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Generalizations of Coarse Properties in Large Scale Spaces

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I am submitting herewith a dissertation written by Kevin Michael Sinclair entitled "Generalizations of Coarse Properties in Large Scale Spaces." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Generalizations of Coarse Properties in Large Scale Spaces

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To my parents, John and Vernice Sinclair, who taught me the importance of education, and to my girlfriend, Danna Sharp, who kept me sane up to this point.

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Abstract

Many results in large scale geometry are proven for a metric space. However, there exists many large scale spaces that are not metrizable. We generalize several concepts to general large scale spaces and prove relationships between them. First we look into the concept of coarse amenability and other variations of amenability on large scale spaces. This leads into the definition of coarse sparsification and connections with coarse amenability. From there, we look into an equivalence of Sako's definition of property A on uniformly locally finite spaces and prove that finite coarse asymptotic definition implies it. As well, we define large scale exactness and prove implications with large scale property A and coarse amenability. We finally look into a stronger concept of bounded geometry on large scale spaces that is a coarse invariant and leads to a way to decompose large scale spaces.

Table of Contents

Chapter 1

Introduction

When introducing concepts in coarse geometry one normally starts by assuming X to be a metric space. On the other hand, it was quickly proved that there exist coarse structures that are not metrizable. Coarse structures were defined by Higson and Roe as a collection of subsets of $X \times X$ called entourages. The original motivation for studying coarse structures was to try to approach the Novikov conjecture which states that the higher signatures of smooth manifolds are homotopy invariants. Since then coarse structures have been utilized in many other fields, and a number of new properties and ideas have arisen. Many of these applications are given in a metric space with bounded geometry. From there, J. Dydak gave an alternate definition of coarse structures denoted as large scale structures. It was shown that coarse structures and large scale structures are in one to one correspondence with each other. Therefore, rather than working with coarse structures, we will prove results for large scale structures. In this dissertation, we will approach such properties only defined for a metric space in order to generalize them to general large scale spaces. In particular we will begin by generalizing the idea of uniform local amenability presented in [BNS+12] and then move onto definitions that are connected to uniform local amenability. We will then prove whether the relations between these concepts still hold in a general coarse space. Additionally, we will look into generalizing the idea of bounded geometry to one that is a coarse invariant. To begin we first need to establish several definitions and concepts used throughout this dissertation.

1.1 Scales

Before defining a large scale structure we must first introduce the concept of scales and how to compare two scales against each other.

Definition 1. *Given a space X, we say $\mathcal{U}$ is a* ***scale*** *of X if $\mathcal{U}$ is a cover of X. In other words $\mathcal{U}$ is a collection of subsets of X such that $\cup_{U\in\mathcal{U}} U = X$.*

Note that for any family $\mathcal{B}$ of X, not necessarily a scale, we can extend it to a scale via the trivial extension $e(\mathcal{B})$ defined as $\mathcal{B} \cup \{\{x\} : x \in X\}$. Therefore, we can think of any family as a scale even if it does not cover the entirety of X.
When defining large scale structures we need a way to compare two scales. This is done through the idea of coarsenings, refinements, and star-refinements.

Definition 2. *Given two scales $\mathcal{U}$ and $\mathcal{V}$, we say $\mathcal{U}$* ***coarsens*** *$\mathcal{V}$, denoted $\mathcal{V} \prec \mathcal{U}$ if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $V \subset U$. Similarly, we say that $\mathcal{V}$* ***refines*** *$\mathcal{U}$.*

In Coarse Geometry we study large scale properties of a space X. As such we need a way to create larger scales from pre-existing one. This is achieved through the act of starring scales.

Definition 3. *Given a set X, $V \subseteq X$, and a collection of subsets $\mathcal{U}$ we define the **star** of V with respect to $\mathcal{U}$, denoted $st(V,\mathcal{U})$, as the union of all sets in $\mathcal{U}$ that have nonempty intersection with V, i.e. $st(V,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap V \neq \emptyset\}$.*

Given two collections of subsets $\mathcal{V}$ and $\mathcal{U}$ we define $st(\mathcal{V},\mathcal{U}) = \{st(V,\mathcal{U}), V \in \mathcal{V}\}$.

*Given two collections of subsets $\mathcal{V}$ and $\mathcal{U}$ we say $\mathcal{V}$ **star refines** $\mathcal{U}$ if $st(\mathcal{V},\mathcal{V}) \prec \mathcal{U}$.*

Stars and star refinement satisfy the following basic properties:

1. If $U \subseteq X$ and $\mathcal{V}$ is a scale of X, then $U \subseteq st(U,\mathcal{V})$.

2. If $\mathcal{U}$ is a collection of subsets of X and $\mathcal{V}$ is a scale of X, then $\mathcal{U} \prec st(\mathcal{U},\mathcal{V})$.

3. If $U \subseteq V$ and $\mathcal{W}$ is any collection of subsets of X, then $st(U,\mathcal{W}) \subseteq st(V,\mathcal{W})$.

4. If $U \prec V$ and $\mathcal{W}$ is any collection of subsets of X, then $st(U,\mathcal{W}) \prec st(V,\mathcal{W})$.

5. If $U \subset X$ and $\mathcal{V}_1, \mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1 \prec \mathcal{V}_2$, then $st(U,\mathcal{V}_1) \prec st(U,\mathcal{V}_2)$.

6. If $\mathcal{U}$ is any collection of subsets of X and $\mathcal{V}_1, \mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1 \prec \mathcal{V}_2$, then $st(\mathcal{U},\mathcal{V}_1) \prec st(\mathcal{U},\mathcal{V}_2)$.

7. If $\mathcal{U}$ is a scale and $\mathcal{V}$ is any collection of subsets of X, then $\mathcal{V} \prec st(\mathcal{U},\mathcal{V})$.

The motivating example for understanding scales and stars of scales is the cover of a metric space by r-balls for any $r > 0$. Let $\mathcal{B}_r$ be a cover of X by r-balls and let $\mathcal{U}$ be any other cover of X. Notice that $st(\mathcal{U},\mathcal{B}_r) = \{B(U,2r) : U \in \mathcal{U}$ where $B(U,2r)$ is the 2r neighborhood of any set U. Therefore, we can think of the act of starring as fattening up any scale to create a new scale that refines the original. Clearly $\mathcal{V} \prec st(\mathcal{V},\mathcal{U})$. This allows us to think of the sets of $st(\mathcal{V},\mathcal{U})$ as neighborhoods to the sets of $\mathcal{V}$. In other words, we can consider $V \in \mathcal{V}$ as points of X with neighborhoods contained in $st(\mathcal{V},\mathcal{U})$.

While the motivating example comes from a metric we will see the definition of a Large Scale Structure is more general in nature [DH06]. However, before we define a Large Scale Structure, or an ls-structure for short, we should first introduce the idea of a coarse structure on X.

1.2 Coarse Structures

Coarse structures were originally introduced by Higson and Roe for use in index theory and signature theory. In particular, coarse structures were defined to provide an approach to the Novikov conjecture, a still open conjecture stating that the higher signatures of smooth manifolds are homotopy invariant [Roe03]. Like scales, metric spaces naturally give rise to coarse structures. However, coarse structures can easily be defined for more general spaces without a metric. Before giving the definition of a coarse structure, we must first introduce some notation.

Definition 4. *Let X be a set and consider the product space $X \times X$.*

1. *The **diagonal** of X to be $\Delta = \{(x,x) : x \in X\}$.*

2. *For $U \subset X \times X$, define the **inverse** of U to be $U^{-1} = \{(y,x) : (x,y) \in U\}$.*

3. *Given two subsets $U, V \in X \times X$, define the product of U and V to be $U \circ V = \{(x,z) : (x,y) \in U, (y,z) \in V$ for some $y \in X\}$.*

4. *For $U \subseteq X \times X$ and for $x \in X$, define $U[x] = \{y \in X : (y,x) \in U\}$.*

Definition 5. *(Roe, 2003) A* ***Coarse Structure*** *on X is a family $\mathcal{U}$ of subsets of $X \times X$ that satisfy the following:*

- $\Delta \in \mathcal{U}$.
- $U \in \mathcal{U}$ *implies* $U^{-1} \in \mathcal{U}$
- *If* $U, V \in \mathcal{U}$ *then* $U \circ V \in \mathcal{U}$
- *If* $U \in \mathcal{U}$ *and* $V \subset U$, *then* $V \in \mathcal{U}$.
- *If* $U, V \in \mathcal{U}$ *then* $U \cup V \in \mathcal{U}$.

The elements of a uniform structure are called either **controlled sets** or **entourages**.

1.3 Large Scale Structures

The definition of coarse structures is analytical in nature, but is not very useful for topologists and geometers. With that in mind large scale structures were defined in [DH06] as a more topological way of looking at coarse properties of a space.

Definition 6. *(J Dydak, 2005) Given a set X. A* ***Large Scale Structure***, *$\mathcal{LSS}_{\mathcal{X}}$ is a nonempty set of families $\mathcal{B}$ of subsets of X (called uniformly bounded or uniformly $\mathcal{LSS}_X$-bounded one $\mathcal{LSS}_X$ is fixed) satisfying the following conditions:*

- $\mathcal{B}_1 \in \mathcal{LSS}_X$ *implies* $\mathcal{B}_2 \in \mathcal{LSS}_X$ *if each element of* $\mathcal{B}_2$ *consisting of more than one point is contained in some element of* $\mathcal{B}_1$.
- $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ *implies* $st(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}_X$.

Alternatively a large scale structure is a nonempty filter of scales on X under reverse star refinement that is closed under refinements of scales.

Elements of the ls-structure are called either **uniformly bounded covers** or **uniformly bounded families**. Note that the first property of our definition gives us that the trivial scale, $\{\{x\} : x \in X\}$ is always in our large scale structure. Therefore if $\mathcal{B} \in \mathcal{LSS}_X$ then all refinements of $e(\mathcal{B})$ also belong to $\mathcal{LSS}_X$. The second condition can be thought of as a generalization of the triangle inequality.

The idea of a uniformly bounded cover is very closely related to the idea of a controlled set in a coarse structure. In [DH06] it was shown that coarse structures and ls-structures are equivalent by the following propositions.

Proposition 1.0.1. *Every ls-structure $\mathcal{LSS}_X$ on X induces a coarse structure $\mathcal{C}$ on X as follows: A subset E of $X \times X$ is declared controlled if and only if there is $B \in \mathcal{LSS}_X$ such that $E \subset \cup_{B \in \mathcal{B}} B \times B$.*

Proposition 1.0.2. *Every coarse structure $\mathcal{C}$ on X induces a large scale structure $\mathcal{LSS}_X$ on X as follows: B is declared uniformly bounded if and only if there is a controlled set E such that $\cup_{B \in \mathcal{B}} B \times B \subset E$.*

Therefore properties of coarse structures can also be defined and proved on large scale structures. As well, we can define functions between large scale structures in a way similar to functions between coarse structures.

Let $f : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). We say that f is uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that given $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$. In other words, if two points are close in the domain they will stay close in the codomain. This idea has been generalized to the idea of closeness between two large scales spaces.

Definition 7. *Let X and Y be large scales spaces. A map $f : X \to Y$ is* ***Large Scale continuous****, or ls-continuous, if for every uniformly bounded cover $\mathcal{U}$ in X, $f(\mathcal{U})$ is a uniformly bounded cover in Y.*

For a large scale continuous function f, we say f is a ***coarse embedding*** *if for every uniformly bounded family $\mathcal{U} \in Y$, $f^{-1}(\mathcal{U}) = \{f^{-1}(U)|U \in \mathcal{U}\}$ is uniformly bounded in X.*

We say f is ***coarsely surjective*** *if there exists a uniformly bounded family $\mathcal{U}$ in Y such that $Y \subseteq st(f(X),\mathcal{U})$.*

Note that if f is not surjective, then $f(\mathcal{U})$ will not be a cover of Y. However, recall that any uniformly bounded family can be trivially extended to a uniformly bounded cover. From the definition it is clear that the composition of two large scale continuous maps is still large scale continuous. In a large scale space we can consider points in a uniformly bounded set as equal. Therefore, we will define a coarse equivalence on the uniformly bounded sets.

Definition 8. *Let X and Y be large scale space. Given maps $f, g : X \to Y$, we say f and g are* ***close*** *if there exists a uniformly bounded cover $\mathcal{V}$ of Y such that for every $x \in X$ there is $V \in \mathcal{V}$ with $f(x), g(x) \in V$.*

A large scale continuous map $f : X \to Y$ between two large spaces is a ***coarse equivalence*** *if there exists a large scale continuous map $g : Y \to X$ such that $f \circ g$ is close to the function id_Y and $g \circ f$ is close to the function id_X. If $f : X \to Y$ is a coarse equivalence then we say X and Y are* ***coarsely equivalent****.*

This is equivalent to saying that $\{\{g \circ f(x), x\} | x \in X\}$ and $\{\{f \circ g(y), y\} | y \in Y\}$ are uniformly bounded families.

