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To the Graduate Council:

I am submitting herewith a dissertation written by James C. Smith entitled "On L_p Solutions of Second Order Linear Differential Equations." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Don Hinton, Major Professor

We have read this dissertation and recommend its acceptance:

Henry Simpson, R. Childers, N. Alskakos

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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We have read this dissertation
and recommend its acceptance:

Hery Simpson

R Childers

N. Alikakos

Accepted for the Council:

Lou Minkel

Associate Vice Chancellor
and Dean of The Graduate School

ON L_p SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

James C. Smith

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ABSTRACT

In this dissertation we study the L_p solutions of second order linear differential equations. The question as to when the equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = f(x), \quad a \leq x < \infty,$$

admits L_p solutions $y(x)$ for arbitrary $f(x)$ in L_p is investigated. We show the condition $\mathrm{Re}(q_1(x)) \geq 1$ or the conditions $\mathrm{Re}(q_1(x)) \geq 0$ and $\mathrm{Im}(q_1(x)) \geq 1$ are sufficient for a L_p solution $y(x)$ to exist.

Functions that bound a solution of the homogeneous equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0, \quad a \leq x < \infty,$$

either above or below, are given for non-oscillatory equations.

An extensive discussion regarding the linear dimension of the set of L_p solutions of $-(q_0y')' + q_1y = 0$ is given. The equation

$$-(x^\beta y'(x))' + (-mx^\gamma)y(x) = 0, \quad 1 \leq x < \infty,$$

is used as an example to illustrate the results.

TABLE OF CONTENTS

CHAPTER	PAGE
0. Introduction	1
1. Existence of L_p Solutions	6
2. Limit-point L_p Criteria for the 2nd-order Equation	42
3. Interval Limit-point L_p Criteria	86
List of References	116
Vita	117

Introduction

Let $q_0(x)$ and $q_1(x)$ be real valued on $[a,\infty)$, $-\infty < a < \infty$, with $q_0(x)$ positive. Suppose $1/q_0(x)$ and $q_1(x)$ are locally Lebesgue integrable in $[a,\infty)$. Consider the differential equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = \lambda y(x) + f(x), \quad a \le x < \infty, \tag{0.1}$$

with λ a complex constant and $f(x)$ complex valued and measurable. If $f(x)$ is Lebesgue integrable on compact intervals $[a,b]$, then for each set of initial conditions

$$y(a) \text{ and } q_0(a)y'(a)$$

equation (0.1) has a unique solution $y(x)$. This solution $y(x)$ is defined by successive approximations with $y(x)$ and $q_0(x)y'(x)$ understood to be absolutely continuous on compact intervals $[a,b]$. The equation is satisfied by $y(x)$ almost everywhere with respect to Lebesgue measure on $[a,\infty)$.

When $f(x)$ is $L_2[a,\infty)$ instead it is of interest to know when equation (0.1) has solutions $y(x)$ such that $y(x)$ is $L_2[a,\infty)$, also. In order to answer this question the homogeneous equation

$$-(q_0(x)y'(x))' + (q_1(x) - \lambda)y(x) = 0, \quad a \le x < \infty, \tag{0.2}$$

is studied. The differential operator

$$Ly(x) = -(q_0(x)y'(x))' + (q_1(x) - \lambda)y(x) \tag{0.3}$$

is defined, with D, the domain of L, taken to be the set

$$\begin{aligned} D = \{y(x) : y(x) \in L_2[a,\infty),\ & y(x) \text{ and } q_0(x)y'(x) \text{ are} \\ & \text{absolutely continuous on intervals } [a,b], \text{ and} \\ & Ly(x) \in L_2[a,\infty)\}. \end{aligned} \tag{0.4}$$

The operator

$$L_0 y(x) \tag{0.5}$$

is defined as $Ly(x)$ restricted to the subdomain

$$D_0 = \{y(x) : y(x) \in D \text{ and } y(a) = q_0(a)y'(a) = 0\}. \tag{0.6}$$

Note that D is the direct sum of D_0 and a two dimensional space. Let $N(L)$ be the null space of $Ly(x)$ and $R(L_0)$ be the range of $L_0 y(x)$.

When $N(L)$ has dimension 2, $R(L_0)$ cannot be extended by adjoining elements of $D(L)$ to $D(L_0)$, since $D(L)$ is the direct sum of $D(L_0)$ and $N(L)$. So, if $f(x)$ is $L_2[a,\infty)$, we need only to check if

$$L_0 y(x) = f(x) \tag{0.7}$$

has a solution $y(x)$ to decide if equation (0.1) has a solution for given initial values. We show in Chapter I that, if $y_1(x)$ and $y_2(x)$ are independent solutions of equation (0.2) satisfying

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1,$$

then the solution of the initial value problem

$$-(q_0(x)y'(x))' + (q_1(x) - \lambda)y(x) = f(x), \quad a \le x < \infty,$$

with

$$y(a) = q_0(a)y'(a) = 0$$

is given by

$$y(x) = \int_a^x (y_1(x)y_2(t) - y_1(t)y_2(x))f(t)dt. \tag{0.8}$$

Using Schwartz's inequality we can bound the integrals $\int_a^x |y_2(t)f(t)|dt$ and $\int_a^x |y_1(t)f(t)|dt$ by $\left(\int_a^x |y_2(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_a^x |f(t)|^2 dt\right)^{\frac{1}{2}}$ and $\left(\int_a^x |y_1(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_a^x |f(t)|^2 dt\right)^{\frac{1}{2}}$. When both $y_1(x)$ and $y_2(x)$ are $L_2[a,\infty)$, we get, using the triangle inequality for L_2, that $y(x)$ is $L_2[a,\infty)$.

When equation (0.2) has solutions that are not $L_2[a,\infty)$ we can still answer this question under certain conditions. The following Theorem holds [0].

Theorem. *If $R(L_0)$ is closed and $N(L)$ has dimension 1, then*

$$R(L_0) = N(L)^{\perp}.$$

($(\cdot)^{\perp}$ is the set of those elements of $L_2[a,\infty)$ that are orthogonal to $(\cdot)$.)

$N(L)$ has finite dimension and hence is closed in $L_2[a,\infty)$. So under the conditions of the above theorem,

$$R(L_0)^{\perp} = N(L)^{\perp\perp} = N(L),$$

and,

$$L_2[a,\infty) = R(L_0) + N(L).$$

(Here $+$ means a direct sum.) When $R(L_0)$ is closed and $N(L)$ has dimension 1, then $R(L_0)$ can be extended by adjoining an element of $D(L)$ to $D(L_0)$.

Theorem. *Suppose that $R(L_0)$ is closed and $N(L)$ has dimension 1 with $y_1(x)$ a non-zero element of $N(L)$. If $f(x)$ is $L_2[a,\infty)$, then equation (0.1) has a unique $L_2[a,\infty)$ solution $y(x)$ satisfying the boundary condition $y_1(a)\overline{y(a)} + (q_0(a)y_1'(a))\overline{(q_0(a)y'(a))} = 0$.*

To see that the Theorem above holds observe the following. Suppose that $y_2(x)$ is an element of $D(L)$ with initial values $y_2(a) = \overline{-q_0(a)y_1'(a)}$ and $q_0(a)y_2'(a) = \overline{y_1(a)}$. Then $y_2(x) \notin N(L)$, since $y_1(x)$ and $y_2(x)$ are linearly independent. Since $L_2[a,\infty) = R(L_0) + N(L)$, we have $Ly_2(x) = h_1(x) + h_2(x)$, with $h_1(x) \in R(L_0)$ and $h_2(x) \in N(L)$. An element of D with the same initial values as $y_2(x)$ cannot be an element of $N(L)$. So $L(y_2(x) - L_0^{-1}h_1(x)) = Ly_2(x) - h_1(x) \not\equiv 0$ and we know $h_2(x) \not\equiv 0$. So any $f(x) \in L_2[a,\infty)$ can be expressed as $f(x) = f_0(x) + \alpha h_2(x)$, for some $f_0(x) \in R(L_0)$ and complex scalar α. Let $y(x) = y_0(x) + \alpha y_2(x)$ with $y_0(x) \in D(L_0)$ such that $L_0 y_0(x) = f_0(x) - \alpha h_1(x)$. Then $Ly(x) = Ly_0(x) + \alpha L y_2(x) =$

$(f_0(x) - \alpha h_1(x)) + (\alpha h_1(x) + \alpha h_2(x)) = f_0(x) + \alpha h_2(x) = f(x)$. Now, $y(x)$ is a solution of $Ly(x) = f(x)$ with initial values $y(a) = \alpha\overline{(-q_0(a)y_1'(a))}$ and $q_0(a)y'(a) = \alpha\overline{(y_1(a))}$. If we have two solutions of (0.1) satisfying the boundary condition, then their difference is a solution of (0.2) and is thus a multiple βy_1 of y_1 since $N(L)$ has dimension 1. Substitution of βy_1 into the boundary condition shows $\beta = 0$.

In this paper we suppose more generally that $f(x)$ is $L_p[a,\infty)$, for p, $1 \le p \le \infty$. The question as to when equation (0.1) has solutions that are $L_p[a,\infty)$, also, is considered. To study this question we define, as before except with $L_2[a,\infty)$ being replaced by $L_p[a,\infty)$, the differential operator

$$Ly(x) = -(q_0(x)y'(x))' + (q_1(x) - \lambda)y(x)$$

with D, the domain of L, taken to be the set

$$\begin{aligned} D = \{y(x) : {} & y(x) \in L_p[a,\infty),\ y(x) \text{ and } q_0(x)y'(x) \text{ are} \\ & \text{absolutely continuous on intervals } [a,b], \text{ and} \\ & Ly(x) \in L_p[a,\infty)\}. \end{aligned} \tag{0.9}$$

The operator

$$L_0 y(x)$$

is defined as $Ly(x)$ restricted to the subdomain

$$D_0 = \{y(x) : y(x) \in D \text{ and } y(a) = q_0(a)y'(a) = 0\}.$$

When the dimension of $N(L)$ is 2 and $f(x)$ is $L_{p/(p-1)}[a,\infty)$, $1 < p < \infty$, we can show the solution $y(x)$ of (0.1) defined by (0.8) is $L_p[a,\infty)$, also. This is the content of Theorem 10 of Chapter I.

When equation (0.2) has solutions that are not $L_p[a,\infty)$, $1 < p < \infty$, we can still answer the question as to when equation (0.1) has $L_p[a,\infty)$ solutions $y(x)$, for $f(x) \in L_p[a,\infty)$, under certain conditions. If we assume conditions on $q_0(x)$ and $q_1(x)$ such that $\langle L_0 y(x), \overline{y(x)}|y(x)|^{p-2}\rangle =$

$\int_a^\infty (L_0 y(x))\overline{y}(x)|y(x)|^{p-2}dx \geq k\int_a^\infty |y(x)|^p dx$, for $y(x) \in D_0$, with k a positive real number, then $R(L_0)$ is closed. When $N(L)$ has dimension 1 we may suspect, as when $p = 2$, that the elements of the form $h_1(x)+h_2(x)$, with $h_1(x) \in R(L_0)$ and $h_2(x) \in N(L)$, span $L_p[a,\infty)$. If the elements $h_1(x)+h_2(x)$ span $L_p[a,\infty)$, we will have, like in the L_2 case, that equation (0.1) has an $L_p[a,\infty)$ solution $y(x)$ satisfying the boundary condition $y_1(a)\overline{y(a)} + (q_0(a)y_1'(a))\overline{(q_0(a)y'(a))} = 0$, for $y_1(x) \in N(L)$. We show two cases, Theorems 11 and 12, by direct calculation.

Many results concerning the L_2 solutions of equation (0.2) are published. Again we can consider what the analogous Theorems about L_p solutions should be. Throughout Chapters I, II, and III we give a number of Theorems about the dimension of $N(L)$.

Chapter 1

Existence of L_p Solutions

1. Let $q_0(x)$ be real valued, positive, and continuously differentiable on $[a, \infty)$, $a \geq 0$. Let $q_1(x)$ be complex valued and locally Lebesgue integrable in $[a, \infty)$. Then the differential equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0 \tag{1.1}$$

will be said to have a $L_p[a, \infty)$ solution, $1 \leq p < \infty$, if it has a solution $y(x)$ such that $\lim_{t\to\infty} \int_a^t |y(x)|^p dx < \infty$. If the differential equation has a solution $y(x)$ such that $\sup_{a\leq x<\infty} |y(x)| < \infty$, then it will be said to have a $L_\infty[a, \infty)$ solution. The equation will be said to be limit-point L_p if it has a solution that is not $L_p[a, \infty)$. If all solutions are $L_p[a, \infty)$ it will be said to be limit-circle L_p.

Conditions on $q_0(x)$ and $q_1(x)$ can be given to insure that L_p solutions exist. In the next section we provide four theorems that guarantee equation (1.1) has a L_p solution.

2. Consider equation (1.1) defined on the interval $[0, \infty)$. In the case that $(\cdot)$ is complex valued, $\mathrm{Re}(\cdot)$ will denote the real part of $(\cdot)$. Likewise $\mathrm{Im}(\cdot)$ will denote the imaginary part.

Theorem 1. *Let $Re(q_1(x)) \geq 1$ on $[0, \infty)$. Then equation (1.1) has a non-trivial solution that is $L_p[0, \infty)$, for $1 \leq p \leq \infty$.*

Proof. First consider when $1 < p < \infty$. Suppose $y(x)$ is a solution of (1.1). Then

$$\begin{aligned} 0 &= \int_0^t (-(q_0(x)y'(x))' + q_1(x)y(x))\overline{y}(x)|y(x)|^{p-2}dx \\ &= \int_0^t -(q_0(x)y'(x))'\overline{y}(x)|y(x)|^{p-2}dx + \int_0^t q_1(x)|y(x)|^p dx. \end{aligned}$$

Integrate the first integral above by parts and use

$$\begin{aligned}\frac{d}{dx}(|y(x)|^{p-2}) &= \frac{d}{dx}((y\overline{y})^{\frac{p}{2}-1}) \\ &= (\frac{p}{2}-1)(y\overline{y})^{\frac{p}{2}-2}(y\overline{y}' + y'\overline{y}) \\ &= (\frac{p}{2}-1)|y|^{p-4}(y\overline{y}' + y'\overline{y}).\end{aligned}$$

This gives

$$\begin{aligned}\int_0^t -(q_0y')'\overline{y}|y|^{p-2}dx = |_0^t - q_0y'\overline{y}|y|^{p-2} + \int_0^t q_0|y'|^2|y|^{p-2}dx \\ + \int_0^t \left(\frac{p}{2}-1\right) q_0 \left(|y'|^2|y|^{p-2} + (y')^2(\overline{y})^2|y|^{p-4}\right) dx.\end{aligned}$$

When $1 < p < 2$ the integral $\int_0^t q_0|y'|^2|y|^{p-2}dx$ is still bounded if $y(x)$ has a zero x_0, $0 < x_0 < t$.

$$\lim_{x\to x_0} y(x)/(x-x_0) = \lim_{x\to x_0} (y(x) - y(x_0))/(x-x_0) = y'(x_0)$$

so that

$$\lim_{x\to x_0} (|y'(x)|^2|y(x)|^{p-2})/(|y'(x_0)|^p|x-x_0|^{p-2}) = 1.$$

In which case for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}\frac{1}{2}\int_{x_0-\varepsilon}^{x_0+\varepsilon} |y'(x_0)|^p|x-x_0|^{p-2}dx < \int_{x_0-\varepsilon}^{x_0+\varepsilon} |y'(x)|^2|y(x)|^{p-2}dx \\ < 2\int_{x_0-\varepsilon}^{x_0+\varepsilon} |y'(x_0)|^p|x-x_0|^{p-2}dx,\end{aligned}$$

with

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} |x-x_0|^{p-2}dx = (2\varepsilon^{p-1})/(p-1).$$

After substituting the expression for $\int_0^t -(q_0y')'\overline{y}|y|^{p-2}dx$ into $\int_0^t -(q_0y')'\overline{y}|y|^{p-2}dx + \int_0^t q_1|y|^p dx$ one obtains

$$\begin{aligned}0 = |_0^t - q_0y'\overline{y}|y|^{p-2} + \frac{p}{2}\int_0^t q_0|y'|^2|y|^{p-2}dx \\ + (\frac{p}{2}-1)\int_0^t q_0(y')^2(\overline{y})^2|y|^{p-4}dx + \int_0^t q_1|y|^p dx. \qquad (2.1)\end{aligned}$$

Now, let $y_1(x)$ and $y_2(x)$ be solutions of (1.1) such that $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 0$, $y_2'(0) = 1$. If we differentiate $y_1(x)(q_0(x)y_2'(x)) - (q_0(x)y_1'(x))y_2(x)$ we obtain

$$\begin{aligned}\frac{d}{dx}[y_1(q_0y_2') - (q_0y_1')y_2] &= y'(q_0y_2') + y_1(q_0y_2')' - (q_0y_1')'y_2 - (q_0y_1')y_2' \\ &= q_0y_1'y_2' + y_1q_1y_2 - q_1y_1y_2 - q_0y_1'y_2' = 0.\end{aligned}$$

Thus $q_0(y_1y_2' - y_1'y_2)$ is constant. In particular

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = q_0(0)(y_1(0)y_2'(0) - y_1'(0)y_2(0)), \tag{2.2}$$

so that y_1 and y_2 cannot both be zero. For each n, $n = 1, 2, \ldots$ take $y_n(x) = c_ny_1(x) + d_ny_2(x)$ where c_n, d_n are complex valued constants satisfying $|c_n|^2 + |d_n|^2 = 1$ and $y_n(n) = c_ny_1(n) + d_ny_2(n) = 0$. Substituting $y_n(x)$ into (2.1) with $t = n$ and using

$$|_0^n - q_0y_n'\overline{y}_n|y_n|^{p-2} = q_0(0)d_n\overline{c}_n|c_n|^{p-2}$$

we get

$$\begin{aligned}0 = q_0(0)d_n\overline{c}_n|c_n|^{p-2} &+ \frac{p}{2}\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx \\ &+ (\frac{p}{2} - 1)\int_0^n q_0(y_n')^2(\overline{y}_n)^2|y_n|^{p-4}dx + \int_0^n q_1|y_n|^p dx.\end{aligned} \tag{2.3}$$

If $1 < p < 2$

$$\mathrm{Re}\left\{(\frac{p}{2} - 1)\int_0^n q_0(y_n')^2(\overline{y}_n)^2|y_n|^{p-4}dx\right\} \geq (\frac{p}{2} - 1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx.$$

In which case the inequality

$$0 \geq \mathrm{Re}\left\{q_0(0)d_n\overline{c}_n|c_n|^{p-2}\right\} + (p-1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx + \mathrm{Re}\left\{\int_0^n q_1|y_n|^p dx\right\}$$

follows. This yields

$$0 \geq -q_0(0) + (p-1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx + \int_0^n |y_n|^p dx. \tag{2.4}$$

If $2 \leq p < \infty$,

$$\mathrm{Re}\left\{(\frac{p}{2}-1)\int_0^n q_0(y_n')^2(\overline{y}_n)^2|y_n|^{p-4}dx\right\} \geq (1-\frac{p}{2})\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx.$$

In which case

$$0 \geq -q_0(0) + \int_0^n q_0|y_n'|^2|y_n|^{p-2}dx + \int_0^n |y_n|^p dx. \tag{2.5}$$

We have now for $1 < p < \infty$ that

$$\int_0^n |y_n|^p dx \leq q_0(0). \tag{2.6}$$

Now, select a subsequence (c_{n_k}, d_{n_k}), $k = 1, 2, \ldots$ of (c_n, d_n) such that (c_{n_k}, d_{n_k}) converges to some (c, d), $|c|^2 + |d|^2 = 1$. Set $y(x) = cy_1(x) + dy_2(x)$. Fix $t > 0$ and consider n_k greater than t. After applying Minkowski's inequality to

$$y = (y - y_{n_k}) + y_{n_k}$$

we have

$$\left(\int_0^t |y|^p dx\right)^{\frac{1}{p}} \leq \left(\int_0^t |y - y_{n_k}|^p dx\right)^{\frac{1}{p}} + \left(\int_0^t |y_{n_k}|^p dx\right)^{\frac{1}{p}}.$$

Applying Minkowski's inequality to

$$y - y_{n_k} = (c - c_{n_k})y_1 + (d - d_{n_k})y_2$$

we have

$$\left(\int_0^t |y - y_{n_k}|^p dx\right)^{\frac{1}{p}} \leq \left(|c - c_{n_k}|^p \int_0^t |y_1|^p dx\right)^{\frac{1}{p}} + \left(|d - d_{n_k}|^p \int_0^t |y_2|^p dx\right)^{\frac{1}{p}}.$$

Letting $k \to \infty$ we get $\left(\int_0^t |y - y_{n_k}|^p dx\right)^{\frac{1}{p}} \to 0$. From (2.6) we have $\int_0^t |y_{n_k}|^p dx \leq q_0(0)$. Hence, it follows

$$\int_0^t |y|^p dx \leq q_0(0). \tag{2.7}$$

To establish the theorem for $p = 1$ let $p \to 1$ in (2.7) to obtain

$$\int_0^t |y| dx \leq q_0(0).$$

For $p = \infty$ use

$$\text{Supremum}_{0\le x\le t}|y(x)| = \lim_{p\to\infty}\left(\int_0^t |y|^p dx\right)^{\frac{1}{p}} \le \lim_{p\to\infty}(q_0(0))^{\frac{1}{p}} = 1.$$

From which $||y||_\infty \le 1$ follows.

Theorem 2. *Let $Re(q_1(x)) \ge 0$ and $Im(q_1(x)) \ge 1$. Then equation (1.1) has a non-trivial solution that is $L_p[0,\infty)$, for $1 < p \le \infty$.*

Proof. Proceed in exactly the same manner as in the proof of Theorem 1 until the equality (2.3),

$$\begin{aligned} 0 = q_0(0)d_n\overline{c}_n|c_n|^{p-2} &+ \frac{p}{2}\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx \\ &+ (\frac{p}{2}-1)\int_0^n q_0(y_n')^2(\overline{y}_n)^2|y_n|^{p-4}dx + \int_0^n q_1|y_n|^p dx, \end{aligned}$$

is obtained. If $1 < p < 2$ the inequality

$$0 \ge -q_0(0) + (p-1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx \tag{2.8}$$

is obtained instead of (2.4). In the case $2 \le p < \infty$ we get

$$0 \ge -q_0(0) + \int_0^n q_0|y_n'|^2|y_n|^{p-2}dx \tag{2.9}$$

instead of (2.5). When $1 < p < 2$,

$$\text{Im}\left\{(\frac{p}{2}-1)\int_0^n q_0(y_n')^2(\overline{y}_n)^2|y_n|^{p-4}dx\right\} \ge (\frac{p}{2}-1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx.$$

In this case

$$0 \ge \text{Im}\left\{q_0(0)d_n\overline{c}_n|c_n|^{p-2}\right\} + (\frac{p}{2}-1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx + \text{Im}\left\{\int_0^n q_1|y_n|^p dx\right\}.$$

From which we get

$$0 \ge -q_0(0) + (\frac{p}{2}-1)\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx + \int_0^n |y_n|^p dx. \tag{2.10}$$

In the case $2 \leq p < \infty$ we have

$$0 \geq -q_0(0) + \left(1 - \frac{p}{2}\right) \int_0^n q_0 |y_n'|^2 |y_n|^{p-2} dx + \int_0^n |y_n|^p dx. \tag{2.11}$$

For $1 < p < 2$ we combine (2.8) and (2.10) to obtain

$$\int_0^n |y_n|^p dx \leq q_0(0) + \left(1 - \frac{p}{2}\right) \int_0^n |y_n'|^2 |y_n|^{p-2} dx \leq q_0(0) + \left(\left(1 - \frac{p}{2}\right) / (p-1)\right) q_0(0),$$

that is

$$\int_0^n |y_n|^p dx \leq (p/(2(p-1)))\, q_0(0). \tag{2.12}$$

For $2 \leq p < \infty$ we combine (2.9) and (2.11) to obtain

$$\int_0^n |y_n|^p dx \leq q_0(0) + \left(\frac{p}{2} - 1\right) \int_0^n q_0 |y_n'|^2 |y_n|^{p-2} dx \leq q_0(0) + \left(\frac{p}{2} - 1\right) q_0(0),$$

that is

$$\int_0^n |y_n|^p dx \leq \frac{p}{2} q_0(0). \tag{2.13}$$

Now, choose $y(x) = cy_1(x) + dy_2(x)$ in the same way that $y(x)$ was chosen in the proof of Theorem 1. It follows that

$$\int_0^t |y|^p dx \leq (p/(2(p-1)))\, q_0(0),$$

for $1 < p < 2$, and

$$\int_0^t |y|^p dx \leq \frac{p}{2} q_0(0),$$

for $2 \leq p < \infty$. The case $p = \infty$ is obtained by considering

$$\text{supremum}_{0 \leq x \leq t} |y(x)| = \lim_{p \to \infty} \left(\int_0^t |y|^p dx \right)^{\frac{1}{p}} \leq \lim_{p \to \infty} \left(\frac{p}{2} q_0(0) \right)^{\frac{1}{p}} = 1$$

as before.

When q_0 is twice continuously differentiable after the change of variable $y = uF$ the equation $-(q_0 y')' + q_1 y = 0$ becomes

$$-(q_0 F)u'' - (q_0' F + 2q_0 F')u' + (q_1 F - q_0' F' - q_0 F'')u = 0.$$

Choosing $F(x) = (q_0(x))^{-\frac{1}{2}}$ the equation assumes the form

$$-u''(x) + g(x)u(x) = 0 \tag{2.14}$$

with

$$g(x) = (q_1(x) - \frac{1}{4}(q_0(x))^{-1}(q_0'(x))^2 + \frac{1}{2}q_0''(x))(q_0(x))^{-1}.$$

Throughout the next section $g(x)$ will be used as notation for this expression. If $w(x)$ is a positive valued and continuous weight function, then

$$\int_0^t w|y|^p dx = \int_0^t w(q_0)^{-\frac{p}{2}}|u|^p dx.$$

For $1 \leq p < \infty$, when

$$\lim_{t\to\infty} \int_0^t w|y|^p dx < \infty$$

we will say that $y(x)$ is $L_p[w; a, \infty)$. For $p = \infty$, $L_\infty[w; 0, \infty)$ will mean the same as $L_\infty[0, \infty)$. Under the assumption that q_0 is twice continuously differentiable we have the following theorem.

Theorem 3. **a)** *For p, $1 < p < \infty$, if $Re(w^{-1}(x)g(x)(q_0(x))^{\frac{p}{2}}) \geq \delta$, where δ is a positive constant, then (1.1) has a non-trivial solution $y(x) = u(x)(q_0(x))^{-\frac{1}{2}}$ that is $L_p[w; 0, \infty)$.*

Theorem 3. **b)** *If $Re(g(x)) \geq 0$ and $Im(w^{-1}(x)g(x)(q_0(x))^{\frac{p}{2}}) \geq \delta$, then (1.1) has a non-trival solution $y(x) = u(x)(q_0(x))^{-\frac{1}{2}}$ that is $L_p[w; 0, \infty)$.*

Proof of a). Suppose $u(x)$ is a solution of (2.14). Multiply the equation by $\overline{u}|u|^{p-2}$ to get

$$0 = -u''\overline{u}|u|^{p-2} + (w^{-1}g(q_0)^{\frac{p}{2}})w(q_0)^{-\frac{p}{2}}|u|^p.$$

Integrate following the procedure used to determine (2.1) to obtain

$$\begin{aligned} 0 = |_0^t - u'\overline{u}|u|^{p-2} + \frac{p}{2}\int_0^t |u'|^2|u|^{p-2}dx \\ + (\frac{p}{2} - 1)\int_0^t (u')^2(\overline{u})^2|u|^{p-4}dx + \int_0^t (w^{-1}g(q_0)^{\frac{p}{2}})w(q_0)^{-\frac{p}{2}}|u|^p dx. \end{aligned} \tag{2.15}$$

Again, as in the proof of Theorem 1, let y_1 and y_2 be solutions of $-(q_0y')' + q_1y = 0$ satisfying $y_1(0) = 1,\ y_1'(0) = 0$ and $y_2(0) = 0,\ y_2'(0) = 1$. Set $y_n(x) = c_n y_1(x) + d_n y_2(x)$ with

$$c_n = y_2(n)/(|y_1(n)|^2 + |y_2(n)|^2)^{\frac{1}{2}},\ d_n = -y_1(n)/(|y_1(n)|^2 + |y_2(n)|^2)^{\frac{1}{2}}$$

so that $y_n(n) = 0$ and $|c_n|^2 + |d_n|^2 = 1$. Put $u_n(x) = (q_0(x))^{\frac{1}{2}} y_n(x)$ with $t = n$ into (2.15) to get

$$\begin{aligned} 0 &= (\frac{1}{2}(q_0(0))^{-\frac{1}{2}} q_0'(0) c_n + (q_0(0))^{\frac{1}{2}} d_n)(q_0(0))^{\frac{1}{2}} \overline{c}_n |(q_0(0))^{\frac{1}{2}} c_n|^{p-2} \\ &+ \frac{p}{2}\int_0^n |u_n'|^2 |u_n|^{p-2} dx + (\frac{p}{2} - 1)\int_0^n (u_n')^2 (\overline{u}_n)^2 |u_n|^{p-4} dx \\ &+ \int_0^n (w^{-1} g(q_0)^{\frac{p}{2}}) w(q_0)^{-\frac{p}{2}} |u_n|^p dx. \end{aligned} \tag{2.16}$$

Using the assumption $\mathrm{Re}(w^{-1}g(q_0)^{\frac{p}{2}}) \geq \delta$ and considering real parts in (2.16) we have, if $1 < p < 2$,

$$0 \geq -K_0 + (p-1)\int_0^n |u_n'|^2 |u_n|^{p-2} dx + \delta \int_0^n w(q_0)^{-\frac{p}{2}} |u_n|^p dx \tag{2.17}$$

with $K_0 = \frac{1}{2}|q_0'(0)|(q_0(0))^{\frac{p}{2}-1} + (q_0(0))^{\frac{p}{2}}$, and if $2 \leq p < \infty$,

$$0 \geq -K_0 + \int_0^n |u_n'|^2 |u_n|^{p-2} dx + \delta \int_0^n w(q_0)^{-\frac{p}{2}} |u_n|^p dx. \tag{2.18}$$

Thus we have

$$\int_0^n w(q_0)^{-\frac{p}{2}} |u_n|^p dx \leq \delta^{-1} K_0. \tag{2.19}$$

Now, as before, select a subsequence $(c_{n_k}, d_{n_k}),\ k = 1, 2, \ldots$ of (c_n, d_n) such that (c_{n_k}, d_{n_k}) converges to some (c, d) with $|c|^2 + |d|^2 = 1$. Set $y(x) = cy_1(x) + dy_2(x)$ and $u(x) = (q_0(x))^{\frac{1}{2}} y(x)$. Fix $t > 0$ and consider $n_k > t$. After applying Minkowski's inequality to

$$w^{\frac{1}{p}}(q_0)^{-\frac{1}{2}} u = w^{\frac{1}{p}}(q_0)^{-\frac{1}{2}}(u - u_{n_k}) + w^{\frac{1}{p}}(q_0)^{-\frac{1}{2}} u_{n_k},$$

we have

$$(\int_0^t w(q_0)^{-\frac{p}{2}} |u|^p dx)^{\frac{1}{p}} \leq (\int_0^t w(q_0)^{-\frac{p}{2}} |u - u_{n_k}|^p dx)^{\frac{1}{p}} + (\int_0^t w(q_0)^{-\frac{p}{2}} |u_{n_k}|^p dx)^{\frac{1}{p}}.$$

Applying Minkowski's inequality to

$$w^{\frac{1}{p}}(q_0)^{-\frac{1}{2}}(u-u_{n_k}) = w^{\frac{1}{p}}(y-y_{n_k}) = w^{\frac{1}{p}}(c-c_{n_k})y_1 + w^{\frac{1}{p}}(d-d_{n_k})y_2,$$

we have

$$(\int_0^t w(q_0)^{-\frac{p}{2}}|u-u_{n_k}|^p dx)^{\frac{1}{p}} \leq (|c-c_{n_k}|^p \int_0^t w|y_1|^p dx)^{\frac{1}{p}} + (|d-d_{n_k}|^p \int_0^t w|y_2|^p dx)^{\frac{1}{p}}.$$

Letting $k \to \infty$ we get $(\int_0^t w(q_0)^{-\frac{p}{2}}|u-u_{n_k}|^p dx)^{\frac{1}{p}} \to 0$. Using (2.19) we have

$$(\int_0^t w(q_0)^{-\frac{p}{2}}|u|^p dx)^{\frac{1}{p}} \leq (\delta^{-1}K_0)^{\frac{1}{p}}.$$

Thus

$$\int_0^t w|y|^p dx \leq \delta^{-1}K_0. \tag{2.20}$$

If a p_0 exists such that for all $p, 1 < p \leq p_0$, the hypothesis of a) holds, then the case $p = 1$ follows. If a p_0 exists such that for all p, $p > p_0$, the hypothesis of a) holds, then the case $p = \infty$ follows.

Proof of b). Proceed as in the proof of a) until (2.16) is obtained. Using the assumption $\mathrm{Re}(g) \geq 0$ and considering real parts in (2.16), if $1 < p < 2$, we get

$$0 \geq -K_0 + (p-1)\int_0^n |u_n'|^2|u_n|^{p-2}dx \tag{2.21}$$

and if $2 \leq p < \infty$ we get

$$0 \geq -K_0 + \int_0^n |u_n'|^2|u_n|^{p-2}dx. \tag{2.22}$$

When $1 < p < 2$ by considering imaginary parts in (2.16),

$$0 \geq -K_0 + (\frac{p}{2}-1)\int_0^n |u_n'|^2|u_n|^{p-2}dx + \delta\int_0^n w(q_0)^{-\frac{p}{2}}|u_n|^p dx \tag{2.23}$$

and when $2 \leq p < \infty$,

$$0 \geq -K_0 + (1-\frac{p}{2})\int_0^n |u_n'|^2|u_n|^{p-2}dx + \delta\int_0^n w(q_0)^{-\frac{p}{2}}|u_n|^p dx. \tag{2.24}$$

We combine (2.21) and (2.23) to obtain

$$\int_0^n w(q_0)^{-\frac{p}{2}}|u_n|^p dx \leq (p/(2(p-1)))\delta^{-1}K_0.$$

We combine (2.22) and (2.24) to obtain

$$\int_0^n w(q_0)^{-\frac{p}{2}}|u_n|^p dx \leq (\frac{p}{2})\delta^{-1}K_0.$$

It follows for $y(x) = u(x)(q_0(x))^{-\frac{p}{2}}$ chosen in the same way as in the proof of a) that, for $1 < p < 2$,

$$\int_0^t w|y|^p dx = \int_0^t w(q_0)^{-\frac{p}{2}}|u|^p dx \leq (p/(2(p-1)))\delta^{-1}K_0$$

and, for $2 \leq p < \infty$,

$$\int_0^t w|y_1|^p dx \leq (p/2)\delta^{-1}K_0.$$

If a p_0 exists such that for all p, $p > p_0$, the hypothesis of b) holds, then the case $p = \infty$ follows.

The interval $[0, \infty)$ in the statement and proofs of Theorems 1,2 and 3 can be replaced by any other interval $[a, \infty)$, $a > -\infty$. For the following example we will assume equation (1.1) $-(q_0y')' + q_1y = 0$ is defined on $[1, \infty)$. Theorem 3 gives us less restrictive bounds on q_1 than Theorem 1 or 2 does for choices of q_0 such as $q_0(x) = \alpha x^\beta$, $q_0(x) = \alpha e^{\beta x}$, etc.

