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Generalizations and Variations of the Zero-Divisor Graph

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To the Graduate Council:

I am submitting herewith a dissertation written by Grace Elizabeth McClurkin entitled "Generalizations and Variations of the Zero-Divisor Graph." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

David F. Anderson, Major Professor

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Generalizations and Variations of the Zero-Divisor Graph

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To my husband, Joshua. Thank you for your love and support throughout this adventure and onto the next one.

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Abstract

We explore generalizations and variations of the zero-divisor graph on commutative rings with identity. A zero-divisor graph is a graph whose vertex set is the nonzero zero-divisors of a ring, wherein two distinct vertices are adjacent if their product is zero. Variations of the zero-divisor graph are created by changing the vertex set, the edge condition, or both. The annihilator graph and the extended zero-divisor graph are both variations that change the edge condition, whereas the compressed graph and ideal-based graph change the vertex set. By combining these concepts, we define and investigate graphs where both the vertex set and edge condition are changed such as compressed annihilator graphs and ideal-based extended zero-divisor graphs.

We then generalize these variations by defining congruence-based versions of the annihilator graph and extended zero-divisor graph. Many of the previous graphs are shown to fit this more general framework. The congruence-based version of a graph has a vertex set equal to the equivalence classes of some multiplicative congruence relation, with the edge condition determined separately. We prove several foundational properties for these and additionally look at the relationships between and within the families of congruence-based graphs.

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Chapter 1

Introduction

Ring theory is an area of study within the field of abstract algebra. Our focus is on commutative rings with nonzero identity. For a general reference on abstract algebra, see [18].

We begin this chapter by introducing relevant graph theory definitions. Several more technical graph theory definitions, which are used in isolated circumstances, are defined as needed in later chapters. Then, we introduce the concept of a zero-divisor graph and highlight several variations which we investigate in detail in later chapters. Lastly, the organization of the dissertation is presented.

1.1 Relevant Graph Theory Concepts

A **graph** is a set of vertices and edges, denoted $G = (V(G), E(G))$, where $V(G)$ denotes the vertex set of G and $E(G) \subseteq V(G) \times V(G)$ denotes the collections of edges of G. If $V(G) = \emptyset$, then G is the **empty graph**, and we write $G = \emptyset$.

Two vertices x and y are called **adjacent** to each other if $(x, y) \in E(G)$. In practice, we often represent an edge $(x, y) \in E(G)$ as $x - y$. We say that a graph G is **undirected** if there is no orientation on the edges, so that edges $x - y$ and $y - x$ are treated as the same. A **loop** is an edge from a vertex to itself; that is, $x - x \in E(G)$. A **multi-edge** is an edge $x - y$ that appears more than once in $E(G)$ (i.e. with multiplicity). A **weighted graph** is a graph where each edge is assigned a value or weight, and an **unweighted graph** has no weights assigned to the edges. A **simple graph** is defined to be an unweighted, undirected graph with no multi-edges and no loops.

We realize graphs in practice as a series of vertices, drawn as boxes or dots, and edges drawn as lines between vertices as illustrated in Figure 1.1.

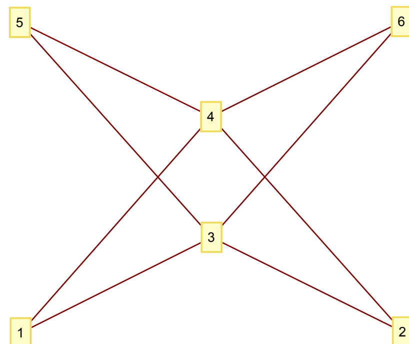


Figure 1.1: A simple graph G with vertices $V(G) = \{1, 2, 3, 4, 5, 6\}$ and edges $E(G) = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (3, 6), (4, 5), (4, 6)\}$.

The edge set of a vertex $x \in V(G)$, denoted $E_G(x)$, is $E_G(x) = \{y \in V(G) \,|\, x{-}y \in E(G)\}$. When dealing with simple graphs, the extended edge set of a vertex $x \in V(G)$, denoted $E^*_G(x)$, is the set of all adjecent vertices as well as any loops that are not included in the graph; that is, $E^*_G(x) = E_G(x) \cup \{x \,|\, x \text{ satisfies the edge condition with itself } \}$. A graph is called **complete** if for every distinct pair of vertices $x, y \in V(G)$, $x - y \in E(G)$. The graph G' is a **subgraph** of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. In addition, we say that G' is an **induced subgraph** of G if for every pair of distinct $x, y \in V(G')$ such that $x - y \in E(G)$, then $x - y \in E(G')$.

For two graphs G and H, a **graph homomorphism** $f : G \to H$ is a map between $V(G)$ and $V(H)$ such that $x - y \in E(G)$ implies that $f(x) - f(y) \in E(H)$. Further, two graphs are **isomorphic** if there exists a bijection $f : V(G) \to V(H)$ such that both f and f^{-1} are graph homomorphisms. In other words, G and H are isomorphic if distinct vertices $x, y \in V(G)$ are adjacent in G if and only if $f(x)$ and $f(y)$ are adjacent in H for some bijection f.

We say a **path** exists between two vertices if there exists a series of edges connecting x and y. A graph G is **connected** if there exists a path between every pair of vertices, otherwise we say G is **disconnected**. The **distance** between distinct vertices $x, y \in V(G)$, denoted by

$d(x,y)$, is defined to be the length of a shortest path between x and y, in particular $d(x,x) = 0$. If no such path exists, $d(x,y) = \infty$. The **diameter** of a graph G, denoted $diam(G)$, is defined as $diam(G) = sup\{d(x,y) \,|\, x, y \in V(G)\}$. That is, the diameter of a graph is the largest distance between any pair of points. Thus, $diam(G) = 0 \iff |V(G)| = 1$, and for $|V(G)| \geq 2$, G is complete $\iff diam(G) = 1$.

A **cycle** is a closed path in a graph; that is, a path that begins and ends at the same vertex. If G contains a cycle, we define the **girth** of G, denoted $gr(G)$, to be the length of a shortest nontrivial cycle in G. If G does not contain any cycles, we say that $gr(G) = \infty$.

Only definitions relevant to the work in this dissertation are presented here; for a more thorough introduction to graph theory, see [15].

1.2 Zero-Divisor Graphs

Let R be a commutative ring with nonzero identity. An element $x \in R$ is a **zero-divisor** if there exists a nonzero $y \in R$ such that $xy = 0$. The set of zero-divisors of a ring R is denoted $Z(R)$ and $Z(R)^* = Z(R) \setminus \{0\}$. A ring R is an integral domain if and only $Z(R) = \{0\}$.

In [13], Beck introduced a graph whose vertex set is the ring R and two distinct vertices were adjacent if their product was zero. Beck's graph highlighted zero-divisors by placing edges between them, but also contained an edge between every element and zero. He was primarily interested in colorings of graphs, and his work was continued in [4].

In [9], Anderson and Livingston introduced a graph associated to the ring R whose vertex set is restricted to $Z(R)^*$, as opposed to the entire ring R.

Definition 1.1. *Let R be a commutative ring with nonzero identity. The* zero-divisor graph*, denoted $\Gamma(R)$, is the simple graph with vertices $Z(R)^*$, such that distinct vertices x and y are adjacent if and only if $xy = 0$.*

Theorem 1.2 highlights several results from Anderson and Livingston's introductory paper on zero-divisor graphs.

Theorem 1.2. *Let R be a commutative ring with nonzero identity.*

1. *$\Gamma(R)$ is finite if and only if R is either finite or an integral domain.*

2. $\Gamma(R)$ *is connected and* $diam(\Gamma(R)) \leq 3$.

3. *If* $\Gamma(R)$ *contains a cycle, then* $gr(\Gamma(R)) \leq 7$.

4. $\Gamma(R)$ *is complete if and only if* $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $xy = 0$ *for all* $x, y \in Z(R)^*$.

Since its definition in [9], zero-divisor graphs have been further studied and extended to other number systems. In addition, there are numerous variations of zero-divisor graphs, some which change the edge condition, vertex set, or both. In the following subsections, we define zero-divisor graphs on semigroups, the annihilator graph, and the extended zero-divisor graph. For a survey on zero-divisor graphs, see [5] .

1.2.1 Semigroups

A **semigroup** is a set with an associative binary operation. Let S be a commutative multiplicative semigroup with zero. The **zero-divisors** of S can be defined similarly to the zero-divisors of a ring R. DeMeyer et al. extends and investigates a semigroup analog to $\Gamma(R)$ in [16].

Definition 1.3. *Let* S *be a commutative multiplicative semigroup with zero. The* zero-divisor graph *of the semigroup, denoted* $\Gamma(S)$*, is the simple graph with vertices* $Z(S)^*$*, such that distinct vertices* x *and* y *are adjacent if and only if* $xy = 0$.

Note that every ring R under multiplication can be viewed as a multiplicative semigroup with zero. In particular, the commutative rings with nonzero identity that we are concerned with throughout this dissertation can be viewed as commutative multiplicative semigroups with zero. Thus, $\Gamma(R)$ is a special case of the semigroup zero-divisor graph $\Gamma(S)$.

1.2.2 Annihilator Graph

The **annihilator** ideal of an element x in a ring R, denoted $ann_R(x)$, is defined as $ann_R(x) = \{r \in R \,|\, rx = 0\}$. Where obvious, the subscript is omitted.

In [12], Badawi introduced a variation of $\Gamma(R)$ called the annihilator graph which has the same edge set as $\Gamma(R)$, but changes the edge condition.

Definition 1.4. *Let R be a commutative ring with nonzero identity. The* annihilator graph*, denoted $AG(R)$, is the simple graph with vertices $Z(R)^*$, such that distinct vertices x and y are adjacent if and only if $ann(xy) \neq ann(x) \cup ann(y)$.*

Theorem 1.5 highlights several results from Badawi's introductory paper on annihilator graphs. A ring R is **reduced** if $x^2 = 0$ implies $x = 0$. For additional results and insights, see [2] and [17].

Theorem 1.5. *Let R be a commutative ring with nonzero identity.*

1. *$AG(R)$ is connected and $diam(AG(R)) \leq 2$.*
2. *$\Gamma(R) \subseteq AG(R)$.*
3. *If R is a reduced commutative ring with $AG(R) \neq \Gamma(R)$, then gr(AG(R))=3.*
4. *If R is a reduced commutative ring that is not an integral domain, then the following are equivalent:*
    - *AG(R) is complete;*
    - *$\Gamma(R)$ is complete;*
    - *R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

1.2.3 Extended Zero-Divisor Graph

In [14], Bennis et al. introduced another variation of $\Gamma(R)$ which changes the edge condition and has the same vertex set as $\Gamma(R)$. Specifically, Bennis relaxes the edge condition by allowing for products of powers of vertices to be zero.

Definition 1.6. *Let R be a commutative ring with nonzero identity. The* extended zero-divisor graph*, denoted $\overline{\Gamma}(R)$, is the simple graph with vertices $Z(R)^*$, such that distinct vertices x and y are adjacent if and only if $x^n y^m = 0$ for $n, m \in \mathbb{Z}^+$ with $x^n \neq 0$ and $y^m \neq 0$.*

Since $\overline{\Gamma}(R)$ deals with nilpotency, the following ring theory definitions are relevant. An element $x \in R$ is called **nilpotent** if there exists some $n \in \mathbb{Z}^+$ such that $x^n = 0$. The set of all nilpotent elements of a ring R is called the **nilradical** and is denoted $Nil(R)$. For a nonzero element $x \in Nil(R)$, n_x denotes the **index of nilpotency** of x; that is n_x is the smallest positive integer such that $x^{n_x} = 0$. The **radical** of an ideal $I \subseteq R$, denoted $\sqrt{I}$, is defined to be $\sqrt{I} = \{r \in R \,|\, r^n \in I \text{ for } n \in \mathbb{Z}^+\}$. In particular, we are interested in $\sqrt{ann(x)} = \{r \in R \,|\, r^n x = 0\}$.

Theorem 1.7 highlights several results from Bennis et al.'s introductory paper on extended zero-divisor graphs.

Theorem 1.7. *Let R be a commutative ring with nonzero identity.*

1. $\overline{\Gamma}(R)$ *is connected with* $diam(\overline{\Gamma}(R)) \leq 3$.

2. $\Gamma(R) \subseteq \overline{\Gamma}(R)$.

3. *If* $\Gamma(R) \neq \overline{\Gamma}(R)$, *then* $\overline{\Gamma}(R)$ *contains a cycle.*

4. $\overline{\Gamma}(R)$ *is complete if and only if either* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $Z(R) = Nil(R)$ *and for every* $x, y \in Z(R)^*$ *we have* $x^{n_x - 1} y^{n_y - 1} = 0$.

1.3 Dissertation Organization

In Chapter 2, we introduce equivalence class based annihilator graphs and equivalence class based extended zero-divisor graphs for several specific equivalence relations. In particular, we are interested in equivalence relations which cause $AG(R)$ and $\overline{\Gamma}(R)$ to compress their vertices based on the original edge set. These particular relations are identified and investigated; in addition, we compare graphs under various equivalence relations. In Chapter 3, we investigate an ideal-based annihilator graph and define an ideal-based extended zero-divisor graph. We prove several properties for each type of graph and detail a method for constructing ideal-based graphs. In Chapter 4, we define congruence-based annihilator graphs and congruence-based extended zero-divisor graphs. These graphs generalize $AG(R)$ and $\overline{\Gamma}(R)$ by creating families of graphs that share the same edge condition and whose vertex set is based off of a congruence relation. In addition to proving several properties for the generalized graphs, we demonstrate relationships between and within graph families. In Chapter 5, we conclude and provide future directions for the work presented.

Chapter 2

Graphs on Equivalence Classes and Compressed Graphs

In this chapter, we consider graphs $G = (V, E)$ whose vertex sets consist of the equivalence classes of the zero-divisors of a ring R under various equivalence relations. Note that the equivalence relation only determines the vertex set of the graph; the edge condition is given separately. An **equivalence relation** on a set R is a binary operation that satisfies the reflexive, symmetric, and transitive properties.

We combine equivalence relations and edge conditions in several ways, but are most interested in equivalence relations that produce a compressed version of the original graph.

Definition 2.1. *Let $G = (V(G), E(G))$ be a simple graph with $x, y \in V(G)$. Let $\sim$ be the equivalence relation given by $x \sim y \iff E^*_G(x) = E^*_G(y)$. Define the graph $G_\sim$ with $V(G_\sim) = \{[x]_\sim \mid x \in V(G)\}$ and an edge between distinct vertices $[x]$ and $[y]$ if $x - y \in E(G)$. We call $G_\sim$ the* ***compression*** *of G since every vertex of G with the same edge set (plus loops) compresses down into a single equivalence class of $G_\sim$.*

Note that since $\sim$ in Definition 2.1 relies on the edge sets of elements of G, which are fixed, the resulting graph $G_\sim$ is unique up to isomorphism. For example, consider the congruence relation $\sim_\Gamma$ given by $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. The zero-divisor graph on the equivalence classes of $Z(R)^*$ under $\sim_\Gamma$ compresses $\Gamma(R)$ since $x \sim_\Gamma y \iff ann_R(x) = ann_R(y) \iff E^*_{\Gamma(R)}(x) = E^*_{\Gamma(R)}(y)$.

