coarsening of covers, and finite unions. In general, the following results are not quite enough to make entropy or capacity a measure, but for a suitably nice cover and space the capacity is a measure, see [Roe03] for example.

Let us first take a look at how the $\mathcal{U}$-capacity and $\mathcal{U}$-entropy work with subsets. Both behave in an intuitive manner, since if $S \subset T$ then one would expect T to be able to "hold more $\mathcal{U}$-balls" than S, which is essentially the $\mathcal{U}$-capacity of T. Also one would expect T to "require more $\mathcal{U}$-balls to cover it" than S would require, which is the $\mathcal{U}$-entropy.

Lemma 4.7. *Let X be a set with $S \subset T \subset X$, and $\mathcal{U}$ a cover of X. Then*

1. $\mathrm{cap}_{\mathcal{U}}(S) \leq \mathrm{cap}_{\mathcal{U}}(T)$, *and*

2. $\mathrm{ent}_{\mathcal{U}}(S) \leq \mathrm{ent}_{\mathcal{U}}(T)$.

Proof.

1. Suppose that $\mathrm{cap}_{\mathcal{U}}(S) = m < \infty$. Thus there are $x_1, \ldots x_m \in S \subset T$ where no two distinct $x_i, x_j \in U$ for any $U \in \mathcal{U}$. Thus $\mathrm{cap}_{\mathcal{U}}(T) \geq m$ since it is the maximum of such points. Next if $\mathrm{cap}_{\mathcal{U}}(S) = \infty$, then the above argument will hold for any $m < \infty$, thus $\mathrm{cap}_{\mathcal{U}}(T) = \infty$ as well.

2. If $\mathrm{ent}_{\mathcal{U}}(T) = n < \infty$ then there are points $x_1, \ldots x_n \in X$ so that T, and hence S, are both covered by $\{\mathrm{st}(\{x_i\}, \mathcal{U})\}_{i=1}^{n}$. So $\mathrm{ent}_{\mathcal{U}}(S) \leq n$ since it is the minimum of such points. Finally if $\mathrm{ent}_{\mathcal{U}}(T) = \infty$, then the inequality holds.

□

Recall that $\mathcal{V}_1$ refines $\mathcal{V}_2$ means that for each $V_1 \in \mathcal{V}_1$ there is a $V_2 \in \mathcal{V}_2$ containing V_1. Using this along with the second parts of Definition 4.2 and Definition 4.3 gives the following easy corollary.

Corollary 4.8. *If X is a set with a cover $\mathcal{U}$, and $\mathcal{V}_1$ and $\mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1$ refining $\mathcal{V}_2$, then*

1. $\mathrm{cap}_{\mathcal{U}}(\mathcal{V}_1) \leq \mathrm{cap}_{\mathcal{U}}(\mathcal{V}_2)$, *and*

2. $\mathrm{ent}_{\mathcal{U}}(\mathcal{V}_1) \leq \mathrm{ent}_{\mathcal{U}}(\mathcal{V}_2)$.

Next we look at the refining of covers for a fixed set S. If $\mathcal{U}$ and $\mathcal{V}$ are covers, the fact that $\mathcal{U}$ refines $\mathcal{V}$ implies that $\mathcal{U}$-balls are smaller than $\mathcal{V}$-balls in some sense. Hence it should come as no surprise that S can hold more $\mathcal{U}$-balls than it can $\mathcal{V}$-balls.

Likewise one would expect S to require more $\mathcal{U}$-balls to cover it than $\mathcal{V}$-balls. This is precisely what the following lemma says.

Lemma 4.9. *Suppose X is a set and $S \subset X$. If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, with $\mathcal{U} \prec \mathcal{V}$, then*

1. $\operatorname{cap}_{\mathcal{U}}(S) \geq \operatorname{cap}_{\mathcal{V}}(S)$, *and*

2. $\operatorname{ent}_{\mathcal{U}}(S) \geq \operatorname{ent}_{\mathcal{V}}(S)$.

Proof. First notice that since $\mathcal{U} \prec \mathcal{V}$ for any $x \in X$, $\operatorname{st}(\{x\},\mathcal{U}) \subset \operatorname{st}(\{x\},\mathcal{V})$. Indeed take $y \in \operatorname{st}(\{x\},\mathcal{U})$, then there is a $U \in \mathcal{U}$ containing both x and y. There is also a $V \in \mathcal{V}$ containing U. Thus $y \in \operatorname{st}(\{x\},\mathcal{V})$.