Example 1) $q_0(x) = \alpha x^\beta$, $\alpha > 0$, $\beta \neq 0$. Then $g(x) = q_1\alpha^{-1}x^{-\beta} + \frac{1}{2}\beta(\frac{1}{2}\beta - 1)x^{-2}$ and $g(x)(q_0(x))^{\frac{p}{2}} = q_1\alpha^{(\frac{p}{2}-1)}x^{(\frac{p}{2}-1)\beta} + \alpha^{\frac{p}{2}}\frac{1}{2}\beta(\frac{1}{2}\beta - 1)x^{\frac{p}{2}\beta-2}$. Theorem 3 a) gives us a $L_p[1, \infty)$ solution exists if $\mathrm{Re}(q_1) \geq \delta\alpha^{(1-\frac{p}{2})}x^{(1-\frac{p}{2})\beta} - \alpha\frac{1}{2}\beta(\frac{1}{2}\beta - 1)x^{\beta-2}$, $\delta > 0$. Theorem 3 b) gives us a $L_p[1, \infty)$ solution exists if $\mathrm{Re}(q_1) \geq -\alpha\frac{1}{2}\beta(\frac{1}{2}\beta - 1)x^{\beta-2}$ and $\mathrm{Im}(q_1) \geq \delta\alpha^{(1-\frac{p}{2})}x^{(1-\frac{p}{2})\beta} - \alpha\frac{1}{2}\beta(\frac{1}{2}\beta - 1)x^{\beta-2}$.

Example 2) $q_0(x) = \alpha e^{\beta x}$, $\alpha > 0$, $\beta \neq 0$. Then $g(x) = q_1\alpha^{-1}e^{-\beta x} + \frac{1}{4}\beta^2$ and $g(x)(q_0(x))^{\frac{p}{2}} = q_1\alpha^{(\frac{p}{2}-1)}e^{(\frac{p}{2}-1)\beta x} + \frac{1}{4}\alpha^{\frac{p}{2}}\beta^2 e^{\frac{p}{2}\beta x}$. Theorem 3 a) gives us a $L_p[0, \infty)$ solution exists if $\mathrm{Re}(q_1) \geq \delta\alpha^{(1-\frac{p}{2})}e^{(1-\frac{p}{2})\beta x} - \frac{1}{4}\alpha\beta^2 e^{\beta x}$, $\delta > 0$. Theorem 3 b) gives us a $L_p[0, \infty)$ solution exists if $\mathrm{Re}(q_1) \geq -\frac{1}{4}\alpha\beta^2 e^{\beta x}$ and $\mathrm{Im}(q_1) \geq \delta\alpha^{(1-\frac{p}{2})}e^{(1-\frac{p}{2})\beta x} - \frac{1}{4}\alpha\beta^2 e^{\beta x}$.

Consider the equation

$$-(x^2y')' + q_1 y = 0$$

defined on $[1, \infty)$, with q_1 real valued. Multiplying by y and integrating we have

$$|_1^t (x^2y')y = \int_1^t x^2(y')^2 dx + \int_1^t q_1 y^2 dx.$$

Integrating by parts

$$\int_1^t y^2 dx = |_1^t y^2 x - \int_1^t 2xyy' dx$$

and applying Schwarz's inequality

$$\int_1^t y^2 dx \leq |_1^t y^2 x + 2\left(\int_1^t x^2(y')^2 dx\right)^{\frac{1}{2}} \left(\int_1^t y^2 dx\right)^{\frac{1}{2}}.$$

After a little more calculation we get

$$\int_1^t x^2(y')^2 dx - \frac{1}{4}\int_1^t y^2 dx \geq |_1^t - \frac{1}{2}y^2 x + \left(\left(\int_1^t x^2(y')^2 dx\right)^{\frac{1}{2}} - \frac{1}{2}\left(\int_1^t y^2 dx\right)^{\frac{1}{2}}\right)^2,$$

and thus

$$|_1^t (x^2y')y \geq |_1^t - \frac{1}{2}y^2 x + \int_1^t (q_1 + \frac{1}{4})y^2 dx.$$

If $y(t) = 0$, then

$$\int_1^t (q_1 + \frac{1}{4})y^2 dx \leq -(y'(1))y(1) - \frac{1}{2}(y(1))^2.$$

Choose a sequence of solutions $y_n(x)$, $n = 1, 2, \ldots$, such that $(y_n(1))^2 + (y_n'(1))^2 = 1$ and $y_n(n) = 0$. As before extract a convergent subsequence to show that a non-trivial solution $y(x)$ exists, such that

$$\int_1^\infty (q_1 + \frac{1}{4})y^2 dx \leq -(y'(1))y(1) - \frac{1}{2}(y(1))^2.$$

It follows if $q_1 \geq -\frac{1}{4} + \delta$, $\delta > 0$, then a $L_2[1, \infty)$ solution exists. In this case, this is a sharper bound than given by Theorem 3. It suggests the following result

Consider equation (1.1) $-(q_0 y')' + q_1 y = 0$ defined on $[1, \infty)$, with q_1 real valued. Let $w(x)$ be a positive and continuously differentiable weight function.

Theorem 4. *If* $q_1(x) \geq \frac{-\alpha(p-1)}{p}\left(\frac{1}{2}(w(x)q_0(x))^{-\frac{1}{2}}(w(x)q_0(x))'\right) + \beta w(x)$, $1 < p < \infty$, *with* $\alpha > 0$, $\beta > 0$ *satisfying* $-(p-1)\frac{\alpha^2}{4} + \beta > 0$, *then (1.1) has a non-trivial* $L_p[w; 1, \infty)$ *solution.*

Proof. Let $y(x)$ be a real solution of (1.1). After integrating

$$0 = (-(q_0 y')' + q_1 y)\, y|y|^{p-2}$$

we have

$$|_1^t q_0 y' y|y|^{p-2} = (p-1)\int_1^t q_0|y'|^2|y|^{p-2}dx + \int_0^t q_1|y|^p dx. \tag{2.25}$$

Integrating by parts

$$\int_1^t \frac{1}{2}(wq_0)^{-\frac{1}{2}}(wq_0)'|y|^p dx = |_1^t (wq_0)^{\frac{1}{2}}|y|^p - \int_1^t p(wq_0)^{\frac{1}{2}} y' y|y|^{p-2}dx.$$

After applying Schwarz's inequality to the last integral

$$\int_1^t \frac{1}{2}(wq_0)^{-\frac{1}{2}}(wq_0)'|y|^p dx \leq |_1^t (wq_0)^{\frac{1}{2}}|y|^p + p\left(\int_1^t q_0|y'|^2|y|^{p-2}dx\right)^{\frac{1}{2}}\left(\int_1^t w|y|^p dx\right)^{\frac{1}{2}}.$$

Using the condition on $q_1(x)$ in (2.25) we have

$$\begin{aligned}
|_1^t q_0 y' y|y|^{p-2} &\geq (p-1)\int_1^t q_0|y'|^2|y|^{p-2}dx \\
&\quad - \frac{\alpha(p-1)}{p}\int_1^t \frac{1}{2}(wq_0)^{-\frac{1}{2}}(wq_0)'|y|^p dx + \beta\int_1^t w|y|^p dx \\
&\geq (p-1)\int_1^t q_0|y'|^2|y|^{p-2}dx - (p-1)\alpha\left(\int_1^t q_0|y'|^2|y|^{p-2}dx\right)^{\frac{1}{2}}\left(\int_1^t w|y|^p dx\right)^{\frac{1}{2}} \\
&\quad + \beta\int_1^t w|y|^p dx - \frac{\alpha(p-1)}{p}|_1^t (wq_0)^{\frac{1}{2}}|y|^p \\
&= (p-1)\left(\left(\int_1^t q_0|y'|^2|y|^{p-2}dx\right)^{\frac{1}{2}} - \frac{\alpha}{2}\left(\int_1^t w|y|^p dx\right)^{\frac{1}{2}}\right)^2 \\
&\quad + \left(-(p-1)\frac{\alpha^2}{4} + \beta\right)\int_1^t w|y|^p dx - \frac{\alpha(p-1)}{p}|_1^t (wq_0)^{\frac{1}{2}}|y|^p.
\end{aligned}$$

Hence

$$\int_1^t w|y|^p dx \leq \left(-(p-1)\frac{\alpha^2}{4} + \beta\right)^{-1}\left(|_1^t q_0 y' y|y|^{p-2} + \frac{\alpha(p-1)}{p}(wq_0)^{\frac{1}{2}}|y|^p\right). \tag{2.26}$$

The proof can be completed by choosing $y_{n_k} = c_{n_k} y_1 + d_{n_k} y_2$ and $y = cy_1 + dy_2$ as before.

Example 3) Return to the case $-(x^2y')' + q_1 y = 0,\ 1 \le x < \infty$. In Theorem 4 take $\alpha = \frac{2}{p}$, $\beta = (p-1)\frac{\alpha^2}{4} + \delta$, $\delta > 0$, to obtain the bound $q_1 \ge \frac{-(p-1)}{p^2} + \delta$. The equation $-(x^{1-a-b}y')' - abx^{-a-b-1}y = 0,\ 1 \le x < \infty$, has solutions $y = x^a$ and $y = x^b$, if $a < b$, or $y = x^a$ and $y = x^a + x^a \ln x$, if $a = b$. For $1 < p < 2$, the choice $a = \frac{-1}{p}$, $b = \frac{(1-p)}{p}$ gives us the equation $-(x^2y')' + \frac{-(p-1)}{p^2}y = 0$. Neither solution $y = x^{\frac{-1}{p}}$ or $y = x^{\frac{(1-p)}{p}}$ is $L_p[1,\infty)$. This shows the constant bound $q_1 \ge \frac{-(p-1)}{p^2} + \delta$ is sharp.

3. For the equation $-(q_0y')' + q_1 y = 0,\ 0 \le x < \infty$, and $p = 2$, (2.8) is

$$0 = -q_0(0)d_n\overline{c}_n + \int_0^n q_0|y_n'|^2dx + \int_0^n q_1|y_n|^2dx.$$

Under the assumption $\mathrm{Im}(q_1) \ge 1$, we get

$$\int_0^n |y_n|^2dx \le q_0(0),$$

and hence a L_2 solution, without any restriction on $\mathrm{Re}(q_1)$. Furthermore if we suppose there exists positive constants k_1, k_2 such that

$$-\mathrm{Re}(q_1/q_0) \le k_1, \quad |(q_0)^{-1}q_0'| \le k_2,$$

then we have the following. Recall $y_n(n) = 0$.

$$\begin{aligned}
\int_0^n |y_n'|^2dx &= \int_0^n (q_0)^{-1}(q_0y_n')\overline{y}_n'dx \\
&= |_0^n y_n'\overline{y}_n - \int_0^n (q_0)^{-1}(q_0y_n')'\overline{y}_n dx - \int_0^n -(q_0)^{-2}q_0'(q_0y_n')\overline{y}_n dx \\
&= |_0^n y_n'\overline{y}_n - \int_0^n (q_0)^{-1}q_1|y_n|^2dx + \int_0^n (q_0)^{-1}q_0'y_n'\overline{y}_n dx \\
&\le \mathrm{Re}(y_n'(0)\overline{y}_n(0)) + k_1\int_0^n |y_n|^2dx + k_2\int_0^n |y_n'||y_n|dx \\
&\le 1 + k_1\int_0^n |y_n|^2dx + k_2\left(\int_0^n |y_n'|^2dx\right)^{\frac{1}{2}}\left(\int_0^n |y_n|^2dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\int_0^n |y_n'|^2dx \le 1 + k_1q_0(0) + k_2(q_0(0))^{\frac{1}{2}}\left(\int_0^n |y_n'|^2dx\right)^{\frac{1}{2}}.$$

It follows $\int_0^n |y_n'|^2 dx < \infty$. In fact

$$\int_0^n |y_n'|^2 dx \leq \left(k_2(q_0(0))^{\frac{1}{2}} + (k_2^2 q_0(0) + 4k_1 q_0(0) + 4)^{\frac{1}{2}}\right)/2.$$

Hence

$$\begin{aligned} |y_n(t)|^2 - |y_n(0)|^2 &= \int_0^t \left(\frac{d}{dx}|y_n(x)|^2\right) dx = \int_0^t (y_n \overline{y}_n' + y_n' \overline{y}_n) dx \\ &\leq 2\left(\int_0^n |y_n|^2 dx\right)^{\frac{1}{2}} \left(\int_0^n |y_n'|^2 dx\right)^{\frac{1}{2}} < \infty. \end{aligned}$$

We have the following.

Theorem 5. *Under the assumptions $Im(q_1) \geq 1$ and there exists positive constants k_1, k_2 such that $-Re(q_1/q_0) \leq k_1$, $|(q_0)^{-1} q_0'| \leq k_2$, equation (1.1) has a bounded $L_2[0,\infty)$ solution.*

More generally for $1 \leq p_0 < \infty$ suppose that $y(x)$ is bounded on $[a,\infty)$, say $\sup_{a \leq x < \infty} |y(x)| \leq B$, and $\int_a^\infty w|y|^{p_0} dx < \infty$, which we will call $y(x)$ bounded $L_{p_0}[w; a, \infty)$. Since $|y(x)|/B \leq 1$, $|y(x)|^p/B^p \leq |y(x)|^{p_0}/B^{p_0}$, for $p > p_0$. In which case $\int_0^t w|y|^p dx \leq B^{(p-p_0)} \int_0^t w|y|^{p_0} dx$. Bounded $L_{p_0}[w; a, \infty)$ implies $L_p[w; a, \infty)$ for all $p > p_0$.

Applying the same reasoning as in Theorem 5 to equation (2.14) $-u'' + gu = 0$ we have the following.

Theorem 6. a) *If $Im(w^{-1} g q_0) \geq \delta$, where δ is a positive constant, then equation (1.1) has a non-trivial solution $y = u(q_0)^{-\frac{1}{2}}$ that is $L_2[w; 0, \infty)$.*

Theorem 6. b) *If in addition we assume that there exists positive constants k_1, k_2, and k_3 such that $-Re(w^{-1} g q_0) \leq k_1$, $-(q_0)^{-1} q_0' w^{-1} \leq k_2$, and $w^{-\frac{1}{2}}(q_0)^{-\frac{1}{2}} \leq k_3$, then equation (1.1) has a non-trivial solution $y = u(q_0)^{-\frac{1}{2}}$ that is bounded $L_2[w; 0, \infty)$.*

proof of a). When $p = 2$, (2.23) and (2.24) are

$$0 \geq -k_0 + \delta \int_0^n w(q_0)^{-1} |u_n|^2 dx.$$

The argument is completed by choosing $y = u(q_0)^{-\frac{1}{2}}$ as in the proof of Theorem 3.

proof of b). When $p = 2$, (2.16)

$$0 = (\frac{1}{2}(q_0(0))^{-\frac{1}{2}} q_0'(0) c_n + (q_0(0))^{\frac{1}{2}} d_n)(q_0(0))^{\frac{1}{2}} \overline{c}_n$$
$$+ \int_0^n |u_n'|^2 dx + \int_0^n (w^{-1} g q_0) w q_0^{-1} |u_n|^2 dx.$$

By considering real parts we have

$$\int_0^n |u_n'|^2 dx \leq k_0 + k_1 \int_0^n w(q_0)^{-1} |u_n|^2 dx$$

and using a),

$$\int_0^n |u_n'|^2 dx \leq k_0 + \delta^{-1} k_0 k_1.$$

Now, for $0 \leq t \leq n$,

$$\begin{aligned}
|y_n(t)|^2 - |y_n(0)|^2 &= \int_0^t \left((q_0)^{-1} u_n \overline{u}_n\right)' dx \\
&= \int_0^t -(q_0)^{-2} q_0' |u_n|^2 dx + \int_0^t (q_0)^{-1} (u_n \overline{u}_n' + u_n' \overline{u}_n) dx \\
&= \int_0^t \left(-(q_0)^{-1} q_0' w^{-1}\right) w(q_0)^{-1} |u_n|^2 dx + \int_0^t w^{-\frac{1}{2}} (q_0)^{-\frac{1}{2}} w^{\frac{1}{2}} (q_0)^{-\frac{1}{2}} (u_n \overline{u}_n' + u_n' \overline{u}_n) dx \\
&\leq k_2 \int_0^t w(q_0)^{-1} |u_n|^2 dx + 2k_3 \int_0^t w^{\frac{1}{2}} (q_0)^{-\frac{1}{2}} |u_n| |u_n'| dx \\
&\leq k_2 \int_0^n w(q_0)^{-1} |u_n|^2 dx + 2k_3 \left(\int_0^n w(q_0)^{-1} |u_n|^2 dx\right)^{\frac{1}{2}} \left(\int_0^n |u_n'|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

The proof is completed by noting that

$$|y(x)| \leq |y(x) - y_{n_k}(x)| + |y_{n_k}(x)| \leq |c - c_{n_k}| |y_1(x)| + |d - d_{n_k}| |y_2(x)| + |y_{n_k}(x)|,$$

where $(c - c_{n_k}) \to 0$, $(d - d_{n_k}) \to 0$, and $|y_{n_k}(x)| \leq B$.

In the proof of Theorem 3 both $\int_0^n w(q_0)^{-\frac{p}{2}} |u_n|^p dx$ and $\int_0^n |u_n'|^2 |u_n|^{p-2} dx$ are bounded by a bound B independent of n. Under the additional assumption that there exists positive constants k_2 and k_3 such that $-(q_0)^{-1} q_0' w^{-1} \leq k_2$ and $w^{-\frac{1}{2}} (q_0)^{-\frac{p}{4}} \leq k_3$ we get that the solution $y =$

$u(q_0)^{-\frac{1}{2}}$ determined in the proof of Theorem 3 is bounded. To see this observe, for $0 \le t \le n$,

$$
\begin{aligned}
|y_n(t)|^p - |y_n(0)|^p &= \int_0^t \left((q_0)^{-\frac{p}{2}}(u_n\overline{u}_n)^{\frac{p}{2}}\right)' dx \\
&= \int_0^t -\frac{p}{2}(q_0)^{-\frac{p}{2}-1}q_0'|u_n|^p dx + \int_0^t (q_0)^{-\frac{p}{2}}\frac{p}{2}(u_n\overline{u}_n)^{\frac{p}{2}-1}(u_n\overline{u}_n' + u_n'\overline{u}_n)dx \\
&\le \frac{p}{2}\int_0^t \left(-(q_0)^{-1}q_0'w^{-1}\right) w(q_0)^{-\frac{p}{2}}|u_n|^p dx \\
&\quad + p\int_0^t w^{-\frac{1}{2}}(q_0)^{-\frac{p}{4}}\left(w^{\frac{1}{2}}(q_0)^{-\frac{p}{4}}|u_n|^{\frac{p}{2}}\right)\left(|u_n'||u_n|^{\frac{p}{2}-1}\right) dx \\
&\le \frac{p}{2}k_2\int_0^t w(q_0)^{-\frac{p}{2}}|u_n|^p dx + pk_3\left(\int_0^t w(q_0)^{-\frac{p}{2}}|u_n|^p dx\right)^{\frac{1}{2}}\left(\int_0^t |u_n'|^2|u_n|^{p-2}dx\right)^{\frac{1}{2}} \\
&\le \left(\frac{p}{2}k_2 + pk_3\right)B.
\end{aligned}
$$

We have the following.

Corollary 3. *Assume the hypothesis of Theorem 3 a) or b). Suppose in addition there exists positive constants k_2 and k_3 such that $-(q_0)^{-1}q_0'w^{-1} \le k_2$ and $w^{-\frac{1}{2}}(q_0)^{-\frac{p}{4}} \le k_3$. Then (1.1) has a bounded $L_p[w;0,\infty)$ solution.*

Example 4) Let $q_0(x)$ be real valued, positive, and continuously differentiable on $(0,1]$. Let $q_1(x)$ be complex valued and locally Lebesgue integrable in $(0,1]$. Let $w(x)$ be a positive, real valued, and continuous weight function on $(0,1]$. Then the differential equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0 \tag{3.1}$$

will be said to have a $L_p(w;(0,1])$ solution, $1 \le p < \infty$, if it has a solution $y(x)$ such that

$$\lim_{\epsilon\to 0+}\int_\epsilon^1 w(x)|y(x)|^p dx < \infty.$$

If we make the change of variable $y(x) = z(t)$ with $x = t^{-1}$, this follows:

$$y'(x) = z^{\cdot}(x^{-1})(-x^{-2}) = -t^2 z^{\cdot}(t),$$

$$y''(x) = z^{\cdot\cdot}(x^{-1})(x^{-4}) + z^{\cdot}(x^{-1})(2x^{-3}) = t^4 z^{\cdot\cdot}(t) + 2t^3 z^{\cdot}(t),$$

and

$$dx = -t^{-2}dt$$

($()^{\cdot}$ denotes differentiation with respect to t). Substituting into (3.1) written as $-q_0y'' - q_0'y' + q_1y = 0$ we have

$$-t^4q_0(t^{-1})z^{\cdot\cdot}(t) - (2t^3q_0(t^{-1}) - t^2q_0'(t^{-1}))z^{\cdot}(t) + q_1(t^{-1})z(t) = 0.$$

After dividing by t^2 we can write the equation as

$$-(t^2q_0(t^{-1})z^{\cdot}(t))^{\cdot} + t^{-2}q_1(t^{-1})z(t) = 0.$$

Setting $\gamma_0(t) = t^2q_0(t^{-1})$ and $\gamma_1(t) = t^{-2}q_1(t^{-1})$ we have

$$-(\gamma_0(t)z^{\cdot}(t))^{\cdot} + \gamma_1(t)z(t) = 0, \quad 1 \le t < \infty. \tag{3.2}$$

The integral

$$\int_{\varepsilon}^{1} w(x)|y(x)|^p dx = \int_{\varepsilon^{-1}}^{1} w(t^{-1})|z(t)|^p(-t^{-2}dt) = \int_{1}^{\varepsilon^{-1}} t^{-2}w(t^{-1})|z(t)|^p dt.$$

Set $\Omega(t) = t^{-2}w(t^{-1})$ so that we seek conditions that give us a $L_p[\Omega; 1, \infty)$ solution of equation (3.2). Let $\Omega(t) = t^{\phi}$, $\gamma_0(t) = \alpha t^{\beta}$, $\alpha > 0$. Then $g(t) = \gamma_1(t)\alpha^{-1}t^{-\beta} + \frac{1}{2}\beta(\frac{1}{2}\beta - 1)t^{-2}$ and $\Omega^{-1}(t)g(t)(\gamma_0(t))^{\frac{p}{2}} = \gamma_1(t)\alpha^{(\frac{p}{2}-1)}t^{(\frac{p}{2}-1)\beta-\phi} + \alpha^{\frac{p}{2}}\frac{1}{2}\beta(\frac{1}{2}\beta - 1)t^{\frac{p}{2}\beta-2-\phi}$. Theorem 3 a) gives us a $L_p[\Omega; 1, \infty)$ solution exists if $\mathrm{Re}(\gamma_1) \ge \delta\alpha^{(1-\frac{p}{2})}t^{(1-\frac{p}{2})\beta+\phi} - \alpha\frac{1}{2}\beta(\frac{1}{2}\beta - 1)t^{\beta-2}$. In the case $\beta < 0$, for some $\gamma_1(t)$ the bound on $\mathrm{Re}(\gamma_1)$ may hold for a fixed p, but not hold as $p \to \infty$. Consider when $\beta < 0$ and $\phi \ge -\frac{p}{2}\beta > 0$. Note that $-(\gamma_0)^{-1}\gamma_0'\Omega^{-1} = -(\alpha t^{\beta})^{-1}(\alpha\beta t^{\beta-1})t^{-\phi} = -\beta t^{-\phi-1} \le -\beta$ and $\Omega^{-\frac{1}{2}}(\gamma_0)^{-\frac{p}{4}} = \alpha^{-\frac{p}{4}}t^{-\frac{1}{2}\phi-\frac{p}{4}\beta} = \alpha^{-\frac{p}{4}}t^{-\frac{1}{2}(\phi+\frac{p}{2}\beta)} \le \alpha^{-\frac{p}{4}}$, for $1 \le t < \infty$. Now, use Corollary 3 to conclude that (3.2) has a bounded $L_p[\Omega; 1, \infty)$ solution $z(t)$. Converting back to the original variable $x = t^{-1}$ we have $q_0(x) = x^2\gamma_0(x^{-1}) = \alpha x^{2-\beta}$, $\gamma_1(t) = x^2q_1(x)$, and $w(x) = x^{-2}\Omega(x^{-1}) = x^{-2-\phi}$. The bound on $\mathrm{Re}(\gamma_1)$ becomes $\mathrm{Re}(x^2q_1(x)) \ge \delta\alpha^{(1-\frac{p}{2})}x^{(\frac{p}{2}-1)\beta-\phi} - \alpha\frac{1}{2}\beta(\frac{1}{2}\beta - 1)x^{-\beta+2}$. Summarizing the preceeding we have this. Let $w(x) = x^{-2-\phi}$, $q_0(x) =$

$\alpha x^{2-\beta}$, $\alpha > 0$ and either $\beta \geq 0$ or $\phi \geq -\frac{p}{2}\beta > 0$. Then (3.1) has a bounded $L_p(w;(0,1])$ solution $y(x)$ if

$$\mathrm{Re}(q_1) \geq \delta\alpha^{(1-\frac{p}{2})}x^{(\frac{p}{2}-1)\beta-\phi-2} - \alpha\frac{1}{2}\beta(\frac{1}{2}\beta-1)x^{-\beta}, \quad 0 < x \leq 1.$$

Similarly we can apply Theorem 3 b). This compares to

$$\mathrm{Re}(q_1) \geq \delta x^{-\phi-2}$$

obtained by applying Theorem 1 instead of Theorem 3.

Let $w(x)$ be a positive and continuous weight function as before. In Theorem 1 replace the condition $\mathrm{Re}(q_1) \geq 1$ by $\mathrm{Re}(w^{-1}q_1) \geq 1$. Then (2.3) can be written as

$$0 = q_0(0)d_n\bar{c}_n|c_n|^{p-2}+\frac{p}{2}\int_0^n q_0|y_n'|^2|y_n|^{p-2}dx+(\frac{p}{2}-1)\int_0^n q_0(y_n')^2(\overline{y}_n)^2|y_n|^{p-4}dx+\int_0^n (w^{-1}q_1)w|y_n|^p dx.$$

In which case

$$\int_0^n w|y_n|^p dx \leq q_0(0)$$

follows instead of (2.6). In Theorem 2 replace $\mathrm{Im}(q_1) \geq 1$ by $\mathrm{Im}(w^{-1}q_1) \geq 1$ to obtain

$$\int_0^n w|y_n|^p dx \leq (p/(2(p-1)))q_0(0)$$

and

$$\int_0^n w|y_n|^p dx \leq \frac{p}{2}q_0(0)$$

instead of (2.12) and (2.13). Hence Theorems 1 and 2 can be restated as follows.

Theorem 1*. *Let $Re(w^{-1}(x)q_1(x)) \geq 1$ on $[0,\infty)$. Then equation (1.1) has a non-trivial solution that is $L_p[w;0,\infty)$, for $1 \leq p \leq \infty$.*

Theorem 2*. *Let $Re(q_1(x)) \geq 0$ and $Im(w^{-1}(x)q_1(x)) \geq 1$. Then equation (1.1) has a non-trivial solution that is $L_p[w;0,\infty)$, for $1 < p \leq \infty$.*

Now, we have the following.

Corollary 1*. *If $Re(q_1(x)) \geq w(x)$ for a positive continuous function w, then (1.1) has a bounded solution. In particular equation (1.1) has a solution $y(x)$ with the property that $|y(x)| \leq 1$ on $[0,\infty)$ and $|y(0)|^2 + |y'(0)|^2 = 1$.*

However, we can prove the stronger result.

Theorem 7. *If $Re(q_1(x)) \geq 0$, then (1.1) has a bounded solution $y(x)$. This solution $y(x)$ has the property that $|y(x)| \leq 1$ on $[0,\infty)$ and $|y(0)|^2 + |y'(0)|^2 = 1$.*

Before we give the proof of Theorem 7 we state some formulas used. We frequently have the need to differentiate a function H of the sort

$$H(x) = \int_a^x h(x,t)dt \tag{3.3}$$

where $h(x,t)$ and $\frac{\partial h}{\partial x}(x,t)$ are continuous. We have

$$H'(x) = h(x,x) + \int_a^x \frac{\partial h}{\partial x}(x,t)dt. \tag{3.4}$$

Consider the inhomogeneous equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = f(x), \quad a \leq x < \infty, \tag{3.5}$$

where q_0, q_1 are as in (1.1) and f is locally Lebesgue integrable in $[a,\infty)$. Suppose that $y_1(x)$ and $y_2(x)$ are two independent solutions of the homogeneous equation $-(q_0(x)y'(x))' + q_1(x)y(x) = 0$ such that $q_0(a)\left(y_1(a)y_2'(a) - y_1'(a)y_2(a)\right) = 1$. From the calculation used to get (2.2) we have

$$q_0(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right) = 1.$$

Let

$$y(x) = cy_1(x) + dy_2(x) + \int_a^x \left(y_1(x)y_2(t) - y_1(t)y_2(x)\right) f(t)dt. \tag{3.6}$$

Using (3.4) to differentiate the last integral we have

$$y'(x) = cy_1'(x) + dy_2'(x) + \int_a^x (y_1'(x)y_2(t) - y_1(t)y_2'(x))f(t)dt$$

and

$$
\begin{aligned}
(q_0(x)y'(x))' &= c(q_0(x)y_1'(x))' + d(q_0(x)y_2'(x))' + q_0(x)\left(y_1'(x)y_2(x) - y_1(x)y_2'(x)\right) f(x) \\
&\quad + \int_a^x \left((q_0(x)y_1'(x))'y_2(t) - y_1(t)(q_0(x)y_2'(x))'\right) f(t)dt \\
&= cq_1(x)y_1(x) + dq_1(x)y_2(x) - f(x) + q_1(x)\int_a^x \left(y_1(x)y_2(t) - y_1(t)y_2(x)\right) f(t)dt,
\end{aligned}
$$

provided x is a Lebesgue point of $f(x)$. Thus $y(x)$ is a solution of (3.5) with initial values

$$y(a) = cy_1(a) + dy_2(a), \quad y'(a) = cy_1'(a) + dy_2'(a).$$

The function

$$G(x,t) = y_1(x)y_2(t) - y_1(t)y_2(x)$$

is called a Green's function for equation (3.5). The solution of (3.5) given by

$$y(x) = \int_a^x G(x,t)f(t)dt$$

has initial values $y(a) = y'(a) = 0$.

The following is called Gronwall's inequality. Let u and v be nonnegative and continuous functions on $[a,b]$ and let c be a nonnegative constant. If $u(x) \le c + \int_a^x u(t)v(t)dt$, $a \le x \le b$, then $u(x) \le c(\exp[\int_a^x v(t)dt])$, for $a \le x \le b$.

Proof of Theorem 7. Suppose for convenience $q_0(0) = 1$. (If required $q_0(x)$ and $q_1(x)$ can be divided by $q_0(0)$.) Let $y_1(x)$ and $y_2(x)$ be solutions of equation (1.1) such that $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 0$, $y_2'(0) = 1$. By (2.2)

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1.$$

A solution $y(x)$ of (1.1) satisfies

$$y(x) = y(0) + \int_0^x (q_0(t))^{-1}(q_0(t)y'(t))dt$$

and

$$q_0(x)y'(x) = q_0(0)y'(0) + \int_0^x q_1(t)y(t)dt.$$

So

$$|y(x)| + |q_0(x)y'(x)| \leq |y(0)| + |q_0(0)y'(0)| + \int_0^x \left((q_0(t))^{-1} + |q_1(t)|\right)\left(|y(t)| + |q_0(t)y'(t)|\right)dt.$$

Making use of Gronwall's inequality we get

$$|y(x)| + |q_0(x)y'(x)| \leq (|y(0)| + |q_0(0)y'(0)|)\exp\left(\int_0^x ((q_0(t))^{-1} + |q_1(t)|)dt\right). \tag{3.7}$$

In particular

$$|y_j(x)| + |q_0(x)y_j'(x)| \leq \exp\left(\int_0^x ((q_0(t))^{-1} + |q_1(t)|)dt\right),$$

for $j = 1, 2$. Choose any positive and continuous function $w(x)$ such that

$$\int_0^\infty w(x)dx \leq 1.$$

Let

$$\alpha_n = (2)^{-n-1}\exp(-2\int_0^n ((q_0(t))^{-1} + |q_1(t)|)dt), \qquad \text{for } n = 1, 2, \ldots.$$

For each n consider the equation

$$-(q_0(x)z'(x))' + (q_1(x) + \alpha_n w(x))z(x) = 0.$$

By Corollary 1* the equation above has a bounded solution $z(x)$ such that $|z(x)| \leq 1$, for $0 \leq x < \infty$, and $|z(0)|^2 + |z'(0)|^2 = 1$. Call this solution $z_n(x)$. $z_n(x)$ is also the solution of the inhomogeneous equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = -\alpha_n w(x)z_n(x)$$

with initial values $y(0) = z_n(0)$, $y'(0) = z_n'(0)$. Using formula (3.6) we get that

$$z_n(x) = c_n y_1(x) + d_n y_2(x) + \int_0^x (y_1(x)y_2(t) - y_1(t)y_2(x))(-\alpha_n w(t)z_n(t))dt,$$

with $c_n = z_n(0)$, $d_n = z_n'(0)$. For $0 \le x \le n$,

$$|z_n(x) - (c_n y_1(x) + d_n y_2(x))| \le \int_0^x (|y_1(x)y_2(t)| + |y_1(t)y_2(x)|)\alpha_n w(t)|z_n(t)|dt$$
$$\le 2\exp(2\int_0^n ((q_0(t))^{-1} + |q_1(t)|)dt)\alpha_n \int_0^n w(t)dt < 2^{-n}.$$

Hence for $0 \le x \le n$,

$$|c_n y_1(x) + d_n y_2(x)| < 1 + 2^{-n}.$$

Since $|c_n|^2 + |d_n|^2 = 1$, we can choose a subsequence (c_{n_k}, d_{n_k}) of (c_n, d_n) such that (c_{n_k}, d_{n_k}) converges as $k \to \infty$ to some (c, d) such that $|c|^2 + |d|^2 = 1$. For $n_k > x$,

$$|cy_1(x) + dy_2(x)| \le |(c - c_{n_k})y_1(x) + (d - d_{n_k})y_2(x)| + |c_{n_k}y_1(x) + d_{n_k}y_2(x)|$$
$$< |c - c_{n_k}||y_1(x)| + |d - d_{n_k}||y_2(x)| + 1 + 2^{-n_k}.$$

Letting $k \to \infty$ we get

$$|cy_1(x) + dy_2(x)| \le 1.$$

$y(x) = cy_1(x) + dy_2(x)$ is a bounded solution of (1.1).

