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Hankel Operators on the Drury-Arveson Space

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To my beautiful wife, Kim. I cannot put into words how much your love means to me. You are infinitely supportive, unconditionally loving, and wonderfully joyful. To my parents, Jim and Mary. All of my success in life would be nonexistent without your infinite support.

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[Mathematics] is security. Certainty. Truth. Beauty. Insight. Structure. Architecture. I see mathematics, the part of human knowledge that I call mathematics, as one thing – one great, glorious thing. -Paul Halmos

Abstract

The Drury-Arveson space, initially introduced in the proof of a generalization of von Neumann's inequality, has seen a lot of research due to its intrigue as a Hilbert space of analytic functions. This space has been studied in the context of Besov-Sobolev spaces, Hilbert spaces with complete Nevanlinna Pick kernels, and Hilbert modules. More recently, McCarthy and Shalit have studied the connections between the Drury-Arveson space and Hilbert spaces of Dirichlet series, and Davidson and Cloutâre have established analogues of classic results of the ball algebra to the multiplier algebra for the Drury-Arveson Space.

The goal of this dissertation is to contribute to this growing body of research by studying the Hankel operators on the Drury-Arveson Space. We begin by establishing basic results regarding the function theoretic properties of the Drury-Arveson space and general properties of Hankel operators. It is then shown that every invariant subspace of the d-shift on the Drury-Arveson space is an at most countable intersection of kernels of Hankel operators. We then prove that if a function and its reciprocal lie in the Drury-Arveson space, then that function must be a cyclic vector. In addition, we prove that each multiplier invariant subspace on the vector-valued Drury-Arveson space is an intersection of kernels of vectorial Hankel operators, and we characterize a special class of symbols which induce a bounded Hankel operator in terms of a Carleson measure condition on the symbol.

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Chapter 1

Introduction

1.1 History

One of the most fruitful ideas of mathematical analysis is to take operators acting on a general Banach or Hilbert space and model them (via unitary equivalence) as operators on a space whose elements are functions. In this way, one hopes that questions about operators can be answered by the techniques of real or complex analysis.

For instance, it is well known that an operator A on a separable space is normal if and only if it is unitarily equivalent to a bounded multiplication operator M_φ, defined by $M_\varphi f = \varphi f$, on $L^2(X, \mu)$ for some finite measure space (X, μ). The basic properties of the operator M_φ can be determined by studying the properties of the function φ. For instance, M_φ is bounded if and only if φ is essentially-bounded and the spectrum of M_φ is precisely the essential-range of φ. Furthermore, one can easily determine equivalent conditions for which M_φ will be Hermitian, unitary, positive, or a projection wholly based upon the properties of φ. In this way, we can recover many of the basic facts about normal operators.

One of the major results which utilizes this line of reasoning is the Sz. Nagy dilation theorem, which implies that every strict contraction on a Hilbert space is unitarily equivalent to the restriction of a backward shift operator (of appropriate multiplicity) to some invariant subspace. These backward shift operators can then be modelled as operators on a vector-valued Hardy space of the unit disc (which has been extensively studied). One of the applications of this theorem is a nice proof of von Neumann's inequality, which states that for any contraction T on some Hilbert space we have that

$$\|p(T)\| \leq \sup_{z \in \mathbb{D}} |p(z)|$$

for any polynomial p. Furthermore, the Sz. Nagy dilation theorem implies that many other operators of interest have representations on the Hardy space, and thus questions about these operators are equivalent to questions regarding functions in the Hardy space.

In 1978, S. W. Drury generalized von Neumann's inequality to the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$. In his paper [13], Drury showed for any fixed $d \in \mathbb{N}$, there is an "archetypal" d-tuple of operators $S = (S_1, ..., S_d)$ acting on some Hilbert Space H^2_d such that

1. $S_i S_j = S_j S_i$ for $i, j \in \{1, ..., d\}$ and
2. $\sum_{i=1}^{d} \|S_i x\|^2 \leq \|x\|^2$ for all $x \in H^2_d$

with property that if $T = (T_1, ..., T_d)$ is a d-tuple of operators on any (other) Hilbert space $\mathcal{H}$ with these properties 1 and 2, then

$$\|Q(T_1, ..., T_d)\| \leq \|Q(S_1, ..., S_d)\|$$

for all analytic polynomials Q of d-variables. Drury's proof is inspired by one of the proofs of the Sz. Nagy dilation theorem mentioned in the previous paragraph. The space H^2_d, on which the tuple S is defined, is known today as the Drury-Arveson space.

Now, Drury initially introduced the space H^2_d as a weighted ℓ^2 space of d-tuples of nonnegative integers. It was William Arveson, twenty years later, who brought the Drury-Arveson space into a more function theoretic setting (see [4]). Since then a substantial amount of research has been done trying to solve the "standard" problems one would generally ask about any Hilbert space of analytic functions. For instance, the multipliers of H_d^2 are characterized by Fabrega and Ortega in [27], the Carleson measures of H_d^2 are characterized by Arcozzi, Rochberg, and Sawyer in [1], and Costea, Sawyer, and Wick proved the Corona theorem for the multiplier algebra of H_d^2 in [12]. However, at this time, there is no characterization of the symbols which induce bounded Hankel operators on H_d^2.

The study of Hankel operators traces back to the 1861 dissertation of Hermann Hankel, in which he studied $n \times n$ matrices $(a_{ij})_{i=1}^n$ which have the property that $a_{ij} = f(i+j)$ for some function $f : \mathbb{N} \to \mathbb{C}$. It is well-known for infinite dimensional matrices of this type when they are finite rank (see [21]), bounded ([26]), and compact (see [17]). Study of these types of objects gives rise to the theory of Hankel operators on the Hardy space and provides a good deal of information regarding the function theory of the Hardy Space. For instance, it is known that a Hankel operator on the Hardy space is bounded if and only if its associated symbol is of bounded mean oscillation, which is equivalent to a certain measure theoretic condition on the symbol. Furthermore, Hankel operators have applications in control theory and vector-valued Hankel operators were used in Pisier's example of an operator that is polynomially bounded but not similar to a contraction.

1.2 Overview

The main goal of this dissertation is to study the space of Hankel symbols $\mathcal{X}(H_d^2)$, which consists of those symbols which induce bounded Hankel operators on the Drury-Arveson space. Although no characterization of $\mathcal{X}(H_d^2)$ is available at this time, the theory of Hankel operators on H_d^2 allows us to prove results regarding invariant subspaces of the "archetypal" tuple of operators introduced by Drury in his 1978 paper and cyclic vectors on H_d^2. Later in this dissertation, we provide a characterization of Hankel symbols which lie in a special class of functions.

Given any Hilbert space $\mathcal{H}$ consisting of holomorphic functions on the unit ball $\mathbb{B}_d$ with the property that the polynomials are dense in $\mathcal{H}$, one can define and study the space of Hankel symbols $\mathcal{X}(\mathcal{H})$ for that space. For several well-known spaces, there is a satisfactory characterization of $\mathcal{X}(\mathcal{H})$. For the Hardy Space $H^2(\partial\mathbb{B}_d)$, Coifman, Rochberg, and Weiss proved in [11] that $\mathcal{X}(H^2(\partial\mathbb{B}_d))$ equals $BMOA(\mathbb{B}_d)$, the functions of bounded mean oscillation on $\mathbb{B}_d$. It also follows from their work that for the weighted Bergman spaces $A^2_\alpha(\mathbb{B}_d)$ with $\alpha > -1$, one has that $\mathcal{X}(A^2_\alpha)$ is the Bloch space, which we will denote by $\mathcal{B}$. Finally, for the Dirichlet space $\mathcal{D}$ of the unit disk, a characterization of $\mathcal{X}(\mathcal{D})$ was given by Arcozzi, Rochberg, Sawyer, and Wick in [2]. The Bloch space, Hardy space, and weighted Bergman spaces will be used frequently throughout this dissertation and more information about these spaces is provided in Section 2.2.

Now, there is a common element between the characterizations of the Hankel symbols for the Hardy space, the weighted Bergman spaces, and the Dirichlet space: they can all be rephrased in terms of a certain measure being a Carleson measure for the underlying space. In Section 2.4, we give an overview of Carleson measures and focus on the space $\mathcal{C}H^2_d$, which consists of those functions $\varphi \in H^2_d$ with the property that $|R^m\varphi(z)|^2(1-|z|^2)^{2m-d}dV(z)$ is a Carleson measure for some (and hence, for all) $m \in \mathbb{N}$ with $2m - d > -1$. The next two results (see Theorem 3.2 and Theorem 7.2) provide evidence that $\mathcal{C}(H^2_d)$ is the correct space to consider when trying to characterize membership in $\chi(H^2_d)$.

Theorem 1.1 (Richter, Sunkes [30]). *$\mathcal{C}(H^2_d) \subseteq \mathcal{X}(H^2_d) \subseteq \mathcal{B}$.*

Theorem 1.2 (Sunkes [34]). *Fix $d \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\beta \in \mathbb{N}_0^{d+1}$ with $\beta_1 = 0$ and $g \in Hol(\mathbb{D})$ be such that the function given by $\varphi(z) = z^\beta g(z_1)$ satisfies $\varphi \in H^2_{d+1}$. Then $\varphi \in \mathcal{X}(H^2_{d+1})$ if and only if $\varphi \in \mathcal{C}H^2_{d+1}$.*

The first inclusion in Theorem 1.1 follows from an argument involving the equivalence of certain bilinear forms on H^2_d and then several applications of the

Carleson measure condition. The second inclusion follows from testing the Hankel condition on the reproducing kernels of H^2_d.

The proof of Theorem 1.2 involves showing that $\varphi \in \chi(H^2_{d+1})$ is equivalent to some condition on the function g, and then showing that $\varphi \in \mathcal{C}H^2_{d+1}$ is equivalent to that same condition on g. The proof of the theorem relies heavily on the characterizations of the Hankel symbols for the Hardy space and weighted Bergman spaces on the unit disk $\mathbb{B}_1$ and also on a decomposition of H^2_d into a certain direct sum of these spaces. Moreover, in Theorem 7.4, we show how to construct all the functions g for which the equivalent conditions of Theorem 1.2 hold.

Now, to each $\varphi \in \mathcal{X}(H^2_d)$, we associate its Hankel operator H_φ. It turns out that $\ker H_\varphi$ is an invariant subspace of the d-shift $(M_{z_1}, ..., M_{z_d})$ for H^2_d. In Theorem 4.3, we prove the following characterization of $\mathrm{Lat}(M_z, H^2_d)$, the collection of all the invariant subspaces of the d-shift, in terms of the kernels of Hankel operators on H^2_d.

Theorem 1.3 (Richter, Sunkes [30]). *Let* $(0) \neq \mathcal{M} \in$ *Lat*(M_z, H^2_d). *Then there are* $\{b_n\}_{n\geq 0} \subseteq \mathcal{X}(H^2_d)$ *such that*

$$\mathcal{M} = \bigcap_{n\geq 0} \ker H_{b_n}.$$

The functions b_n come from the density of finite linear combinations of the reproducing kernels of H^2_d. Using a result of McCullough and Trent (in [24]), one shows that each b_n is a multiplier for H^2_d. For a function $b \in H^2_d$, we have that b is a multiplier for H^2_d if and only if b is bounded and $b \in \mathcal{C}H^2_d$ (see Theorem 2.3 or [27]). Thus by Theorem 1.1 above, each multiplier for H^2_d must lie in $\mathcal{X}(H^2_d)$. The argument in Theorem 1.3 is based on a proof by Luo and Richter in [23]. A vector-valued version of this theorem is the content of Chapter 5.

A function $f \in H^2_d$ is said to be cyclic for H^2_d if the set of all polynomial multiples of f is dense in H^2_d. For the Hardy space on the unit disk, the cyclic vectors are simply the outer functions. For the weighted Bergman spaces and the Dirichlet space of the unit ball, there is no characterization of cyclic vectors; although, for the Dirichlet space, there is a conjecture by Brown and Shields (see [7]). For H^2_d, we prove the following in Theorem 4.4.

Theorem 1.4 (Richter, Sunkes [30]). *If $f, \dfrac{1}{f} \in H^2_d$, then f is cyclic in H^2_d.*

The analogue of this theorem for the Dirichlet space $\mathcal{D}$ of $\mathbb{B}_1$ was proven in [23] by means of cut-off functions and an integral formula for the Dirichlet norm of an outer function proven by Carleson. The proof of Theorem 1.4 is valid in the context of the Dirichlet space and avoids these technicalities.

Of course, after proving a theorem like this, it is only natural to explore when it is the case that a function $f \in H^2_d$ satisfies $\frac{1}{f} \in H^2_d$. To this end, we prove the following theorem (see Theorem 4.6).

Theorem 1.5 (Richter, Sunkes [30]). *If $f \in H^2_d \cap \mathcal{B}$, and if there is a $c > 0$ such that $|f(z)| \geq c$ for all $z \in \mathbb{B}_d$, then $\frac{1}{f} \in H^2_d$ and f is cyclic for H^2_d.*

Here $\mathcal{B}$ is the Bloch space mentioned earlier. The Bloch space contains the multipliers of H^2_d, the bounded functions on $\mathbb{B}_d$, and by virture of Theorem 1.1, those functions in $\mathcal{C}H^2_d$. Because this theorem applies to multipliers for H^2_d, it also generalizes what is commonly referred to as the "one-variable Corona theorem" for H^2_d. The proof of Theorem 1.5 involves utilizing the realization that H^2_d can be identified as a Besov-Sobolev space, and then relies on the Bloch condition to estimate the norms of the summands that appear when taking a higher order derivative of $\dfrac{1}{f}$.

1.3 Basic Notation

Before we start in earnest, we will introduce fundamental notation that will be used throughout this dissertation.

For $d \in \mathbb{N}$, let $\mathbb{B}_d = \left\{ z = (z_1, ..., z_d) \in \mathbb{C}^d | \sum_{i=1}^d |z_i|^2 < 1 \right\}$ and let $\mathrm{Hol}(\mathbb{B}_d)$ denote the collection of complex-valued functions f on $\mathbb{B}_d$ which are holomorphic, or complex differentiable, on $\mathbb{B}_d$.

For a d-tuple of nonnegative integers $\alpha = (\alpha_1, ..., \alpha_d)$, we will write $\alpha! = \prod_{i=1}^d \alpha_i!$ and $|\alpha| = \sum_{i=1}^d \alpha_i$. If we are further given a point $z = (z_1, ..., z_d) \in \mathbb{C}^d$, then we will write $z^\alpha = \prod_{i=1}^d z_i^{\alpha_i}$. With this notation in place, for every $f \in \mathrm{Hol}(\mathbb{B}_d)$, we can find coefficients $\widehat{f}(\alpha) \in \mathbb{C}$ which satisfy

$$f(z) = \sum_{\alpha} \widehat{f}(\alpha) z^\alpha$$

where the sum is taken over all d-tuples of nonnegative integers. Moreover, for $n \in \mathbb{N}$, we can write

$$f_n(z) = \sum_{|\alpha|=n} \widehat{f}(\alpha) z^\alpha$$

and we have

$$f(z) = \sum_{n=0}^{\infty} f_n(z).$$

This is called the homogeneous expansion of f.

For two positive quantities $A = A(x_1, ..., x_n)$ and $B = B(x_1, ..., x_n)$ which depend on the variables $x_1, ..., x_n$, we will write $A \lesssim B$ to mean that there is a constant $C > 0$, independent of $x_1, ..., x_n$, such that $A(x_1, ..., x_n) \leq CB(x_1, ..., x_n)$ for any choice of $x_1, ..., x_n$. For example, it follows from the convexity of the real-valued map $x \mapsto x^2$ that $(x + y)^2 \leq 2(x^2 + y^2)$ for all $x, y \in \mathbb{R}$. We can (and will) write this as $(x + y)^2 \lesssim x^2 + y^2$. In the case that $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$.

Chapter 2

The Drury-Arveson Space

2.1 Definition

For $d \in \mathbb{N}$, the Drury-Arveson space H^2_d is the Hilbert space of holomorphic functions $f(z) = \sum_\alpha \widehat{f}(\alpha) z^\alpha \in \mathrm{Hol}(\mathbb{B}_d)$ equipped with the norm

$$\|f\|^2_{H^2_d} = \sum_\alpha \frac{\alpha!}{|\alpha|!} |\widehat{f}(\alpha)|^2 < \infty.$$

Given such an $f \in H^2_d$ and $\lambda \in \mathbb{B}_d$, an application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} |f(\lambda)| &\leq \sum_\alpha |\widehat{f}(\alpha)||\lambda^\alpha| \\ &\leq \left(\sum_\alpha \frac{\alpha!}{|\alpha|!} |\widehat{f}(\alpha)|^2 \right)^{\frac{1}{2}} \left(\sum_\alpha \frac{|\alpha|!}{\alpha!} |\lambda^\alpha|^2 \right)^{\frac{1}{2}} \\ &= \frac{\|f\|_{H^2_d}}{1 - |\lambda|^2}. \end{aligned}$$

This implies that the evaluation functional $f \mapsto f(\lambda)$ is bounded, and therefore the Riesz Representation theorem implies that for every $\lambda \in \mathbb{B}_d$ there is a unique vector $k_\lambda \in H^2_d$ which satisfies $f(\lambda) = \langle f, k_\lambda \rangle_{H^2_d}$. These vectors k_λ are called the

reproducing kernels of H^2_d, and we say that H^2_d is a **reproducing kernel Hilbert space**. In fact, given $\lambda \in \mathbb{B}_d$, we have that

$$k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle} = \sum_\alpha \frac{|\alpha|!}{\alpha!} z^\alpha \overline{\lambda}^\alpha$$

which can be easily verified by using the definition of the inner product on H^2_d. **Note:** Although we will talk about other reproducing kernels throughout this dissertation, we will reserve the notation k_λ to mean the reproducing kernel for H^2_d.

An alternate norm for H^2_d can be given in terms of a Sobolev-type norm. In particular, given an $m \in \mathbb{N}$ such that $2m - d > -1$, we have that $f \in H^2_d$ if and only if

$$\int_{\mathbb{B}_d} |R^m f(z)|^2 (1 - |z|^2)^{2m-d} dv(z) < \infty.$$

Here dv denotes Lebesgue measure on $\mathbb{B}_d$ and $Rf = \sum_{i=1}^d z_i \frac{\partial f}{\partial z_i}$ is called the **radial derivative** of f. To see this, it is enough to check that this is true for the monomials. Noting that $R(z^\alpha) = |\alpha| z^\alpha$, we have

$$\begin{aligned} \int_{\mathbb{B}_d} |R^m(z^\alpha)|^2 (1 - |z|^2)^{2m-d} dV(z) &= |\alpha|^{2m} \int_{\mathbb{B}_d} |z^\alpha|^2 (1 - |z|^2)^{2m-d} dV(z) \\ &= |\alpha|^{2m} \frac{\alpha! \Gamma(1 + 2m)}{\Gamma(1 + 2m + |\alpha|)} \\ &\sim \frac{\alpha!}{|\alpha|!}. \end{aligned} \tag{2.1}$$

(Here we have used Lemma 2.1.1 in the next section.)

2.2 Spaces Related to H^2_d

We will frequently return to the fact that H^2_d is a reproducing kernel Hilbert space. We now recall some of the basic properties for general reproducing kernel Hilbert spaces on $\mathbb{B}_d$ (For the purposes of this dissertation, we fix our domain to be the unit ball $\mathbb{B}_d$, although these spaces can be studied in a more general context).

If $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$ is a reproducing kernel Hilbert space with reproducing kernels l_λ, then the set

$$\mathcal{S} = \left\{ \sum_{i=1}^{n} \alpha_i l_{\lambda_i} \mid \alpha_1, ..., \alpha_n \in \mathbb{C} \text{ and } \lambda_1, ..., \lambda_n \in \mathbb{B}_d \right\}$$

is dense in $\mathcal{H}$. In fact, if f is perpendicular to l_λ for all $\lambda \in \mathbb{B}_d$, then $f(\lambda) = \langle f, l_\lambda \rangle = 0$, so that f is identically equal to the zero function.

Using the reproducing property of the l_λ, one can compute a nice expression for the norm of an arbitrary element in $\mathcal{S}$:

$$\left\| \sum_{i=1}^{n} \alpha_i l_{\lambda_i} \right\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \langle l_{\lambda_i}, l_{\lambda_j} \rangle = \sum_{i=1}^{n} \alpha_i \overline{\alpha_j} l_{\lambda_i}(\lambda_j) \tag{2.2}$$

This implies that for any choice of $\lambda_1, ..., \lambda_n \in \mathbb{C}$, that the matrix $(l_{\lambda_i}(\lambda_j))_{i,j=1}^n$ is positive definite. In this case, we say that the function $l : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$ given by $l(z, \lambda) = l_\lambda(z)$ is a **positive definite function** and we write $l \gg 0$. Moreover, given any positive definite function $l : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}$, we can define a reproducing kernel Hilbert space with the function l_λ as its reproducing kernels, which will denote by $\mathcal{H}(l)$.