Throughout the chapter, we investigate how various combinations of equivalence relations and edge conditions relate to each other. We are most interested in determining the conditions for which the graphs are naturally isomorphic, i.e. the map given by $f : [x]_{\sim_1} \mapsto [x]_{\sim_2}$ a well-defined graph isomorphism between $G_{\sim_1}$ and $G_{\sim_2}$. For convenience, we write $G_{\sim_1} = G_{\sim_2}$ when a natural isomorphism exists and $G_{\sim_1} \neq G_{\sim_2}$ when no such isomorphism exists.

2.1 Zero-Divisor Graphs on Equivalence Classes

Let $\sim_\Gamma$ be the equivalence relation on R given by $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. The zero-divisor graph whose vertex set is the equivalence classes of $Z(R)^*$ under $\sim_\Gamma$ was first introduced by Mulay in [20]. Note that multiplication is well-defined between equivalence classes of $\sim_\Gamma$.

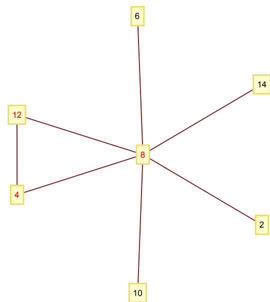
Definition 2.2. *Let R be a commutative ring with nonzero identity. Let $\sim_\Gamma$ be the equivalence relation on R given by $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. The* ***zero-divisor graph on the equivalence classes of*** $\sim_\Gamma$*, denoted $\Gamma_{E_\Gamma}(R)$, is the graph with vertices $Z(R/\sim_\Gamma)^* = \{[x]_{\sim_\Gamma} \,|\, x \in Z(R/\sim_\Gamma)^*\} = \{[x]_{\sim_\Gamma} \,|\, x \in R\} \setminus \{[0]_{\sim_\Gamma}, [1]_{\sim_\Gamma}\}$, such that distinct vertices $[x]_{\sim_\Gamma}$ and $[y]_{\sim_\Gamma}$ are adjacent if and only if $[x]_{\sim_\Gamma}[y]_{\sim_\Gamma} = [xy]_{\sim_\Gamma} = [0]_{\sim_\Gamma}$.*

Classically this graph is denoted $\Gamma_E(R)$, but we have added the subscript Γ to specify its relationship to $\sim_\Gamma$, and we propagate this notation for other equivalence based graphs.

As illustrated in Figure 2.1, $\Gamma_{E_\Gamma}(R)$ compresses $\Gamma(R)$ according to Definition 2.1. That is to say, any vertices with the same extended edge set in $\Gamma(R)$ are compressed down into a single vertex represented by an equivalence class in $\Gamma_{E_\Gamma}(R)$. Recall that since $\Gamma(R)$ is simple, loops are not included in the graphs. However since loops are taken into account when compressing $\Gamma(R)$ to $\Gamma_{E_\Gamma}(R)$ (i.e., x may be in $ann(x)$), we highlight the existence of loops with red vertices in the graph, such as vertices $12, 4 \in V(\Gamma(\mathbb{Z}_{16}))$ in Figure 2.1 .

In [24], Spiroff and Wickham prove several foundational properties of $\Gamma_{E_\Gamma}(R)$, including:

- $\Gamma_{E_\Gamma}(R)$ is connected and $diam(\Gamma_{E_\Gamma}(R)) \leq 3$.
- If $|V(\Gamma_{E_\Gamma}(R))| \geq 3$, then $\Gamma_{E_\Gamma}(R)$ is not a complete graph.



(a) $\Gamma(\mathbb{Z}_{16})$

(b) $\Gamma_{E_\Gamma}(\mathbb{Z}_{16})$

Figure 2.1: $\Gamma_{E_\Gamma}(\mathbb{Z}_{16})$ compresses the vertices of $\Gamma(\mathbb{Z}_{16})$

For further results and examples concerning $\Gamma_{E_\Gamma}(R)$, see [6].

2.2 Annihilator Graph on Equivalence Classes

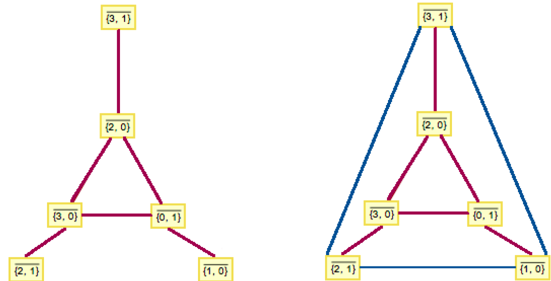
Analogous to defining $AG(R)$ with the vertex set of $\Gamma(R)$ and changing the edge condition, we define $AG_{E_\Gamma}(R)$ with the vertex set of $\Gamma_{E_\Gamma}(R)$ and applying the edge condition of $AG(R)$.

Definition 2.3. *Let R be a commutative ring with nonzero identity. Let $\sim_\Gamma$ be the equivalence relation on R given by $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. The* ***annihilator graph on the equivalence classes of*** $\sim_\Gamma$*, denoted $AG_{E_\Gamma}(R)$, is the graph with vertices $Z(R/\sim_\Gamma)^*$, such that distinct vertices $[x]_{\sim_\Gamma}$ and $[y]_{\sim_\Gamma}$ are adjacent if and only if*

$$ann_{R/\sim_\Gamma}[xy]_{\sim_\Gamma} \neq ann_{R/\sim_\Gamma}([x]_{\sim_\Gamma}) \cup ann_{R/\sim_\Gamma}([y]_{\sim_\Gamma}).$$

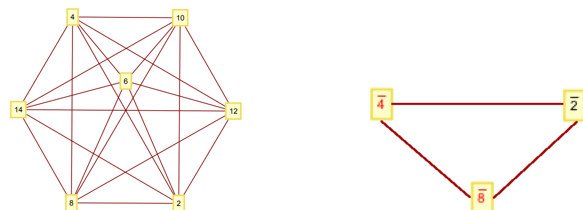
Proposition 2.4. *Let R be a commutative ring with nonzero identity. Then $\Gamma_{E_\Gamma}(R) \subseteq AG_{E_\Gamma}(R)$.*

Proof. Let $[x]_{\sim_\Gamma}, [y]_{\sim_\Gamma} \in Z(R/{\sim_\Gamma})^*$ be distinct vertices with $[x]_{\sim_\Gamma} - [y]_{\sim_\Gamma} \in E(\Gamma_{\sim_\Gamma}(R))$, i.e. $[xy]_{\sim_\Gamma} = [0]_{\sim_\Gamma}$. Thus $ann_{R/\sim_\Gamma}([xy]_{\sim_\Gamma}) = ann_{R/\sim_\Gamma}([0]_{\sim_\Gamma}) = R/{\sim_\Gamma} \ni [1]_{\sim_\Gamma}$. In contrast since $[x]_{\sim_\Gamma}, [y]_{\sim_\Gamma} \neq [0]_{\sim_\Gamma}$, $[1]_{\sim_\Gamma} \notin ann_{R/\sim}([x]_{\sim_\Gamma}) \cup ann_{R/\sim_\Gamma}([y]_{\sim_\Gamma})$ as both are missing the identity element. Therefore, $ann_{R/\sim_\Gamma}([xy]_{\sim_\Gamma}) \neq ann_{R/\sim_\Gamma}([x]_{\sim_\Gamma}) \cup ann_{R/\sim_\Gamma}([y]_{\sim_\Gamma})$, and $[x]_{\sim_\Gamma} - [y]_{\sim_\Gamma}$ is an edge of $AG_{\sim_\Gamma}(R)$. □



(a) $\Gamma_{E_\Gamma}(\mathbb{Z}_6 \times \mathbb{Z}_3)$ **(b)** $AG_{E_\Gamma}(\mathbb{Z}_6 \times \mathbb{Z}_3)$

Figure 2.2: An example of the containment described in Proposition 2.4 for $\mathbb{Z}_6 \times \mathbb{Z}_3$



(a) $AG(\mathbb{Z}_{16})$ **(b)** $AG_{E_\Gamma}(\mathbb{Z}_{16})$

Figure 2.3: $AG_{E_\Gamma}(\mathbb{Z}_{16})$ does not compress $AG(\mathbb{Z}_{16})$ because the compression of $AG(\mathbb{Z}_{16})$ is a single point since $AG(\mathbb{Z}_{16})$ is complete.

$\Gamma_{E_\Gamma}(R)$ is classically called the compressed zero-divisor graph of $\Gamma(R)$, however we do not refer to $AG_{E_\Gamma}(R)$ as the compressed graph of $AG(R)$ as it does not satisfy Definition 2.1. This is illustrated in Figure 2.3 for $\mathbb{Z}_{16}$. We define a new equivalence relation on R based on the annihilator edge condition to create a compression of $AG(R)$.

Definition 2.5. *Let R be a commutative ring with nonzero identity. Let $\sim_{AG}$ be the equivalence relation on R given by $x \sim_{AG} y \iff E^*_{AG(R)}(x) = E^*_{AG(R)}(y)$, where $E^*_{AG(R)}(x) = \{r \in R \,|\, ann_R(rx) \neq ann_R(r) \cup ann_R(x)\}$. The **annihilator graph on the equivalence classes of** $\sim_{AG}$, denoted $AG_{E_{AG}}(R)$, is the graph with vertices $Z(R/\sim_{AG})^* = \{[x]_{\sim_{AG}} \,|\, x \in R\} \setminus \{[0]_{\sim_{AG}}, [1]_{\sim_{AG}}\}$, such that distinct vertices $[x]_{\sim_{AG}}$ and $[y]_{\sim_{AG}}$ are adjacent if and only if $ann_{R/\sim_{AG}}[xy]_{\sim_{AG}} \neq ann_{R/\sim_{AG}}([x]_{\sim_{AG}}) \cup ann_{R/\sim_{AG}}([y]_{\sim_{AG}})$.*

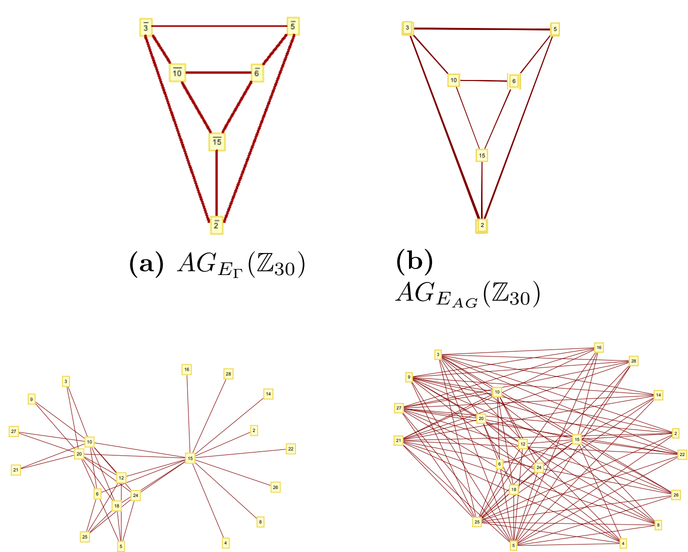
The graph $AG_{\sim_{AG}}(R)$ is (by construction) the compression of $AG(R)$ since every vertex of $AG(R)$ with the same edge set (plus loops) compresses down into a single equivalence class of $AG_{\sim_{AG}}(R)$. Note that $[0]_{\sim_{AG}} = \{0\}$, $[1]_{\sim_{AG}} = R \setminus Z(R)$, and the other equivalence classes form a partition of $Z(R)^*$. We conjecture that multiplication on equivalence classes of R under $\sim_{AG}$ is well-defined, but a rigorous proof is not available as yet.

Proposition 2.6. *Let R be a commutative ring with nonzero identity. If $\Gamma(R) = AG(R)$, then $AG_{E_\Gamma}(R) = AG_{E_{AG}}(R)$.*

Proof. Suppose $\Gamma(R) = AG(R)$. Recall, we denote the extended edge sets of a vertex $x \in Z(R)^*$ in $\Gamma(R)$ and $AG(R)$ by $E^*_{\Gamma(R)}(x) = \{r \in Z(R)^* \mid rx = 0\}$ and $E^*_{AG(R)}(x) = \{r \in Z(R)^* \mid ann(rx) \neq ann(r) \cup ann(x)\}$, respectively. Since $\Gamma(R) = AG(R)$, $E^*_{\Gamma(R)}(x) = E^*_{AG(R)}(x)$ for every $x \in Z(R)^*$. Since $AG_{E_\Gamma}(R)$ and $AG_{E_{AG}}(R)$ have the same edge condition, it suffices to show that the $AG_{E_\Gamma}(R)$ and $AG_{E_{AG}}(R)$ have the same vertex set to conclude that $AG_{E_\Gamma}(R) = AG_{E_{AG}}(R)$.

Let $x, y \in Z(R)^*$. Then $x \sim_{AG} y \iff E^*_{AG(R)}(x) = E^*_{AG(R)}(y) \iff E^*_{\Gamma(R)}(x) = E^*_{\Gamma(R)}(y) \iff ann(x) = ann(y) \iff x \sim_\Gamma y$. Hence, the equivalence classes of $Z(R)^*$ under $\sim_{AG}$ and $\sim_\Gamma$ are identical. Therefore, $AG_{E_\Gamma}(R)$ and $AG_{E_{AG}}(R)$ have the same vertex set and edge condition, so $AG_{E_\Gamma}(R) = AG_{E_{AG}}(R)$. $\square$

The converse of Proposition 2.6 does not hold for $R = \mathbb{Z}_{30}$ as illustrated in Figure 2.4.



(a) $AG_{E_\Gamma}(\mathbb{Z}_{30})$ **(b)** $AG_{E_{AG}}(\mathbb{Z}_{30})$

(c) $\Gamma(\mathbb{Z}_{30})$ **(d)** $AG(\mathbb{Z}_{30})$

Figure 2.4: $AG_{E_\Gamma}(\mathbb{Z}_{30}) = AG_{E_{AG}}(\mathbb{Z}_{30})$, but $\Gamma(\mathbb{Z}_{30}) \neq AG(\mathbb{Z}_{30})$.

Theorem 2.7. *Let p be a prime and $n, m \in \mathbb{Z}^+$ with $m \geq 3$. If p^m divides n, then $AG_{E_\Gamma}(\mathbb{Z}_n) \neq AG_{E_{AG}}(\mathbb{Z}_n)$.*

First, we show that $AG_{E_\Gamma}(\mathbb{Z}_{p^m}) \neq AG_{E_{AG}}(\mathbb{Z}_{p^m})$ for p prime and $m \geq 3$. Then by using the structure of graphs of direct products, we show that $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m}) \neq AG_{E_{AG}}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$. Next, we extend this result to any finite direct product that includes a $\mathbb{Z}_{p^m}$ term. Lastly, we obtain the result of Theorem 2.7.

Lemma 2.7.1. *Let p be a prime and $m \in \mathbb{Z}^+$ with $m \geq 3$. Then $AG_{E_\Gamma}(\mathbb{Z}_{p^m}) \neq AG_{E_{AG}}(\mathbb{Z}_{p^m})$.*

Proof. Let $R = \mathbb{Z}_{p^m}$, where p is a prime and $m \in \mathbb{Z}^+$ with $m \geq 3$. Since $AG_{E_\Gamma}(R)$ and $AG_{E_{AG}}(R)$ have the same edge condition, it is enough to show that they have different sized vertex sets to show the graphs are not equivalent.