Example 5) Let $b_0(x)$ be positive and continuous and $b_1(x)$ real and continuous on $[0, \infty)$. Let $b_2(x)$ be complex valued and locally Lebesgue integrable in $[0, \infty)$. Consider the differential equation

$$b_0(x)y''(x) + b_1(x)y'(x) + b_2(x)y(x) = 0, \quad 0 \le x < \infty. \tag{3.8}$$

After dividing the equation by $-b_0(x)$ and then multiplying by the integrating factor

$$\exp\left(\int_0^x b_1(t)(b_0(t))^{-1}dt\right),$$

the equation can be written in the form $-(q_0y')' + q_1y = 0$, with

$$q_0(x) = \exp\left(\int_0^x b_1(b_0)^{-1}dt\right) \text{ and } q_1(x) = -(b_2(b_0)^{-1})\exp\left(\int_0^x b_1(b_0)^{-1}dt\right).$$

Applying Theorem 7 we have that (3.8) has a bounded solution $y(x)$ if

$$\mathrm{Re}(b_2(x)) \leq 0.$$

Corollary 7. *Let $\eta(x)$ be positive and twice continuously differentiable on $[0,\infty)$. If $q_0\eta'' + q_0'\eta' - Re(q_1)\eta \leq 0$, then (1.1) has a non-trivial solution $y(x)$ such that $\eta^{-1}(x)y(x)$ is bounded. Furthermore, if $\eta(x)$ is $L_p[w;0,\infty)$ then so is $y(x)$.*

Proof. Let

$$y(x) = \eta(x)z(x).$$

Substitute $y = \eta z$, $y' = \eta z' + \eta' z$, and $y'' = \eta z'' + 2\eta' z' + \eta'' z$ into $q_0 y'' + q_0' y' - q_1 y = 0$. We obtain

$$q_0\eta z'' + (2q_0\eta' + q_0'\eta)z' + (q_0\eta'' + q_0'\eta' - q_1\eta)z = 0. \tag{3.9}$$

Using the conclusion of Example 5) we have that equation (3.9) has a bounded solution $z(x)$ if

$$q_0\eta'' + q_0'\eta' - \mathrm{Re}(q_1)\eta \leq 0.$$

$y(x) = \eta(x)z(x)$ is a solution of (1.1). If $\eta(x)$ is bounded then so is $y(x)$. If $\eta(x)$ is $L_p[w;0,\infty)$, then

$$\lim_{t\to\infty}\int_0^t w|y|^p dx = \lim_{t\to\infty}\int_0^t w\eta^p|z|^p dx < \infty.$$

Example 6) Let $\eta(x) = e^{-rx}$. Then (1.1) has a solution $y(x)$ with

$$e^{rx}y(x)$$

bounded if

$$\mathrm{Re}(q_1) \geq \eta^{-1}(q_0\eta'' + q_0'\eta') = r^2 q_0 - rq_0'.$$

Applying Corollary 7 to equation (2.14) we get a solution such that

$$(q_0)^{\frac{1}{2}}\eta^{-1}y = \eta^{-1}u$$

is bounded if

$$\mathrm{Re}(q_1) \geq \eta^{-1}\eta'' q_0 + \frac{1}{4}(q_0)^{-1}(q_0')^2 - \frac{1}{2}q_0''.$$

For $q_0 = e^{kx}$ and $\eta = e^{-rx}$ we have that (1.1) has a solution $y(x)$ with

$$e^{(\frac{k}{2}+r)x}y(x)$$

bounded if

$$\mathrm{Re}(q_1) \geq (r^2 - \frac{1}{4}k^2)e^{kx}.$$

Example 7) Let the interval be $[1,\infty)$ instead of $[0,\infty)$. For $\eta(x) = x^{\Delta}$, (1.1) has a solution $y(x)$ with

$$x^{-\Delta}y(x)$$

bounded if

$$\mathrm{Re}(q_1) \geq \eta^{-1}(q_0\eta'' + q_0'\eta') = \Delta(q_0(\Delta-1)x^{-2} + q_0'x^{-1}).$$

When $q_0 = \alpha x^{\beta}$, this is

$$\mathrm{Re}(q_1) \geq \alpha\Delta((\Delta-1)+\beta)x^{\beta-2}.$$

If

$$\mathrm{Re}(q_1) \geq \eta^{-1}\eta'' q_0 + \frac{1}{4}(q_0)^{-1}(q_0')^2 - \frac{1}{2}q_0'' = \alpha(\Delta(\Delta-1) - \frac{1}{2}\beta(\frac{1}{2}\beta - 1))x^{\beta-2},$$

then (1.1) has a solution $y(x)$ with

$$\alpha^{\frac{1}{2}}x^{\frac{\beta}{2}-\Delta}y(x)$$

bounded.

4. A Theorem due to Halvorsen [1] and generalized by Hinton [2] gives conditions that are sufficient for equation (1.1) to be limit-circle L_2. It is easily modified by substituting the exponent p for the exponent 2 to get conditions sufficient for (1.1) to be limit-circle L_p, $1 \leq p \leq \infty$.

Theorem (Halvorsen, Hinton). *Equation (1.1) is limit-circle L_2 if there exists a real function $\eta(x) > 0$ and a real number $c > 0$ such that*

i) η, $q_0\eta'$ *are locally absolutely continuous on* $[a, \infty)$

ii) $\int_a^\infty \frac{dx}{q_0\eta^2} = \infty$

iii) $\int_a^\infty \eta^2(x)e^{g(x)}dx < \infty$, *where* $g(x) = \frac{1}{c}\int_a^x h(\xi)d\xi$, *and* $h = |-q_1 + \frac{(q_0\eta')'}{\eta} - \frac{c^2}{q_0\eta^4}|\eta^2$.

Theorem 8. *Equation (1.1) is limit-circle $L_p[w; a, \infty)$, for $1 \leq p < \infty$, if there exists a real function $\eta(x) > 0$ and a real number $c > 0$ such that*

i) η, $q_0\eta'$ *are locally absolutely continuous on* $[a, \infty)$

ii) $\int_a^\infty \frac{dx}{q_0\eta^2} = \infty$

iii) $\int_a^\infty w(x)\eta^p(x)e^{\frac{p}{2}g(x)}dx < \infty$, *where* $g(x) = \frac{1}{c}\int_a^x h(\xi)d\xi$, *and* $h = |-q_1 + \frac{(q_0\eta')'}{\eta} - \frac{c^2}{q_0\eta^4}|\eta^2$.

The case $p = \infty$ holds if instead of iii) we have $\eta(x)$ and $g(x)$ are bounded.

Proof. Suppose $y(x)$ is a solution of (1.1) and i), ii), and iii) hold. We will show

$$\int_a^\infty w(x)|y(x)|^p dx < \infty.$$

Set

$$z(t) = \eta^{-1}(x)y(x),$$

with

$$t = \int_a^x \frac{ds}{q_0(s)\eta^2(s)}.$$

(()$^{\cdot}$ as before denotes differentiation with respect to t.)

$$z^{\cdot} = \left(\frac{y}{\eta}\right)' / \frac{dt}{dx} = \left(\frac{y}{\eta}\right)' q_0\eta^2 = (\eta(q_0y') - (q_0\eta')y)$$

and

$$
\begin{aligned}
z^{\cdot\cdot} &= (\eta(q_0 y') - (q_0\eta')y)'/\frac{dt}{dx} \\
&= (\eta(q_0 y')' + \eta'(q_0 y') - (q_0\eta')'y - (q_0\eta')y')q_0\eta^2 \\
&= (\eta(q_1 y) - (q_0\eta')'y)q_0\eta^2 \\
&= q_0\eta^4(q_1 - \frac{(q_0\eta')'}{\eta})(\eta^{-1}y) \\
&= q_0\eta^4(q_1 - \frac{(q_0\eta')'}{\eta})z.
\end{aligned}
$$

Therefore

$$z^{\cdot\cdot}(t) + f(t)z(t) = 0$$

with

$$f(t) = (q_0\eta^4(-q_1 + \frac{(q_0\eta')'}{\eta}))(x).$$

Set

$$E(t) = (|z^{\cdot}(t)|^2 + c^2|z(t)|^2)/2.$$

It follows that

$$
\begin{aligned}
E^{\cdot}(t) &= (z^{\cdot}\overline{z}^{\cdot} + z^{\cdot\cdot}\overline{z} + c^2(z\overline{z}^{\cdot} + z^{\cdot}\overline{z}))/2 \\
&= (z^{\cdot}(\overline{z}^{\cdot\cdot} + c^2\overline{z}) + \overline{z}(z^{\cdot\cdot} + c^2 z))/2 \\
&= (z^{\cdot}(-\overline{f(t)} + c^2)\overline{z} + \overline{z}(-f(t) + c^2)z)/2.
\end{aligned}
$$

We have

$$|E^{\cdot}| \leq |z^{\cdot}(cz)|\frac{1}{c}|c^2 - f(t)|.$$

Since

$$|z^{\cdot}(cz)| \leq (|z^{\cdot}|^2 + c^2|z|^2)/2$$

we get

$$|E^{\cdot}| \leq E\frac{1}{c}|c^2 - f(t)|.$$

Thus

$$\frac{E(t)}{E(0)} = \exp\int_0^t \frac{E^{\cdot}(s)}{E(s)}ds \le \exp\int_0^t \frac{1}{c}|c^2 - f(s)|ds$$
$$= \exp\int_a^x \frac{1}{c}|c^2 - q_0\eta^4(-q_1 + \frac{(q_0\eta')'}{\eta})|\frac{1}{q_0\eta^2}d\xi = \exp\frac{1}{c}\int_a^x h(\xi)d\xi = e^{g(x)}.$$

Therefore

$$\frac{E(t)}{E(0)} = \frac{|z^{\cdot}(t)|^2 + c^2|z(t)|^2}{2E(0)} \le e^{g(x)}.$$

From which we get

$$|z(t)|^p \le (2E(0))^{\frac{p}{2}}c^{-p}e^{\frac{p}{2}g(x)}.$$

Hence

$$\int_a^\infty w(x)|y(x)|^p dx = \int_a^\infty w(x)\eta^p(x)|z(t)|^p dx \le (2E(0))^{\frac{p}{2}}c^{-p}\int_a^\infty w(x)\eta^p(x)e^{\frac{p}{2}g(x)}dx < \infty.$$

Example 8) Limit-circle L_p condition for $q_0(x) = x^\beta$, $-q_1(x) = mx^\gamma$. Let $\eta(x) = x^\Delta$. Condition ii) is satisfied for $a > 0$ and $\frac{1}{q_0\eta^2} = x^{-(\beta+2\Delta)}$, if $\beta + 2\Delta \le 1$. Choose $c^2 = m > 0$ and $-\beta - 4\Delta = \gamma$ so that $-q_1 - \frac{c^2}{q_0\eta^4} = 0$. Then $h = |(q_0\eta')'\eta| = |\Delta(\beta + \Delta - 1)|x^{\beta+2\Delta-2}$. $g(x) = \frac{1}{c}\int_a^x h(\xi)d\xi$ is bounded if $\beta + 2\Delta < 1$. When $g(x)$ is bounded condition iii) is satisfied if $\int_a^\infty \eta^p dx = \int_a^\infty x^{p\Delta}dx < \infty$. This holds if $\Delta < -\frac{1}{p}$. Hence ii) and iii) hold if $\beta + 2\Delta < 1$ and $\Delta < -\frac{1}{p}$. We have two cases (both with $m > 0$).

Case 1) Equation is limit-circle L_p if $\beta \le 1 + \frac{2}{p}$ and $\gamma > \frac{4}{p} - \beta$.

To see this check that $\gamma = -\beta - 4\Delta > \frac{4}{p} - \beta$ gives $\Delta < -\frac{1}{p}$. $\Delta < -\frac{1}{p}$ and $\beta \le 1 + \frac{2}{p}$ gives $\beta + 2\Delta < \beta - \frac{2}{p} \le 1$.

Case 2) Equation is limit-circle L_p if $\beta > 1 + \frac{2}{p}$ and $\gamma > \beta - 2$.

To see this check that $\gamma = -\beta - 4\Delta > \beta - 2$ gives $\Delta < \frac{1}{2}(1 - \beta)$ so that $\beta + 2\Delta < 1$. $\Delta < \frac{1}{2}(1 - \beta)$ and $\beta > 1 + \frac{2}{p}$ gives $\Delta < -\frac{1}{p}$.

When $\beta > 1 + \frac{2}{p}$ and $\gamma = \beta - 2$, let $\Delta = \frac{1}{2}(1 - \beta)$. Condition ii) is satisfied and $h = |-q_1 + \frac{(q_0\eta')'}{\eta} - \frac{c^2}{q_0\eta^4}|\eta^2 = |mx^\gamma + \Delta(\Delta - 1 + \beta)x^{\beta-2} - c^2x^{-\beta-4\Delta}|x^{2\Delta} = |m - \Delta^2 - c^2|x^{-1}$.

For simplicity assume $a = 1$ so that $g(x) = \frac{1}{c}|m - \Delta^2 - c^2| \ln x$ and $e^{\frac{p}{2}g(x)} = x^{\frac{p}{2c}|m-\Delta^2-c^2|}$. We have $\int_1^\infty \eta^p e^{\frac{p}{2}g} dx = \int_1^\infty x^{p\Delta + \frac{p}{2c}|m-\Delta^2-c^2|} dx < \infty$ for $p\Delta + \frac{p}{2c}|m - \Delta^2 - c^2| < -1$. That is when $c^2 + (\frac{2}{p} + 2\Delta)c + \Delta^2 < m < c^2 - (\frac{2}{p} + 2\Delta)c + \Delta^2$. The left hand side is minimal for $c = -(\frac{1}{p} + \Delta)$. This gives that the equation is limit-circle $L_p[1, \infty)$ if $m > c^2 + (\frac{2}{p} + 2\Delta)c + \Delta^2 \geq \frac{-2\Delta}{p} - \frac{1}{p^2} = (p\beta - (p+1))/p^2$.

Case 3) Equation is limit-circle L_p if $\beta > 1 + \frac{2}{p}$, $\gamma = \beta - 2$, and $m > (p\beta - (p+1))/p^2$.

When $\beta \leq 1$ and $\gamma > -\beta$, let $\eta = x^\Delta$ with $\gamma = -\beta - 4\Delta$ and take $c^2 = m > 0$. Then $\Delta < 0$ and $\beta + 2\Delta < 1$. Thus $h = |\Delta(\Delta - 1 + \beta)| x^{\beta + 2\Delta - 2}$ and $g(x)$ is bounded. In fact since $\eta(x) \to 0$ as $x \to \infty$ it follows for all solutions $y(x)$ of the equation $y(x) \to 0$ as $x \to \infty$. When $\beta > 1$ and $\gamma > \beta - 2$, let $\eta = x^\Delta$ with $\gamma = -\beta - 4\Delta$ and take $c^2 = m > 0$. Then $\Delta < \frac{1}{2}(1 - \beta) < 0$ and again for all solutions $y(x)$ of the equation $y(x) \to 0$ as $x \to \infty$.

Case 4) Equation is limit-circle L_∞ if either $\beta \leq 1$ and $\gamma > -\beta$ or $\beta > 1$ and $\gamma > \beta - 2$. In particular for all solutions $y(x)$ of the equation $y(x) \to 0$ as $x \to \infty$.

5. In this section we will consider the inhomogeneous equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = w(x)f(x), \quad a \leq x < \infty \tag{5.1}$$

where the conditions on $q_0(x)$, $q_1(x)$ are the same as for equation (1.1), $w(x)$ is a positive and continuous weight function, and $f(x)$ is locally Lebesgue integrable in $[a, \infty)$. Making use of formula (3.6) we get that if $y_1(x)$ and $y_2(x)$ are two solutions of (1.1) such that $q_0(y_1 y_2' - y_1' y_2) = 1$, then

$$y(x) = cy_1(x) + dy_2(x) + \int_a^x (y_1(x)y_2(t) - y_1(t)y_2(x))w(t)f(t)dt \tag{5.2}$$

is a solution of (5.1) for each choice of c, d. If we let $c = 0$ and $d = \int_a^b wy_1 f dt$, for $a \leq x \leq b$,

$$y(x) = y_1(x) \int_a^x wy_2 f dt + y_2(x) \int_x^b wy_1 f dt.$$

If y_1 is $L_{p/(p-1)}[w; a, \infty)$ and f is $L_p[w; a, \infty)$ then letting $b \to \infty$ we have that $\int_x^\infty wy_1 f dt$ is

convergent to a complex number for each x. In which case

$$y(x) = y_1(x)\int_a^x wy_2 f dt + y_2(x)\int_x^\infty wy_1 f dt, \quad a \le x < \infty, \tag{5.3}$$

is a solution of (5.1) with initial values

$$y(a) = y_2(a)\int_a^\infty wy_1 f dt \text{ and } y'(a) = y_2'(a)\int_a^\infty wy_1 f dt.$$

Theorem 10. *If f is $L_{p/(p-1)}[w; a, \infty)$ and (1.1) is limit-circle $L_p[w; a, \infty)$, then all solutions of (5.1) are $L_p[w; a, \infty)$.*

Proof. From (5.2) all solutions of (5.1) are of the form

$$y(x) = y_1(x)\left(c + \int_a^x wy_2 f dt\right) + y_2(x)\left(d - \int_a^x wy_1 f dt\right).$$

Note

$$\left|\int_a^x wy_j f dt\right| \le \left(\int_a^\infty w|y_j|^p dt\right)^{\frac{1}{p}} \left(\int_a^\infty w|f|^{p/(p-1)} dt\right)^{(p-1)/p}, \quad j = 1, 2,$$

and apply Minkowski's inequality to get $y(x)$ is $L_p[w; a, \infty)$.

Theorem 11. *Suppose that the conditions of Theorem 1* hold for $q_0(x)$ and $q_1(x)$ and $f(x)$ is $L_p[w; 0, \infty)$ for some p, $1 < p < \infty$. Then the inhomogeneous equation (5.1) $-(q_0 y')' + q_1 y = wf$ has a non-trivial solution that is $L_p[w; 0, \infty)$.*

Proof. Let

$$z_{1n}(x) = c_n y_1(x) + d_n y_2(x) \text{ and } z_{2n}(x) = (-\overline{d}_n/q_0(0))y_1(x) + (\overline{c}_n/q_0(0))y_2(x)$$

where $y_1(x)$, $y_2(x)$, c_n, and d_n are chosen as in the proof of Theorem 1. Let

$$f_j(t) = \begin{cases} f(t), & 0 \le t < j, \quad \text{for } j = 1, 2, \ldots. \\ 0, & j \le t \end{cases}$$

Then

$$q_0(z_{1n}z_{2n}' - z_{1n}'z_{2n}) = q_0(0)(c_n(\overline{c}_n/q_0(0)) - d_n(-\overline{d}_n/q_0(0))) = |c_n|^2 + |d_n|^2 = 1$$

and

$$z_{nj}(x) = z_{1n}(x)\int_0^x wz_{2n}f_j\,dt + z_{2n}(x)\int_x^\infty wz_{1n}f_j\,dt$$

is a solution of

$$-(q_0y')' + q_1y = wf_j$$

with initial values

$$z_{nj}(0) = \left(\int_0^j wz_{1n}f\,dt\right)(-\bar{d}_n/q_0(0)) \text{ and } z'_{nj}(0) = \left(\int_0^j wz_{1n}f\,dt\right)(\bar{c}_n/q_0(0)).$$

Multiply both sides of $-(q_0y')'+q_1y = wf_j$ by $\bar{y}|y|^{p-2}$ and integrate. From the same calculation used to obtain (2.1) we get

$$|_0^t - q_0y'\bar{y}|y|^{p-2} + \frac{p}{2}\int_0^t q_0|y'|^2|y|^{p-2}dx + (\frac{p}{2}-1)\int_0^t q_0(y')^2(\bar{y})^2|y|^{p-4}dx$$
$$+\int_0^t q_1|y|^p\,dx = \int_0^t wf_j\bar{y}|y|^{p-2}dx. \tag{5.4}$$

Consider $n \geq j$ so that

$$z_{nj}(n) = z_{1n}(n)\int_0^n wz_{2n}f_j\,dt + z_{2n}(n)\int_n^\infty wz_{1n}f_j\,dt = 0.$$

Substitute $y = z_{nj}$ and $t = n$ into (5.4) to get

$$q_0(0)z'_{nj}(0)\bar{z}_{nj}(0)|z_{nj}(0)|^{p-2} + \frac{p}{2}\int_0^n q_0|z'_{nj}|^2|z_{nj}|^{p-2}dx + (\frac{p}{2}-1)\int_0^n q_0(z'_{nj})^2(\bar{z}_{nj})^2|z_{nj}|^{p-4}dx$$
$$+\int_0^n q_1|z_{nj}|^p dx = \int_0^n wf_j\bar{z}_{nj}|z_{nj}|^{p-2}dx. \tag{5.5}$$

Considering the real parts of each side of (5.5) we have if $1 < p < 2$,

$$\mathrm{Re}(\int_0^n wf_j\bar{z}_{nj}|z_{nj}|^{p-2}dx) \geq -(q_0(0))^{1-p}|\int_0^j wz_{1n}f\,dt|^p|c_n||d_n|^{p-1}$$
$$+(p-1)\int_0^n q_0|z'_{nj}|^2|z_{nj}|^{p-2}dx + \int_0^n w|z_{nj}|^p dx$$

and if $2 \leq p < \infty$,

$$\mathrm{Re}\left(\int_0^n wf_j\bar{z}_{nj}|z_{nj}|^{p-2}dx\right) \geq -(q_0(0))^{1-p}|\int_0^j wz_{1n}f\,dt|^p|c_n||d_n|^{p-1}$$
$$+\int_0^n q_0|z'_{nj}|^2|z_{nj}|^{p-2}dx + \int_0^n w|z_{nj}|^p dx.$$

Applying Holder's inequality to $\int_0^n wf_j\overline{z}_{nj}|z_{nj}|^{p-2}dx$ we get

$$\left(\int_0^j w|f|^p dt\right)^{\frac{1}{p}}\left(\int_0^n w|z_{nj}|^p dx\right)^{(p-1)/p} \geq \mathrm{Re}\left(\int_0^n wf_j\overline{z}_{nj}|z_{nj}|^{p-2}dx\right).$$

From which

$$\int_0^n w|z_{nj}|^p dx \leq A_{nj} + B_{nj}\left(\int_0^n w|z_{nj}|^p dx\right)^{(p-1)/p},$$

with

$$A_{nj} = (q_0(0))^{1-p}|\int_0^j wz_{1n}fdt|^p \text{ and } B_{nj} = \left(\int_0^j w|f|^p dt\right)^{\frac{1}{p}}$$

follows. Set

$$I_{nj} = \int_0^n w|z_{nj}|^p dx.$$

The inequality $I \leq A+BI^{(p-1)/p}$ is equivalent to $I^{(p-1)/p}(I^{\frac{1}{p}}-B) \leq A$. If $I > A^{p/(p-1)}B^{-p/(p-1)}$ and $I > 2^pB^p$ then $I^{(p-1)/p} > AB^{-1}$ and $I^{\frac{1}{p}} - B > 2B - B = B$ so that $I^{(p-1)/p}(I^{\frac{1}{p}} - B) > (AB^{-1})B = A$ violating $I \leq A + BI^{(p-1)/p}$. Thus

$$I_{nj} \leq \max\{A_{nj}^{p/(p-1)}B_{nj}^{-p/(p-1)},\ 2^pB_{nj}^p\} = J_{nj}.$$

Select a subsequence (c_{n_k}, d_{n_k}), $k = 1, 2, \ldots$ of (c_n, d_n) such that (c_{n_k}, d_{n_k}) converges to some (c, d) with $|c|^2 + |d|^2 = 1$ as $k \to \infty$. Set

$$z_1(x) = cy_1(x) + dy_2(x) \text{ and } z_2(x) = (-\overline{d}/q_0(0))y_1(x) + (\overline{c}/q_0(0))y_2(x).$$

Let

$$z^j(x) = z_1(x)\int_0^x wz_2f_jdt + z_2(x)\int_x^\infty wz_1f_jdt.$$

$z_1(x)$ is chosen the same way as the solution $y(x)$ in Theorem 1* so $z_1(x)$ is $L_{p/(p-1)}[w; 0, \infty)$ and

$$\int_x^\infty wz_1fdt$$

converges for each x. Fix $t > 0$ and consider $n_k > t$. Then

$$\int_0^t w|z_{n_kj}|^p dx \leq J_{n_kj}$$

and

$$(\int_0^t w|z^j|^p dx)^{\frac{1}{p}} \leq (\int_0^t w|z^j - z_{n_{kj}}|^p dx)^{\frac{1}{p}} + (\int_0^t w|z_{n_{kj}}|^p dx)^{\frac{1}{p}}.$$

First we estimate $J_{n_{kj}}$.

$$|\int_0^j w z_{1n_k} f dt| \leq (\int_0^j w|z_{1n_k}|^{p/(p-1)} dt)^{(p-1)/p} (\int_0^j w|f|^p dt)^{\frac{1}{p}},$$

$$(\int_0^j w|z_{1n_k}|^{p/(p-1)} dt)^{(p-1)/p} \leq (\int_0^j w|z_{1n_k} - z_1|^{p/(p-1)} dt)^{(p-1)/p} + (\int_0^j w|z_1|^{p/(p-1)} dt)^{(p-1)/p},$$

and

$$(\int_0^j w|z_{1n_k} - z_1|^{p/(p-1)} dt)^{(p-1)/p} \leq |c_{n_k} - c|(\int_0^j w|y_1|^{p/(p-1)} dt)^{(p-1)/p} + |d_{n_k} - d|(\int_0^j w|y_2|^{p/(p-1)} dt)^{(p-1)/p}.$$

So

$$\begin{aligned} |\int_0^j w z_{1n_k} f dt| \leq (\int_0^j w|f|^p dt)^{\frac{1}{p}} ((\int_0^j w|z_1|^{p/(p-1)} dt)^{(p-1)/p} \\ + |c_{n_k} - c|(\int_0^j w|y_1|^{p/(p-1)} dt)^{(p-1)/p} + |d_{n_k} - d|(\int_0^j w|y_2|^{p/(p-1)} dt)^{(p-1)/p}). \end{aligned}$$

Choose k_ε (k_ε depends on j, $\varepsilon > 0$) sufficiently large so that for $k \geq k_\varepsilon$,

$$|\int_0^t w z_{1n_k} f dt| \leq (\int_0^\infty w|f|^p dt)^{\frac{1}{p}} ((\int_0^\infty w|z_1|^{p/(p-1)} dt)^{(p-1)/p} + \varepsilon).$$

We have then

$$A_{n_k j} \leq (q_0(0))^{1-p} (\int_0^\infty w|f|^p dt)((\int_0^\infty w|z_1|^{p/(p-1)} dt)^{(p-1)/p} + \varepsilon)^p$$

and

$$B_{n_k j} = (\int_0^j w|f|^p dt)^{\frac{1}{p}} \leq (\int_0^\infty w|f|^p dt)^{\frac{1}{p}}.$$

Thus

$$\begin{aligned} J_{n_k j} \leq \max \Bigg\{ (q_0(0))^{-1} \left(\int_0^\infty w|f|^p dt\right)^{p/(p-1)} \left(\left(\int_0^\infty w|z_1|^{p/(p-1)} dt\right)^{(p-1)/p} + \varepsilon \right)^{p^2/(p-1)} \cdot \\ (\int_0^j w|f|^p dt)^{-1/(p-1)},\ 2^p (\int_0^\infty w|f|^p dt) \Bigg\} = J_j, \end{aligned}$$

for $k \geq k_\varepsilon$.

$$z^j(x) = (-c\overline{d}/q_0(0))G_1(x) + (-|d|^2/q_0(0))G_2(x) + (|c|^2/q_0(0))G_3(x) + (\overline{c}d/q_0(0))G_4(x)$$

with

$$\begin{aligned}
G_1(x) &= y_1(x)\int_0^x wy_1 f_j\,dt + y_2(x)\int_x^\infty wy_2 f_j\,dt,\\
G_2(x) &= y_2(x)\int_0^x wy_1 f_j\,dt + y_1(x)\int_x^\infty wy_2 f_j\,dt,\\
G_3(x) &= y_1(x)\int_0^x wy_2 f_j\,dt + y_2(x)\int_x^\infty wy_1 f_j\,dt, \text{ and}\\
G_4(x) &= y_2(x)\int_0^x wy_2 f_j\,dt + y_1(x)\int_x^\infty wy_1 f_j\,dt.
\end{aligned}$$

Also,

$$\begin{aligned}
|z^j(x) - z_{n_k j}(x)| \leq\ & |-c\overline{d}/q_0(0) + c_{n_k}\overline{d}_{n_k}/q_0(0)||G_1(x)|\\
& + |-|d|^2/q_0(0) + |d_{n_k}|^2/q_0(0)||G_2(x)| + ||c|^2/q_0(0) - |c_{n_k}|^2/q_0(0)||G_3(x)|\\
& + |\overline{c}d/q_0(0) - \overline{c}_{n_k}d_{n_k}/q_0(0)||G_4(x)|.
\end{aligned}$$

Applying Minkowski's inequality we get that

$$\int_0^t w|z^j - z_{n_k j}|^p dx \to 0 \text{ as } k \to \infty.$$

Thus

$$\int_0^t w|z^j|^p dx \leq J_j. \tag{5.6}$$

Now, let

$$y(x) = z_1(x)\int_0^x wz_2 f\,dt + z_2(x)\int_x^\infty wz_1 f\,dt.$$

Then $y(x)$ is a solution of equation (5.1) with initial values

$$y(0) = (-\overline{d}/q_0(0))\int_0^\infty wz_1 f\,dt \text{ and } y'(0) = (\overline{c}/q_0(0))\int_0^\infty wz_1 f\,dt.$$

We have $J_j \geq J_{j+1}$ so that

$$\int_0^t w|z^j|^p dx \leq J$$

for a positive constant J. We have

$$(\int_0^t w|y|^p dx)^{\frac{1}{p}} \leq (\int_0^t w|y - z^j|^p dx)^{\frac{1}{p}} + (\int_0^t w|z^j|^p dx)^{\frac{1}{p}}.$$

$$y(x) - z^j(x) = z_1(x)\int_0^x wz_2(f - f_j)dx + z_2(x)\int_x^\infty wz_1(f - f_j)ds.$$

For $j > t \geq x$,

$$\int_0^x wz_2(f - f_j)ds = 0 \text{ and } \int_x^\infty wz_1(f - f_j)ds = \int_j^\infty wz_1 f ds.$$

So for $j > t \geq x$

$$y(x) - z^j(x) = (\int_j^\infty wz_1 f ds)z_2(x)$$

and

$$(\int_0^t w|y - z^j|^p dx)^{\frac{1}{p}} = |\int_j^\infty wz_1 f ds|(\int_0^t w|z_2|^p dx)^{\frac{1}{p}}.$$

Since $\int_j^\infty wz_1 f ds \to 0$ as $j \to \infty$ we have

$$\int_0^t w|y|^p dx \leq J. \tag{5.7}$$

Theorem 12. *Suppose that the conditions of Theorem 2* hold for $q_0(x)$ and $q_1(x)$ and $f(x)$ is $L_p[w; 0, \infty)$ for some p, $1 < p < \infty$. Then the inhomogeneous equation (5.1) $-(q_0 y')' + q_1 y = wf$ has a non-trivial solution that is $L_p[w; 0, \infty)$.*

Proof. Begin as in the proof of Theorem 11 until (5.5)

$$\begin{aligned} q_0(0)z'_{nj}(0)\overline{z}_{nj}(0)|z_{nj}(0)|^{p-2} &+ \frac{p}{2}\int_0^n q_0|z'_{nj}|^2|z_{nj}|^{p-2}dx \\ &+ (\frac{p}{2} - 1)\int_0^n q_0(z'_{nj})^2(\overline{z}_{nj})^2|z_{nj}|^{p-4}dx \\ &+ \int_0^n q_1|z_{nj}|^p dx = \int_0^n wf_j\overline{z}_{nj}|z_{nj}|^{p-2}dx \end{aligned}$$

has been reached. Considering the real parts of each side of (5.5), if $1 < p < 2$,

$$\text{Re}(\int_0^n wf_j\overline{z}_{nj}|z_{nj}|^{p-2}dx) \geq -(q_0(0))^{1-p}|\int_0^j wz_{1n}fdt|^p|c_n||d_n|^{p-1} + (p-1)\int_0^n q_0|z'_{nj}|^2|z_{nj}|^{p-2}dx$$

and, if $2 \le p < \infty$,

$$\mathrm{Re}(\int_0^n w f_j \overline{z}_{nj} |z_{nj}|^{p-2} dx) \ge -(q_0(0))^{1-p} |\int_0^j w z_{1n} f dt|^p |c_n| |d_n|^{p-1} + \int_0^n q_0 |z'_{nj}|^2 |z_{nj}|^{p-2} dx.$$

From which, for $1 < p < 2$,

$$(p-1) \int_0^n q_0 |z'_{nj}|^2 |z_{nj}|^{p-2} dx \le A_{nj} + B_{nj} (\int_0^n w |z_{nj}|^p dx)^{(p-1)/p}$$

and, for $2 \le p < \infty$,

$$\int_0^n q_0 |z'_{nj}|^2 |z_{nj}|^{p-2} dx \le A_{nj} + B_{nj} (\int_0^n w |z_{nj}|^p dx)^{(p-1)/p}$$

with

$$A_{nj} = (q_0(0))^{1-p} |\int_0^j w z_{1n} f dt|^p \text{ and } B_{nj} = (\int_0^j w |f|^p dt)^{\frac{1}{p}}$$

follows. Considering imaginary parts of each side of (5.5) we have

$$\begin{aligned} \mathrm{Im}(\int_0^n w f_j \overline{z}_{nj} |z_{nj}|^{p-2} dx) \ge & -(q_0(0))^{1-p} |\int_0^j w z_{1n} f dt|^p |c_n| |d_n|^{p-1} \\ & - |\frac{p}{2} - 1| \int_0^n q_0 |z'_{nj}|^2 |z_{nj}|^{p-2} dx + \int_0^n w |z_{nj}|^p dx. \end{aligned}$$

Thus

$$\int_0^n w |z_{nj}|^p dx \le A_{nj} + |\frac{p}{2} - 1| \int_0^n q_0 |z'_{nj}|^2 |z_{nj}|^{p-2} dx + B_{nj} (\int_0^n w |z_{nj}|^p dx)^{(p-1)/p}.$$

We get then, if $1 < p < 2$,

$$\int_0^n w |z_{nj}|^p dx \le (1 + |\frac{p}{2} - 1| / (p-1)) A_{nj} + (1 + |\frac{p}{2} - 1| / (p-1)) B_{nj} (\int_0^n w |z_{nj}|^p dx)^{(p-1)/p}$$

and, if $2 \le p < \infty$,

$$\int_0^n w |z_{nj}|^p dx \le (1 + |\frac{p}{2} - 1|) A_{nj} + (1 + |\frac{p}{2} - 1|) B_{nj} (\int_0^n w |z_{nj}|^p dx)^{(p-1)/p}.$$

The proof is completed in the same fashion as the proof of Theorem 11 with the terms A_{nj}, B_{nj} being replaced by $(p/(2(p-1)))A_{nj}$, $(p/(2(p-1)))B_{nj}$ or $(p/2)A_{nj}$, $(p/2)B_{nj}$ depending on

whether $1 < p < 2$ or $2 \leq p < \infty$ and $z_1(x)$ chosen in the same way as the solution $y(x)$ in Theorem 2*.

Let $\eta(x)$ be positive and twice continuously differentiable. Let $y(x) = \eta(x)z(x)$ and substitute into the equation

$$-(q_0 y')' + q_1 y = w\eta^{p-1} f, \quad 0 \leq x < \infty, \tag{5.8}$$

to get

$$-q_0\eta z'' - (2q_0\eta' + q_0\eta)z' + (q_1\eta - q_0\eta'' - q_0'\eta')z = w\eta^{p-1} f.$$

After multiplying both sides by η we can rewrite the equation as

$$-(q_0\eta^2 z')' + (q_1\eta - q_0\eta'' - q_0'\eta')\eta z = w\eta^p f. \tag{5.9}$$

Since

$$\int_0^t w|y|^p dx = \int_0^t w\eta^p |z|^p dx$$

a $L_p[w\eta^p; 0, \infty)$ solution z of (5.9) gives us a $L_p[w; 0, \infty)$ solution y of (5.8). Applying Theorems 11 and 12 to (5.9) we have the following.

Corollary 13. *Let $f(x)$ be $L_p[w(x)\eta^p(x); 0, \infty)$.*

a) If for p, $1 < p < \infty$, $Re(q_1) \geq w\eta^{p-2} + \eta^{-1}(q_0\eta'' + q_0'\eta')$, then (5.8) has non-trivial solution $y(x) = \eta(x)z(x)$ that is $L_p[w; 0, \infty)$.

b) If $Re(q_1) \geq \eta^{-1}(q_0\eta'' + q_0'\eta')$ and $Im(q_1) \geq w\eta^{p-2} + \eta^{-1}(q_0\eta'' + q_0'\eta')$, then (5.8) has a non-trivial solution $y(x) = \eta(x)z(x)$ that is $L_p[w; 0, \infty)$.

Chapter II

Limit-point L_p Criteria for the 2nd Order Equation

6. A variety of conditions that imply equation (1.1) is limit-point L_2 are known. In this chapter we give generalizations to L_p of methods due to Weyl [3], Kurss [4], Levinson [5], and Hartman and Wintner [6]. A section with interval criteria for limit-point L_p like the limit-point L_2 criteria of Evans and Zettl [7] is included, in Chapter III.

7. We begin with a limit-point criteria of Weyl. Consider as in Chapter I the differential equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0, \quad a \leq x < \infty, \tag{7.1}$$

with the conditions on $q_0(x)$ and $q_1(x)$ the same as for equation (1.1) except that $q_1(x)$ is real valued. If

$$-q_1(x) \leq k$$

for a real constant k, then observe the following. Choose $\lambda_0 \geq k$. Then $-(q_0y')' + q_1y = -\lambda_0 y$ if and only if $-(q_0y')' + (q_1 + \lambda_0)y = 0$. In which case

$$(q_0(x)y'(x)) = (q_0(a)y'(a)) + \int_a^x (q_1(s) + \lambda_0)y(s)ds$$

where

$$(q_1(s) + \lambda_0) \geq -k + \lambda_0 \geq 0.$$

So the solution with

$$y(a) = 0, \quad q_0(a)y'(a) = 1$$

satisfies

$$q_0(x)y'(x) = 1 + \int_a^x (q_1(s) + \lambda_0)y(s)ds \geq 1 + \int_a^x (\lambda_0 - k)y(s)ds$$

and

$$y(x) = \int_a^x (q_0(s))^{-1}ds + \int_a^x (q_0(\tau))^{-1} \int_a^\tau (q_1(s) + \lambda_0)y(s)dsd\tau.$$

This implies $y(x)$ is positive and non-decreasing on $a < x < \infty$ and $y(x)$ is not $L_2[a,\infty)$. (Likewise $y(x)$ is not $L_p[a,\infty)$ for $1 \le p < \infty$.) This yields that the equation $-(q_0(x)y'(x))' + q_1(x)y(x) = 0$ is limit-point L_2 for the following reason.