When dealing with reproducing kernels, we will commonly exploit the following result from [3], which is an excellent survey of the theory of reproducing kernels.

Theorem 2.1 (in [3]). *Let l^1 and l^2 be positive definite functions on $\mathbb{B}_d$. Then $l^1 l^2$ is positive definite on $\mathbb{B}_d$.*

There is a special class of reproducing kernels that we will frequently use, so we will have a special notation for them. For $\gamma > 0$, we will define $\mathcal{H}_\gamma$ to be the reproducing kernel Hilbert space of functions in $\text{Hol}(\mathbb{B}_d)$ with reproducing kernel

$$k_\lambda^\gamma(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\gamma}$$

Note that in keeping with our assertion that k^1_λ is the reproducing kernel for H^2_d, we have that $H^2_d = \mathcal{H}_1$. Furthermore, we will write $\|f\|_\gamma$ for the norm of a function $f \in H_\gamma$; in particular, we will write $\|f\|_1 = \|f\|_{H^2_d}$.

This scale of spaces contains several well-known spaces. In particular, we have that $\mathcal{H}_d$ is the Hardy Space $H^2(\partial\mathbb{B}_d)$ and

$$\|f\|_d^2 = \sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\zeta)|^2 d\sigma(\zeta)$$

where $\mathbb{S}^n = \{z = (z_1, ..., z_d) \in \mathbb{C}^d \mid \sum_{i=1}^d |z_i|^2 = 1\}$ and $d\sigma$ is Lebesgue measure on $\mathbb{S}^n$. For $\beta > -1$, we have that $\mathcal{H}_{d+1+\beta}$ is the weighted Bergman space $A^2_\beta(\mathbb{B}_d)$ and

$$\|f\|^2_{d+1+\beta} = c_\beta \int_{\mathbb{B}_d} |f(z)|^2 (1 - |z|^2)^\beta dv(z).$$

where

$$c_\beta = \frac{\Gamma(d+1+\beta)}{n!\Gamma(\beta+1)}$$

is a normalization constant so that $\|1\|_{d+1+\beta} = 1$.

We will frequently need to work with the Bergman space norms. To this end, we have the following two lemmas.

Lemma 2.1.1 (see Lemma 1.11 in [37])**.** *Let $\alpha = (\alpha_1, ..., \alpha_d)$ be a d-tuple of nonnegative integers and suppose that $\eta > -1$. Then*

$$\int_{\mathbb{B}_d} |z^\alpha|^2 (1 - |z|^2)^\eta dV(z) = \frac{\alpha!\Gamma(d+1+\eta)}{\Gamma(d+1+\eta+|\alpha|)}$$

Lemma 2.1.2. *Let $f(z) = \sum_\alpha \widehat{f}(\alpha) z^\alpha \in Hol(\mathbb{B}_d)$. For $\eta > -1$, we have that*

$$\|f\|^2_{d+1+\eta} = \sum_\alpha \frac{\alpha!\Gamma(d+1+\eta)}{\Gamma(d+1+\eta+|\alpha|)} |\widehat{f}(\alpha)|^2$$

We note that

$$k_\lambda^\gamma(z) = \frac{1}{(1-\langle z,\lambda\rangle)^\gamma} = \sum_{n=0}^\infty a_{n,\gamma} \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{\lambda}^\alpha \tag{2.3}$$

where

$$a_{n,\gamma} = \frac{\Gamma(\gamma+n)}{n!\Gamma(\gamma)}.$$

From this it follows that for a function $f(z) = \sum_\alpha \widehat{f}(\alpha) z^\alpha$, we have

$$\|f\|_\gamma^2 = \sum_{n=0}^\infty \frac{1}{a_{n,\gamma}} \sum_{|\alpha|=n} \frac{\alpha!}{|\alpha|!} |\widehat{f}(\alpha)|^2$$

In particular, if write $f(z) = \sum_{k=0}^\infty f_k(z)$ for the homogeneous expansion for f, then we have that

$$\|f\|_\gamma^2 = \sum_{n=0}^\infty \frac{1}{a_{n,\gamma}} \|f_n\|_1^2 \tag{2.4}$$

Using this and mimicking Calculation 2.1 gives us the following proposition, which we will frequently use.

Proposition 1. *Let $\gamma > 0$ and $n \in \mathbb{N}$. Then $f \in H_\gamma$ if and only if $R^n f \in H_{\gamma+2n}$.*

We will also need to know how the radial derivative R acts with the inner product $\langle \cdot, \cdot \rangle_\gamma$. Let $f(z) = \sum_k f_k(z), g(z) = \sum_k g_k(z) \in \mathrm{Hol}(\mathbb{B}_d)$ be written with respect to their homogeneous expansions, and note that $Rf(z) = \sum_k k f_k(z)$. Therefore, we have that

$$\langle Rf, g\rangle_\gamma = \langle f, Rg\rangle_\gamma = \sum_{k=0}^\infty \frac{k}{c_{k,\gamma}} \langle f_k, g_k\rangle_1 \tag{2.5}$$

where we have used Equation 2.4.

Another space that will be important for us is the Bloch space $\mathcal{B}$, which is the Banach space of those functions $f \in \mathrm{Hol}(\mathbb{B}_d)$ with the property that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{B}_d} |Rf(z)|(1-|z|^2) < \infty.$$

The property that we will frequently use regarding $\mathcal{B}$ is given by the following proposition.

Proposition 2 (Theorem 3.5 in [37])**.** *Let $f \in Hol(\mathbb{B}_d)$. Then the following are equivalent:*

1. *$f \in \mathcal{B}$*
2. *There is an $m \in \mathbb{N}$ such that*

$$\sup_{z \in \mathbb{B}_d} |R^m f(z)|(1 - |z|^2)^m < \infty.$$

3. *For all $m \in \mathbb{N}$ we have that*

$$\sup_{z \in \mathbb{B}_d} |R^m f(z)|(1 - |z|^2)^m < \infty.$$

2.3 Multipliers

For a Hilbert space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$, we say that a function $b \in \mathcal{H}$ is a **multiplier** of $\mathcal{H}$, and we write $b \in \mathcal{M}(\mathcal{H})$, if for every $f \in \mathcal{H}$, we have that $fb \in \mathcal{H}$.

If we assume that $\mathcal{H}$ is a reproducing kernel Hilbert space, then to each $b \in \mathcal{M}(\mathcal{H})$ we can associate a bounded **multiplication operator** $M_b : \mathcal{H} \to \mathcal{H}$ given by $M_b f = bf$ that is colloquially called multiplication by b. To see this operator is in fact bounded, suppose that $f_n, f, g \in \mathcal{H}$ satisfy $f_n \to f$ and $M_b f_n \to g$. Since point evaluations are bounded in $\mathcal{H}$, for every $\lambda \in \mathbb{B}_d$, we have that $f_n(\lambda) \to f(\lambda)$ and $b(\lambda) f_n(\lambda) \to g(\lambda)$. But this implies that $b(\lambda) f(\lambda) = g(\lambda)$ so that $bf = g$. The claim now follows by the closed graph theorem. We can therefore define the **multiplier norm** of a function $b \in H^2_d$ as $\|b\|_{\mathcal{M}} = \|M_b\|$, where the second norm here denotes the usual operator norm.

If $\mathcal{H}(l) \subseteq \mathrm{Hol}(\mathbb{B}_d)$ is a reproducing kernel Hilbert space and $b \in \mathcal{M}(\mathcal{H}(l))$, then we have the relation $M_b^* l_\lambda = \overline{b(\lambda)} l_\lambda$ for any $\lambda \in \mathbb{B}_d$. To see this, let $g \in \mathcal{H}(l)$. Then

$$\begin{aligned} \langle g, M_b^* l_\lambda \rangle &= \langle bg, l_\lambda \rangle \\ &= b(\lambda) g(\lambda) \\ &= b(\lambda) \langle g, l_\lambda \rangle \\ &= \langle g, \overline{b(\lambda)} l_\lambda \rangle \end{aligned}$$

and the claim follows.

Now, because finite linear combinations of reproducing kernels are dense in $\mathcal{H}(l)$, we see that the operator M_b is bounded if and only if

$$\left\| M_b^* \left(\sum_{i=1}^n \alpha_i l_{\lambda_i} \right) \right\|^2 \lesssim \left\| \sum_{i=1}^n \alpha_i l_{\lambda_i} \right\|^2$$

for every choice of $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $\lambda_1, ..., \lambda_n \in \mathbb{B}_d$. Using the fact that $M_b^* l_\lambda = \overline{b(\lambda)} l_\lambda$ and repeating Calculation 2.2, we see that this is equivalent to

$$\sum_{i=1}^n \alpha_i \overline{\alpha_j} \overline{b(\lambda_i)} b(\lambda_j) l_{\lambda_i}(\lambda_j) \lesssim \sum_{i=1}^n \alpha_i \overline{\alpha_j} l_{\lambda_i}(\lambda_j).$$

This implies that there is a constant $C > 0$ such that the function

$$B(z, \lambda) = (C - b(z)\overline{b(\lambda)}) l_\lambda(z)$$

is positive definite. In fact, with virtually the same calculations, we have the following useful proposition.

Proposition 3. *Let $\mathcal{H}(l^1), \mathcal{H}(l^2) \subseteq Hol(\mathbb{B}_d)$ be reproducing kernel Hilbert Spaces and let $b \in Hol(\mathbb{B}_d)$. Then the formal adjoint of the operator operator $M_b : \mathcal{H}(l^1) \to \mathcal{H}(l^2)$ satisfies*

$$M_b^* l_\lambda^2 = \overline{b(\lambda)} l_\lambda^1$$

for all $\lambda \in \mathbb{B}_d$*. Furthermore,* M_b *is bounded if and only if there is a constant* $C > 0$ *such that the function*

$$B(z, \lambda) = Cl^2_\lambda(z) - b(z)\overline{b(\lambda)}l^1_\lambda(z)$$

is positive definite.

Now, let $b \in \mathcal{M}(\mathcal{H}(l))$. We have shown that for every $\lambda \in \mathbb{B}_d$ that $\overline{b(\lambda)}$ is an eigenvalue of $M_b : \mathcal{H}(l) \to \mathcal{H}(l)$. Therefore, we have that

$$\sup_{\lambda \in \mathbb{B}_d} |b(\lambda)| \leq \|M_b\|.$$

This implies that $\mathcal{M}(\mathcal{H}(l)) \subseteq H^\infty(\mathbb{B}_d)$, where $H^\infty(\mathbb{B}_d)$ is the algebra of those functions $f \in \mathrm{Hol}(\mathbb{B}_d)$ which satisfy

$$\|f\|_\infty = \sup_{z \in \mathbb{B}_d} |f(z)| < \infty.$$

For the Hardy Space $\mathcal{H}_d = H^2(\mathbb{B}_d)$ and the weighted Bergman spaces $\mathcal{H}_{d+1+\alpha} = A^2_\alpha(\mathbb{B}_d)$ with $\alpha > -1$, it is easy to see that the multipliers of these spaces are precisely the $H^\infty(\mathbb{B}_d)$ functions (largely because their norms involve the integral of the function).

Unfortunately, for the Drury-Arveson space H^2_d, the situation is not as simple. In fact, Arveson gave an example of a bounded function which does not lie in H^2_d, and thus, in order to characterize multipliers for H^2_d, another condition is needed. The characterization of the multipliers of H^2_d involves certain measures being what is referred to as a Carleson measure.

2.4 Carleson Measures

If $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ is a Hilbert space, then a positive measure μ on $\mathbb{B}_d$ is called a **Carleson measure** for $\mathcal{H}$ if the inclusion $i : \mathcal{H} \to L^2(\mu)$ is bounded. This is equivalent to the existence of a constant $C > 0$ which satisfies

$$\int_{\mathbb{B}_d} |f(z)|^2 d\mu(z) \leq C^2 \|f\|_{\mathcal{H}}^2$$

for all $f \in \mathcal{H}$. The smallest such C which appears above (which is equal to the norm of the inclusion i) is called the **Carleson measure norm** of μ and which we will denote by $\|\mu\|_{\text{Carleson}}$.

Now, if $b \in \text{Hol}(\mathbb{B}_d)$, $m \in \mathbb{N}$, and $\beta > -1$, then we see that the measure $|R^m b(z)|^2(1-|z|^2)^\beta dv(z)$ is a Carleson measure for H_γ for some $\gamma > 0$ if and only if the multiplication operator $M_{R^m b} : \mathcal{H}_\gamma \to \mathcal{H}_{d+1+\beta}$ is bounded. By means of Proposition 3, we can show that the boundedness several of these types of Carleson measures are equivalent. The proof of this fact was given in [8]; however, we will provide an alternate proof here based on the theory of reproducing kernel Hilbert spaces. Before giving this proof, we prove three lemmas which illustrate how the use of positive definiteness can be useful for proving results about multipliers for the space H_γ with $\gamma > 0$.

Lemma 2.1.3. *Let $\gamma, \eta > 0$ and suppose that $b \in Hol(\mathbb{B}_d)$ is such that the operator $M_b : \mathcal{H}_\gamma \to \mathcal{H}_\eta$ is bounded. Then*

$$\sup_{z \in \mathbb{B}_d} |b(z)|^2 (1 - |z|^2)^{\eta - \gamma} < \infty.$$

Proof. By Proposition 3, we have that there is a constant $C > 0$ such that

$$\frac{C}{(1 - \langle z, \lambda \rangle)^\eta} - \frac{b(z)\overline{b(\lambda)}}{(1 - \langle z, \lambda \rangle)^\gamma} \gg 0$$

The result now follows from the resulting inequality by testing this positive definiteness for $z = \lambda$. □

Lemma 2.1.4. *Let $\eta > -1$ and let $\beta < 1+\eta$. Then for any $b \in Hol(\mathbb{B}_d)$, the operator $M_b : \mathcal{H}_{d+1+\eta-\beta} \to \mathcal{H}_{d+1+\eta}$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}_d} |b(z)|^2(1-|z|^2)^\beta < \infty.$$

Proof. Let C be equal to the supremum in the statement of the corollary.

($\Rightarrow$) If M_b is bounded, then we have that $C < \infty$ by Lemma 2.1.3.

($\Leftarrow$) Since $\eta > -1$, we have that $\mathcal{H}_{d+1+\eta}$ is equal to the weighted Bergman space $A^2_\eta(\mathbb{B}_d)$. Therefore, we have that

$$\begin{aligned} \|fb\|^2_{d+1+\eta} &= c_\eta \int_{\mathbb{B}_d} |f(z)b(z)|^2(1-|z|^2)^\eta dV(z) \\ &\leq Cc_\eta \int_{\mathbb{B}_d} |f(z)|^2(1-|z|^2)^{\eta-\beta} dV(z) \\ &= \frac{Cc_\eta}{c_{\eta-\beta}} \|f\|^2_{d+1+\eta-\beta}. \end{aligned}$$

We note that for the last equality we needed that $\eta - \beta > -1$. $\square$

Note that taking $\beta = 0$ in the previous corollary yields a proof of the fact that $\mathcal{M}(\mathcal{H}_{d+1+\eta}) = H^\infty(\mathbb{B}_d)$ for $\eta > -1$.

Lemma 2.1.5. *Let $\eta, \gamma > 0$ and suppose that $M_b : \mathcal{H}_\gamma \to \mathcal{H}_\eta$ is bounded. Then for any $\epsilon > 0$, we have that the operator $M_b : \mathcal{H}_{\gamma+\epsilon} \to \mathcal{H}_{\eta+\epsilon}$ is bounded.*

Proof. Let $\epsilon > 0$. By Proposition 3, we have that there is a constant $C > 0$ such that

$$\frac{C}{(1-\langle z,\lambda\rangle)^\eta} - \frac{b(z)\overline{b(\lambda)}}{(1-\langle z,\lambda\rangle)^\gamma} \gg 0.$$

Then, Theorem 2.1, we have that

$$\frac{C}{(1-\langle z,\lambda\rangle)^{\eta+\epsilon}} - \frac{b(z)\overline{b(\lambda)}}{(1-\langle z,\lambda\rangle)^{\gamma+\epsilon}} \gg 0$$

since $k^\epsilon_\lambda \gg 0$. The result now follows by another application of Proposition 3. $\square$

Theorem 2.2 (Corollary 3.12 in [8]). *Let $\gamma > 0$. For a function $b \in H_\gamma$, the following are equivalent:*

1. *the operator $M_{R^m b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2m}$ is bounded for some $m \in \mathbb{N}$.*

2. *the operator $M_{R^m b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2m}$ is bounded for all $m \in \mathbb{N}$.*

Proof. It is clear that 2 implies 1, therefore we need only show that 1 implies 2.

Fix $m \in \mathbb{N}$ and suppose that $M_{R^m b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2m}$ is bounded. By Lemma 2.1.5, we have that $M_{R^m b} : \mathcal{H}_{\gamma+2} \to \mathcal{H}_{\gamma+2(m+1)}$ is bounded. By the product rule, we have that $fR^{m+1}b = R(fR^m b) - RfR^m b$, and therefore

$$\begin{aligned}
\|fR^{m+1}b\|^2_{\mathcal{H}_{\gamma+2(m+1)}} &\lesssim \|R(fR^m b)\|^2_{\mathcal{H}_{\gamma+2(m+1)}} + \|RfR^m b\|^2_{\mathcal{H}_{\gamma+2(m+1)}} \\
&\lesssim \|fR^m b\|^2_{\mathcal{H}_{\gamma+2m}} + \|Rf\|^2_{\mathcal{H}_{\gamma+2}} \\
&\lesssim \|f\|^2_\gamma.
\end{aligned}$$

This implies that $M_{R^{m+1}b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2(m+1)}$ is bounded. In fact, by induction, we actually have that $M_{R^{m+k}b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2(m+k)}$ is bounded for all $k \in \mathbb{N}$. Therefore it suffices to show that $M_{Rb} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2}$ is bounded.

Now, choose $M \in \mathbb{N}$ large so that $2(M - m + 1) > d - \gamma$. Then we have that

$$\|fRb\|^2_{\mathcal{H}_{\gamma+2}} \approx \|R^M(fRb)\|^2_{\mathcal{H}_{\gamma+2(M+1)}} \lesssim \sum_{k=0}^{M} \|R^k f R^{M-k+1} b\|^2_{\mathcal{H}_{\gamma+2(M+1)}}$$

It now suffices to prove the appropriate boundedness of the individual terms in the above summation.

If $M - m + 1 \geq k$, then we have that

$$\|R^k f R^{M-k+1} b_n\|^2_{\mathcal{H}_{\gamma+2(M+1)}} \lesssim \|R^k f\|^2_{\mathcal{H}_{\gamma+2k}} \approx \|f\|^2_{\mathcal{H}_\gamma}$$

since the multiplication operator $M_{R^{M-k+1}b} : \mathcal{H}_{\gamma+2k} \to \mathcal{H}_{\gamma+2(M+1)}$ is bounded since $M - k + 1 \geq m$.

Now suppose that $k \geq M - m + 1$, which implies that $2k + \gamma > d$. By Lemma 2.1.3, the boundedness of $M_{R^m b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2m}$ implies that

$$\sup_{z\in\mathbb{B}_d} |R^m b(z)|^2(1-|z|^2)^{2m} < \infty$$

so that $f \in \mathcal{B}$ by Proposition 2. Therefore, by the same proposition, we have that

$$\sup_{z\in\mathbb{B}_d} |R^n b(z)|^2(1-|z|^2)^{2n} < \infty$$

for all $n \in \mathbb{N}$ and therefore by taking $n = M-k+1$, $\eta = \gamma+2(M+1)-(d+1)$, and $\beta = 2(M-k+1)$ in Lemma 2.1.4, we have that the operator $M_{R^{M-k+1}b} : \mathcal{H}_{\gamma+2k} \to \mathcal{H}_{\gamma+2(M+1)}$ is bounded (**Note:** For this we need $\beta < \eta + 1$, which is equivalent to $\gamma + 2k > d$). Finally, we have that

$$\|R^k f R^{M-k+1} b_n\|^2_{\mathcal{H}_{\gamma+2(M+1)}} \lesssim \|R^k f\|^2_{\mathcal{H}_{\gamma+2k}} \approx \|f\|^2_{\mathcal{H}_\gamma}$$

and the result follows. □

In light of this result, we will define $\mathcal{CH}_\gamma$ to be the set of all those functions in H_γ which satisfy condition 2 of Theorem 2.2. Furthermore, using this theorem, we can finally prove the characterization of multipliers for H^2_d, and more generally, the multipliers of H_γ for $\gamma > 0$.