1. If $x_1, \ldots x_m \in S$ are points so that $x_j \notin \operatorname{st}(\{x_i\},\mathcal{V})$ for $i \neq j$, then $x_j \notin \operatorname{st}(\{x_i\},\mathcal{U})$ either.

2. If $x_1, \ldots x_m \in X$ are points so that $\{\operatorname{st}(\{x_i\},\mathcal{U})\}_{i=1}^n$ covers S, then $\{\operatorname{st}(\{x_i\},\mathcal{V})\}_{i=1}^n$ also covers S.

□

Our next lemma describes how capacity and entropy behave with respect to finite unions. One would like $\operatorname{cap}_{\mathcal{U}}(S \cup T) = \operatorname{cap}_{\mathcal{U}}(S) + \operatorname{cap}_{\mathcal{U}}(T)$ for subset S and T as this would (almost) make $\mathcal{U}$-capacity into a measure. However in general this is not the case. It can happen if $\mathcal{U}$ is "small" enough cover with respect to the spacing of the points of a set, see Chapter 3 of [Roe03].

Lemma 4.10. *Let X be a space, $\mathcal{U}$ a cover of X, and $S, T \subset X$. Then*

1. $\operatorname{cap}_{\mathcal{U}}(S \cup T) \leq \operatorname{cap}_{\mathcal{U}}(S) + \operatorname{cap}_{\mathcal{U}}(T)$, *and*

2. $\operatorname{ent}_{\mathcal{U}}(S \cup T) \leq \operatorname{ent}_{\mathcal{U}}(S) + \operatorname{ent}_{\mathcal{U}}(T)$.

Proof. 1. First suppose $\operatorname{cap}_{\mathcal{U}}(S \cup T) = m < \infty$. So there are points $x_1, \ldots x_m \in S \cup T$ where no two distinct $x_i, x_j \in U$ for any $U \in \mathcal{U}$. Then let

$$S' = \{s_1, \ldots s_k\} = \{x_1 \ldots x_m\} \cap S$$

and

$$T' = \{t_1, \dots t_l\} = \{x_1 \dots x_m\} \cap T.$$

Notice that $m \leq k + l$, and for distinct $s_i, s_j \in S'$ we have that s_i and s_j are not in any $U \in \mathcal{U}$. Thus $k \leq \mathrm{cap}_{\mathcal{U}}(S)$. Likewise $l \leq \mathrm{cap}_{\mathcal{U}}(T)$. Finally this shows

$$\mathrm{cap}_{\mathcal{U}}(S \cup T) = m \leq k + l \leq \mathrm{cap}_{\mathcal{U}}(S) + \mathrm{cap}_{\mathcal{U}}(T).$$

Next suppose $\mathrm{cap}_{\mathcal{U}}(S \cup T) = \infty$. First notice this implies $S \cup T$ has infinitely many points, and also that S, T, or both have infinitely many points. There are infinitely many points $x_1, x_2, \dots \in S \cup T$ so that no two distinct $x_i, x_j \in U$ for any $U \in \mathcal{U}$. If infinitely many of these are in S, then $\mathrm{cap}_{\mathcal{U}}(S) = \infty$ and we are done. If only finitely many are in S, then infinitely many must be in T and again we are done.

2. First note that if either $\mathrm{ent}_{\mathcal{U}}(S)$ or $\mathrm{ent}_{\mathcal{U}}(T)$ is infinite then the inequality holds. So suppose $\mathrm{ent}_{\mathcal{U}}(S) = n_1$ and $\mathrm{ent}_{\mathcal{U}}(T) = n_2$, meaning that S can be covered by n_1 sets of the form $\mathrm{st}(x,\mathcal{U})$ and T can be covered by n_2 sets of the same form. Then $S \cup T$ is covered by $n_1 + n_2$ elements of the form $\mathrm{st}(\{x\},\mathcal{U})$, but $\mathrm{ent}_{\mathcal{U}}(S \cup T)$ is the minimal such covering. This shows

$$\mathrm{ent}_{\mathcal{U}}(S \cup T) \leq \mathrm{ent}_{\mathcal{U}}(S) + \mathrm{ent}_{\mathcal{U}}(T).$$

□

Of course the above theorem holds for finitely many unions.