Theorem. *Suppose that equation (1.1) is limit-circle L_2 and that $r(x)$ is a measurable function on $[a,\infty)$ such that $|r(x)| \le B$ for a positive constant B. Then the equation $-(q_0(x)y'(x))' + (q_1(x) + r(x))y(x) = 0$, $a \le x < \infty$, is also limit-circle L_2.*

By the theorem above, if $-(q_0y')' + q_1y = 0$ was limit-circle L_2, then so would be $-(q_0y')' + (q_1 + \lambda_0)y = 0$.

Theorem (Weyl). *If $-q_1(x) \le k$ for a real constant k, then the equation (7.1) is limit-point $L_2[a,\infty)$.*

We can state the following in addition.

Theorem 14. *If $-q_1(x) \le 0$, then the equation (7.1) is limit-point $L_p[a,\infty)$ for $1 \le p < \infty$.*

Example 9) Consider equation (7.1) with $q_0(x) = x^2$, $1 \le x < \infty$. For $p > 2$ use Example 8) *Case 3)* of Chapter I with $\beta = 2$ to get that the equation

$$-(x^2y')' + q_1y = 0, \quad 1 \le x < \infty,$$

is limit-circle $L_p[1,\infty)$ if

$$-q_1(x) = m > (p-1)/p^2.$$

It is undecided at this point as to whether $-(x^2y')' + q_1y = 0$ is limit-point L_p for $-q_1(x) \le (p-1)/p^2$. The following theorem gives the answer that $-(x^2y')' + q_1y = 0$ is limit-point $L_p[1,\infty)$, for $1 \le p < \infty$, if

$$-q_1(x) \le (p-1)/p^2.$$

The case with $p = 2$ is a theorem due to Kurss.

Theorem. *Equation (7.1) is limit-point $L_p[a,\infty)$, for p, $1 \le p \le \infty$, if there exists a real function $\eta(x) > 0$ on $[a,\infty)$ such that*

(i) *$\eta(x)$ and $q_0(x)\eta'(x)$ are locally absolutely continuous*

(ii) $-q_1(x) \le -(q_0(x)\eta'(x))'/\eta(x)$

(iii) $\int_a^\infty \eta^p(x)dx = \infty$, *if* $1 \le p < \infty$, *or* $\sup \eta(x) = \infty$, *if* $p = \infty$.

Proof. Let $y(x)$ be a solution of equation (7.1). Define

$$\phi(x) = y(x)/\eta(x).$$

$\phi(x)$ satisfies the differential equation

$$-(q_0(x)\eta^2(x)\phi'(x))' + (q_1(x) - (q_0(x)\eta'(x))'/\eta(x))\eta^2(x)\phi(x) = 0. \tag{7.2}$$

To see this let

$$y(x) = \eta(x)\phi(x).$$

Then

$$y' = \eta\phi' + \eta'\phi, \quad q_0y' = \frac{1}{\eta}(q_0\eta^2\phi') + (q_0\eta')\phi,$$

and

$$(q_0y')' = \frac{1}{\eta}(q_0\eta^2\phi')' - \frac{\eta'}{\eta^2}(q_0\eta^2\phi') + (q_0\eta')\phi' + (q_0\eta')'\phi = \frac{1}{\eta}(q_0\eta^2\phi')' + (q_0\eta')'\phi.$$

Substituting into

$$-(q_0y')' + q_1y = 0$$

we get

$$-\frac{1}{\eta}(q_0\eta^2\phi')' - (q_0\eta')'\phi + q_1\eta\phi = 0.$$

After multiplying the last equation by η we have (7.2). If condition (ii) holds, then there exists a solution $\phi_1(x)$ of equation (7.2) such that

$$\phi_1(a) = 0, \quad \phi_1(x) > 0 \text{ on } (a,\infty), \text{ and } \phi_1(x) \text{ is non-decreasing.}$$

To see this let $\phi_1(x)$ be the solution of (7.2) with initial values

$$\phi_1(a) = 0, \quad q_0(a)\phi_1'(a) = 1.$$

Then

$$\begin{aligned} q_0(x)\eta^2(x)\phi_1'(x) &= q_0(a)\eta^2(a)\phi_1'(a) + \int_a^x (q_1(s) - (q_0(s)\eta'(s))'/\eta(s))\eta^2(s)\phi_1(s)ds \\ &\geq q_0(a)\eta^2(a)\phi_1'(a) \geq \eta^2(a), \end{aligned}$$

since

$$(q_1(s) - (q_0(s)\eta'(s))'/\eta(s))\eta^2(s) \geq 0.$$

Now, let $y_1(x) = \eta(x)\phi_1(x)$. $y_1(x)$ is a solution of (7.1) and if $1 \leq p < \infty$,

$$\int_a^\infty |y_1(x)|^p dx = \int_a^\infty \eta^p(x)|\phi_1(x)|^p dx = \infty$$

or if $p = \infty$,

$$\sup_{a \leq x < \infty} |y_1(x)| = \sup_{a \leq x < \infty} \eta(x)\phi_1(x) = \infty.$$

Returning to Example 9) let $\eta(x) = x^{-\frac{1}{p}}$, for p, $1 \leq p < \infty$. Then $\int_1^\infty \eta^p dx = \int_1^\infty x^{-1}dx = \infty$ and $(x^2\eta')'/\eta = -(p-1)/p^2$.

Applying the previous theorem we have that $-(x^2y')' + q_1y = 0$, $1 \leq x < \infty$, *is limit-point* $L_p[1,\infty)$, *if* $-q_1(x) \leq -(x^2\eta')'/\eta = (p-1)/p^2$. The case $p = \infty$ is handled by letting $\eta(x) = x^\Delta$, $\Delta > 0$, to get $(x^2\eta')'/\eta = \Delta(\Delta+1)$. Choosing $\Delta = -\frac{1}{2} + \sqrt{\frac{1}{4} + k}$, for $k > 0$, $\Delta(\Delta+1) = k$.

We have $-(x^2y')' + q_1y = 0$, $1 \leq x < \infty$, *has an unbounded solution if* $-q_1(x) \leq -k$, *for a positive constant* k.

If $q_1(x)$ is complex valued we can do the following. Let $y(x)$ be a solution of equation (1.1) and define

$$\phi(x) = y(x)/\eta(x).$$

As before $\phi(x)$ satisfies the equation (7.2)

$$-(q_0\eta^2\phi')' + (q_1 - (q_0\eta')'/\eta)\eta^2\phi = 0.$$

Let

$$\psi(x) = |\phi(x)|^2.$$

Then

$$\psi' = \phi\overline{\phi}' + \phi'\overline{\phi}, \quad q_0\eta^2\psi' = \phi(q_0\eta^2\overline{\phi}') + (q_0\eta^2\phi')\overline{\phi},$$

and

$$\begin{aligned}(q_0\eta^2\psi')' &= \phi(q_0\eta^2\overline{\phi}')' + \phi'(q_0\eta^2\overline{\phi}') + (q_0\eta^2\phi')\overline{\phi}' + (q_0\eta^2\phi')'\overline{\phi} \\ &= \phi(\overline{q}_1 - (q_0\eta')'/\eta)\eta^2\overline{\phi} + 2q_0\eta^2|\phi'|^2 + (q_1 - (q_0\eta')'/\eta)\eta^2\phi\overline{\phi} \\ &= 2q_0\eta^2|\phi'|^2 + 2(\mathrm{Re}(q_1) - (q_0\eta')'/\eta)\eta^2\psi.\end{aligned}$$

Choose the solution $\phi_1(x)$ of (7.2) with

$$\phi_1(a) = 1 \text{ and } q_0(a)\eta^2(a)\phi_1'(a) = 1.$$

Then for

$$\psi_1(x) = |\phi_1(x)|^2, \quad \psi_1(a) = 1 \text{ and } q_0(a)\eta^2(a)\psi_1'(a) = 2.$$

We have

$$\begin{aligned}q_0(x)\eta^2(x)\psi_1'(x) &= q_0(a)\eta^2(a)\psi_1'(a) \\ &\quad + 2\int_a^x (q_0(s)\eta^2(s)|\phi_1'(s)|^2 + (\mathrm{Re}(q_1(s)) - (q_0(s)\eta'(s))'/\eta(s))\eta^2(s)\psi_1(s))ds.\end{aligned}$$

Hence if

$$-\mathrm{Re}(q_1(x)) \le -(q_0(s)\eta'(s))'/\eta(s)$$

we have

$$\psi_1'(x) > 0 \text{ so that } \psi_1(x) \text{ is non-decreasing.}$$

It follows

$$\psi_1(x) \ge 1 \text{ and } |y_1(x)| = \eta(x)|\phi_1(x)| = \eta(x)(\psi_1(x))^{\frac{1}{2}} \ge \eta(x).$$

Theorem 15. *If there exists a real function $\eta(x) > 0$ on $[a, \infty)$ such that $\eta(x)$ and $q_0(x)\eta'(x)$ are locally absolutely continuous and $-Re(q_1(x)) \leq -(q_0(x)\eta'(x))'/\eta(x)$, then equation (1.1) has a solution $y(x)$ such that $|y(x)| \geq \eta(x)$ on $[a, \infty)$.*

Again consider equation (7.1)

$$-(q_0 y')' + q_1 y = 0, \quad a \leq x < \infty.$$

Let $y_1(x)$ and $y_2(x)$ be two real solutions whose wronskian

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = c \neq 0.$$

Set

$$r(x) = (y_1^2(x) + y_2^2(x))^{\frac{1}{2}}.$$

Then

$$r' = \frac{1}{2}(y_1^2 + y_2^2)^{-\frac{1}{2}}(2y_1y_1' + 2y_2y_2') = r^{-1}(y_1y_1' + y_2y_2'),$$

$$q_0 r' = r^{-1}(y_1(q_0y_1') + y_2(q_0y_2')),$$

and

$$\begin{aligned}(q_0r')' &= r^{-1}(y_1(q_0y_1')' + y_2(q_0y_2')') + r^{-1}(y_1'(q_0y_1') + y_2'(q_0y_2')) - r^{-2}r'(y_1(q_0y_1') + y_2(q_0y_2')) \\ &= r^{-1}q_1(y_1^2 + y_2^2) + r^{-1}q_0((y_1')^2 + (y_2')^2) - r^{-2}r'q_0(y_1y_1' + y_2y_2') \\ &= q_1 r + q_0 r^{-3}(((y_1')^2 + (y_2')^2)(y_1^2 + y_2^2) - (y_1y_1' + y_2y_2')^2).\end{aligned}$$

$$\begin{aligned}((y_1')^2 + (y_2')^2)(y_1^2 + y_2^2) - (y_1y_1' + y_2y_2')^2 &= (y_1')^2y_1^2 + (y_1')^2y_2^2 + (y_2')^2y_1^2 + (y_2')^2y_2^2 \\ &\quad - y_1^2(y_1')^2 - 2y_1y_1'y_2y_2' - y_2^2(y_2')^2 \\ &= (y_1')^2y_2^2 + (y_2')^2y_1^2 - 2y_1y_1'y_2y_2' \\ &= (y_1y_2' - y_1'y_2)^2 = (q_0)^{-2}c^2.\end{aligned}$$

Thus

$$(q_0 r')'/r = q_1 + (q_0)^{-1} c^2 r^{-4}.$$

We have then $q_1(x)$ and $r(x)$ are related by

$$q_1(x) = (q_0(x)r'(x))'/r(x) - c^2/(q_0(x)r^4(x)).$$

Write $y_1(x)$ and $y_2(x)$ in the form

$$y_1(x) = r(x)\cos\theta(x) \text{ and } y_2(x) = r(x)\sin\theta(x).$$

Now,

$$y_1' = -r(\sin\theta)\theta' + r'\cos\theta, \quad y_2' = r(\cos\theta)\theta' + r'\sin\theta,$$

and

$$\begin{aligned} y_1 y_2' - y_1' y_2 &= r^2(\cos\theta)^2\theta' + rr'(\cos\theta)(\sin\theta) + r^2(\sin\theta)^2\theta' - rr'(\cos\theta)(\sin\theta) \\ &= r^2(\cos^2\theta + \sin^2\theta)\theta' = r^2\theta'. \end{aligned}$$

So

$$q_0(x)r^2(x)\theta'(x) = c \text{ and } \theta(x) = \theta(a) + c\int_a^x (q_0(s))^{-1}(r(s))^{-2}ds.$$

Conversely suppose that $r(x)$ is a positive real function such that $r(x)$ and $q_0(x)r'(x)$ are locally absolutely continuous on $[a,\infty)$. Let

$$\theta(x) = \theta(a) + c\int_a^x (q_0(s))^{-1}(r(s))^{-2}ds$$

for real numbers $\theta(a)$ and $c \neq 0$. For

$$y_1(x) = r(x)\cos\theta(x) \text{ and } y_2(x) = r(x)\sin\theta(x), \quad y_1 y_2' - y_1' y_2 = r^2\theta'$$

and so

$$q_0(y_1 y_2' - y_1' y_2) = c.$$

Also,

$$\begin{aligned} q_0 y_1' &= -r(\sin\theta)(q_0\theta') + (q_0 r')\cos\theta \\ &= -r(\sin\theta)(cr^{-2}) + (q_0 r')\cos\theta \\ &= -cr^{-1}\sin\theta + (q_0 r')\cos\theta \end{aligned}$$

and

$$\begin{aligned} (q_0 y_1')' &= -cr^{-1}(\cos\theta)\theta' + cr^{-2}r'\sin\theta - (q_0 r')(\sin\theta)\theta' + (q_0 r')'\cos\theta \\ &= -cr^{-1}(\cos\theta)(cq_0^{-1}r^{-2}) + cr^{-2}r'\sin\theta - (q_0 r')(\sin\theta)(cq_0^{-1}r^{-2}) + (q_0 r')'\cos\theta \\ &= -c^2 q_0^{-1} r^{-3}\cos\theta + (q_0 r')'\cos\theta \\ &= \left((q_0 r')'/r - c^2/(q_0 r^4)\right) r\cos\theta. \end{aligned}$$

Likewise

$$(q_0 y_2')' = \left((q_0 r')'/r - c^2/(q_0 r^4)\right) r\sin\theta.$$

So we get that the equation

$$-(q_0(x)y'(x))' + ((q_0(x)r'(x))'/r(x) - c^2/(q_0(x)r^4(x)))y(x) = 0, \quad a \le x < \infty, \tag{7.3}$$

has independent solutions

$$y_1(x) = r(x)\cos\theta(x) \text{ and } y_2(x) = r(x)\sin\theta(x)$$

with

$$\theta(x) = \theta(a) + c\int_a^x (q_0(s))^{-1}(r(s))^{-2}ds$$

for real numbers $\theta(a)$ and c such that $c \neq 0$.

Now, if $r(x)$ is a positive real function such that $r(x)$ and $q_0(x)r'(x)$ are locally absolutely continuous and if

$$\int_a^\infty (q_0(s))^{-1}(r(s))^{-2}ds < \infty,$$

do the following. For $c > 0$, choose b_0, $a \leq b_0 < \infty$, such that

$$c \int_{b_0}^{\infty} (q_0(s))^{-1}(r(s))^{-2} ds < \frac{\pi}{2}.$$

For $b_0 \leq x < \infty$, define

$$\eta(x) = (\cos(c \int_{b_0}^{\infty} q_0^{-1} r^{-2} ds))^{-1} r(x) \cos\theta(x)$$

with

$$\theta(x) = c \int_{b_0}^{x} q_0^{-1} r^{-2} ds.$$

Then $\eta(x) > r(x)$ and

$$(q_0\eta')'/\eta = (q_0(r\cos\theta)')'/(r\cos\theta) = (q_0 r')'/r - c^2/(q_0 r^4).$$

If for some b, $b_0 \leq b < \infty$,

$$-\mathrm{Re}(q_1(x)) \leq -(q_0 r')'/r - c^2/(q_0 r^4),$$

for $b \leq x < \infty$, then apply Theorem 15 on the interval $[b, \infty)$ to obtain the following.

Corollary 15. *Suppose that there exists a positive real function $r(x)$ such that $r(x)$ and $q_0(x)r'(x)$ are locally absolutely continuous on $[a, \infty)$ and $\int_a^\infty (q_0(s))^{-1}(r(s))^{-2} ds < \infty$. Suppose there exists b, $a \leq b < \infty$, and $c > 0$ such that $c\int_b^\infty (q_0(s))^{-1}(r(s))^{-2} ds < \frac{\pi}{2}$ and $-Re(q_1(x)) \leq -(q_0(x)r'(x))'/r(x) + c^2/(q_0(x)r^4(x))$, for $b \leq x < \infty$. Then equation (1.1) has a solution $y(x)$ such that $|y(x)| > r(x)$, for $b \leq x < \infty$.*

Example 10 Let $q_0(x) = x^\beta$, $1 \leq x < \infty$. For $r(x) = x^\Delta$ and $c > 0$, $(q_0(x)r'(x))'/r(x) - c^2/(q_0(x)r^4(x)) = \Delta(\Delta - 1 + \beta)x^{\beta-2} - c^2 x^{-\beta-4\Delta}$. *When $\Delta \leq \frac{1}{2}(1-\beta)$ we can apply Theorem 15 with $\eta(x) = x^\Delta$ to get that $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \leq x < \infty$, has a solution $y(x)$ such that $|y(x)| \geq x^\Delta$ if $-Re(q_1(x)) \leq -(x^\beta \eta'(x))'/\eta(x) = -\Delta(\Delta - 1 + \beta)x^{\beta-2}$.* When $\Delta > \frac{1}{2}(1-\beta)$, $\int_1^\infty (q_0(s))^{-1}(r(s))^{-2} ds = \int_1^\infty s^{-\beta-2\Delta} ds < \infty$ since $-\beta - 2\Delta < -\beta - 2\frac{1}{2}(1-\beta) = -1$. Also,

$c\int_b^\infty s^{-\beta-2\Delta}ds = c(\beta+2\Delta-1)^{-1}b^{-\beta-2\Delta+1} < \frac{\pi}{2}$ for $b^{\beta+2\Delta-1} > \frac{2}{\pi}c(\beta+2\Delta-1)^{-1}$, or equivalently for $b > (2c/(\pi(\beta+2\Delta-1)))^{1/(\beta+2\Delta-1)}$. *We can apply Corollary 15 to get that* $-(x^\beta y')'+q_1 y = 0$, $1 \le x < \infty$, *has a solution* $y(x)$ *such that* $|y(x)| > x^\Delta$ *for* $x > (2c/(\pi(\beta+2\Delta-1)))^{1/(\beta+2\Delta-1)}$, *if* $-Re(q_1(x)) \le -\Delta(\Delta-1+\beta)x^{\beta-2} + c^2x^{-\beta-4\Delta}$.

However when $\Delta \le \frac{1}{2}(1-\beta)$ we cannot expect for $-q_1(x) \le -\Delta(\Delta-1+\beta)x^{\beta-2} + c^2x^{-\beta-4\Delta}$ that the equation $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \le x < \infty$, has a solution $y(x)$ such that $|y(x)| \ge x^\Delta$ for $x > b$, for any $1 \le b < \infty$. This we can see because the equation

$$-(x^\beta y'(x))' + (\Delta(\Delta-1+\beta)x^{\beta-2} - c^2x^{-\beta-4\Delta})y(x) = 0, \quad 1 \le x < \infty,$$

has solutions

$$y(x) = Ax^\Delta \cos\theta(x) + Bx^\Delta \sin\theta(x),$$

with

$$\theta(x) = \theta(a) + c\int_a^x s^{-\beta-2\Delta}ds.$$

Since $-\beta - 2\Delta \ge -\beta - 2\frac{1}{2}(1-\beta) = -1$,

$$\theta(x) \to \infty \text{ as } x \to \infty$$

and thus $y(x)$ has infinitely many zeros on $[1,\infty)$.

Letting $\Delta = -\frac{1}{p}$ *we have that the equation* $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \le x < \infty$, *is limit-point* $L_p[1,\infty)$ *if* $\beta \le 1 + \frac{2}{p}$ *and* $-Re(q_1(x)) \le \frac{1}{p^2}(p\beta - (p+1))x^{\beta-2}$ *or if* $\beta > 1 + \frac{2}{p}$ *and* $-Re(q_1(x)) \le \frac{1}{p^2}(p\beta - (p+1))x^{\beta-2} + c^2x^{\frac{4}{p}-\beta}$.

The equation

$$-(x^\beta y'(x))' + (-\frac{1}{p^2}(p\beta - (p+1))x^{\beta-2} - c^2x^{\frac{4}{p}-\beta})y(x) = 0, \quad 1 \le x < \infty,$$

has solutions

$$y_1(x) = x^{-\frac{1}{p}}\cos\theta(x) \text{ and } y_2(x) = x^{-\frac{1}{p}}\sin\theta(x),$$

with

$$\theta(x) = \theta(a) + c\int_a^x x^{-\beta+\frac{2}{p}}ds,$$

and is limit-point $L_p[1,\infty)$.

By comparison we may expect the equation $-(x^\beta y')' + q_1 y = 0$, $1 \le x < \infty$, is limit-point $L_p[1,\infty)$ if $-\mathrm{Re}(q_1(x)) \le \frac{1}{p^2}(p\beta - (p+1))x^{\beta-2} + c^2 x^{\frac{4}{p}-\beta}$, for any β.

Let $q_0(x) = e^{kx}$, $0 \le x < \infty$. For $r(x) = e^{\ell x}$ and $c > 0$, $(q_0(x)r'(x))'/r(x) - c^2/(q_0(x)r^4(x)) = \ell(\ell+k)e^{kx} - c^2 e^{-(k+4\ell)x}$. *When $\ell \le -\frac{1}{2}k$ we can apply Theorem 15 with $\eta(x) = e^{\ell x}$ to get that , if $-Re(q_1(x)) \le -\ell(\ell+k)e^{kx}$, then the equation $-(e^{kx}y'(x))' + q_1(x)y(x) = 0$, $0 \le x < \infty$, has a solution $y(x)$ such that $|y(x)| \ge e^{\ell x}$. When $\ell > -\frac{1}{2}k$ we can apply Corollary 15 to get that, if $-Re(q_1(x)) \le -\ell(\ell+k)e^{kx} + c^2 e^{-(k+4\ell)}$, then the equation $-(e^{kx}y'(x))' + q_1(x)y(x) = 0$, $0 \le x < \infty$, has a solution $y(x)$ such that $|y(x)| > e^{\ell x}$ for $x > (k+2\ell)^{-1}\ln(2c/(\pi(k+2\ell)))$.*

8. In Example 10) for $\beta \le 1 + \frac{2}{p}$ we had obtained the result that $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \le x < \infty$, is limit-point $L_p[1,\infty)$ if $-q_1(x) \le \frac{1}{p^2}(p\beta - (p+1))x^{\beta-2}$. Though, we might expect a better bound such as $-q_1(x) \le \frac{1}{p^2}(p\beta - (p+1))x^{\beta-2} + c^2 x^{\frac{4}{p}-\beta}$. The following theorem due to Levinson obtains this bound when $p = 2$.

Theorem (Levinson) *Equation (7.1) is limit-point $L_2[a,\infty)$ if there exists a positive and continuously differentiable function $M(x)$ on $[a,\infty)$ and a positive constant k such that*

(i) $-q_1(x) \le kM(x)$

(ii) $\int_a^\infty \frac{1}{\sqrt{q_0(x)M(x)}}dx = \infty$

(iii) $|\sqrt{q_0(x)}M'(x)/(M(x))^{\frac{3}{2}}| \le k$.

For $\beta \le 2$, let $M(x) = x^{2-\beta}$ on the interval $[1,\infty)$. Then

$$\int_1^\infty \frac{1}{\sqrt{q_0(x)M(x)}}dx = \int_1^\infty \frac{1}{\sqrt{x^\beta x^{2-\beta}}}dx = \int_1^\infty \frac{1}{x}dx = \infty$$

and

$$|\sqrt{q_0(x)}M'(x)/(M(x))^{\frac{3}{2}}| = |x^{\frac{\beta}{2}}(2-\beta)x^{1-\beta}/x^{\frac{3}{2}(2-\beta)}| = (2-\beta)x^{\beta-2} \leq (2-\beta).$$

Taking $k > (2-\beta)$ the bound

$$-q_1(x) \leq kx^{2-\beta}$$

follows.

We obtained the following generalization for $p > 2$. Let $w(x)$ be a non-negative and continuous weight function on $[a, \infty)$. If the equation

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0, \quad a \leq x < \infty,$$

has a solution $y(x)$ such that

$$\lim_{t\to\infty} \int_a^t w(x)|y(x)|^p dx = \infty,$$

then the equation will be said to be limit-point $L_p[w(x); a, \infty)$.

Theorem 16. *Let $w(x)$ be a non-negative and continuous weight function on $[a, \infty)$ and let $p > 2$. Equation (1.1) is limit-point $L_p[w(x); a, \infty)$ if there exists a positive and continuously differentiable function $M(x)$ on $[a, \infty)$ and a positive constant k such that*

(i) $-Re(q_1(x)) \leq k(w(x))^{\frac{2}{p}}M(x)$ *and* $|Im(q_1(x))| \leq k(w(x))^{\frac{2}{p}}M(x)$

(ii) *there exists a measurable real function $f(x) \geq 0$ such that $\int_a^t (f(x))^{p/(p-2)}dx < \infty$ for $t < \infty$ and* $\limsup_{t\to\infty} \int_a^t ((w(x))^{\frac{1}{p}}/\sqrt{q_0(x)M(x)})f(x)dx/(\int_a^t (f(x))^{p/(p-2)}dx)^{(p-2)/p} = \infty$

(iii) $|(1-\frac{p}{2})q_0'(x)/\sqrt{q_0(x)M(x)} - \frac{p}{2}\sqrt{q_0(x)}M'(x)/(M(x))^{\frac{3}{2}}| \leq k(w(x))^{\frac{1}{p}}$

(iv) $\int_a^\infty w(x)(\int_a^x (q_0(t)/M(t))^{-p/(2(p-1))}dt)^{p-1}dx = \infty.$

The first part of the proof we state as a Lemma. Let $y(x)$ be a solution of equation (1.1).

Lemma 16. Under the hypothesis that (i), (iii), and (iv) hold, if $\int_a^\infty w(x)|y(x)|^p dx < \infty$, *then* $\int_a^\infty (q_0(x)/M(x))^{\frac{p}{2}}|y'(x)|^p dx < \infty$.

Proof. Indicate $\mathrm{Re}(y(x))$ by $u(x)$ and $\mathrm{Im}(y(x))$ by $v(x)$ so that

$$y(x) = u(x) + iv(x)$$

with $u(x)$ and $v(x)$ real valued. Suppose that

$$\int_a^\infty w(x)|y(x)|^p dx < \infty$$

so

$$\int_a^\infty w(x)|u(x)|^p dx < \infty \text{ and } \int_a^\infty w(x)|v(x)|^p dx < \infty.$$

Separate real and imaginary parts of the equation $-(q_0(x)y'(x))' + q_1(x)y(x) = 0$ to get

$$-(q_0(x)u'(x))' + \mathrm{Re}(q_1(x))u(x) = \mathrm{Im}(q_1(x))v(x)$$

and

$$-(q_0(x)v'(x))' + \mathrm{Re}(q_1(x))v(x) = -\mathrm{Im}(q_1(x))u(x).$$

Since $u'(x)$ is continuous, if $u'(x_0) > 0$, then there exists a $\delta > 0$ such that $u'(x) > 0$ for $x \in [a, \infty)$ and $|x - x_0| < \delta$. So, if $u'(x) > 0$,

$$\begin{aligned}\frac{d}{dx}(q_0(x)u'(x))^{p-1} &= (p-1)(q_0(x)u'(x))^{p-2}(q_0(x)u'(x))' \\ &= (p-1)(q_0(x)u'(x))^{p-2}(\mathrm{Re}(q_1(x))u(x) - \mathrm{Im}(q_1(x))v(x)).\end{aligned}$$

Likewise the set $\{x : u'(x) < 0\}$ is open relative to $[a, \infty)$. So, if $u'(x) < 0$,

$$\begin{aligned}\frac{d}{dx}(q_0(x)|u'(x)|)^{p-1} = \frac{d}{dx}(-q_0(x)u'(x))^{p-1} &= (p-1)(-q_0(x)u'(x))^{p-2}(-q_0(x)u'(x))' \\ &= -(p-1)(-q_0(x)u'(x))^{p-2}(\mathrm{Re}(q_1(x))u(x) - \mathrm{Im}(q_1(x))v(x)).\end{aligned}$$

When $u'(x) > 0$,

$$q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2} = (q_0(x)u'(x))^{p-1},$$

and when $u'(x) < 0$,

$$q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2} = -(q_0(x)|u'(x)|)^{p-1}.$$

In either case, that is when $u'(x) \neq 0$,

$$\frac{d}{dx}(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) = (p-1)(q_0(x)|u'(x)|)^{p-2}(\mathrm{Re}(q_1(x))u(x) - \mathrm{Im}(q_1(x))v(x)).$$

When $u'(x_0) = 0$,

$$\begin{aligned}
&\frac{d}{dx}(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2})|_{x=x_0} \\
&= \lim_{x\to x_0} \frac{q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2} - q_0(x_0)u'(x_0)(q_0(x_0)|u'(x_0)|)^{p-2}}{x - x_0} \\
&= \lim_{x\to x_0} \frac{q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}}{x - x_0} \\
&= \lim_{x\to x_0} \frac{q_0(x)u'(x) - q_0(x_0)u'(x_0)}{x - x_0}(q_0(x)|u'(x)|)^{p-2} \\
&= (q_0(x)u'(x))'|_{x=x_0}(q_0(x_0)|u'(x_0)|)^{p-2} = 0.
\end{aligned}$$

Hence

$$\frac{d}{dx}(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) = (p-1)(q_0(x)|u'(x)|)^{p-2}(\mathrm{Re}(q_1(x))u(x) - \mathrm{Im}(q_1(x))v(x)).$$

Express

$$u(x)u'(x)(q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^{p-2}$$

as

$$u(x)((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}).$$

Then

$$
\begin{aligned}
&(u(x)u'(x)(q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^{p-2})' \\
&\quad = u(x)((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2})' \\
&\qquad + u(x)((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})'(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) \\
&\qquad + u'(x)((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) \\
&\quad = (p-1)((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})(\mathrm{Re}(q_1(x))(q_0(x)|u'(x)|)^{p-2}(u(x))^2 \\
&\qquad - \mathrm{Im}(q_1(x))(q_0(x)|u'(x)|)^{p-2}u(x)v(x) \\
&\qquad + ((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})'(q_0(x)u(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) \\
&\qquad + ((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})(q_0(x)(u'(x))^2(q_0(x)|u'(x)|)^{p-2}) \\
&\quad = (p-1)((\mathrm{Re}(q_1(x))/M(x))(u(x))^2(q_0(x)/M(x))^{\frac{p}{2}-1}|u'(x)|^{p-2} \\
&\qquad - (\mathrm{Im}(q_1(x))/M(x))u(x)v(x)(q_0(x)/M(x))^{\frac{p}{2}-1}|u'(x)|^{p-2}) \\
&\qquad + (q_0(x))^{p-1}((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})'(u(x)u'(x)|u'(x)|^{p-2}) \\
&\qquad + (q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^p.
\end{aligned}
$$

After rearranging terms and integrating we have the following.

$$
\begin{aligned}
&\int_a^t (q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^p dx \\
&\quad = |_a^t u(x)u'(x)(q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^{p-2} \\
&\qquad + (p-1)\int_a^t (-\mathrm{Re}(q_1(x))/M(x))|u(x)|^2(q_0(x)/M(x))^{\frac{p}{2}-1}|u'(x)|^{p-2}dx \\
&\qquad + (p-1)\int_a^t (\mathrm{Im}(q_1(x))/M(x))u(x)v(x)(q_0(x)/M(x))^{\frac{p}{2}-1}|u'(x)|^{p-2}dx \\
&\qquad - \int_a^t (q_0(x))^{p-1}((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})'u(x)u'(x)|u'(x)|^{p-2}dx. \qquad (8.1)
\end{aligned}
$$

Given that condition (i) holds we have that

$$
-\mathrm{Re}(q_1(x))/M(x) \le k(w(x))^{\frac{2}{p}} \text{ and } |\mathrm{Im}(q_1(x))|/M(x) \le k(w(x))^{\frac{2}{p}}
$$

so

$$
(p-1)\int_a^t (-\mathrm{Re}(q_1)/M)|u|^2(q_0/M)^{\frac{p}{2}-1}|u'|^{p-2}dx \le (p-1)k\int_a^t w^{\frac{2}{p}}|u|^2(q_0/M)^{\frac{p}{2}-1}|u'|^{p-2}dx
$$

and

$$(p-1)\int_a^t (\mathrm{Im}(q_1)/M)uv(q_0/M)^{\frac{p}{2}-1}|u'|^{p-2}dx \le (p-1)k\int_a^t w^{\frac{2}{p}}|u||v|(q_0/M)^{\frac{p}{2}-1}|u'|^{p-2}dx.$$

Applying Holder's inequality with exponents $\frac{p}{2}$ and $p/(p-2)$ we get

$$\int_a^t w^{\frac{2}{p}}|u|^2(q_0/M)^{\frac{p}{2}-1}|u'|^{p-2}dx \le \left(\int_a^t w|u|^p dx\right)^{\frac{2}{p}}\left(\int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx\right)^{(p-2)/p}.$$

With exponents p, p, and $p/(p-2)$ we get

$$\int_a^t w^{\frac{2}{p}}|u||v|(q_0/M)^{\frac{p}{2}-1}|u'|^{p-2}dx \le \left(\int_a^t w|u|^p dx\right)^{\frac{1}{p}}\left(\int_a^t w|v|^p dx\right)^{\frac{1}{p}}\left(\int_a^t (q_0/M)|u'|^p dx\right)^{(p-2)/p}.$$

$$\begin{aligned}((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})' &= (q_0(x))^{1-p}\frac{p}{2}(q_0(x)/M(x))^{\frac{p}{2}-1}(M(x)q_0'(x) - M'(x)q_0(x))/(M(x))^2 \\ &\quad + (1-p)(q_0(x))^{-p}q_0'(x)(q_0(x)/M(x))^{\frac{p}{2}} \\ &= (q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}-1}((1-\frac{p}{2})(q_0'(x)/M(x)) \\ &\quad - \frac{p}{2}(q_0(x)M'(x)/(M(x))^2)).\end{aligned}$$

Thus

$$\begin{aligned}(q_0(x))^{p-1}((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})' &= (q_0(x)/M(x))^{\frac{p}{2}-1}((1-\frac{p}{2})(q_0'(x)/M(x)) \\ &\quad - \frac{p}{2}(q_0(x)M'(x)/(M(x))^2)) \\ &= (q_0(x)/M(x))^{(p-1)/2}((1-\frac{p}{2})(q_0'(x)/\sqrt{q_0(x)M(x)}) \\ &\quad - \frac{p}{2}(\sqrt{q_0(x)}M'(x)/(M(x))^{\frac{3}{2}})).\end{aligned}$$

Given that condition (iii) holds we have that

$$|(q_0(x))^{p-1}((q_0(x))^{1-p}(q_0(x)/M(x))^{\frac{p}{2}})'| \le k(w(x))^{\frac{1}{p}}(q_0(x)/M(x))^{(p-1)/2}.$$

So

$$-\int_a^t (q_0)^{p-1}((q_0)^{1-p}(q_0/M)^{\frac{p}{2}})'uu'|u'|^{p-2}dx \le k\int_a^t w^{\frac{1}{p}}|u|(q_0/M)^{(p-1)/2}|u'|^{p-1}dx.$$

Applying Holder's inequality with exponents p and $p/(p-1)$ we get

$$\int_a^t w^{\frac{1}{p}}|u|(q_0/M)^{(p-1)/2}|u'|^{p-1}dx \leq \left(\int_a^t w|u|^p dx\right)^{\frac{1}{p}} \left(\int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx\right)^{(p-1)/p}.$$

Putting the preceeding inequalities into equality (8.1) we have

$$\begin{aligned}\int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx \leq\ & |_a^t u(x)u'(x)(q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^{p-2} \\ & + (p-1)k\left(\int_a^t w|u|^p dx\right)^{\frac{2}{p}} \left(\int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx\right)^{(p-2)/p} \\ & + (p-1)k\left(\int_a^t w|u|^p dx\right)^{\frac{1}{p}} \left(\int_a^t w|v|^p dx\right)^{\frac{1}{p}} \left(\int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx\right)^{(p-2)/p} \\ & + k\left(\int_a^t w|u|^p dx\right)^{\frac{1}{p}} \left(\int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx\right)^{(p-1)/p}. \end{aligned} \tag{8.2}$$

Now, if

$$\lim_{t\to\infty} \int_a^t (q_0/M)^{\frac{p}{2}}|u'|^p dx = \infty$$

then from the assumption

$$\int_a^\infty w|u|^p dx < \infty \text{ and } \int_a^\infty w|v|^p dx < \infty$$

and (8.2) we could conclude that

$$\lim_{t\to\infty} |_a^t u(x)u'(x)(q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^{p-2} = \infty.$$

In which case

$$u(x)u'(x)(q_0(x)/M(x))^{\frac{p}{2}}|u'(x)|^{p-2} \geq 1,$$

for t sufficiently large, say $t \geq b$. Since

$$u(t)u'(t) > 0,$$

for $t \geq b$, either $u(t) > 0$ and $u'(t) > 0$ or $u(t) < 0$ and $u'(t) < 0$, for $t \geq b$. Consider if $u(t) > 0$ and $u'(t) > 0$. Then

$$u(t)u'(t)(q_0(t)/M(t))^{\frac{p}{2}}|u'(t)|^{p-2} = u(t)(u'(t))^{p-1}(q_0(t)/M(t))^{\frac{p}{2}} \geq 1$$

and so

$$(u(t))^{1/(p-1)}u'(t) \geq (q_0(t)/M(t))^{-p/(2(p-1))},$$

for $t \geq b$. Hence

$$\begin{aligned}(u(x))^{p/(p-1)} &= (u(b))^{p/(p-1)} + \frac{p}{(p-1)}\int_b^x (u(t))^{1/(p-1)}u'(t)dt \\ &\geq (u(b))^{p/(p-1)} + \frac{p}{(p-1)}\int_b^x (q_0(t)/M(t))^{-p/(2(p-1))}dt \\ &\geq \frac{p}{(p-1)}\int_b^x (q_0(t)/M(t))^{-p/(2(p-1))}dt.\end{aligned}$$

From which

$$\int_b^\infty w(x)|u(x)|^p dx \geq (p/(p-1))^{p-1}\int_b^\infty w(x)\left(\int_b^x (q_0(t)/M(t))^{-p/(2(p-1))}dt\right)^{p-1} dx$$

would follow. By condition (iv)

$$\int_b^\infty w(x)\left(\int_b^x (q_0(t)/M(t))^{-p/(2(p-1))}dt\right)^{p-1} dx = \infty,$$

which would contradict

$$\int_b^\infty w(x)|u(x)|^p dx < \infty.$$

(If $u(t) < 0$ and $u'(t) < 0$ replace $u(x)$ by $-u(x)$ in the previous argument so that $u(t) > 0$ and $u'(t) > 0$ instead.) Hence we can conclude that if

$$\int_a^\infty w|u|^p dx < \infty$$

and if conditions (i), (iii), and (iv) hold, then

$$\int_a^\infty (q_0(x)/M(x))|u'(x)|^p dx < \infty.$$

By the exact same sort of reasoning we can conclude that

$$\int_a^\infty (q_0(x)/M(x))^{\frac{p}{2}}|v'(x)|^p dx < \infty.$$

It follows that

$$\int_a^\infty (q_0(x)/M(x))^{\frac{p}{2}}|y'(x)|^p dx < \infty$$

and Lemma 16 is proved.