Theorem 2.3 (Theorem 3.7 in [27]). *Let $\gamma > 0$. A function $b \in \mathcal{M}(\mathcal{H}_\gamma)$ if and only if $b \in H^\infty \cap \mathcal{CH}_\gamma$.*

Proof. ($\Rightarrow$) Suppose that $b \in \mathcal{M}(\mathcal{H}_\gamma)$. We have mentioned the necessity of $b \in H^\infty$ in Section 2.3. If $b \in \mathcal{M}(\mathcal{H}_\gamma)$, then we have that $M_b : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is bounded. By Lemma 2.1.5, we have that $M_b : \mathcal{H}_{\gamma+2} \to \mathcal{H}_{\gamma+2}$ is bounded. Therefore, using

$fRb = R(fb) - (Rf)b$, we have that

$$\|fRb\|_{\gamma+2} \leq \|R(fb)\|_{\gamma+2} + \|(Rf)b\|_{\gamma+2} \lesssim \|fb\|_{\gamma} + \|(Rf)\|_{\gamma+2} \lesssim \|f\|_{\gamma}$$

which shows that $M_{Rb} : \mathcal{H}_{\gamma} \to \mathcal{H}_{\gamma+2}$ is bounded. Therefore $b \in \mathcal{CH}_{\gamma}$.

($\Leftarrow$) Suppose that $b \in H^{\infty} \cap \mathcal{CH}_{\gamma}$. Let $m \in \mathbb{N}$ be large so that $\gamma + 2m > d$. Then we have that

$$\|fb\|_{\gamma} \approx \|R^m(fb)\|_{\gamma+2m} \lesssim \sum_{k=0}^{m} \|R^k f R^{m-k} b\|_{\gamma+2m}.$$

Since $b \in H^{\infty}$, we have that

$$\|(R^m f)b\|_{\gamma+2m} \lesssim \|R^m f\|_{\gamma+2m} \approx \|f\|_{\gamma}$$

since $M_b : \mathcal{H}_{\gamma+2m} \to \mathcal{H}_{\gamma+2m}$ is bounded if and only if $b \in H^{\infty}$ by Lemma 2.1.3 (take $\beta = 0$ and $\eta = \gamma + 2m - (d+1)$). If $k < m$, then since $b \in \mathcal{CH}_{\gamma}$, we have that $M_{R^{m-k}b} : \mathcal{H}_{\gamma} \to \mathcal{H}_{\gamma+2(m-k)}$ is bounded. Therefore, by Lemma 2.1.5, we have that $M_{R^{m-k}b} : \mathcal{H}_{\gamma+2k} \to \mathcal{H}_{\gamma+2m}$ is bounded and therefore

$$\|R^k f R^{m-k} b\|_{\gamma+2m} \lesssim \|R^k f\|_{\gamma+2k} \approx \|f\|_{\gamma}.$$

Hence we have that $\|fb\|_{\gamma} \lesssim \|f\|_{\gamma}$ so that $b \in \mathcal{M}(\mathcal{H}_{\gamma})$. $\square$

We mention there that for $\gamma \geq d$, it is the case that $\mathcal{M}(\mathcal{H}_{\gamma}) = H^{\infty}$, so that the requirement that the function be in $\mathcal{C}(\mathcal{H}_{\gamma})$ is not needed in these cases (in fact, it turns out that $H^{\infty} \subseteq \mathcal{CH}_{\gamma}$ in these cases). That this is true for $\gamma > d$ can be seen by taking $\beta = 0$ in Lemma 2.1.4. The case when $\gamma = d$ follows along the same type of proof as used in Lemma 2.1.4.

Chapter 3

Hankel Operators

3.1 Definition

Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ be a Hilbert space such $\mathrm{Hol}(\overline{\mathbb{B}_d})$ is dense in $\mathcal{H}$. We then write $b \in \mathcal{X}(\mathcal{H})$ and call b a **Hankel symbol** of $\mathcal{H}$ if there is a constant $C > 0$ such that

$$|\langle fg, b\rangle| \leq C\|f\|\|g\| \tag{3.1}$$

holds for all $f, g \in \mathrm{Hol}(\overline{\mathbb{B}_d})$. We will write $\|b\|_{\mathcal{X}(\mathcal{H})}$ for the smallest such C that can appear in the above inequality.

Now, let $\overline{\mathcal{H}}$ be the Hilbert space that equals $\mathcal{H}$ as a set but is equipped with the inner product

$$\langle \overline{f}, \overline{g}\rangle_{\overline{\mathcal{H}}} = \langle g, f\rangle_{\mathcal{H}}.$$

If $b \in \mathcal{X}(\mathcal{H})$, then we have that the sesquilinear form given by $(f, \overline{g}) \mapsto \langle fg, b\rangle$ extends to be bounded on $\mathcal{H} \times \overline{\mathcal{H}}$. Therefore, we can find a bounded operator $H_b : \mathcal{H} \to \overline{\mathcal{H}}$, which we will call the **Hankel operator** with symbol b, which satisfies the relation

$$\langle H_b f, \overline{g}\rangle_{\overline{\mathcal{H}}} = \langle fg, b\rangle_{\mathcal{H}}.$$

If $\mathcal{H} = \mathcal{H}(l)$ is a reproducing kernel Hilbert space, then one can easily find several elements of $\mathcal{X}(\mathcal{H})$. In particular if $\lambda \in \mathbb{B}_d$ and if $f, g \in \mathrm{Hol}(\overline{\mathbb{B}_d})$, then

$$|\langle fg, l_\lambda \rangle| = |f(\lambda)||g(\lambda)| \leq \|f\|\|g\|\|l_\lambda\|^2.$$

Therefore $l_\lambda \in \mathcal{X}(\mathcal{H})$ with $\|l_\lambda\|_{\mathcal{X}(\mathcal{H})} \leq \|l_\lambda\|^2$.

If $\mathcal{H}$ turns out to be a Banach algebra (this is true for some weighted Dirichlet spaces on $\mathbb{D}$, for instance), then $\mathcal{X}(\mathcal{H}) = \mathcal{H}$ by the Cauchy-Schwarz inequality and the standard Banach algebra inequality. However, there are several spaces, the Drury-Arveson space included, that are not algebras. In order to have a context in which to talk about products of functions in Hilbert spaces that are not algebras, we will now introduce the concept of a weakly-factored space.

3.2 Weakly Factored Spaces

Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ be a Hilbert space. We then define the **weakly-factored space**

$$\mathcal{H} \odot \mathcal{H} = \left\{ \sum_{i=1}^{\infty} g_i h_i \mid g_i, h_i \in \mathcal{H} \text{ and } \sum_{i=1}^{\infty} \|f_i\|\|g_i\| \right\}.$$

We can define a norm on $\mathcal{H} \odot \mathcal{H}$ by

$$\|f\|_* = \inf \left\{ \sum_i \|g_i\|\|h_i\| \mid f = \sum_i g_i h_i \text{ where } g_i, h_i \in \mathcal{H} \right\}.$$

The following is a special case of Theorem 1.3 in [29], which illustrates the connection bewteen $H^2_d \odot H^2_d$ and $\mathcal{X}(H^2_d)$.

Theorem 3.1. *For every $b \in \mathcal{X}(H^2_d)$, define the map $L_b : H^2_d \to \mathbb{C}$ by*

$$L_b(h) = \langle h, b \rangle.$$

Then L_b extends to be bounded on $H^2_d \odot H^2_d$, and furthermore, the map $b \mapsto L_b$ is a conjugate linear isometric isomorphism of $\mathcal{X}(H^2_d)$ onto $(H^2_d \odot H^2_d)^*$.

3.3 Equivalence of Bilinear Forms

For the Drury-Arveson space $H^2_d = \mathcal{H}_1$, the inequality 3.1 is equivalent to the boundedness of several other bilinear forms.

Lemma 3.1.1. *Let $b \in H^2_d$. Then the following are equivalent:*

(a) $b \in \mathcal{X}(H^2_d)$,

(b) there is an integer $n \geq 0$ and a $C > 0$ such that

$$|\langle \varphi\psi, R^n b\rangle_{n+1}| \leq C\|\varphi\|_1\|\psi\|_1 \text{ for all } \varphi, \psi \in Hol(\overline{\mathbb{B}_d}),$$

(c) for all integers $n \geq 0$ there is a $C > 0$ such that

$$|\langle \varphi\psi, R^n b\rangle_{n+1}| \leq C\|\varphi\|_1\|\psi\|_1 \text{ for all } \varphi, \psi \in Hol(\overline{\mathbb{B}_d}).$$

Proof. It is trivial that (c) implies (a) and that (a) implies (b), hence we only need to show the implication $(b) \Rightarrow (c)$. This will follow, if we show that for each integer $n \geq 0$ there is a $c > 0$ such that

$$\left|\langle \varphi\psi, R^n b\rangle_{n+1} - \frac{1}{n+1}\langle \varphi\psi, R^{n+1} b\rangle_{n+2}\right| \leq c\|\varphi\|_1\|\psi\|_1\|b\|_1. \tag{3.2}$$

Let $f \in \mathrm{Hol}(\overline{\mathbb{B}_d})$, and let $b = \sum_{k\geq 0} b_k$ and $f = \sum_{k\geq 0} f_k$ be the homogeneous expansions of b and f. Since $b \in \mathcal{H}_1$, we have that $R^n b \in \mathcal{H}_{1+2n}$ for each $n \geq 0$, and thus the series

$$\langle f, R^n b\rangle_{n+1} = \sum_{k\geq 0} \frac{k^n}{a_{k,n+1}}\langle f_k, b_k\rangle_1$$

converges absolutely.

Since

$$\frac{a_{k,n+2}}{a_{k,n+1}} = \frac{\left(\frac{\Gamma(n+k+2)}{k!\Gamma(n+2)}\right)}{\left(\frac{\Gamma(n+k+1)}{k!\Gamma(n+1)}\right)} = 1 + \frac{k}{n+1}$$

one easily proves from Equation 2.4 that

$$\langle f, R^n b\rangle_{n+1} = \langle f, R^n b\rangle_{n+2} + \frac{1}{n+1}\langle f, R^{n+1} b\rangle_{n+2}.$$

Thus

$$\begin{aligned}
\left|\langle f, R^n b\rangle_{n+1} - \frac{1}{n+1}\langle f, R^{n+1}b\rangle_{n+2}\right| &\leq \sum_{k\geq 0} \frac{k^n}{a_{k,n+2}}\|f_k\|_1\|b_k\|_1 \\
&\leq \|f\|_2 \left(\sum_{k\geq 0} \frac{a_{k,2}k^{2n}}{a_{k,n+2}^2}\|b_k\|_1^2\right)^{1/2} \\
&\leq c\|f\|_2\|b\|_2 \\
&\leq c\|f\|_2\|b\|_1
\end{aligned}$$

for some $c > 0$. The second to last inequality followed since for each n we have $a_{k,n+1} \sim (k+1)^n$ as $k \to \infty$ (see e.g. [35], p. 58.) Thus there is a $c > 0$ such that for all $k \geq 0$ one has $\frac{a_{k,2}k^{2n}}{a_{k,n+2}^2} \leq \frac{c}{k+1}$.

In [29] (see Theorem 1.4), it was shown that for any reproducing kernel Hilbert space $\mathcal{H}(k)$ with reproducing kernel k one has a contractive inclusion $\mathcal{H}(k) \odot \mathcal{H}(k) \subseteq \mathcal{H}(k^2)$. We apply this with $k = k^1$, the Drury-Arveson kernel, to obtain $\|\varphi\psi\|_2 \leq \|\varphi\|_1\|\psi\|_1$ for all $\varphi, \psi \in \mathrm{Hol}(\overline{\mathbb{B}_d})$. Inequality 3.2 then follows by substituting $f = \varphi\psi$ in the earlier estimate. □

Theorem 3.2.

$$\mathcal{M}(H_d^2) \subseteq \mathcal{C}(H_d^2) \subseteq \mathcal{X}(H_d^2) \subseteq \mathcal{B}.$$

Proof. It follows from Theorem 2.3 that $\mathcal{M}(H_d^2) \subseteq \mathcal{C}(H_d^2)$.

Now, from Formula 2.3 we have that

$$k_\lambda^2(z) = \sum_{n=0}^{\infty}(n+1)\sum_{|\alpha|=n}\frac{|\alpha|!}{\alpha!}z^\alpha\overline{\lambda}^\alpha = \sum_\alpha \frac{|\alpha||\alpha|!}{\alpha!}z^\alpha\overline{\lambda}^\alpha + k_\lambda(z).$$

If $b \in H_d^2$, then writing $b(z) = \sum_\alpha \widehat{b}(\alpha)z^\alpha$, we have that

$$Rb(\lambda) = \sum_\alpha |\alpha|\widehat{b(\alpha)}\lambda^\alpha = \sum_\alpha \frac{\alpha!}{|\alpha|!}\widehat{b(\alpha)}\frac{|\alpha||\alpha|!}{\alpha!}\lambda^\alpha = \langle b, k_\lambda(k_\lambda - 1)\rangle.$$

If, additionaly, $b \in \mathcal{X}(H_d^2)$, then we have that

$$\begin{aligned} |Rb(\lambda)| &= |\langle k_\lambda(k_\lambda - 1), b\rangle| \\ &\leq \|b\|_{\mathcal{X}(H_d^2)}\|k_\lambda\|\|k_\lambda - 1\| \\ &= \|b\|_{\mathcal{H}(H_d^2)}\|k_\lambda\|\sqrt{\|k_\lambda\|^2 - 1} \\ &\leq \frac{\|b\|_{\mathcal{X}(H_d^2)}}{1 - |\lambda|^2}. \end{aligned}$$

Thus we see that

$$\sup_{z\in\mathbb{B}_d}|Rb(z)|(1 - |\lambda|^2) \leq \|b\|_{\mathcal{X}(H_d^2)}$$

so that $b \in \mathcal{B}$ by Proposition 2. Thus $\mathcal{X}(H_d^2) \subseteq \mathcal{B}$.

It is left to show that $\mathcal{C}(H_d^2) \subseteq \mathcal{X}(H_d^2)$. To this end, choose m large enough so that $2m - d > -1$. By Lemma 3.1.1, it suffices to show that

$$|\langle fg, R^{3m}b\rangle_{1+3m}| \lesssim \|f\|_1\|g\|_1$$

and by Formula 2.5, this is equivalent to

$$|\langle R^{2m}(fg), R^m b\rangle_{1+3m}| \lesssim \|f\|_1\|g\|_1.$$

Furthermore, we have that

$$\langle R^{2m}(fg), R^m b\rangle_{1+3m} = \sum_{k=0}^{2m} \left(\begin{array}{c} 2m \\ k \end{array} \right) \langle R^k f R^{2m-k} g, R^m b\rangle_{1+3m}.$$

It now suffices to show the boundedness of each of the terms in the sum. By symmetry, we may also assume that $0 \leq k \leq m$. Then we have that $2(m+k) - d > -1$ and $2(2m-k) - d > -1$ and since $1 + 3m > d$, we have that

$$\begin{aligned} |\langle R^k f R^{2m-k} g, R^n b\rangle_{3m+1}| &\lesssim \int_{\mathbb{B}_d} |R^k f R^{2m-k} g R^m b|(1-|z|^2)^{3m-d} dV \\ &\lesssim \left(\int_{\mathbb{B}_d} |R^k f R^m b|^2 (1-|z|^2)^{2(m+k)-d} dV \right)^{1/2} \\ &\quad \left(\int_{\mathbb{B}_d} |R^{2m-k} g|^2 (1-|z|^2)^{2(2m-k)-d} dV \right)^{1/2} \\ &\approx \|R^k f R^m b\|_{2(m+k)+1} \|R^{2m-k} g\|_{2(2m-k)+1} \\ &\lesssim \|f\|_1 \|g\|_1, \end{aligned}$$

Here we have used Theorem 2.2 and recalled the equivalent norms from Section 2.1.

□

Chapter 4

Characterization of Lat(M_z, H^2_d)

4.1 Motivation

We now turn to the characterization of the invariant subspaces of the d-tuple of operators $M_z = (M_{z_1}, ..., M_{z_d})$ on H^2_d. We will write $\mathcal{M} \in \text{Lat}(M_z, H^2_d)$ if $\mathcal{M}$ is a closed subspace of H^2_d which satisfies $z_i\mathcal{M} \subseteq \mathcal{M}$ for all $i \in \{1, ..., d\}$. Our motivation is a classical theorem of Beurling that gives a characterization of $\text{Lat}(M_z, H^2_1)$.

Theorem 4.1 (Beurling). *Let $(0) \neq \mathcal{M}$ be an invariant subspace for M_z on $H^2_1 = H^2(\mathbb{D})$. Then there is a function $\Theta \in \mathcal{M}$ which satisfies*

1. $\Theta \in \mathcal{M} \ominus z\mathcal{M}$.
2. $\lim_{r\to 1^-} |\Theta(re^{i\theta})| = 1$ *for a.e.* $e^{i\theta} \in \mathbb{T}$.
3. $\mathcal{M} = \Theta H^2(\mathbb{D})$.

The proof of Beurling's Theorem can be given using the isometric identification of functions $f \in H^2(\mathbb{D})$ with their radial limit function

$$f^*(e^{i\theta}) = \lim_{r\to 1^{-1}} f(re^{i\theta})$$

(see Chapter 17 of [32]). We note that f^* exists for a.e. $e^{i\theta} \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and that the limit exists in a nontangential region about the point $e^{i\theta}$ (again, for a.e. $e^{i\theta}$).

Now, if $b \in \mathcal{X}(H^2(\mathbb{D}))$, then we have that

$$\langle H_b(zf), \overline{g}\rangle_{\overline{H^2(\mathbb{D})}} = \langle zfg, b\rangle_{H^2(\mathbb{D})} = \langle H_b f, \overline{zg}\rangle_{\overline{H^2(\mathbb{D})}}. \tag{4.1}$$

This implies that if $H_b f = 0$ then $H_b(zf) = 0$; that is, we have that $\ker H_b \in \mathrm{Lat}(M_z, H^2(\mathbb{D}))$. Using the identification in the previous paragarph, we can use Beurling's theorem to show that every $\mathcal{M} \in \mathrm{Lat}(M_z, H^2(\mathbb{D}))$ is actually the kernel of a Hankel operator.

Proposition 4. *Let $(0) \neq \mathcal{M}$ be an invariant subspace for M_z on $H^2_1 = H^2(\mathbb{D})$. Then there is $b \in \mathcal{X}(H^2_1)$ which satisfies $\mathcal{M} = \ker H_b$.*

Proof. Let $b = M_z^*\Theta$, where Θ is the function given to us by Beurling's Theorem for $\mathcal{M}$. Then for polynomials $f, g \in H^2(\mathbb{D})$, we have that

$$\begin{aligned} |\langle fg, b\rangle_{H^2(\mathbb{D})}| &= |\langle zfg, \Theta\rangle_{H^2(\mathbb{D})}| \\ &\leq \int_0^{2\pi} |e^{i\theta} f(e^{i\theta}) g(e^{i\theta}) \Theta(e^{i\theta})| \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} |f(e^{i\theta}) g(e^{i\theta})| \frac{d\theta}{2\pi} \\ &\leq \|f\|_{H^2(\mathbb{D})} \|g\|_{H^2(\mathbb{D})} \end{aligned}$$

where the last inequality follows by Hölder's inequality. Thus $b \in \mathcal{X}(H^2(\mathbb{D}))$.

Now, let $f \in \mathcal{M}$. Then for any $n \in \mathbb{N}_0$, we have that

$$\langle H_b f, \overline{z}^n\rangle_{\overline{H^2(\mathbb{D})}} = \langle z^n f, b\rangle_{H^2(\mathbb{D})} = \langle z^{n+1} f, \Theta\rangle_{H^2(\mathbb{D})} = 0. \tag{4.2}$$

Therefore $\mathcal{M} \subseteq \ker H_b$.