Theorem 1.1. *A ls-continuous map $f : X \to Y$ is a large scale equivalence if and only if f is a coarse embedding and is coarsely surjective.*

Proof. Assume that $f : X \to Y$ is a large scale equivalence. First we will show that f is a coarse embedding. Let $\mathcal{U}$ be a uniformly bounded cover of Y and let $g : Y \to X$ be the function such that $g \circ f : X \to X$ is close to the identity id_X. Therefore, there exists a uniformly bounded family $\mathcal{V}$ of X such that for every $x \in X$ there exists a $V \in \mathcal{V}$ with $x \in V$ and $g \circ f(x) \in V$. Let $\mathcal{U}$ be a uniformly bounded family in Y. Since g is large scale continuous, then $g(\mathcal{U}) = \{g(U) : U \in \mathcal{U}\}$ is a uniformly bounded family in X. We claim the uniformly bounded family $\mathcal{W} = st(g(\mathcal{U}), \mathcal{V})$ coarsens $f^{-1}(\mathcal{U})$.

Let $U \in \mathcal{U}$. Consider any $x \in X$ such that $f(x) = y \in U$. From the definition of coarse equivalence, there exists a $V_y \in \mathcal{V}$ such that $\{g \circ f(x), x\} \subseteq V_y$. Therefore, $\{g(y), x\} \subseteq V_y$ for all $x \in X$ with $f(x) = y$, or $\{g(y), f^{-1}(y)\} \subseteq V_y$.

Now, for $U \in \mathcal{U}$, $f^{-1}(U) = \cup_{y \in U} f^{-1}(y) \subseteq \cup_{y \in U} V_y$ for V_y defined above. However, for $y \in U$ we know $\{g(y), x\} \in V_y$ for all $x \in X$ with $f(x) = y$, so $V_y \cap g(U) \neq \emptyset$. Thus for all $y \in U$ $V_y \subseteq st(g(U), \mathcal{V})$. Therefore $f^{-1}(U) \subseteq st(g(U), \mathcal{V})$, and from the definition of a large scale structure we have $f^{-1}(\mathcal{U})$ is a uniformly bounded family, and f is a coarse embedding.

Next we will show that f is coarsely surjective. From the definition of coarse equivalence, there exists a uniformly bounded cover $\mathcal{U}$ of Y such that for every $y \in Y$ there exists a $U \in \mathcal{U}$ with $\{f \circ g(y), y\} \subseteq U$. Arbitrarily take $y \in Y$. Then there exists a $U \in \mathcal{U}$ with $\{f \circ g(y), y\} \subseteq U$. Therefore $y \in st(f \circ g(Y), \mathcal{U})$. Therefore $Y \subseteq st(f \circ g(Y), \mathcal{U})$, and since

$f \circ g(Y) \subseteq f(X)$ we have that $Y \subseteq st(f \circ g(X), \mathcal{U})$, and f is coarsely surjective.

Now, suppose f is a large scale continuous map that is a coarse embedding and is coarsely surjective. First we will construct a large scale continuous function $g : Y \to X$. From coarse surjectivity, we know there exists a uniformly bounded family $\mathcal{U}$ in Y such that $Y \subseteq st(f(X), \mathcal{U})$. Thus, for every $y \in Y$ there exists a $U_y \in \mathcal{U}$ with $U_y \cap f(X) \neq \emptyset$. Choose an $x_y \in X$ such that $f(x_y) \in U_y$. Therefore define $g : Y \to X$ with $g(y) = x_y$.

Claim: $f \circ g$ is close to id_Y. Let $y\ inY$. From the construction of g, there exists $U_y \in \mathcal{U}$ with $y \in U_y$, and $f \circ g(y) \in U_y$. Thus $\{f \circ g(y), y\} \subseteq U_y$ and thus $f \circ g$ is close to id_Y. Let $y \in Y$. Claim: $g \circ f$ is close to id_X. From coarse embeddability we know $f^{-1}(\mathcal{U})$ is a uniformly bounded family in X. Choose $x \in X$. We know from coarse surjectivity that there exists $U_{f(x)} \in \mathcal{U}$ with both $f(x) \in U_{f(x)}$ and $f \circ g \circ f(x) \in U_{f(x)}$ from the construction of the function g. Therefore $\{g \circ f(x), x\} \subseteq f^{-1}(U_f(x)) = \{z \in X | f(z) \in U_{f(x)}\}$, and $g \circ f$ is close to id_X.

Finally we will show that g is large scale continuous. Let $\mathcal{U}$ be a uniformly bounded family of Y. Since $f \circ g$ is close to id_Y, then there exists a uniformly bounded family $\mathcal{V}$ of Y such that for every $y \in Y$ there exists $V \in \mathcal{V}$ with $\{f \circ g(y), y\} \subseteq V$. Let $U \in \mathcal{U}$. For every $y \in U$, there exists $V \in \mathcal{V}$ such that $\{f \circ g(y), y\} \subseteq V$. Therefore, for every $U \in \mathcal{U}$, $f \circ g(U) \subset st(U, \mathcal{V})$, and we have that $f \circ g(\mathcal{U})$ is uniformly bounded in Y. Finally, since f is a coarse embedding $f^{-1} \circ f \circ g(\mathcal{U})$ is a uniformly bounded family and for all $U \in \mathcal{U}$, $g(U) \subseteq f^{-1} \circ f \circ g(U)$. This means that $g(\mathcal{U})$ is uniformly bounded in X from the definition of large scale structure.

$\square$

This result will be used many times when showing a property is a coarse invariant.

Chapter 2

Generalizations of Coarse Properties

2.1 Coarse Amenability

Amenability of groups was introduced by Von Neumann who showed that the Banach-Tarski paradox is caused by the lack of amenability of the group of isometries of the 3D-space $\mathbb{R}^3$. Amenability now is used in many areas of mathematics and admits many different definitions. One of the most common definitions of amenability is a geometric interpretation due to Følner for finitely generated groups. Let G be a finitely generated group with a word length metric and let a subset $A \subseteq G$ be given. We define the R-**boundary** of A in the following way

$$\partial_R A = \{g \in G \setminus A \mid d(g, A) \leq R\}$$

Følner's definition of amenability compares the cardinality of finite sets against their boundaries.

Definition 9. *A finitely generated group G is* ***amenable*** *if for every $R > 0$ and $\epsilon > 0$ there exists a finite set $F \subseteq G$ such that*

$$\frac{|\partial_R F|}{|F|} \leq \epsilon.$$

Such finite sets are referred to as **Følner sets**.

From there analogs of amenability were defined for various coarse spaces. Cencelj, Dydak, and Vavpetic [CDV12] came up with a definition of coarsely amenable spaces by dualizing the concept of paracompactness.

Definition 10. *X is* ***large scale weakly paracompact*** *if for all $r, s > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X of Lebesgue number at least s such that every r-ball $B(x, r)$ is contained in only finitely many elements of $\mathcal{U}$.*

The motivation for the definition of Coarse Amenability came as a result of analyzing expanders

Definition 11. *The horizon $hor(A,\mathcal{U}) = \{U_i \mid A \cap U_i \neq \emptyset, U_i \in \mathcal{U}\}$.*

Definition 12. *X is* ***coarsely amenable*** *if for each $s > r > 0$ and each $\epsilon > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X such that for each $x \in X$ the horizon $hor(B(x, s),\mathcal{U})$ is finite and*

$$\frac{|hor(B(x, r),\mathcal{U})|}{|hor(B(x, s),\mathcal{U})|} > 1 - \epsilon.$$

Since not all large scale spaces have a metric we need a definition for Coarse Amenability that relies solely on covers rather than r-balls. The generalization turns out to be simpler in nature and will imply the above definition in the metric space case.

Definition 13. *Let X have a large scale structure. X is* ***coarsely amenable*** *if for each uniformly bounded cover $\mathcal{W}$ of X and for each $\epsilon > 0$, there exists a uniformly bounded cover $\mathcal{U}$ of X such that for all $x \in X$ $hor(st(x, \mathcal{W}), \mathcal{U})$ is finite and*

$$\frac{|hor(x,\mathcal{U})|}{|hor(st(x,\mathcal{W}),\mathcal{U})|} > 1 - \epsilon$$

First we will show this definition is a coarse invariant.

Theorem 2.1. *Let X and Y be large scale spaces, and let $f : X \to Y$ be a coarse embedding. Then if Y is coarsely amenable, then X is also coarsely amenable.*

Proof. Let $\mathcal{W}$ be a uniformly bounded cover of X and let $\epsilon > 0$ be given. Then, $f(\mathcal{W})$ is a uniformly bounded cover of Y and there exists a uniformly bounded cover $\mathcal{U}$ of Y such that for any $x \in X$ $|hor(st(f(x), f(\mathcal{W})),\mathcal{U})|$ is finite and

$$\frac{|hor(f(x),\mathcal{U})|}{|hor(st(f(x), f(\mathcal{W})),\mathcal{U})|} > 1 - \epsilon.$$

Since f is a coarse embedding we know $f^{-1}(\mathcal{U})$ is a uniformly bounded family of X. First suppose $U \in hor(f(x),\mathcal{U})$. Thus, $f(x) \in U$ and therefore $x \in f^{-1}(U)$ and $f^{-1}(U) \in hor(x, f^{-1}(\mathcal{U}))$. Therefore $|hor(x, f^{-1}(\mathcal{U}))| \geq |hor(f(x),\mathcal{U})|$.

Secondly, notice that if $f^{-1}(U) \in hor(st(x,\mathcal{W}), f^{-1}(\mathcal{U}))$ then there exists a $W \in \mathcal{W}$ and a $z \in X$ with $x \in W$ and $z \in W \cap f^{-1}(U)$. Therefore, $f(x) \in f(W)$ and $f(y) \in f(W) \cap U$. This gives us that $U \in hor(st(f(x), f(\mathcal{W})),\mathcal{U})$, and $|hor(st(x,\mathcal{W}), f^{-1}(\mathcal{U}))| \leq$

$|hor(st(f(x), f(\mathcal{W})), \mathcal{U})|$. Putting this together we get

$$\frac{|hor(x, f^{-1}(U))|}{|hor(st(x, \mathcal{W}), f^{-1}(\mathcal{U}))|} \geq \frac{|hor(f(x), \mathcal{U})|}{|hor(st(f(x), f(\mathcal{W})), \mathcal{U})|} > 1 - \epsilon$$

and X is coarsely amenable. $\square$

This general definition of coarse amenability relies only on points in X. We can however define a general coarse amenability in a very similar way to the metric case.

Theorem 2.2. *X is coarsely amenable if and only if for all $\epsilon > 0$ and uniformly bounded covers $\mathcal{V}, \mathcal{W}$ with $\mathcal{V}$ refining $\mathcal{W}$ then there exists a uniformly bounded $\mathcal{U}$ of X such that for all $x \in X$, the horizon, $hor(st(x, \mathcal{W}), \mathcal{U})$ is finite and*

$$\frac{|hor(st(x, \mathcal{V}), \mathcal{U})|}{|hor(st(x, \mathcal{W}), \mathcal{U})|} > 1 - \epsilon$$

Proof. Suppose X is coarsely amenable. Let $\mathcal{V}$ be a uniformly bounded cover and let $\epsilon > 0$. Let $\mathcal{V}$ be the trivial cover of X. Thus $\mathcal{V}$ refines $\mathcal{W}$ trivially, and by assumption there exists a uniformly bounded cover $\mathcal{U}$ such that, for all $x \in X$ $hor(st(x, \mathcal{W}), \mathcal{U})$ is finite, and

$$\frac{|hor(x, \mathcal{U})|}{|hor(st(x, \mathcal{W}), \mathcal{U})|} = \frac{|hor(st(x, \mathcal{V}), \mathcal{U})|}{|hor(st(x, \mathcal{W}), \mathcal{U})|} > 1 - \epsilon$$

Now, assume the second condition. Let $\epsilon > 0$ and let $\mathcal{V}$ and $\mathcal{W}$ be two uniformly bounded covers with $\mathcal{V}$ refining $\mathcal{W}$. By assumption, there exists a uniformly bounded cover $\mathcal{U}$ such that, for all $x \in X$, $hor(st(x, \mathcal{W}), \mathcal{U})$ is finite and

$$\frac{|hor(x, \mathcal{U})|}{|hor(st(x, \mathcal{W}), \mathcal{U})|} > 1 - \epsilon$$

If $U \in hor(x, \mathcal{U})$ then $x \in U$. Since $\mathcal{V}$ is a cover of X, then $x \in st(x, \mathcal{V})$ so $U \cap st(x, \mathcal{V}) \neq \emptyset$ and thus $U \in hor(st(x, \mathcal{V}), \mathcal{U})$ and $hor(x, \mathcal{U}) \subset hor(st(x, \mathcal{V}), \mathcal{U})$. Next, if $U \in hor(st(x, \mathcal{V}), \mathcal{U})$, then there exists a $V \in \mathcal{V}$ such that $U \cap V \neq \emptyset$. Since $\mathcal{V}$ refines $\mathcal{W}$, there exists a $W \in \mathcal{W}$ such that $V \subset W$. Therefore, $W \cap U \neq \emptyset$, $U \in hor(st(x, \mathcal{W}), \mathcal{U})$, and $hor(st(x, \mathcal{V}, \mathcal{U}) \subset hor(st(x, \mathcal{W}, \mathcal{U})$. Putting this together we get

$$|hor(x, \mathcal{U})| < |hor(st(x, \mathcal{V}), \mathcal{U})| < |hor(st(x, \mathcal{W}), \mathcal{U})|$$

and therefore

$$\frac{|hor(st(x, \mathcal{V}), \mathcal{U})|}{|hor(st(x, \mathcal{W}), \mathcal{U})|} \geq \frac{|hor(x, \mathcal{U})|}{|hor(st(x, \mathcal{W}), \mathcal{U})|} > 1 - \epsilon$$

$\square$

2.2 Coarse Uniform Local Amenability with respect to all probability measures

A version of local uniform amenability was defined and generalized by Brodzki, Niblo, Spakula, Willett, and Wright [BNS+12]. The inspiration for this definition was to create a form of amenability that is preserved by passing to subspaces.