Now, suppose in addition condition (ii) holds and the equation (1.1) $-(q_0(x)y'(x))'+q_1(x)y(x) = 0$, $a \le x < \infty$, is limit-circle $L_p[w(x); a, \infty)$. Let $y_1(x)$ and $y_2(x)$ be two independent solutions of (1.1) with

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1.$$

After multiplying both sides of $q_0(y_1y_2' - y_1'y_2) = 1$ by

$$((w(x))^{\frac{1}{p}}/\sqrt{q_0(x)M(x)})f(x)$$

and integrating we have

$$\int_a^t ((w^{\frac{1}{p}}y_1)((q_0/M)^{\frac{1}{2}}y_2')f - ((q_0/M)^{\frac{1}{2}}y_1')(w^{\frac{1}{p}}y_2)f)dx = \int_a^t (w^{\frac{1}{p}}/\sqrt{q_0M})fdx.$$

By applying Holder's inequality with exponents p, p, and $p/(p-2)$ we can bound the left hand side of the above by

$$\left(\int_a^t w|y_1|^p dx\right)^{\frac{1}{p}} \left(\int_a^t (q_0/M)^{\frac{p}{2}}|y_2'|^p dx\right)^{\frac{1}{p}} \left(\int_a^t f^{p/(p-2)}dx\right)^{(p-2)/p}$$
$$+ \left(\int_a^t (q_0/M)^{\frac{p}{2}}|y_1'|^p dx\right)^{\frac{1}{p}} \left(\int_a^t w|y_2|^p dx\right)^{\frac{1}{p}} \left(\int_a^t f^{p/(p-2)}dx\right)^{(p-2)/p}.$$

And thus

$$\int_a^t (w^{\frac{1}{p}}/\sqrt{q_0M})fdx/\left(\int_a^t f^{p/(p-2)}dx\right)^{(p-2)/p} \le \left(\int_a^t w|y_1|^p dx\right)^{\frac{1}{p}} \left(\int_a^t (q_0/M)^{\frac{p}{2}}|y_2'|^p dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_a^t (q_0/M)^{\frac{p}{2}}|y_1'|^p dx\right)^{\frac{1}{p}} \left(\int_a^t w|y_2|^p dx\right)^{\frac{1}{p}}.$$

By Lemma 16, if

$$\int_a^\infty w|y_1|^p dx < \infty \text{ and } \int_a^\infty w|y_2|^p dx < \infty,$$

then

$$\int_a^\infty (q_0/M)^{\frac{p}{2}}|y_1'|^p dx < \infty \text{ and } \int_a^\infty (q_0/M)^{\frac{p}{2}}|y_2'|^p dx < \infty.$$

This contradicts

$$\limsup_{t\to\infty} \int_a^t (w^{\frac{1}{p}}/\sqrt{q_0 M}) f dx / \left(\int_a^t f^{p/(p-2)} dx\right)^{(p-2)/p} = \infty.$$

Hence equation (1.1) is not limit-circle $L_p[w; a, \infty)$.

Example 11) Let $q_0(x) = x^\beta$, $1 \le x < \infty$, and $\beta \le 1 + \frac{2}{p}$. Use $M(x) = x^{\frac{4}{p}-\beta}$ with $w(x) \equiv 1$ in Theorem 16. Condition (iii) is satisfied since

$$|q_0'(x)/\sqrt{q_0(x)M(x)}| = |\beta x^{\beta-1}/\sqrt{x^\beta x^{\frac{4}{p}-\beta}}| = |\beta| x^{\beta-1-\frac{2}{p}} \le |\beta|$$

and

$$\begin{aligned} |\sqrt{q_0(x)}M'(x)/(M(x))^{\frac{3}{2}}| &= |\sqrt{x^\beta}(\frac{4}{p} - \beta)x^{\frac{4}{p}-\beta-1}/(x^{\frac{4}{p}-\beta})^{\frac{3}{2}}| \\ &= |\frac{4}{p} - \beta| x^{\beta-1-\frac{2}{p}} \le |\frac{4}{p} - \beta|. \end{aligned}$$

Condition (iv) is satisfied since $w(x) \equiv 1$. Condition (ii) is satisfied since for $f(x) = x^{-(p-2)/p}$,

$$(1/\sqrt{q_0(x)M(x)})f(x) = (1/\sqrt{x^\beta x^{\frac{4}{p}-\beta}})x^{-(p-2)/p} = x^{-\frac{2}{p}}x^{-1+\frac{2}{p}} = x^{-1}$$

and

$$(f(x))^{p/(p-2)} = (x^{-(p-2)/p})^{p/(p-2)} = x^{-1}$$

so that

$$\int_1^t (1/\sqrt{q_0 M}) f dx / (\int_1^t f^{p/(p-2)} dx)^{(p-2)/p} = \int_1^t x^{-1} dx / (\int_1^t x^{-1} dx)^{(p-2)/p} = (\ln t)^{\frac{2}{p}}.$$

Condition (i) is satisfied for

$$-\mathrm{Re}(q_1(x)) \le k x^{\frac{4}{p}-\beta} \text{ and } |\mathrm{Im}(q_1(x))| \le k x^{\frac{4}{p}-\beta}.$$

We have the following when $p > 2$.

The equation $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \le x < \infty$, *for* $\beta \le 1 + \frac{2}{p}$, *is limit-point* $L_p[1,\infty)$ *if there exists a constant* $k > 0$ *such that* $-Re(q_1(x)) \le kx^{\frac{4}{p}-\beta}$ *and* $|Im(q_1(x))| \le kx^{\frac{4}{p}-\beta}$.

When $1 < p < 2$, we are still left with this question. For $\beta \le 1 + \frac{2}{p}$, is the equation $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \le x < \infty$ limit-point $L_p[1,\infty)$ if $-q_1(x) \le kx^{\frac{4}{p}-\beta}$? We show a less satisfactory result that for $\beta < 1 + \frac{2}{p}$ and $-q_1(x) = mx^\gamma$, $m > 0$, and $\beta - 2 < \gamma \le \frac{4}{p} - \beta$, the equation is limit-point $L_p[1,\infty)$.

Theorem 17. *Let* $\eta(x) > 0$ *be a real function on* $[a,\infty)$ *such that* $\eta(x)$ *and* $q_0(x)\eta'(x)$ *are locally absolutely continuous. Let* $y(x)$ *be a solution of equation (1.1). Define* $\phi(x) = y(x)/\eta(x)$ *and* $E(x) = ((q_0(x)\eta^2(x)|\phi'(x)|)^2 + c^2|\phi(x)|^2)/2$ *for a positive constant* c. *Then* $E(x) \le E(a)\exp(\frac{1}{c}\int_a^x h(t)dt)$ *with* $h(t) = |q_1(t) - (q_0(t)\eta'(t))'/\eta(t) + c^2/(q_0(t)\eta^4(t))|\eta^2(t)$. *Furthermore*
$|y(x)| \le \frac{\sqrt{2}}{c}\sqrt{E(a)}\eta(x)\exp(\frac{1}{2c}\int_a^x h(t)dt)$ *and* $|y'(x)| \le (1/(q_0(x)\eta(x)) + \frac{\sqrt{2}}{c}\eta'(x))E(a)\exp(\frac{1}{2c}\int_a^x h(t)dt)$.

Proof. Since $\phi(x)$ satisfies equation (7.2),

$$-(q_0(x)\eta^2(x)\phi'(x))' + (q_1(x) - (q_0(x)\eta'(x))'/\eta(x) + c^2/(q_0(x)\eta^4(x)))\eta^2(x)\phi(x) = 0$$

and

$$E(x) = ((q_0(x)\eta^2(x)\phi'(x))(q_0(x)\eta^2(x)\overline{\phi}'(x)) + c^2\phi(x)\overline{\phi}(x))/2.$$

So

$$\begin{aligned} E' &= ((q_0\eta^2\phi')(q_0\eta^2\overline{\phi}')' + (q_0\eta^2\phi')'(q_0\eta^2\overline{\phi}') + c^2\phi\overline{\phi}' + c^2\phi'\overline{\phi})/2 \\ &= ((q_0\eta^2\phi')(\overline{q}_1 - (q_0\eta')'/\eta)\eta^2\overline{\phi} + (q_1 - (q_0\eta')'/\eta)\eta^2\phi(q_0\eta^2\overline{\phi}') + c^2\phi\overline{\phi}' + c^2\phi'\overline{\phi})/2 \\ &= ((q_0\eta^4(\overline{q}_1 - (q_0\eta')'/\eta) + c^2)\phi'\overline{\phi} + (q_0\eta^4(q_1 - (q_0\eta')'/\eta) + c^2)\phi\overline{\phi}')/2. \end{aligned}$$

Thus

$$
\begin{aligned}
|E'| &\le |q_0\eta^4(q_1-(q_0\eta')'/\eta)+c^2||\phi'||\phi| = \frac{1}{c}|q_1-(q_0\eta')'/\eta+c^2/(q_0\eta^4)|\eta^2(q_0\eta^2|\phi'|)(c|\phi|) \\
&\le \frac{1}{c}|q_1-(q_0\eta')'/\eta+c^2/(q_0\eta^4)|\eta^2((q_0\eta^2|\phi'|)^2+c^2|\phi|^2)/2 \\
&= \frac{1}{c}hE.
\end{aligned}
$$

So

$$
E'(t)/E(t) \le \frac{1}{c}h(t), \quad \ln(E(x)/E(a)) = \int_a^x (E'(t)/E(t))dt \le \frac{1}{c}\int_a^x h(t)dt,
$$

and

$$
E(x) \le E(a)\exp\left(\frac{1}{c}\int_a^x h(t)dt\right).
$$

Also,

$$
|\phi(x)| \le \frac{\sqrt{2}}{c}\sqrt{E(x)} \le \frac{\sqrt{2}}{c}\sqrt{E(a)}\exp\left(\frac{1}{2c}\int_a^x h(t)dt\right)
$$

and

$$
|y(x)| = \eta(x)|\phi(x)| \le \frac{\sqrt{2}}{c}\sqrt{E(a)}\eta(x)\exp\left(\frac{1}{2c}\int_a^x h(t)dt\right).
$$

Likewise,

$$
|\phi'(x)| \le (1/(q_0(x)\eta^2(x)))\sqrt{E(x)} \le (1/(q_0(x)\eta^2(x)))\sqrt{E(a)}\exp\left(\frac{1}{2c}\int_a^x h(t)dt\right)
$$

and

$$
|y'(x)| = |\eta(x)\phi'(x)+\eta'(x)\phi(x)| \le (1/(q_0(x)\eta(x))+\frac{\sqrt{2}}{c}\eta'(x))\sqrt{E(a)}\exp\left(\frac{1}{2c}\int_a^x h(t)dt\right).
$$

Theorem 18. *Let $\eta(x) > 0$ be a real function on $[a,\infty)$ such that $\eta(x)$ and $q_0(x)\eta'(x)$ are locally absolutely continuous and let $w(x)$ be a non-negative and continuous weight function. Suppose there exists a non-negative measurable real function $g(x)$ on $[a,\infty)$ such that $\int_a^t |g(x)|^{p/(p-1)}dx < \infty$, for $a \le t < \infty$, and*

$$
\limsup_{t\to\infty} \int_a^t (w(x))^{\frac{1}{p}} g(x)dx \Big/ \left(\int_a^t (g(x)|1/\eta(x)+\tfrac{\sqrt{2}q_0(x)\eta'(x)}{c}|)^{p/(p-1)}\exp(p/(2c(p-1))\int_a^x h(s)ds)dx\right)^{(p-1)/p}
$$

$= \infty,$

for $h(x) = |q_1(x) - (q_0(x)\eta'(x))'/\eta(x) + c^2/(q_0(x)\eta^4(x))|\eta^2(x)$, *with* c *a positive real number. Then equation (1.1) is limit-point* $L_p[w; a, \infty)$.

Proof. Suppose to the contrary that equation (1.1) is limit-circle $L_p[w; a, \infty)$. Let $y_1(x)$ and $y_2(x)$ be two independent solutions of (1.1) such that

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1.$$

Multiply both sides of the equality above by

$$(w(x))^{\frac{1}{p}} g(x)$$

and integrate to get

$$\int_a^t ((w^{\frac{1}{p}} y_1)(gq_0 y_2') - (gq_0 y_1')(w^{\frac{1}{p}} y_2))dx = \int_a^t w^{\frac{1}{p}} g dx.$$

Define

$$\phi_j(x) = y_j(x)/\eta(x)$$

and

$$E_j(x) = ((q_0(x)\eta^2(x)|\phi_j'(x)|)^2 + c^2|\phi_j(x)|^2)/2, \quad \text{for } j = 1, 2.$$

Use Theorem 17 to get

$$|y_j'| \leq (1/(q_0\eta) + \frac{\sqrt{2}}{c}\eta')E_j(a)\exp\left(\frac{1}{2c}\int_a^x h ds\right).$$

Thus

$$|gq_0 y_j'| \leq g|\frac{1}{\eta} + \frac{\sqrt{2}}{c}q_0\eta'|E_j(a)\exp\left(\frac{1}{2c}\int_a^x h ds\right).$$

Use Holder's inequality with exponents p and $p/(p-1)$ to get

$$\begin{aligned}
&|\int_a^t ((w^{\frac{1}{p}} y_1)(gq_0 y_2') - (gq_0 y_1')(w^{\frac{1}{p}} y_2))dx| \\
&\leq \left(\int_a^t w|y_1|^p dx\right)^{\frac{1}{p}} E_2(a)\left(\int_a^t (g|\frac{1}{\eta} + \frac{\sqrt{2}}{c}q_0\eta'|)^{p/(p-1)}\exp\left(p/(2c(p-1))\int_a^x h(s)ds\right)dx\right)^{(p-1)/p} \\
&\quad + \left(\int_a^t w|y_2|^p dx\right)^{\frac{1}{p}} E_1(a)\left(\int_a^t (g|\frac{1}{\eta} + \frac{\sqrt{2}}{c}q_0\eta'|)^{p/(p-1)}\exp\left(p/(2c(p-1))\int_a^x h(s)ds\right)dx\right)^{(p-1)/p}.
\end{aligned}$$

It follows that

$$\int_a^t w^{\frac{1}{p}} g dx / \left(\int_a^t (g|\frac{1}{\eta} + \frac{\sqrt{2}}{c} q_0 \eta'|)^{p/(p-1)} \exp\left(p/(2c(p-1)) \int_a^x h(s)ds \right) dx \right)^{(p-1)/p}$$
$$\leq \left(\int_a^t w|y_1|^p dx \right)^{\frac{1}{p}} E_2(a) + \left(\int_a^t w|y_2|^p dx \right)^{\frac{1}{p}} E_1(a).$$

Then

$$\limsup_{t\to\infty} \int_a^t w^{\frac{1}{p}} g dx / \left(\int_a^t (g|1/\eta + \frac{\sqrt{2}}{c} q_0 \eta'|)^{p/(p-1)} \exp\left(p/(2c(p-1)) \int_a^x h(s)ds \right) dx \right)^{(p-1)/p} = \infty$$

contradicts

$$\int_a^\infty w|y_1|^p dx < \infty \text{ and } \int_a^\infty w|y_2|^p dx < \infty.$$

Example 12) Let $1 < p < 2$, $\beta < 1 + \frac{2}{p}$, $q_0(x) = x^\beta$, and $-q_1(x) = mx^\gamma$, with $m > 0$ and $\beta - 2 < \gamma \leq \frac{4}{p} - \beta$. Consider the equation

$$-(x^\beta y'(x))' + (-mx^\gamma)y(x) = 0, \quad 1 \leq x < \infty.$$

Set $c = \sqrt{m}$ and $\eta(x) = x^{-(\beta+\gamma)/4}$ so $mx^\gamma = c^2/(x^\beta \eta^4(x))$. Take $g(x) = x^{-1}$. Then

$$g|1/\eta + \sqrt{2}q_0\eta'/c| = x^{-1}|x^{(\beta+\gamma)/4} + (\sqrt{2}x^\beta(-(\beta+\gamma)/4)x^{-(\beta+\gamma)/4-1})/c|$$
$$\leq x^{(\beta+\gamma)/4-1} + |\sqrt{2}((\beta+\gamma)/4)/c|x^{-(\beta+\gamma)/4+\beta-2}.$$

Since $\gamma > \beta - 2$, $(\beta+\gamma)/4 - 1 > (\beta+\beta-2)/4 - 1 = \frac{1}{2}\beta - \frac{3}{2}$ and $-(\beta+\gamma)/4 + \beta - 2 < -(\beta+\beta-2)/4 + \beta - 2 = \frac{1}{2}\beta - \frac{3}{2}$. So

$$x^{(\beta+\gamma)/4-1} + |\sqrt{2}((\beta+\gamma)/4)/c|x^{-(\beta+\gamma)/4+\beta-2} \leq (1 + |\sqrt{2}((\beta+\gamma)/4)/c|)x^{(\beta+\gamma)/4-1}.$$

Since $\gamma \leq \frac{4}{p} - \beta$,

$$x^{(\beta+\gamma)/4-1} \leq x^{(\beta+\frac{4}{p}-\beta)/4-1} = x^{(1-p)/p}.$$

Combining the preceeding inequalities

$$g|1/\eta + \sqrt{2}q_0\eta'/c| \leq (1 + |\sqrt{2}((\beta+\gamma)/4)/c|)x^{(1-p)/p}$$

and

$$(g|1/\eta + \sqrt{2}q_0\eta'/c|)^{p/(p-1)} \leq (1 + |\sqrt{2}((\beta+\gamma)/4)/c|)^{p/(p-1)}x^{-1}.$$

Thus

$$\int_1^t (g|1/\eta + \sqrt{2}q_0\eta'/c|)^{p/(p-1)}dx \leq (1 + |\sqrt{2}((\beta+\gamma)/4)/c|)^{p/(p-1)} \ln t.$$

It follows

$$\begin{aligned}
&\left(\int_1^t (g|1/\eta + \sqrt{2}q_0\eta'/c|)^{p/(p-1)} \exp\left(p/(2c(p-1))\int_1^x hds\right)dx\right)^{(p-1)/p} \\
&\leq \left(\int_1^t (g|1/\eta + \sqrt{2}q_0\eta'/c|)^{p/(p-1)} \exp\left(p/(2c(p-1))\int_1^\infty hds\right)dx\right)^{(p-1)/p} \\
&\leq \left(\exp(1/(2c)\int_1^\infty hds)\right)\left(1 + |\sqrt{2}((\beta+\gamma)/4)/c|\right)(\ln t)^{(p-1)/p}.
\end{aligned}$$

Now,

$$\begin{aligned}
h(s) &= |q_1(s) - (q_0(s)\eta'(s))'/\eta(s) + c^2/(q_0(s)\eta^4(s))|\eta^2(s) \\
&= |-mx^\gamma - (x^\beta(-(\beta+\gamma)/4)x^{-(\beta+\gamma)/4-1})'/x^{-(\beta+\gamma)/4} + mx^\gamma|x^{-(\beta+\gamma)/2} \\
&= |-(-(\beta+\gamma)/4)(-(\beta+\gamma)/4+\beta-1)\,x^{-(\beta+\gamma)/4+\beta-2}/x^{-(\beta+\gamma)/4}|x^{-(\beta+\gamma)/2} \\
&= |((\beta+\gamma)/4)(-(\beta+\gamma)/4+\beta-1)|x^{-(\beta+\gamma)/2+\beta-2}.
\end{aligned}$$

Since $\gamma > \beta - 2$, $-(\beta+\gamma)/2 + \beta - 2 < -(\beta+\beta-2)/2 + \beta - 2 = -1$. Thus

$$\int_1^\infty h(s)ds < \infty.$$

Take $w(x) \equiv 1$. Then

$$\int_1^t w^{\frac{1}{p}} g dx = \int_1^t x^{-1}dx = \ln t$$

and

$$\begin{aligned}
&\int_1^t w^{\frac{1}{p}} g dx / \left(\int_1^t (g|1/\eta + \sqrt{2}q_0\eta'/c|)^{p/(p-1)} \exp\left(p/(2c(p-1))\int_1^x hds\right)dx\right)^{(p-1)/p} \\
&\geq \ln t / \left(\exp(1/(2c)\int_1^\infty hds)\right)\left(1 + |\sqrt{2}((\beta+\gamma)/4)/c|\right)(\ln t)^{(p-1)/p} \\
&= (\ln t)^{\frac{1}{p}} / \left(\exp(1/(2c)\int_1^\infty hds)\right)\left(1 + |\sqrt{2}((\beta+\gamma)/4)/c|\right).
\end{aligned}$$

Since

$$\lim_{t\to\infty}(\ln t)^{\frac{1}{p}} = \infty,$$

the hypotheses of Theorem 18 are satisfied for the following choice of $q_0(x)$, $q_1(x)$, $\eta(x)$, c, $g(x)$, and $w(x)$: $q_0(x) = x^\beta$, with $\beta < 1 + \frac{2}{p}$, $1 < p < 2$; $-q_1(x) = mx^\gamma$, with $m > 0$ and $\beta - 2 < \gamma \leq \frac{4}{p} - \beta$; $\eta(x) = x^{-(\beta+\gamma)/4}$ and $c = \sqrt{m}$; and $w(x) \equiv 1$.

Hence the equation

$$-(x^\beta y'(x))' + (-mx^\gamma)y(x) = 0, \quad 1 \leq x < \infty,$$

is limit-point $L_p[1,\infty)$ *for* $\beta < 1 + \frac{2}{p}$, $1 < p < 2$, $m > 0$, *and* $\beta - 2 < \gamma \leq \frac{4}{p} - \beta$.

9. Consider the equation

$$y''(x) + q(x)y(x) = 0, \quad a \leq x < \infty, \tag{9.1}$$

with $q(x)$ real valued and locally Lebesgue integrable in $[a,\infty)$. Hartman and Wintner showed equation (9.1) is limit-point $L_2[a,\infty)$, if $\int_a^x q_+(s)ds \leq kx^3$, for a positive constant k. $q_+(s)$ denotes the positive part of $q(s)$, that is $q_+(s) = \begin{cases} q(s) & \text{if } q(s) \geq 0 \\ 0 & \text{if } q(s) < 0 \end{cases}$. In this section we prove a generalization of Hartman and Wintner's result for equation (7.1) and L_p, with $p \geq 2$.

First we need to generalize a Lemma due to Lyapunov.

(Lyapunov's inequality) *Let* $y(x)$ *be a non-trivial solution of* $y''(x) + f(x)y(x) = 0$, *with* $y(a) = y(b) = 0$ *and* $f(x)$ *continuous on* $[a,b]$. *Then* $\int_a^b |f(x)|dx > 4/(b-a)$.

The inequality above is proved by showing that $y(x)$ satisfies $y(x) = \int_a^b G(x,s)f(s)y(s)ds$ for $G(x,s) = \begin{cases} (s-a)(b-x)/(b-a), & \text{for } a \leq s \leq x \\ (x-a)(b-s)/(b-a), & \text{for } x \leq s \leq b \end{cases}$. After this, observe $|G(x,s)| \leq (b-a)/4$ with equality only at $x = s = (a+b)/2$. Let $y_m = \max_{a\leq x\leq b}|y(x)|$. Then $y_m \leq \int_a^b |G(x,s)||f(s)|y_m ds$ so $1 \leq \int_a^b |G(x,s)||f(s)|ds \leq ((b-a)/4)\int_a^b |f(s)|ds$. $\int_a^b |f(s)|ds \geq 4/(b-a)$ follows.

Lyapunov's inequality can be proven in the same way when $f(x)$ is only assumed to be complex valued and Lebesgue integrable on $[a, b]$. Furthermore the following holds.

Lemma. *Suppose that $y(x)$ is a non-trivial solution of the equation $y''(x) + f(x)y(x) = 0$, with $y(a) = y(b) = 0$. Then $\int_a^b |f(x)|dx > 4/(b-a)$. If $f(x)$ is real valued and a and b are consecutive zeros of $y(x)$, then $\int_a^b f_+(x)dx > 4/(b-a)$.*

Suppose $y(x)$ is a solution of (1.1) and set

$$z(t) = y(x)$$

with

$$t = \int_a^x (q_0(s))^{-1} ds.$$

Then

$$z^{\cdot}(t) = (y(x))'/(dt/dx) = y'(x)/(q_0(x))^{-1} = q_0(x)y'(x)$$

and

$$z^{\cdot\cdot}(t) = (q_0(x)y'(x))'/(dt/dx) = q_0(x)(q_0(x)y'(x))'.$$

($()^{\cdot}$ denotes the differentiation with respect to t.) Therefore

$$z^{\cdot\cdot}(t) + f(t)z(t) = 0, \quad 0 \le t < \infty, \tag{9.2}$$

with $f(t) = -q_0(x)q_1(x)$. Suppose x_1 and x_2 are zeros of $y(x)$ so that

$$t_1 = \int_a^{x_1} (q_0(s))^{-1} ds \text{ and } t_2 = \int_a^{x_2} (q_0(s))^{-1} ds$$

are zeros of $z(t)$. Applying the previous lemma we have that

$$\int_{t_1}^{t_2} |-q_0(x)q_1(x)|dt > 4/(t_2 - t_1). \tag{9.3}$$

If $q_1(x)$ is real valued and x_1 and x_2 are consecutive zeros of $y(x)$, then

$$\int_{t_1}^{t_2} (-q_0(x)q_1(x))_+ dt > 4/(t_2 - t_1). \tag{9.4}$$

Making the change of variable $t = \int_a^x (q_0(s))^{-1} ds$ we have

$$dt = (q_0(x))^{-1} dx \text{ and } t_2 - t_1 = \int_{x_1}^{x_2} (q_0(s))^{-1} ds.$$

So inequality (9.3) changes to

$$\int_{x_1}^{x_2} |q_1(x)| dx > 4 / \int_{x_1}^{x_2} (q_0(s))^{-1} ds. \tag{9.5}$$

Inequality (9.4) changes to

$$\int_{x_1}^{x_2} (-q_1(x))_+ dx > 4 / \int_{x_1}^{x_2} (q_0(s))^{-1} ds. \tag{9.6}$$

We have the following.

Lemma 19. *Suppose that $y(x)$ is a non-trivial solution of equation (1.1) with $y(x_1) = y(x_2) = 0$, $a \leq x_1 < x_2 < \infty$. Then $\int_{x_1}^{x_2} |q_1(x)| dx > 4 / \int_{x_1}^{x_2} (q_0(x))^{-1} dx$. If $q_1(x)$ is real valued and x_1 and x_2 are consecutive zeros of $y(x)$, then $\int_{x_1}^{x_2} (-q_1(x))_+ dx > 4 / \int_{x_1}^{x_2} (q_0(x))^{-1} dx$.*

Theorem (Hartman and Wintner). *Suppose $y(x)$ is a non-trivial solution of equation (9.1) and that $N(t)$ is the number of zeros of $y(x)$ on the interval $[a, t]$. Then $N(t) < \frac{1}{2}[(t - a) \int_a^t (q(s))_+ ds]^{\frac{1}{2}} + 1$.*

We have the generalization to equation (1.1).

Theorem 19. *Suppose that $y(x)$ is a non-trivial solution of equation (1.1) and $N(t)$ is the number of zeros of $y(x)$ on $[a, t]$. Then $N(t) < \frac{1}{2}[(\int_a^t (q_0(s))^{-1} ds) \int_a^t |q_1(s)| ds]^{\frac{1}{2}} + 1$. If $q_1(x)$ is real valued, then $N(t) < \frac{1}{2}[(\int_a^t (q_0(s))^{-1} ds) \int_a^t (-q_1(s))_+ ds]^{\frac{1}{2}} + 1$.*

Theorem (Hartman and Wintner). *Suppose that $\int_a^x (q(s))_+ ds \leq kx^3$ for a positive constant k. Then equation (9.1) is limit-point $L_2[a, \infty)$.*

We have the following generalization to equation (7.1).