Conversely, suppose that $f \in \ker H_b \cap \mathcal{M}^\perp$. By the previous calculation, we have that $z^n f \perp \Theta$ for all $n \in \mathbb{N}$. Since $f \in \mathcal{M}^\perp$, we have that $f \perp z^n\Theta$ for all $n \in \mathbb{N}_0$. By the identification of functions in $H^2(\mathbb{D})$ with the radial limit functions in $L^2(\mathbb{T})$, we therefore have that the Fourier coefficients of the function $f^*\overline{\theta}^*$ are all identically zero. Therefore we have that $f^*\overline{\theta}^* = 0$ a.e. on $\mathbb{T}$. But since $|\theta^*| = 1$ a.e. on $\mathbb{T}$, we conclude that $f^* \equiv 0$ on $\mathbb{T}$. Hence $f \equiv 0$ and the result follows. □

The above proof relies on the fact that the Hardy space of the unit disk $H^2(\mathbb{D})$ can be isometrically identified with a closed subspace of $L^2(\mathbb{T})$, which allows for one to use the tools from the theory of Fourier series. Unfortunately for $d \geq 2$, this technique no longer works. In fact, in [5], Arveson proved that for $d \geq 2$ there is no positive measure μ on $\mathbb{C}^d$ which satisfies

$$\|p\|^2_{H^2_d} = \int_{\mathbb{C}} |p(z)|^2 d\mu(z)$$

for every polynomial p. In order to generalize Beurling's Theorem to H^2_d for $d \geq 2$, we will rely on techniques from the theory of complete Nevanlinna Pick kernels, which are discussed in the next section.

Another difficulty for the case when $d \geq 2$ is that there are invariant subspaces for M_z on H^2_d which are not the kernel of a Hankel operator. We now prove a proposition which gives a characterization of the kernels of Hankel operators and use it to prove this assertion. As a consequence of this proposition, we see that kernels of Hankel operators lie in $\mathrm{Lat}(M_z, H^2_d)$ (although a proof similar to 4.2 also works as well).

Proposition 5. *If $b \in \mathcal{X}(H^2_d)$, then $\ker H_b = [b]_*^\perp$, where we have written $[b]_*$ for the smallest closed subspace of H^2_d that contains b and is invariant under $M^*_{z_i}$ for $i = 1, ..., d$.*

Proof. Given an analytic polynomial p, we have that

$$\langle H_b f, \overline{p}\rangle_{\overline{H^2_d}} = \langle pf, b\rangle_{H^2_d} = \langle f, p(M^*_z)b\rangle_{H^2_d}$$

where for a multiindex $\alpha = (\alpha_1, ..., \alpha_d)$, we define $(M_z^*)^\alpha = \prod_{i=1} M^*_{z_i^{\alpha_i}}$. The result now easily follows by noting that elements of the form $p(M_z^*)b$ are dense in $[b]_*$. $\square$

Now, let $d \geq 2$ and let

$$\mathcal{M} = \left\{ f \in H_d^2 \mid f(0) = \frac{\partial f}{\partial z_i}(0) = 0 \text{ for } i = 1, ..., d \right\}$$

By considering the Taylor series expansion of f about the origin, it is easy to see that if $f \in \mathcal{M}$, then $z_i f \in \mathcal{M}$ for $i = 1, ..., d$ and therefore $\mathcal{M} \in \mathrm{Lat}(M_z, H_d^2)$. By the orthogonality of the monomials in H_d^2, we have that

$$\mathcal{M}^\perp = \left\{ a + \sum_{i=1}^{d} b_i z_i \mid a, b_1, ..., b_d \in \mathbb{C} \right\}$$

so that dim $\mathcal{M}^\perp = d + 1$. An elementary calculation using the orthogonality of the monomials in H_d^2, shows that $M^*_{z_i}\left(a + \sum_{i=1}^d b_i z_i\right) \in \mathbb{C}$, so that for any $b \in \mathcal{M}^\perp$, we have that dim $[b]_* \leq 2$. Therefore when $d \geq 2$, there is no $b \in \mathcal{X}(H_d^2)$, which satisfies $\mathcal{M} = [b]_*^\perp$. Hence by Proposition 5, we see that $\mathcal{M}$ is not the kernel of a Hankel operator.

4.2 Characterization of Invariant Subspaces of M_z for H_d^2

Although there are invariant subspaces of M_z on H_d^2 which are not kernels of Hankel operators for $d \geq 2$, it turns out that every such invariant subspace is an at most countable *intersection* of kernels of Hankel operators. In order to find the symbols of these Hankel operators, we rely on the theory of complete Nevanlinna Pick kernels.

A reproducing kernel l on $\mathbb{B}_d$ is called a **complete Nevanlinna Pick (CNP) kernel** if $l_0(z) = 1$ for all $z \in \mathbb{B}_d$ and

$$1 - \frac{1}{l_\lambda(z)} \gg 0.$$

Although we are going to focus on one result from the field, the theory of CNP kernels is a fruitful, well-established area of mathematics, laden with wonderful and useful results. One notes that for the Drury-Arveson kernel k_λ that

$$1 - \frac{1}{k_\lambda(z)} = \langle z, \lambda \rangle \gg 0$$

so that k_λ is a CNP kernel. The following result of McCullough and Trent gives a characterization of the projections onto multiplier invariant subspaces of reproducing kernel Hilbert spaces with a CNP kernel.

Theorem 4.2 (from [25])**.** *Let k be a complete Nevanlinna Pick kernel, and let $\mathcal{M}$ be a multiplier invariant subspace of $\mathcal{H}(k)$. Then there is a sequence $\{\varphi_n\} \subseteq \mathcal{M}(\mathcal{H}(k)) \cap \mathcal{M}$ which satisfy*

$$P_{\mathcal{M}} = \sum_n M_{\varphi_n} M_{\varphi_n}^*$$

where the convergence is in the strong operator topology, and $P_{\mathcal{M}}$ denotes the projection onto $\mathcal{M}$.

The following lemma will be a useful tool in proving that certain symbols induce bounded Hankel operators.

Lemma 4.2.1. *Let $\mathcal{M} \in Lat(M_z, H_d^2)$. Then the function*

$$l_\lambda(z) = \frac{(P_{\mathcal{M}} k_\lambda)(z)}{k_\lambda(z)}$$

is positive definite and $\mathcal{H}(l) \subseteq \mathcal{M}(H_d^2)$. In particular $l_\lambda \in \mathcal{M}(H_d^2)$ for all $\lambda \in \mathbb{B}_d$.

Proof. By Theorem 4.2, there is a sequence $\{\varphi_n\} \subseteq \mathcal{M}(H^2_d) \cap \mathcal{M}$ which satisfies

$$P_{\mathcal{M}} = \sum_n M_{\varphi_n} M^*_{\varphi_n}$$

Applying this relation to the reproducing kernels k_λ, and using that $M^*_\varphi k_\lambda = \overline{\varphi(\lambda)} k_\lambda$ we obtain

$$P_{\mathcal{M}} k_\lambda = \sum_n \overline{b_n(\lambda)} b_n k_\lambda.$$

Therefore we conclude that the function

$$l_\lambda(z) = \frac{(P_{\mathcal{M}} k_\lambda)(z)}{k_\lambda(z)} = \sum_n \overline{b_n(\lambda)} b_n(z)$$

is positive definite and therefore we can construct $\mathcal{H}(l)$. If $\{e_n\}_n$ is an orthonormal basis of $\mathcal{H}(l)$, then

$$l_\lambda = \sum_n \langle l_\lambda, e_n \rangle_{\mathcal{H}(l)} e_n = \sum_n \overline{e_n(\lambda)} e_n.$$

Now, for $n \in \mathbb{N}$ and $\lambda \in \mathbb{B}_d$, set $T_n k_\lambda = \overline{e_n(\lambda)} k_\lambda$ and extend T_n onto the set $\mathcal{D}$ of finite linear combinations of reproducing kernels k_λ. Since

$$\begin{aligned}
\sum_n \langle T_n k_\lambda, T_n k_z \rangle &= \sum_n \overline{e_n(\lambda)} e_n(z) k_\lambda(z) \\
&= l_\lambda(z) k_\lambda(z) \\
&= (P_{\mathcal{M}} k_\lambda)(z) \\
&= \langle P_{\mathcal{M}} k_\lambda, k_z \rangle \\
&= \langle P_{\mathcal{M}} k_\lambda, P_{\mathcal{M}} k_z \rangle
\end{aligned}$$

we conclude for any $f \in \mathcal{D}$ we have $\sum_n \|T_n f\|^2 = \|P_{\mathcal{M}} f\|^2$. Therefore each T_n extends to be bounded on all of $\mathcal{H}(k)$. Furthermore, for every $g \in \mathcal{H}(k)$, we have that

$$\begin{aligned}(T_n^* g)(\lambda) &= \langle T_n^* g, k_\lambda \rangle \\ &= e_n(\lambda)\langle g, k_\lambda \rangle \\ &= e_n(\lambda) g(\lambda) \\ &= (M_{e_n} g)(\lambda)\end{aligned}$$

so that $T_n = M_{e_n}$ and hence $e_n \in \mathcal{M}(k)$. Given an arbitrary $0 \neq f \in \mathcal{H}(l)$, we can find an orthonormal basis of $\mathcal{H}(l)$ which contains f (with possible renormalization), and thus the proof is complete. □

Theorem 4.3. *Let $d \in \mathbb{N}$ and let $(0) \neq \mathcal{M} \in Lat(M_z, H^2_d)$. Then the exists a sequence $\{b_n\}_{n=1}^\infty \subseteq \mathcal{X}(H^2_d)$ which satisfies*

$$\mathcal{M} = \bigcap_{n=1}^{\infty} ker\ H_{b_n}.$$

Proof. Given a countable dense subset $\{\lambda_n\}_n \subseteq \mathbb{B}_d$, the set $\{k_{\lambda_n}\}$ is dense in H^2_d. Therefore, we have that

$$\mathcal{M}^\perp = \bigvee_n [P_{\mathcal{M}^\perp} k_{\lambda_n}]_*$$

and hence

$$\mathcal{M} = \bigcap_n [P_{\mathcal{M}^\perp} k_{\lambda_n}]_*^\perp.$$

By Proposition 3.2, if we can show that $P_{\mathcal{M}^\perp} k_\lambda \in \mathcal{M}(H^2_d)$, then we are done. However, we have that

$$P_{\mathcal{M}^\perp} k_\lambda = k_\lambda - \frac{P_{\mathcal{M}} k_\lambda}{k_\lambda} k_\lambda \in \mathcal{M}(H^2_d) \subseteq \mathcal{X}(H^2_d)$$

by Lemma 4.2.1. □

The proof of the above theorem does not rely on the norm of H^2_d. In fact, one easily sees how this result can generalize to reproducing kernel Hilbert spaces with a CNP kernel which satisfy the property that multipliers induce bounded Hankel forms on the space.

4.3 Cyclic Vectors in H^2_d

Let $X \subseteq \mathrm{Hol}(\mathbb{B}_d)$ be a Banach space such that M_z is bounded. For a function $f \in X$, let $[f]$ be the smallest member of $\mathrm{Lat}(M_z, X)$ which contains f. If the polynomials are dense in X, one easily checks that

$$[f] = \overline{\{pf \mid p \text{ is a polynomial}\}}.$$

We will say that f is cyclic if $[f] = X$. Note that the polynomials are dense in the space X if and only if 1 is cyclic for X.

In order to aid our discussion of cyclic vectors, we will use the weakly-factored space defined in Section 3.2. Furthermore, we will use the notation

$$H^1_d = H^2_d \odot H^2_d.$$

and for a subspace $\mathcal{M} \subseteq H^2_d$, we will write $\mathrm{clos}_{H^1_d}\mathcal{M}$ for the closure of $\mathcal{M}$ in H^1_d. Our first result establishes that cyclicity in H^2_d is equivalent to cyclicity in H^1_d.

Lemma 4.3.1. *Let $\mathcal{M} \in Lat(M_z, H^2_d)$. Then*

$$\mathcal{M} = H^2_d \cap clos_{H^1_d}\mathcal{M}.$$

Consequently, a function $f \in H^2_d$ is cyclic in H^2_d if and only if it is cyclic in H^1_d.

Proof. It is clear that $\mathcal{M} \subseteq H^2_d \cap \mathrm{clos}_{H^1_d}\mathcal{M}$. If $f \notin \mathcal{M}$, then by Theorem 4.3 there is a $b \in \mathcal{X}(H^2_d)$ such that $\mathcal{M} \subseteq \ker H_b$ and $f \notin \ker H_b$. Thus there is a multiindex

α such that $\langle H_b f, \overline{z^\alpha}\rangle_{\overline{H^2_d}} \neq 0$, otherwise we would have $H_b f = 0$. If $f \in \text{clos}_{H^1_d}\mathcal{M}$, then we futher have that there are $f_n \in \mathcal{M}$ such that $f_n \to f$ in H^1_d. This implies that $z^\alpha f_n \to z^\alpha f$ in H^1_d. Hence $0 = \langle H_b f_n, \overline{z^\alpha}\rangle_{\overline{H^2_d}} \to \langle H_b f, \overline{z^\alpha}\rangle_{\overline{H^2_d}} \neq 0$ by Theorem 3.1. This contradiction shows that $f \notin \text{clos}_{H^1_d}\mathcal{M}$ and the result follows. $\square$

Now that we have established a connection between the weakly factored space H^1_d and cyclicity, we have a tool that can be used to prove cyclicity results regarding products of functions in H^2_d. The Drury-Arveson space is *not* an algebra; however, there are nontrivial ways (i.e. more than just being multipliers) in which a product of two H^2_d functions can again be in H^2_d.

For instance, it turns out that $H^2_d \cap H^\infty$ is an algebra which is contained in H^2_d. If $f, g \in H^2_d \cap H^\infty$, then it is clear that $fg \in H^\infty$. Furthermore, if one takes $n \in \mathbb{N}$ such that $2n - d > -1$, then we have that

$$R^{2n}(fg) = \sum_{k=0}^{2n} \binom{2n}{k} R^k f R^{2n-k} g \in H_{4n+1}.$$

so that $fg \in H^2_d$. To verify this, it suffices to show that $R^k f R^{2n-k} \in H_{4n+1}$, and by symmetry, it is enough to show this for $0 \leq k \leq n$. Since $f \in H^\infty$, we have that f is in the Bloch space $\mathcal{B}$ and therefore for every $k \in \mathbb{N}$, we have that

$$\sup_{z \in \mathbb{B}_d} |R^k f(z)|(1 - |z|^2)^k < \infty.$$

Thus $\|R^k f R^{2n-k} g\|_{4n+1} \lesssim \|R^{2n-k} g\|_{2(2n-k)+1} \approx \|g\|_1$. Here we have used that $2(2n - k)+1 \geq 2n+1 > d$, which implies that all of the norms in the previous inequalities are actually Bergman norms, so that we may just use the Bloch condition as a pointwise estimate.

As another example, we have that if $f, g \in \mathcal{C}H^2_d$, then $fg \in H^2_d$. To see this, we have

$$\|R(fg)\|_3 \leq \|gRf\|_3 + \|fRg\|_3 \lesssim \|f\|_1 + \|g\|_1 < \infty$$

so that $\|fg\|_1 < \infty$.

Theorem 4.4. *Let $f, g \in H^2_d$.*

(a) If $fg \in H^2_d$, then $fg \in [f] \cap [g]$.

(b) If $fg \in H^2_d$ and if f is cyclic in H^2_d, then $[fg] = [g]$.

(c) If $fg \in H^2_d$, then fg is cyclic in H^2_d, if and only if both f and g are cyclic in H^2_d.

(d) If $f, 1/f \in H^2_d$, then f is cyclic in H^2_d.

Proof. (a) Suppose that f, g and $fg \in H^2_d$. Since the polynomials are dense in H^2_d, there is a sequence p_n of polynomials such that $p_n \to f$ in H^2_d. Then $\|p_n g - fg\|_* \leq \|p_n - f\|_{H^2_d} \|g\|_{H^2_d}$, hence $p_n g \to fg$ in H^1_d. Thus

$$fg \in H^2_d \cap \operatorname{clos}_{H^1_d}[g] = [g].$$

Similarly $fg \in [f]$.

(b) Now additionally suppose that f is cyclic in H^2_d. By (a) we have $[fg] \subseteq [g]$ and thus it suffices to show that $g \in [fg]$. Since f is cyclic, there is a sequence of polynomials p_n such that $p_n f \to 1$ in H^2_d. Then as in part (a) of the proof it follows that $p_n fg \to g$ in H^1_d. Hence $g \in H^2_d \cap \operatorname{clos}_{H^1_d}[fg] = [fg]$, again by Theorem 4.3.1.

(c) If fg is cyclic in H^2_d, then by (a) $H^2_d = [fg] \subseteq [f] \cap [g]$. Hence both f and g must be cyclic. Conversely, if both f and g are cyclic, then by (b) we have that $[fg] = [g] = H^2_d$.

(d) follows from (c) by taking $g = 1/f$. □

4.4 Functions that are bounded below

In view of part (d) of Theorem 4.4, it is a natural question to ask under what conditions on a function $f \in H^2_d$ can one guarantee that $\frac{1}{f} \in H^2_d$. If $f \in \mathcal{M}(H^2_d)$

and $\frac{1}{f} \in H^\infty$, then the Corona theorem for $\mathcal{M}(H^2_d)$ implies that $\frac{1}{f} \in \mathcal{M}(H^2_d)$. Along these lines, we have the following theorem.

Theorem 4.5. *Let $\gamma > 0$. If $f \in H_\gamma \cap \mathcal{B}$ and $\frac{1}{f} \in H^\infty$, then $\frac{1}{f} \in H_\gamma$.*

The proof of this theorem will rely on the equivalence that $f \in H_\gamma$ if and only if $R^n f \in H_{\gamma+2n}$ for some (and hence all) $n \in \mathbb{N}$. Therefore will need to work with $R^m\left(\frac{1}{f}\right)$ for potentially large m, and for this, we will exploit Faa di Bruno's formula for the higher order derivatives of a composition.

Let $f \in \mathrm{Hol}(\mathbb{B}_d)$ and $m \in \mathbb{N}$. We define A_m to be the set of all m-tuples $\eta = (\eta_1, ..., \eta_m)$ which satisfy $\sum_{i=1}^m i\eta_i = m$, and we write

$$T_\eta(f) = \prod_{i=1}^m (R^i f)^{\eta_i}.$$

Then Faa di Bruno's formula (see [20]) gives

$$R^m\left(\frac{1}{f}\right) = \sum_{\eta \in A_m} \frac{m!}{\eta!} \frac{(-1)^{|\eta|}|\eta|!}{f^{|\eta|+1}} \prod_{j=1}^m \left(\frac{1}{j!}\right)^{\eta_j} T_\eta(f).$$

Our approach will be to take m large enough so that $\mathcal{H}_{\gamma+2m}$ is a weighted Bergman space and estimate the norm of each term in the sum individually. Furthermore, since the multipliers for the weighted Bergman spaces are those functions which lie in H^∞, to prove Theorem 4.5 it suffices to prove the following lemma.

Lemma 4.5.1. *Let $\gamma > 0$ and $m \in \mathbb{N}$. If $f \in \mathcal{H}_\gamma \cap \mathcal{B}$, then*

$$\|T_\eta(f)\|_{\gamma+2m} < \infty$$

for any m-tuple $\eta = (\eta_1, ..., \eta_m) \in A_m$.

Proof. Fix $\eta = (\eta_1, ..., \eta_m) \in A_m$. For $j \in \mathbb{N}$, let B_j be those $|\eta|$-tuples $\beta = (\beta_1, ..., \beta_{|\eta|})$ which satisfy $|\beta| = j$. Then by writing powers as products of single terms, one sees that there is a function $g : \{1, ..., |\eta|\} \to \{1, ..., m\}$ which satisfies $T_\eta(f) = \prod_{i=1}^{|\eta|} R^{g(i)} f$. Note that this implies that $\sum_{i=1}^{|\eta|} g(i) = \sum_{i=1}^{m} i\eta_i = m$.

Now choose $j \geq \frac{d|\eta|}{2}$, and for each $\beta \in B_j$ choose an index i_β such that $\beta_{i_\beta} \geq \frac{j}{|\eta|} \geq d/2$. This is possible since $|\beta| = j$. Then $2(\beta_{i_\beta} + g(i_\beta)) + \gamma > d$. Since $f \in \mathcal{B}$, for any $n \in \mathbb{N}$ we have that $(1 - |z|^2)^n |R^n f(z)|$ is bounded in $\mathbb{B}_d$ (see Proposition 2), hence there is a $C > 0$ such that

$$(1 - |z|^2)^{(j+m)} \prod_{i=1}^{|\eta|} |R^{\beta_i + g(i)} f(z)| \leq C(1 - |z|^2)^{\beta_{i_\beta} + g(i_\beta)} |R^{\beta_{i_\beta} + g(i_\beta)} f(z)|.$$

Here we also used that $\sum_{i=1}^{|\eta|} \beta_i + g(i) = j + m$.