Claim 1: $|V(AG_{E_\Gamma}(R))| \geq 2$.

The vertex set of $AG_{E_\Gamma}(R)$ is the set of equivalence classes of the zero-divisors of R under the equivalence relation $\sim_\Gamma$ given by $x \sim_\Gamma y \iff ann(x) = ann(y)$. Notice that $ann(p) = \{r \in R \,|\, pr = 0\} = \{r \in R \,\big|\, p^{m-1} \mid r\}$ and $ann(p^2) = \{r \in R \,|\, p^2 r = 0\} = \{r \in R \,\big|\, p^{m-2} \mid r\}$. Therefore $[p]_{\sim_\Gamma}$ and $[p^2]_{\sim_\Gamma}$ are distinct vertices of $\Gamma_{E_\Gamma}(R)$ since $ann(p) \neq ann(p^2)$.

Claim 2: $|V(AG_{E_{AG}}(R))| = 1$.

For $i \in \mathbb{Z}^+$ with $i < m$, define $S_i \subseteq Z(R)^*$ by $S_i = \{r \in Z(R)^* \,\big|\, p^i \mid r \text{ and } p^j \nmid r \ \forall j \geq i\}$. For $R = \mathbb{Z}_{p^m}$, $Z(R)^* = \bigcup_{i=1}^{m-1} S_i$. Fix $i \in \mathbb{Z}^+$ with $i < m$. Let $s_i \in S_i$, then $ann(s_i) = \{r \in R \,\big|\, rs_i = 0\} = \{r \in R \,\big|\, p^{m-i} \mid r\}$. Let $s_j \in S_j$, then $ann(s_j) = \{r \in R \,\big|\, rs_j = 0\} = \{r \in R \,\big|\, p^{m-j} \mid r\}$.

Suppose that $i+j < m$. Notice $ann(s_i s_j) = \{r \in R \,\big|\, r(s_i s_j) = 0\} = \{r \in R \,\big|\, p^{m-(i+j)} \mid r\}$. Hence, $ann(s_i s_j) \neq ann(s_i) \cup ann(s_j)$ and $s_i - s_j$ is an edge in $AG(R)$. Suppose that $i+j \geq m$. Then $s_i s_j = 0$, and $ann(s_i s_j) = R \neq ann(s_i) \cup ann(s_j)$ and $s_i - s_j$ is an edge in $AG(R)$.

Since every vertex of $AG(R)$ is of the form s_i for some $i \in \mathbb{Z}^+$ with $i \leq m$, we have that $AG(R)$ is complete. Thus the compressed annihilator graph, $AG_{E_{AG}}(R)$, contains a single vertex.

Therefore $AG_{E_\Gamma}(\mathbb{Z}_{p^m}) \neq AG_{E_{AG}}(\mathbb{Z}_{p^m})$ as they have a different number of vertices.

□

Toward extending this result to finite direct products of rings, we organize the construction of the edge set and vertex set of the components of the direct product from their components. We state these relationships for a direct product of two rings, and they can be rephrased for any finite direct product.

Remark 2.8. *Let R and S be commutative rings with nonzero identity. The vertex set of $\Gamma(R\times S)$ is $V(\Gamma(R\times S)) = \{(x,0)\,|\,x \in R\}\cup\{(0,y)\,|\,y \in S\}\cup\{(x,y)\,|\,x \in Z(R)^*\,,\,y \in Z(S)^*\}$. Let $(x_1,y_1),(x_2,y_2) \in V(\Gamma(R\times S))$. Then $(x_1,y_1)-(x_2,y_2)$ is an edge of $\Gamma(R\times S)$ if and only if one of the following scenarios occurs:*

1. *$x_1 = 0$ and $y_2 = 0$;*
2. *$x_2 = 0$ and $y_1 = 0$;*
3. *$x_1 - x_2$ is an edge or loop of $\Gamma(R)$ and either $y_1 = 0$ or $y_2 = 0$;*
4. *$y_1 - y_2$ is an edge or loop of $\Gamma(S)$ and either $x_1 = 0$ or $x_2 = 0$;*
5. *$x_1 - x_2$ is an edge or loop of $\Gamma(R)$ and $y_1 - y_2$ edge or loop of $\Gamma(S)$.*

Recall that the vertex set of $\Gamma(R\times S)$ is the same as $AG(R\times S)$ and the edge construction can be easily rephrased for $AG(R\times S)$. Similarly, an analogous construction for $\Gamma_{\sim_\Gamma}(R\times S)$ can be rephrased for $AG_{\sim_\Gamma}(R\times S)$. We can use the construction of vertices and edges of $\Gamma(R\times S)$ in conjunction with the ideas laid out in the proof of Lemma 2.7.1 to extend this idea to direct products.

Lemma 2.8.1. *Let p be a prime and $n,m \in \mathbb{Z}^+$ with $m \geq 3$. Then $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m}) \neq AG_{E_{AG}}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$.*

Proof. Since $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$ and $AG_{E_{AG}}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$ have the same edge condition, it is enough to show that they have different vertex sets to show the graphs are not equivalent.

As we saw in Claim 1 of Lemma 2.7.1, $[p]_{\sim_\Gamma}$ and $[p^2]_{\sim_\Gamma}$ are distinct vertices of $AG_{E_\Gamma}(\mathbb{Z}_{p^m})$. Combining this result with the edge construction of $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$ in Remark 2.8, we have that $[(0,p)]_{\sim_\Gamma}$ and $[(0,p^2)]_{\sim_\Gamma}$ are distinct vertices of $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$.

From Claim 2 of Lemma 2.7.1, we know that $AG(\mathbb{Z}_{p^m})$ is complete, so all vertices of the form $(0, p^i)$ with $i \in \mathbb{Z}^+$ and $i < m$ condense into a single vertex of the graph $AG_{E_{AG}}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$.

We have shown that the graphs have different vertex sets, specifically because $[(0,p)]_{\sim_\Gamma}$ and $[(0,p^2)]_{\sim_\Gamma}$ are distinct vertices of $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$, where as their counterparts $[(0,p)]_{\sim_{AG}}$ and $[(0,p^2)]_{\sim_{AG}}$ condense into a single vertex of the graph $AG_{E_{AG}}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$.

As such, $AG_{E_\Gamma}(\mathbb{Z}_n \times \mathbb{Z}_{p^m}) \neq AG_{E_{AG}}(\mathbb{Z}_n \times \mathbb{Z}_{p^m})$. □

The reasoning in the proof for Lemma 2.8.1 can be extended to any finite direct product where $\mathbb{Z}_{p^m}$ is a component of the direct product.

Corollary 2.8.1. *Let R be the direct product of some finite commutative rings, $\mathbb{Z}_{n_i}$ for some $n_i \in \mathbb{Z}^+$. Let p be a prime and $m \in \mathbb{Z}^+$ with $m \geq 3$. If $\mathbb{Z}_{p^m}$ is a component of the direct product, then $AG_{E_\Gamma}(R) \neq AG_{E_{AG}}(R)$.*

Proof of Theorem 2.7

Proof. Consider $\mathbb{Z}_n$ with n divisible by p^m for some prime p and $m \in \mathbb{Z}^+$ the largest power of p that divides n with $m \geq 3$. Since n is divisible by p^m, $\mathbb{Z}_{p^m}$ is a component of the direct product decomposition of $\mathbb{Z}_n$. By Lemma 2.8.1, $AG_{E_\Gamma}(\mathbb{Z}_n) \neq AG_{E_{AG}}(\mathbb{Z}_n)$. □

2.3 Extended Zero-Divisor Graph on Equivalence Classes

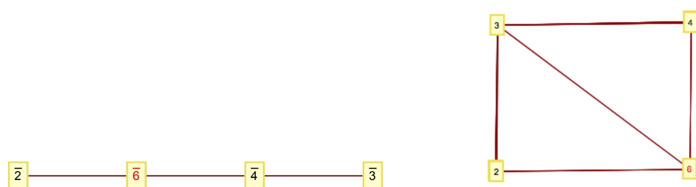
Analogous to defining $\overline{\Gamma}(R)$ with the vertex set of $\Gamma(R)$ and changing the edge condition, we define $\overline{\Gamma}_{E_\Gamma}(R)$ with the vertex set of $\Gamma_{E_\Gamma}(R)$ and applying the edge condition of $\overline{\Gamma}(R)$.

Definition 2.9. *Let R be a commutative ring with nonzero identity. Let $\sim_\Gamma$ be the equivalence relation on R given by $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. The* ***extended zero-divisor graph on the equivalence classes of*** *$\sim_\Gamma$, denoted $\overline{\Gamma}_{E_\Gamma}(R)$, is the graph with vertices $Z(R/\sim_\Gamma)^* = \{[x]_{\sim_\Gamma} \mid x \in R\} \setminus \{[0]_{\sim_\Gamma}, [1]_{\sim_\Gamma}\}$, such that distinct vertices $[x]_{\sim_\Gamma}$ and $[y]_{\sim_\Gamma}$ are adjacent if and only if three exists an $n, m \in \mathbb{Z}^+$ such that $[x]^n_{\sim_\Gamma}[y]^m_{\sim_\Gamma} = [0]_{\sim_\Gamma}$, $[x]^n_{\sim_\Gamma} \neq [0]_{\sim_\Gamma}$, and $[y]^m_{\sim_\Gamma} \neq [0]_{\sim_\Gamma}$.*

Proposition 2.10. *Let R be a commutative ring with nonzero identity. Then $\Gamma_{E_\Gamma}(R) \subseteq \overline{\Gamma}_{E_\Gamma}(R)$.*

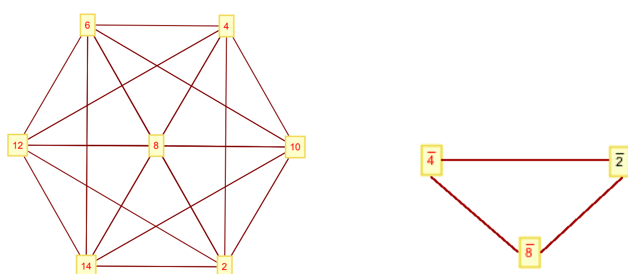
Proof. Notice that $V(\Gamma_{E_\Gamma}(R)) = V(\overline{\Gamma}_{E_\Gamma}(R))$, so to show containment it suffices to show that each edge of $\Gamma_{E_\Gamma}(R)$ is also an edge of $\overline{\Gamma}_{E_\Gamma}(R)$. Let $[x]_{\sim_\Gamma}, [y]_{\sim_\Gamma} \in V(\Gamma_{E_\Gamma}(R))$ such that $[x]_{\sim_\Gamma} - [y]_{\sim_\Gamma}$ is an edge of $\Gamma_{E_\Gamma}(R)$, so $[x]_{\sim_\Gamma}[y]_{\sim_\Gamma} = [0]_{\sim_\Gamma}$. Clearly the edge condition of the extended zero-divisor graph is satisfied with $n = m = 1$ since $[x]_{\sim_\Gamma}[y]_{\sim_\Gamma} = [0]_{\sim_\Gamma}$, $[x]_{\sim_\Gamma} \neq [0]_{\sim_\Gamma}$, and $[y]_{\sim_\Gamma} \neq [0]_{\sim_\Gamma}$. Thus $[x]_{\sim_\Gamma} - [y]_{\sim_\Gamma}$ is an edge of $\overline{\Gamma}_{\sim_\Gamma}(R)$. $\square$

The containment described in Proposition 2.10 is illustrated in Figure 2.5 for $R = \mathbb{Z}_{12}$.



(a) $\Gamma_{E_\Gamma}(\mathbb{Z}_{12})$ **(b)** $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{12})$

Figure 2.5: $\Gamma_{E_\Gamma}(\mathbb{Z}_{12}) \subseteq \overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{12})$



(a) $\overline{\Gamma}(\mathbb{Z}_{16})$ **(b)** $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{16})$

Figure 2.6: $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{16})$ does not compress the vertices of $\overline{\Gamma}(\mathbb{Z}_{16})$ based on their edge sets.

$\overline{\Gamma}_{E_\Gamma}(R)$ does not compress $\overline{\Gamma}(R)$ as illustrated in Figure 2.6. In this case, if $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{16})$ created a compression of the vertices of $\overline{\Gamma}(\mathbb{Z}_{16})$ it would contain only one vertex as $\overline{\Gamma}(R)$ is complete. We define a new equivalence relation on R based on the extended edge condition to create a compression of $\overline{\Gamma}(R)$.

Definition 2.11. *Let R be a commutative ring with nonzero identity. Let $\sim_{\overline{\Gamma}}$ be the equivalence relation on R given by $x \sim_{\overline{\Gamma}} y \iff E^*_{\overline{\Gamma}(R)}(x) = E^*_{\overline{\Gamma}(R)}(y)$, where $E^*_{\overline{\Gamma}(R)}(x) = \{r \in$*

$R \,|\, \exists\, n,m \in \mathbb{Z}^+$ *such that* $x^n r^m = 0, x^n \neq 0,$ *and* $r^m \neq 0\}$. *The* ***extended zero-divisor graph on the equivalence classes of*** $\sim_{\overline{\Gamma}}$, *denoted* $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$, *is the graph with vertices* $Z(R/\sim_{\overline{\Gamma}})^* = \{[x]_{\sim_{\overline{\Gamma}}} \,|\, x \in R\} \setminus \{[0]_{\sim_{\overline{\Gamma}}}, [1]_{\sim_{\overline{\Gamma}}}\}$, *such that distinct vertices* $[x]_{\sim_{\overline{\Gamma}}}$ *and* $[y]_{\sim_{\overline{\Gamma}}}$ *are adjacent if and only if there exist* $n,m \in \mathbb{Z}^+$ *such that* $[x]^n_{\sim_{\overline{\Gamma}}}[y]^m_{\sim_{\overline{\Gamma}}} = [0]_{\sim_{\overline{\Gamma}}}$, $[x]^n_{\sim_{\overline{\Gamma}}} \neq [0]_{\sim_{\overline{\Gamma}}}$ *and* $[y]^m_{\sim_{\overline{\Gamma}}} \neq [0]_{\sim_{\overline{\Gamma}}}$.

The graph $\overline{\Gamma}_{\sim_{\overline{\Gamma}}}(R)$ is (by construction) the compression of $\overline{\Gamma}(R)$ since every vertex of $\overline{\Gamma}(R)$ with the same edge set (plus loops) compresses down into a single equivalence class of $\overline{\Gamma}_{\sim_{\overline{\Gamma}}}(R)$. Note that $[0]_{\sim_{\overline{\Gamma}}} = \{0\}$, $[1]_{\sim_{\overline{\Gamma}}} = R \setminus Z(R)$, and the other equivalence classes form a partition of $Z(R)^*$. We conjecture that multiplication on equivalence classes of R under $\sim_{\overline{\Gamma}}$ is well-defined, but a rigorous proof is not available as yet.

Proposition 2.12. *Let R be a commutative ring with nonzero identity. If* $\Gamma(R) = \overline{\Gamma}(R)$, *then* $\overline{\Gamma}_{E_{\Gamma}}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$.