4.5 Universal Bounded Geometry

In this section, we will define what we mean by the Universal Bounded Geometry based on a gauge (gauge also defined below). In order to achieve this, we first need some more definitions and a few more preliminary results. We are interested in the case when the $\mathcal{U}$-capacity, and hence the $\mathcal{U}$-entropy, of another cover is finite.

Eventually, we will want the stars of such covers to also have finite $\mathcal{U}$-capacity. In order to ensure that stars of covers with finite capacity themselves have finite capacity, we have the following definition.

Definition 4.11. Let $\mathcal{U}$ be a cover of a set X, and $\mathcal{V}$ be a collection of subsets of X. The **star capacity of $\mathcal{V}$ with respect to $\mathcal{U}$**, is

$$\begin{aligned}\operatorname{stcap}_{\mathcal{U}}(\mathcal{V}) &= \operatorname{cap}_{\mathcal{U}}\left(\operatorname{st}\left(\operatorname{st}(\mathcal{D}_X,\mathcal{U}),\mathcal{V}\right)\right) \\ &= \sup_{x\in X}\left\{\operatorname{cap}_{\mathcal{U}}(\operatorname{st}(\operatorname{st}(\{x\},\mathcal{U}),\mathcal{V}))\right\}.\end{aligned}$$

The idea of the star capacity of $\mathcal{V}$ with respect to $\mathcal{U}$ is to take each point $x \in X$ and create a $\mathcal{U}$-ball around x. Then take a $\mathcal{V}$-ball about this $\mathcal{U}$-ball. We are then interested in how many $\mathcal{U}$-balls this $\mathcal{V}$-ball can hold, in other words the $\mathcal{U}$-capacity.

Definition 4.12. Let X be a set and $\mathcal{U}$ a cover of X. A collection of subsets $\mathcal{V}$ of X is **uniform with respect to $\mathcal{U}$** if $\operatorname{stcap}_{\mathcal{U}}(\mathcal{V})$ is finite.

Definition 4.13. A **gauge** for a set X is a cover $\mathcal{U}$ which is uniform with respect to itself. That is $\operatorname{stcap}_{\mathcal{U}}(\mathcal{U})$ is finite.

Proposition 4.14 gives a somewhat trivial example of a gauge. It should be noted however that not any cover can be a gauge, which was the error Roe made.

Proposition 4.14. *Let X be a set, then the cover of X by singletons, $\mathcal{D}_X$ is a gauge.*

Proof. We wish to show that

$$\operatorname{stcap}_{\mathcal{D}_x}(\mathcal{D}_X) = \sup_{x\in X}\{\operatorname{cap}_{\mathcal{D}_X}(\operatorname{st}(\operatorname{st}(\{x\},\mathcal{D}_X),\mathcal{D}_x))\} < \infty. \tag{4.1}$$

First notice that for any $x \in X$, $\operatorname{st}(\{x\},\mathcal{D}_x) = \{x\}$. Thus by applying this twice, Equation (4.1) becomes

$$\operatorname{stcap}_{\mathcal{D}_x}(\mathcal{D}_X) = \sup_{x\in X}\{\operatorname{cap}_{\mathcal{D}_X}(\{x\})\}.$$

But the $\mathcal{D}_x$-capacity of a singleton is 1. Thus

$$\operatorname{stcap}_{\mathcal{D}_x}(\mathcal{D}_X) = 1 < \infty,$$

showing that $\mathcal{D}_X$ is a gauge. $\square$

The following is a very useful proposition for us. It is, along with Proposition 4.16, what essentially allows us to show that star capacity works well with stars.

Proposition 4.15. *If X is a set with subset S, $\mathcal{U}$ a cover, and $\mathcal{V}$ uniform with respect to $\mathcal{U}$, then $\operatorname{cap}_{\mathcal{U}}(\operatorname{st}(S,\mathcal{V})) \leq \operatorname{ent}_{\mathcal{U}}(S) \cdot \operatorname{stcap}_{\mathcal{U}}(\mathcal{V})$.*

Proof. If $\operatorname{ent}_{\mathcal{U}}(S)$ is infinite then the inequality holds, so suppose $\operatorname{ent}_{\mathcal{U}}(S) = n < \infty$. Thus

$$S \subset \bigcup_{i=1}^{n} \operatorname{st}(\{x_i\},\mathcal{U})$$

for some $x_1, \dots x_n \in X$. Thus using Proposition 2.5, and Proposition 2.6 we have

$$\operatorname{st}(S,\mathcal{V}) \subset \operatorname{st}\left(\bigcup_{i=1}^{n} \operatorname{st}(\{x_i\},U),\mathcal{V}\right) \subset \bigcup_{i=1}^{n} \operatorname{st}(\operatorname{st}(\{x_i\},\mathcal{U}),\mathcal{V}).$$