Finally, by the Leibniz rule, we have that

$$\begin{aligned} \|T_\eta(f)\|_{\gamma+2m} &\approx \|R^j T_\eta(f)\|_{\gamma+2(m+j)} \\ &= \left\| \sum_{\beta \in B_j} \frac{j!}{\beta!} \prod_{i=1}^{|\eta|} R^{\beta_i + g(i)} f \right\|_{\gamma+2(m+j)} \\ &\lesssim \sum_{\beta \in \beta_j} \left\| \prod_{i=1}^{|\eta|} R^{\beta_i + g(i)} f \right\|_{\gamma+2(m+j)} \\ &\lesssim \sum_{\beta \in \beta_j} \left\| R^{\beta_{t_\beta} + g(t_\beta)} f \right\|_{\gamma+2(\beta_{t_\beta} + g(t_\beta))} \end{aligned}$$

and the result follows since $f \in \mathcal{H}_\gamma$. □

Combining Theorem 4.4 and Theorem 4.5, we obtain the following result.

Theorem 4.6. *If $f \in H_d^2 \cap \mathcal{B}$, and if there is $c > 0$ such that $|f(z)| \geq c$ for all $z \in \mathbb{B}_d$, then $\frac{1}{f} \in H_d^2$ and f is cyclic in H_d^2.*

As mentioned earlier, we have that in the special case of the previous theorem where $f \in \mathcal{M}(H_d^2)$ and $1/f \in H^\infty$, then by the one function case of Corona theorem for $\mathcal{M}(H_d^2)$ one has $1/f \in \mathcal{M}(H_d^2)$. That result was also proved by Fang and Xia in

[14]. The next theorem establishes the same conclusion in the context of H_γ, $\gamma > 0$, and without assuming that f be bounded. The proof is also significantly shorter than the one found in [14].

Lemma 4.6.1. *If $0 < \gamma < \beta$, then $\mathcal{CH}_\gamma \subseteq \mathcal{CH}_\beta$.*

Proof. Set $\varepsilon = \beta - \gamma > 0$ and let $b \in \mathcal{CH}_\gamma$. By Theorem 2.2 with $n = 1$ this implies that $M_{Rb} : H_\gamma \to H_{\gamma+2}$ is bounded. By Proposition 3, this is equivalent to the existence of $C > 0$ such that $Ck_w^{\gamma+2}(z) - Rb(z)\overline{Rb(w)}k_w^\gamma(z)$ is positive definite. We multiply this by $k_w^\varepsilon(z)$ and apply Theorem 2.1 to obtain that $Ck_w^{\beta+2}(z) - Rb(z)\overline{Rb(w)}k_w^\beta(z)$ is positive definite. This implies that $b \in \mathcal{CH}_\beta$. □

Theorem 4.7. *If $f \in \mathcal{CH}_\gamma$, and if there is a constant $c > 0$ such that $|f(z)| \geq c$ for all $z \in \mathbb{B}_d$, then $\frac{1}{f}$ is a multiplier for $\mathcal{H}_\gamma$.*

Proof. If $f \in \mathcal{CH}_\gamma$, then by Lemma 4.6.1 we have that $f \in \mathcal{CH}_{\gamma+n}$ for all $n \in \mathbb{N}_0$. Since for large n the space $H_{\gamma+n}$ is a weighted Bergman space, the hypothesis implies that $1/f$ is a multiplier of $H_{\gamma+n+2}$ for sufficiently large n. Thus the theorem will follow inductively from the claim: If $f \in \mathcal{CH}_{\gamma+n}$ and if $1/f$ is a multiplier of $H_{\gamma+n+2}$, then $1/f$ is a multiplier of $H_{\gamma+n}$.

Let $g \in H_{\gamma+n}$. If $\frac{1}{f} \in H_{\gamma+n}$ and $\frac{1}{f} \in \mathcal{M}(H_{\gamma+n+2})$, then

$$\begin{aligned}
\left\|\frac{g}{f}\right\|_{\gamma+n} &\approx \left\|R\left(\frac{g}{f}\right)\right\|_{\gamma+n+2} \\
&\leq \left\|\frac{Rg}{f}\right\|_{\gamma+n+2} + \left\|\frac{gRf}{f^2}\right\|_{\gamma+n+2} \\
&\lesssim \|Rg\|_{\gamma+n+2} + \|gRf\|_{\gamma+n+2} \\
&\lesssim \|g\|_{\gamma+n}.
\end{aligned}$$

Here the last inequality followed from Theorem 2.3. □

Chapter 5

Vector Valued Invariant Subspaces

In this chapter, we adapt Theorem 4.3 into a vector-valued situation. Most of the proofs in this chapter are analogous to one variable results from earlier in this dissertation. We will prove the results here for completeness and to point out the changes that must be made.

5.1 Definitions

Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ be a Hilbert space and let $\mathcal{E}$ be an arbitrary separable Hilbert space. We can then construct the space $\mathcal{H} \otimes \mathcal{E}$ in the standard manner, but we will think of this space as a **vector valued space of analytic functions**. In particular, if $f \in \mathcal{H}$ and $x \in \mathcal{E}$, then for a point $\lambda \in \mathbb{B}_d$, we will define point evaluation on an elementary tensor of $\mathcal{H} \otimes \mathcal{E}$ by

$$(f \otimes x)(\lambda) = f(\lambda)x. \tag{5.1}$$

If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$ and $\{f_m\}$ is an orthonormal basis for $\mathcal{E}$, then $\{e_n \otimes f_m\}$ forms an orthonormal basis for $\mathcal{H} \otimes \mathcal{E}$. Then, for a general element $F \in \mathcal{H} \otimes \mathcal{E}$, we have that

$$F = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle F, e_n \otimes f_m \rangle e_n \otimes f_m$$

and therefore by Equation 5.1, we have that

$$F(\lambda) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle F, e_n \otimes f_m\rangle e_n(\lambda) f_m. \tag{5.2}$$

Now if $F \in \mathcal{H} \otimes \mathcal{E}$, then for every $x \in \mathcal{E}$, we can define a scalar valued function $F_x \in \mathrm{Hol}(\mathbb{B}_d)$ by

$$F_x(\lambda) = \langle F(\lambda), x\rangle.$$

In fact, by the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \|F_x\|^2 &= \sum_{n=0}^{\infty}\left|\sum_{m=0}^{\infty}\langle F, e_n \otimes f_m\rangle\langle f_m, x\rangle\right|^2 \\ &\leq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}|\langle F, e_n \otimes f_m\rangle|^2\right)\left(\sum_{m=0}^{\infty}|\langle f_m, x\rangle|^2\right) \\ &= \|F\|^2\|x\|^2 \end{aligned}$$

so that $F_x \in \mathcal{H}$.

If $\{e_n\}$ and $\{f_m\}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{E}$, respectively, then by Equation 5.2, we have that

$$F(\lambda) = \sum_{m=0}^{\infty} F_{f_m}(\lambda) f_m$$

and furthermore that

$$\|F\|^2 = \sum_{m=0}^{\infty}\|F_{f_m}\|^2.$$

If we further assume that $\mathcal{H}$ is a reproducing kernel space with reproducing kernels l_λ, then $\mathcal{H} \otimes \mathcal{E}$ is a reproducing kernel Hilbert space with reproducing kernels $l_\lambda \otimes x$ (with $x \in \mathcal{E}$). If $f \in \mathcal{H}$, $x, y \in \mathcal{E}$, and $\lambda \in \mathbb{B}_d$, then

$$\langle f \otimes y, l_\lambda \otimes x\rangle = \langle f, l_\lambda\rangle\langle y, x\rangle = \langle f(\lambda)y, x\rangle = (f \otimes y)_x(\lambda).$$

It thus follows that $F_x(\lambda) = \langle F, l_\lambda \otimes x\rangle$ for arbitrary $F \in \mathcal{H} \otimes \mathcal{E}$.

Furthermore, as before, we have the set

$$\left\{ \sum_{i=1}^{n} \alpha_i k_{\lambda_i} \otimes x_i \mid \alpha_i \in \mathbb{C}, \lambda_i \in \mathbb{B}_d, x_i \in \mathcal{E} \right\}$$

is dense in $\mathcal{H} \otimes \mathcal{E}$. Again, using the reproducing property of these elements, the norm of an element in the above set is given by

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i k_{\lambda_i} \otimes x_i \right\|^2_{\mathcal{H}\otimes\mathcal{E}} &= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha_i \alpha_j} \langle k_{\lambda_i} \otimes x_i, k_{\lambda_j} \otimes x_j \rangle_{\mathcal{H}\otimes\mathcal{E}} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} k_{\lambda_i}(\lambda_j) \langle x_i, x_j \rangle_{\mathcal{E}}. \end{aligned}$$

This says that if we define the function $L : \mathbb{B}_d \times \mathcal{E} \times \mathbb{B}_d \times \mathcal{E} \to \mathbb{C}$ given by $L(\lambda, x, z, y) = k_\lambda(z)\langle x, y\rangle$, then for any choice of $\lambda_1, ..., \lambda_n \in \mathbb{B}_d$ and $x_1, ..., x_n \in \mathcal{E}$, the matrix $(L(\lambda_i, x_i, \lambda_j, x_j))_{i,j=1}^{n}$ is positive definite. Any such function L which satisfies this property, we will call a **positive definite function** and we will write $L \gg 0$.

5.2 Multipliers on Vector Valued Reproducing Kernel Hilbert Spaces

Given a sequence of functions $B = \{b_n\} \subseteq \mathrm{Hol}(\mathbb{B}_d)$, one can study the collective behavior of how the elements of B act on a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(l)$ by multiplication. To this end, we will define the **column multiplication operator** $CM_B : \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}$ to be the operator given by

$$CM_B f = \sum_{n=0}^{\infty} b_n f \otimes e_n.$$

However, we can also define the **row multiplication operator** $RM_b : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H}$ to be the operator given by

$$RM_b \left(\sum_n f_n \otimes e_n \right) = \sum_n b_n f_n.$$

Both of these operators will be important for this chapter; however, we will focus on the operator CM_B, because the boundedness of CM_B will imply the boundedness of RM_B (see Theorem 5.1.1).

One notes that if CM_B is bounded, then for each $f \in \mathcal{H}$, we have that

$$\sum_{n=0}^{\infty} \|b_n f\|_{\mathcal{H}}^2 = \left\| \sum_{n=0}^{\infty} b_n f \otimes e_n \right\|_{\mathcal{H}\otimes\mathcal{E}}^2 \leq \|CM_b\| \|f\|_{\mathcal{H}}^2$$

so that each b_n is a multiplier for $\mathcal{H}$ and the operators M_{b_n} are uniformly bounded.

Given a sequence of functions $B = \{b_n\} \subseteq \mathrm{Hol}(\mathbb{B}_d)$, we will identify B with the function

$$B = \sum_n b_n \otimes e_n$$

so that, as in the previous section, we have that

$$B(\lambda) = \sum_n b_n(\lambda) e_n$$

and $B_x(\lambda) = \langle B(\lambda), x \rangle$ for every $x \in \mathcal{E}$. Now, if $\mathcal{L}^1 = \mathcal{H}(l^1), \mathcal{L}^2 = \mathcal{H}(l^2) \subseteq \mathrm{Hol}(\mathbb{B}_d)$ are reproducing kernel Hilbert spaces, then for the operator $CM_B : \mathcal{L}^1 \to \mathcal{L}^2 \otimes \mathcal{E}$

(defined in the analogous way), we have

$$\begin{aligned}
\langle CM_B^*(l_\lambda^2 \otimes x), g\rangle_{\mathcal{L}^2} &= \langle l_\lambda^2 \otimes x, CM_B g\rangle_{\mathcal{L}^2\otimes\mathcal{E}} \\
&= \left\langle l_\lambda^2 \otimes x, \sum_n b_n g \otimes e_n \right\rangle_{\mathcal{L}^2\otimes\mathcal{E}} \\
&= \sum_n \langle l_\lambda^2, b_n g\rangle_{\mathcal{L}^2} \langle x, e_n\rangle_{\mathcal{E}} \\
&= \sum_n \overline{b_n(\lambda) g(\lambda)} \langle x, e_n\rangle_{\mathcal{E}} \\
&= \overline{g(\lambda) B_x(\lambda)} \\
&= \langle \overline{B_x(\lambda)} l_\lambda^1, g\rangle_{\mathcal{L}^1}.
\end{aligned}$$

Therefore $CM_B^*(l_\lambda^2 \otimes x) = \overline{B_x(\lambda)} l_\lambda^1$, as in the discussion preceeding Proposition 3. Furthermore, we have the following:

Proposition 6. *Let $\mathcal{L}^1 = \mathcal{H}(l^1), \mathcal{L}^2 = \mathcal{H}(l^2) \subseteq Hol(\mathbb{B}_d)$ be reproducing kernel Hilbert spaces. Then the operator $CM_B : \mathcal{L}^1 \to \mathcal{L}^2 \otimes \mathcal{E}$ is bounded if and only if there is a constant $C > 0$ such that*

$$\overline{B_x(\lambda)} B_y(z) l_\lambda^1(z) << C l_\lambda^2(z) \langle x, y\rangle$$

holds. Furthermore $\|CM_B\|^2$ is equal to the infimum over all possible C's that satisfy the above condition.

Proof. By the density of linear combinations of elements of the form $l_\lambda^2 \otimes x$ in $\mathcal{L}^2 \otimes \mathcal{E}$, we have that CM_B is bounded if and only if there is a constant $C > 0$ such that

$$\left\| CM_B^* \left(\sum_{i=1}^n \alpha_i l_\lambda^2 \otimes x_i \right) \right\|^2 \leq C \left\| \sum_{i=1}^n \alpha_i l_\lambda^2 \otimes x_i \right\|^2 \tag{5.3}$$

for any choice of $\alpha_1, ..., \alpha_n \in \mathbb{C}$, $\lambda_1, ..., \lambda_n \in \mathbb{B}_d$, and $x_1, ..., x_n \in \mathcal{E}$. By the discussion preceding this lemma, we have that

$$\begin{aligned} \left\| CM_B^* \left(\sum_{i=1}^n \alpha_i l_{\lambda_i}^2 \otimes x_i \right) \right\|^2 &= \left\| \sum_{i=1}^n \alpha_i \overline{B_{x_i}(\lambda_i)} l_{\lambda_i}^2 \right\|^2 \\ &= \sum_{i,j=1}^n \alpha_i \overline{\alpha_j} \overline{B_{x_i}(\lambda_i)} B_{x_j}(\lambda_j) l_{\lambda_i}(\lambda_j). \end{aligned}$$

Furthermore, we have that

$$\left\| \sum_{i=1}^n \alpha_i l_{\lambda_i} \otimes x_i \right\|^2 = \sum_{i,j=1}^\infty \alpha_i \overline{\alpha_j} l_{\lambda_i}(\lambda_j) \langle x_i, x_j \rangle$$

and the result now follows by virtue of the inequality 5.3. □

Recall the definition of the reproducing kernel Hilbert Space $\mathcal{H}_\gamma$ for $\gamma > 0$ given in Section 2.2. We have the following analog of Lemma 2.1.5.

Proposition 7. *Let $\gamma, \eta > 0$ and suppose that $CM_B : \mathcal{H}_\gamma \to \mathcal{H}_\eta \otimes \mathcal{E}$ is bounded. Then for any $\epsilon > 0$, we have that $CM_B : \mathcal{H}_{\gamma+\epsilon} \to \mathcal{H}_{\eta+\epsilon} \otimes \mathcal{E}$ is bounded with*

$$\|CM_B\|_{\mathcal{B}(\mathcal{H}_{\gamma+\epsilon}, \mathcal{H}_{\eta+\epsilon} \otimes \mathcal{E})} \leq \|CM_B\|_{\mathcal{B}(\mathcal{H}_\gamma, \mathcal{H}_\eta \otimes \mathcal{E})}$$

Proof. By Proposition 6, we have that if $CM_B : \mathcal{H}_\gamma \to \mathcal{H}_\eta \otimes \mathcal{E}$ is bounded, then there is a constant $C > 0$ such that

$$\frac{\overline{B_x(\lambda)} B_y(z)}{(1 - \langle z, \lambda \rangle)^\gamma} << C \frac{\langle x, y \rangle}{(1 - \langle z, \lambda \rangle)^\eta}. \tag{5.4}$$

Since $\frac{1}{(1 - \langle z, \lambda \rangle)^\epsilon} \gg 0$, Theorem 2.1 implies that

$$\frac{\overline{B_x(\lambda)} B_y(z)}{(1 - \langle z, \lambda \rangle)^{\gamma+\epsilon}} << C \frac{\langle x, y \rangle}{(1 - \langle z, \lambda \rangle)^{\eta+\epsilon}}.$$

The result follows by another application of Proposition 6. □

For a sequence $B = \{b_n\} \subseteq \mathrm{Hol}(\mathbb{B})$ and a positive m, define $R^m B = \{R^m b_n\}_{n=1}^{\infty}$. Then we have the following analog of an implication from Theorem 2.2. We note that we do not prove a full analog of Theorem 2.2 because we do not have a vector-valued analog of Proposition 2.

Theorem 5.1. *Suppose that there exists and $m \in \mathbb{N}_0$ such that the operator $CM_{R^m B} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2m} \otimes \mathcal{E}$ is bounded (Here R^0 is just the identity operator). Then for all $k \in \mathbb{N}$ with $k \geq m$, we have that the operator $CM_{R^m B} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2k} \otimes \mathcal{E}$ is bounded.*

Proof. Fix $m \in \mathbb{N}$ and suppose that $M_{R^m B} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2m} \otimes \mathcal{E}$ is bounded. By Proposition 7, we have that $M_{R^m b} : \mathcal{H}_{\gamma+2} \to \mathcal{H}_{\gamma+2(m+1)} \otimes \mathcal{E}$ is bounded. By the product rule, we have that $fR^{m+1}b_n = R(fR^m b_n) - RfR^m b_n$, and therefore

$$\begin{aligned}
\|CM_{R^{m+1}B} f\|^2_{\mathcal{H}_{\gamma+2(m+1)}\otimes\mathcal{E}} &= \sum_{n=0}^{\infty} \|fR^{m+1}b_n\|^2_{\mathcal{H}_{\gamma+2(m+1)}} \\
&\lesssim \sum_{n=0}^{\infty} \|R(fR^m b_n)\|^2_{\mathcal{H}_{\gamma+2(m+1)}} + \sum_{n=0}^{\infty} \|RfR^m b_n\|^2_{\mathcal{H}_{\gamma+2(m+1)}} \\
&\lesssim \sum_{n=0}^{\infty} \|fR^m b_n\|^2_{\mathcal{H}_{\gamma+2m}} + \|Rf\|^2_{\mathcal{H}_{\gamma+2}} \\
&\lesssim \|f\|^2_\gamma.
\end{aligned}$$

This implies that $M_{R^{m+1}b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2(m+1)} \otimes \mathcal{E}$ is bounded. In fact, by induction, we actually have that $M_{R^{m+k}b} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+2(m+k)} \otimes \mathcal{E}$ is bounded for all $k \in \mathbb{N}$. The result now follows. □

For any sequence $B = \{b_n\}$ which satisfies that the operator $CM_{RB} : \mathcal{H}_1 \to \mathcal{H}_3 \otimes \mathcal{E}$ is bounded, we will write $B \in CC(H^2_d \otimes \mathcal{E})$, because this is a column Carleson condition. It follows from a proof of Trent, that that the column Carleson condition implies the analogous row Carleson measure condition. We include the proof here for completeness.

Lemma 5.1.1. *(see [36]). Let $\eta > -1$ and let $\mathcal{H} \subseteq Hol(\mathbb{B}_d)$ be a Banach Space. Then the inequality*

$$\sum_n \|fb_n\|^2_{\mathcal{H}_{d+1+\eta}} \lesssim \|f\|^2$$

implies the inequality

$$\left\|\sum_n f_n b_n\right\|^2_{\mathcal{H}_{d+1+\eta}} \lesssim \sum_n \|f_n\|^2.$$

In particular, we have that the boundedness of $CM_B : \mathcal{H} \to \mathcal{H}_{d+1+\eta} \otimes \mathcal{E}$ implies the boundedness of $RM_B : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H}_{d+1+\eta}$.