Proof. Suppose $\Gamma(R) = \overline{\Gamma}(R)$. Recall, we denote the extended edge sets of a vertex $x \in Z(R)^*$ in $\Gamma(R)$ by $E^*_{\Gamma(R)}(x) = \{r \in Z(R)^* \,|\, rx = 0\}$ and the edge set in $\overline{\Gamma}(R)$ by $E^*_{\overline{\Gamma}(R)}(x) = \{r \in R \,|\, \exists\, n,m \in \mathbb{Z}^+$ such that $x^n r^m = 0, x^n \neq 0,$ and $r^m \neq 0\}$. Since $\overline{\Gamma}_{E_{\Gamma}}(R)$ and $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$ have the same edge condition, it suffices to show that $\overline{\Gamma}_{E_{\Gamma}}(R)$ and $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$ have the same vertex set to conclude that $\overline{\Gamma}_{E_{\Gamma}}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$. Notice that since $\Gamma(R) = \overline{\Gamma}(R)$, we have $E^*_{\Gamma(R)}(x) = E^*_{\overline{\Gamma}(R)}(x)$ for every vertex x.

Let $x, y \in Z(R)^*$. Then $x \sim_{\overline{\Gamma}} y \iff E^*_{\overline{\Gamma}(R)}(x) = E^*_{\overline{\Gamma}(R)}(y) \iff E^*_{\Gamma(R)}(x) = E^*_{\Gamma(R)}(y) \iff ann(x) = ann(y) \iff x \sim_{\Gamma} y$. Hence the equivalence classes of $Z(R)^*$ under $\sim_{\overline{\Gamma}}$ and $\sim_{\Gamma}$ are identical. Therefore, $\overline{\Gamma}_{E_{\Gamma}}(R)$ and $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$ have the same vertex set and edge condition, so $\overline{\Gamma}_{E_{\Gamma}}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$. □

Remark 2.13. *Let R be a commutative ring with nonzero identity. If* $\Gamma(R) = \overline{\Gamma}(R)$, *then* $\Gamma_{E_{\Gamma}}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$.

Proof. The application of Definition 2.1 acts identically on both $\Gamma(R)$ and $\overline{\Gamma}(R)$. Thus $\Gamma_{E_{\Gamma}}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$.

□

We now apply results from Bennis et al. [14] to show that the converse of Proposition 2.12 holds.

Proposition 2.14. *Let R be a commutative ring with nonzero identity. If $\Gamma_{E_\Gamma}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$, then $\Gamma(R) = \overline{\Gamma}(R)$.*

Proof. Suppose $\Gamma_{E_\Gamma}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$. For any element $x \in Z(R)^*$, we may assume that x belongs to some equivalence class under either equivalence relation; so without loss of generality, we can say $[x]_{\sim_\Gamma}$ and $[x]_{\sim_{\overline{\Gamma}}}$ are vertices of $\Gamma_{E_\Gamma}(R)$ and $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$, respectively.

In order to use [Theorem 2.1, [14]], we must show the following two conditions hold.

Claim 1: If $Nil(R) \neq 0$, then every nonzero nilpotent element has index 2.

Suppose there exists an $x \in R$ such that $x \neq 0$ and $x \in Nil(R)$. This means there exists some smallest $m \in \mathbb{Z}^+$ such that $x^m = 0$. Suppose that $m \geq 3$. By Lemma 2.2 in [14], $ann(x) \subsetneq ann(x^n)$ for every integer $n \geq 2$. Since $x \neq 0$, we may assume that $2 \leq n \leq m$. Let $y \in ann(x^n) \setminus ann(x)$; then $yx^n = 0$ and $xy \neq 0$. It follows that $[x]_{\sim_{\overline{\Gamma}}} - [y]_{\sim_{\overline{\Gamma}}}$ is an edge of $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$, but $[x]_{\sim_\Gamma} - [y]_{\sim_\Gamma}$ is not an edge of $\Gamma_{E_\Gamma}(R)$. This contradicts $\Gamma_{E_\Gamma}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$. Hence $m < 3$ and since $x \neq 0$, $m = 2$.

Claim 2: For every $x \in Z(R)$, $\sqrt{ann(x)} \setminus Nil(R) \subseteq ann(x)$.

Let $x \in Z(R)^*$. Suppose there exists a $y \in R$ such that $x \neq y$ and $y \in \sqrt{ann(x)} \setminus Nil(R)$. This means $y^n x = 0$ for some $n \in Z^+$ and $y^m \neq 0$ for all $m \in \mathbb{Z}^+$. It follows that $[x]_{\sim_{\overline{\Gamma}}} - [y]_{\sim_\Gamma}$ is an edge in $\overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$, so $[x]_{\sim_\Gamma} - [y]_{\sim_\Gamma}$ is an edge of $\Gamma_{E_\Gamma}(R)$. Thus, $xy = 0$ and $y \in ann(x)$.

Therefore by [Theorem 2.1, [14]], $\Gamma(R) = \overline{\Gamma}(R)$. □

For $R = \mathbb{Z}_n$, $V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_n))$ is heavily influenced by the prime factorization of n. As such, the following result lays out some specific properties of $\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_n)$.

Proposition 2.15.

1. *Let p be some prime and $m \in \mathbb{Z}^+$. Then $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_{p^m}))| = 1$.*

2. *Let $n = p_1 \cdot p_2 \cdots p_r$ for distinct primes $p_1, p_2, ..., p_r$. Then $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(Z_n))| = \sum_{i=1}^{r} \binom{r}{i} - 1$.*

3. *Let $n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_r^{m_r}$ for distinct primes $p_1, p_2, ..., p_r$ and $m_1, m_2, ..., m_r \in \mathbb{Z}^+$ with at least one $m_i \geq 2$. Then $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_n))| = \sum_{i=1}^{r} \binom{r}{i}$.*

Proof. 1. Let p be a prime and $m \in \mathbb{Z}^+$. Then $\overline{\Gamma}(\mathbb{Z}_{p^m})$ is complete, so the compression under $\sim_{\overline{\Gamma}}$ is a single vertex. Thus, $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_{p^m}))| = 1$.

2. Let $n = p_1 \cdot p_2 \cdots p_r$ for distinct primes $p_1, p_2, ..., p_r$. Then $V(\overline{\Gamma}(\mathbb{Z}_n)) = Z(\mathbb{Z}_n)^* = \{x \in \mathbb{Z}_n \,|\, x$ is divisible by some product of p_i's$\}$. Under the equivalence relation $\sim_\Gamma$, vertices are compressed based on annihilators or what send a vertex to zero. In the context of $\mathbb{Z}_n$ with the extended-zero divisor graph condition, a product is zero if it contains all the primes $p_1, ...p_r$. Thus under $\sim_\Gamma$, the vertices are naturally compressed based on which primes that they are divisible by. To count the number of vertices in $\overline{\Gamma}_{E_{\overline{\Gamma}}}(Z_n)$, we can sum up all the ways to choose primes, resulting in $\sum_{i=1}^{r} \binom{r}{i}$ vertices. Since an element containing r primes equals zero, such an element is not a vertex of the graph, and we subtract 1 from our sum. Thus, $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(Z_n))| = \sum_{i=1}^{r} \binom{r}{i} - 1$.

3. Let $n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_r^{m_r}$ for distinct primes $p_1, p_2, ..., p_r$ and $m_1, m_2, ..., m_r \in \mathbb{Z}^+$ with at least one $m_i \geq 2$. The argument is similar to that presented in (2) above. Indeed under the extended edge condition, vertices with the same primes to differing powers compress into the same equivalence class under $\sim_{\overline{\Gamma}}$. Additionally since at least one $m_i \geq 2$, an element with r primes does not equal zero, so such an element appears as a vertex of the graph. Thus, $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_n))| = \sum_{i=1}^{r} \binom{r}{i}$. □

Corollary 2.15.1. *Let p be a prime and $m \in \mathbb{Z}^+$ with $m \geq 3$. Then $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{p^m}) \neq \overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_{p^m})$.*

Proof. To show the graphs are not equal, it is enough to show that they have different numbers of vertices. Recall that, $V(\Gamma_{E_\Gamma}(\mathbb{Z}_{p^m})) = V(\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{p^m}))$ since they are graphs under the same equivalence relation which determines the vertex set.

By Claim 1 of the proof of Lemma 2.7.1, $|V(\Gamma_{E_\Gamma}(\mathbb{Z}_{p^m}))| \geq 2$. It follows that $|V(\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{p^m}))| \geq 2$. By Proposition 2.15, we know that $|V(\overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_{p^m}))| = 1$. Thus the graphs have different sized vertex sets, and therefore $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_{p^m}) \neq \overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_{p^m})$. □

In the proof of the following proposition, we follow a similar argument as the one used for Theorem 2.7.

Proposition 2.16. *Let p be a prime and $m, n \in \mathbb{Z}^+$ with $m \geq 3$. If p^m divides n, then $\overline{\Gamma}_{E_\Gamma}(\mathbb{Z}_n) \neq \overline{\Gamma}_{E_{\overline{\Gamma}}}(\mathbb{Z}_n)$.*

Chapter 3

Ideal-based Graphs

In this chapter we consider ideal-based graphs, which generalize the concept of a zero-divisor graph and its variations to any ideal in the ring. Towards this end, we begin by introducing Redmond's ideal-based zero-divisor graph,. Then, we successively define and investigate properties of ideal-based versions of the annihilator graph and the extended zero-divisor graph.

3.1 Ideal-based Zero-Divisor Graphs

The ideal-based zero-divisor graph was first introduced by Redmond in [22]. This graph generalizes the notion of a zero-divisor graph to any ideal in the ring, with $\Gamma(R)$ being a special case when considering the zero ideal.

Definition 3.1. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal. The **ideal-based zero-divisor graph**, denoted $\Gamma_I(R)$, is the graph with vertices $\{x \in R \backslash I \mid xy \in I \text{ for some } y \in R \setminus I\}$, such that distinct vertices x and y are adjacent if and only if $xy \in I$.*

In [22], Redmond showed several foundational results for a commutative ring R with nonzero identity, including:

- $\Gamma_I(R)$ is connected with $diam(\Gamma_I(R)) \leq 3$.
- If $\Gamma_I(R)$ contains a cycle, then $gr(\Gamma_I(R)) \leq 7$.

For further results and examples concerning $\Gamma_I(R)$, see [10].

3.2 Ideal-based Annihilator Graphs

Analogous to defining $AG(R)$ with the vertex set of $\Gamma(R)$ and changing the edge condition, we define $AG_I(R)$ with the vertex set of $\Gamma_I(R)$ and applying the edge condition of $AG(R)$. This concept was first defined and briefly discussed in [1]. The classic annihilator graph, $AG(R)$, compares annihilators of elements and their products to create edges. To generalize this idea in context, we consider an analog to the annihilator of an element.

Definition 3.2. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal. The* ***ideal-based annihilator graph****, denoted $AG_I(R)$, is the graph with vertices $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$, and two distinct vertices x and y are adjacent if and only if $(I : (x)(y)) \neq (I : (x)) \cup (I : (y))$, where $(I : (x)) = \{r \in R \mid r(x) \subseteq I\} = \{r \in R \mid rx \in I\}$.*

Since $AG_I(R)$ combines aspects of both Badawi's $AG(R)$ and Redmond's $\Gamma_I(R)$, we prove analogs to several results from [12] and [22].

3.2.1 Generalizing Results for $AG(R)$ to $AG_I(R)$

Proposition 3.3. *Let R be a commutative ring with nonzero identity and let I be an ideal of R. Take $V(AG_I(R)) = \{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$.*

1. *Let x, y be distinct elements such that $x, y \in V(AG_I(R))$. Then $x - y$ is not an edge of $AG_I(R)$ if and only if $(I : (xy)) = (I : (x))$ or $(I : (xy)) = (I : (y))$.*

2. *If $x - y$ is an edge of $\Gamma_I(R)$ for some distinct $x, y \in V(AG_I(R))$, then $x - y$ is an edge of $AG_I(R)$. In particular, if P is a path in $\Gamma_I(R)$, then P is a path in $AG_I(R)$.*

3. *If $x - y$ is not an edge of $AG_I(R)$ for some distinct $x, y \in V(AG_I(R))$, then $(I : (x)) \subseteq (I : (y))$ or $(I : (y)) \subseteq (I : (x))$.*

4. *If $(I : (x)) \nsubseteq (I : (y))$ and $(I : (y)) \nsubseteq (I : (x))$ for some distinct $x, y \in V(AG_I(R))$, then $x - y$ is an edge of $AG_I(R)$.*

5. *If $d_{AG_I(R)}(x, y) = 3$ for some distinct $x, y \in V(AG_I(R))$, then $x - y$ is an edge of $AG_I(R)$.*

6. *If $x - y$ is not an edge of $AG_I(R)$ for some distinct $x, y \in V(AG_I(R)$, then there is a $w \in V(AG_I(R))$ such that $x - w - y$ is a path in $\Gamma_I(R)$, and hence $x - w - y$ is also a path in $AG_I(R)$.*

Proof. Let R be a commutative ring with nonzero identity and let I be an ideal of R. Let $V(AG_I(R)) = \{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$.

1. ($\Rightarrow$) Suppose that $x - y$ is not an edge of $AG_I(R)$. Then $(I : (xy)) = (I : (x)) \cup (I : (y))$ by definition. Since $(I : (xy))$ is a union of two ideals, we have that $(I : (xy)) = (I : (x))$ or $(I : (xy)) = (I : (x))$.
   ($\Leftarrow$) Suppose that $(I : (xy)) = (I : (x))$ or $(I : (xy)) = (I : (y))$. Then $(I : (xy)) = (I : (x)) \cup (I : (y))$, so $x - y$ is not an edge of $AG_I(R)$.

2. Suppose that $x - y$ is an edge of $\Gamma_I(R)$ for some distinct $x, y \in V(AG_I(R))$. Then $xy \in I$ and hence $(I : (xy)) = R$. Since $x \notin I$ and $y \notin I$, $(I : (x)) \neq R$ and $(I : (y)) \neq R$. Thus $x - y$ is an edge of $AG_I(R)$. It is clear then that a path in $\Gamma_I(R)$ would also be a path in $AG_I(R)$.

3. Suppose that $x - y$ is not an edge of $AG_I(R)$ for some distinct $x, y \in V(AG_I(R))$. Then $(I : (xy)) = (I : (x)) \cup (I : (y))$. Since $(I : (xy))$ is the union of two ideals, we have $(I : (x)) \subseteq (I : (y))$ or $(I : (y)) \subseteq (I : (x))$.

4. Follows from (3).

5. Suppose that $d_{AG_I(R)}(x, y) = 3$ for some distinct $x, y \in V(AG_I(R))$. Then $(I : (x)) \nsubseteq (I : (y))$ and $(I : (y)) \nsubseteq (I : (x))$. Hence $x - y$ is an edge of $AG_I(R)$ by (4).

6. Suppose that $x - y$ is not an edge of $AG_I(R)$ for some distinct $x, y \in V(AG_I(R))$. Then there is a $w \in (I : (x)) \cap (I : (y))$ such that $w \notin R \setminus I$ by (3). Since $xy \in R \setminus I$, we have $w \in V(AG_I(R)) \setminus \{x, y\}$. Hence $x - w - y$ is a path in $\Gamma_I(R)$, and thus $x - w - y$ is a path in $AG_I(R)$ by (2).