And hence by using Lemma 4.7 and Lemma 4.10 it is the case that

$$\begin{aligned}\operatorname{cap}_{\mathcal{U}}(\operatorname{st}(S,\mathcal{V})) &\leq \operatorname{cap}_{\mathcal{U}}\left(\bigcup_{i=1}^{n} \operatorname{st}(\operatorname{st}(\{x_i\},\mathcal{U}),\mathcal{V}\right) \\ &\leq \sum_{i=1}^{n} \operatorname{cap}_{\mathcal{U}}(\operatorname{st}(\operatorname{st}(\{x_i\},\mathcal{U}),\mathcal{V})) \\ &\leq n \cdot \operatorname{stcap}_{\mathcal{U}}(\mathcal{V}) = \operatorname{ent}_{\mathcal{U}}(S) \cdot \operatorname{stcap}_{\mathcal{U}}(\mathcal{V}).\end{aligned}$$

$\square$

Proposition 4.16. *Let $\mathcal{U}$ be a gauge for a set X, and $\mathcal{V}_1, \dots, \mathcal{V}_n$ uniform with respect to $\mathcal{U}$. Then for $x \in X$,*

$$\operatorname{cap}_{\mathcal{U}}(\operatorname{st}(\{x\},\mathcal{U},\mathcal{V}_1,\dots,\mathcal{V}_n)) \leq \prod_{i=1}^{n} \operatorname{stcap}_{\mathcal{U}}(\mathcal{V}_i).$$

Proof. We proceed by induction on n. For $n = 1$ we have

$$\text{cap}_{\mathcal{U}}(\text{st}(\{x\},\mathcal{U},\mathcal{V}_1)) \leq \sup_{y\in X}\{\text{cap}_{\mathcal{U}}(\text{st}(\{y\},\mathcal{U},\mathcal{V}_1)\} = \text{stcap}_{\mathcal{U}}(\mathcal{V}_1).$$

Then suppose the result holds for $n = k$, and notice that by using Proposition 4.15,

$$\text{cap}_{\mathcal{U}}(\text{st}\{x\},\mathcal{U},\mathcal{V}_1\ldots,\mathcal{V}_k,\mathcal{V}_{k+1}) \leq \text{ent}_{\mathcal{U}}(\text{st}\{x\},\mathcal{U},\mathcal{V}_1,\ldots,\mathcal{V}_k)\cdot\text{stcap}_{\mathcal{U}}(\mathcal{V}_{k+1}). \quad (4.2)$$

But by the induction hypothesis we have

$$\text{ent}_{\mathcal{U}}(\text{st}\{x\},\mathcal{U},\mathcal{V}_1,\ldots,\mathcal{U}_k) \leq \text{cap}_{\mathcal{U}}(\text{st}\{x\},\mathcal{U},\mathcal{V}_1,\ldots,\mathcal{V}_k) \leq \prod_{i=1}^{k}\text{stcap}_{\mathcal{U}}(\mathcal{V}_i). \quad (4.3)$$

Finally from Equation (4.2) and Equation (4.3)

$$\text{cap}_{\mathcal{U}}(\text{st}(\{x\},\mathcal{U},\mathcal{V}_1,\ldots,\mathcal{V}_k,\mathcal{V}_{k+1})) \leq \prod_{i=1}^{k+1}\text{stcap}_{\mathcal{U}}(\mathcal{V}_i),$$

so the result holds for all n. □

Corollary 4.17. *Let $\mathcal{U}$ be a gauge for a set X and $n \geq 1$ then for any $x \in X$*

$$\text{cap}_{\mathcal{U}}(\text{st}(\{x\},\underset{0}{\mathcal{U}},\underset{1}{\mathcal{U}}\ldots,\underset{n}{\mathcal{U}})) \leq (\text{stcap}_{\mathcal{U}}(\mathcal{U}))^n,$$

where $\underset{i}{\mathcal{U}}$ denotes the index.

We are now finally in a position to define the universal bounded geometry based on a gauge $\mathcal{U}$. Again notice that $\mathcal{U}$ is a gauge, and hence must be uniform with respect to itself.