Proof. By a clever application of the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
\left\|\sum_n f_n b_n\right\|^2_{\mathcal{H}_{d+1+\eta}} &= c_\eta \int_{\mathbb{B}_d} \left|\sum_n f_n(z) b_n(z)\right|^2 (1-|z|^2)^\eta dV(z) \\
&= c_\eta \int_{\mathbb{B}_d} \sum_{n,m} f_n(z)\overline{b_m(z) f_m(z)} b_n(z)(1-|z|^2)^\eta dV(z) \\
&\leq \int_{\mathbb{B}_d} \left(\sum_{n,m} |f_n(z) b_m(z)|^2\right)^{\frac{1}{2}} \left(\sum_{n,m} |f_m(z) b_n(z)|^2\right)^{\frac{1}{2}} (1-|z|^2)^\eta dV(z) \\
&= \sum_n \int_{\mathbb{B}_d} \sum_m |b_m(z) f_n(z)|^2 (1-|z|^2)^\eta dV(z) \\
&\lesssim \sum_n \|f_n\|^2.
\end{aligned}$$

□

5.3 Vector-Valued Hankel Operators

Given a sequence $B = \{b_n\} \subseteq H^2_d$, we define the **column Hankel operator** $CH_B : H^2_d \to \overline{H^2_d} \otimes \mathcal{E}$ by

$$CH_B f = \sum_n H_{b_n} f \otimes e_n$$

and the **row Hankel operator** $RH_B : H^2_d \otimes \mathcal{E} \to \overline{H^2_d}$ by

$$RH_B = \left(\sum_n f_n \otimes e_n \right) = \sum_n H_{b_n} f_n.$$

Here, for a symbol $\varphi \in H^2_d$, the operator $H_\varphi : H^2_d \to \overline{H^2_d}$ is the usual Hankel operator which satisfies the following inner product relationship:

$$\langle H_b f, \overline{g} \rangle_{\overline{H^2_d}} = \langle fg, b \rangle_{H^2_d}.$$

As previously mentioned, the column multiplication operator implies the boundedness of the row multiplication operator, but the converse is not true in general (see [36]). Our first lemma shows that the boundedness of the column Hankel operator and the row Hankel operator are equivalent, and furthermore, we obtain a vector valued version of Theorem 3.1.1.

Theorem 5.2. *The following are equivalent*

1. $CH_B : H^2_d \to \overline{H^2_d} \otimes \mathcal{E}$ *is bounded*

2. $RH_B : H^2_d \otimes \mathcal{E} \to \overline{H^2_d}$ *is bounded*

3. *There is an* $m \in \mathbb{N}$ *such that*

$$\left| \sum_{n=0}^{\infty} \langle fg, R^m b_n \rangle_{1+m} \right| \lesssim \|f\| \left(\sum_{n=0}^{\infty} \|g_n\|^2_{\mathcal{H}_1} \right)^{\frac{1}{2}}$$

4. *For all* $m \in \mathbb{N}$, *we have that*

$$\left| \sum_{n=0}^{\infty} \langle fg, R^m b_n \rangle_{1+m} \right| \lesssim \|f\| \left(\sum_{n=0}^{\infty} \|g_n\|^2_{\mathcal{H}_1} \right)^{\frac{1}{2}}$$

Proof. We have that

$$\begin{aligned}
\left\langle f, (CH_B)^* \left(\sum_m \overline{g_m} \otimes e_m \right) \right\rangle_{H^2_d} &= \sum_{m,n} \langle H_{b_n} f \otimes e_n, \overline{g_m} \otimes e_m \rangle_{\overline{H^2_d} \otimes \mathcal{E}} \\
&= \sum_n \langle H_{b_n} f, \overline{g_n} \rangle_{\overline{H^2_d}} \\
&= \sum_n \langle f g_n, b_n \rangle_{H^2_d} \\
&= \left\langle \sum_n H_{b_n} g_n, \overline{f} \right\rangle_{\overline{H^2_d}} \\
&= \left\langle f, \overline{RH_B \left(\sum_n g_n \otimes e_n \right)} \right\rangle_{H^2_d}.
\end{aligned}$$

It follows that 1 and 2 are equivalent.

Furthermore, by considering the third equality above, we have also shown here that CH_B (and hence RH_B) is bounded if and only if

$$\left| \sum_{n=0}^{\infty} \langle f g_n, b_n \rangle_1 \right| \lesssim \|f\|_1 \left(\sum_{n=0}^{\infty} \|g_n\|_1^2 \right)^{\frac{1}{2}}.$$

Therefore 1 implies 3 and 4 implies 1. If we can show that 3 and 4 are equivalent, we are done.

In the proof of Theorem 3.1.1, it was shown that for $\varphi, \psi, b \in H^2_d$ there is a constant $c > 0$ such that

$$\left| \langle \varphi\psi, R^m b \rangle_{m+1} - \frac{1}{m+1} \langle \varphi\psi, R^{m+1} b \rangle_{m+2} \right| \leq c \|\varphi\|_1 \|\psi\|_1 \|b\|_1. \tag{5.5}$$

Therefore, we have that

$$I := \left| \sum_{n=0}^{\infty} \langle f g_n, R^m b_n \rangle_{n+1} - \frac{1}{m+1} \sum_{n=0}^{\infty} \langle f g_n, R^{m+1} b_n \rangle_{m+1} \right|$$

satisfies

$$I \le c \sum_{n=0}^{\infty} \|f\|_1 \|g_n\|_1 \|b_n\|_1 \le \|f\|_1 \left(\sum_{n=0}^{\infty} \|g_n\|_1^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \|b_n\|_1^2 \right)^{\frac{1}{2}}$$

As in Theorem 3.1.1, it follows that 3 and 4 are equivalent. ☐

Theorem 5.3. *If $B \in \mathcal{CC}(H_d^2)$, then the operator $CH_B : \mathcal{H}_d^2 \otimes \mathcal{E} \to \overline{\mathcal{H}_d^2}$ is bounded.*

Proof. Let $m \in \mathbb{N}$ be large so that $2m+1 > d$. By Theorem 5.2, to show that boundedness of $CH_B : \mathcal{H}_d^2 \otimes \mathcal{E} \to \mathcal{H}_d^2$, it suffices to show that

$$\left| \sum_{n=0}^{\infty} \langle fg, R^{3m} b_n \rangle_{1+3m} \right| \lesssim \|f\| \left(\sum_{n=0}^{\infty} \|g_n\|_1^2 \right)^{\frac{1}{2}}.$$

We have that

$$\begin{aligned} \left| \sum_{n=0}^{\infty} \langle fg, R^{3m} b_n \rangle_{1+3m} \right| &= \left| \sum_{n=0}^{\infty} \langle R^{2m}(fg), R^m b_n \rangle_{1+3m} \right| \\ &\le \sum_{k=0}^{2m} \begin{pmatrix} 2m \\ k \end{pmatrix} \left| \sum_{n=0}^{\infty} \langle R^k f R^{2m-k} g, R^m b_n \rangle_{1+3m} \right| \end{aligned}$$

and it therefore suffices to get the appropriate boundedness on each of the inner summations.

Now, we note by a combination of Theorem 5.1 and Lemma 7, since $B \in \mathcal{CC}(H_d^2)$, we have that the operator $CM_{R^m B} : \mathcal{H}_k \to \mathcal{H}_{2m+k} \otimes \mathcal{E}$ is bounded for all $k, m \in \mathbb{N}$ This, by Proposition 5.1.1, implies that $RM_{R^m B} : \mathcal{H}_k \otimes \mathcal{E} \to \mathcal{H}_{2m+k}$ is bounded for all $k, m \in \mathbb{N}$. To this end, let

$$I_k = \left| \sum_{n=0}^{\infty} \langle R^k f R^{2m-k} g, R^m b_n \rangle_{1+3m} \right|$$

If $m \geq k$, then $2(2m-k)-d > -1$ and $2(m+k)-d > -1$ and therefore we have

$$
\begin{aligned}
I_k &\lesssim \sum_{n=0}^{\infty} \int_{\mathbb{B}_d} |R^k f R^{2m-k} g_n R^m b_n| (1-|z|^2)^{3m-d} dV(z) \\
&\leq \left(\sum_{n=0}^{\infty} \int_{\mathbb{B}_d} |R^k f R^m b_n|^2 (1-|z|^2)^{2(m+k)-d} dV(z) \right)^{\frac{1}{2}} \\
&\quad \left(\sum_{n=0}^{\infty} \int_{\mathbb{B}_d} |R^{2m-k} g_n|^2 (1-|z|^2)^{2(2m-k)-d} dV(z) \right)^{\frac{1}{2}} \\
&\lesssim \|R^k f\|_{2k+1} \left(\sum_{n=0}^{\infty} \|R^{2m-k} g_n\|_{2(2m-k)+1}^2 \right)^{\frac{1}{2}} \\
&\approx \|f\|_1 \left(\sum_{n=0}^{\infty} \|g_n\|_1^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

where have used the boundedness of the operator $CM_{R^m B} : \mathcal{H}_{1+2k} \to \mathcal{H}_{1+2(m+k)}$.

Now, if $k \geq m$, then $2k - d > -1$ and

$$
\begin{aligned}
I_k &\lesssim \int_{\mathbb{B}_d} |R^k f| \left| \sum_{n=0}^{\infty} R^{3m-k} g_n R^m b_n \right| (1-|z|^2)^{2m-d} dV(z) \\
&\leq \left(\int_{\mathbb{B}_d} |R^k f|^2 (1-|z|^2)^{2k-d} dV(z) \right)^{\frac{1}{2}} \\
&\quad \left(\int_{\mathbb{B}_d} \left| \sum_{n=0}^{\infty} R^{2m-k} g_n R^m b_n \right|^2 (1-|z|^2)^{2(3m-k)-d} dV(z) \right)^{\frac{1}{2}} \\
&\leq \|R^k f\|_{1+2k}^2 \left(\sum_{n=0}^{\infty} \|R^{2m-k} g_n\|_{1+2(2m-k)}^2 \right)^{\frac{1}{2}} \\
&\approx \|f\|_1 \left(\sum_{n=0}^{\infty} \|g_n\|_1^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

where have used the boundedness of the operator $RM_{R^m B} : \mathcal{H}_{1+2(3m-k)} \otimes \mathcal{E} \to \mathcal{H}_{1+2(2m-k)}$. The result follows.

□

5.4 Vector-Valued Hankel Forms

Now, if $B = \{b_n\}$ is a sequence such that $CH_B : \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}$ is bounded, then we would like to find a symbol which induces the operator (as in the scalar-valued situation).

If f is multiplier of $\mathcal{H}$, then we can extend $M_f : \mathcal{H} \to \mathcal{H}$ to the bounded operator $M_f : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H} \otimes \mathcal{E}$ given by

$$M_f \left(\sum_n g_n \otimes e_n \right) = \sum_n f g_n \otimes e_n$$

Then, given an element $\sum_n g_n \otimes e_n \in \mathcal{H} \otimes \mathcal{E}$ and a multiplier f, we have that

$$\begin{aligned}
\left\langle f \left(\sum_n g_n \otimes e_n \right), \sum_n b_n \otimes e_n \right\rangle &= \left\langle \sum_n f g_n \otimes e_n, \sum_n b_n \otimes e_n \right\rangle \\
&= \sum_n \langle f g_n, b_n \rangle \\
&= \sum_n \langle H_{b_n} f, \overline{g_n} \rangle \\
&= \left\langle CH_B f, \sum_n \overline{g_n} \otimes e_n \right\rangle
\end{aligned}$$

Thus we see that the associated symbol should be $B = \sum_n b_n \otimes e_n$.

For $B = \sum_n^\infty b_n \otimes e_n \in \mathcal{H} \otimes \mathcal{E}$, we will write $B \in \mathcal{X}(\mathcal{H} \otimes \mathcal{E})$ if $CH_{\{b_n\}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{E}$ is bounded. Instead of writing $CH_{\{b_n\}}$, we will simply write CH_B. Using the previous equations, we have that $B \in \mathcal{X}(\mathcal{H} \otimes \mathcal{E})$ if and only if

$$|\langle fg, B \rangle| \lesssim \|g\|_{\mathcal{H} \otimes \mathcal{E}} \|f\|_{\mathcal{H}}$$

for all mutlipliers f of $\mathcal{H}$ and $g \in \mathcal{H} \otimes \mathcal{E}$. We note that we generally consider spaces in which mulitplication by z is bounded and the polynomials are dense, and therefore we can just verify this inequality for all polynomials f.

For $B \in \mathcal{H} \otimes \mathcal{E}$, let $[B]_*$ denote the smallest closed subspace of $\mathcal{H} \otimes \mathcal{E}$ which contains B that is invariant under the action of $M_\varphi^* : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H} \otimes \mathcal{E}$ for all scalar-valued multipliers of $\mathcal{H}$. Then we have the corresponding analog of Proposition 5.

Proposition 8. *Let $B \in \mathcal{X}(\mathcal{H} \otimes \mathcal{E})$. Then* $\ker H_B = [B]_*^\perp$.

Proof. From the paragraph proceeding the definition of $\mathcal{X}(\mathcal{H} \otimes \mathcal{E})$, we have that

$$\left\langle \left(\sum_n g_n \otimes e_n \right), M_f^* \left(\sum_n b_n \otimes e_n \right) \right\rangle = \left\langle CH_B f, \sum_n \overline{g_n} \otimes e_n \right\rangle$$

The result follows by noting that elements of the form $M_f^* \left(\sum_n b_n \otimes e_n \right)$ are dense in $[B]_*$. ☐

Example: Suppose now that $\mathcal{H} = \mathcal{H}(l) \subseteq \mathrm{Hol}(\mathbb{B}_d)$. Fix $x \in \mathcal{E}$ and $\lambda \in \mathcal{B}_d$. Let $\eta = k_\lambda \otimes x$ for some $\lambda \in \mathbb{D}$ and $x \in \mathcal{E}$. Then

$$|\langle \varphi g, l_\lambda \otimes x \rangle| = |\varphi(\lambda) g_x(\lambda)| \leq \|\varphi\| \|g\| \|l_\lambda\|^2 \|x\|$$

so that $l_\lambda \otimes x$ is Hankel.

5.4.1 The Generalized Result

In this section, we provide the full statement of the result of McCullough and Trent; however, we first need to introduce the notion of an operator valued multiplier.

Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ be a Hilbert space. Further, let $\mathcal{E}$ and $\mathcal{D}$ be separable Hilbert spaces, and let $\mathcal{B}(\mathcal{E}, \mathcal{D})$ denote the bounded linear operators from $\mathcal{E}$ into $\mathcal{D}$. If $\phi : \mathbb{B}_d \to \mathcal{B}(\mathcal{E}, \mathcal{D})$ is a map such that $\Phi : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H} \otimes \mathcal{D}$ given by

$$\Phi(f)(\lambda) = \phi(\lambda) f(\lambda)$$

is bounded, then we say that ϕ is an operator-valed multiplier.

If $\mathcal{H} = \mathcal{H}(l)$ is a reproducing kernel Hilbert space, then for $g \in \mathcal{H}$, $x \in \mathcal{E}$ and $y \in \mathcal{D}$, we have that

$$\begin{aligned} \langle g \otimes x, \Phi^*(k_\lambda \otimes y)\rangle &= \langle \Phi(g \otimes x), k_\lambda \otimes y\rangle \\ &= \langle \Phi(g \otimes x)(\lambda), y\rangle \\ &= \langle \phi(\lambda) g(\lambda) x, y\rangle \\ &= \langle g(\lambda) x, \phi^*(\lambda) y\rangle \\ &= \langle g \otimes x, k_\lambda \otimes \phi^*(\lambda) y\rangle. \end{aligned}$$

Therefore we have that $\Phi^*(k_\lambda \otimes y) = \phi^*(\lambda) y \otimes k_\lambda$.

Theorem 5.4 (from [25])**.** *Let l be a complete Nevanlinna Pick kernel, let $\mathcal{E}$ be a separable Hilbert space, and let $\mathcal{M} \subseteq \mathcal{H}(l) \otimes \mathcal{E}$ be a multiplier invariant subspace. Then there is an auxiliary Hilbert Space X and an inner multplication operator $\Phi : \mathcal{H}(l) \otimes X \to \mathcal{H}(l) \otimes \mathcal{E}$ such that $\mathcal{M} = \Phi(\mathcal{H}(l) \otimes \mathcal{E})$ and $P_{\mathcal{M}} = \Phi^*\Phi$.*

Recall that the reproducing kernel of H^2_d, which we denoted k_λ, is a complete Nevanlinna pick kernel.

Lemma 5.4.1. *If the multiplication operator $\Phi : H^2_d \otimes X \to \mathcal{H} \otimes \mathcal{E}$ is bounded, then for $h \in \mathcal{M}(H^2_d)$ and $x \in \mathcal{E}$, we have that $\Phi(h \otimes x) \in \mathcal{X}(H^2_d \otimes \mathcal{E})$.*

Proof. We note that for $f \in H^2_d$ and $y \in \mathcal{E}$ that

$$\begin{aligned} \left(\Phi(fh \otimes x)\right)_y(\lambda) &= \langle \Phi(fh \otimes x)(\lambda), y\rangle \\ &= \langle f(\lambda) h(\lambda) \phi(\lambda) x, y\rangle \\ &= f(\lambda)\langle h(\lambda)\phi(\lambda) x, y\rangle \\ &= f(\lambda)(\Phi(h \otimes x))_y(\lambda). \end{aligned}$$

Then we have that

$$\begin{aligned}\sum_n \|f(\Phi(h\otimes x))_{e_n}\|^2 &= \sum_n \|(\Phi(fh\otimes x))_{e_n}\|^2 \\ &= \|\Phi(fh\otimes x)\|^2 \\ &\lesssim \|fh\|^2\|x\|^2 \\ &\leq \|M_h\|^2\|x\|^2\|f\|^2\end{aligned}$$

It follows that the operator $CM_B : \mathcal{H} \to \mathcal{H}\otimes\mathcal{E}$ where $B = \{(\Phi(h\otimes x))_{e_n}\}$ is bounded. Therefore $CH_B : H^2_d \otimes \mathcal{E} \to \overline{H^2_d}$ is bounded by Theorems 5.1 and 5.3 and we conclude that $\Phi(h\otimes x) \in \mathcal{X}(H^2_d \otimes \mathcal{E})$. □

5.5 The Main Result

Theorem 5.5. *Let $\mathcal{M} \subseteq H^2_d \otimes \mathcal{E}$ be a multiplier invariant subspace. Then there are symbols $\eta_n \subseteq H^2_d \otimes \mathcal{E}$ such that H_{η_n} is bounded and*

$$\mathcal{M} = \bigcap_{n=0}^{\infty} \ker H_{\eta_n}$$

Proof. Since H^2_d has a Nevanlinna-Pick kernel, there is an auxiliary Hilbert space H^2_d and an inner operator-valued multiplier $\Phi : \mathcal{H} \otimes H^2_d \to \mathcal{H} \otimes \mathcal{E}$ which satisfies $P_{\mathcal{M}} = \Phi\Phi^*$ by Theorem 5.4. For $\lambda \in \mathbb{D}$ and $x \in \mathcal{E}$, by the discussion at the beginning of Section 5.4.1, we have that $P_{\mathcal{M}}(k_\lambda \otimes x) = \Phi(k_\lambda \otimes \phi^*(\lambda)x)$. Since k_λ is a multiplier, it follows by Lemma 5.4.1 that $P_{\mathcal{M}}(k_\lambda \otimes \varphi^*(\lambda)x) \in \mathcal{X}(H^2_d \otimes \mathcal{E})$. Therefore $\eta_{\lambda,x} = P_{\mathcal{M}^\perp}(k_\lambda \otimes x) = k_\lambda \otimes x - P_{\mathcal{M}}(k_\lambda \otimes x) \in \mathcal{X}(H^2_d \otimes \mathcal{E})$. Then we have that

$$\mathcal{M}^\perp = \bigvee_{\lambda,x} [P_{\mathcal{M}^\perp}(k_\lambda \otimes x)]_*$$

from which it follows that

$$\mathcal{M} = \bigcap_{\lambda,x} [P_{\mathcal{M}^\perp}(k_\lambda \otimes x)]_*^\perp = \bigcap_{\lambda,x} \ker H_{\eta_{\lambda,x}},$$

as desired. □

Chapter 6

Mixed Hankel Forms

6.1 Introduction

The purpose of this section is to discuss a generalization of the following formula, proven by Rochberg and Wu.

Theorem 6.1 (see [31]). *Suppose that g is an analytic function on $\mathbb{D}$ and let η, σ and τ be real numbers which satisfy $\sigma, \tau > -1$, $\eta \geq 0$ and min$(\sigma, \tau) + 2 > m$. Then the quantity*

$$\int_{\mathbb{D}} |g^{'}(z)|^2 (1 - |z|^2)^{\eta} dA(z)$$

is comparable to

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z) - g(w)|^2}{|1 - z\overline{w}|^{4+\sigma+\tau-\eta}} (1 - |z|^2)^{\sigma} (1 - |w|^2)^{\tau} dA(z) dA(w)$$

Using this formula, Rochberg and Wu characterize the symbols which induce bounded (small) Hankel operators on weighted Dirichlet spaces and also provide results regarding atomic decompositions of such symbols. Using a different technique, Blasi and Pau in [6] generalized this result to Besov-Sobolev Banach spaces on $\mathbb{D}$. Recently, Liu, Chacon, and Lou in [22] gave a similar type of formula for the Dirichlet type spaces $D(\mu)$.