$\square$

The next two theorems address foundational graph properties of diameter and girth. In this instance, $AG_I(R)$ behaves more like $AG(R)$, which has stricter diameter and girth constraints than $\Gamma_I(R)$.

Theorem 3.4. *Let R be a commutative ring with nonzero identity and $|V(AG_I(R))| \geq 2$. Then $AG(R)$ is connected and diam$(AG_I(R)) \leq 2$.*

Proof. This follows directly from Proposition 3.3 (6). $\square$

Theorem 3.5. *Let R be a commutative ring with nonzero identity and I a radical ideal of R, and suppose that $AG_I(R) \neq \Gamma_I(R)$. Then $gr(AG_I(R)) = 3$. Furthermore, there is a cycle C of length three in $AG_I(R)$ such that each edge of C is not an edge of $\Gamma_I(R)$.*

To prove this result, we need the following Lemma.

Lemma 3.5.1. *Let R be a commutative ring with nonzero identity , and suppose that $x - y$ is an edge of $AG_I(R)$ that is not an edge of $\Gamma_I(R)$ for some distinct $x, y \in V(AG_I(R))$. If $xy^2 \notin I$ and $x^2y \notin I$, then there is a $w \in V(AG_I(R))$ such that $x - w - y$ is a path in $AG_I(R)$ that is not a path in $\Gamma_I(R)$, and hence $C : x - w - y - x$ is a cycle in $AG_I(R)$ of length three and each edge of C is not an edge of $\Gamma_I(R)$.*

Proof. Suppose that $x - y$ is an edge of $AG_I(R)$ that is not an edge of $\Gamma_I(R)$. Then $xy \notin I$ and there is a $w \in (I : (xy)) \setminus ((I : (x)) \cup (I : (y)))$.
Suppose $w \in \{x, y\}$. Then $w(xy) \in I \Rightarrow x^2y \in I$ or $xy^2 \in I$, which is a contradiction. Thus $w \notin \{x, y\}$.
Notice that $y \in (I : (xw)) \setminus ((I : (x)) \cup (I : (w)))$, so $x - w$ is an edge of $AG_I(R)$. Similarly, $x \in (I : (yw)) \setminus ((I : (y)) \cup (I : (w)))$, so $y - w$ is an edge of $AG_I(R)$. Hence, $x - w - y - x$ is a cycle in $AG_I(R)$. Moreover, since $xw \notin I$ and $yw \notin I$, we have that $x - w - y$ is not a path in $\Gamma_I(R)$.

$\square$

Proof of Theorem 3.5

Since $AG_I(R) \neq \Gamma_I(R)$, there exists an edge $x - y$ of $AG_I(R)$ that is not an edge of $\Gamma_I(R)$ for some distinct $x, y \in V(AG_I(R))$. Since I is a radical ideal, we have that

$(xy)^2 \notin I \Rightarrow x^2y \notin I$ and $xy^2 \notin I$. Thus by Lemma 3.5.1, there is a cycle of length three in $AG_I(R)$ such that each edge is not an edge of $\Gamma_I(R)$.

3.2.2 Generalizing Results for $\Gamma_I(R)$ to $AG_I(R)$

Theorem 3.6. *[Proposition 1.2 / Theorem 1.3, [1]] Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal.*

- *Let $x + I$ and $y + I$ be distinct elements in R/I. Then $x + I$ is adjacent to $y + I$ in $AG(R/I)$ if and only if x is adjacent to y in $AG_I(R)$.*
- *If $x + i$ is adjacent to $y + i$ in $AG_I(R)$, for some $i \in I$, then all elements of $x + I$ and $y + I$ are adjacent in $AG_I(R)$.*

With Theorem 3.6, we detail an efficient construction of $AG_I(R)$. Afterward, we apply the construction to an example to clarify notation and details.

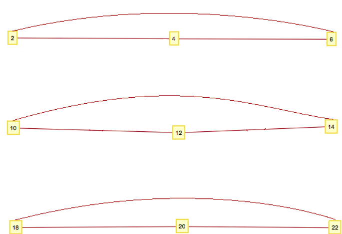
Remark 3.7. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(R/I)$. For each $i \in I$, define a graph AG_i with vertices $\{a_\lambda + i \,|\, \lambda \in \Lambda\}$, where distinct vertices $a_\lambda + i$ and $a_\beta + i$ are adjacent in AG_i if and only if $a_\lambda + I$ and $a_\beta + I$ are adjacent in $AG(R/I)$. We define the graph $AG_I(R)$ to have as its vertex set the vertices of $\bigcup AG_i$ and as its edge set: (1) all edges contained in AG_i for each $i \in I$, (2) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ and $a_\beta + I$ are adjacent in $AG(R/I)$, (3) for $\lambda \in \Lambda$ and distinct $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $(I : (a_i^2)) \neq (I : (a_i))$.*

Example 3.8. *Let $R = \mathbb{Z}_{24}$ and consider the ideal $I = (8)$. We aim to construct $AG_{(8)}(\mathbb{Z}_{24})$ using the process in Remark 3.7.*

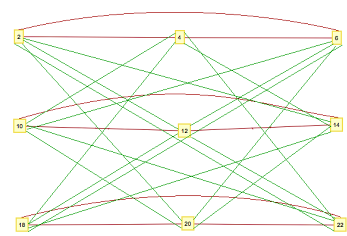
Begin by creating $|I| = |(8)| = 3$ copies of $AG(\mathbb{Z}_{24}/(8)) \cong AG(\mathbb{Z}_8)$, which can be seen in Figure 3.1 (a); these copies correspond to the $AG_i(R)$ and cover the vertex set. Notice that when constructed in this fashion, the vertices $\{a_\lambda + i\}_{i\in I}$ create columns. For example, the center column, 6, 14, and 22, corresponds to $\{6 + i\}_{i\in I}$.

Next we need to add in edges as detailed in Remark 3.7. We already have the edges present in the copies of $AG(\mathbb{Z}_8)$, so (1) in Remark 3.7 is complete. We add in edges satisfying (2) *in Remark 3.7 by extending a connection within a copy of* $AG(\mathbb{Z}_8)$ *to the entire column. For example,* $2-4$ *is an edge in the first copy of* $AG(\mathbb{Z}_8)$ *, so we add in an edge between* 2 *and every element in the column containing* 4*. The addition of edges between columns is illustrated by the green edges in Figure 3.1* (b)*.*

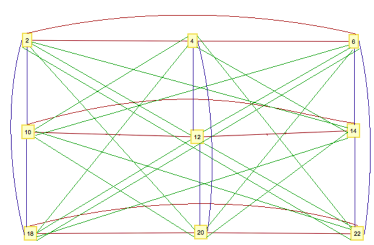
We add in edges satisfying (3) in Remark 3.7 by extending a loop within the first copy of $AG(\mathbb{Z}_8)$ *to the entire column. Applying (3) in Remark 3.7 to the leftmost column, we want to determine whether* $(I : (2^2)) = (I : (2))$*. Notice* $(I : (2^2)) = (I : (4)) = \{2, 4, 6, 10, 12, 14, 18, 20, 22\}$ *and* $(I : (2)) = \{4, 12, 20\}$*, so* $(I : (2^2)) \neq (I : (2))$*. Hence, we add in an edge between every element in the column containing* 2*; this is known as a connected column. The addition of edges in connected columns satisfying (3) in Remark 3.7 is illustrated by the blue edges in Figure 3.1* (c)*.*



(a) 3 copies of $AG(\mathbb{Z}_8)$



(b) Add in Edges Between Columns



(c) Add in Connected Column Edges

Figure 3.1: Constructing $AG_{(8)}(\mathbb{Z}_{24})$

3.3 Ideal-based Extended Zero-Divisor Graphs

Analogous to defining $\overline{\Gamma}(R)$ with the vertex set of $\Gamma(R)$ and changing the edge condition, we define $\overline{\Gamma}_I(R)$ with the vertex set of $\Gamma_I(R)$ and applying the edge condition of $\overline{\Gamma}(R)$.

Definition 3.9. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal. The **ideal-based extended zero-divisor graph**, denoted $\overline{\Gamma}_I(R)$, is the graph with vertices $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$, such that distinct vertices x and y are adjacent if and only if there exists an $n, m \in \mathbb{Z}^+$ such that $x^n y^m \in I$, $x^n \notin I$, and $y^m \notin I$.*

Since $\overline{\Gamma}_I(R)$ combines aspects of both $\overline{\Gamma}(R)$ and $\Gamma_I(R)$, we prove analogs to several results found in [14] and [22].

3.3.1 Generalizing Resuls for $\overline{\Gamma}(R)$ to $\overline{\Gamma}_I(R)$

The first result describes conditions under which equality is obtained. Recall, the radical of an ideal is $\sqrt{I} = \{x \in R \,|\, x^n \in I$ for some $n \in Z^+\}$. The smallest positive integer n for which $x^n \in I$ is called the radical index of x in I, denoted $n_I(x)$.

Theorem 3.10. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal. The following statements are equivalent:*

1. *$\overline{\Gamma}_I(R) = \Gamma_I(R)$.*
2. *R satisfies the two following conditions:*
   (i) If $\sqrt{I} \neq \{0\}$, then every element $x \in R \setminus I$ with $x \in \sqrt{I}$ has index 2.
   (ii) For every $x \in R \setminus I$ with $x \notin \sqrt{I}$, $(I : (x^2)) = (I : (x))$.
3. *R satisfies the two following conditions:*
   (i) If $\sqrt{I} \neq \{0\}$, then every element $x \in R \setminus I$ with $x \in \sqrt{I}$ has index 2.
   (ii) For every $x \in R \setminus I$, $\sqrt{(I : (x))} \setminus \sqrt{I} \subseteq (I : (x))$.

Lemma 3.10.1. *Let R be a commutative ring with nonzero identity and let $x \in R \setminus I$.*

1. *If $x \in \sqrt{I}$, then $(I : (x)) \subsetneq (I : (x^n))$ for every $n \geq 2$.*
2. *If $x \notin \sqrt{I}$, then we have the equivalence:*
   $(I : (x^2)) = (I : (x))$ if and only if $(I : (x^n)) = (I : (x))$ for every integer $n \geq 2$.

Proof. Let R be a ring and let $x \in R \setminus I$.

1. Let $x \in \sqrt{I}$. Suppose $n_I(x) = 2$, then for every integer $n \geq 2$, $(I : (x^n)) = (I : (I)) = R \supsetneq (I : (x))$. Suppose $n_I(x) \geq 3$ and (for contradiction) that there exists an $n \geq 2$ such that $(I : (x^n)) = (I : (x))$. Since for $n \geq n_I(x)$ we know $(I : (x^n)) = (I : (I)) = R$, we must have $2 \leq n \leq n_I(x)$. It follows that $x^{n_I(x)-n} \in (I : (x^n)) = (I : (x))$. So $x^{n_I(x)-n}x = x^{n_I(x)-n+1} \in I$, which contradicts the definition of $n_I(x)$ since $2 \leq n_I(x) - n + 1 \leq n_I(x) - 1$.

2. Let $x \notin \sqrt{I}$.

   ($\Rightarrow$) Suppose $(I : (x^2)) = (I : (x))$ and let $n \geq 2$. Towards induction, assume $(I : (x^{n-1})) = (I : (x))$. Let $y \in (I : (x^n))$, which means $yx^n \in I$. Rewriting this as $yx(x^{n-1}) \in I$, we have $yx \in (I : (x^{n-1}))$. By our induction hypothesis, $(I : (x^{n-1})) = (I : (x))$, so $yx \in (I : (x))$, which means $yx^2 \in I$. Hence $y \in (I : (x^2)) = (I : (x))$. Thus, $(I : (x^n)) \subseteq (I : (x))$ for every integer $n \geq 2$.

   ($\Leftarrow$) If $(I : (x^n)) = (I : (x))$ for every integer $n \geq 2$, then clearly $(I : (x^2)) = (I : (x))$.

□

Proof of Theorem 3.10.

(1) $\Rightarrow$ (2) Suppose that $\overline{\Gamma}_I(R) = \Gamma_I(R)$.

(i) Let $\sqrt{I} \neq \{0\}$. Suppose there exists an $x \in \sqrt{I}$ such that $x \notin I$ with $n_I(x) \geq 3$. By Lemma 3.10.1, $(I : (x)) \subsetneq (I : (x^n))$ for every $n \geq 2$. We may assume $2 \leq n \leq n_I(x)$. Consider an element $y \in (I : (x^n)) \setminus (I : (x))$; then $x^n y \in I$ and $xy \notin I$, which contradicts our initial assumption that $\overline{\Gamma}_I(R) = \Gamma_I(R)$. Thus, $n_I(x) = 2$.

(ii) Let $x \in R \setminus I$ with $x \notin \sqrt{I}$. Since $(I : (x)) \subset (I : (x^2))$, it remains to show $(I : (x^2)) \subseteq (I : (x))$. Let $y \in (I : (x^2))$ with $x \neq y$. Then x and y are adjacent in $\overline{\Gamma}_I(R)$, so by our initial assumption x and y are adjacent in $\Gamma_I(R)$. Hence $xy \in I$, and we have that $y \in (I : x)$. Thus, $(I : (x^2)) = (I : (x))$.

(2) $\Rightarrow$ (3) Let $x \in R \setminus I$ and $y \in \sqrt{(I : (x))} \setminus \sqrt{I}$. Then there exists an $n \in \mathbb{Z}^+$ such that $y^n x \in I$. So, $x \in (I : (y^n)) = (I : (y))$ by Lemma 3.10.1, and we have $xy \in I$. Thus $\sqrt{(I : (x))} \setminus \sqrt{I} \subseteq (I : (x))$.

(3) $\Rightarrow$ (1) Let x and y be two adjacent vertices in $\overline{\Gamma}_I(R)$. Then there exist $n, m \in \mathbb{Z}^+$ such that $x^n y^m \in I$ with $x^n \notin I$ and $y^m \notin I$. Three possible cases occur:

Case 1: If $x, y \in \sqrt{I}$, then $n_I(x) = n_I(y) = 2$ (by 3(i)), so , then $n = m = 1$. Thus, x and y are adjacent vertices in $\Gamma_I(R)$.

Case 2: If $x \notin \sqrt{I}$ and $y \in \sqrt{I}$, then $m = 1$ (by 3(i)). Hence, $x \in \sqrt{(I : (y))} \setminus \sqrt{I}$ and $\sqrt{(I : (y))} \setminus \sqrt{I} \subseteq (I : (y))$ by hypothesis, so $xy \in I$. Thus, x and y are adjacent vertices in $\Gamma_I(R)$.

Case 3: If $x \notin \sqrt{I}$ and $y \notin \sqrt{I}$, then $x \in \sqrt{(I : (y^m))} \setminus \sqrt{I} \subseteq (I : (y^m))$, so $xy^m \in I$. Hence, $y \in \sqrt{(I : (x))} \setminus \sqrt{I} \subseteq (I : (x))$, so $xy \in I$. Thus, x and y are adjacent vertices in $\Gamma_I(R)$.

Therefore, $\overline{\Gamma}_I(R) = \Gamma_I(R)$. □

3.3.2 Generalizing Results for $\Gamma_I(R)$ to $\overline{\Gamma}_I(R)$

The following result illustrates the relationship between $\overline{\Gamma}(R/I)$ and $\overline{\Gamma}_I(R)$. The proof is analogous to results in [22] and [1] and will be omitted here.