Theorem 4.18. *Let X be a set with a gauge $\mathcal{U}$, and let $\mathcal{LSS}$ be the collection of all $\mathcal{V}$ so that $\mathcal{V}$ is uniform with respect to $\mathcal{U}$ ($\mathcal{V}$ is a collection of subsets of X). Then $(X,\mathcal{LSS})$ is a large scale structure, known as the* ***universal bounded geometry large scale structure based on the gauge $\mathcal{U}$***.

Proof. Notice that since $\mathcal{U}$ is a gauge, $\mathcal{U}$ is uniform with respect to itself. Hence $\mathcal{U} \in \mathcal{LSS}$, showing, for one thing, that $\mathcal{LSS}$ is not trivial. We now show that $\mathcal{LSS}$ satisfies the definition of a large scale structure.

1. Let $\mathcal{B}_1 \in \mathcal{LSS}$ and $\mathcal{B}_2$ be a collection of subsets of X so that each non-singleton element of $\mathcal{B}_2$ is contained in some element of $\mathcal{B}_1$ (see Definition 2.9). Since $\mathcal{B}_1 \in \mathcal{LSS}$, $\text{stcap}_{\mathcal{U}}(\mathcal{B}_1)$ is finite. Notice that $\mathcal{B}_2$ refines $\mathcal{B}_1$, so for each $x \in X$

   $$\text{st}(\text{st}(\{x\},\mathcal{U}),\mathcal{B}_2) \subset \text{st}(\text{st}(\{x\},\mathcal{U}),\mathcal{B}_1).$$

   Hence by Lemma 4.7

   $$\text{cap}_{\mathcal{U}}(\text{st}(\text{st}(\{x\},\mathcal{U}),\mathcal{B}_2)) \leq \text{cap}_{\mathcal{U}}(\text{st}(\text{st}(\{x\},\mathcal{U}),\mathcal{B}_1)).$$

   Since $x \in X$ was arbitrary, and by properties of sup,

   $$\begin{aligned} \text{stcap}_{\mathcal{U}}(\mathcal{B}_2) &= \sup_{x\in X}\{\text{cap}_{\mathcal{U}}(\text{st}(\text{st}(\{x\},\mathcal{U}),\mathcal{B}_2)\} \\ &\leq \sup_{x\in X}\{\text{cap}_{\mathcal{U}}(\text{st}(\text{st}(\{x\},\mathcal{U}),\mathcal{B}_1)\} \\ &= \text{stcap}_{\mathcal{U}}(\mathcal{B}_1), \end{aligned}$$

   which shows $\mathcal{B}_2 \in \mathcal{LSS}$.

2. Suppose that $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}$, then $\text{stcap}_{\mathcal{U}}(\mathcal{B}_1), \text{stcap}_{\mathcal{U}}(\mathcal{B}_2) < \infty$. Also notice that for each $x \in X$,

   $$\text{st}(\{x\},\mathcal{U}, \text{st}(\mathcal{B}_1, \mathcal{B}_2)) \subset \text{st}(\{x\},\mathcal{U}, \mathcal{B}_2, \mathcal{B}_1, \mathcal{B}_2).$$

   And so using Lemma 4.7,

   $$\text{cap}_{\mathcal{U}}(\text{st}(\{x\},\mathcal{U}, \text{st}(\mathcal{B}_1, \mathcal{B}_2))) \leq \text{cap}_{\mathcal{U}}(\text{st}(\{x\},\mathcal{U}, \mathcal{B}_2, \mathcal{B}_1, \mathcal{B}_2)).$$

Also by Proposition 4.16,

$$\mathrm{cap}_{\mathcal{U}}(\mathrm{st}(\{x\},\mathcal{U},\mathcal{B}_2,\mathcal{B}_1,\mathcal{B}_2)) \leq \mathrm{stcap}_{\mathcal{U}}(\mathcal{B}_2)\cdot \mathrm{stcap}_{\mathcal{U}}(\mathcal{B}_1)\cdot \mathrm{stcap}_{\mathcal{U}}(\mathcal{B}_2) < \infty.$$

As this was true for arbitrary $x \in X$, we have

$$\begin{aligned}\mathrm{stcap}_{\mathcal{U}}(\mathrm{st}(\mathcal{B}_1,\mathcal{B}_2)) &= \sup_{x\in x}\{\mathrm{st}(\{x\},\mathcal{U},\mathrm{st}(\mathcal{B}_1,\mathcal{B}_2))\} \\ &\leq \mathrm{stcap}(\mathcal{B}_1)\cdot \mathrm{stcap}(\mathcal{B}_2)^2 \\ &< \infty.\end{aligned}$$

Thus $\mathrm{st}(\mathcal{B}_1,\mathcal{B}_2) \in \mathcal{LSS}$, showing that $\mathcal{LSS}$ is in fact a large scale structure on X.