Definition 14. *Let X be a metric space. X is said to be* ***uniformly locally amenable*** *if for all $R, \epsilon > 0$, there exists $S > 0$ such that for any finite subset F of X there exists $E \subseteq X$ such that $diam(E) \leq S$ and*

$$|\partial_R E \cap F| < \epsilon |E \cap F|.$$

Uniform local amenability, or ULA, means that all finite subsets of a space X must be amenable with Følner sets of uniform size. This definition of local uniform amenability is equivalent to the following more analytical characterization:

For all R, and for all $\epsilon > 0$ there exists $S > 0$ such that for all finite subsets F of X there exists a function $\phi \in l^1(X)$ such that

- $diam(supp(\phi)) \leq S$
- the following inequality holds

$$\sum_{x \in F} \sum_{\substack{y \in F \\ d(x,y) \leq R}} |\phi(x) - \phi(y)| < \epsilon \sum_{x \in F} |\phi(x).$$

These definitions can be used to prove equivalences of several coarse properties. However, in order to accomplish this a stronger version of local uniform amenability must be introduced where probability measures are used to localize.

Definition 15. *Let X be a space. X is said to have uniform local amenability with respect to probability measures, or ULA_μ if for all $R, \epsilon > 0$ there exists $S > 0$ such that for all probability measures μ on X there exists a finite subset $E \subseteq X$ such that*

- $diam(E) \leq S$

- *the following inequality holds*

$$\mu(\partial_R E) < \epsilon\mu(E).$$

Again, the definition for ULA_μ is equivalent to the following more analytical approach: For all $R, \epsilon > 0$ there exists $S > 0$ such that for all probability measures μ on X there exists a function $\phi \in l^1(X)$ such that

- $diam(supp(\phi)) \leq S$
- the following inequality holds

$$\sum_{x \in supp(\mu)} \mu(x) \sum_{\substack{y \in supp(\mu) \\ d(x,y) \leq R}} |\phi(x) - \phi(y)| < \epsilon \sum_{x \in supp(\mu)} \mu(x)|\phi(x)|.$$

Both ULA and ULA_μ are coarse invariants and it has been shown that ULA_μ implies ULA. However, both definitions require both a metric and the subsets E to be finite subsets. Finite subsets can be considered as simply points in large scale spaces. Therefore, these concepts are not completely clear in large scale spaces, but can be generalized to definitions that fit the idea behind large scale spaces.

Definition 16. *If $\mathcal{U}$ is an ls-cover of X, then $\partial_{\mathcal{U}}(A) = (st(A,\mathcal{U}) \setminus A)$*

Definition 17. *A set E is $\mathcal{U}$-bounded if $E \subset U$ for some $U \in \mathcal{U}$.*

Definition 18. *Let X be a large scale space. We say X has coarse uniform local amenability with respect to all probability measures, or coarse ULA_μ, if for each $\mathcal{U}, \epsilon > 0$ there exists $\mathcal{V}$ such that for all probability measures μ on X there exists a set $E \subset X$ such that:*

- *E is $\mathcal{V}$-bounded*

- *the following inequality hold*

$$\mu(\partial_{\mathcal{U}}(E)) < \epsilon \cdot \mu(E)$$

The first thing to notice in the large scale generalization is that the subset E need not be finite. We will first show that this definition still passes to subsets as originally intended and is still a coarse invariant.

Theorem 2.3. *Let X be a set with coarse uniform local amenability with respect to all probability measures. If $Y \subseteq X$ then Y also has coarse ULA_μ.*

Proof. Suppose X an ls-space has coarse ULA_μ and let $Y \subset X$. Let $\mathcal{U}$ be a uniformly bounded cover of Y, and trivially extend $\mathcal{U}$ to a uniformly bounded cover $\mathcal{U}'$ on X. Thus there exists a $\mathcal{V}'$ on X such that for all probability measures ϕ on X, there exists $E' \subset X$ such that E' is $\mathcal{V}'$-bounded and $\phi(\partial_{\mathcal{U}'}(E')) < \epsilon \cdot \phi(E')$

Now, define $\mathcal{V} = \mathcal{V}' \cap Y$ to be a uniformly bounded cover on Y. Let μ be a probability measure on Y and extend it trivially to a probability measure ϕ on X. Therefore, there exists $E' \subset X$ such that E' is $\mathcal{V}'$-bounded and $\phi(\partial_{\mathcal{U}'}(E')) < \epsilon \cdot \phi(E')$. If $E' \cap Y = \emptyset$ then $\phi(E') = 0 \leq \phi(\partial_{\mathcal{U}'}(E'))$ which is a contradiction, so $E' \cap Y \neq \emptyset$ and we can define $E = E' \cap Y$. Clearly, E is (V)-bounded by construction. Next, note that $E' = E \cup \{\{x\} | x \in E' \setminus Y\}$, and $\mathcal{U}' = \mathcal{U} \cup \{\{x\} | x \in X \setminus Y\}$. Therefore $\partial_{\mathcal{U}'}(E') = st(\mathcal{U}', E') \setminus E' = st(\mathcal{U}, E) \setminus E = \partial_{\mathcal{U}}(E)$ So,

$$\mu(\partial_{\mathcal{U}}(E)) = \phi(\partial_{\mathcal{U}}(E)) = \phi(\partial_{\mathcal{U}'}(E')) < \epsilon\phi(E') = \epsilon\phi(E' \cap Y) = \epsilon\mu(E)$$

Since μ was chosen arbitrarily, $\mathcal{V}$ meets our condition for all probability measures and Y has $ls - ULA_{\mu}$. $\square$

Before showing this is a coarse property, we first need to prove a basic property of stars of sets.

Lemma 2.3.1. *Let X be a set, $E \subseteq X$ and $\mathcal{U}$ family in X. Then $st(E,\mathcal{U}) \subset f^{-1}[st(f(E), f(\mathcal{U}))]$*

Proof. Let $x \in st(E,\mathcal{U})$. Therefore $x \in U$ with $U \in \mathcal{U}, U \cap E \neq \emptyset$. These conditions give us that $f(x) \in f(U), f(U) \in f(\mathcal{U})$, and $f(U) \cap f(E) \neq \emptyset$. By definition, $f(x) \in st(f(E), f(\mathcal{U}))$. and thus $x \in f^{-1}(st(f(E), f(\mathcal{U})))$

Let $\epsilon > 0, \mathcal{U}$ an ls-cover of X. Since f is a coarse equivalence then $f(\mathcal{U})$ is an ls-cover of Y, and thus we have an ls-cover $\mathcal{V}'$ of Y such that for all probability measures ϕ on Y there exists an E' that is $\mathcal{V}'$-bounded and $\phi(\partial_{f(\mathcal{U})} E') < \epsilon \phi(E')$. Consider $\mathcal{V} = f^{-1}(\mathcal{V}')$ which is an ls-cover since f is a coarse equivalence. $\square$

Theorem 2.4. *Let X be a large scale space, Y be a large scale space with coarse ULA_{μ} and let $f : X \to Y$ be a coarse equivalence. Then X has coarse ULA_{μ}*

Proof. Now let μ be a probability measure on X. Let ϕ be the push-forward measure for μ, a probability measure on Y. Thus there exists an E' that is $\mathcal{V}'$-bounded and $\phi(\partial_{f(\mathcal{U})} E') < \epsilon \phi(E')$. Define E to be $f^{-1}(E')$.

Since E' is $\mathcal{V}'$-bounded, then by construction E is $\mathcal{V}$-bounded Using our lemma, we can show that $\mu(\partial_{\mathcal{U}}(E)) < \epsilon \mu(E)$ directly:

$$\mu(\partial_{\mathcal{U}}(E)) = \mu(st(f^{-1}(E'),\mathcal{U}) \setminus f^{-1}(E')) \leq \mu(f^{-1}(st(E', f(\mathcal{U}))) \setminus f^{-1}(E')) =$$

$$= \mu(f^{-1}[st(E', f(\mathcal{U})) \setminus E']) = \phi(\partial_{f(\mathcal{U})}(E')) < \epsilon\phi(E') = \epsilon\mu(f^{-1}(E')) = \epsilon\mu(E)$$

Since μ was chosen arbitrarily, $\mathcal{V}$ satisfies our properties for all probability measures on X and thus X has $ls - ULA_\mu$. □

Theorem 2.5. *Let X be a space with a large scale structure. If X is coarsely amenable, then X has coarse ULA_μ.*

Proof. Suppose X is coarsely amenable. Let $\mathcal{V}$ be a large scale cover on X and $\epsilon > 0$. Not pick a large scale cover $\mathcal{U} = \{U_s\}_{s\in S}$ such that for all $x \in X$:

$$\frac{|hor(x,\mathcal{U})|}{|hor(st(x,\mathcal{V}),\mathcal{U})|} > \frac{1}{1+\epsilon}$$

By rearranging, we get the following inequality:

$$|hor(st(x,\mathcal{V}),\mathcal{U})| - |hor(x,\mathcal{U})| < \epsilon|hor(x,\mathcal{U})|$$

Let μ be a probability measure on X with finite support. Note that if we fix $x \in X$, then $U_s \in hor(st(x,\mathcal{V}),\mathcal{U}) \setminus hor(x,\mathcal{U})$ if and only if $\cup_{x\in V_i} V_i \cap U_s \neq \emptyset$ and $x \notin U_s$ which is the condition for $x \in \partial_{\mathcal{V}}(U_s)$

Now Consider $\sum_{s\in S}\mu(\partial_{\mathcal{V}}(U_s))$. By summing over $x \in X$, the previous statement gives us that

$$\sum_{s\in S}\mu(\partial_{\mathcal{V}}(U_s)) = \sum_{x\in X}\mu(x)(|hor(st(x,\mathcal{V}),\mathcal{U})| - |hor(x,\mathcal{U})|) <$$

$$< \sum_{x\in X}\mu(x)\epsilon|hor(x,\mathcal{U})| = \sum_{s\in S}\epsilon\mu(U_s)$$

Thus there must exist a $U_t \in \mathcal{U}$ such that

$$\mu(\partial_{\mathcal{V}}(U_t)) < \epsilon\mu(U_t)$$

Clearly U_t is $\mathcal{U}$-bounded, and therefore X has coarse ULA_μ. □

2.3 Coarse Sparsification

Metric sparsification was introduced by Guoliang Yu to study the operator localization property on metric spaces.

Definition 19. *Let X be a space. Then X has the **metric sparsification property (MSP)** if there exists $c > 0$ such that for all $R > 0$ there exists $S > 0$ such that for all probability measures μ on X there exists a subset $\Omega \subseteq X$ equipped with a decomposition*

$$\Omega = \sqcup_i \Omega_i$$

such that

- $\mu(\Omega) \geq c$
- *for all i, $diam(\Omega_i) \leq S$*
- *for all $i \neq j$, $d(\Omega_i, \Omega_j) > R$.*

In [CTWY07] it is proved that if X has MSP for some constant $c > 0$, then it has MSP for any $0 < c < 1$. Again, this definition depends on a metric being defined. Therefore, we will define coarse sparsification for general large scale spaces that only depends on properties of the large scale structure.