Proposition 3.11. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal.*

1. *Let $x + I$ and $y + I$ be distinct elements in R/I. Then $x + I$ is adjacent to $y + I$ in $\overline{\Gamma}(R/I)$ if and only if x is adjacent to y in $\overline{\Gamma}_I(R)$.*

2. *If $x + i$ is adjacent to $y + i$ in $\overline{\Gamma}_I(R)$, for some $i \in I$, then all elements of $x + I$ and $y + I$ are adjacent in $\overline{\Gamma}_I(R)$.*

With Proposition 3.11, we detail an efficient construction of $\overline{\Gamma}_I(R)$. Afterward, we apply the construction to an example to clarify the notation and details.

Remark 3.12. *Let R be a commutative ring with nonzero identity and $I \subseteq R$ an ideal. Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\overline{\Gamma}(R/I)$. For each $i \in I$, define a graph $\overline{G}_i$ with vertices $\{a_\lambda + i \,|\, \lambda \in \Lambda\}$, where distinct vertices $a_\lambda + i$ and $a_\beta + i$ are adjacent in $\overline{G}_i$ if and only if $a_\lambda + I$ and $a_\beta + I$ are adjacent in $\overline{\Gamma}(R/I)$. We define the graph $\overline{\Gamma}_I(R)$ to have as its vertex set the vertices of $\bigcup \overline{G}_i$ and as its edge set: (1) all edges contained in $\overline{G}_i$ for each $i \in I$, (2) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$*

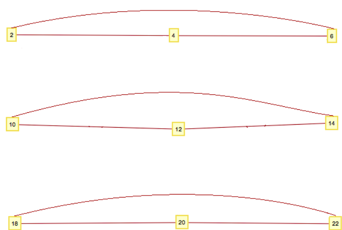
if and only if $a_\lambda + I$ and $a_\beta + I$ are adjacent in $\overline{\Gamma}(R/I)$, (3) for $\lambda \in \Lambda$ and distinct $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\lambda + i$ if and only if $a_\lambda \in \sqrt{I}$.

Example 3.13. *Let $R = \mathbb{Z}_{24}$ and consider the ideal $I = (8)$. We aim to construct $\overline{\Gamma}_{(8)}(\mathbb{Z}_{24})$.*

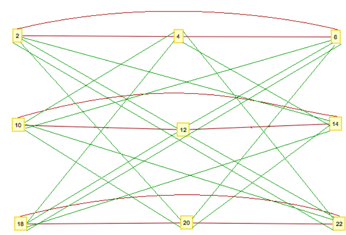
Begin by creating $|I| = |(8)| = 3$ copies of $\overline{\Gamma}(\mathbb{Z}_{24}/(8)) \cong \overline{\Gamma}(\mathbb{Z}_8)$, which can be seen in (a) of Figure 3.2; these copies correspond to the $\overline{\Gamma}_i(R)$ and cover the vertex set.

Next we need to add in edges as detailed in Remark 3.12. We already have the edges present in the copies of $\overline{\Gamma}(\mathbb{Z}_8)$, so (1) in Remark 3.12 is complete. We add in edges satisfying (2) in Remark 3.12 by extending a connection within a copy of $\overline{\Gamma}(\mathbb{Z}_8)$ to the entire column. The addition of edges between columns is illustrated by the green edges in (b) of Figure 3.2.

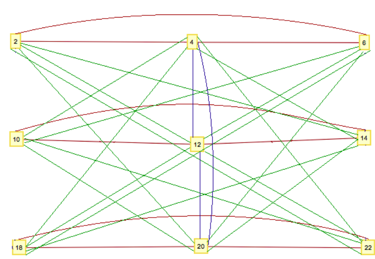
We add in edges satisfying (3) in Remark 3.12 by extending a loop within the first copy of $\overline{\Gamma}(\mathbb{Z}_8)$ to the entire column. Applying (3) in Remark 3.12 to the center column, we want to determine whether $4^2 \in (8)$. Since $4^2 = 16 \in (8)$, the center column is connected. Neither outside columns are connected since $2^2 = 4 \notin (8)$ and $6^2 = 36 \equiv 12(mod\ 24) \notin (8)$. The addition of edges in connected columns satisfying (3) in Remark 3.12 is illustrated by the blue edges in (c) of Figure 3.2.



(a) 3 copies of $\overline{\Gamma}(\mathbb{Z}_8)$



(b) Add in Edges Between Columns



(c) Add in Connected Column Edges

Figure 3.2: Constructing $AG_{(8)}(\mathbb{Z}_{24})$

Chapter 4

Congruence-based Graphs

In this chapter, we define and investigate three families of graphs based on various multiplicative congruence relations, namely congruence-based zero-divisor graphs, congruence-based annihilator graphs, and congruence-based extended zero-divisor graphs. After a brief introduction to multiplicative congruence relations, we successively demonstrate that $\Gamma(R)$, $AG(R)$, $\overline{\Gamma}(R)$, and the variations discussed in Chapters 2 and 3 can be represented as congruence-based graphs. Lastly, we prove several foundational properties for each family and discuss relationships between and within the three families of congruence-based graphs.

4.1 Multiplicative Congruence Relations

Let R be a commutative ring with nonzero identity. A **multiplicative congruence relation** $\sim$ on R is an equivalence relation on the multiplicative monoid R with the added property that given $x, y, z, w \in R$ with $x \sim y$ and $z \sim w$, we have $xz \sim yw$.

The next example introduces three pertinent multiplicative congruence relations.

Example 4.1. *Let R be a commutative ring with nonzero identity.*

1. *Let $\sim_=$ be the relation on R given by $x \sim_= y \iff x = y$. It is clear that $\sim_=$ is a multiplicative congruence relation.*

2. *Let $\sim_\Gamma$ be the relation on R given by $s \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. Already, $\sim_\Gamma$ was introduced as an equivalence relation on R in Chapter 2. Toward showing $\sim_\Gamma$*

is also multiplicative, let $x, y, z, w \in R$ with $x \sim_\Gamma y$ and $z \sim_\Gamma w$. Then $ann_R(x) = ann_R(y)$ and $ann_R(z) = ann_R(w)$. Without loss of generality, it suffices to show $ann_R(xz) \subseteq ann_R(yw)$. Let $r \in ann_R(xz)$, then $0 = r(xz) = (rx)z$, which implies that $rx \in ann_R(z) = ann_R(w)$. It follows that $0 = (rx)w = rw(x)$, which implies that $rw \in ann_R(x) = ann_R(y)$. Hence $0 = (rw)y = r(wy)$ and $r \in ann(wy)$ as desired. Therefore, $\sim_\Gamma$ is a multiplicative congruence relation.

3. *Let $J \subseteq R$ be a semigroup ideal. Let $\sim_J$ be the relation on R given by $x \sim_J y \iff x = y$ or $x, y \in J$. Then $\sim_J$ is a multiplicative congruence relation on R by properties of ideals. In particular, $\sim_I$ is a multiplicative congruence relation on R for $I \subseteq R$, an ideal of the ring.*

We denote the set of congruence relations on a commutative ring R with nonzero identity by $C(R) = \{\sim \mid \sim \text{ is a multiplicative congruence relation on } R\}$.

A partial ordering is induced on $\sim \in C(R)$ by viewing each element $\sim \in C(R)$ as a subset of $R \times R$, where $(x, y) \in \sim \iff x \sim y$; specifically, $\sim_1 \leq \sim_2 \iff \sim_1 \subseteq \sim_2$. In other words, given $\sim_1, \sim_2 \in C(R)$ with $\sim_1 \leq \sim_2$, $x \sim_1 y$ implies $x \sim_2 y$ for $x, y \in R$. Also if $\sim_1 \leq \sim_2$, then $[x]_{\sim_1} \subseteq [x]_{\sim_2}$.

Since $\sim$ is a multiplicative congruence relation, the product $[x]_\sim [y]_\sim = [xy]_\sim$ is well-defined and associative. Thus $R/\sim = \{[x]_\sim \mid x \in R\}$ is a semigroup under the multiplication defined above. Moreover, $R/\sim$ is a commutative semigroup with zero element $[0]_\sim$ and nonzero identity element $[1]_\sim$.

Toward generalizing zero-divisor graphs, we discuss the $[0]_\sim$ of various congruence relations.

Theorem 4.2. *[Theorem 2.6, [19]] Let R be a commutative ring with nonzero identity, and let J be a nonempty subset of R. Then, J is a semigroup ideal of R if and only if $J = [0]_\sim$ for some multiplicative congruence relation $\sim$ on R.*

The following example characterizes the semigroup ideals $J = [0]_\sim$ for the three congruence relations introduced in Example 4.1.

Example 4.3. *Let R be a commutative ring with nonzero identity.*

1. $[0]_{\sim_=} = \{0\}$.

2. $[0]_{\sim_\Gamma} = \{r \in R \,|\, ann_R(r) = ann_R(0)\} = \{r \in R \,|\, ann_R(r) = R\} = \{0\}$.

3. *Let $J \subseteq R$ be a semigroup ideal. Then we have that* $[0]_{\sim_J} = \{r \in R \,|\, 0 = r$ *or* $0, r \in J\} = \{r \in R \,|\, r \in J\} = J$. *In particular for $I \subseteq R$ a ring ideal, we have that* $[0]_{\sim_I} = I$.

Define the following subset of $C(R)$ for every semigroup ideal $J \subseteq R$:

$$C_J(R) = \{\sim \in C(R) \,|\, [0]_\sim = J\}.$$

As every semigroup ideal corresponds to $[0]_\sim$ for some $\sim \in C(R)$, $C(R)$ is the disjoint union of $C_J(R)$, specifically $C(R) = \bigcup\{C_J(R) \,|\, J \text{ is a semigroup ideal of } R\}$.

For additional examples of multiplicative congruence relations and discussion of $C(R)$, see [19] and [8].

4.2 Congruence-based Zero-Divisor Graphs

The congruence-based zero-divisor graph was first introduced by Anderson and Lewis in [8] and is constructed as follows. Let $\sim$ be a multiplicative congruence relation on the commutative ring with nonzero identity R. Then $R/\sim$ is a commutative semigroup with nonzero identity and zero. Using the zero-divisor structure on semigroups defined in [16], we define a congruence-based zero-divisor graph.

Definition 4.4. *Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. The **congruence-based zero-divisor graph**, denoted $\Gamma_\sim(R) = \Gamma(R/\sim)$, is the graph with vertices $Z(R/\sim)^* = Z(R/\sim) \setminus \{[0]_\sim\}$, such that distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if $[x]_\sim[y]_\sim = [xy]_\sim = [0]_\sim$ (i.e. $[x]_\sim - [y]_\sim$ is an edge iff $xy \sim 0$).*

In the following example, we illustrate $\Gamma_\sim(R)$ for several specific congruence relations. Recall, we say two graphs are equivalent when a natural isomorphism exists between them.

Example 4.5. *Let R be a commutative ring with nonzero identity.*

1. *Let $\sim_=$ be the multiplicative congruence relation $x \sim_= y \iff x = y$. It follows that $\Gamma_{\sim_=}(R) = \Gamma(R)$.*

2. *Let $\sim_\Gamma$ be the multiplicative congruence relation $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. It follows that $\Gamma_{\sim_\Gamma}(R) = \Gamma_{E_\Gamma}(R)$, where $\Gamma_{E_\Gamma}(R)$ is the compressed zero-divisor graph discussed in Chapter 2.*

3. *Let $I \subseteq R$ be an ideal, and let $\sim_I$ be the multiplicative congruence relation $x \sim_I y \iff$ either $x = y$ or $x, y \in I$. It follows that $\Gamma_{\sim_I}(R) = \Gamma_I(R)$, where $\Gamma_I(R)$ is the ideal-based zero-divisor graph discussed in Chapter 3.*

We relay the following foundational results from [8] and [19].

Proposition 4.6. *[Theorem 2,7, [8]] Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R.*

- *$\Gamma_\sim(R)$ is connected and diam$(\Gamma_\sim(R)) \leq 3$.*
- *If $\Gamma_\sim(R)$ contains a cycle, then gr$(\Gamma_\sim(R)) \leq 4$.*

Proposition 4.7. *[Theorem 2.8,[19]] Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. Then the following statements are equivalent*

1. *For all $x, y \in R$, $xy \sim 0$ implies that $xy = 0$.*

2. *For all $x, y \in R$, $x \sim y$ implies that $ann_R(x) = ann_R(y)$.*

3. *$\sim \leq \sim_\Gamma$, where $\sim_\Gamma$ is defined as in Examples 4.5.*

4. *$[0]_\sim = \{0\}$.*

The next series of results involve two graph properties, *complemented* and *uniquely complemented*, defined below, and the total quotient ring of R/I, $T(R/I)$. These results hold for congruence relations $C_I(R)$, where I is a proper radical ideal of R.

Definition 4.8. *Let a and b be distinct vertices of the graph G. Then we say:*

- *$a \leq b$ if and only if a and b are not adjacent and each vertex adjacent to b is also adjacent to a;*
- *$a \perp b$ if and only if a and b are adjacent, with no other vertex adjacent to both a and b;*
- *$a \approx b$ if and only $a \leq b$ and $b \leq a$.*

*The graph G is **complemented** if for each vertex a of G, there exists a vertex b of G such that $a \perp b$. The graph G is **uniquely complemented** if G is complemented and if $a \perp b$ and $a \perp c$, then $b \approx c$.*

Remark 4.9. *In context, it follows that if $\sim \in C_I(R)$ and $a, b \in Z(R/\sim)^*$, we have:*

- *$a \approx b$ if and only if $(I : (a)) - \{a\} = (I : (b)) - \{b\}$*
- *$a \perp b$ if and only $(I : (a)) \cap (I : (b)) \subseteq \{I, a, b\}$.*

Theorem 4.10. *Let R be a commutative ring with nonzero identity and let I be a proper, radical ideal of R. Let $\sim \in C_I(R)$. Then the following are equivalent:*

1. *$\Gamma_\sim(R)$ is complemented.*
2. *$\Gamma_\sim(R)$ is uniquely complemented.*
3. *$\Gamma(R/I)$ is complemented.*
4. *$\Gamma(R/I)$ is uniquely complemented.*
5. *$T(R/I)$ is von Neumann regular.*

Lemma 4.10.1. *Let R be a commutative ring with nonzero identity and $\sim \in C_I(R)$, where I is a proper, radical ideal of R. Let $a, b, c \in Z(R/\sim)^*$. If $a \perp b$ and $a \perp c$, then $b \approx c$. Thus $\Gamma_\sim(R)$ is uniquely complemented if and only if $\Gamma_\sim(R)$ is complemented.*

Proof. Let R be a commutative ring with $\sim \in C_I(R)$, where I is a radical ideal of R. Let $a, b, c \in Z(R/\sim)^*$. Suppose $a \perp b$ and $b \perp c$.

We first need to show that there is no edge between b and c in $\Gamma_\sim(R)$. Recall, that b and c are adjacent in $\Gamma_\sim(R)$ if and only if $[bc]_\sim = [0]_\sim = I$; so b and c are adjacent in $\Gamma_\sim(R)$ if and only if $bc \in I$.