□

Example 4.19. Let X be a discrete infinite set, and define $\mathcal{U}$ to be the cover by singleton sets: $\mathcal{U} = \mathcal{D}_X$. Then the universal bounded geometry large scale structure based on $\mathcal{U}$ is the same as the one given in Example 4.1.

Chapter 5

Localization

5.1 Finite Localization

Our goal in this section is to define a finite localization. In order to do this, we must first recall from Chapter 3 that a $\mathcal{U}$-chain is a sequence of points $x_0, \ldots x_n$ so that for each $i = 0, \ldots n-1$, there is a $U_i \in \mathcal{U}$ containing both x_i, and x_{i+1}. Next we will also need a few more definitions, and the next few may be found in [DW16].

Definition 5.1. For a set X and a cover $\mathcal{U}$ of X. Define an equivalence relation on X with respect to $\mathcal{U}$ by $x \sim_{\mathcal{U}} y$ if there is a finite $\mathcal{U}$ chain $x = x_0, \ldots x_n = y$. A **$\mathcal{U}$-component** of X is an equivalence class of $\sim_{\mathcal{U}}$.

Definition 5.2. A **coarse component** of a point x in a large scale space $(X, \mathcal{LSS})$ is the union of all all its $\mathcal{U}$-components where $\mathcal{U} \in \mathcal{LSS}$.

Definition 5.3. Let $(X, \mathcal{LSS})$ be a large scale space. A subset B_0 of X is **weakly bounded** if the intersection of B_0 with each coarse component of X is bounded.

Our definition of a finite localization will use weakly bounded sets. We want finite localizations to actually be large scale structures, so we must show that weakly bounded sets work well with finite unions and stars. The next two propositions do precisely this.

Proposition 5.4. *Let $(X, \mathcal{LSS})$ be a large scale structure. If B_1 and B_2 are weakly bounded subsets X, then $B_1 \cup B_2$ is also weakly bounded.*

Proof. Let C be a coarse component of X. Then $V_1 = B_1 \cap C$ and $V_2 = B_2 \cap C$ are bounded set, meaning that $V_1 \in \mathcal{V}_1$ and $V_2 \in \mathcal{V}_2$ for some $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{LSS}$. Take $x \in V_1$ and $y \in V_2$. As x, y are both in the same component C, there is a $\mathcal{U} \in \mathcal{LSS}$ and a $\mathcal{U}$-chain $x = x_0, \ldots x_n = y$. Then $W = \text{st}(V_1, \underset{1}{\mathcal{U}}, \ldots, \underset{n}{\mathcal{U}})$ contains V_1 and the point y. Thus $\text{st}(W, \mathcal{V}_2)$ contains V_1 as $V_1 \subset W$ and also $V_2 \in \mathcal{V}_2$ as V_2 intersects W at y. Hence $V_1 \cup V_2 \subset \text{st}(W, \mathcal{V}_2)$, which is a bounded set. Therefore $V_1 \cup V_2$ is bounded as well. Since C was an arbitrary coarse component, $B_1 \cup B_2$ is weakly bounded. □

Proposition 5.5. *Let $(X, \mathcal{LSS})$ be a large scale structure, B a weakly bounded subset of X, and $\mathcal{V}$ a uniformly bounded family. Then $\text{st}(B, \mathcal{V})$ is weakly bounded.*

Proof. Let C be a coarse component of X, and notice that $C \cap B$ is bounded. Then

$$C \cap \text{st}(B, \mathcal{V}) = \text{st}(C \cap B, \mathcal{V})$$

is bounded as well. Hence $\text{st}(B, \mathcal{V})$ is weakly bounded. □

We are now in a position to both define a finite localization of a large scale structure, and the show it is in fact a large scale structure (Proposition 5.7). The proof of Proposition 5.7 follows closely that of Example 4.1, the only difference is that the bounded sets take some care.