Definition 20. *$\mathcal{W}$ is $\mathcal{U}$-disjoint in X if each element of $\mathcal{V}$ intersects at most one element of $\mathcal{W}$. In other words, for $W \in \mathcal{W}$, $st(W, \mathcal{V}) \cap W_i = \emptyset$ for all $W \neq W_i \in \mathcal{W}$*

Definition 21. *X is an ls-space. X has **coarse sparsification** if there is $c > 0$ such that for any ls-cover $\mathcal{U}$ of X, there exists an ls-cover $\mathcal{V}$ such that for all probability measures μ on X there exists a set $\Omega \subset X$ with $\Omega = \sqcup_i \Omega_i$ such that:*

- *$\mu(\Omega) \geq c$*
- *each Ω_i is $\mathcal{V}$-bounded*
- *$st(\Omega_i, \mathcal{U}) \cap \Omega_j = \emptyset$ when $i \neq j$ ($\{\Omega_i\}_{i \geq 1}$ is $\mathcal{U}$-disjoint, an ls-family).*

We will show this definition also passes to subsets and is a coarse invariant, but first we must establish a property of $\mathcal{U}$-disjointness.

Proposition 2.5.1. *Suppose $f : X \to Y$, $\mathcal{U}$ is a cover of X, $\mathcal{V}$ coarsens $f(\mathcal{U})$. If $\mathcal{W}$ is $\mathcal{V}$-disjoint in Y, then $f^{-1}(\mathcal{W})$ is $\mathcal{U}$-disjoint.*

Proof. Suppose by contradiction that $f^{-1}(\mathcal{W})$ is not $\mathcal{U}$-disjoint. If not, then there exists W_1 and $W_2 \in \mathcal{W}$ such that $f^{-1}(W_i) \cap U \neq \emptyset$.
Thus $W_i = f(f^{-1}(W_i)) \cap f(U) \neq \emptyset$. but $\mathcal{V}$ coarsens $f(\mathcal{U})$, so $\exists V \in \mathcal{V}$ such that $f(U) \subset V$ and thus $W_1 \cap V \neq \emptyset$ and $W_2 \cap V \neq \emptyset$ which contradicts the fact that $\mathcal{W}$ is $\mathcal{V}$-disjoint in Y. □

Theorem 2.6. *Let X and Y be large scale spaces and let $f : X \to Y$ be a coarse embedding. If Y satisfies coarse sparsification then X also satisfies coarse sparsification.*

Proof. Let Y be a space satisfying coarse sparsification, and let $f : X \to Y$ be a coarse equivalence. Let $\mathcal{U}$ be an ls-cover, μ be a probability measure. Since f is ls-continuous, then $f(\mathcal{U})$ is uniformly bounded. Now, let $\mathcal{V}$ be an ls-cover that coarsens $\mathcal{U}$ and define a probability measure ψ to be $\psi(B) = \mu(f^{-1}(B))$, the "pushforward measure". Now, since Y has coarse sparsification, there is a $c > 0$ and a family $\Omega' = \bigcup_{i=1}^{\infty} \Omega'_i$ such that

- $\{\Omega_i\}_{i \geq 0}$ is $\mathcal{V}$-disjoint, an ls-family
- $\psi(\Omega) > c$

Define $\Omega_i = f^{-1}(\Omega'_i)$.

By the lemma, Ω_i is $\mathcal{U}$-disjoint, an ls-family, and

$\mu(\Omega) = \mu(f^{-1}(\Omega')) = \psi(\Omega') \geq c.$

Therefore, X has coarse sparsification. □

Theorem 2.7. *Let X be an ls-space satisfying the coarse sparsification property and let $Y \subseteq X$. Then Y also satisfies the coarse sparsification property.*

Proof. Let X satisfy coarse sparsification and let $Y \subseteq X$. Let $\mathcal{U}$ an ls-cover of Y, μ be a probability measure on Y, and let $\epsilon > 0$

Extend $\mathcal{U}$ trivially to $\mathcal{V}$ by adding one point sets. Thus we have $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$

$\{\Omega_i\}_{i \geq i}$ is $\mathcal{U}$-disjoint, an ls-family,

$\mu(\Omega) > c$. The probability for all points outside of Y is 0, and any element of the cover of X outside of $\mathcal{U}$ is just a point, so $\Omega \cap Y$ satisfies our properties for coarse sparsification. □

In [BNS+12] it was shown that ULA_μ is equivalent to metric sparsification. Therefore, for the large scale versions of these properties we also expect the concepts to be equivalent for the definitions to be useful.

Theorem 2.8. *Let X be a large scale space. X has coarse uniform local amenability with respect to all probability measures if and only if X has coarse sparsification.*

Proof. Let X be an ls-space with Coarse Sparsification Property. Let $\mathcal{U}$ be an ls-cover, $\epsilon > 0$, and μ be a probability measure on X. Define $c = \frac{1}{1+\epsilon}$. Thus there exists an ls-cover $\mathcal{V}$ for μ and a set Ω with a decomposition like in the definition. Since each Ω_i are $\mathcal{U}$ disjoint, then

$$\sum \mu(st(\Omega_i, \mathcal{U}) \leq \mu(X) = 1 \leq \frac{1}{c}\mu(\Omega) = \frac{1}{c}\sum \mu(\Omega_i).$$

Thus by a counting argument, there exists some i such that

$$\mu(\partial_{\mathcal{U}}(\Omega_i)) + \mu(\Omega_i) = \mu(st(\Omega_i, \mathcal{U})) \leq \frac{1}{c}\mu(\Omega_i)$$

which gives us

$$\mu(\partial_{\mathcal{U}}(\Omega_i)) \leq (\frac{1}{c} - 1)\mu(\Omega_i) = \epsilon\mu(\Omega_i)$$

Since each Ω_i is $\mathcal{V}$-bounded, we can set $E = \Omega_i$ to show that X has coarse ULA_μ.

Now, assume that X has coarse ULA_μ. We will first show that X satisfies the conditions for coarse sparsification property on all probability measures with finite support. Let $\mathcal{U}$ be an ls-cover of X. Let μ be a probability measure with finite support. Fix $\epsilon \geq 0$. Thus there exists a set E and an ls-cover $\mathcal{V}$ such that E is $\mathcal{V}$-bounded and $\mu(\partial_{\mathcal{U}}(E)) < \epsilon \cdot \mu(E)$. Define $\mu_1 = \mu$

and define F_1=supp(μ_1). Define $E_1 = E \cap F_1$ which is $\mathcal{V}$-bounded and $\mu(\partial_{\mathcal{U}}(E_1)) < \epsilon \cdot \mu(E_1)$ Define $F_2 = F_1 \setminus st(E_1,\mathcal{U})$ and rescale μ_1 to μ_2 to be a probability measure on F_2. Thus there exists some set E_2 with $\mu_2(\partial_{\mathcal{U}}(E_2)) < \epsilon \cdot \mu_2(E_2)$

Note that μ_2 is just a rescaling of μ, so the following holds

$\mu(\partial_{\mathcal{U}}(E_2) \cap F_2) < \epsilon \cdot \mu(E_2)$.

Define $F_3 = F_1 \setminus (st(E_1,\mathcal{U}) \cup st(E_2,\mathcal{U}))$, and continue on in this process until it terminates as supp(μ) is finite. Thus we have sets $E_1, ..., E_n$ that is $\mathcal{U}$ disjoint, and each E_i is $\mathcal{V}$-bounded. As well, $\mu(\partial_{\mathcal{U}}(E_i) \cap F_i) < \epsilon \cdot \mu(E_i)$.

Thus define $\Omega_i = E_i$ and $\cup\Omega_i = \Omega$ we have that

$$1 = \mu(X) = \mu(F_1) = \sum \mu(\Omega_i) + \mu(\partial_{\mathcal{U}}(\Omega_i) \cap F_i) < \sum (1+\epsilon)\mu(\Omega_i) = (1+\epsilon)\mu(\Omega)$$

Thus we have that $\mu(\Omega) \geq \frac{1}{1+\epsilon}$. Since ϵ works for all probability measures with finite support, then setting $c = \frac{1}{1+\epsilon}$ shows that $c < \mu(\Omega)$.

Now, let ψ be any probability measure on X. Thus for any $\delta > 0$,there exists a probability measure μ with finite support such that $|\psi(\Omega) - \mu(\Omega)| \leq \delta$. Therefore by letting δ tend towards 0 the same Ω that works for probability measures with finite support will result in $\mu(\Omega) \geq c$ for all probability measures μ on X. Therefore X has coarse sparsification property. $\square$

Chapter 3

Property A

3.1 Property A on Large Scale Structures

Property A was defined by Guoliang Yu in [Yu00] to approach the Baum-Connes conjecture. It can be viewed as a weaker version of amenability and is a sufficient condition for many properties such as coarse embeddability into a Hilbert Space.

Definition 22. *Let X be a uniformly discrete metric space. X has* ***property A*** *if for every $\epsilon > 0$ and $R > 0$ there exists a collection of finite subsets $\{A_x\}_{x \in X}, A_x \subseteq X \times \mathbb{N}$ for every $x \in X$, and a constant $S > 0$ such that*

- $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} \leq \epsilon$ *when* $d(x, y) \leq R$
- $A_x \subseteq B(x, S) \times \mathbb{N}$

These conditions make the sets A_x and A_y be almost equal if $d(x, y) \leq R$ and disjoint if $d(x, y) \geq 2S$. From there Hiroki Sako defined property A in [Sak13] for uniformly locally finite coarse spaces.

Definition 23. *A coarse space* $(X, \mathcal{C})$ *is said to be* ***uniformly locally finite*** *if for every controlled set* $T \in \mathcal{C}$ *satisfies the inequality* $sup_{x \in X}|T[x]| < \infty$*.*

When giving this definition, Sako states that a metric space with a uniformly locally finite coarse structure is called a metric space with bounded geometry.

Definition 24. *(Sako Definition) A uniformly locally finite coarse space* $(X, \mathcal{C})$ *is said to have Property A if for every positive number* ϵ *and every controlled set* $T \in \mathcal{C}$*, there exists a controlled set* $S \in \mathcal{C}$ *and a subset* $A^{\mathcal{C}} \subset S \times \mathbb{N}$ *such that*

- *For* $x \in X$, $A_x^{\mathcal{C}} = \{(y, n) \in X \times \mathbb{N}; (x, y, n) \in A^{\mathcal{C}}\}$ *is finite*
- $\Delta_X \times \{1\} \subset A^{\mathcal{C}}$*, where* Δ_X *is the diagonal subset of* X^2*.*
- $|A_x^{\mathcal{C}} \Delta A_y^{\mathcal{C}}| < \epsilon |A_x^{\mathcal{C}} \cap A_y^{\mathcal{C}}|$ *if* $(x, y) \in T$*.*

Since there is a one to one correspondence between Coarse structures and Large Scale structures, it would benefit us to have an equivalent definition of property A for Large Scale structures with bounded geometry.

Definition 25. *Let* X *be a space and let* $\mathcal{LS}$ *be a Large Scale structure on* X *with bounded geometry.* $(X, \mathcal{LS})$ *is said to have Property A if for every* $\epsilon > 0$ *and every uniformly bounded family* $\mathcal{U} \in \mathcal{LS}$*, there exists* $\mathcal{V} \in \mathcal{LS}$ *and a family of finite subsets* $\{A_x^{\mathcal{LS}}\}$ *of* $X \times \mathbb{N}$ *such that*

- $A_x^{\mathcal{LS}} \subset st(x, \mathcal{V}) \times \mathbb{N}$
- $(x, 1) \in A_x^{\mathcal{LS}}$
- $|A_x^{\mathcal{LS}} \Delta A_y^{\mathcal{LS}}| < \epsilon |A_x^{\mathcal{LS}} \cap A_y^{\mathcal{LS}}|$ *if* $y \in st(x, \mathcal{U})$*.*

From an earlier proposition we have that coarse structures and large scale structures are equivalent constructions. As such for our definition to be useful it should be that if a uniformly locally finite coarse space $(X, \mathcal{C})$ has Property A, then the induced large scale structure should also have Property A and the other implication should hold as well. We will prove that this is indeed the case.

Theorem 3.1. *Let X be a large scale space with bounded geometry. A uniformly locally finite coarse space $(X, \mathcal{C})$ has Property A, then the induced large scale structure $(X, \mathcal{LS})$ has Property A. As well, if $(X, \mathcal{LS})$ has Property A, then the induced coarse structure $(X, \mathcal{C})$ also has Property A.*

Proof. First, let X be a large scale space, and suppose there exists a large scale structure, $\mathcal{LS}$ such that $(X, \mathcal{LS})$ has Property A. Let $\mathcal{C}$ be the coarse structure induced by $\mathcal{LS}$. Let $\epsilon > 0$ and $T \in \mathcal{C}$. Thus, there exists a uniformly bounded family $\mathcal{U} \in \mathcal{LS}$ with $T \subset \cup_{U \in \mathcal{U}} U \times U$. Let $\mathcal{V}$ be the uniformly bounded family satisfying the definition of Property A for the above ϵ and $\mathcal{U}$. Define $S := \cup_{V \in \mathcal{V}} V \times V$ which is obviously a controlled set. Define $A^{\mathcal{C}} := \cup_{x \in X} \{x\} \times A_x^{\mathcal{LS}}$.