Suppose b and c are adjacent in $\Gamma_\sim(R)$, then $bc \in I$. Since $a \perp b$ and $a \perp c$, we must have that either $b = c$ or $b = a$. If $b = c$, then $c^2 \in I$ since $b \perp c$. Since I is radical, $c \in I$, which is a contradiction. We reach a similar contradiction if $b = a$, therefore $bc \notin I$ and there is no edge between b and c in $\Gamma_\sim(R)$.

Next we need to show that every vertex adjacent to b is also adjacent to c and vice versa. Let $d \in Z(R/\sim)^*$ such that there is an edge between d and b. If $dc \notin I$, then we have $(dc)a = d(ca) \in I$ and $(dc)b = (db)c \in I$, which is a contradiction since $a \perp b$. Therefore $dc \in I$, and $c \leq b$. Similarly, we can show $b \leq c$. Thus, $b \approx c$ if $a \perp b$ and $b \perp c$.

The conclusion follows from the work done above and the definitions of complemented and uniquely complemented. $\square$

Lemma 4.10.2. *Let R be a commutative ring with nonzero identity and $\sim \in C_I(R)$, where I is a proper, radical ideal of R. Then $\Gamma_\sim(R)$ is complemented if and only if $\Gamma(R/I)$ is complemented.*

Proof. Let R be a commutative ring with nonzero identity and I a proper, radical ideal of R and $\sim \in C_I(R)$.

($\Rightarrow$) Suppose $\Gamma_\sim(R)$ is complemented. Let $x + I$ be a vertex of $\Gamma(R/I)$. It follows that $[x]_\sim$ is a vertex of $\Gamma_\sim(R)$. Since $\Gamma_\sim(R)$ is complemented, there exists a vertex $[y]_\sim$ of $\Gamma_\sim(R)$ such that $[x]_\sim \perp [y]_\sim$. We want to show that $x + I \perp y + I$ in $\Gamma(R/I)$. First we need to show that $x + I$ and $y + I$ are distinct vertices in $\Gamma(R/I)$. Suppose for a contradiction that $x + I = y + I$, then we have $x - y = i \in I$. Equivalently, $x(x - y) = xi \in I$, so it follows that $x^2 = xi + xy \in I$. Since, I is a radical ideal, we have that $x^2 \in I$ implies $x \in I$, which is a contradiction. Therefore $x + I \neq y + I$. Second, we must show that $x + I$ and $y + I$ are adjacent in $\Gamma(R/I)$. This follows directly from their adjacency in $\Gamma_\sim(R)$. Lastly, we need to show that no other vertex is adjacent to both $x + I$ and $y + I$. Suppose for a contradiction that there exists a vertex $z + I$ adjacent to both $x + I$ and $y + I$ in $\Gamma(R/I)$. Notice that since I is a radical ideal, $[x]_\sim$, $[y]_\sim$, and $[z]_\sim$ are all distinct vertices of $\Gamma_\sim(R)$. So, we have $[x]_\sim$ and $[z]_\sim$ are adjacent, similarly $[y]_\sim$ and $[z]_\sim$ are adjacent, which is a contradiction since $[x]_\sim \perp [y]_\sim$. Therefore, $x + I \perp y + I$ and $\Gamma(R/I)$ is complemented.

($\Leftarrow$) Suppose $\Gamma(R/I)$ is complemented. Let $[x]_\sim$ be a vertex of $\Gamma_\sim(R)$. It follows that $x+I$ is a vertex of $\Gamma(R/I)$. Since $\Gamma(R/I)$ is complemented, there exists a vertex $y+I$ of $\Gamma(R/I)$ such that $x+I \perp y+I$. Since these vertices are distinct in $\Gamma(R/I)$, we have that $[x]_\sim$ and $[y]_\sim$ are distinct vertices of $\Gamma_\sim(R)$ that are adjacent. We want to show that $[x]_\sim \perp [y]_\sim$ in $\Gamma_\sim(R)$. Suppose for a contradiction that there exists a vertex $[z]_\sim$ of $\Gamma_\sim(R)$ such that $[z]_\sim$ is adjacent to both $[x]_\sim$ and $[y]_\sim$. Since I is a radical ideal, we have that $x+I$, $y+I$, and $z+I$ are distinct vertices of $\Gamma(R/I)$, which lead us to a contradiction similar to that in the forward direction since $x+I \perp y+I$. Therefore, $[x]_\sim \perp [y]_\sim$ and $\Gamma_\sim(R)$ is complemented. □

Proof of Theorem 4.10

(1) $\iff$ (2) Lemma 4.10.1

(2) $\iff$ (3) Lemma 4.10.2

(3) $\iff$ (4) $\iff$ (5) [Theorem 4.9, [23]] □

4.3 Congruence-based Annihilator Graph

Let $\sim$ be a multiplicative congruence relation on the commutative ring with nonzero identity R. The congruence-based annihilator graph is defined similarly to $\Gamma_\sim(R)$. Since $R/\sim$ is a commutative semigroup with nonzero identity, we use the annihilator graph structure on semigroups defined in [3] to define a congruence-based annihilator graph.

Definition 4.11. *Let R be a commutative ring with nonzero identity and let $\sim$ be a multiplicative congruence relation on R. The* ***congruence-based annihilator graph****, denoted $AG_\sim(R)$, is the graph with vertices $Z(R/\sim)^* = Z(R/\sim) \setminus \{[0]_\sim\}$, such that two distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if*

$$ann_{R/\sim}([x]_\sim[y]_\sim) \neq ann_{R/\sim}([x]_\sim) \cup ann_{R/\sim}([y]_\sim),$$

where $ann_{R/\sim}([x]_\sim) = \{[z]_\sim \in R/\sim \mid [x]_\sim[z]_\sim = [0]_\sim\}$

Example 4.12. *Let R be a commutative ring with nonzero identity.*

1. *Let* $\sim_=$ *be the multiplicative congruence relation* $x \sim_= y \iff x = y$. *Then* $AG_{\sim_=}(R) = AG(R)$, *because*

$$ann_{R/\sim_=}([xy]_{\sim_=}) \neq ann_{R/\sim_=}([x]_{\sim_=}) \cup ann_{R\sim_=}([y]_{\sim_=})$$

$$\iff ann_R(xy) \neq ann_R(x) \cup ann_R(y).$$

2. *Let* $\sim_\Gamma$ *be the multiplicative congruence relation* $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$. *Then* $AG_{\sim_\Gamma}(R) = AG_{E_\Gamma}(R)$.

3. *Let* $\sim_{AG}$ *be the equivalence relation given by* $x \sim_{AG} y \iff E^*_{AG(R)}(x) = E^*_{AG(R)}(y)$, *where* $E^*_{AG(R)}(x) = \{r \in R \,|\, ann(rx) \neq ann(r) \cup ann(x)\}$. *We expect that* $AG_{\sim_{AG}}(R) = AG_{E_{AG}}(R)$, *but it is thus far unknown whether* $\sim_{AG}$ *is multiplicative.*

4. *Let* $I \subseteq R$ *be an ideal and let* $\sim_I$ *be the multiplicative congruence relation* $x \sim_I y \iff$ *either* $x = y$ *or* $x, y \in I$.

   Claim: $AG_{\sim_I}(R) = AG_I(R)$

   Under this congruence relation,

$$ann_{R/\sim_I}([x]_{\sim_I}) = \{z \in R \mid xz \sim_I 0\}.$$

   By definition and since $0 \in I$, $xz \sim_I 0 \iff xz = 0$ *or* $xz, 0 \in I \iff xz \in I$. *Applying this, we obtain:*

$$ann_{R/\sim_I}([x]_\sim) = \{z \in R \mid xz \in I\} = (I : (x)).$$

   Consequently,

$$ann_{R/\sim_I}([xy]_{\sim_I}) \neq ann_{R/\sim_I}([x]_{\sim_I}) \cup ann_{R/\sim_I}([y]_{\sim_I})$$

$$\iff (I : (xy)) \neq (I : (x)) \cup (I : (y)).$$

The following results extend the work in [12] about $AG(R)$ to $AG_\sim(R)$. The proofs are omitted since they are analogous to those presented in [12].

Proposition 4.13. *Let R be a commutative ring with nonzero identity and $|Z(R/\sim)^*| \geq 2$. Then $AG_\sim(R)$ is connected and $diam(AG_\sim(R)) \leq 2$.*

Proposition 4.14. *Let R be a reduced commutative ring with nonzero identity. Suppose $AG_\sim(R) \neq \Gamma_\sim(R)$. Then $gr(AG_\sim(R)) = 3$. Moreover, there is a cycle C of length three in $AG_\sim(R)$ such that each edge of C is not an edge of $\Gamma_\sim(R)$.*

For the following result, recall that a semigroup ideal J is called **prime** if for any $xy \in J$, it follows that $x \in J$ or $y \in J$.

Proposition 4.15. *Let R be a commutative ring with nonzero identity, and let $\sim \in C(R)$. Then $AG_\sim(R) = \emptyset$ if and only if $[0]_\sim$ is a prime semigroup ideal of R.*

4.4 The Congruence-based Extended Zero-Divisor Graph

Let $\sim$ be a multiplicative congruence relation on the commutative ring with nonzero identity R. The congruence-based extended zero-dvisor graph is defined similarly to $\Gamma_\sim(R)$. Since $R/\sim$ is a commutative semigroup with nonzero identity, we use the extended graph structure on semigroups to define a congruence-based zero-divisor graph.

Definition 4.16. Let S be a commutative semigroup with nonzero identity. Define the **extended zero-divisor graph on** S, denoted by $\overline{\Gamma}(S)$, as the simple graph with vertices $Z(S)^*$, such that distinct vertices $x, y \in Z(S)^*$ are adjacent if and only if there exists $n, m \in \mathbb{Z}^+$ such that $x^n y^m = 0$, $x^n \neq 0$, and $y^m \neq 0$.

Definition 4.17. *Let R be a commutative ring with nonzero identity and let $\sim$ be a multiplicative congruence relation on R. The **congruence-based extended zero-divisor graph**, denoted $\overline{\Gamma}_\sim(R)$, is the graph with vertices $Z(R/\sim)^* = Z(R/\sim) \setminus \{[0]_\sim\}$, such that two distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if there exists $n, m \in \mathbb{Z}^+$ such that $[x]_\sim^n [y]_\sim^m = [0]_\sim$, $[x]_\sim^n \neq [0]_\sim$, and $[y]_\sim^m \neq [0]_\sim$.*

Example 4.18. *Let R be a commutative ring with nonzero identity.*

1. *Let $\sim_=$ be the multiplicative congruence relation $x \sim_= y \iff x = y$. Then $\overline{\Gamma}_{\sim_=}(R) = \overline{\Gamma}(R)$.*

2. *Let $\sim_\Gamma$ be the multiplicative congruence relation $x \sim_\Gamma y \iff ann_R(x) = ann_R(y)$, where $ann(x) = \{r \in R \mid xr = 0\}$. Then $\overline{\Gamma}_{\sim_\Gamma}(R) = \overline{\Gamma}_{E_\Gamma}(R)$.*

3. *Let $\sim_{\overline{\Gamma}}$ be the equivalence relation given by $x \sim_{\overline{\Gamma}} y \iff E^*_{\overline{\Gamma}(R)}(x) = E^*_{\overline{\Gamma}(R)}(y)$, where $E^*_{\overline{\Gamma}(R)}(x) = \{r \in R \mid x^n r^m = 0$, where $n, m \in \mathbb{Z}^+$, $x^n \neq 0$, and $r^m \neq 0\}$. We expect that $\overline{\Gamma}_{\sim_{\overline{\Gamma}}}(R) = \overline{\Gamma}_{E_{\overline{\Gamma}}}(R)$, but it is thus far unknown whether $\sim_{\overline{\Gamma}}$ is multiplicative.*

4. *Let $I \subseteq R$ be an ideal and $\sim_I$ the multiplicative congruence relation $x \sim_I y \iff$ either $x = y$ or $x, y \in I$. Under $\sim_I$, we have that $[x]^m_{\sim_I}[y]^n_{\sim_I} = [0]_{\sim_I} \iff x^n y^m \in I$ So, $\overline{\Gamma}_{\sim_I}(R) = \overline{\Gamma}_I(R)$.*

The following results extend the work in [14] about $\overline{\Gamma}(R)$ to $\overline{\Gamma}_\sim(R)$. The proofs are omitted since they are analogous to those presented in [14].

Proposition 4.19. *Let R be a commutative ring with nonzero identity. Then $\overline{\Gamma}_\sim(R)$ is connected and $diam(\overline{\Gamma}_\sim(R)) \leq 3$.*

Proposition 4.20. *Let R be a reduced commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. Then $\overline{\Gamma}_\sim(R) = \emptyset$ if and only if $[0]_\sim = R$ or $[0]_\sim$ is a prime semigroup ideal of R.*

4.5 Relationships Between Congruence-based Families of Graphs

For a fixed a congruence relation $\sim$, the vertex set for each of the congruence-based graphs $\Gamma_\sim(R)$, $AG_\sim(R)$, and $\overline{\Gamma}_\sim(R)$ is the same, namely $Z(R/\sim)^*$. Consequently, we investigate containment or equality amongst the graphs by looking at the edge sets.

Proposition 4.21. *Let R be a commutative ring with nonzero identity and let $\sim$ be a multiplicative congruence relation on R. Then $\Gamma_\sim(R) \subseteq AG_\sim(R)$.*

Proof. Let $[x]_\sim, [y]_\sim \in Z(R/\sim)^*$ be distinct vertices with $[x]_\sim - [y]_\sim \in E(\Gamma_\sim(R))$, i.e. $[xy]_\sim = [0]_\sim$. Thus $ann_{R/\sim}([xy]_\sim) = ann_{R/\sim}([0]_\sim) = R/\sim \ni [1]_\sim$. In contrast, since $[x]_\sim, [y]_\sim \neq [0]_\sim$, $[1]_\sim \notin ann_{R/\sim}([x]_\sim) \cup ann_{R/\sim}([y]_\sim)$ as both are missing the identity element.

Therefore, $ann_{R/\sim}([xy]_\sim) \neq ann_{R/\sim}([x]_\sim) \cup ann_{R/\sim}([y]_\sim)$, and $[x]_\sim - [y]_\sim$ is an edge of $AG_\sim(R)$
. ☐

Proposition 4.22. *Let R be a commutative ring with nonzero identity and let $\sim$ be a multiplicative congruence relation on R. Then $\Gamma_\sim(R) \subseteq \overline{\Gamma}_\sim(R)$.*

Proof. Let $[x]_\sim, [y]_\sim \in Z(R/\sim)^*$ be distinct elements. Suppose that $[x]_\sim - [y]_\sim$ is an edge of $\Gamma_\sim(R)$. Then we have that $[xy]_\sim = [0]_\sim$, which can be phrased as $[x]_\sim[y]_\sim = [0]_\sim$ where $[x]_\sim, [y]_\sim \neq [0]_\sim$. Hence $[x]_\sim - [y]_\sim$ is an edge of $\overline{\Gamma}_\sim(R)$ and $\Gamma_\sim(R) \subseteq \overline{\Gamma}_\sim(R)$.

☐

The following results require the concept of a **2-absorbing** ideal [11], and generalize a result from [21].

Definition 4.23. An ideal $I \not\subseteq R$ is a **2-absorbing ideal** if for any $a, b, c \in R$ such that $abc \in I$ we have either $ab \in I$, $bc \in I$, or $ac \in I$.