Definition 5.6. Let $(X, \mathcal{LSS})$ be a large scale structure. A **finite localization** $(X, \mathcal{LSS}_F)$ **of** $(X, \mathcal{LSS})$ is defined as all $\mathcal{U} \in \mathcal{LSS}$ so that there is a weakly bounded set $B(\mathcal{U})$ and number $N(\mathcal{U})$ such that for any $U \in \mathcal{U}$, $\text{card}(U \setminus B(\mathcal{U})) < N(\mathcal{U})$ and each $x \in X \setminus B(\mathcal{U})$ belongs to at most $N(\mathcal{U})$ elements of $\mathcal{U}$.

We will also introduce some notation. If $\mathcal{U}$ is a collection of subsets of X, we denote by $B(\mathcal{U})$ the weakly bounded set associated to $\mathcal{U}$ in a given finite localization.

Proposition 5.7. *If $(X, \mathcal{LSS})$ is a large scale structure and $(X, \mathcal{LSS}_F)$ a finite localization, then $\mathcal{LSS}_F$ is in fact a large scale structure on X.*

Proof. First suppose $\mathcal{U}_1 \in \mathcal{LSS}_F$ and $\mathcal{U}_2$ has the property that every non-singleton element of $\mathcal{U}_2$ is contained in some element of $\mathcal{U}_1$. Since $\mathcal{U}_1 \in \mathcal{LSS}_F$ there is a weakly bounded set $B(\mathcal{U}_1)$ and a number $N(\mathcal{U}_1)$ satisfying Definition 5.6. Take the weakly bounded set for $\mathcal{U}_2$ to be the same as that for $\mathcal{U}_1$, that is, $B(\mathcal{U}_2) = B(\mathcal{U}_1)$, and take $N(\mathcal{U}_2) = N(\mathcal{U}_1) \cdot 2^{N(\mathcal{U}_1)}$. The remainder of the proof that $\mathcal{U}_2 \in \mathcal{LSS}_F$ is the same as the proof following Example 4.1.

Next suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are bounded families of $\mathcal{LSS}$. We show that $\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2) \in \mathcal{LSS}$. Let $B(\mathcal{U}_i)$ and $N(\mathcal{U}_i)$ be the weakly bounded set and bounding number for $\mathcal{U}_i$ respectively for $i = 1, 2$. Take the weakly bounded set for $\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2)$ to be

$$B_3 = B(\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2)) := B(\mathcal{U}_1) \cup \mathrm{st}(B(\mathcal{U}_1), \mathcal{U}_2) \cup B(\mathcal{U}_2),$$

and take $N(\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2)) = N(\mathcal{U}_1)^2 \cdot N(\mathcal{U}_2)^2$ as in Example 4.1. Notice that B_3 is weakly bounded by using Proposition 5.4 and Proposition 5.5.

Consider some $\mathrm{st}(U_1, \mathcal{U}_2) \in \mathrm{st}(\mathcal{U}_1, \mathcal{U}_2)$. We first show that $\mathrm{card}(\mathrm{st}(U_1, \mathcal{U}_2)) < N(\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2))$. We know that $U_1 \setminus B_3$ contains at most $N(\mathcal{U}_1)$ points, as $B_3 \supset B(\mathcal{U}_1)$. Next U_1 may intersect infinitely many elements of $\mathcal{U}_2$; however, any $U_2 \in \mathcal{U}_2$ intersecting U_1 which also intersects $B(\mathcal{U}_1)$ will get thrown out completely as $B_3 \supset \mathrm{st}(B(\mathcal{U}_1), \mathcal{U}_2)$. Hence only elements of $\mathcal{U}_2$ which intersect the (at most) $N(B_1)$ elements of $U_1 \setminus B_3$ will contribute to the number of elements in $\mathrm{st}(U_1, \mathcal{U}_2) \setminus B_3$. Then each of the at most $N(\mathcal{U}_1)$ points in $U_1 \setminus B_3$ is in at most $N(\mathcal{U}_2)$ elements of $\mathcal{U}_2$ and each such $U_2 \in \mathcal{U}_2$ has $\mathrm{card}(U_2 \setminus B_3) < N(\mathcal{U}_2)$. Thus

$$\mathrm{card}(\mathrm{st}(U_1, \mathcal{U}_2)) < N(\mathcal{U}_1) \cdot N(\mathcal{U}_2)^2 \leq N(\mathcal{U}_1)^2 \cdot N(\mathcal{U}_2)^2 = N(\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2)).$$

The final part, showing that each $x \in X \setminus B_3$ belongs to at most $N(\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2))$ elements of $\mathrm{st}(\mathcal{U}_1, \mathcal{U}_2)$, proceeds exactly as the corresponding part of the proof of Example 4.1. □

5.2 Asymptotic Dimension and Finite Localization

It would be nice to say that if $\text{asdim}(X, \mathcal{LSS}) \leq n$, then a finite localization $(X, \mathcal{LSS}_F)$ has $\text{asdim}(X, \mathcal{LSS}_F) \leq n$ as well. However this is not the case. Consider the following example.