Show $A^{\mathcal{C}} \subset S \times \mathbb{N}$: Given $(x, y, n) \in A^{\mathcal{C}}$ then $(x, y) \in \{x\} \times A_x^{\mathcal{LS}}$ or $y \in A_x^{\mathcal{LS}} \subset st(x, \mathcal{V})$ Thus, there exists a $V \in \mathcal{V}$ such that $x, y \in V$. Therefore $(x, y) \in V \times V$, and $A^{\mathcal{C}} \subset S \times \mathbb{N}$.

Show for $x \in X$, $A_x^{\mathcal{C}} = \{(y, n) \in X \times \mathbb{N}; (x, y, n) \in A\} = A_x^{\mathcal{LS}}$: Let $x \in X$ and consider $(y, n) \in A_x^{\mathcal{C}}$. By definition $(y, n) \in A_x^{\mathcal{LS}}$. Similarly, if $(z, m) \in A_x^{\mathcal{LS}}$, then $(x, z, m) \in A^{\mathcal{C}}$, and therefore $(z, m) \in A_x^{\mathcal{C}}$, so $A_x^{\mathcal{C}} = A_x^{\mathcal{LS}}$

By definition of $A_x^{\mathcal{LS}}$, and $A^{\mathcal{C}}$ we easily get

- For $x \in X$, $A_x^{\mathcal{C}} = \{(y, n) \in X \times \mathbb{N}; (x, y, n) \in A^{\mathcal{C}}\}$ is finite

- $\Delta_X \times \{1\} \subset A^{\mathcal{C}}$, where Δ_X is the diagonal subset of X^2.

The third property also follows easily as if $(x, y) \in T$, then there exists $U \in \mathcal{U}$ with $x, y \in U$. Thus, $y \in st(x, \mathcal{U})$, so we have

$$\frac{|A_x^{\mathcal{C}} \Delta A_y^{\mathcal{C}}|}{|A_x^{\mathcal{C}} \cap A_y^{\mathcal{C}}|} = \frac{|A_x^{\mathcal{LS}} \Delta A_y^{\mathcal{LS}}|}{|A_x^{\mathcal{LS}} \cap A_y^{\mathcal{LS}}|} < \epsilon$$

and $(X, \mathcal{C})$ has Property A.

Next, assume that $(X, \mathcal{C})$ has Property A then $(X, \mathcal{LS})$ has Property A, where $\mathcal{LS}$ is the large scale structure induced by $\mathcal{C}$. Let $\epsilon > 0$ and $\mathcal{U}$ be a uniformly bounded cover. Thus there exists a controlled set $T \in \mathcal{C}$ such that $\cup_{U \in \mathcal{U}} U \times U \subset T$. Let S be the controlled set and $A \subset S \times \mathbb{N}$ satisfying Property A for ϵ and T.

Define $\mathcal{V} = \{V_x\}_{x \in X}$ where $V_x = \{y \in X; (y, n) \in A_x, n \in \mathbb{N}\}$. It needs to be shown that $\mathcal{V}$ is uniformly bounded: given $(y, z) \in \cup_{V \in \mathcal{V}} V \times V$. Then, for some $x \in X, (x, y, n), (x, z, m) \in A^{\mathcal{C}} \subset S \times \mathbb{N}$ for some $n, m \in \mathbb{N}$. More importantly, $(x, y), (x, z) \in S$ meaning $(y, x) \in S^{-1}$ and thus $(y, z) \in S^{-1} \circ S$ which is a controlled set. Since (y, z) was chosen arbitrarily, we have that $\mathcal{V}$ is uniformly bounded.

Note that $st(x, \mathcal{V}) = \{y \in X; (y, n) \in A_z^{\mathcal{C}}$ and $(x, m) \in A_z^{\mathcal{C}}$ for $z \in X\}$. since $(x, 1) \in A_x^{\mathcal{C}}$, then $A_x^{\mathcal{C}} \subset st(x, \mathcal{V}$. Setting $A_x^{\mathcal{LS}} = A_x^{\mathcal{C}}$ it is easy to show:

- $A_x^{\mathcal{LS}} \subset st(x, \mathcal{V}) \times \mathbb{N}$
- $(x, 1) \in A_x^{\mathcal{LS}}$

are true. Finally, if $y \in st(x,\mathcal{U})$, then there exists $U \in \mathcal{U}$ such that $x, y \in U$ which means $(x,y) \in \cup_{U\in\mathcal{U}} U \times U \subset T$ and the following holds:

$$\frac{|A_x^{\mathcal{LS}} \Delta A_y^{\mathcal{LS}}|}{|A_x^{\mathcal{LS}} \cap A_y^{\mathcal{LS}}|} = \frac{|A_x^{\mathcal{C}} \Delta A_y^{\mathcal{C}}|}{|A_x^{\mathcal{C}} \cap A_y^{\mathcal{C}}|} < \epsilon$$

and $(X, \mathcal{LS})$ has Property A. □

Theorem 3.2. *Let X and Y be large scale spaces with bounded geometry with large scale structures $\mathcal{LS}_X$ and $\mathcal{LS}_Y$ respectively. Let $f : X \to Y$ be a large scale equivalence. If $(Y, \mathcal{LS}_Y)$ has property A, then $(X, \mathcal{LS}_X)$ also has property A.*

Proof. Let $\mathcal{U}$ be a uniformly bounded family of X and let $\epsilon > 0$ be given. Let $N = sup_{y\in Y}|f^{-1}(y)|$. Let $\mathcal{W}$ be the uniformly bounded family of Y and $\{B_x\}$ the finite subsets of $Y \times \mathbb{N}$ from the definition of property A for $f(\mathcal{U})$ and $\frac{\epsilon}{N}$. Finally, let $\mathcal{V} = f^{-1}(\mathcal{W})$ and for each $x \in X$ define

$$A_x = \{(z,n) \in X \times \mathbb{N} | (f(z), n) \in B_{f(x)}\}.$$

From the definition of $\mathcal{V}$ it holds trivially that $A_x \subset st(x, \mathcal{V}) \times \mathbb{N}$. As well, it is clear that $(x, 1) \in A_x$.

Finally, we check that if $y \in st(x,\mathcal{U})$ then $|A_x \triangle A_y| < \epsilon |A_x \cap A_y|$. If $y \in st(x,\mathcal{U})$, then $f(y) \in st(f(x), f(\mathcal{U}))$, and

$$|B_{f(x)} \triangle B_{f(y)} < \frac{\epsilon}{N} |B_{f(x)} \cap B_{f(y)}|.$$

From the construction of A_x's it follows that $|A_x \triangle A_y| \leq N|B_{f(x)} \triangle B_{f(y)}|$ and $|A_x \cap A_y| \geq |B_{f(x)} \cap B_{f(y)}|$. Putting this all together we get

$$|A_x \triangle A_y| \leq N|B_{f(x)} \triangle B_{f(y)}| \leq N\frac{\epsilon}{N}|B_{f(x)} \cap B_{f(y)}| \leq \epsilon|A_x \cap A_y|.$$

and therefore $(X, \mathcal{LS}_X)$ has property A. $\square$

Since these two definitions are equivalent, we will drop the superscripts from the collections $\{A_x\}_{x \in X}$. The large scale definition of property A allows us to approach relationships with other coarse properties in a more Topological way. As such, we will use this definition to show that finite asymptotic dimension implies the large scale definition of property A.

Asymptotic Dimension can be thought of as a large scale geometric version of he covering dimension. One of the first useful definitions involved the concept of R-multiplicity of a scale on a metric space. For a scale $\mathcal{U}$, the R-multiplicity of $\mathcal{U}$ on a metric space is the smallest positive integer n such that for every $x \in X$ the ball $B(x, R)$ intersects at most n elements of $\mathcal{U}$.

Definition 26. *Suppose that X is a metric space. The asymptotic dimension of X is the smallest positive integer n such that for every $R > 0$ there exists a uniformly bounded cover $\mathcal{U}$ with R-multiplicity $n+1$. It is denoted by asdim$X = n$.*

From there, asymptotic dimension was generalized for large scale structures.

Definition 27. *Let X be a large scale space with large scale structure $\mathcal{LS}$. We say $asdim(X, \mathcal{LS}) \leq n$ if for every uniformly bounded cover $\mathcal{U}$ in $\mathcal{LS}$ there exists a uniformly bounded cover $\mathcal{V}$ such that $\mathcal{V}$ is a coarsening of $\mathcal{U}$ with multiplicity at most $n+1$.*

Since the trivial cover of X is uniformly bounded, this definition gives us that there exists a uniformly bounded cover $\mathcal{B}$ such that each $x \in X$ is contained in at most $n+1$ elements of $\mathcal{B}$. It can be shown that $asdim(X, \mathcal{LS}) \leq n$ if $\mathcal{LS}$ can be generated by a uniformly bounded family $\mathcal{B}$ such that the multiplicity of $\mathcal{B}$ is at most $n+1$ [DH06]. In [CDV15] a different characterization of asymptotic dimension was given.

Theorem 3.3. *The following conditions are equivalent for any coarse space X and any integer $N \geq 0$:*

1. *$asdim X \leq n$*
2. *for every $\epsilon > 0$ and every uniformly bounded cover $\mathcal{U}$ of X there is a $(\mathcal{U}, \epsilon)$-partition of unity $f : X \to K^{(n)}$.*
3. *for every uniformly bounded cover $\mathcal{U}$ of X there is a $(\mathcal{U}, \infty)$-partition of unity $f : X \to K^{(n)}$.*

As for the metric case, we will show that finite asymptotic dimension for coarse spaces implies the general large scale definition of property A. First we will introduce some terminology and a lemma. Given a uniformly bounded cover $\mathcal{U}$ for all positive integers n, define $st^n(\mathcal{U})$ in the following way:

$st^0(\mathcal{U}) = \mathcal{U}$

$st^n(\mathcal{U}, st^{n-1}(\mathcal{U}))$

Lemma 3.3.1. *If $y \in st(x,\mathcal{U})$, then $st(y, st^{n-1}(\mathcal{U})) \subseteq st(x, st^n(\mathcal{U}))$.*

Proof. Let $z \in st(y, st^{n-1}(\mathcal{U}))$. Then there exists $U \in \mathcal{U}$ such that $z.y \in st(U, st^{n-2}(\mathcal{U}))$. Since $y \in st(x,\mathcal{U})$ there exists $U' \in \mathcal{U}$ with $x, y \in U'$. Therefore we have

$$U' \cap st(U, st^{n-2}(\mathcal{U})) \neq \emptyset$$

and thus $z \in st(U, st^{n-2}(\mathcal{U})) \subset st(U', st(\mathcal{U}, st^{n-2}(\mathcal{U})))$.

Putting this all together we then have

$$x, z \in st(U', st(\mathcal{U}, st^{n-2})) = st(U', st^{n-1}(\mathcal{U})) \subseteq st^n(\mathcal{U})$$

which shows $z \in st(x, st^n(\mathcal{U}))$. □

Theorem 3.4. *Let $(X, \mathcal{LS})$ be a large scale space. If $asdim(X, \mathcal{LS})$ is finite, then $(X, \mathcal{LS})$ has property A.*

Proof. Let $asdim(X, \mathcal{LS}) = k$. Let $\epsilon > 0$ be given and $\mathcal{U}$ an arbitrary uniformly bounded family. Then $st(\{x\}_{x\in X}, st^n(\mathcal{U}))$ is a uniformly bounded family. From the definition of finite asymptotic dimension, let $\mathcal{V}$ be a uniformly bounded family such that $st(\{x\}_{x\in X}, st^n(\mathcal{U})) \prec \mathcal{V}$ with multiplicity at most $k+1$. For each $V \in \mathcal{V}$ arbitrarily pick$z \in V$ and denote it z_V. and define a uniformly bounded family $\mathcal{W} = st(st(\{x\}_{x\in X}, st^n(\mathcal{U})), \mathcal{V})$

For each $x \in X$ define A_x in the following way:

$$A_x := \{(x, 1)\} \cup \{(z_V, m) \mid st(x, st^m(\mathcal{U})) \cap V \neq \emptyset, V \not\subset st(x, st^m(\mathcal{U}), 1 \leq m \leq n\}.$$

Then each A_x is finite due to the multiplicity of $\mathcal{V}$ against $st(x, st^n(\mathcal{U}))$ is at most $k+1$ for all $x \in X$. As well, it follows simply that $A_x \subseteq st(x, \mathcal{W}) \times \mathbb{N}$ and $(x, 1) \in A_x$. All that remains is to check what happens when $y \in st(x,\mathcal{U})$.