Proposition 4.24. *Let R be a commutative ring with nonzero identity and let $\sim$ be a multiplicative congruence relation on R. Then $AG_\sim(R) = \Gamma_\sim(R)$ if and only if $[0]_\sim$ is a 2-absorbing ideal of $R/\sim$.*

Proof. ($\Rightarrow$) Suppose $[0]_\sim$ is a 2-absorbing ideal of $R/\sim$. Let $[x]_\sim$, $[y]_\sim \in Z(R/\sim)^*$ such that there exists an edge between $[x]_\sim$ and $[y]_\sim$ in $AG_\sim(R)$. This means that $ann_{R/\sim}([xy]_\sim) \neq ann_{R/\sim}([x]_\sim) \cup ann_{R/\sim}([y]_\sim)$, so there exists some $[z]_\sim \in R/\sim$ such that

$$[z]_\sim \in ann_{R/\sim}([xy]_\sim) \setminus (ann_{R/\sim}([x]_\sim) \cup ann_{R/\sim}([y]_\sim))$$

$$\Rightarrow [zxy]_\sim = [0]_\sim \text{ and } [zx]_\sim \neq [0]_\sim,\ [zy]_\sim \neq [0]_\sim.$$

Since $[0]_\sim$ is a 2-absorbing ideal, it follows that $[xy]_\sim = [0]_\sim$, which means that there exists an edge between $[x]_\sim$ and $[y]_\sim$ in $\Gamma_\sim(R)$. Therefore, $\Gamma_\sim(R) = AG_\sim(R)$.

($\Leftarrow$) Suppose that $\Gamma_\sim(R) = AG_\sim(R)$. Let $[x]_\sim$, $[y]_\sim$, and $[z]_\sim$ be elements of $Z(R/\sim)^*$ such that $[xyz]_\sim = [0]_\sim$ with $[xy]_\sim \neq [0]_\sim$ and $[xz]_\sim \neq [0]_\sim$. It follows that $[x]_\sim \in ann_{R/\sim}([yz]_\sim) \setminus (ann_{R/\sim}([y]_\sim) \cup ann_{R/\sim}([z]_\sim))$, which means that there exists an edge between $[y]_\sim$ and $[z]_\sim$

in $AG_\sim(R)$. Since the two graphs are equal, we have that $[xy]_\sim = [0]_\sim$. Therefore, $[0]_\sim$ is a 2-absorbing ideal.

$\square$

4.6 Relationships Within Congruence-based Families of Graphs

Let R be commutative ring with nonzero identity and J a semigroup ideal of R. In this section, we consider pairs $\sim_1, \sim_2 \in C_J(R)$ such that $\sim_1 \leq \sim_2$. We use the natural surjective homomorphism between $R/\sim_1$ and $R/\sim_2$ to create surjective maps from $AG_{\sim_1}(R)$ to $AG_{\sim_2}(R)$ and from $\overline{\Gamma}_{\sim_1}(R)$ to $\overline{\Gamma}_{\sim_2}(R)$. We invesitgate conditions under which these maps are graph homomorphisms.

Since $\Gamma_\sim(R)$, $AG_\sim(R)$, and $\overline{\Gamma}_\sim(R)$ share the same vertex set, we utilize the following result from [8].

Proposition 4.25. *[Lemma 2.4, [8]] Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. For any $\sim_1, \sim_2 \in C_J(R)$, $[x]_{\sim_1} \in Z(R/\sim_1)^* \iff [x]_{\sim_2} \in Z(R/\sim_2)^*$.*

Remark 4.26. *Let R be a commutative ring with nonzero identity and let $J \subseteq R$ be a semigroup ideal. Let $\sim_1, \sim_2 \in C_J(R)$ with $\sim_1 \leq \sim_2$. Consider the surjective homomorphism $f : R/\sim_1 \to R/\sim_2$ given by $f([x]_{\sim_1}) = [x]_{\sim_2}$, which maps $[0]_{\sim_1}$ to $[0]_{\sim_2}$ and $[1]_{\sim_1}$ to $[1]_{\sim_2}$.*

The surjective semigroup homomorphism $f([x]_{\sim_1}) = [x]_{\sim_2}$ induces the surjective maps $G : AG_{\sim_1}(R) \to AG_{\sim_2}(R)$ given by $G = f|_{Z(R/\sim_1)^*}$ and $H : \overline{\Gamma}_{\sim_1}(R) \to \overline{\Gamma}_{\sim_2}(R)$ given by $H = f|_{Z(R/\sim_1)^*}$ on the vertex sets. The maps G and H will be extended to edge sets.

Theorem 3.1 in [8] states for $[x]_{\sim_1}, [y]_{\sim_2} \in V(\Gamma_{\sim_1}(R))$ with $[x]_{\sim_1} - [y]_{\sim_2} \in E(\Gamma_{\sim_1}(R))$, then either $f[x_{\sim_1}] - f[y_{\sim_1}] \in E(\Gamma_{\sim_2}(R))$ or $f[x_{\sim_1}] = f[y_{\sim_1}]$. This characterization is used to verify that both G and H are graph homomorphisms when J is a radical semigroup ideal. Recall, a semigroup ideal $J \subseteq R$ is called **radical** if for any $x \in R$ such that $x^2 \in J$, then $x \in J$.

Theorem 4.27. *Let R be a commutative ring with nonzero identity and let $J \subseteq R$ be a semigroup ideal. Let $\sim_1, \sim_2 \in C_J(R)$ with $\sim_1 \leq \sim_2$. Let $G : AG_{\sim_1}(R) \to AG_{\sim_2}(R)$ be the surjective map given by $G([x]_{\sim_1}) = [x]_{\sim_2}$. Then G is a graph homomorphism if J is a radical semigroup ideal.*

Proof. Let $\sim_1, \sim_2 \in C_J(R)$ with $\sim_1 \leq \sim_2$. Note that $[0]_{\sim_1} = J = [0]_{\sim_2}$. Let $[x]_{\sim_1} - [y]_{\sim_1}$ be an edge of $AG_{\sim_1}(R)$, so $ann_{R/\sim}([x]_\sim [y]_\sim) \neq ann_{R/\sim}([x]_\sim) \cup ann_{R/\sim}([y]_\sim)$. Hence there exists $[r]_{\sim_1} \in Z(R/\sim_1)^*$ satisfying $[rxy]_{\sim_1} = [0]_{\sim_1}$, $[rx]_{\sim_1} \neq 0$, and $[ry]_{\sim_1} \neq 0$.

Claim 1: $[rxy]_{\sim_2} = [0]_{\sim_2}$

By Theorem 3.1 in [8], $[rxy]_{\sim_1} = [0]_{\sim_1}$ impies either $[rxy]_{\sim_2} = [0]_{\sim_2}$ or $[r]_{\sim_2} = [xy]_{\sim_2}$. Suppose that $[r]_{\sim_2} = [xy]_{\sim_2}$. Since $\sim_2$ is multiplicative, we have $r^2 \sim_2 rxy$. Notice $[rxy]_{\sim_1} = [0]_{\sim_1}$ implies $rxy \in [0]_{\sim_1} = [0]_{\sim_2}$, so $[rxy]_{\sim_2} = [0]_{\sim_2}$. Combining these details, we have $[r^2]_{\sim_2} = [0]_{\sim_2}$, so $r^2 \in [0]_{\sim_2} = J$. Since J is radical, it follows that $r \in J = [0]_{\sim_1}$, which is a contradiction as $[r]_{\sim_1}$ was chosen to be non-zero. Thus, $[rxy]_{\sim_2} = [0]_{\sim_2}$.

Claim 2: $[rx]_{\sim_1} \neq 0$ and $[ry]_{\sim_1} \neq 0$

Suppose, toward a contradiction, that $[rx]_{\sim_2} = [0]_{\sim_2}$. Since $[0]_{\sim_2} = J$, we have that $rx \in J = [0]_{\sim_1}$, which implies that $rx \in [0]_{\sim_1}$. Hence $[rx]_{\sim_1} = [0]_{\sim_1}$, which is a contradiction as $[rx]_{\sim_1} \neq [0]_{\sim_1}$. Thus, $[rx]_{\sim_2} \neq [0]_{\sim_2}$. We can similarly show that $[ry]_{\sim_2} \neq [0]_{\sim_2}$.

Since there exists $[r]_{\sim_2} \in Z(R/\sim_2)^*$ such that $[rxy]_{\sim_2} = [0]_{\sim_2}$, $[rx]_{\sim_2} \neq [0]_{\sim_2}$, and $[ry]_{\sim_2} \neq [0]_{\sim_2}$, we have an edge $[x]_{\sim_2} - [y]_{\sim_2}$ in $AG_{\sim_2}(R)$. Therefore, $G : AG_{\sim_1}(R) \to AG_{\sim_2}(R)$ is a graph homomorphism □

Theorem 4.28. *Let R be a commutative ring with nonzero identity and let $J \subseteq R$ be a semigroup ideal. Let $\sim_1, \sim_2 \in C_J(R)$ with $\sim_1 \leq \sim_2$. Let $H : \overline{\Gamma}_{\sim_1}(R) \to \overline{\Gamma}_{\sim_2}(R)$ be the surjective map given by $H([x]_{\sim_1}) = [x]_{\sim_2}$. Then H is a graph homomorphism if J is a radical semigroup ideal.*

Proof. Let $\sim_1, \sim_2 \in C_J(R)$ such that $\sim_1 \leq \sim_2$. Let $[x]_{\sim_1} - [y]_{\sim_1}$ be an edge of $\overline{\Gamma}_{\sim_1}(R)$, so there exists $n, m \in \mathbb{Z}^+$ such that $[x]^n_{\sim_1}[y]^m_{\sim_1} = [0]_{\sim_1}, [x]^n_{\sim_1} \neq [0]_{\sim_1}$, and $[y]^m_{\sim_1} \neq [0]_{\sim_1}$. Recall $[x]^n_\sim = [x^n]_\sim$, so we can rephrase the adjacency condition as there exists $n, m \in \mathbb{Z}^+$ such that $[x^n y^m]_{\sim_1} = [0]_{\sim_1}, [x^n]_{\sim_1} \neq [0]_{\sim_1}$, and $[y^m]_{\sim_1} \neq [0]_{\sim_1}$.

Claim 1: $[x^n y^m]_{\sim_2} = [0]_{\sim_2}$

By Theorem 3.1 in [8], $[x^ny^m]_{\sim_1} = [0]_{\sim_1}$ implies either $[x^ny]^m_{\sim_2} = [0]_{\sim_2}$ or $[x^n]_{\sim_2} = [y^m]_{\sim_2}$. A similar argument to that presented in the proof of Theorem 4.27 can be used here. Thus, $[x^ny^m]_{\sim_2} = [0]_{\sim_2}$.

Claim 2: $[x^n]_{\sim_2} \neq [0]_{\sim_2}$ and $[y^m]_{\sim_2} \neq [0]_{\sim_2}$

Suppose, toward a contradiction, that $[x^n]_{\sim_2} = [0]_{\sim_2}$. Since $[0]_{\sim_2} = J$, we have that $x^n \in J = [0]_{\sim_1}$, which implies that $x^n \in [0]_{\sim_1}$. Hence $[x^n]_{\sim_1} = [0]_{\sim_1}$, which is a contradiction as $[x^n]_{\sim_1} \neq [0]_{\sim_1}$. Thus, $[x^n]_{\sim_2} \neq [0]_{\sim_2}$. We can similarly show that $[y^n]_{\sim_2} \neq [0]_{\sim_2}$.

Since there exists $n, m \in \mathbb{Z}^+$ such that $[x^ny^m]_{\sim_2} = [0]_{\sim_1}, [x^n]_{\sim_2} \neq [0]_{\sim_2}$, and $[y^m]_{\sim_2} \neq [0]_{\sim_2}$. Therefore, $H : \overline{\Gamma}_{\sim_1}(R) \to \overline{\Gamma}_{\sim_2}(R)$ is a graph homomorphism $\square$

Chapter 5

Conclusion

5.1 Summary

This dissertation introduces vertex set variations to Badawi's annihilator graph, $AG(R)$, and Bennis et al.'s extended zero-divisor graph, $\overline{\Gamma}(R)$.

In Chapter 2, we define compressed versions of $AG(R)$ and $\overline{\Gamma}(R)$ via equivalence relations. We prove several results determining conditions under which graphs built from various equivalence relations are equivalent. We also classify the size of the vertex set of the compression of $\overline{\Gamma}(\mathbb{Z}_n)$.

In Chapter 3, we define and investigate the ideal-based versions of $AG(R)$ and $\overline{\Gamma}(R)$. In particular, we generalize several results for $AG(R)$ and $\overline{\Gamma}(R)$ to their ideal-based counterparts and detail an efficient construction for ideal-based graphs.

In Chapter 4, we define congruence-based versions of $AG(R)$ and $\overline{\Gamma}(R)$. In addition to generalizing several foundational properties, we determine conditions for which the natural surjective homomorphism from $R/\sim_1$ to $R/\sim_2$ with $\sim_1 \leq \sim_2$ extends to a graph homomorphism.

5.2 Future Directions

This dissertation introduces several new kinds of zero-divisor type graphs. Though we investigate a great deal here, the adherence of many properties proven for earlier graphs remain untouched for the new varieties. The following specific directions and questions are germane to the work presented here.

- Investigating the structure of $AG_{\sim_{\overline{\Gamma}}}(R)$ and $\overline{\Gamma}_{\sim_{AG}}(R)$ in a fashion analogous to the graphs introduced in Chapter 2.
- Under what conditions is the compression of $\overline{\Gamma}(\mathbb{Z}_n)$ complete?
- Investigating the structure of $\overline{\Gamma}(R)$ for rings other than $\mathbb{Z}_n$.
- Exploring properties of $AG_I(R)$ and $\overline{\Gamma}_I(R)$ for specific ideals (e.g.radical, prime, maximal).
- Theorem 4.10 is stated for ring ideals. Can this result be extended to semigroup ideals?
- Can we establish a relationship between $\sim$ on a ring R and $\sim_S$ on the localization R_S that allows us to extend results from [7]?

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Vita

Grace McClurkin was born in Youngstown, Ohio on July 21, 1989. She attended St. Rose grade school and Ursuline High School where she introduced to the logic of mathematics and proofs by Mrs. De Chant and Mr. Nalepa.

Grace McClurkin completed her undergraduate education at Saint Mary's College in South Bend, Indiana. It was there that she further developed her love of mathematics and teaching as she learned from the wonderful and caring math department. There she obtained a Bachelor's degree in mathematics with an additional major in French in 2011. Besides her coursework, Grace studied abroad in Dijon, France and tutored undergraduate mathematics and French.

Immediately after graduation, Grace began to pursue her mathematics PhD at the University of Tennessee, Knoxville (UTK) while working as a mathematics teaching associate. During her time at UTK, Grace was involved in the creating materials for an inverted college algebra course under the direction of Melissa Peery and worked with Joshua Mike to design an inverted multivariable calculus course with Mathematica, receiving a grant for the latter project. Grace studied algebra and topology, passing preliminary exams in both. Her oral exam was presented under the direction of Dr. David Anderson and involved complemented graphs and Von Neumann regular rings. After her oral exam and continuing under the direction of Dr. David Anderson, Grace worked on new variations and generalizations of the zero-divisor graph.