Theorem 20. *Let $q_0(x) = x^\beta$ on $[a, \infty)$ with $\beta \leq 1$ and $a > 0$. Suppose that $\int_a^x (-q_1(s))_+ ds \leq kx^{3-\beta}$ for a positive constant k. Then equation (7.1), $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $a \leq x < \infty$, is limit-point $L_2[a, \infty)$.*

Proof. As in the proof of Hartman's and Wintner's Theorem let $y_1(x)$ and $y_2(x)$ be two real solutions of equation (7.1) such that

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1.$$

Write $y_1(x)$ and $y_2(x)$ in the form

$$y_1(x) = r(x)\cos\theta(x) \text{ and } y_2(x) = r(x)\sin\theta(x).$$

Then

$$q_0(x)r^2(x)\theta'(x) = 1$$

and

$$\int_a^x r^2(s)ds = \int_a^x (q_0(s))^{-1}(\theta'(s))^{-1}ds.$$

Equation (7.1) is limit-point $L_2[a, \infty)$ if and only if

$$\int_a^\infty r^2(s)ds = \infty.$$

Use the idea that

$$((q_0(x))^{-\frac{1}{2}}(\theta'(x))^{-\frac{1}{2}} - \mu(\theta'(x))^{\frac{1}{2}})^2 = (q_0(x))^{-1}(\theta'(x))^{-1} - 2\mu(q_0(x))^{-\frac{1}{2}} + \mu^2\theta'(x) \geq 0.$$

Then

$$(q_0(x))^{-1}(\theta'(x))^{-1} \geq 2\mu(q_0(x))^{-\frac{1}{2}} - \mu^2\theta'(x)$$

and

$$\int_b^x r^2(s)ds \geq 2\mu\int_b^x (q_0(s))^{-\frac{1}{2}}ds - \mu^2(\theta(x) - \theta(b)),$$

for $a \leq b < x < \infty$. Let $N(x)$ be the number of zeros of $y_2(x)$ in the interval $[a, x]$. Then

$$\theta(x) - \theta(a) \leq \pi(N(x) + 1).$$

By Theorem 19,

$$N(x) < \frac{1}{2}\left[\left(\int_a^x (q_0(s))^{-1}ds\right)\left(\int_a^x (-q_1(s))_+ ds\right)\right]^{\frac{1}{2}} + 1.$$

So

$$N(x) < \frac{1}{2}\left[\left(\int_a^x s^{-\beta}ds\right)kx^{3-\beta}\right]^{\frac{1}{2}} + 1 = \frac{1}{2}k^{\frac{1}{2}}\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 1$$

and

$$\theta(x) - \theta(a) \leq \pi\left(\frac{1}{2}k^{\frac{1}{2}}\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 2\right).$$

And so, for $x \geq 1$, there exists a positive constant M such that

$$\theta(x) - \theta(b) \leq \theta(x) - \theta(a) \leq M\left(\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 1\right).$$

Thus for $\mu > 0$, $x \geq 1$,

$$\int_b^x r^2(s)ds \geq 2\mu\int_b^x s^{-\frac{\beta}{2}}ds - \mu^2 M\left(\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 1\right).$$

For

$$\mu = \int_b^x s^{-\frac{\beta}{2}}ds \Big/ \left(M\left(\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 1\right)\right)$$

we have

$$\int_b^x r^2(s)ds \geq \left(\int_b^x s^{-\frac{\beta}{2}}ds\right)^2 \Big/ \left(M\left(\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 1\right)\right).$$

If $\beta < 1$,

$$\int_b^x s^{-\frac{\beta}{2}}ds = (1-\frac{\beta}{2})^{-1}(x^{1-\frac{\beta}{2}} - b^{1-\frac{\beta}{2}}) \geq \frac{1}{2}(1-\frac{\beta}{2})^{-1}x^{1-\frac{\beta}{2}}, \text{ for } x \geq 2^{1/(1-\frac{\beta}{2})}b,$$

and

$$\int_a^x s^{-\beta}ds = (1-\beta)^{-1}(x^{1-\beta} - a^{1-\beta}) \leq (1-\beta)^{-1}x^{1-\beta}.$$

Then

$$\begin{aligned}\left(\int_b^x s^{-\frac{\beta}{2}}ds\right)^2 \Big/ \left(M\left(\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{\beta}{2}} + 1\right)\right) &\geq \frac{1}{4}(1-\frac{\beta}{2})^{-2}x^{2-\beta} \Big/ \left(M\left((1-\beta)^{-\frac{1}{2}}x^{\frac{1}{2}-\frac{\beta}{2}}x^{\frac{3}{2}-\frac{\beta}{2}} + 1\right)\right) \\ &= \frac{1}{4}(1-\frac{\beta}{2})^{-2}x^{2-\beta} \Big/ \left(M\left((1-\beta)^{-\frac{1}{2}}x^{2-\beta} + 1\right)\right).\end{aligned}$$

Thus

$$\lim_{x\to\infty}\int_b^x r^2(s)ds \geq \lim_{x\to\infty}\frac{1}{4}(1-\frac{\beta}{2})^{-2}x^{2-\beta}/\left(M\left((1-\beta)^{-\frac{1}{2}}x^{2-\beta}+1\right)\right)$$
$$=(1-\beta)^{\frac{1}{2}}/(4M(1-\frac{\beta}{2})^2).$$

This is a positive value independent of b and hence

$$\int_a^\infty r^2(s)ds=\infty.$$

If $\beta=1$,

$$\int_b^x s^{-\frac{\beta}{2}}ds=2(x^{\frac{1}{2}}-b^{\frac{1}{2}})\geq x^{\frac{1}{2}},\text{ for } x\geq 4b,$$

and

$$\int_a^x s^{-\beta}ds=\ln(x/a).$$

Then

$$\int_b^x r^2(s)ds\geq\left(\int_b^x s^{-\frac{\beta}{2}}ds\right)^2/\left(M\left(\left(\int_a^x s^{-\beta}ds\right)^{\frac{1}{2}}x^{\frac{3}{2}-\frac{\beta}{2}}+1\right)\right)\geq x/(M((\ln(x/a))^{\frac{1}{2}}x+1)),\text{ for } x\geq 4b.$$

Now,

$$\begin{aligned}\int_a^\infty r^2(s)ds&=\sum_{n=1}^\infty\int_{4^{n-1}a}^{4^na}r^2(s)ds\\&\geq\sum_{n=1}^\infty 4^na/(M((\ln(4^na/a))^{\frac{1}{2}}4^na+1))\\&=\sum_{n=1}^\infty 4^na/(M((n\ln 4)^{\frac{1}{2}}4^na+1))\\&=\sum_{n=1}^\infty 1/(M(n^{\frac{1}{2}}(\ln 4)^{\frac{1}{2}}+1/(4^na))).\end{aligned}$$

By comparison with the series

$$\sum_{n=1}^\infty 1/n^{\frac{1}{2}}$$

we have that

$$\int_a^\infty r^2(s)ds=\infty.$$

Theorem 21. *Let $q_0(x) = x^\beta$ on $[a, \infty)$, with $-(1+\frac{2}{p})/(\frac{p}{2}-1) < \beta < 1$, $a > 0$, and let $p > 2$. Suppose that $\int_a^x (-q_1(s))_+ ds \le kx^{\frac{4}{p}-\beta+1}$ for a positive constant k. Then the equation (7.1), $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $a \le x < \infty$, is limit-point $L_p[a, \infty)$.*

Proof. Let $y_1(x)$ and $y_2(x)$ be two real solutions of equation (7.1) such that

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1.$$

Write $y_1(x)$ and $y_2(x)$ in the form

$$y_1(x) = r(x)\cos\theta(x) \text{ and } y_2(x) = r(x)\sin\theta(x).$$

Then

$$q_0(x)r^2(x)\theta'(x) = 1$$

and

$$r^p(x) = (q_0(x)\theta'(x))^{-\frac{p}{2}}.$$

Also,

$$r(x) = (y_1^2(x) + y_2^2(x))^{\frac{1}{2}}$$

so that by Minkowski's inequality

$$\left(\int_a^x r^p(s)ds\right)^{\frac{2}{p}} = \left(\int_a^x (y_1^2(s) + y_2^2(s))^{\frac{p}{2}} ds\right)^{\frac{2}{p}} \le \left(\int_a^x |y_1(s)|^p ds\right)^{\frac{2}{p}} + \left(\int_a^x |y_2(s)|^p ds\right)^{\frac{2}{p}}.$$

Thus equation (7.1) is limit-point $L_p[a, \infty)$ if and only if

$$\int_a^\infty r^p(x)dx = \infty.$$

Let $N(x)$ be the number of zeros of $y_2(x)$ in the interval $[a, x]$. Then

$$\theta(x) - \theta(a) \le \pi(N(x) + 1).$$

By Theorem 19,

$$N(x) < \frac{1}{2}\left[\left(\int_a^x (q_0(s))^{-1} ds\right)\left(\int_a^x (-q_1(s))_+ ds\right)\right]^{\frac{1}{2}} + 1.$$

So

$$N(x) < \frac{1}{2}\left[\left(\int_a^x s^{-\beta}ds\right)kx^{\frac{4}{p}-\beta+1}\right]^{\frac{1}{2}} + 1.$$

If $\beta < 1$, then

$$\int_a^x s^{-\beta}ds = (1-\beta)^{-1}(x^{1-\beta} - a^{1-\beta}) \leq (1-\beta)^{-1}x^{1-\beta}$$

and

$$N(x) < \frac{1}{2}(1-\beta)^{-\frac{1}{2}}K^{\frac{1}{2}}x^{\frac{2}{p}-\beta+1} + 1.$$

So, for $a \leq b < x < \infty$,

$$\theta(x) - \theta(b) \leq \theta(x) - \theta(a) < \pi(\frac{1}{2}(1-\beta)^{-\frac{1}{2}}K^{\frac{1}{2}}x^{\frac{2}{p}-\beta+1} + 2).$$

Consider $b \geq 1$ so that

$$\theta(x) \leq Mx^{\frac{2}{p}-\beta+1}$$

for a positive constant M independent of b. Use the idea that

$$\left((q_0(x))^{-\frac{p}{4}}(\theta'(x))^{-\frac{p}{4}} - \mu(q_0(x))^{\frac{p}{4}}(\theta'(x))^{\frac{1}{2}}\right)^2 = (q_0(x))^{-\frac{p}{2}}(\theta'(x))^{-\frac{p}{2}} - 2\mu(\theta'(x))^{-\frac{p}{4}+\frac{1}{2}}$$
$$+ \mu^2(q_0(x))^{\frac{p}{2}}\theta'(x) \geq 0.$$

Then

$$r^p(x) = (q_0(x))^{-\frac{p}{2}}(\theta'(x))^{-\frac{p}{2}} \geq 2\mu(\theta'(x))^{-\frac{p}{4}+\frac{1}{2}} - \mu^2(q_0(x))^{\frac{p}{2}}\theta'(x)$$

and

$$\int_b^x r^p(s)ds \geq 2\mu\int_b^x(\theta'(s))^{-\frac{p}{4}+\frac{1}{2}}ds - \mu^2\int_b^x(q_0(s))^{\frac{p}{2}}\theta'(s)ds.$$

Define, for $\delta > 0$, the set E_δ by

$$E_\delta = \left\{s : s \in [b,x] \text{ and } \theta'(s) > Mx^{\frac{2}{p}-\beta+1+\delta}/(x-b)\right\}.$$

Since

$$\int_b^x \theta'(s)ds < Mx^{\frac{2}{p}-\beta+1},$$

It follows

$$m(E_\delta) < x^{-\delta}(x-b).$$

(Here $m(\cdot)$ indicates Lebesgue measure). Thus, for

$$E_\delta^c = \left\{ s : s \in [b,x] \text{ and } \theta'(s) \le M x^{\frac{2}{p}-\beta+1+\delta}/(x-b) \right\},$$

$$m(E_\delta^c) = (x-b) - m(E_\delta) > (x-b) - x^{-\delta}(x-b) = (1-x^{-\delta})(x-b).$$

And

$$\begin{aligned}
\int_b^x (\theta'(s))^{-\frac{p}{4}+\frac{1}{2}} ds &\ge \int_{E_\delta^c} (\theta'(s))^{-\frac{p}{4}+\frac{1}{2}} ds \\
&\ge \int_{E_\delta^c} (M x^{\frac{2}{p}-\beta+1+\delta}/(x-b))^{-\frac{p}{4}+\frac{1}{2}} ds \\
&= (M x^{\frac{2}{p}-\beta+1+\delta}/(x-b))^{-\frac{p}{4}+\frac{1}{2}} m(E_\delta^c) \\
&> (M x^{\frac{2}{p}-\beta+1+\delta}/(x-b))^{-\frac{p}{4}+\frac{1}{2}} (1-x^{-\delta})(x-b) \\
&= (M)^{-\frac{p}{4}+\frac{1}{2}} x^{(\frac{2}{p}-\beta+1+\delta)(-\frac{p}{4}+\frac{1}{2})} (x-b)^{\frac{p}{4}+\frac{1}{2}} (1-x^{-\delta}).
\end{aligned}$$

If $x \ge 2b$,

$$(x-b)^{\frac{p}{4}+\frac{1}{2}} \ge (\frac{1}{2})^{\frac{p}{4}+\frac{1}{2}} x^{\frac{p}{4}+\frac{1}{2}}$$

so then

$$\int_b^x (\theta'(s))^{-\frac{p}{4}+\frac{1}{2}} ds \ge (\frac{1}{2})^{\frac{p}{4}+\frac{1}{2}} (M)^{-\frac{p}{4}+\frac{1}{2}} x^{(\frac{2}{p}-\beta+\delta)(-\frac{p}{4}+\frac{1}{2})+1} (1-x^{-\delta}).$$

Also,

$$\begin{aligned}
\int_b^x (q_0(s))^{\frac{p}{2}} \theta'(s) ds &= \int_b^x s^{(\frac{p}{2})\beta} \theta'(s) ds \\
&= |_b^x s^{(\frac{p}{2})\beta} \theta(s) - \int_b^x (\frac{p}{2}) \beta s^{(\frac{p}{2})\beta - 1} \theta(s) ds.
\end{aligned}$$

$$\begin{aligned}
x^{(\frac{p}{2})\beta}\theta(x) - b^{(\frac{p}{2})\beta}\theta(b) &\le x^{(\frac{p}{2})\beta}\theta(x) \\
&\le x^{(\frac{p}{2})\beta} M x^{\frac{2}{p}-\beta+1} \\
&= M x^{(\frac{p}{2}-1)\beta+1+\frac{2}{p}}
\end{aligned}$$

and

$$\int_b^x (\frac{p}{2})\beta s^{(\frac{p}{2})\beta-1}\theta(s)ds \le \int_b^x (\frac{p}{2})|\beta| s^{(\frac{p}{2})\beta-1} M s^{\frac{2}{p}-\beta+1} ds$$
$$= (\frac{p}{2})|\beta| M((\frac{p}{2}-1)\beta+1+\frac{2}{p})^{-1}(x^{(\frac{p}{2}-1)\beta+1+\frac{2}{p}} - b^{(\frac{p}{2}-1)\beta+1+\frac{2}{p}})$$
$$\le (\frac{p}{2})|\beta| M((\frac{p}{2}-1)\beta+1+\frac{2}{p})^{-1} x^{(\frac{p}{2}-1)\beta+1+\frac{2}{p}}.$$

Thus

$$\int_b^x (q_0(s))^{\frac{p}{2}}\theta'(s)ds \le M(1+(\frac{p}{2})|\beta|((\frac{p}{2}-1)\beta+1+\frac{2}{p})^{-1})x^{(\frac{p}{2}-1)\beta+1+\frac{2}{p}}.$$

Now, for $b \ge 1$, $x \ge 2b$, and $\mu > 0$,

$$\int_b^x r^p(s)ds \ge 2\mu \int_b^x (\theta'(s))^{-\frac{p}{4}+\frac{1}{2}} ds - \mu^2 \int_b^x (q_0(s))^{\frac{p}{4}}\theta'(s)ds \ge 2\mu A - \mu^2 B,$$

with

$$A = (\frac{1}{2})^{\frac{p}{4}+\frac{1}{2}}(M)^{-\frac{p}{4}+\frac{1}{2}} x^{(\frac{2}{p}-\beta+\delta)(-\frac{p}{4}+\frac{1}{2})+1}(1-x^{-\delta})$$

and

$$B = M(1+(\frac{p}{2})|\beta|((\frac{p}{2}-1)\beta+1+\frac{2}{p})^{-1})x^{(\frac{p}{2}-1)\beta+1+\frac{2}{p}}.$$

For $\mu = A/B$ we get

$$\int_b^x r^p(s)ds \ge A^2/B = Cx^{-\delta(\frac{p}{2}-1)}(1-x^{-\delta})^2,$$

with

$$C = (\frac{1}{2})^{\frac{p}{2}+1}(M)^{-\frac{p}{2}}/(1+(\frac{p}{2})|\beta|((\frac{p}{2}-1)\beta+1+\frac{2}{p})^{-1}).$$

Choose $\delta = \ln 2/\ln x$ so that $x^{-\delta} = \frac{1}{2}$. Then we have

$$\int_b^x r^p(s)ds \ge C(\frac{1}{2})^{\frac{p}{2}-1}(\frac{1}{2})^2 = C(\frac{1}{2})^{\frac{p}{2}+1}.$$

So

$$\int_b^\infty r^p(s)ds \ge C(\frac{1}{2})^{\frac{p}{2}+1}$$

which is a positive value independent of b. Hence

$$\int_a^\infty r^p(s)ds = \infty.$$

If $\beta = 1$ and $2 < p \leq 4$, then we can add the following.

(Theorem). *Let $2 < p \leq 4$. Suppose that $\int_a^x (-q_1(s))_+ ds \leq kx^{\frac{4}{p}}$ for a positive constant k. Then the equation (7.1), $-(xy'(x))' + q_1(x)y(x) = 0$, $a \leq x < \infty$, is limit-point $L_p[a, \infty)$.*

Proof. We proceed as in the proof of the case when $\beta < 1$ except for the following changes.

$$N(x) < \frac{1}{2}\left[\left(\int_a^x s^{-1} ds\right) kx^{\frac{4}{p}}\right]^{\frac{1}{2}} + 1 = \frac{1}{2}k^{\frac{1}{2}}(\ln(x/a))^{\frac{1}{2}}x^{\frac{2}{p}} + 1$$

and

$$\theta(x) - \theta(b) \leq \theta(x) - \theta(a) \leq \pi\left(\frac{1}{2}k^{\frac{1}{2}}(\ln(x/a))^{\frac{1}{2}}x^{\frac{2}{p}} + 2\right).$$

Consider $b \geq \max\{ae, 1\}$ and $0 < a \leq b < x < \infty$, so that

$$\theta(x) \leq M(\ln(x/a))^{\frac{1}{2}}x^{\frac{2}{p}}$$

for a positive constant M independent of b. Define, for $\delta > 0$,

$$E_\delta = \left\{s : s \in [b, x] \text{ and } \theta'(s) > M(\ln(x/a))^{\frac{1}{2}}x^{\frac{2}{p}+\delta}/(x-b)\right\}$$

and

$$E_\delta^c = \left\{s : s \in [b, x] \text{ and } \theta'(s) \leq M(\ln(x/a))^{\frac{1}{2}}x^{\frac{2}{p}+\delta}/(x-b)\right\}.$$

Then, for $x \geq 2b$,

$$\begin{aligned}\int_b^x (\theta'(s))^{-\frac{p}{4}+\frac{1}{2}} ds &\geq \int_{E_\delta^c} (\theta'(s))^{-\frac{p}{4}+\frac{1}{2}} ds \\ &\geq (\frac{1}{2})^{\frac{p}{4}+\frac{1}{2}}(M)^{-\frac{p}{4}+\frac{1}{2}}(\ln(x/a))^{\frac{1}{2}(-\frac{p}{4}+\frac{1}{2})}x^{\frac{1}{p}+\frac{p}{4}-\frac{1}{2}\delta(\frac{p}{2}-1)}(1 - x^{-\delta}).\end{aligned}$$

Also,

$$\int_b^x (q_0(s))^{\frac{p}{2}}\theta'(s)ds \leq M(1 + (\frac{p}{2})(\frac{p}{2} + \frac{2}{p})^{-1})(\ln(x/a))^{\frac{1}{2}}x^{\frac{p}{2}+\frac{2}{p}}.$$

Thus for $b \geq \max\{ae, 1\}$, $x \geq 2b$, and $\mu > 0$,

$$\int_b^x r^p(s)ds \geq 2\mu A - \mu^2 B$$

with

$$A = (\frac{1}{2})^{\frac{p}{4}+\frac{1}{2}}(M)^{-\frac{p}{4}+\frac{1}{2}}(\ln(x/a))^{\frac{1}{2}(-\frac{p}{4}+\frac{1}{2})}x^{\frac{1}{p}+\frac{p}{4}-\frac{1}{2}\delta(\frac{p}{2}-1)}(1-x^{-\delta})$$

and

$$B = M(1+(\frac{p}{2})(\frac{p}{2}+\frac{2}{p})^{-1})(\ln(x/a))^{\frac{1}{2}}x^{\frac{p}{2}+\frac{2}{p}}.$$

For $\mu = A/B$,

$$\int_b^x r^p(s)ds \geq A^2/B = C(\ln(x/a))^{-\frac{p}{4}}x^{-\delta(\frac{p}{2}-1)}(1-x^{-\delta})^2$$

with

$$C = (\frac{1}{2})^{\frac{p}{2}+1}(M)^{-\frac{p}{4}}/(1+(\frac{p}{2})(\frac{p}{2}+\frac{2}{p})^{-1}).$$

Choose $\delta = \ln 2/\ln x$ so that $x^{-\delta} = \frac{1}{2}$ and

$$\int_b^x r^p(s)ds \geq C(\ln(x/a))^{-\frac{p}{4}}(\frac{1}{2})^{\frac{p}{2}+1},$$

for $b \geq \max\{ae, 1\}$ and $x \geq 2b$. Thus, for $2^{n-1}a \geq \max\{ae, 1\}$,

$$\int_{2^{n-1}a}^{2^n a} r^p(s)ds \geq C(\frac{1}{2})^{\frac{p}{2}+1}(\ln(2^n))^{-\frac{p}{4}} = C(\frac{1}{2})^{\frac{p}{2}+1}(\ln 2)^{-\frac{p}{4}}n^{-\frac{p}{4}}.$$

Hence, for k such that $2^{k-1}a \geq \max\{ae, 1\}$,

$$\int_a^\infty r^p(s)ds \geq \sum_{n=k}^\infty \int_{2^{n-1}a}^{2^n a} r^p(s)ds \geq C(\frac{1}{2})^{\frac{p}{2}+1}(\ln 2)^{-\frac{p}{4}}\sum_{n=k}^\infty n^{-\frac{p}{4}}.$$

For $p \leq 4$,

$$\sum_{n=k}^\infty n^{-\frac{p}{4}} \text{ diverges}$$

and so

$$\int_a^\infty r^p(s)ds = \infty.$$

Another set of conditions that imply equation (7.1) is limit-point L_p were found, for $1 < p < 2$.

Theorem 22. *Consider the equation (7.1)* $-(q_0(x)y'(x))' + q_1(x)y(x) = 0$, $a \leq x < \infty$, *and let* $1 < p < 2$. *Suppose that there exist positive and continuously differentiable functions* $\rho(x)$ *and* $g(x)$ *on* $[a, \infty)$ *and a positive real number* k *such that*

(i) $-\rho(x)q_1(x) \leq k$ *and* $(\rho(x))^{-\frac{1}{2}}|\rho'(x)|(q_0(x))^{\frac{1}{2}} \leq k$

(ii) $|g'(x)| \leq k$ *and* $g(x)(q_0(x)\rho(x))^{-\frac{1}{2}} \leq k$

(iii) $\int_a^\infty (\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}(g(x))^{(2-p)/p}dx = \infty.$

Then the equation (7.1) is limit-point $L_p[a,\infty)$.

Proof. Suppose that $y(x)$ is a non-trivial solution of the equation (7.1). If $y(x) > 0$, then

$$\rho(x)(q_0(x)y'(x))y(x)|y(x)|^{p-2} = \rho(x)(q_0(x)y'(x))(y(x))^{p-1}$$

and

$$\begin{aligned}(\rho(x)(q_0(x)y'(x))y(x)|y(x)|^{p-2})' &= \rho(x)(q_0(x)y'(x))(p-1)(y(x))^{p-2}y'(x)\\ &\quad + \rho(x)(q_0(x)y'(x))'(y(x))^{p-1}\\ &\quad + \rho'(x)(q_0(x)y'(x))(y(x))^{p-1}\\ &= (p-1)\rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}\\ &\quad + \rho(x)q_1(x)|y(x)|^p\\ &\quad + \rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}.\end{aligned}$$

If $y(x) < 0$, then

$$\begin{aligned}\rho(x)(q_0(x)y'(x))y(x)|y(x)|^{p-2} &= \rho(x)(q_0(x)y'(x))y(x)(-y(x))^{p-2}\\ &= -\rho(x)(q_0(x)y'(x))(-y(x))^{p-1}\end{aligned}$$

and

$$\begin{aligned}(\rho(x)(q_0(x)y'(x))y(x)|y(x)|^{p-2})' &= -\rho(x)(q_0(x)y'(x))(p-1)(-y(x))^{p-2}(-y'(x))\\ &\quad - \rho(x)(q_0(x)y'(x))'(-y(x))^{p-1}\\ &\quad - \rho'(x)(q_0(x)y'(x))(-y(x))^{p-1}\\ &= (p-1)\rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}\\ &\quad + \rho(x)q_1(x)|y(x)|^p\\ &\quad + \rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}.\end{aligned}$$

By Theorem 19, $y(x)$ has a finite number of zeros on the interval $[a,t]$, for $a < t < \infty$. Either of two possibilities could occur. The first is that $y(x)$ has only a finite number, if any, of zeros on $[a,\infty)$. The second is that $y(x)$ has an infinite number of zeros on $[a,\infty)$. In the first case let n be the number of zeros $y(x)$ has on $[a,\infty)$. If $n = 0$, then $y(x)$ is either strictly positive or strictly negative on (a,∞). In which case

$$\begin{aligned}(p-1)\rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2} = {} & (\rho(x)(q_0(x)y'(x))y(x)|y(x)|^{p-2})' \\ & - \rho(x)q_1(x)|y(x)|^p \\ & - \rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}\end{aligned}$$

and

$$\begin{aligned}(p-1)\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx = {} & |_a^t \rho(x)q_0(x)y'(x)y(x)|y(x)|^{p-2} \\ & + \int_a^t -\rho(x)q_1(x)|y(x)|^p dx \\ & + \int_a^t -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx,\end{aligned}$$

for $a < t < \infty$. If $n > 0$, then $y(x)$ has zeros, say $t_1, t_2, \ldots, t_n$, with

$$a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < t_{n+1} = t,$$

for $t_n < t < \infty$. (If $n = 1$, this is $a = t_0 \leq t_1 < t_2 = t$.) In which case $y(x)$ is either strictly positive or strictly negative on the interiors of each of the intervals

$$[a,t_1], [t_1,t_2], \ldots, [t_{n-1},t_n], [t_n,t].$$

Since on the interiors of the intervals

$$\begin{aligned}(p-1)\rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2} = {} & (\rho(x)(q_0(x)y'(x))y(x)|y(x)|^{p-2})' \\ & - \rho(x)q_1(x)|y(x)|^p - \rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2},\end{aligned}$$

we have

$$
\begin{aligned}
(p-1)\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx &= (p-1)\sum_{k=1}^{n+1}\int_{t_{k-1}}^{t_k} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx \\
&= \sum_{k=1}^{n+1} |_{t_{k-1}}^{t_k} \rho(x)q_0(x)y'(x)y(x)|y(x)|^{p-2} \\
&\quad + \sum_{k=1}^{n+1}\int_{t_{k-1}}^{t_k} -\rho(x)q_1(x)|y(x)|^p dx \\
&\quad + \sum_{k=1}^{n+1}\int_{t_{k-1}}^{t_k} -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx.
\end{aligned}
$$

Since $y(t_k) = 0$, for $k = 1, 2, \ldots, n$ we get

$$
\begin{aligned}
(p-1)\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx &= |_a^t \rho(x)q_0(x)y'(x)y(x)|y(x)|^{p-2} \\
&\quad + \int_a^t -\rho(x)q_1(x)|y(x)|^p dx \\
&\quad + \int_a^t -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx.
\end{aligned}
$$

In the second case let $t_1, t_2, \ldots, t_n, \ldots$ with

$$a = t_0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots$$

be all the zeros of $y(x)$ on $[a, \infty)$. We have that

$$\lim_{n\to\infty} t_n = \infty.$$

On the interiors of the intervals

$$[a, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n], \ldots$$

$y(x)$ is either strictly positive or strictly negative. In which case

$$
\begin{aligned}
(p-1)\int_a^{t_n} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx &= (p-1)\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx \\
&= \sum_{k=1}^{n} |_{t_{k-1}}^{t_k} \rho(x)q_0(x)y'(x)y(x)|y(x)|^{p-2} \\
&\quad + \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} -\rho(x)q_1(x)|y(x)|^p dx \\
&\quad + \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx \\
&= -\rho(a)q_0(a)y'(a)y(a)|y(a)|^{p-2} + \int_a^{t_n} -\rho(x)q_1(x)|y(x)|^p dx \\
&\quad + \int_a^{t_n} -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx.
\end{aligned}
$$

Assume that condition (i) holds and that $y(x)$ is $L_p[a,\infty)$, in particular that

$$\int_a^{\infty} |y(x)|^p dx \le M < \infty,$$

for a positive real number M. By (i)

$$-\rho(x)q_1(x) \le k,$$

so that

$$\int_a^{t} -\rho(x)q_1(x)|y(x)|^p dx \le kM,$$

and by (ii)

$$
\begin{aligned}
|-\rho'(x)q_0(x)| &= |\rho'(x)|q_0(x) = (\rho(x))^{-\frac{1}{2}}|\rho'(x)|(q_0(x))^{\frac{1}{2}}(\rho(x)q_0(x))^{\frac{1}{2}} \\
&\le k(\rho(x)q_0(x))^{\frac{1}{2}},
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_a^{t} -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx &\le \int_a^{t} |\rho'(x)|q_0(x)|y'(x)||y(x)|^{p-1}dx \\
&= \int_a^{t} (\rho(x))^{-\frac{1}{2}}|\rho'(x)|(q_0(x))^{\frac{1}{2}}((\rho(x)q_0(x))^{\frac{1}{2}}|y'(x)||y(x)|^{(p-2)/2})(|y(x)|^{\frac{2}{2}})dx \\
&\le k\left(\int_a^{t} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}}\left(\int_a^{t} |y(x)|^p dx\right)^{\frac{1}{2}} \\
&\le kM^{\frac{1}{2}}\left(\int_a^{t} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}}.
\end{aligned}
$$

In the first case, when $y(x)$ has finitely many zeros on $[a, \infty)$,

$$\begin{aligned}(p-1)\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx &= |_a^t \rho(x)q_0(x)y'(x)y(x)|y(x)|^{p-2} \\ &\quad + \int_a^t -\rho(x)q_1(x)|y(x)|^p dx \\ &\quad + \int_a^t -\rho'(x)q_0(x)y'(x)y(x)|y(x)|^{p-2}dx \\ &\leq |_a^t \rho(x)q_0(x)y'(x)y(x)|y(x)|^{p-2} \\ &\quad + kM + kM^{\frac{1}{2}}\left(\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}}.\end{aligned}$$

If it were that

$$\lim_{t\to\infty}\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx = \infty,$$

then it would follow that

$$\lim_{t\to\infty} \rho(t)q_0(t)y'(t)y(t)|y(t)|^{p-2} = \infty.$$

In which case there exists b, $a \leq b < \infty$, such that

$$y'(t)y(t) > 0,$$

for $t \geq b$. Then $y(t) > 0$ and $y'(t) > 0$ or $y(t) < 0$ and $y'(t) < 0$, for $t \geq b$. In either case $|y(t)|$ would be increasing on $b \leq t < \infty$,

$$\text{violating } \int_a^\infty |y(x)|^p dx < \infty.$$

Hence, in the first case, if condition (i) holds and $\int_a^\infty |y(x)|^p dx < \infty$, then

$$\int_a^\infty \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx < \infty.$$

In the second case, when $y(x)$ has infinitely many zeros on $[a, \infty)$,

$$\begin{aligned}(p-1)\int_a^{t_n} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx &= -\rho(a)q_0(a)y'(a)y(a)|y(a)|^{p-2} \\ &\quad + \int_a^{t_n} -\rho(x)q_1(x)|y(x)|^p dx \\ &\quad + \int_a^{t_n} -\rho'(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx \\ &\leq -\rho(a)q_0(a)y'(a)y(a)|y(a)|^{p-2} + kM \\ &\quad + kM^{\frac{1}{2}}\left(\int_a^{t_n} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}}.\end{aligned}$$

Using

$$\int_a^\infty \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx = \lim_{n\to\infty}\int_a^{t_n} \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx$$

and the inequality above we can conclude that in the second case, also, if condition (i) holds and $\int_a^\infty |y(x)|^p dx < \infty$, then

$$\int_a^\infty \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx < \infty.$$

Assume, in addition, that condition (ii) holds. We have that

$$(|y(x)|^p)' = p|y(x)|^{p-2}y(x)y'(x)$$

so

$$(g(x)|y(x)|^p)' = g(x)p|y(x)|^{p-2}y(x)y'(x) + g'(x)|y(x)|^p.$$

$$\begin{aligned} g(t)|y(t)|^p &= g(a)|y(a)|^p + \int_a^t (g(x)|y(x)|^p)'dx \\ &\le g(a)|y(a)|^p + \int_a^t pg(x)|y'(x)||y(x)|^{p-1}dx \\ &\quad + \int_a^t |g'(x)||y(x)|^p dx. \end{aligned}$$

$$\begin{aligned} \int_a^t g(x)|y'(x)||y(x)|^{p-1}dx &= \int_a^t g(x)(\rho(x)q_0(x))^{-\frac{1}{2}}((\rho(x)q_0(x))^{\frac{1}{2}}|y'(x)||y(x)|^{(p-2)/2})(|y(x)|^{p/2})dx \\ &\le k\left(\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}}\left(\int_a^t |y(x)|^p dx\right)^{\frac{1}{2}} \end{aligned}$$

and

$$\int_a^t |g'(x)||y(x)|^p dx \le k\int_a^t |y(x)|^p dx.$$

This gives us

$$g(t)|y(t)|^p \le L, \quad a \le t < \infty,$$

with L a positive constant independent of t.

Now, suppose that the equation (7.1) is limit-circle $L_p[a,\infty)$. (We will show this supposition contradicts the inclusion of condition (iii).) Let $y_1(x)$ and $y_2(x)$ be two independent real solutions of (7.1) such that

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1,$$

$$\int_a^\infty |y_1(x)|^p dx < \infty, \text{ and } \int_a^\infty |y_2(x)|^p dx < \infty.$$

Then we have

$$\begin{aligned}
&(\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}|y_1(x)y_2(x)|^{(p-2)/2} \\
&= (\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}|y_1(x)y_2(x)|^{(p-2)/2}q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) \\
&= y_1(x)|y_1(x)|^{\frac{p}{2}-1}(\rho(x)q_0(x))^{\frac{1}{2}}|y_2'(x)||y_2(x)|^{(p-2)/2} \\
&\quad - y_2(x)|y_2(x)|^{\frac{p}{2}-1}(\rho(x)q_0(x))^{\frac{1}{2}}|y_1'(x)||y_1(x)|^{(p-2)/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_a^t (\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}|y_1(x)y_2(x)|^{(p-2)/2}dx &\le \left(\int_a^t |y_1(x)|^p dx\right)^{\frac{1}{2}} \left(\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}} \\
&\quad + \left(\int_a^t |y_2(x)|^p dx\right)^{\frac{1}{2}} \left(\int_a^t \rho(x)q_0(x)|y'(x)|^2|y(x)|^{p-2}dx\right)^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

Furthermore,

$$(\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}(g(x))^{(2-p)/2} = (\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}|y_1(x)y_2(x)|^{(p-2)/2}(g(x)|y_1(x)y_2(x)|^{\frac{p}{2}})^{(2-p)/p}.$$

From the previous estimation

$$g(x)|y_1(x)y_2(x)|^{\frac{p}{2}} = (g(x)|y_1(x)|^p)^{\frac{1}{2}}(g(x)|y_2(x)|^p)^{\frac{1}{2}} \le L,$$

for a positive constant L. This would yield

$$\begin{aligned}
\int_a^\infty (\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}(g(x))^{(2-p)/2}dx &\le (L)^{(2-p)/p}\int_a^\infty (\rho(x))^{\frac{1}{2}}(q_0(x))^{-\frac{1}{2}}|y_1(x)y_2(x)|^{(p-2)/2}dx \\
&< \infty,
\end{aligned}$$

contradicting condition (iii).

Chapter III

Interval Limit-point L_p Criteria

10. Make the substitution

$$M(x) = (W(x))^{-2}$$

into the theorem of Levinson. Conditions (i), (ii), and (iii) can be rewritten as

(i) $-q_1(x)(W(x))^2 \leq k$

(ii) $\int_a^\infty (q_0(x))^{-\frac{1}{2}} W(x)dx = \infty$

(iii) $|(q_0(x))^{\frac{1}{2}} W'(x)| \leq \frac{1}{2}k$.

An observation by Read is that allowing $W(x)$ to be non-negative instead of positive yields interval-type criteria.

Theorem 23. *Equation (7.1) is limit-point $L_2[a,\infty)$ if there exists a non-negative and locally absolutely continuous function $W(x)$ on $[a,\infty)$ and a positive constant k such that*

(i) $-q_1(x)(W(x))^2 \leq k$

(ii) $\int_a^\infty (q_0(x))^{-\frac{1}{2}} W(x)dx = \infty$

(iii) $(q_0(x))^{\frac{1}{2}} |W'(x)| \leq k$ *almost everywhere with respect to Lebesgue measure.*

Evans and Zettl proved a variety of interval limit-point L_2 criteria for the operator $My = -(q_0 y')' + q_1 y$ in [7]. The following interval criteria was found by adapting the construction of the function $w(x)$ used in the proof of their Corollary 2.

Theorem 24. *Let I_m, $m = 1, 2, \ldots$ be a sequence of disjoint closed intervals in $[a,\infty)$, $Q_m = \int_{I_m} (q_0(x))^{-\frac{1}{2}} dx$, and let R_m, $m = 1, 2, \ldots$ be a sequence of positive real numbers. Suppose the following conditions hold for a positive real number k.*

(a) $-q_1(x) R_m^2 Q_m^2 \leq k$, *for* $x \in I_m$, $m = 1, 2, \ldots$

(b) $\sum_{m=1}^{\infty} R_m Q_m^2 = \infty$

(c) $R_m \leq k$, *for each* m.

Then equation (7.1) is limit-point $L_2[a, \infty)$.