Since $st(\{x\}_{x\in X}, st^n(\mathcal{U})) \prec \mathcal{V}$ there exists $V \in \mathcal{V}$ with $st(x, st^n(\mathcal{U})) \subseteq V$. From the construction of A_x for this fixed $V \in mathcalV$ we have $(z_V, m) \in A_x$ for $1 \leq m \leq n$. Additionally, we have $y \in V$ and from our lemma we have $(z_V, m) \in A_y$ for $1 \leq m \leq n-1$. Therefore, $|A_x \cap A_y| \geq n - 1$.

On the other hand, consider $(z_V, l) \in A_x \triangle A_y$. Without loss of generality suppose $(z_V, l) \in A_x \setminus A_y$. Therefore we know that $V \cap st(x, st^l(\mathcal{U})) \neq \emptyset$ and $V \not\subset st(x, st^l(\mathcal{U}))$. On the other hand, we know that either $V \cap st(y, st^l(\mathcal{U})) = \emptyset$ or $V \subset st(y, st^l(\mathcal{U}))$.

First suppose that $V \cap st(y, st^l(\mathcal{U})) = \emptyset$. From the previous lemma we have that $st(x, st^l(\mathcal{U})) \subset st(y, st^{l+1}(\mathcal{U}))$, so $V \cap st(y, st^{l+1}(\mathcal{U})) \neq \emptyset$. On the other hand, suppose $V \subset st(y, st^l(\mathcal{U}))$. Again, by the above lemma we have $V \subset st(y, st^l(\mathcal{U})) \subseteq st(x, st^{l+1}(\mathcal{U}))$. Therefore any (z_V, l) can only be contained in $A_x \setminus A_y$ for at most 2 distinct values of l. Each A_x can only have at most $k+1$ distinct z_V's so $|A_x \triangle A_y| \leq 2(2(k+1)+1)$. Putting both inequalities together we get that

$$\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} \leq \frac{4k+6}{n-1}$$

the numerators is independent of our choice of n, so to all we have to do is choose an n great enough so that $\frac{4k+6}{n-1} < \epsilon$ to satisfy the inequality. Therefore the collection $\{A_x\}_{x\in X}$ satisfies the requirements and $(X, \mathcal{LS})$ has property A. $\square$

3.2 Coarse Exactness

For metric space, exactness was introduced by Dardalet and Guentner to replace property A. It has been shown that for metric spaces with bounded geometry, exactness and property A are equivalent [HMS14].

Definition 28. *(Exact Metric Space) A metric space X is **exact** if $\forall R > 0$ and $\epsilon > 0 \exists$ a partition of unity $(\phi_i)_{i\in I}$ on X subordinated to a cover $\mathcal{V} = (V_i)_{i\in I}$ such that:*

- *$\forall x, y \in X$ with $d(x, y) \leq R$, $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$.*
- *The cover $\mathcal{V} = (V_i)_{i\in I}$ is a uniformly bounded family.*

Since a Large Scale Space need not be a metric space, we need a new definition of exact that only relies on uniformly bounded families.

Definition 29. *A space X with a large scale structure is **coarse exact** if for every uniformly bounded covers $\mathcal{U}$ and $\epsilon > 0$ there exists a partition of unity $(\phi_i)_{i\in I}$ on X subordinate to a cover $\mathcal{V} = (V_i)_{i\in I}$ such that:*

- *$\forall x, y \in X$ with $x, y \in U$ for $U \in \mathcal{U}$, $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$.*
- *The cover $\mathcal{V} = (V_i)_{i\in I}$ is a uniformly bounded family.*

If X is a metric space, the general definition of exactness will imply the original definition. However, we can now show implications between general coarse properties.

Theorem 3.5. *Let X be an ls-space with coarse amenability. Then X is also a coarse exact space.*

Proof. Let X be a space with coarse amenability. Given a uniformly bounded cover, $\mathcal{V}$ and $\epsilon > 0$, define a uniformly bounded cover $\mathcal{U}$ of X such that $hor(st(x,\mathcal{V}),\mathcal{U})$ is finite and

$$\frac{|hor(x,\mathcal{U})|}{|hor(st(x,\mathcal{V}),\mathcal{U})|} > 1 - \frac{\epsilon}{4}$$

. Define $\psi^{\mathcal{V}}_{U_s}$ to be the following function:

$$\psi^{\mathcal{V}}_{U_s}(x) = \left\{ \begin{array}{c} 0, U_s \cap st(x,\mathcal{V}) = \emptyset \\ \\ 1, U_s \cap st(x,\mathcal{V}) \neq \emptyset \end{array} \right\}$$

Then define the partition of unity to be the following:

$$\phi_s(x) = \frac{\psi^{\mathcal{V}}_{U_s}(x)}{\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)}$$

Suppose $x, y \in V$ for $V \in \mathcal{V}$.

$$\sum_{s\in S} |\phi_s(x) - \phi_s(y)| = \sum_{s\in S} |\frac{\psi^{\mathcal{V}}_{U_s}(x)}{\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)} - \frac{\psi^{\mathcal{V}}_{U_s}(y)}{\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)}| =$$

$$= \sum_{s\in S} \frac{|\psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)] - \psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)] + \psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)] - \psi^{\mathcal{V}}_{U_s}(y)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)]|}{|(\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x))(\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y))|} \leq$$

$$\leq \sum_{s\in S} \frac{|\psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)] - \psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)]|}{|[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)]|} + \sum_{s\in S} \frac{\psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)] - \psi^{\mathcal{V}}_{U_s}(y)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)]|}{|[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)]|} =$$

$$= \sum_{s\in S} \frac{|\psi^{\mathcal{V}}_{U_s}(x)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x) - \psi^{\mathcal{V}}_{U_t}(y)]|}{|[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)]|} + \sum_{s\in S} \frac{|\psi^{\mathcal{V}}_{U_s}(x) - \psi^{\mathcal{V}}_{U_s}(y)[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)]|}{|[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)]|} \leq$$

$$\leq \frac{[\sum_{s\in S} \psi^{\mathcal{V}}_{U_s}(x)][\sum_{t\in S} |\psi^{\mathcal{V}}_{U_t}(x) - \psi^{\mathcal{V}}_{U_t}(y)|]}{|[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)]|} + \frac{[\sum_{s\in S} |\psi^{\mathcal{V}}_{U_s}(x) - \psi^{\mathcal{V}}_{U_s}(y)|][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)]}{|[\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(x)][\sum_{t\in S} \psi^{\mathcal{V}}_{U_t}(y)]|} =$$

$$= \frac{2\sum_{s\in S}|\psi^{\mathcal{V}}_{U_s}(x) - \psi^{\mathcal{V}}_{U_s}(y)|}{\sum_{t\in S}\psi^{\mathcal{V}}_{U_t}(y)}$$

$|\psi^{\mathcal{V}}_{U_s}(x) - \psi^{\mathcal{V}}_{U_s}(y)| = 1$ in the following two cases.

- $U_s \cap st(x,\mathcal{V}) \neq \emptyset$ and $U_s \cap st(y,\mathcal{V}) = \emptyset$ $(x \in st(y,\mathcal{V}))$

- $U_s \cap st(x,\mathcal{V}) = \emptyset$ and $U_s \cap st(y,\mathcal{V}) \neq \emptyset$ $(y \in st(x,\mathcal{V}))$

Thus either $U_s \in \{U \mid U \cap st(x,\mathcal{V}) \neq \emptyset, x \notin U, U \in \mathcal{U}\}$ or $U_s \in \{U \mid U \cap st(y,\mathcal{V}) \neq \emptyset, y \notin U, U \in \mathcal{U}\}$. Note that $|\{U \mid U \cap st(x,\mathcal{V}) \neq \emptyset, x \notin U, U \in \mathcal{U}\}| = |hor(st(x,\mathcal{W}),\mathcal{U})| - |hor(x,\mathcal{U})|$.

Without loss of generality let $|hor(st(y,\mathcal{V}),\mathcal{U})| - |hor(y,\mathcal{U})| \geq |hor(st(x,\mathcal{V}),\mathcal{U})| - |hor(x,\mathcal{U})|$. We then have the following inequality:

$$\frac{2\sum_{s\in S}|\psi^{\mathcal{V}}_{U_s}(x) - \psi^{\mathcal{V}}_{U_s}(y)|}{\sum_{t\in S}\psi^{\mathcal{V}}_{U_t}(y)} \leq \frac{4(|hor(st(y,\mathcal{V}),\mathcal{U})| - |hor(y,\mathcal{U})|)}{|hor(st(y,\mathcal{V}),\mathcal{U})|} < 4\frac{\epsilon}{4} = \epsilon$$

And therefore X is an exact space as the support of ϕ_s is contained in $U_s \in \mathcal{U}$, a uniformly bounded family. □

Theorem 3.6. *Let X be a large scale space with bounded geometry. If $(X, \mathcal{LS})$ has Property A, then X is a coarse exact space.*

Proof. Let X be a space with a large scale structure and bounded geometry. Let $\epsilon > 0$ and $\mathcal{U}$ be a uniformly bounded cover. Thus, for $\frac{\epsilon}{2}$ and $\mathcal{U}$, let $\mathcal{V}$ be the uniformly bounded cover such that the following holds:

- $A^{\mathcal{LS}}_x \subset st(x,\mathcal{V}) \times \mathbb{N}$

- $(x, 1) \in A_x^{\mathcal{LS}}$

- $|A_x^{\mathcal{LS}} \Delta A_y^{\mathcal{LS}}| < \epsilon |A_x^{\mathcal{LS}} \cap A_y^{\mathcal{LS}}|$ if $y \in st(x,\mathcal{U})$.

Define a function $\psi_x : X \to \mathbb{R}$ to be the following:

$$\psi_x(y) = |\{x\} \times \mathbb{N} \cap A_y|$$

Then define $\phi_x(y) = \frac{\psi_x(y)}{\sum_{x\in X} \psi_x(y)}$.

We claim that $supp(\phi_x)$ is a uniformly bounded family

to prove this, define $B_x = \{y \in X; (y, n) \in A_x\}$ for some $n \in \mathbb{N}$ Since $st(\{x\}_{x\in X}, \mathcal{V})$ is a uniformly bounded cover and $A_x \subset st(x, \mathcal{V}) \times \mathbb{N}$, we know the collection $\{B_x\}_{x\in X}$ is a uniformly bounded cover of X. Next note that $supp(\phi_x) = \{y \in X; x \in B_y\} \subset st(x, \{B_x\}_{x\in X})$ since $y \in B_y$. Therefore each $supp(\phi_x)$ is contained in $st(x, \{B_x\}_{x\in X})$ which implies that $\{supp(\phi_x)\}_{x\in X}$ is a uniformly bounded family.

Next, we claim that $\{\phi_x\}_{x\in X}$ is a partition of unity.

Clearly $\sum_{z\in x} \frac{\psi_z(x)}{\sum_{z\in X}\psi_z(x)} = \frac{|A_x|}{|A_x|} = 1$. As well, the support of each function is finite.

Finally, we will show that $\forall x, y \in X$ with $x, y \in U$ for $U \in \mathcal{U}$, $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$.

Given $x, y \in U \in \mathcal{U}$, then $y \in st(x,\mathcal{U})$. Consider the following inequalities: $\sum_{z\in X} |\phi_z(x) - \phi_z(y)| = \sum_{z\in X} |\frac{\psi_z(x)}{\sum_{z\in X}\psi_z(x)} - \frac{\psi_z(y)}{\sum_{z\in X}\psi_z(y)}|$

$\leq \sum_{z\in X} \frac{|\psi_z(x)[\sum_{z\in X}\psi_z(y)] - \psi_z(x)[\sum_{z\in X}\psi_z(x)]|}{|[\sum_{z\in X}\psi_z(x)][\sum_{z\in X}\psi_z(y)]|} + \sum_{z\in X} \frac{\psi_z(x)[\sum_{z\in X}\psi_z(x)] - \psi_z(y)][\sum_{z\in X}\psi_z(x)]|}{|[\sum_{z\in X}\psi_z(x)][\sum_{z\in X}\psi_z(y)]|} =$

$\leq \frac{[\sum_{z\in X}\psi_z(x)][\sum_{z\in X}|\psi_z(x) - \psi_z(y)|]}{|[\sum_{z\in X}\psi_z(x)][\sum_{z\in X}\psi_z(y)]|} + \frac{[\sum_{z\in X}|\psi_z(x) - \psi_z(y)|][\sum_{z\in X}\psi_z(x)]}{|[\sum_{z\in X}\psi_z(x)][\sum_{z\in X}\psi_z(y)]|} =$

$$= \frac{2\displaystyle\sum_{z\in X} |\psi_z(x) - \psi_z(y)|}{\displaystyle\sum_{z\in X} \psi_z(y)}$$

First of all, Note that $\sum_{z\in X}\psi_z(x) = |A_x|$. Secondly, for a fixed $z \in X$, $|\psi_z(x) - \psi_z(y)| = ||\{z\} \times \mathbb{N} \cap A_x| - |\{z\} \times \mathbb{N} \cap A_y|| \leq |\{z\} \times \mathbb{N} \cap A_x \Delta A_y|$

Thus, the above inequality becomes

$$2\frac{\sum_{z\in X}|\psi_z(x) - \psi_z(y)|}{\sum_{z\in X}\psi_z(y)} \leq 2\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} \leq 2\frac{\epsilon}{2} = \epsilon$$

and therefore, if $x, y \in U \in \mathcal{U}$, $|\phi_i(x) - \phi_i(y)| \leq \epsilon$.