An analog of such a formula for the Drury-Arveson space in higher than two dimensions was published in [10] and [18]; however, in Fang and Xia's paper [15] it was noted that the results of these papers were incorrect. Both papers follow the technique of proof used in the Rochberg and Wu paper, but unfortunately contain the same combinatorial error, rendering the results invalid. Upon correcting the mistake, one finds that the formula of Rochberg and Wu generalizes to the scale containing the Hardy Space on $\mathbb{B}_d$ and the weighted Bergman spaces on $\mathbb{B}_d$. Furthermore, the generalized formula does *not* extend to give a formula for the norm of the Drury-Arveson space.

In this section, we prove a generalization of the Rochberg and Wu formula for holomorphic functions on $\mathbb{B}_d$. Instead of correcting the combinatorial formula and following the proof of Rochberg and Wu, we will give an alternate proof based on a result of [28]. We then prove a result about Hankel forms that will be used in the next section.

6.2 Preliminaries

We now collect preliminaries required for the remainder of the section. These results are standard to the field and can be found in [37].

For each point $0 \neq a \in \mathbb{B}_d$, we define the function

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1-|a|^2}Q_a(z)}{1 - \langle z, a\rangle}$$

where P_a is the orthogonal projection from $\mathbb{C}^n$ onto the linear multiples of a and $Q_a = 1 - P_a$. We define $\varphi_0(z) = -z$. These maps are involutive automorphisms of $\mathbb{B}_d$; that is, they are biholomorphic and satisfy $\varphi_a(\varphi_a(z)) = z$. The following Lemma provides a change of variable formula involving these automorphisms.

Lemma 6.1.1 (Proposition 1.13 in [37]). *Let $\alpha > -1$. Then if $f \in A^1_\alpha$, we have that*

$$\int_{\mathbb{B}_d} f \circ \varphi_a(z) dv_\alpha(z) = \int_{\mathbb{B}_d} f(z) \frac{(1-|a|^2)^{d+1+\alpha}}{|1-\langle z, a\rangle|^{2(d+1+\alpha)}} dv_\alpha(z)$$

As a corollary to the following theorem, we will obtain estimates for the Hardy Space norm and Bergman space norms of powers of the functions

$$k_\lambda(z) = \frac{1}{1-\langle z, \lambda\rangle}.$$

Theorem 6.2 (Theorem 1.12 in [37]). *Suppose that c is real and $t > -1$. Then the integrals*

$$I_c(z) = \int_{\mathbb{S}_d} \frac{d\sigma(\zeta)}{|1-\langle z, \zeta\rangle|^{d+c}}$$

and

$$J_{c,t}(z) = \int_{\mathbb{B}_d} \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^{d+1+t+c}} dv(w)$$

have the following asymptotic properties.

1. *If $c < 0$, then both I_c and $J_{c,t}$ are bounded.*

2. *If $c = 0$ then*

   $$I_c(z) \sim J_{c,t}(z) \sim \log \frac{1}{1-|z|^2}$$

   as $|z| \to 1^-$

3. *If $c > 0$ then*

   $$I_c(z) \sim J_{c,t}(z) \sim (1-|z|^2)^{-c}$$

   as $|z| \to 1^-$

Lemma 6.2.1. *If η satisfies $\eta p > d$, then*

$$\|k_\lambda^\eta\|^p_{H^p(\mathbb{B}_d)} \sim (1-|\lambda|^2)^{d-p\eta}$$

and if $\alpha > -1$ *and* $\eta p - (d+1+\alpha) > 0$*, then*

$$\|k_\lambda^\eta\|_{A_\alpha^2(\mathbb{B}_d)}^p \sim (1-|\lambda|^2)^{d+1+\alpha-p\eta}$$

For a function $f \in \mathrm{Hol}(\mathbb{B}_d)$, we define the **invariant gradient** of f to be

$$\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$$

where $\varphi_z : \mathbb{B}_d \to \mathbb{B}_d$ is the involutive automorphism of $\mathbb{B}_d$ which interchanges z and 0 mentioned earlier and ∇ is the standard gradient. The invariant gradient satisfies the nice property that for any automorphism φ of $\mathbb{B}_d$, we have

$$|\nabla(\tilde{f \circ} \varphi)(z)| = |(\tilde{\nabla} f) \circ \varphi(z)|$$

Furthermore, we can obtain an equivalent norm for the weighted Bergman spaces interms of invariant graident.

Lemma 6.2.2 (Theorem 2.16 in [37])**.** *Let* $\alpha > -1$*,* $p > 0$*, and let* $f \in$ *Hol*$(\mathbb{B}_d)$*. Then*

$$\|f\|_{A_\alpha^p}^p \approx \int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p (1-|z|^2)^\alpha dV(z).$$

Lemma 6.2.3. *Suppose* $p > 0$*,* $\alpha > -1$ *and* $m = (m_1, ..., m_d)$ *is a d-tuple of nonnegative integers. Then there is a constant* $C > 0$ *such that*

$$\left|\tilde{\nabla} f(z)\right| \leq C\|f \circ \varphi_z - f(z)\|_{A_\alpha^p}$$

for all $f \in$ *Hol*$(\mathbb{B}_d)$ *and all* $z \in \mathbb{B}_d$*.*

Proof. By the Möbius invariance of $\tilde{\nabla}$, we have that $|\tilde{\nabla} f(z)| = |\tilde{\nabla} f \circ \varphi_z(0)| = |\tilde{\nabla}(f \circ \varphi_z)(0)|$. Therefore it suffices to prove the result when $z = 0$. Since $f \in \mathrm{Hol}(\mathbb{B}_d)$, we conclude by subharmonicity that

$$\left|\frac{\partial f}{\partial z_i}(0)\right|^p \le \|f - f(0)\|^p_{A^p_\alpha}$$

and the result follows. □

6.3 Proof of the Formula

Our starting point for the generalization is the following lemma, proven recently by Pau and Zhao. To condense the calculations, for $\alpha > -1$, we will write $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$, where c_α is the normalization constant attached to the norm of $A^2_\alpha(\mathbb{B}_d)$.

Lemma 6.2.4 (from [28]). *Let $1 < p < \infty$, $\alpha > -1$, and $d + 1 + \alpha < b$. Then*

$$\int_{\mathbb{B}_d} \frac{|f(z) - f(a)|^p}{|1 - \langle z, a\rangle|^b} dv_\alpha(z) \lesssim \int_{\mathbb{B}_d} \frac{|\tilde{\nabla} f(a)|^p}{|1 - \langle z, a\rangle|^b} dv_\alpha(z)$$

for any $f \in Hol(\mathbb{B}_d)$ and $a \in \mathbb{B}_d$.

Theorem 6.3. *Let $0 < p < \infty$, $\alpha, \beta > -1$ and η be real numbers which satisfy and $\eta < min\{\alpha, \beta\}$. Then given any $f \in Hol(\mathbb{B}_d)$, the quantities*

$$I_{\alpha,\beta,\eta} := \int_{\mathbb{B}_d}\int_{\mathbb{B}_d} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w\rangle|^{d+1+\alpha+\beta-\eta}} dv_\alpha(z) dv_\beta(w)$$

and

$$\int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p dv_\eta(z)$$

are comparable.

Proof. Since $\eta < \beta$, we may apply Lemma 6.2.4 and Fubini's Theorem to get

$$I_{\alpha,\beta,\eta} \lesssim \int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p \int_{\mathbb{B}_d} \frac{dv_\beta(w)}{|1 - \langle z, w\rangle|^{d+1+\alpha+\beta-\eta}} dv_\alpha(z)$$

Then since $\eta < \alpha$, we may apply Proposition 6.2 to obtain

$$I_{\alpha,\beta,\gamma} \lesssim \int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p dv_\eta(z).$$

Choose $\sigma > -1$ so that $2\sigma > \alpha + \beta - \eta - d - 1$. By Lemma 6.2.3, we have that

$$\int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p d_\eta(z) \lesssim \int_{\mathbb{B}_d} \int_{\mathbb{B}_d} |f \circ \varphi_z(w) - f(z)|^p dv_\sigma(w) dv_\eta(z)$$

Applying the change of variables $u = \varphi_z(w)$ (see Lemma 6.1.1), we obtain

$$\int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p dv_\eta(z) \lesssim \int_{\mathbb{B}_d} \int_{\mathbb{B}_d} |f(w) - f(z)|^p \frac{(1-|w|^2)^{d+1+\sigma}}{|1-\langle z, w\rangle|^{2(d+1+\sigma)}} dv_\sigma(w) dv_\eta(z)$$

The integral on the right hand side of the inequality is equal to

$$\frac{c_\sigma c_\eta}{c_\alpha c_\beta} \int_{\mathbb{B}_d} \int_{\mathbb{B}_d} \frac{|f(w) - f(z)|^p}{|1-\langle z, w\rangle|^{d+1+\alpha+\beta-\eta}} \frac{(1-|w|^2)^{d+1+2\sigma-\alpha}(1-|z|^2)^{\eta-\beta}}{|1-\langle z, w\rangle|^{d+1+2\sigma-\alpha-\beta+\eta}} dv_\alpha(w) dv_\beta(z).$$

Finally, using the estimate

$$|1-\langle z, w\rangle| \geq 1 - |\langle z, w\rangle| \geq 1 - |z| \geq \frac{1}{2}(1-|z|^2)$$

and the inequality $d + 1 + 2\sigma - \alpha - \beta + \eta > 0$, we conclude that

$$\int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p dv_\eta(z) \lesssim \int_{\mathbb{B}_d} \int_{\mathbb{B}_d} \frac{|f(w) - f(z)|^p}{|1-\langle z, w\rangle|^{d+1+\alpha+\beta-\eta}} dv_\alpha(w) dv_\beta(z).$$

as desired. □

In view of Lemma 6.2.2, this result gives us another way to write the norm of a function in a weighted Bergman space A^2_α. In [33] it was for shown that for $f \in \mathrm{Hol}(\mathbb{B}_d)$, we have that

$$\|f\|^2_{H^2(\mathbb{B}_d)} \approx \int_{\mathbb{B}_d} \frac{|\tilde{\nabla} f(z)|^2}{1-|z|^2} dv(z).$$

Furthermore, the spaces of functions which satisfy

$$\int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^p dv_\alpha(z) < \infty$$

have been studied in their own right (see for instance [16]).

6.4 Application to Mixed Bergman Hankel forms

We now aim to characterize certain situations in which a function φ satisfies the inequality

$$|\langle fg, \varphi\rangle_{A^2_\omega(\mathbb{B}_d)}| \leq \|f\|_{A^2_\alpha(\mathbb{B}_d)}\|g\|_{A^2_\beta(\mathbb{B}_d)}$$

for varying ω, α, β. We refer to these as "mixed Hankel forms" because of the varying spaces involved. Our main tool for this section will be Theorem 6.3, but we will also make use of the Bergman projections. For $\alpha > -1$, for a function $f \in L^2((1-|z|^2)^\alpha dV)$, we define

$$(P_\alpha f)(z) = c_\alpha \int_{\mathbb{B}_d} \frac{f(w)}{(1-\langle z, w\rangle)^{d+1+\alpha}} dv_\alpha(z) = \langle f, k_z^{d+1+\alpha}\rangle_{A^2_\alpha(\mathbb{B}_d)}.$$

This is the projection from $L^2((1-|z|^2)^\alpha dV)$ onto $A^2_\alpha(\mathbb{B}_d)$ (see Chapter 2 of [37]).

Theorem 6.4. *Let $\alpha, \beta, \eta, \omega > -1$ satisfy the inequalities $2w > \alpha + \eta$ and $\alpha + \beta + \eta - 2\omega > -1$. Let $\varphi \in Hol(\mathbb{B}_d)$. If for all $f, g \in Hol(\overline{\mathbb{B}_d})$, the inequality*

$$|\langle fg, \varphi\rangle_{A^2_\omega(\mathbb{B}_d)}| \leq \|f\|_{A^2_\alpha(\mathbb{B}_d)}\|g\|_{A^2_\beta(\mathbb{B}_d)}$$

holds, then

$$\|P_\omega(\overline{f}\varphi)\|_{A^2_\eta} \lesssim \|f\|_{A^2_{\alpha+\beta+\eta-2\omega}}.$$

Proof. Since $\eta > -1$, we can find an $\epsilon > 0$ such that $d+2+\alpha+\eta > 2\epsilon > d+1+\alpha$. Then, by the assumption on φ and Lemma 6.2, we have that

$$\begin{aligned} |P_\omega(\overline{f}\varphi)(\lambda)|^2 &= |\langle \varphi, f k_\lambda^{d+1+\omega}\rangle_{A^2_\omega(\mathbb{B}_d)}|^2 \\ &\lesssim \|k_\lambda^\epsilon\|^2_{A^2_\alpha(\mathbb{B}_d)} \|f k_\lambda^{d+1+\omega-\epsilon}\|^2_{A^2_\beta(\mathbb{B}_d)} \\ &\lesssim (1-|\lambda|^2)^{d+1+\alpha-2\epsilon} \|f k_\lambda^{d+1+\omega-\epsilon}\|^2_{A^2_\beta(\mathbb{B}_d)} \end{aligned}$$

Integrating these inequalities over $\mathbb{B}_d$ against the measure $dv_\eta(\lambda)$, we obtain

$$\begin{aligned} \int_{\mathbb{B}_d} |P_\omega(\overline{f}\varphi)(\lambda)|^2 dv_\eta(\lambda) &\lesssim \int_{\mathbb{B}_d}\int_{\mathbb{B}_d} \frac{|f(z)|^2}{|1-\langle z,w\rangle|^{2(d+1+\omega-\epsilon)}} dv_\beta(z) dv_{d+1+\alpha+\eta-2\epsilon}(\lambda) \\ &\approx \int_{\mathbb{B}_d} |f(z)|^2 \int_{\mathbb{B}_d} \frac{(1-|\lambda|^2)^{d+1+\alpha+\eta-2\epsilon}}{|1-\langle z,w\rangle|^{2(d+1+\omega-\epsilon)}} dv(\lambda) dv_\beta(z) \\ &\lesssim \int_{\mathbb{B}_d} |f(z)|^2 dv_{\alpha+\beta+\eta-2\omega}(z). \end{aligned}$$

Here we have used Lemma 6.2 and our use of the Lemma is justified by the inequalities $d+1+\alpha+\eta-2\epsilon > -1$ and $2\omega > \alpha+\eta$. □

Theorem 6.5. *Let $\alpha, \beta, \eta, \omega > -1$ satisfy max$\{\beta, \eta\} < 2\omega - \alpha$ and let $\varphi \in Hol(\mathbb{B}_d)$. If for all $f, g \in Hol(\overline{\mathbb{B}_d})$, the inequality*

$$|\langle fg, \varphi\rangle_{A^2_\omega(\mathbb{B}_d)}| \leq \|f\|_{A^2_\alpha(\mathbb{B}_d)} \|g\|_{A^2_\beta(\mathbb{B}_d)}$$

holds, then

$$\|(I-P_\omega)(\overline{f}\varphi)\|_{A^2_\eta} \lesssim \int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{\alpha+\beta+\eta-2\omega} dv(z)$$

Proof. By the assumption on the parameters, we can find an $\epsilon > 0$ which satisfies

$$d+1+\alpha < 2\epsilon < \min\{d+1+2\omega-\beta, d+2+\alpha+\eta\}.$$

Then, by the assumption on φ and Lemma 6.2, we have that

$$\begin{aligned}|(1-P_\omega)(\overline{f}\varphi)(\lambda)|^2 &= |\langle \varphi, (f-f(\lambda))k_\lambda^{d+1+\omega}\rangle_{A^2_\omega(\mathbb{B}_d)}|^2 \\ &\lesssim \|k_\lambda^\epsilon\|^2_{A^2_\alpha(\mathbb{B}_d)} \|(f-f(\lambda))k_\lambda^{d+1+\omega-\epsilon}\|^2_{A^2_\beta(\mathbb{B}_d)} \\ &\lesssim (1-|\lambda|^2)^{d+1+\alpha-2\epsilon} \|(f-f(\lambda))k_\lambda^{d+1+\omega-\epsilon}\|^2_{A^2_\beta(\mathbb{B}_d)}\end{aligned}$$

Integrating these inequalities over $\mathbb{B}_d$ against the measure $dv_\eta(\lambda)$, we obtain

$$\begin{aligned}\int_{\mathbb{B}_d} |(I-P_\omega)(\overline{f}\varphi)(\lambda)|^2 dv_\eta(\lambda) &\lesssim \int_{\mathbb{B}_d}\int_{\mathbb{B}_d} \frac{|f(z)-f(\lambda)|^2}{|1-\langle z,w\rangle|^{2(d+1+\omega-\epsilon)}} dv_\beta(z) dv_{d+1+\alpha+\eta-2\epsilon}(\lambda) \\ &\lesssim \int_{\mathbb{B}_d} |\tilde{\nabla} f(z)|^2 (1-|z|^2)^{\alpha+\beta+\eta-2\omega} dv(z)\end{aligned}$$

Here we have used Theorem 6.3 and our use is justified by the inequalities $d+1+\alpha+\eta-2\epsilon > -1$ and $\alpha+\beta+\eta-2\omega < \min\{d+1+\alpha+\eta-2\omega, \beta\}$. □

Theorem 6.6. *Let $\alpha, \beta, \eta, \omega > -1$ satisfy the inequalities max$\{\beta, \eta\} < 2\omega - \alpha$ and $\alpha+\beta+\eta-2\omega > -1$. Let $\varphi \in Hol(\mathbb{B}_d)$. Then the following are equivalent*

1. *The inequality*

$$|\langle fg, \varphi\rangle_{A^2_\omega(\mathbb{B}_d)}| \lesssim \|f\|_{A^2_\alpha(\mathbb{B}_d)} \|g\|_{A^2_\beta(\mathbb{B}_d)}$$

   holds for all $f, g \in Hol(\mathbb{B}_d)$

2. *the measure*

$$|\varphi(z)|^2 (1-|z|^2)^\eta dv(z)$$

   is a Carleson measure for $A^2_{\alpha+\beta+\eta-2\omega}(\mathbb{B}_d)$.

3. *The quantity*

$$\sup_{z\in\mathbb{B}_d} (1-|z|^2)^{2\omega-\alpha-\beta} |\varphi(z)|^2$$

   is finite.

Proof. (1 ⇒ 2) By the previous two theorems, we have that

$$\|f\varphi\|_{A^2_\eta(\mathbb{B}_d)} \leq \|P_\omega(\overline{f}\varphi)\|_{A^2_\eta(\mathbb{B}_d)} + \|(I-P_\omega)(\overline{f}\varphi)\|_{A^2_\eta(\mathbb{B}_d)} \lesssim \|f\|_{A^2_{\alpha+\beta+\eta-2\omega}}$$

(2 ⇒ 3) If the measure $|\varphi(z)|^2(1-|z|^2)^\eta dv(z)$ is a Carleson measure for $A^2_{\alpha+\beta+\eta-2\omega}(\mathbb{B}_d)$, then the multiplication operator $M_\varphi : A^2_{\alpha+\beta+\eta-2\omega}(\mathbb{B}_d) \to A^2_\eta(\mathbb{B}_d)$ is bounded. Therefore Lemma 2.1.4, we have

$$\sup_{z\in\mathbb{B}_d}(1-|z|^2)^{2\omega-\alpha-\beta}|\varphi(z)|^2 \leq C$$

(3 ⇒ 1) If we let C equal the supremum that appears in 3), then we have

$$\begin{aligned} |\langle fg,\varphi\rangle_{A^2_\omega(\mathbb{B}_d)}| &\lesssim \int_{\mathbb{B}d} |f(z)g(z)\varphi(z)|(1-|z|^2)^\omega dv(z) \\ &\leq \sqrt{C}\int_{\mathbb{B}_d} |f(z)g(z)|(1-|z|^2)^{\frac{\alpha+\beta}{2}} dv(z) \\ &\lesssim \|f\|_{A^2_\alpha(\mathbb{B}_d)}\|g\|_{A^2_\beta(\mathbb{B}_d)} \end{aligned}$$

which proves the result. □

In case when one of the norms is the Hardy space norm (i.e. when $\alpha = -1$), we have the following.

Theorem 6.7. *Let $\beta, \eta > -1$ and let $\varphi \in Hol(\mathbb{B}_d)$. Then the following are equivalent*

1. *The inequality*

$$|\langle fg,\varphi\rangle_{A^2_{\frac{\beta+\eta}{2}}(\mathbb{B}_d)}| \lesssim \|f\|_{H^2(\mathbb{B}_d)}\|g\|_{A^2_\beta(\mathbb{B}_d)}$$

   holds for all $f, g \in Hol(\overline{\mathbb{B}_d})$

2. *the measure*

$$|\varphi(z)|^2(1-|z|^2)^\eta dv(z)$$

   is a Carleson measure for $H^2(\mathbb{B}_d)$

Proof. Throughout this proof, let $\omega = \frac{\beta+\eta}{2}$.