Proof. Let $I_m = [\alpha_m, \beta_m]$, for $m = 1, 2, \ldots$. For $m = 1, 2, \ldots$ define

$$W_m(x) = \begin{cases} \int_{\alpha_m}^{x} (q_0(s))^{-\frac{1}{2}} ds, & \alpha_m \leq x \leq \gamma_m \\ \int_{x}^{\beta_m} (q_0(s))^{-\frac{1}{2}} ds, & \gamma_m \leq x \leq \beta_m \\ 0, & \text{otherwise} \end{cases}$$

with γ_m being defined by $\int_{\alpha_m}^{\gamma_m} (q_0(s))^{-\frac{1}{2}} ds = \int_{\gamma_m}^{\beta_m} (q_0(s))^{-\frac{1}{2}} ds = \frac{1}{2} Q_m$. Let $W(x) = \sum_{m=1}^{\infty} R_m W_m(x)$. We show that if conditions (a), (b), and (c) hold, then conditions (i), (ii), and (iii) of Theorem 23 also hold with $W(x)$ and k as above. For $x \in I_m$,

$$W(x) = R_m W_m(x)$$

where

$$W_m(x) \leq \int_{\alpha_m}^{\gamma_m} (q_0(s))^{-\frac{1}{2}} ds = \int_{\gamma_m}^{\beta_m} (q_0(s))^{-\frac{1}{2}} ds = \frac{1}{2} Q_m.$$

If x is not contained in I_m, for any m, then $W(x) = 0$. So $(W(x))^2 \leq \frac{1}{4} R_m^2 Q_m^2$, for $x \in I_m$, and $(W(x)) = 0$, if x is not contained in any interval I_m. In which case $-q_1(x) R_m^2 Q_m^2 \leq k$, for $x \in I_m$, implies

$$-q_1(x)(W(x))^2 \leq -q_1(x) \left(\frac{1}{4} R_m^2 Q_m^2 \right) \leq \frac{1}{4} k \leq k,$$

for $x \in I_m$ and $q_1(x) \leq 0$. If $q_1(x) > 0$, then the inequality

$$-q_1(x)(W(x))^2 \leq k$$

still holds. Thus (a) implies (i).

$$\begin{aligned} \int_a^{\infty} (q_0(x))^{-\frac{1}{2}} W(x) dx &= \int_a^{\infty} (q_0(x))^{-\frac{1}{2}} \left(\sum_{m=1}^{\infty} R_m W_m(x) \right) dx \\ &= \sum_{m=1}^{\infty} \int_a^{\infty} (q_0(x))^{-\frac{1}{2}} R_m W_m(x) dx \\ &= \sum_{m=1}^{\infty} R_m \int_{\alpha_m}^{\beta_m} (q_0(x))^{-\frac{1}{2}} W_m(x) dx. \end{aligned}$$

Now,

$$\begin{aligned}\int_{\alpha_m}^{\beta_m}(q_0(x))^{-\frac{1}{2}}W_m(x)dx &= \int_{\alpha_m}^{\gamma_m}(q_0(x))^{-\frac{1}{2}}W_m(x)dx \\ &\quad + \int_{\gamma_m}^{\beta_m}(q_0(x))^{-\frac{1}{2}}W_m(x)dx \\ &= \int_{\alpha_m}^{\gamma_m}(q_0(x))^{-\frac{1}{2}}\left(\int_{\alpha_m}^{x}(q_0(s))^{-\frac{1}{2}}ds\right)dx \\ &\quad + \int_{\gamma_m}^{\beta_m}(q_0(x))^{-\frac{1}{2}}\left(\int_{x}^{\beta_m}(q_0(s))^{-\frac{1}{2}}ds\right)dx.\end{aligned}$$

To evaluate the integral $\int_{\alpha_m}^{\gamma_m}(q_0(x))^{-\frac{1}{2}}\left(\int_{\alpha_m}^{x}(q_0(s))^{-\frac{1}{2}}ds\right)dx$ let

$$F(x) = \int_{\alpha_m}^{x}(q_0(s))^{-\frac{1}{2}}ds.$$

Then

$$F'(x) = (q_0(x))^{-\frac{1}{2}}$$

and

$$\begin{aligned}\int_{\alpha_m}^{\gamma_m}(q_0(x))^{-\frac{1}{2}}\left(\int_{\alpha_m}^{x}(q_0(s))^{-\frac{1}{2}}ds\right)dx &= \int_{\alpha_m}^{\gamma_m}F(x)F'(x)dx \\ &= |_{\alpha_m}^{\gamma_m}\frac{1}{2}(F(x))^2 \\ &= \frac{1}{2}\left(\int_{\alpha_m}^{\gamma_m}(q_0(s))^{-\frac{1}{2}}ds\right)^2 \\ &= \frac{1}{8}Q_m^2.\end{aligned}$$

Likewise

$$\int_{\gamma_m}^{\beta_m}(q_0(x))^{-\frac{1}{2}}\left(\int_{x}^{\beta_m}(q_0(s))^{-\frac{1}{2}}ds\right)dx = \frac{1}{8}Q_m^2.$$

So

$$\int_{\alpha_m}^{\beta_m}(q_0(x))^{-\frac{1}{2}}W_m(x)dx = \frac{1}{4}Q_m^2$$

and we have

$$\int_{a}^{\infty}(q_0(x))^{-\frac{1}{2}}W(x)dx = \sum_{m=1}^{\infty}R_m\int_{\alpha_m}^{\beta_m}(q_0(x))^{-\frac{1}{2}}W_m(x)dx = \frac{1}{4}\sum_{m=1}^{\infty}R_mQ_m^2 = \infty.$$

Thus (b) implies (ii). We have

$$W_m'(x) = \begin{cases} (q_0(x))^{-\frac{1}{2}}, & \alpha_m < x < \gamma_m \\ -(q_0(x))^{-\frac{1}{2}}, & \gamma_m < x < \beta_m \\ 0, & x < \alpha_m \text{ or } x > \beta_m \end{cases}$$

and $W_m'(x)$ not defined at the points α_m, γ_m, and β_m. Now

$$W'(x) = \sum_{m=1}^{\infty} R_m W_m'(x).$$

So

$$|W'(x)| = |\sum_{m=1}^{\infty} R_m W_m'(x)| = \sum_{m=1}^{\infty} R_m |W_m'(x)| = R_m |W_m'(x)| = R_m (q_0(x))^{-\frac{1}{2}},$$

for $\alpha_m < x < \gamma_m$ or $\gamma_m < x < \beta_m$, $m = 1, 2, \ldots$, with $|W'(x)| = 0$ otherwise except at the points α_m, γ_m, and β_m, $m = 1, 2, \ldots$, where $W'(x)$ is undefined. Thus

$$(q_0(x))^{-\frac{1}{2}} |W'(x)| = R_m,$$

for $x \in I_m$ except at the points α_m, γ_m, and β_m, $m = 1, 2, \ldots$, and

$$(q_0(x))^{-\frac{1}{2}} |W'(x)| = 0$$

otherwise. Thus condition (c) that

$$R_m \leq k, \quad m = 1, 2, \ldots,$$

implies condition (iii), that

$$(q_0(x))^{-\frac{1}{2}} |W'(x)| \leq k$$

almost everywhere with respect to Lebesgue measure. Hence, if conditions (a), (b), and (c) hold, we can apply Theorem 23 with

$$W(x) = \sum_{m=1}^{\infty} R_m W_m(x)$$

to conclude that the equation (7.1) is limit-point $L_2[a, \infty)$.

In the next example we use this inequality to estimate Q_m. For real numbers x, y, and z

$$\begin{cases} (x+y)^z - x^z \geq yzx^{z-1}, & \text{if } x > 0, y \geq 0, \text{ and } z \geq 1 \\ (x+y)^z - x^z \geq yz(x+y)^{z-1}, & \text{if } x > 0, y \geq 0, \text{ and } 0 \leq z < 1. \end{cases} \tag{10.1}$$

To see that inequality (10.1) holds do the following. Let

$$F(y) = (x+y)^z - x^z, \quad G(y) = yzx^{z-1},$$

and

$$H(y) = yz(x+y)^{z-1}.$$

When $z \geq 1$,

$$F(0) = G(0) = 0 \text{ and } F'(y) = z(x+y)^{z-1} \geq zx^{z-1} = G'(y).$$

So

$$F(y) = F(0) + \int_0^y F'(s)ds \geq G(0) + \int_0^y G'(s)ds = G(y).$$

When $0 \leq z < 1$,

$$F(0) = H(0) = 0 \text{ and } z(z-1) \leq 0,$$

so that

$$F'(y) = z(x+y)^{z-1} \geq z(x+y)^{z-1} + yz(z-1)(x+y)^{z-2} = H'(y).$$

So

$$F(y) = F(0) + \int_0^y F'(s)ds \geq H(0) + \int_0^y H'(s)ds = H(y).$$

Example 13) Let $q_0(x) = x^\beta$, with $\beta \leq 1$, for $1 \leq x < \infty$. Let ε, δ, and k be real numbers such that $0 < \varepsilon < 1$, $\delta \geq 0$, $\varepsilon + \delta \leq 1$, and $k > 0$. Let $I_m = [m+\delta, m+\delta+\varepsilon]$, for $m = 1, 2, \ldots$.

If $-q_1(x) \leq kx^{2-\beta}$, for $x \in I_m$, $m = 1, 2, \ldots$, then the equation (7.1) $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \leq x < \infty$, is limit-point $L_2[a, \infty)$.

We show this by applying Theorem 24 with $R_m = (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}/Q_m$ and k replaced by $\max\{k, \varepsilon^{-1}\}$. The intervals I_m, $m = 1, 2, \ldots$ are disjoint and of length ε. $Q_m = \int_{I_m} (q_0(x))^{-\frac{1}{2}} dx = \int_{m+\delta}^{m+\delta+\varepsilon} x^{-\beta/2} dx = \left(1 - \frac{\beta}{2}\right)^{-1} ((m+\delta+\varepsilon)^{1-\beta/2} - (m+\delta)^{1-\beta/2})$. By inequality (10.1), if $\beta \leq 0$, $(m+\delta+\varepsilon)^{1-\frac{\beta}{2}} - (m+\delta)^{1-\frac{\beta}{2}} \geq \varepsilon(1-\frac{\beta}{2})(m+\delta)^{-\frac{\beta}{2}}$, and if $0 < \beta \leq 1$, $(m+\delta+\varepsilon)^{1-\frac{\beta}{2}} - (m+\delta)^{1-\frac{\beta}{2}} \geq \varepsilon(1-\frac{\beta}{2})(m+\delta+\varepsilon)^{-\frac{\beta}{2}}$. If $\beta \leq 0$, $Q_m = (1-\frac{\beta}{2})^{-1}((m+\delta+\varepsilon)^{1-\frac{\beta}{2}} - (m+\delta)^{1-\frac{\beta}{2}}) \geq \varepsilon(m+\delta)^{-\frac{\beta}{2}}$,

and if $0 < \beta \leq 1$, $Q_m \geq \varepsilon(m+\delta+\varepsilon)^{-\frac{\beta}{2}}$. So, if $\beta \leq 0$, $R_m = (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}/Q_m \leq (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}/(\varepsilon(m+\delta)^{-\frac{\beta}{2}}) = \varepsilon^{-1}(m+\delta)^{\frac{\beta}{2}}(m+\delta+\varepsilon)^{\frac{\beta}{2}-1} \leq \varepsilon^{-1}$, and if $0 < \beta \leq 1$, $R_m \leq (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}/(\varepsilon(m+\delta+\varepsilon)^{-\frac{\beta}{2}}) = \varepsilon^{-1}(m+\delta+\varepsilon)^{\beta-1} \leq \varepsilon^{-1}$. Now, if $\beta \leq 0$, $R_m Q_m^2 = (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}Q_m \geq \varepsilon(m+\delta+\varepsilon)^{\frac{\beta}{2}-1}(m+\delta)^{-\frac{\beta}{2}} = \varepsilon((m+\delta+\varepsilon)/(m+\delta))^{\frac{\beta}{2}-1}(m+\delta)^{-1} \geq \varepsilon(1+\varepsilon/(1+\delta))^{\frac{\beta}{2}-1}(m+\delta)^{-1}$, and if $0 < \beta \leq 1$, $R_m Q_m^2 = (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}Q_m \geq \varepsilon(m+\delta+\varepsilon)^{\frac{\beta}{2}-1}(m+\delta+\varepsilon)^{-\frac{\beta}{2}} = \varepsilon(m+\delta+\varepsilon)^{-1}$. We have then, for $\beta \leq 1$, $\sum_{m=1}^{\infty} R_m Q_m^2 = \infty$. $R_m^2 Q_m^2 = (m+\delta+\varepsilon)^{\beta-2}$ so that $-q_1(x)R_m^2Q_m^2 \leq k$, for $x \in I_m = [m+\delta, m+\delta+\varepsilon]$, is $-q_1(x) \leq k(m+\delta+\varepsilon)^{2-\beta}$, for $m+\delta \leq x \leq m+\delta+\varepsilon$. Thus, if $-q_1(x) \leq kx^{2-\beta}$, for $x \in I_m$, $m = 1, 2, \ldots$, then $-q_1(x) \leq k(m+\delta+\varepsilon)^{\beta-2}$ and $-q_1(x)R_m^2Q_m^2 \leq k$, for $x \in I_m$, $m = 1, 2, \ldots$. Hence we have that for $k \geq \varepsilon^{-1}$ and $R_m = (m+\delta+\varepsilon)^{\frac{\beta}{2}-1}/Q_m$

(a) $-q_1(x)R_m^2Q_m^2 \leq k$, for $x \in I_m$, $m = 1, 2, \ldots$

(b) $\sum_{m=1}^{\infty} R_m Q_m^2 = \infty$

(c) $R_m \leq k$, for each m.

Make the substitution

$$M(x) = (q_0(x))^{(p-2)/p}(W(x))^{-2}$$

into the statement of conditions (i), (ii), and (iii) of Theorem 16. Then conditions (i), (ii), and (iii) can be rewritten as

(i) $-\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p}(W(x))^2 \leq k(w(x))^{\frac{2}{p}}$ and $|\mathrm{Im}(q_1(x))|(q_0(x))^{(2-p)/p}(W(x))^2 \leq k(w(x))^{\frac{2}{p}}$

(ii) there exists a measurable real function $f(x) \geq 0$ such that $\int_a^t (f(x))^{p/(p-2)}dx < \infty$, for $t < \infty$, and $\limsup_{t\to\infty} \int_a^t (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p}W(x)f(x)dx / \left(\int_a^t (f(x))^{p/(p-2)}dx\right)^{(p-2)/p} = \infty$

(iii) $|(2-p)(q_0(x))^{(1-p)/p}q_0'(x)W(x) + p(q_0(x))^{\frac{1}{p}}W'(x)| \leq k(w(x))^{\frac{1}{p}}$.

Again, allowing $W(x)$ to be non-negative instead of positive yields interval type criteria.

Theorem 25. *Let $w(x)$ be a non-negative and continuous weight function on $[a,\infty)$ and let $p > 2$. Equation (1.1) is limit-point $L_p[w(x);a,\infty)$ if there exists a non-negative and locally absolutely continuous function $W(x)$ on $[a,\infty)$ and a positive real number k such that*

(i) $-Re(q_1(x))(q_0(x))^{(2-p)/p}(W(x))^2 \leq k(w(x))^{\frac{2}{p}}$ *and* $|Im(q_1(x))|(q_0(x))^{(2-p)/p}(W(x))^2 \leq k(w(x))^{\frac{2}{p}}$

(ii) *there exists a measurable real function $f(x) \geq 0$ such that $\int_a^t (f(x))^{p/(p-2)}dx < \infty$, for $t < \infty$, and* $\limsup_{t\to\infty} \int_a^t (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p}W(x)f(x)dx / \left(\int_a^t (f(x))^{p/(p-2)}dx\right)^{(p-2)/p} = \infty$

(iii) $|(2-p)(q_0(x))^{(1-p)/p}q_0'(x)W(x) + p(q_0(x))^{\frac{1}{p}}W'(x)| \leq k(w(x))^{\frac{1}{p}}$ *almost everywhere with respect to Lebesgue measure on $[a,\infty)$*

(iv) *there exists a sequence of points $t_1, t_2, \ldots$ with $a \leq t_1 < t_2 < t_3 < \ldots$ and $\lim_{n\to\infty} t_n = \infty$ such that $W(t_n) = 0$, for $n = 1, 2, \ldots$.*

The proof is done in nearly the same way as the proof of Theorem 16.

Lemma 25. *Let $y(x)$ be a solution of equation (1.1). Under the hypothesis that (i), (iii), and (iv) hold, if $\int_a^\infty w(x)|y(x)|^p dx < \infty$, then $\int_a^\infty q_0(x)(W(x))^p|y'(x)|^p dx < \infty$.*

Proof. Proceed exactly as in the proof of Lemma 16 until the result

$$\frac{d}{dx}(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) = (p-1)(q_0(x)|u'(x)|)^{p-2}(\mathrm{Re}(q_1(x))u(x) - \mathrm{Im}(q_1(x))v(x))$$

has been obtained. Express

$$u(x)u'(x)(q_0(x)(W(x))^p)|u'(x)|^{p-2}$$

as

$$u(x)((q_0(x))^{2-p}(W(x))^p)(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}).$$

Then

$$
\begin{aligned}
&(u(x)u'(x)(q_0(x)(W(x))^p)|u'(x)|^{p-2})' \\
&\quad = u(x)((q_0(x))^{2-p}(W(x))^p)(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2})' \\
&\qquad + u(x)((q_0(x))^{2-p}(W(x))^p)'(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) \\
&\qquad + u'(x)((q_0(x))^{2-p}(W(x))^p)(q_0(x)u'(x)(q_0(x)|u'(x)|)^{p-2}) \\
&\quad = (p-1)\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p}(W(x))^2(u(x))^2(q_0(x)(W(x))^p)^{(p-2)/p}|u'(x)|^{p-2} \\
&\qquad - (p-1)\mathrm{Im}(q_1(x))(q_0(x))^{(2-p)/p}(W(x))^2(u(x)v(x)(q_0(x)(W(x))^p)^{(p-2)/p}|u'(x)|^{p-2} \\
&\qquad + (q_0(x))^{p-1}((q_0(x))^{2-p}(W(x))^p)'u(x)u'(x)|u'(x)|^{p-2} \\
&\qquad + q_0(x)(W(x))^p|u'(x)|^p.
\end{aligned}
$$

The inequality above is understood to hold almost everywhere with respect to Lebesgue measure. In particular everywhere except at the points where $W'(x)$ is undefined. After rearranging terms and integrating we have

$$
\begin{aligned}
&\int_a^t q_0(x)(W(x))^p|u'(x)|^p dx \\
&\quad = |_a^t u(x)u'(x)q_0(x)(W(x))^p|u'(x)|^{p-2} \\
&\qquad + (p-1)\int_a^t (-\mathrm{Re}(q_1(x)))(q_0(x))^{(2-p)/p}(W(x))^2(u(x))^2(q_0(x)(W(x))^p)^{(p-2)/p}|u'(x)|^{p-2}dx \\
&\qquad + (p-1)\int_a^t (\mathrm{Im}(q_1(x)))(q_0(x))^{(2-p)/p}(W(x))^2u(x)v(x)(q_0(x)(W(x))^p)^{(p-2)/p}|u'(x)|^{p-2}dx \\
&\qquad - \int_a^t (q_0(x))^{p-1}((q_0(x))^{2-p}(W(x))^p)'u(x)u'(x)|u'(x)|^{p-2}dx. \qquad (10.2)
\end{aligned}
$$

Given that condition (i) holds,

$$
-\mathrm{Re}(q_1)(q_0)^{(2-p)/p}W^2 \le kw^{\frac{2}{p}} \text{ and } |\mathrm{Im}(q_1)|(q_0)^{(2-p)/p}W^2 \le kw^{\frac{2}{p}}.
$$

So

$$
\begin{aligned}
&(p-1)\int_a^t (-\mathrm{Re}(q_1))(q_0)^{(2-p)/p}(W)^2(u)^2(q_0(W)^p)^{(p-2)/p}|u'|^{p-2}dx \\
&\quad \le (p-1)k\int_a^t (w)^{\frac{2}{p}}|u|^2(q_0(W)^p)^{(p-2)/p}|u'|^{p-2}dx \\
&\quad \le (p-1)k\left(\int_a^t w|u|^p dx\right)^{\frac{2}{p}}\left(\int_a^t q_0(W)^p|u'|^p dx\right)^{(p-2)/p}.
\end{aligned}
$$

and

$$
\begin{aligned}
&(p-1)\int_a^t (\mathrm{Im}(q_1))(q_0)^{(2-p)/p}(W)^2 uv(q_0(W)^p)^{(p-2)/p}|u'|^{p-2}dx \\
&\le (p-1)k\int_a^t (w)^{\frac{2}{p}}|u||v|(q_0(W)^p)^{(p-2)/p}|u'|^{p-2}dx \\
&\le (p-1)k\left(\int_a^t w|u|^p dx\right)^{\frac{1}{p}}\left(\int_a^t w|v|^p dx\right)^{\frac{1}{p}}\left(\int_a^t q_0(W)^p|u'|^p dx\right)^{(p-2)/p}.
\end{aligned}
$$

Now

$$
\begin{aligned}
(q_0(x))^{p-1}((q_0(x))^{2-p}(W(x))^p)' &= (q_0(x))^{p-1}((q_0(x))^{2-p}p(W(x))^{p-1}W'(x) \\
&\quad + (2-p)(q_0(x))^{1-p}q_0'(x)(W(x))^p) \\
&= ((2-p)(q_0(x))^{(1-p)/p}q_0'(x)W(x) \\
&\quad + p(q_0(x))^{\frac{1}{p}}W'(x))(q_0(x))^{(p-1)/p}(W(x))^{p-1}.
\end{aligned}
$$

So, by condition (iii),

$$
|(q_0)^{p-1}((q_0)^{2-p}(W)^p)'| \le k(w)^{\frac{1}{p}}(q_0)^{(p-1)/p}(W)^{p-1}
$$

almost everywhere on $[a,\infty)$, except where W' is undefined. We have then

$$
\begin{aligned}
-\int_a^t (q_0)^{p-1}((q_0)^{2-p}(W)^p)'uu'|u'|^{p-2}dx &\le \int_a^t |(q_0)^{p-1}((q_0)^{2-p}(W)^p)'||u||u'|^{p-1}dx \\
&\le k\int_a^t ((w)^{\frac{1}{p}}|u|)((q_0)^{(p-1)/p}(W)^{p-1}|u'|^{p-1})dx \\
&\le k\left(\int_a^t w|u|^p dx\right)^{\frac{1}{p}}\left(\int_a^t q_0(W)^p|u'|^p dx\right)^{(p-1)/p}.
\end{aligned}
$$

Inserting the inequalities above into (10.2) we get

$$
\begin{aligned}
\int_a^t q_0(x)(W(x))^p|u'(x)|^p dx &\le |_a^t u(x)u'(x)q_0(x)(W(x))^p|u'(x)|^{p-2} \\
&\quad + (p-1)k\left(\int_a^t w(x)|u(x)|^p dx\right)^{\frac{2}{p}}\left(\int_a^t q_0(x)(W(x))^p|u'(x)|^p dx\right)^{(p-2)/p} \\
&\quad + (p-1)k\left(\int_a^t w(x)|u(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^t w(x)|v(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^t q_0(x)(W(x))^p|u'(x)|^p dx\right)^{(p-2)/p} \\
&\quad + k\left(\int_a^t w(x)|u(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^t q_0(x)(W(x))^p|u'(x)|^p dx\right)^{(p-1)/p}. \qquad (10.3)
\end{aligned}
$$

Suppose that

$$\int_a^\infty w(x)|y(x)|^p dx \leq B < \infty$$

for a positive real number B. Then

$$\int_a^\infty w(x)|u(x)|^p dx \leq B \text{ and } \int_a^\infty w(x)|v(x)|^p dx \leq B,$$

also. For t_n such that

$$W(t_n) = 0,$$

we have from (10.3) that

$$\int_a^{t_n} q_0(x)(W(x))^p|u'(x)|^p dx \leq 2(p-1)kB^{\frac{2}{p}} \left(\int_a^{t_n} q_0(x)(W(x))^p|u'(x)|^p dx\right)^{(p-2)/p}$$
$$+ kB^{\frac{1}{p}} \left(\int_a^{t_n} q_0(x)(W(x))^p|u'(x)|^p dx\right)^{(p-1)/p}.$$

After dividing both sides of the inequality above by

$$\left(\int_a^{t_n} q_0(W)^p|u'|^p dx\right)^{(p-2)/p}$$

we have

$$\left(\int_a^{t_n} q_0(W)^p|u'|^p dx\right)^{\frac{2}{p}} \leq 2(p-1)kB^{\frac{2}{p}} + kB^{\frac{1}{p}} \left(\int_a^{t_n} q_0(W)^p|u'|^p dx\right)^{\frac{1}{p}}.$$

Letting

$$A = \left(\int_a^{t_n} q_0(W)^p|u'|^p dx\right)^{\frac{1}{p}}$$

this is

$$A^2 \leq 2(p-1)kB^{\frac{2}{p}} + kB^{\frac{1}{p}}A$$

or

$$A^2 - kB^{\frac{1}{p}}A - 2(p-1)kB^{\frac{2}{p}} \leq 0.$$

Using the quadratic formula we can determine

$$A \leq (kB^{\frac{1}{p}} + ((kB^{\frac{1}{p}})^2 + 8(p-1)kB^{\frac{2}{p}})^{\frac{1}{2}})/2 = ((k + (k^2 + 8(p-1)k)^{\frac{1}{2}})/2)B^{\frac{1}{p}}$$

or that

$$\int_a^{t_n} q_0(W)^p |u'|^p dx \leq ((k + (k^2 + 8(p-1)k)^{\frac{1}{2}})/2)^p B.$$

Assuming condition (iv) holds,

$$t_n \to \infty \text{ as } n \to \infty,$$

so

$$\int_a^\infty q_0(x)(W(x))^p |u'(x)|^p dx = \lim_{n\to\infty} \int_a^{t_n} q_0(x)(W(x))^p |u'(x)|^p dx$$
$$\leq ((k + (k^2 + 8(p-1)k)^{\frac{1}{2}})/2)^p B.$$

By the same reasoning

$$\int_a^\infty q_0(x)(W(x))^p |v'(x)|^p dx < \infty.$$

It follows

$$\int_a^\infty q_0(x)(W(x))^p |y'(x)|^p dx < \infty$$

and Lemma 25 is proved.

Now, suppose in addition (ii) holds and the equation (1.1)

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0, \quad a \leq x < \infty,$$

is limit-circle $L_p[w; a, \infty)$. Let $y_1(x)$ and $y_2(x)$ be two independent solutions of (1.1) with

$$q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1.$$

After multiplying both sides of $q_0(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) = 1$ by

$$(w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p} W(x) f(x)$$

and integrating we have

$$\int_a^t ((w(x))^{\frac{1}{p}} y_1(x))((q_0(x))^{\frac{1}{p}} W(x) y_2'(x)) f(x) - ((q_0(x))^{\frac{1}{p}} W(x) y_1'(x))((w(x))^{\frac{1}{p}} y_2(x)) f(x)) dx$$
$$= \int_a^t (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W(x) f(x) dx.$$

By applying Holder's inequality with exponents p, p, and $p/(p-2)$ we can bound the lefthand side of the equality above by

$$\left(\int_a^t w|y_1|^p dx\right)^{\frac{1}{p}} \left(\int_a^t q_0(W)^p |y_2'|^p dx\right)^{\frac{1}{p}} \left(\int_a^t f^{p/(p-2)} dx\right)^{(p-2)/p}$$
$$+ \left(\int_a^t q_0(W)^p |y_1'|^p dx\right)^{\frac{1}{p}} \left(\int_a^t w|y_2|^p dx\right)^{\frac{1}{p}} \left(\int_a^t f^{p/(p-2)} dx\right)^{(p-2)/p}.$$

And thus

$$\int_a^t w^{\frac{1}{p}} (q_0)^{(1-p)/p} W f dx / \left(\int_a^t f^{p/(p-2)} dx\right)^{(p-2)/p} \leq \left(\int_a^t w|y_1|^p dx\right)^{\frac{1}{p}} \left(\int_a^t q_0(W)^p |y_2'|^p dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_a^t q_0(W)^p |y_1'|^p dx\right)^{\frac{1}{p}} \left(\int_a^t w|y_2|^p dx\right)^{\frac{1}{p}}.$$

By Lemma 25, if

$$\int_a^\infty w|y_1|^p dx < \infty \text{ and } \int_a^\infty w|y_2|^p dx < \infty,$$

then

$$\int_a^\infty q_0(W)^p |y_1'|^p dx < \infty \text{ and } \int_a^\infty q_0(W)^p |y_2'|^p dx < \infty.$$

This contradicts

$$\limsup_{t\to\infty} \int_a^t w^{\frac{1}{p}} (q_0)^{(1-p)/p} W f dx / \left(\int_a^t f^{p/(p-2)} dx\right)^{(p-2)/p} = \infty.$$

Hence equaiton (1.1) is not limit-circle $L_p[w(x); a, \infty)$ and Theorem 25 is proved.

Theorem 26 *Let $w(x)$ be a non-negative and continuous weight function on $[a, \infty)$ and let $p > 2$. Let $I_m = [\alpha_m, \beta_m]$, $m = 1, 2, \ldots$ be a sequence of disjoint intervals contained in $[a, \infty)$ such that $\alpha_m \to \infty$ as $m \to \infty$, and let μ_m be the length of I_m for each $m = 1, 2, \ldots$. Let $Q_m = \int_{I_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} dx$, and let R_m and S_m, $m = 1, 2, \ldots$ be two sequences of positive real numbers. Suppose the following conditions hold for a positive real number k.*

(a) $-Re(q_1(x))(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k(w(x))^{\frac{2}{p}}$ *and* $|Im(q_1(x))|(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k(w(x))^{\frac{2}{p}}$, *for $x \in I_m$, $m = 1, 2, \ldots$*

(b) $\limsup_{n\to\infty} \sum_{m=1}^{n} R_m Q_m^2 S_m / \left(\sum_{m=1}^{n} \mu_m (S_m)^{p/(p-2)}\right)^{(p-2)/p} = \infty$

(c) $R_m (q_0(x))^{(2-p)/p} \le k$ *and* $R_m Q_m (q_0(x))^{(1-p)/p} |q_0'(x)| \le k(w(x))^{\frac{1}{p}}$ *for* $x \in I_m$, $m = 1, 2, \ldots$.

Then equation (1.1) is limit-point $L_p[w(x); a, \infty)$.

Proof. For $m = 1, 2, \ldots$ define

$$W_m(x) = \begin{cases} \int_{\alpha_m}^{x} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds, & \alpha_m \le x \le \gamma_m \\ \int_{x}^{\beta_m} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds, & \gamma_m \le x \le \beta_m \\ 0, & \text{otherwise} \end{cases}$$

with γ_m being defined by

$$\int_{\alpha_m}^{\gamma_m} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds = \int_{\gamma_m}^{\beta_m} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds = \frac{1}{2} Q_m.$$

Let

$$W(x) = \sum_{m=1}^{\infty} R_m W_m(x)$$

and

$$f(x) = \begin{cases} S_m, & \text{if } x \in I_m,\ m = 1, 2, \ldots \\ 0, & \text{if } x \text{ is not contained in any } I_m. \end{cases}$$

We show that if conditions (a), (b), and (c) hold, then conditions (i), (ii), and (iii) of Theorem 25 also hold with $W(x)$ defined as above and k enlarged to $(\frac{3}{2}p - 1)k$. Condition (iv) of Theorem 25 holds since $W(x) = 0$, for $\beta_m < x < \alpha_{m+1}$, and $\beta_m \to \infty$ as $m \to \infty$. For $x \in I_m$,

$$W(x) = R_m W_m(x)$$

where

$$\begin{aligned} W_m(x) &\le \int_{\alpha_m}^{\gamma_m} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds \\ &= \int_{\gamma_m}^{\beta_m} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds \\ &= \frac{1}{2} Q_m. \end{aligned}$$

If x is not contained in I_m, for any m, then $W(x) = 0$. So

$$(W(x))^2 \leq \frac{1}{4} R_m^2 Q_m^2,$$

for $x \in I_m$, and

$$(W(x)) = 0,$$

if x is not contained in any interval I_m. In which case

$$-\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k(w(x))^{\frac{2}{p}},$$

for $x \in I_m$, implies

$$\begin{aligned} -\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p}(W(x))^2 &\leq -\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p}(\frac{1}{4} R_m^2 Q_m^2) \\ &\leq \frac{1}{4} k(w(x))^{\frac{2}{p}} \\ &\leq k(w(x))^{\frac{2}{p}}, \end{aligned}$$

if $\mathrm{Re}(q_1(x)) \leq 0$, and

$$|\mathrm{Im}(q_1(x))|(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k(w(x))^{\frac{2}{p}},$$

for $x \in I_m$, implies

$$\begin{aligned} |\mathrm{Im}(q_1(x))|(q_0(x))^{(2-p)/p}(W(x))^2 &\leq |\mathrm{Im}(q_1(x))|(q_0(x))^{(2-p)/p}(\frac{1}{4} R_m^2 Q_m^2) \\ &\leq \frac{1}{4} k(w(x))^{\frac{2}{p}} \\ &\leq k(w(x))^{\frac{2}{p}}. \end{aligned}$$

If $\mathrm{Re}(q_1(x)) > 0$, then the inequality $-\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p}(W(x))^2 \leq k(w(x))^{\frac{2}{p}}$ still holds. Thus (a) implies (i).

$$\begin{aligned} \int_a^{\beta_m} (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p} W(x) f(x) dx &= \sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p} W(x) f(x) dx \\ &= \sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p} W(x) S_m dx \\ &= \sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p} R_m W_m(x) S_m dx \\ &= \sum_{m=1}^{n} R_m S_m \int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p} W_m(x) dx. \end{aligned}$$

Now,

$$\int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx = \int_{\alpha_m}^{\gamma_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx + \int_{\gamma_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx.$$

We have

$$\begin{aligned}
&\int_{\alpha_m}^{\gamma_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx \\
&= \int_{\alpha_m}^{\gamma_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} \left(\int_{\alpha_m}^{x} (w(x))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds \right) dx \\
&= |_{\alpha_m}^{\gamma_m} \frac{1}{2} \left(\int_{\alpha_m}^{x} (w(x))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds \right)^2 \\
&= \frac{1}{8} Q_m^2.
\end{aligned}$$

Likewise

$$\begin{aligned}
&\int_{\gamma_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx \\
&= \int_{\alpha_m}^{\gamma_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} \left(\int_{\gamma_m}^{x} (w(s))^{\frac{1}{p}} (q_0(s))^{(1-p)/p} ds \right) dx.
\end{aligned}$$

So

$$\int_{\gamma_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx = \frac{1}{8} Q_m^2,$$

also, and

$$\begin{aligned}
\int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx &= \int_{\alpha_m}^{\gamma_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx \\
&\quad + \int_{\gamma_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx \\
&= \frac{1}{4} Q_m^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{a}^{\beta_n} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W(x) f(x)dx &= \sum_{m=1}^{n} R_m S_m \int_{\alpha_m}^{\beta_m} (w(x))^{\frac{1}{p}} (q_0(x))^{(1-p)/p} W_m(x)dx \\
&= \frac{1}{4} \sum_{m=1}^{n} R_m Q_m^2 S_m.
\end{aligned}$$

Also

$$\begin{aligned}\int_a^{\beta_n}(f(x))^{p/(p-2)}dx &= \sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}(f(x))^{p/(p-2)}dx\\ &= \sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}(S_m)^{p/(p-2)}dx\\ &= \sum_{m=1}^{n}(\beta_m-\alpha_m)(S_m)^{p/(p-2)}\\ &= \sum_{m=1}^{n}\mu_m(S_m)^{p/(p-2)}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_a^{\beta_n}(w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p}W(x)f(x)dx\Big/\left(\int_a^{\beta_n}(f(x))^{p/(p-2)}dx\right)^{(p-2)/p}\\ &= \frac{1}{4}\sum_{m=1}^{n}R_mQ_m^2S_m\Big/\left(\sum_{m=1}^{n}\mu_m(S_m)^{p/(p-2)}\right)^{(p-2)/p}.\end{aligned}$$

So

$$\limsup_{n\to\infty}\sum_{m=1}^{n}R_mQ_m^2S_m\Big/\left(\sum_{m=1}^{n}\mu_m(S_m)^{p/(p-2)}\right)^{(p-2)/p}=\infty$$

implies that

$$\limsup_{n\to\infty}\int_a^{\beta_n}(w)^{\frac{1}{p}}(q_0)^{(1-p)/p}Wfdx\Big/\left(\int_a^{\beta_n}(f)^{p/(p-2)}dx\right)^{(p-2)/p}=\infty.$$

It follows that condition (b) implies condition (ii). We have

$$W_m'(x)=\begin{cases}(w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p}, & \alpha_m<x<\gamma_m\\ -(w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p}, & \gamma_m<x<\beta_m\\ 0, & x<\alpha_m \text{ or } x>\beta_m\end{cases}$$

and $W_m'(x)$ not defined at the points α_m, γ_m, and β_m.

$$W'(x)=\sum_{m=1}^{\infty}R_mW_m'(x).$$

So

$$|W'(x)|=|\sum_{m=1}^{\infty}R_mW_m'(x)|=\sum_{m=1}^{\infty}R_m|W_m'|=R_m(w(x))^{\frac{1}{p}}(q_0(x))^{(1-p)/p},$$

for $\alpha_m < x < \gamma_m$ or $\gamma_m < x < \beta_m$, $m = 1, 2, \ldots$, and

$$|W'(x)| = 0$$

otherwise except at the points α_m, γ_m, and β_m, $m = 1, 2, \ldots$, where $W'(x)$ is undefined. Thus

$$(q_0(x))^{\frac{1}{p}}|W'(x)| = R_m(w(x))^{\frac{1}{p}}(q_0(x))^{(2-p)/p},$$

for $x \in I_m$, except at the points α_m, γ_m, and β_m, $m = 1, 2, \ldots$, and

$$(q_0(x))^{\frac{1}{p}}|W'(x)| = 0$$

otherwise. Thus condition (c), that

$$R_m(q_0(x))^{(2-p)/p} \leq k \text{ and } R_m Q_m(q_0(x))^{(1-p)/p}|q_0'(x)| \leq k(w(x))^{\frac{1}{p}},$$

for $x \in I_m$, $m = 1, 2, \ldots$, implies that

$$(q_0(x))^{\frac{1}{p}}|W'(x)| = R_m(w(x))^{\frac{1}{p}}(q_0(x))^{(2-p)/p} \leq k(w(x))^{\frac{1}{p}}$$

and

$$(q_0(x))^{(1-p)/p}|q_0'(x)|W(x) \leq \frac{1}{2}R_m Q_m(q_0(x))^{(1-p)/p}|q_0'(x)| \leq \frac{1}{2}k(w(x))^{\frac{1}{p}},$$

for $x \in I_m$, $m = 1, 2, \ldots$, except at the points α_m, γ_m, and β_m, $m = 1, 2, \ldots$, where $W'(x)$ and the inequality $(q_0(x))^{\frac{1}{p}}|W'(x)| \leq k(w(x))^{\frac{1}{p}}$ is undefined.

$$(q_0(x))^{\frac{1}{p}}|W'(x)| = 0 \text{ and } (q_0(x))^{(1-p)/p}|q_0'(x)|W(x) = 0,$$

if x is not contained in I_m for any m. So

$$\begin{aligned}
&|(2-p)(q_0(x))^{(1-p)/p}q_0'(x)W(x) + p(q_0(x))^{\frac{1}{p}}W'(x)| \\
&\quad \leq (p-2)(q_0(x))^{(1-p)/p}|q_0'(x)|W(x) + p(q_0(x))^{\frac{1}{p}}|W'(x)| \\
&\quad \leq \frac{1}{2}(p-2)k(w(x))^{\frac{1}{p}} + pk(w(x))^{\frac{1}{p}} = (\frac{3}{2}p-1)k(w(x))^{\frac{1}{p}},
\end{aligned}$$

except at the points α_m, γ_m, and β_m, $m = 1, 2, \ldots$. We have condition (c) implies condition (iii), that

$$|(2-p)(q_0(x))^{(1-p)/p} q_0'(x) W(x) + p(q_0(x))^{\frac{1}{p}} W'(x)| \leq (\frac{3}{2}p - 1) k (w(x))^{\frac{1}{p}},$$

almost everywhere with respect to Lebesgue measure. Hence we can apply Theorem 25 to conclude that the equation (1.1) is limit-point $L_p[w(x); a, \infty)$.