Therefore, Property A implies Coarse Exactness.

$\square$

Our next goal is to show that if a large scale space with bounded geometry is a coarse exact space, then it has coarse ULA_μ. However, before we prove this, we will first prove some basic properties of coarse uniform local amenability with respect to all probability measures.

Lemma 3.6.1. *Coarse uniform local amenability with respect to probability measures is equivalent to the following definition:*

For each $\mathcal{U}, \epsilon > 0$ there exists $\mathcal{V}$ such that for all probability measures μ on X with finite support there exists a set $E \subset X$ such that:

- *E is $\mathcal{V}$-bounded*
- *the following inequality hold*

$$\mu(\partial_{\mathcal{U}}(E)) < \epsilon \cdot \mu(E)$$

Proof. This follows easily from the proof of coarse equivalence between coarse sparsification and coarse ULA_μ. $\square$

Lemma 3.6.2. *Let X be a space with a large scale structure. If for all uniformly bounded covers $\mathcal{U}$ and all $\epsilon > 0$ there exists a uniformly bounded cover $\mathcal{V}$ such that for all probability measures μ on X there exists a function $l^1(X)$ such that the following holds:*

- *diam(supp(ϕ)) is $\mathcal{V}$-bounded.*
- *the following inequality holds*

$$\sum_{x \in supp(\mu)} \mu(x) \sum_{y \in supp(\mu), y \in st(x, \mathcal{U})} |\phi(x) - \phi(y)| < \epsilon \sum_{s \in supp(\mu)} \mu(x)|\phi(x)|$$

Then X has Coarse Uniform local amenability with respect to probability measures

Proof. Let $\mathcal{U}$ be a uniformly bounded cover, $\epsilon > 0$ and let μ be a probability measure. Thus, there exists a uniformly bounded cover, $\mathcal{V}$, independent of μ, and there exists $\psi \in l^1(X)$ satisfying the conditions above. Define $\phi = |\psi|$ so that ϕ is a non-negative function. Take $F_1 = supp(\phi)$ and define a sequence of subsets $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n$ such that $\phi = \sum_{i=1}^n a_i \chi_{F_i}$ for a_i a non-negative number, and χ the indicator function. Thus we can rewrite our inequality to be:

$$\sum_{i=1}^{n} \sum_{x \in supp(\mu)} \mu(x) \sum_{y \in supp(\mu), y \in st(x, \mathcal{U})} |\chi_{F_i}(x) - \chi_{F_i}(y)| < \epsilon \sum_{i=1}^{n} a_i \sum_{x \in supp(\mu)} \mu(x)|\chi_{F_i}(x)|$$

Therefore, for some fixed i, the following inequality must hold.

$$\sum_{x\in supp(\mu)} \mu(x) \sum_{y\in supp(\mu), y\in st(x,\mathcal{U})} |\chi_{F_i}(x) - \chi_{F_i}(y)| < \epsilon \sum_{x\in supp(\mu)} \mu(x)|\chi_{F_i}(x)| = \epsilon\mu(F_i)$$

Notice that if $x \in \partial_{\mathcal{U}}(F_i)$, then $x \in V_j \setminus F_i$ with $V_j \cap F_i \neq \emptyset$. Choose $y \in V_j \cap F_i$ and thus $\mu(x)|\chi_{F_i}(x) - \chi_{F_i}(y)| = \mu(x)$. If we set $E = F_i$ we get that

$$\mu(\partial_{\mathcal{U}}(E)) \leq \sum_{x\in supp(\mu)} \mu(x) \sum_{y\in supp(\mu), y\in st(x,\mathcal{U})} |\chi_{F_i}(x) - \chi_{F_i}(y)| < \epsilon\mu(x)$$

Putting this together with the fact that $F_i \subset supp(\phi)$, which is $\mathcal{V}$-bounded, we have that X has coarse ULA_μ. □

Theorem 3.7. *Let X be a space with a large scale structure and bounded geometry. Then if X is a Coarse Exact space, it has Coarse Uniform Local Amenability for all probability measures.*

Proof. Let X be a space with a large scale structure. Given a uniformly bounded cover $\mathcal{U}$ and $\epsilon > 0$. Then there exists a partition of unity on X, $(\phi)_{i\in I}$, subordinated to a cover $\mathcal{V} = (V_i)_{i\in I}$, such that:

1. $\forall x, y \in X$ with $x, y \in U$ for $U \in \mathcal{U}$, $\sum_{i\in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$. 2. The cover $\mathcal{V} = (V_i)_{i\in I}$ is a uniformly bounded family.

Let μ be a probability measure with finite support, and for $\mathcal{V}$ let M be the maximum cardinality of any element of $st(\mathcal{U},\mathcal{U})$.

We have the following inequalities:

$$\sum_{i\in I}\sum_{x\in supp(\mu)}\mu(x)\sum_{y\in supp(\mu),y\in st(x,\mathcal{U})}|\phi_i(x)-\phi_i(y)| = \sum_{x\in supp(\mu)}\mu(x)\sum_{y\in supp(\mu),y\in st(x,\mathcal{U})}\sum_{i\in I}|\phi_i(x)-\phi_i(y)| <$$

$$< \sum_{x\in supp(\mu)}\mu(x)\sum_{y\in supp(\mu),y\in st(x,\mathcal{U})}\epsilon \leq M\epsilon\sum_{x\in supp(\mu)}\mu(x) = \sum_{i\in I}M\epsilon\sum_{x\in supp(\mu)}\mu(x)|\phi_i(x)|.$$

Therefore, for some fixed $i\in I$ the following inequality must hold:

$$\sum_{x\in supp(\mu)}\mu(x)\sum_{y\in supp(\mu),y\in st(x,\mathcal{U})}|\phi_i(x)-\phi_i(y)| < M\epsilon\sum_{x\in supp(\mu)}\mu(x)|\phi_i(x)|.$$

$\square$

Corollary 3.7.1. *Let X be a large scale space with bounded geometry. If $(X,\mathcal{LS})$ has Property A, then X has Coarse Uniform Local Amenability with respect to all probability measures.*

Chapter 4

Coarse Generalization of Bounded Geometry

4.1 Bounded Scale Measure

Definition 30. *Given a scale $\mathcal{U}$ on X and given a subset A of X, an **$\mathcal{U}$-net** in A is a maximal subset B of A with respect to the following property: no two elements of B belong to the same element of the scale $\mathcal{U}$.*

Definition 31. *X has **Bounded Scale Measure** if there exists a scale $\mathcal{U}$ such that for all scales $\mathcal{V}$ there exists a natural number $u(\mathcal{V})$ such that for all $V \in \mathcal{V}$ any $\mathcal{U}$-net of V has at most $u(\mathcal{V})$ elements.*

Theorem 4.1. *X has Bounded Scale Measure if and only if there exists a scale $\mathcal{U}$ such that for all scales $\mathcal{V}$ there exists a natural number $n(\mathcal{V})$ so that for all $V \in \mathcal{V}$ there is a $\mathcal{U}$-net of V which has at most $n(\mathcal{V})$ elements.*

Proof. First suppose there exists a scale $\mathcal{U}$ such that for all scales $\mathcal{V}$ there exists a natural number $u(\mathcal{V})$ such that for all $V \in \mathcal{V}$ any $\mathcal{U}$-net of V has at most $u(\mathcal{V})$ elements. Let $\mathcal{V}$ be a scale of X and let $V \in \mathcal{V}$. Take $x \in V$ and then choose all points of X $x = x_1, x_2, ..., x_k$ such that if $i \neq j$ $\{x_i, x_j\} \notin U$ for all $U \in \mathcal{U}$. By construction this is a $\mathcal{U}$-net in V, and thus $k \leq u(\mathcal{V})$ as otherwise we have a $\mathcal{U}$-net with more than $(\mathcal{V})$ elements. Thus taking our $n(\mathcal{V}) = u(\mathcal{V})$ and the collection $\{x_i\}_{1 \leq i \leq k}$ satisfies our condition.

Conversely, suppose there exists a scale $\mathcal{W}$ such that for all scales $\mathcal{V}$ there exist a natural number $n(\mathcal{V})$ such that for all $V \in \mathcal{V}$ there is a $\mathcal{W}$-net, B, which has at most $n(\mathcal{V})$ elements. Define $\mathcal{U} = st(\mathcal{W}, \mathcal{W})$, and let $\mathcal{V}$ be any scale of X. Let A be any $\mathcal{U}$-net in V and B be the $\mathcal{W}$-net with $|B| \leq n(\mathcal{V})$.

We claim that $|A \setminus B| \leq n(\mathcal{V})$. By contradiction assume $|A \setminus B| > n(\mathcal{V})$. By maximality of B, given any $x_i \in A \setminus B$, there exists $y_i \in B$ such that $\{x_i, y_i\} \subset W_i$ for $W_i \in \mathcal{W}$. If for $i \neq j$, $y_i = y_j$, then $\{x_i, x_j\} \subset st(W_i, \mathcal{W})$ which is a contradiction. Take a collection $\{x_i\}$ of $n(\mathcal{V})$ elements of $A \setminus B$, and associate with each x_i the unique $y_i \in B$ and form the collection $\{y_i\} = B$. Since $|A \setminus B| > n(\mathcal{V})$ there exists $x \in (A \setminus B) \setminus \{x_i\}_{1 \leq i \leq n(\mathcal{V})}$, but by construction given any $y \in B$ $\{x, y\} \notin W$ for all all $W \in \mathcal{W}$. Otherwise, if there exists a $y \in B$ with $\{x, y\} \subset W$, then $y = y_i$ for some i and $\{x, x_i\} \subset st(W_i, \mathcal{W})$. This means B is maximal and therefore not a $\mathcal{W}$-net, which is a contradiction. Thus $|A \setminus B| \leq n(\mathcal{V})$, and any $\mathcal{U}$-net must have less than $2n(\mathcal{V}) + 1$ elements which satisfies Definition. □

Theorem 4.2. *A space X has Bounded Scale Measure if and only if there exists a scale $\mathcal{U}$ such that for all scales $\mathcal{V}$ there exists a natural number $k(\mathcal{V})$ such that for all $V \in \mathcal{V}$, $V \subseteq \cup_{i=1}^{k(\mathcal{V})} U_i$ for a collection $U_i \in \mathcal{U}$.*

Proof. Let$X$ have Bounded Scale Measure, and let $\mathcal{V}$ be a scale of X. Let $\mathcal{U}$ and $u(\mathcal{V})$ be defined as they are in Definition 1. Define $\mathcal{W} = st(\mathcal{U}, \mathcal{U})$. Take a $\mathcal{U}$-net of $V \in \mathcal{V}$, and call it B. We know $|B| \leq u(\mathcal{V})$. For each $b \in B$, pick $U_b \in \mathcal{U}$ such that $b \in U_b$. Define $W_b = st(U_b, \mathcal{U})$.

First, we claim that $V \subseteq \cup_{i=1}^{u(\mathcal{V})} W_{b_i}$.

Let $x \in V$. Then if $x \in B$, by construction there exists a $U_x \in \mathcal{U}$ such that $x \in U_x \subseteq \cup_{i=1}^{u(\mathcal{V})} W_{b_i}$. If $x \notin B$ then by maximality of B there exists $b \in B$ such that $\{x, b\} \subset U$ for some $U \in \mathcal{U}$. This means $x \in st(U_b, \mathcal{U})$ and $x \in \cup_{i=1}^{u(\mathcal{V})} W_{b_i}$. Therefore, $V \subseteq \cup_{i=1}^{u(\mathcal{V})} W_{b_i}$ Letting $\mathcal{W}$ be our designated scale and letting $k(\mathcal{V}) = u(\mathcal{V})$ satisfies our condition.

Conversely, suppose there exists a scale $\mathcal{U}$ such that for all scales $\mathcal{V}$ there exists a natural number $k(\mathcal{V})$ such that for all $V \in \mathcal{V}$, $V \subset \cup_{i=1}^{k(\mathcal{V})} U_i$ for a collection $U_i \in \mathcal{U}$. Let $\mathcal{V}$ be an arbitrary scale of X. Let$V \in \mathcal{V}$ and let B be a $\mathcal{U}$-net of V.