Example 5.8. Let

$$X = \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n = \left\{ \left(n^2 + \frac{p}{2^n}, \frac{q}{2^n} \right) : p, q \in \{0, 1, \ldots 2^n\} \right\}.$$

We view X as a subset of $\mathbb{R}^2$, and so has the large scale structure induced by the metric. Notice that $\text{asdim}(X) = 0$ for essentially the same reason that the set $\{n^2 : n \in \mathbb{N}\}$ has asymptotic dimension 0. We will describe a finite localization $(X, \mathcal{LSS}_F)$ which has non-zero asymptotic dimension. We do this by finding a uniformly bounded cover $\mathcal{W} \in \mathcal{LSS}_F$ so that if $\mathcal{V} \in \mathcal{LSS}_F$ is a coarsening of $\mathcal{W}$, then $\dim(\mathcal{V}) > 0$.

We first take the finite localization $\mathcal{LSS}_F$ given by $\mathcal{U} \in \mathcal{LSS}_F$ if there is a number $N(\mathcal{U}) < \infty$ so that for each $U \in \mathcal{U}$, $\text{card}(U) \leq N(\mathcal{U})$ and each $x \in X$ is in at most $N(\mathcal{U})$ elements of $\mathcal{U}$. This is a finite localization where the weakly bounded set for each $\mathcal{U}$ is the empty-set.

Now we define

$$\mathcal{W} = \left\{ W_n^{(p,q)} \right\}_{n \in \mathbb{N}}^{p,q \in \{0, \ldots, 2^n - 1\}},$$

where the $W_n^{(p,q)}$ is the 2×2 square defined as

$$W_n^{(p,q)} = \left\{ \left(n^2 + \frac{p}{2^n}, \frac{q}{2^n} \right), \left(n^2 + \frac{p}{2^n}, \frac{q+1}{2^n} \right), \left(n^2 + \frac{p+1}{2^n}, \frac{q}{2^n} \right), \left(n^2 + \frac{p+1}{2^n}, \frac{q+1}{2^n} \right) \right\}.$$

First notice that $\mathcal{W}$ is a cover for X. Next notice it is uniformly bounded since it refines the cover $B(n^2, 2)$. Finally $\mathcal{W} \in \mathcal{LSS}_F$ since each point of X is in at most 4 elements of $\mathcal{W}$, and each $W_n^{(p,q)}$ contains at most 4 elements.

Now let $\mathcal{V} \in \mathcal{LSS}_F$ be a coarsening of $\mathcal{W}$, with bounding number $N(\mathcal{V}) < \infty$. By way of contradiction, suppose the $\dim(\mathcal{V}) = 0$, that is $\mathcal{V}$ is a disjoint collection — if $V_1 \neq V_2 \in \mathcal{V}$ then $V_1 \cap V_2 = \emptyset$.

Since $\mathcal{V}$ is a disjoint collection and coarsens $\mathcal{W}$, if $W \in \mathcal{W}$ intersects some $V \in \mathcal{V}$, then $W \subset V$. Then if $V \in \mathcal{V}$ intersects S_n for some n it must be the case that $S_n \subset V$, otherwise there is some $W \in \mathcal{W}$ intersecting V and $V \setminus S_n$. Next since $\mathcal{V}$ is a cover, for each n there is a $V_n \in \mathcal{V}$ containing S_n. However this is a contradiction, since $\mathrm{card}(V_n)$ grows arbitrary large as n increases, but it also must be less than $N(\mathcal{V})$. Thus no disjoint $\mathcal{V}$ exists in $\mathcal{LSS}_F$, showing $\mathrm{asdim}(X, \mathcal{LSS}_F) > 0$.

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Vita

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After he and Anna graduated from Brigham Young University, Ryan attended graduate school in mathematics at the University of Tennessee. While at the University of Tennessee, Ryan has worked on his PhD under the direction of Dr. Jerzy Dydak. Ryan and Anna currently have three children, James, Ashlyn, and Andrew Jensen.

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