Example 14) Let $q_0(x) = x^\beta$, with $\beta \leq \frac{2}{p}$, for $1 \leq x < \infty$. Let ε, δ, and k be real numbers such that $0 < \varepsilon < 1$, $\delta \geq 0$, $\varepsilon + \delta \leq 1$, and $k > 0$. Let $I_m = [m + \delta, m + \delta + \varepsilon]$, for $m = 1, 2, \ldots$.

If $-Re(q_1(x)) \leq kx^{\frac{4}{p}-\beta}$ *and* $|Im(q_1(x))| \leq kx^{\frac{4}{p}-\beta}$, *for* $x \in I_m$, $m = 1, 2, \ldots$, *then the equation (1.1)* $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \leq x < \infty$, *is limit-point* $L_p[a, \infty)$, *for* $p > 2$.

We show this by applying Theorem 26 with $R_m = (m + \delta + \varepsilon)^{((p-1)/p)\beta - \frac{2}{p}}/Q_m$, $S_m = (m + \delta + \varepsilon)^{(2-p)/p}$, $\mu_m = \varepsilon$, $w(x) \equiv 1$, and k chosen large enough.

First we do the case when $\beta \leq 0$. For $R_m = (m+\delta+\varepsilon)^{((p-1)/p)\beta - \frac{2}{p}}/Q_m$, $(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 = (x^\beta)^{(2-p)/p}(m + \delta + \varepsilon)^{2((p-1)/p)\beta - \frac{4}{p}}$. For $x \in I_m$, $(x^\beta)^{(2-p)/p} \leq (m + \delta + \varepsilon)^{((2-p)/p)\beta}$. So, for $x \in I_m$, $(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq (m + \delta + \varepsilon)^{((2-p)/p)\beta}(m + \delta + \varepsilon)^{2((p-1)/p)\beta - \frac{4}{p}} = (m + \delta + \varepsilon)^{\beta - \frac{4}{p}}$. Thus, if $-\mathrm{Re}(q_1(x)) \leq kx^{\frac{4}{p}-\beta}$, for $x \in I_m$, then $-\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq kx^{\frac{4}{p}-\beta}(m + \delta + \varepsilon)^{\beta - \frac{4}{p}} \leq k(m + \delta + \varepsilon)^{\frac{4}{p}-\beta}(m + \delta + \varepsilon)^{\beta - \frac{4}{p}} = k$, for $x \in I_m$. Likewise, if $|\mathrm{Im}(q_1(x))| \leq kx^{\frac{4}{p}-\beta}$, for $x \in I_m$, then $|\mathrm{Im}(q_1(x))|(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k$, for $x \in I_m$. Thus condition (a) holds.

$Q_m = \int_{I_m} (q_0(x))^{(1-p)/p} dx = \int_{m+\delta}^{m+\delta+\varepsilon} x^{((1-p)/p)\beta} dx = (1 + ((1-p)/p)\beta)^{-1}((m+\delta+\varepsilon)^{1+((1-p)/p)\beta} - (m + \delta)^{1+((1-p)/p)\beta})$. By inequality (10.1), $Q_m \geq \varepsilon(m + \delta)^{((1-p)/p)\beta}$. Now, $R_m Q_m^2 = (m + \delta + \varepsilon)^{((p-1)/p)\beta - \frac{2}{p}} Q_m \geq \varepsilon(m + \delta + \varepsilon)^{((p-1)/p)\beta - \frac{2}{p}}(m + \delta)^{((1-p)/p)\beta} = \varepsilon((m + \delta)/(m + \delta + \varepsilon))^{((1-p)/p)\beta}(m + \delta + \varepsilon)^{-\frac{2}{p}} = \varepsilon(1 - \varepsilon/(m + \delta + \varepsilon))^{((1-p)/p)\beta}(m + \delta + \varepsilon)^{-\frac{2}{p}} \geq \varepsilon(1 - \varepsilon/(1 + \delta + \varepsilon))^{((1-p)/p)\beta}(m + \delta + \varepsilon)^{-\frac{2}{p}}$. So, for $S_m = (m + \delta + \varepsilon)^{(2-p)/p}$, $R_m Q_m^2 S_m \geq \varepsilon(1 - \varepsilon/(1 + \delta + \varepsilon))^{((1-p)/p)\beta}(m + \delta + \varepsilon)^{-\frac{2}{p}}(m + \delta + \varepsilon)^{(2-p)/p} = \varepsilon(1 - \varepsilon/(1 + \delta + \varepsilon))^{((1-p)/p)\beta}(m + \delta + \varepsilon)^{-1}$ and $\sum_{m=1}^n R_m Q_m^2 S_m \geq \varepsilon(1 - \varepsilon/(1 + \delta + \varepsilon))^{((1-p)/p)\beta} \sum_{m=1}^n (m + \delta + \varepsilon)^{-1}$. $\sum_{m=1}^n \mu_m (S_m)^{p/(p-2)} =$

$\sum_{m=1}^{n} \varepsilon((m+\delta+\varepsilon)^{(2-p)/p})^{p/(p-2)} = \varepsilon \sum_{m=1}^{n} (m+\delta+\varepsilon)^{-1}$. Thus

$\sum_{m=1}^{n} R_m Q_m^2 S_m / \left(\sum_{m=1}^{n} \mu_m (S_m)^{p/(p-2)}\right)^{(p-2)/p} \geq$

$\varepsilon^{\frac{2}{p}}(1-\varepsilon/(1+\delta+\varepsilon))^{((1-p)/p)\beta}(\sum_{m=1}^{n}(m+\delta+\varepsilon)^{-1})^{\frac{2}{p}}$ and condition (b) holds.

For $x \in I_m$, $R_m(q_0(x))^{(2-p)/p} = ((m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}/Q_m)(x^\beta)^{(2-p)/p} \leq ((m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}/(\varepsilon(m+\delta)^{((1-p)/p)\beta}))(m+\delta+\varepsilon)^{((2-p)/p)\beta} = \varepsilon^{-1}(m+\delta+\varepsilon)^{(\frac{1}{p})\beta-\frac{2}{p}}(m+\delta)^{((p-1)/p)\beta} = \varepsilon^{-1}(1-\varepsilon/(m+\delta+\varepsilon))^{((p-1)/p)\beta}(m+\delta+\varepsilon)^{\beta-\frac{2}{p}} \leq \varepsilon^{-1}(1-\varepsilon/(1+\delta+\varepsilon))^{((p-1)/p)\beta}(1+\delta+\varepsilon)^{\beta-\frac{2}{p}}$. For $x \in I_m$, $R_m Q_m(q_0(x))^{(1-p)/p}|q_0'(x)| = (m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}(x^\beta)^{(1-p)/p}|\beta x^{\beta-1}| = |\beta|(m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}x^{(\frac{1}{p})\beta-1} \leq |\beta|(m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}(m+\delta)^{(\frac{1}{p})\beta-1} = |\beta|(1-\varepsilon/(m+\delta+\varepsilon))^{(\frac{1}{p})\beta-1}(m+\delta+\varepsilon)^{\beta-\frac{2}{p}-1} \leq |\beta|(1-\varepsilon/(1+\delta+\varepsilon))^{(\frac{1}{p})\beta-1}(1+\delta+\varepsilon)^{\beta-\frac{2}{p}-1}$. Thus condition (c), that $R_m(q_0(x))^{(2-p)/p} \leq k$ and $R_m Q_m(q_0(x))^{(1-p)/p}|q_0'(x)| \leq k$, for $x \in I_m$, holds with k enlarged, if needed, to

$\max\left\{\varepsilon^{-1}(1-\varepsilon/(1+\delta+\varepsilon))^{((p-1)/p)\beta}(1+\delta+\varepsilon)^{\beta-\frac{2}{p}-1}, |\beta|(1-\varepsilon/(1+\delta+\varepsilon))^{(\frac{1}{p})\beta-1}(1+\delta+\varepsilon)^{\beta-\frac{2}{p}-1}\right\}$.

Now, we do the case when $0 < \beta \leq \frac{2}{p}$.

$(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 = (x^\beta)^{(2-p)/p}(m+\delta+\varepsilon)^{2((p-1)/p)\beta-\frac{4}{p}}$. For $x \in I_m$, $(x^\beta)^{(2-p)/p} \leq (m+\delta)^{((2-p)/p)\beta}$. So $(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq (m+\delta)^{((2-p)/p)\beta}(m+\delta+\varepsilon)^{2((p-1)/p)\beta-\frac{4}{p}} = (1-\varepsilon/(m+\delta+\varepsilon))^{((2-p)/p)\beta}(m+\delta+\varepsilon)^{\beta-\frac{4}{p}} \leq (1-\varepsilon/(1+\delta+\varepsilon))^{((2-p)/p)\beta}(m+\delta+\varepsilon)^{\beta-\frac{4}{p}}$. Thus, if $-\mathrm{Re}(q_1(x)) \leq k x^{\frac{4}{p}-\beta}$, for $x \in I_m$, then $-\mathrm{Re}(q_1(x))(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k(1-\varepsilon/(1+\delta+\varepsilon))^{((2-p)/p)\beta} x^{\frac{4}{p}-\beta}(m+\delta+\varepsilon)^{\beta-\frac{4}{p}} \leq k(1-\varepsilon/(1+\delta+\varepsilon))^{((2-p)/p)\beta}(m+\delta+\varepsilon)^{\frac{4}{p}-\beta}(m+\delta+\varepsilon)^{\beta-\frac{4}{p}} = k(1-\varepsilon/(1+\delta+\varepsilon))^{((2-p)/p)\beta}$, for $x \in I_m$. Likewise, if $|\mathrm{Im}(q_1(x))| \leq k x^{\frac{4}{p}-\beta}$, for $x \in I_m$, then $|\mathrm{Im}(q_1(x))|(q_0(x))^{(2-p)/p} R_m^2 Q_m^2 \leq k(1-\varepsilon/(1+\delta+\varepsilon))^{((2-p)/p)\beta}$, for $x \in I_m$. Thus condition (a) holds.

$Q_m = (1+((1-p)/p)\beta)^{-1}((m+\delta+\varepsilon)^{1+((1-p)/p)\beta} - (m+\delta)^{1+((1-p)/p)\beta})$. By inequality (10.1), $Q_m \geq \varepsilon(m+\delta+\varepsilon)^{((1-p)/p)\beta}$. Now, $R_m Q_m^2 = (m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}} Q_m \geq \varepsilon(m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}(m+\delta+\varepsilon)^{((1-p)/p)\beta} = \varepsilon(m+\delta+\varepsilon)^{-\frac{2}{p}}$. So, for $S_m = (m+\delta+\varepsilon)^{(2-p)/p}$, $R_m Q_m^2 S_m \geq \varepsilon(m+\delta+\varepsilon)^{-\frac{2}{p}}(m+\delta+\varepsilon)^{(2-p)/p} = \varepsilon(m+\delta+\varepsilon)^{-1}$ and $\sum_{m=1}^{n} R_m Q_m^2 S_m = \varepsilon \sum_{m=1}^{n}(m+\delta+\varepsilon)^{-1}$. $\sum_{m=1}^{n} \mu_m(S_m)^{p/(p-2)} = \varepsilon \sum_{m=1}^{n}(m+\delta+\varepsilon)^{-1}$. Thus $\sum_{m=1}^{n} R_m Q_m^2 S_m / (\sum_{m=1}^{n} \mu_m(S_m)^{p/(p-2)})^{(p-2)/p} =$

$\varepsilon^{\frac{2}{p}}(\sum_{m=1}^{n}(m+\delta+\varepsilon)^{-1})^{\frac{2}{p}}$ and condition (b) holds.

For $x \in I_m$, $R_m(q_0(x))^{(2-p)/p} = ((m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}/Q_m)(x^\beta)^{(2-p)/p} \leq ((m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}/(\varepsilon(m+\delta+\varepsilon)^{((1-p)/p)\beta}))(m+\delta)^{((2-p)/p)\beta} = \varepsilon^{-1}(m+\delta+\varepsilon)^{2((p-1)/p)\beta-\frac{2}{p}}(m+\delta)^{((2-p)/p)\beta} = \varepsilon^{-1}(1-\varepsilon/(m+\delta+\varepsilon))^{((2-p)/p)\beta}(m+\delta+\varepsilon)^{\beta-\frac{2}{p}} \leq \varepsilon^{-1}(1-\varepsilon/(1+\delta+\varepsilon))^{((2-p)/p)\beta}(1+\delta+\varepsilon)^{\beta-\frac{2}{p}}$. For $x \in I_m$, $R_mQ_m(q_0(x))^{(1-p)/p}|q_0'(x)| = |\beta|((m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}x^{(\frac{1}{p})\beta-1}$. Since $\beta \leq \frac{2}{p}$, $(\frac{1}{p})\beta - 1 \leq (\frac{1}{p})(\frac{2}{p}) - 1 < -\frac{1}{2}$ and so, for $x \in I_m$, $x^{(\frac{1}{p})\beta-1} \leq (m+\delta)^{(\frac{1}{p})\beta-1}$. We get, for $x \in I_m$, $R_mQ_m(q_0(x))^{(1-p)/p}|q_0'(x)| \leq |\beta|((m+\delta+\varepsilon)^{((p-1)/p)\beta-\frac{2}{p}}(m+\delta)^{(\frac{1}{p})\beta-1} = |\beta|(1-\varepsilon/(m+\delta+\varepsilon))^{(\frac{1}{p})\beta-1}(m+\delta+\varepsilon)^{\beta-\frac{2}{p}-1} \leq |\beta|(1-\varepsilon/(1+\delta+\varepsilon))^{(\frac{1}{p})\beta-1}(1+\delta+\varepsilon)^{\beta-\frac{2}{p}-1}$. Thus condition (c) holds. We can apply Theorem 26 to get our result.

In the next section we make use of the following interval criteria.

Example 15) *Let $q_0(x) = x^\beta$, with $\beta \leq 1$, for $1 \leq x < \infty$. Let ε, δ, and k be real numbers such that $0 < \varepsilon < 1$, $\delta \geq 0$, $\varepsilon + \delta \leq 1$, and $k > 0$. Let $I_m = [m+\delta, m+\delta+\varepsilon]$, for $m = 1, 2, \ldots$. If $-Re(q_1(x)) \leq kx^{2-\beta}$ and $|Im(q_1(x))| \leq kx^{2-\beta}$, then the equation (1.1) $-(x^\beta y'(x))' + q_1(x)y(x) = 0$, $1 \leq x < \infty$, is limit-point $L_2[a,\infty)$.*

This is proven by showing that if equation (1.1) is considered instead of equation (7.1) in Theorem 23 then condition (i) can be replaced by $-\mathrm{Re}(q_1(x))(W(x))^2 \leq k$ and $|\mathrm{Im}(q_1(x))|(W(x))^2 \leq k$. It follows that equation (1.1) can be considered in Theorem 24 with condition (a) being replaced by $-\mathrm{Re}(q_1(x))R_m^2Q_m^2 \leq k$ and $|\mathrm{Im}(q_1(x))|R_m^2Q_m^2 \leq k$, for $x \in I_m$, $m = 1, 2, \ldots$. Example 15) follows in the same way as Example 13) did.

11. When

$$\mathrm{Im}(q_1(x)) = \mu,$$

for a non-zero real number μ, the equation (1.1)

$$-(q_0(x)y'(x))' + q_1(x)y(x) = 0, \quad a \leq x < \infty,$$

always has a non-trivial solution that is $L_2[a,\infty)$. When $p \neq 2$, the equation (1.1) may not have any non-trivial $L_p[a,\infty)$ solutions, even though $\mathrm{Im}(q_1(x)) = \mu \neq 0$.

When $1 < p < 2$ consider the Euler equation

$$-(x^2y'(x))' + (\lambda + i\mu)y(x) = 0, \quad 1 \leq x < \infty, \tag{11.1}$$

where λ and $\mu \neq 0$ are real numbers.

$$y_1(x) = x^{-\frac{1}{2}+(\frac{1}{4}+\lambda+i\mu)^{\frac{1}{2}}} \text{ and } y_2(x) = x^{-\frac{1}{2}-(\frac{1}{4}+\lambda+i\mu)^{\frac{1}{2}}}$$

are independent solutions of equation (11.1). When $1 < p < 2$, no solutions

$$y(x) = cy_1(x) + dy_2(x)$$

of $-(x^2y')' + (\lambda + i\mu)y = 0$, $1 \leq x < \infty$, are $L_p[1,\infty)$, if

$$-\frac{1}{2} - \mathrm{Re}((\frac{1}{4} + \lambda + i\mu)^{\frac{1}{2}}) \geq -\frac{1}{p} \text{ and } -\frac{1}{2} + \mathrm{Re}((\frac{1}{4} + \lambda + i\mu)^{\frac{1}{2}}) \geq -\frac{1}{p}.$$

For a complex number $A + iB$, the following formula holds when $B \geq 0$.

$$(A + iB)^{\frac{1}{2}} = (2)^{-\frac{1}{2}}(((A^2 + B^2)^{\frac{1}{2}} + A)^{\frac{1}{2}} + i((A^2 + B^2)^{\frac{1}{2}} - A)^{\frac{1}{2}}) \tag{11.2}$$

Consider when $\mu > 0$. Then

$$\mathrm{Re}((\frac{1}{4} + \lambda + i\mu)^{\frac{1}{2}}) = (2)^{-\frac{1}{2}}(((\frac{1}{4} + \lambda)^2 + \mu^2)^{\frac{1}{2}} + (\frac{1}{4} + \lambda))^{\frac{1}{2}}$$

and equation (11.1) has no $L_p[1,\infty)$ solutions if

$$-\frac{1}{2} - \mathrm{Re}((\frac{1}{4} + \lambda + i\mu)^{\frac{1}{2}}) = -\frac{1}{2} - (2)^{-\frac{1}{2}}(((\frac{1}{4} + \lambda)^2 + \mu^2)^{\frac{1}{2}} + (\frac{1}{4} + \lambda))^{\frac{1}{2}} \geq -\frac{1}{p}.$$

That is when

$$(2)^{-\frac{1}{2}}(((\frac{1}{4} + \lambda)^2 + \mu^2)^{\frac{1}{2}} + (\frac{1}{4} + \lambda))^{\frac{1}{2}} \leq \frac{1}{p} - \frac{1}{2}$$

or equivalently

$$((\frac{1}{4} + \lambda)^2 + \mu^2)^{\frac{1}{2}} \leq 2(\frac{1}{p} - \frac{1}{2})^2 - (\frac{1}{4} + \lambda),$$

$$\mu^2 \leq 4(\frac{1}{p}-\frac{1}{2})^2((\frac{1}{p}-\frac{1}{2})^2-(\frac{1}{4}+\lambda)),$$

and

$$\lambda \leq -\frac{1}{4}+(\frac{1}{p}-\frac{1}{2})^2-\mu^2/(4(\frac{1}{p}-\frac{1}{2})^2).$$

Hence for

$$\mu > 0 \text{ and } \lambda \leq -\frac{1}{4}+(\frac{1}{p}-\frac{1}{2})^2-\mu^2/(4(\frac{1}{p}-\frac{1}{2})^2)$$

the equation (11.1) has no non-trivial $L_p[1,\infty)$ solutions.

To finish this section we will show equations

$$-y''(x)+q(x)y(x)=i\mu y(x), \quad a \leq x < \infty, \tag{11.3}$$

with $q(x)$ real valued and μ a non-zero real number, can be constructed so that no non-trivial $L_2[a,\infty)$ solutions that are bounded in absolute value exist. Furthermore, for a given $p > 2$, the equation (11.3) can be chosen so that no non-trivial $L_p[a,\infty)$ solutions exist. We begin by transforming equation (11.3) into polar form. For a solution $y(x)$ of equation (11.3) let

$$y(x)=r(x)e^{i\theta(x)}$$

so that

$$y'(x)=(r'(x)+ir(x)\theta'(x))e^{i\theta(x)}$$

and

$$y''(x)=(r''(x)-r(x)(\theta'(x))^2+i(2r'(x)\theta'(x)+r(x)\theta''(x)))e^{i\theta(x)}.$$

Substitute the expressions for y and y'' into $-y''+qy=i\mu y$ to get

$$(-r''+r(\theta')^2-i(2r'\theta'+r\theta'')+qr)e^{i\theta}=i\mu re^{i\theta}. \tag{11.4}$$

Equate real and imaginary parts to obtain

$$-r''+r(\theta')^2+qr=0 \text{ and } 2r'\theta'+r\theta''+\mu r=0. \tag{11.5}$$

Under the assumption r is positive valued and twice continuously differentiable the dependence of θ' on r can be determined by rewritting

$$2r'\theta' + r\theta'' + \mu r = 0$$

as

$$(\theta'(x))' + 2(r'(x)/r(x))\theta'(x) + \mu = 0 \tag{11.6}$$

and solving for $\theta'(x)$. The general solution of

$$v'(x) + b(x)v(x) + c(x) = 0, \quad a \leq x < \infty,$$

is given by

$$v(x) = e^{-\int_a^x b(s)ds}(v(a) - \int_a^x c(s)e^{\int_a^s b(u)du}ds). \tag{11.7}$$

Use formula (11.7) to solve equation (11.6) to obtain

$$\theta'(x) = (r(x))^{-2}((r(a))^2\theta'(a) - \mu\int_a^x (r(s))^2 ds) \tag{11.8}$$

and

$$\theta(x) = \theta(a) + \int_a^x \theta'(s)ds.$$

Now, it can be seen that

$$y(x) = r(x)e^{i\theta(x)}$$

solves the initial value problem

$$-y''(x) + q(x)y(x) = i\mu y(x),$$

with

$$q(x) = r''(x)/r(x) - (\theta'(x))^2,$$

subject to

$$y(a) = r(a)e^{i\theta(a)} \text{ and } y'(a) = (r'(a) + ir(a)\theta'(a))e^{i\theta(a)}.$$

The idea at this point is to construct $r(x)$ in a way so that

$$\int_a^\infty (r(s))^2 ds < \infty, \quad |r(x)| \text{ is unbounded on } a \le x < \infty,$$

$$\lim_{t\to\infty} \int_a^t (r(s))^p ds = \infty, \text{ and } q(x) = r''(x)/r(x) - (\theta'(x))^2$$

satisfies interval criteria to be in the limit-point $L_2[a,\infty)$ condition. The function defined by

$$g(t) = \begin{cases} 2t^3 - t^4, & 0 \le t \le \frac{3}{2} \\ 2(3-t)^3 - (3-t)^4, & \frac{3}{2} < t \le 3 \end{cases}$$

is used to make the construction of $r(x)$.

$g(t)$ has the properties: $g(t)$ is positive on $0 < t < 3$; $g(t)$, $g'(t)$, and $g''(t)$ are continuous on $0 \le t \le 3$; $g(\frac{3}{2}) = 3^3/2^4$, $g(0) = g'(0) = g''(0) = 0$, and $g(3) = g'(3) = g''(3) = 0$. Also, $\int_0^3 (g(t))^2 dt = 2\int_0^{\frac{3}{2}} (g(t))^2 dt = 2\int_0^{\frac{3}{2}} (4t^6 - 4t^7 + t^8) dt = (\frac{3}{2})^7 (\frac{1}{14})$.

Similarly, for $p > 2$, we can conclude using the theory of calculus that

$$\int_0^3 (g(t))^p dt$$

is a positive real number. Set

$$g_n(t) = \begin{cases} ng(3(n)^{p+1}t - 3(n)^{p+2}), & n \le t \le n + (n)^{-(p+1)} \\ 0, & \text{otherwise} \end{cases}$$

and

$$h(t) = \sum_{n=1}^\infty g_n(t).$$

We have that

$$h(n + (\frac{1}{2})(n)^{-(p+1)}) = g_n(n + (\frac{1}{2})(n)^{-(p+1)}) = ng(\frac{3}{2}) = n(3^3/2^4).$$

In addition,

$$\begin{aligned} \int_1^\infty (h(s))^2 ds &= \sum_{n=1}^\infty n^2 \int_n^{n+(n)^{-(p+1)}} (g_n(s))^2 ds \\ &= \sum_{n=1}^\infty n^2 (1/(3(n)^{p+1})) \int_0^3 (g(s))^2 ds \\ &= (\frac{1}{3}) \left(\int_0^3 (g(s))^2 ds \right) \sum_{n=1}^\infty n^{1-p} \end{aligned}$$

and

$$\begin{aligned}\int_1^\infty (h(s))^p\,ds &= \sum_{n=1}^\infty n^p \int_n^{n+(n)^{-(p+1)}} (g_n(s))^p\,ds \\ &= \sum_{n=1}^\infty n^p(1/(3(n)^{p+1})) \int_0^3 (g(s))^p\,ds \\ &= (\tfrac{1}{3})\left(\int_0^3 (g(s))^p\,ds\right)\sum_{n=1}^\infty n^{-1}.\end{aligned}$$

Since

$$\sum_{n=1}^\infty n^{1-p} \text{ converges}$$

and

$$\sum_{n=1}^\infty n^{-1} \text{ diverges}$$

we have that

$$\int_a^\infty (h(s))^2 ds < \infty \text{ and } \lim_{t\to\infty}\int_a^t (h(s))^p\,ds = \infty.$$

Note the integral comparison

$$n^{1-p} < \int_{n-1}^n s^{1-p}\,ds$$

to get

$$\sum_{n=k+1}^\infty n^{1-p} < \int_k^\infty s^{1-p}\,ds = (1/(p-2))k^{2-p}.$$

$$\int_{k+1}^\infty (h(s))^2 ds = (\tfrac{1}{3})\left(\int_0^3 (g(s))^2 ds\right)\sum_{n=k+1}^\infty n^{1-p} < (\tfrac{1}{3})\left(\int_0^3 (g(s))^2 ds\right)(1/(p-2))k^{2-p}$$

and for $t \geq 1$

$$\begin{aligned}\int_t^\infty (h(s+2))^2 ds &\leq \int_{[t]}^\infty (h(s+2))^2 ds \\ &= \int_{[t]+2}^\infty (h(s))^2 ds \\ &< (\tfrac{1}{3})\left(\int_0^3 (g(s))^2 ds\right)(1/(p-2))([t]+1)^{2-p} \\ &\leq (\tfrac{1}{3})\left(\int_0^3 (g(s))^2 ds\right)(1/(p-2))t^{2-p}.\end{aligned}$$

($[t]$ means the greatest integer less than or equal to t.) Put

$$r(x) = h(x+2) + x^{(1-p)/2}, \quad 1 \le x < \infty,$$

and determine $\theta(x)$ by

$$\theta(1) \text{ and } \theta'(x) = (r(x))^{-2}(\theta'(1) - \mu \int_1^x (r(s))^2 ds)$$

where

$$\theta'(1) = \mu \int_1^\infty (r(s))^2 ds.$$

Write

$$\theta'(x) = \mu(r(x))^{-2} \int_x^\infty (r(s))^2 ds.$$

Now,

$$\int_x^\infty (r(s))^2 ds = \int_x^\infty ((h(s+2))^2 + 2(h(s+2))s^{(1-p)/2} + s^{(1-p)})ds,$$

$$\int_x^\infty (h(s+2))^2 ds \le (\frac{1}{3}) \left(\int_0^3 (g(s))^2 ds \right) (1/(p-2))x^{2-p},$$

$$\int_x^\infty s^{(1-p)} ds = (1/(p-2))x^{2-p},$$

and

$$\int_x^\infty (h(s+2))s^{(1-p)/2} ds \le \left(\int_x^\infty (h(s+2))^2 ds \right)^{\frac{1}{2}} \left(\int_x^\infty s^{(1-p)} ds \right)^{\frac{1}{2}}.$$

It follows

$$\int_x^\infty (r(s))^2 ds \le (1 + ((\frac{1}{3}) \left(\int_0^3 (g(s))^2 ds \right))^{\frac{1}{2}})^2 (1/(p-2))x^{2-p}.$$

In particular we have that

$$\theta'(x) \le \mu(1 + ((\frac{1}{3}) \left(\int_0^3 (g(s))^2 ds \right))^{\frac{1}{2}})^2 (1/(p-2))(r(x))^{-2} x^{2-p}.$$

On the intervals $I_m = [m + \frac{1}{2}, m+1]$, $m = 1, 2, \ldots,$

$$h(x+2) = 0$$

so that

$$r(x) = x^{(1-p)/2}.$$

In which case

$$\theta'(x) \leq \mu(1 + ((\frac{1}{3})\left(\int_0^3 (g(s))^2 ds\right))^{\frac{1}{2}})^2(1/(p-2))x$$

and

$$r''(x)/r(x) = ((p^2-1)/4)x^{-2}.$$

Hence we have

$$-q(x) = -r''(x)/r(x)+(\theta'(x))^2 \leq -((p^2-1)/4)x^{-2}+\mu^2(1+((\frac{1}{3})\left(\int_0^3 (g(s))^2 ds\right))^{\frac{1}{2}})^4(1/(p-2))^2x^2.$$

More specifically

$$-q(x) \leq kx^2$$

for some positive constant k, for

$$x \in I_m, \quad m = 1, 2, \ldots .$$

Thus the equation (11.3) satisfies the interval criteria of Example 15). It has been established that this equation

$$-y''(x) + q(x)y(x) = i\mu y(x), \quad 1 \leq x < \infty, \tag{11.9}$$

with

$$q(x) = r''(x)/r(x) - (\theta'(x))^2, \quad r(x) = h(x+2) + x^{(1-p)/2},$$

and

$$\theta'(x) = \mu(r(x))^{-2}\int_x^\infty (r(s))^2 ds,$$

has no non-trivial $L_2[1,\infty)$ solution that is bounded in absolute value as $x \to \infty$. We continue on to show that equation (11.9) has no non-trivial $L_p[1,\infty)$ solution. The solution given by $y(x) = r(x)e^{i\theta(x)}$ is not L_p since

$$\left(\int_1^x (h(s+2))^p ds\right)^{\frac{1}{p}} \leq \left(\int_1^x (r(s))^p ds\right)^{\frac{1}{p}} + \left(\int_1^x s^{((1-p)/2)p} ds\right)^{\frac{1}{p}}$$

and

$$h(s+2) \text{ is not } L_p[1,\infty) \text{ with } \int_1^\infty s^{((1-p)/2)p} ds < \infty.$$

The proof is completed by contradicting the assumption a $L_p[1,\infty)$ solution $z(x)$ of equation (11.9), that is independent of $y(x)$, exists. This is done as follows. Assuming such a solution $z(x)$ exists we could say

$$y(x)z'(x) - y'(x)z(x) = c,$$

where c is a non-zero constant. In addition, we could apply Lemma 25 with $w(x) \equiv 1$ and $W(x)$ defined in the following way.

$$W_m(x) = \begin{cases} (1/(m+1))(x-(m+\frac{1}{2})), & m+\frac{1}{2} \le x \le m+\frac{3}{4} \\ (1/(m+1))(m+1-x), & m+\frac{3}{4} \le x \le m+1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$W(x) = \sum_{m=1}^{\infty} W_m(x).$$

Now

$$W_m(x) \le (\frac{1}{4})(1/(m+1)),$$

for $x \in I_m = [m+\frac{1}{2}, m+1]$, $m = 1, 2, \ldots$, and so

$$W(x) \le (\frac{1}{4})x^{-1}.$$

In particular the conclusion of Lemma 25 would hold with $W(x) \le (\frac{1}{4})x^{-1}$. That is

$$W(x) \le (\frac{1}{4})x^{-1} \text{ and } \int_1^\infty (W(x))^p |z'(x)|^p dx < \infty.$$

Since

$$W(x) \equiv 0 \text{ when } x \in [m, m+1] \backslash I_m = [m, m+\frac{1}{2}],$$

we only consider

$$W(x)(y(x)z'(x) - y'(x)z(x)) = cW(x)$$

when $x \in I_m$, for some m. When $x \in I_m$,

$$r(x) = x^{(1-p)/2}, \quad r'(x) = ((1-p)/2)x^{-(1+p)/2}, \text{ and } \theta'(x) \leq Mx$$

for a constant $M > 0$. This gives

$$\begin{aligned} W(x)|y'(x)| &= W(x)|r'(x) + ir(x)\theta'(x)| \\ &\leq (\frac{1}{4})x^{-1}(|(1-p)/2|x^{-(1+p)/2} + x^{(1-p)/2}(Mx)) \\ &\leq M_1 x^{(1-p)/2}, \end{aligned}$$

for

$$M_1 = (\frac{1}{4})(|(1-p)/2| + M).$$

Also,

$$W(x)|y(x)z'(x)| = x^{(1-p)/2}W(x)|z'(x)|.$$

Thus

$$\begin{aligned} \left|\int_1^\infty W(x)(y(x)z'(x) - y'(x)z(x))dx\right| &\leq \left|\int_1^\infty W(x)|y(x)z'(x)|dx\right| \\ &\quad + \left|\int_1^\infty W(x)|y'(x)||z(x)|dx\right| \\ &\leq \int_1^\infty x^{(1-p)/2}W(x)|z'(x)|dx \\ &\quad + M_1 \int_1^\infty x^{(1-p)/2}|z(x)|dx. \end{aligned}$$

Applying Holders inequality with exponents $p/(p-1)$ and p to the last two integrals gives us

$$\int_1^\infty x^{(1-p)/2}W(x)|z'(x)|dx \leq \left(\int_1^\infty x^{-\frac{p}{2}}dx\right)^{(p-1)/p} \left(\int_1^\infty (W(x))^p|z'(x)|^p dx\right)^{\frac{1}{p}} < \infty$$

and

$$\int_1^\infty x^{(1-p)/2}|z(x)|dx \leq \left(\int_1^\infty x^{-\frac{p}{2}}dx\right)^{(p-1)/p} \left(\int_1^\infty |z(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

The integral

$$\int_1^\infty W(x)dx = \sum_{m=1}^\infty \int_{m+\frac{1}{2}}^{m+1} W_m(x)dx = \sum_{m=1}^\infty (\frac{1}{16}(1/(m+1)) \text{ diverges.}$$

The contradiction is obtained by considering

$$\int_1^\infty W(x)(y(x)z'(x) - y'(x)z(x))dx = \int_1^\infty cW(x)dx.$$

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