We claim that $|B| \leq k(\mathcal{V})$. By contradiction assume $|B| > k(\mathcal{V})$. We know there exists $U_i \in \mathcal{U}$ such that $V \subseteq \cup_{i=1}^{k(\mathcal{V})} U_i$. Given any $x, y \in B$, then $x, y \notin U_i$ for all $1 \leq i \leq k(\mathcal{V})$, but each $x \in B$ must be in some U_i as the U_i's cover V. For each U_i, assign the unique $x_i \in B$ such that $x_i \in U_i$. By assumption $|B| > k(\mathcal{V})$, so there exists $x \in B \setminus \{x_i\}_{1 \leq i \leq k(\mathcal{V})}$. Since B is a $\mathcal{U}$-net, then $x \notin U_i$ for all i with $1 \leq i \leq k(\mathcal{V})$. However this means $x \notin V$ and thus B is not a $\mathcal{U}$-net which is a contradiction. Therefore for any $\mathcal{U}$-net in V, $|B| \leq k(\mathcal{V})$. Setting $u(\mathcal{V}) = k(\mathcal{V})$ and letting $\mathcal{U}$ from Definition 3 be our designated scale satisfies Definition. $\square$

Proposition 4.2.1. *Let X be a space with bounded scale measure. Then any subset of X also has bounded scale measure.*

Proof. Let X be a space with bounded scale measure, and let Y be a subset of X. Let $\mathcal{U}'$ be the scale of X. Define a new scale of Y to be $\mathcal{U} = \{U' \cap Y | U' \in \mathcal{U}'\}$. Let $\mathcal{V}$ be a scale of Y and for $V \in \mathcal{V}$, let B_V be a $\mathcal{U}$-net of V. Define $\mathcal{V}'$ to be the trivial extension of $\mathcal{V}$ to a scale of X and $u(\mathcal{V}')$ be the natural number from the definition of bounded geometry for X. If $V' \subset X \setminus Y$, then V must be a point and a $\mathcal{U}'$-net must also be a point. Otherwise V' is an element of $\mathcal{V}$, and a $\mathcal{U}$-net of V' is also a $\mathcal{U}'$-net of V'. This is because given any $x, y \in B_{V'} \subset Y$, we know that for all $U' \in \mathcal{U}', x, y \notin U' \cap Y$. Note that $x, y \in Y$, so therefore $x, y \notin U'$ for all $U' \in \mathcal{U}'$. Therefore, for any scale $\mathcal{V}$ of Y, a $\mathcal{U}$-net B is a $\mathcal{U}'$-net and by assumption, $|B| \leq u(\mathcal{V}')$. Thus for any scale $\mathcal{V}$ of Y, setting $u(\mathcal{V}) = u(\mathcal{V}')$, where $\mathcal{V}'$ is the trivial extension of $\mathcal{V}$ over X, shows Y has bounded scale measure. $\square$

Theorem 4.3. *Let X and Y be ls-spaces and let $f : X \to Y$ be a large scale equivalence. Then X has Bounded Scale Measure if and only if Y has Bounded Scale Measure.*

Proof. Let X be a space with bounded scale measure. Let $\mathcal{U}$ be the scale from the definition of bounded geometry. Let$f : X \to Y$ be a scale equivalence. Therefore $f(\mathcal{U})$ is a scale in Y. Now, let $\mathcal{V}$ be any scale of Y. For arbitrary $V \in \mathcal{V}$ let B_V be a $f(\mathcal{U})$-net of V. We know $f^{-1}(\mathcal{V})$ is a scale in X. We will show $f^{-1}(B_V)$ is a $\mathcal{U}$-net of $f^{-1}(V)$. By contradiction assume there exists $x, y \in f^{-1}(B_V)$ such that there is some $U \in \mathcal{U}$ with $x, y \in U$. However, this implies $f(x), f(y) \in f(U)$, but $f(x), f(y) \in B_V$ which contradicts that B_V is a $f(\mathcal{U})$-net. Next assume there is some $z \in f^{-1}(V) \setminus f^{-1}(B_V)$ such that for all $x \in B_V$ there does not exist a $U \in \mathcal{U}$ with $x, z \in U$. However, this implies there is not $f(U) \in f(\mathcal{U})$ with $f(x), f(z) \in f(U)$. By maximality of B_V, $f(z) \in B$ which contradicts that fact that $z \notin f^{-1}(B_V)$. Therefore $f^{-1}(B_V)$ is a $\mathcal{U}$-net of $f^{-1}(V)$. Since X has bounded geometry we know there exists a natural number $u(f^{-1}(\mathcal{V}))$ such that $|f^{-1}(B_V)| \leq u(f^{-1}(\mathcal{V}))$. As well, since f is a function $|B_V| \leq |f^{-1}(B_V)|$. Finally, setting $u(\mathcal{V}) = u(f^{-1}(\mathcal{V}))$ gives us that for any $\mathcal{U}$-net of $V \in \mathcal{V}$ has at most $u(\mathcal{V})$ elements and thus Y has bounded scale measure.

Conversely, let Y be a space with Bounded Scale Measure and let $f : X \to Y$ be a large scale equivalence function. Let $\mathcal{U}$ be a scale in Y such that for all scales $\mathcal{V}$ of Y there exists a natural number $k(\mathcal{V})$ so that for all $V \in \mathcal{V}$, $V \subseteq \cup_{i=1}^{k(\mathcal{V})} U_i$ for a collection $U_i \in \mathcal{U}$. Since f is a large scale equivalence, then $f^{-1}(\mathcal{U})$ is a uniformly bounded cover of X. Take any uniformly bounded cover $\mathcal{V}$ of X. Then $f(\mathcal{V})$ is a cover of Y. From the definition, there exists a collection of elements $U_i \in \mathcal{U}$ with $f(\mathcal{V}) \subset \cup_{i=1}^{k(f(\mathcal{V}))} U_i$. Thus $V \subset f^{-1}(f(V)) \subset \cup_{i=1}^{k(f(\mathcal{V}))} f^{-1}(U_i)$. Since $\mathcal{V}$ was chosen arbitrarily, $f^{-1}(\mathcal{U})$ with $k(\mathcal{V}) = k(f(\mathcal{V}))$ satisfies Definition 3, and X has bounded scale measure. $\square$

Proposition 4.3.1. *Let X be a large scale space. If X has bounded geometry, then X has bounded scale measure.*

Proof. Define $\mathcal{U}$ to be the trivial cover of X. Let $\mathcal{V}$ be any ls-cover of X. Since X has bounded geometry, then for any $V \in \mathcal{V}$, $|V| \leq M_{\mathcal{V}}$. Therefore, $V = \bigcup_{i=1}^{M_{\mathcal{V}}} x_i$, and letting $k(\mathcal{V}) = M_{\mathcal{V}}$ gives us X has bounded scale measure. $\square$

4.2 Decomposition of ls-spaces

Decomposition complexity is a generalization of finite asymptotic dimension. The metric space definition of asymptotic dimension, given in [PWY12], was used to show the following result:

For every $R > 0$ there exists $n + 1$ families $\mathcal{U}_i$ of subsets of X, $i = 0, 1, ..., n$ and $S > 0$ such that each $\mathcal{U}_i$ is R-disjoint, S-bounded and the families $\mathcal{U}_i$ cover X.

This concept gave rise to finite decomposition complexity and asymptotic property C amongst other definitions. Similarly, the large scale generalization gives rise to a pseudo-metric definition on a coarse space and the following results:

Proposition 4.3.2. *Suppose $\mathcal{LS}$ is a large scale structure on a set X. If $n \geq 0$, then the following conditions are equivalent:*

1. *for every uniformly bounded family $\mathcal{B}$ in X there is a uniformly bounded family $\mathcal{B}'$ on X of which $\mathcal{B}$ is a refinement such that the multiplicity on $\mathcal{B}'$ is at most $n+1$.*
2. *for every uniformly bounded family $\mathcal{B}$ in X there is a decomposition of X as $X_0 \cup ... \cup X_n$ such that the family of $\mathcal{B}$-components of each X_i is uniformly bounded.*

We now show that large scale spaces that satisfy a nice star property have a similar decomposition of the space.

Theorem 4.4. *Let X be a large scale space and $\mathcal{U}$ be a large scale cover. If every element of the third star of $\mathcal{U}$ can be covered by n elements of $\mathcal{U}$, then*

$X = X_1 \cup ... \cup X_n$, where each X_i is a union of $\mathcal{U}$-disjoint subsets that refines $st(\mathcal{U},\mathcal{U})$

Proof. This will be proved by induction using the following sets; Define K_n to be the maximal subset of $\mathcal{U}$ such that $U, V \in K_n$ if $st(U,\mathcal{U})$ and $st(V,\mathcal{U})$ are $\mathcal{U}$-disjoint. To show such a set exists, we will use Zorn's lemma with a partial order of inclusion. Suppose we have a totally ordered chain of subsets C_i such that $U, V \in C_i$ if $st(U,\mathcal{U})$ and $st(V,\mathcal{U})$ are $\mathcal{U}$-disjoint. We will show that $C = \cup C_i$ is an upper bound of the chain that satisfies our properties. Let $U, V \in C$.Since the chain of C_i's are totally ordered, then there exists some C_j such that

$U, V \in C_j$. Thus $st(U,\mathcal{U})$ and $st(V,\mathcal{U})$ are $\mathcal{U}$-disjoint. Therefore, a maximal element of X exists with respect to stars of the elements being $\mathcal{U}$-disjoint. Define $X_n = \cup_{V \in K_n} st(V,\mathcal{U})$.

For the base case, suppose every element of the third star of $\mathcal{U}$ can be covered by 1 element of $\mathcal{U}$. Define K_1 to be the maximal subset of $\mathcal{U}$ such that $U, V \in K_n$ if $st(U,\mathcal{U})$ and $st(V,\mathcal{U})$ are $\mathcal{U}$-disjoint, and define $X_1 = \cup_{V \in K_n} st(V,\mathcal{U})$.

We will show that $X = X_1$. Let $x \in X$. Since $\mathcal{U}$ is a cover of X, then for some $U \in \mathcal{U}$, $x \in U$. If $U \in K_1$, then $x \in X_1$. If $U \notin K_1$, then for some $V \in K_1$, $st(U,\mathcal{U}) \cap st(V,\mathcal{U}) \neq \emptyset$. Moreover, $st^3(U,\mathcal{U}) \cap V \neq \emptyset$. Since $st^3(U,\mathcal{U})$ can be covered by 1 element, say U' of $\mathcal{U}$, then $U' \cap V \neq \emptyset$, and $x \in st^3(U,\mathcal{U}) \subset U' \subset X_1$ and we have shown $X = X_1$.

For the inductive step suppose that every element of the third star of $\mathcal{U}$ can be covered by n elements of $\mathcal{U}$. We will show that for X_n as defined above, $X \setminus X_n$ has the property that each third star of $\mathcal{U}$ can be covered by $n-1$ elements of $\mathcal{U}$ restricted to $X \setminus X_n$.

Let $U \in \mathcal{U}$ restricted to $X \setminus X_n$. From the hypothesis we know $st^3(U,\mathcal{U}) \subset \cup_{i=1}^{n} U_i$ with $U_i \in \mathcal{U}$. Since $U \notin K_n$, then there exists $V \in K_n$ such that $st(U,\mathcal{U}) \cap st(V,\mathcal{U}) \neq \emptyset$,which implies $st^3(U,\mathcal{U}) \cap V \neq \emptyset$. Therefore, $(\cup_{i=1}^{n} U_i) \cap V \neq \emptyset$ and for some $1 \leq i \leq n$, $U_i \cap V \neq \emptyset$ or $U_i \subset st(V,\mathcal{U}) \subset X_n$. Since $U \in X \setminus X_n$, then $U \subset U_1 \cup, ..., \cup U_{i-1} \cup U_{i+1} \cup, ... \cup U_n$ and therefore $X \setminus X_n$ has the property that each third star of $\mathcal{U}$ can be covered by $n-1$ elements of $\mathcal{U}$ restricted to $X \setminus X_n$.

Thus by induction $X \setminus X_n$ can be split into $n-1$ subsets, where each X_i is a union of $\mathcal{U}$-disjoint subsets that refines $st(\mathcal{U},\mathcal{U})$. Thus $X = X_1 \cup ... \cup X_{n-1} \cup X_n$ with each X_i a union of $\mathcal{U}$-disjoint sets that clearly refine $st(\mathcal{U},\mathcal{U})$. $\square$

Clearly spaces with bounded scale measure has the property listed above, and therefore a space having bounded scale measure can be decomposed in this way.

This leads us to ask about connections, if any, between bounded scale measure, finite asymptotic dimension and asymptotic property C.

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Vita

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