($\Rightarrow$) Since

$$\langle fg, \varphi\rangle_{A^2_w} = \langle g, P_w(\overline{f}g)\rangle_{A^2_w}$$

we conclude that the assumed inequality is equivalent to the inequality

$$\|P_\omega(\overline{f}g)\|_{A^2_\eta} \lesssim \|f\|_{H^2(\mathbb{B}_d)}$$

by duality (see Theorem 2.12 in [37]). By choosing $\alpha = -1$, and β, η, ω as they are in this proof, one checkts that Theorem 6.5 is still valid, and therefore, we have that

$$\|(1 - P_\omega)(\overline{f}\varphi)\|_{A^2_\eta} \lesssim \|f\|_{H^2(\mathbb{B}_d)}.$$

This direction follows now exactly as the implication 1) implies 2) in Theorem 6.6.

($\Leftarrow$) We have

$$\begin{aligned} |\langle fg, \varphi\rangle_{A^2_\omega(\mathbb{B}_d)}| &\leq \int_{\mathbb{B}_d}\int_{\mathbb{B}_d} |f(z)g(z)\varphi(z)| dv_\omega(z) \\ &\lesssim \|f\varphi\|_{A^2_\eta(\mathbb{B}_d)}\|g\|_{A^2_\beta(\mathbb{B}_d)} \\ &\lesssim \|f\|_{H^2(\mathbb{B}_d)}\|g\|_{A^2_\beta(\mathbb{B}_d)} \end{aligned}$$

□

Chapter 7

A Characterization of Certain Hankel Symbols for H^2_d

In this chapter, we provide a connection between Carleson measures (introduced in Section 2.4) and Hankel operators. We characterize when a special type of symbol lies in $\mathcal{X}(H^2_d)$ in terms of a Carleson measure condition.

7.1 History

For several well-known spaces of holomorphic functions $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$, membership of a function φ in $\mathcal{X}(\mathcal{H})$ is equivalent to a Carleson measure condition on φ. In particular, we have that $b \in \mathcal{X}(H^2(\partial\mathbb{B}_d)) = BMOA(\mathbb{B}_d)$ (see [11]). It is well known that membership in $BMOA(\mathbb{B}_d)$ is equivalent to the measure $|Rb|^2(1-|z|^2)dV_{d+1}(z)$ being a Carleson measure for $H^2(\partial\mathbb{B}_d)$ (see Theorem 5.14 in [37]). For the weighted Bergman space $A^2_\alpha(\mathbb{B}_d)$, which for $\alpha > -1$ is the Hilbert space of functions $f \in \mathrm{Hol}(\mathbb{B}_d)$ which satisfy

$$\|f\|^2_{A^2(\mathbb{B}_d)} = \frac{\Gamma(d+\alpha+1)}{d!\Gamma(\alpha+1)} \int_{\mathbb{B}_{d+1}} |f(z)|^2(1-|z|^2)^\alpha dV(z) < \infty,$$

we similarly have that $\varphi \in \mathcal{X}(A^2_\alpha(\mathbb{B}_d))$ if and only if φ is in the Bloch space

$$\mathcal{B} = \left\{ b \in \mathrm{Hol}(\mathbb{B}_d) \mid \sup_{z \in \mathbb{B}_{d+1}} |Rb(z)|(1-|z|^2) < \infty \right\}.$$

(again, see see [11]). Furthermore, one can show that $\varphi \in \mathcal{B}$ if and only if $|R\varphi|^2(1-|z|^2)^{\alpha+2}dV(z)$ is a Carleson measure for $A^2_\alpha(\mathbb{B}_d)$. Finally, for the Dirichlet space

$$\mathcal{D} = \left\{ f \in \mathrm{Hol}(\mathbb{D}) \mid \int_{\mathbb{D}} |f^{'}(z)|^2 dA(z) < \infty \right\}$$

we have that $\varphi \in \mathcal{X}(\mathcal{D})$ if and only if $|\varphi^{'}(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$ (see [2]).

It is often the case that proving the boundedness of a Hankel operator under the assumption that a given measure is a Carleson measure is relatively straightforward (in fact, in [2] they refer to this as the "easy direction" of their proof). One sees that being a Carleson measure involves having absolute values inside an integral, while the definition of $\mathcal{X}(\mathcal{H})$ usually involves having absolute values outside of an integral. Therefore, it is easy to see that this direction is set up for (and frequently uses) tools like Hölder's inequality. The reverse implication, however, is generally much more subtle and requires a very intricate geometric analysis of Carleson measures and the underlying Hilbert space (see [2], [9], [11]). A geometric characterization of the Carleson measures for H^2_d is proven in [1], but at this time, there is no complete characterization of $\mathcal{X}(H^2_d)$ in terms of Carleson measures.

The main result in this chapter takes a certain type of symbol $\varphi \in H^2_{d+1}$ and reduces both problems of inducing an appropriate Carleson measure and inducing a bounded Hankel operator into problems on Hilbert spaces of analytic functions on $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. We then rely on the well-known theory mentioned above to prove our results.

7.2 Decomposition of H^2_d into Function Spaces of One Variable

Let $f = \sum_n a_n z^n \in \mathrm{Hol}(\mathbb{D})$ and let $\beta = (\beta_1, ..., \beta_{d+1}) \in \mathbb{N}_0^{d+1}$ be a $(d+1)$-tuple with $\beta_1 = 0$. Consider the function $g : \mathbb{B}_{d+1} \to \mathbb{C}$ given by $g_\beta(z) = z^\beta f(z_1)$. If $|\beta| = 0$, then we have that

$$\|g_\beta\|^2_{H^2_d} = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2_{H^2(\mathbb{D})}.$$

Furthermore, if $|\beta| \neq 0$, then we have

$$\|g_\beta\|^2_{H^2_{d+1}} = \frac{\beta!}{|\beta|!} \sum_{n=0}^{\infty} |a_n|^2 \frac{|\beta|!n!}{(|\beta|+n)!} = \frac{\beta!}{|\beta|!} \|f\|_{A^2_{|\beta|-1}(\mathbb{D})}.$$

(see Lemma 2.1.2). Letting

$$\mathcal{S}_{d+1} = \{\alpha = (\alpha_1, ..., \alpha_{d+1}) \in \mathbb{N}_0^{d+1} \mid \alpha_1 = 0\},$$

It follows that for every $f \in H^2_{d+1}$, we may write

$$f = \sum_{\alpha \in \mathcal{S}_{d+1}} f_\alpha z^\alpha,$$

where the f_α are holomorphic functions which depend only on the variable z_1 such that

$$\|f\|^2_{H^2_{d+1}} = \sum_{\alpha \in \mathcal{S}_{d+1}} \frac{\alpha!}{|\alpha|!} \|f_\alpha\|^2_{A^2_{|\alpha|-1}(\mathbb{D})}$$

When convenient, we will write $A^2_{-1}(\mathbb{D}) = H^2(\mathbb{D})$, as in the above formula.

Throughout this chapter, when we are given $\beta \in \mathcal{S}_{d+1}$ and a function $f \in \mathrm{Hol}(\mathbb{D})$, we will just write $z^\beta f$ to mean the function g_β discussed in the previous paragraph; that is, we will think of the function f as a holomorphic function on $\mathbb{B}_{d+1}$ which only depends on the variable z_1. With this identification, we have that $H^2(\mathbb{D})$ embeds isometrically into H^2_{d+1} and for $k \in \mathbb{N}$ and the weighted Bergman spaces $A^2_k(\mathbb{D})$ embed into H^2_{d+1} contractively for $k \in \mathbb{N}$.

The main theorem of this chapter makes a connection between Carleson measures and Hankel forms for those functions that can appear as $f_\alpha z^\alpha$ for some function in $f \in H^2_{d+1}$ with respect to this decomposition. By Theorem 3.2, we have that for any $b \in H^2_{d+1}$ and any $m \in \mathbb{N}$ such that $2m-(d+1) > -1$ that $|R^m b|^2(1-|z|^2)^{2m-(d+1)}dV$ being a Carleson measure for H^2_{d+1} implies that $b \in \mathcal{X}(H^2_{d+1})$; therefore, we will focus on the other implication for these particular symbols.

7.3 Reduction of the Hankel Condition

In this section, we characterize the symbols of the type $z^\beta \varphi \in H^2_{d+1}$ that are elements of $\mathcal{X}(H^2_{d+1})$ in terms of a one-dimensional Carleson measure condition involving the function φ. To motivate this, let $\varphi \in H^2(\mathbb{D}) \subseteq H^2_{d+1}$. Given arbitrary functions $f = \sum_{\alpha \in \mathcal{S}_{d+1}} f_\alpha z^\alpha$ and $g = \sum_{\gamma \in \mathcal{S}_{d+1}} g_\gamma z^\gamma \in H^2_{d+1}$, written with respect to the grading of H^2_{d+1}, we see by exploiting orthogonality of the monomials in H^2_{d+1} that

$$\langle fg, \varphi \rangle_{H^2_{d+1}} = \langle f_0 g_0, \varphi \rangle_{H^2(\mathbb{D})}. \tag{7.1}$$

By choosing f and g to be scalar-valued functions in $H^2(\mathbb{D}) \subseteq H^2_{d+1}$ and using the fact that this inclusion is isometric, one easily sees that $\mathcal{X}(H^2_{d+1}) \cap H^2(\mathbb{D}) \subseteq \mathcal{X}(H^2(\mathbb{D}))$. Furthermore, since $\|f_0\|_{H^2(\mathbb{D})} \leq \|f\|_{H^2_{d+1}}$ for any $f \in H^2_{d+1}$, from (2.1) we get $\mathcal{X}(H^2(\mathbb{D})) \subseteq \mathcal{X}(H^2_{d+1}) \cap H^2(\mathbb{D})$. Therefore, it follows that $\varphi \in H^2(\mathbb{D}) \subseteq H^2_{d+1}$ is in $\mathcal{X}(H^2_{d+1})$ if and only if $\varphi \in BMOA(\mathbb{D})$. As mentioned earlier, this is equivalent to the measure $|R\varphi(z)|^2(1-|z|^2)dA(z)$ being a Carleson measure for $H^2(\mathbb{D})$.

The main theorem in this section follows the same type of proof strategy as above; however, for arbitrary $\beta \in \mathcal{S}_{d+1}$, we note that (7.1) becomes

$$\langle fg, z^\beta \varphi \rangle_{H^2_{d+1}} = \sum_{\alpha+\gamma=\beta} \frac{\beta!}{|\beta|!} \langle f_\alpha g_\gamma, \varphi \rangle_{A^2_{|\beta|-1}(\mathbb{D})} \tag{7.2}$$

so that we must deal with a sum of Hankel forms instead of a single one. Nonetheless, the appropriate boundedness of this sum is equivalent to the appropriate boundedness of the summands corresponding to $\alpha = 0$ or $\gamma = 0$, which, in turn, is equivalent to a one-dimensional Carleson measure condition on the function φ.

Theorem 7.1. *Let $\beta \in \mathcal{S}_{d+1}$ with $|\beta| \neq 0$, and $\varphi \in Hol(\mathbb{D})$ be such that $z^\beta \varphi \in H^2_{d+1}$. Then the following are equivalent*

1. $z^\beta \varphi \in \mathcal{X}(H^2_{d+1})$.

2. *There is a constant $C > 0$ such that for all $f, g \in Hol(\overline{\mathbb{D}})$,*

$$|\langle fg, \varphi \rangle_{A^2_{|\beta|-1}(\mathbb{D})}| \leq C\|f\|_{A^2_{s-1}(\mathbb{D})}\|g\|_{A^2_{t-1}(\mathbb{D})}$$

   holds whenever $s, t \in \mathbb{N}_0$ with $s + t = |\beta|$.

3. *There is a constant $C > 0$ such that for all $f, g \in Hol(\overline{\mathbb{D}})$,*

$$|\langle fg, \varphi \rangle_{A^2_{|\beta|-1}(\mathbb{D})}| \leq C\|f\|_{A^2_{|\beta|-1}(\mathbb{D})}\|g\|_{H^2(\mathbb{D})}.$$

4. *The measure $|\varphi(z)|^2(1 - |z|^2)^{|\beta|-1}dA(z)$ is Carleson for $H^2(\mathbb{D})$.*

Proof. (1) $\Rightarrow$ (3) If $z^\beta \varphi \in \mathcal{X}(H^2_{d+1})$, then for any $f, g \in \text{Hol}(\overline{\mathbb{D}})$,

$$\begin{aligned} |\langle fg, \varphi \rangle_{A^2_{|\beta|-1}(\mathbb{D})}| &= \frac{|\beta|!}{\beta!}|\langle f z^\beta g, z^\beta \varphi \rangle_{H^2_{d+1}}| \\ &\leq \frac{|\beta|!}{\beta!}\|z^\beta \varphi\|_{\mathcal{X}(H^2_{d+1})}\|f\|_{H^2_{d+1}}\|z^\beta g\|_{H^2_{d+1}} \\ &= \|z^\beta \varphi\|_{\mathcal{X}(H^2_{d+1})}\|f\|_{H^2(\mathbb{D})}\|g\|_{A^2_{|\beta|-1}(\mathbb{D})} \end{aligned}$$

so that we may choose $C = \|z^\beta \varphi\|_{\mathcal{X}(H^2_{d+1})}$.

(3) $\Leftrightarrow$ (4). This follows immediately from Theorem 6.7.

(4) $\Rightarrow$ (2) If $|\varphi(z)|^2(1-|z|^2)^{|\beta|-1}dA(z)$ is Carleson for $H^2(\mathbb{D})$, then the operator $M_b : H^2(\mathbb{D}) \to A^2_{|\beta|-1}(\mathbb{D})$ is bounded. It follows by Lemma 2.1.3 (with $d=1, \gamma = 1, \eta = |\beta|-1$) that $C := \sup_{z\in\mathbb{D}} |\varphi(z)|(1-|z|^2)^{\frac{|\beta|}{2}} < \infty$. For $0 < t < |\beta|$, we have

$$\begin{aligned}
|\langle fg, \varphi\rangle_{A^2_{|\beta|-1}(\mathbb{D})}| &\lesssim \int_{\mathbb{D}} |f(z)g(z)\varphi(z)|(1-|z|^2)^{|\beta|-1}dA(z) \\
&\leq C\int_{\mathbb{D}} |f(z)g(z)|(1-|z|^2)^{\frac{|\beta|}{2}-1}dA(z) \\
&\leq C\|f\|_{A^2_{t-1}(\mathbb{D})}\|g\|_{A^2_{(|\beta|-t)-1}(\mathbb{D})}
\end{aligned}$$

The cases when $t=0$ and $t=|\beta|$ follow since (3) is equivalent to (4).

(2) $\Rightarrow$ (1) If (2) holds, then for any $f = \sum_\alpha z^\alpha f_\alpha, g = \sum_\gamma z^\gamma g_\gamma \in H^2_{d+1}$,

$$\begin{aligned}
|\langle fg, z^\beta\varphi\rangle_{H^2_{d+1}}| &\leq \sum_{\alpha+\gamma=\beta} |\langle z^\alpha f_\alpha z^\gamma g_\gamma, z^\beta \varphi\rangle_{H^2_{d+1}}| \\
&= \frac{\beta!}{|\beta|!}\sum_{\alpha+\gamma=\beta} |\langle f_\alpha g_\gamma, \varphi\rangle_{A^2_{|\beta|-1}(\mathbb{D})}| \\
&\leq \frac{C\beta!}{|\beta|!}\sum_{\alpha+\gamma=\beta} \|f_\alpha\|_{A^2_{|\alpha|-1}(\mathbb{D})}\|g_\gamma\|_{A^2_{|\gamma|-1}(\mathbb{D})} \\
&\leq \frac{C\beta!}{|\beta|!}\left(\sum_{\alpha+\gamma=\beta} \|f_\alpha\|^2_{A^2_{|\alpha|-1}(\mathbb{D})}\right)^{\frac{1}{2}}\left(\sum_{\alpha+\gamma=\beta} \|g_\gamma\|^2_{A^2_{|\gamma|-1}(\mathbb{D})}\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{H^2_{d+1}}\|g\|_{H^2_{d+1}},
\end{aligned}$$

Thus $z^\beta\varphi \in \mathcal{X}(H^2_{d+1})$. □

We note that (2) can be replaced with

(2′) There is a constant $C > 0$ such that

$$|\langle fg, \varphi\rangle_{A^2_{|\beta|-1}}| \leq C\|f\|_{A^2_{s-1}(\mathbb{D})}\|g\|_{A^2_{t-1}(\mathbb{D})}$$

for any *nonnegative real numbers* s, t which satisfy $s + t = \beta$.

because $(2^{'})$ clearly implies (3) and the proof of (4) implies (2) is valid for nonnegative real numbers, since Lemma 2.1.3 is valid for any appropriate real numbers.

Using Theorem 2.2, we can combine the paragraph preceding Theorem 7.1 and Theorem 7.1 to obtain the following Corollary.

Corollary 7.1.1. *Let $\beta \in \mathcal{S}_{d+1}$ and let $\varphi \in Hol(\mathbb{D})$ be such that $z^\beta \varphi \in H^2_{d+1}$. Then $z^\beta \varphi \in \mathcal{X}(H^2_{d+1})$ if and only if*

$$|R\varphi|^2(1-|z|^2)^{|\beta|+1}dA(z)$$

is a Carleson measure for $H^2(\mathbb{D})$.

7.4 Hankel implies Carleson

Now, note that if $\varphi \in \text{Hol}(\mathbb{B}_{d+1})$ depends only on the variable z_1, then

$$\begin{aligned} R^m(z^\beta\varphi) &= \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} (R^k z^\beta)(R^{m-k}\varphi) \\ &= z^\beta \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} |\beta|^k (R^{m-k}\varphi) \\ &= z^\beta(|\beta|+R)^m\varphi \end{aligned}$$

so that $R^m(z^\beta\varphi)$ is again z^β multiplied by a function which depends only on the variable z_1. The following lemma allows us to exploit this observation and reduce certain integrals over $\mathbb{B}_{d+1}$ into integrals over $\mathbb{D}$.

Lemma 7.1.1. *Let $f \in Hol(\mathbb{D})$. Then, for any $\alpha > -1$ and $\beta \in \mathcal{S}_{d+1}$ we have that*

$$\|z^\beta f\|_{A^2_\alpha(\mathbb{B}_{d+1})} = \|z^\beta\|_{A^2_\alpha(\mathbb{B}_{d+1})}\|f\|_{A^2_{d+\alpha+|\beta|}(\mathbb{D})}$$

Proof. Writing $f = \sum_n \widehat{f}(n) z^n$, we have by orthogonality and Lemma 2.1.2

$$\begin{aligned}
\|z^\beta f\|^2_{A^2_\alpha(\mathbb{B}_{d+1})} &= \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \|z_1^n z^\beta\|^2_{A^2_\alpha(\mathbb{B}_{d+1})} \\
&= \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \frac{n!\beta!\Gamma(d+\alpha+2)}{\Gamma(d+n+|\beta|+\alpha+2)} \\
&= \frac{\beta!\Gamma(d+\alpha+2)}{\Gamma(d+|\beta|+\alpha+2)} \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \frac{n!\Gamma(d+|\beta|+\alpha+2)}{\Gamma(d+n+|\beta|+\alpha+2)} \\
&= \|z^\beta\|^2_{A^2_\alpha(\mathbb{B}_{d+1})} \|f\|^2_{A^2_{d+\alpha+|\beta|}(\mathbb{D})}
\end{aligned}$$

□

The following lemma regards a Carleson measure that appears in the proof of the main theorem.

Lemma 7.1.2. *Let $\beta \in \mathcal{S}_{d+1}$ and let $\varphi \in Hol(\mathbb{D})$ be such that $z^\beta \varphi \in H^2_{d+1}$. If $z^\beta \varphi \in \mathcal{X}(H^2_{d+1})$, then $|(|\beta| + R)^m \varphi|^{2m+|\beta|-1} dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$ for all $m \in \mathbb{N}$.*

Proof. If $\varphi \in \mathcal{X}(H^2_{d+1})$, then Corollary 7.1.1 implies that the measure $|R\varphi|^2(1-|z|^2)dA(z)$ is Carleson measure for $H^2(\mathbb{D})$. It follows immediately from Theorem 2.2 that the measure $|R^m\varphi|^2(1-|z|^2)^{2m-1}dA(z)$ is Carleson measure for $H^2(\mathbb{D})$ for all $m \in \mathbb{N}$.

If $\beta \in \mathcal{S}_{d+1}$ with $|\beta| \neq 0$, we have that the measure $|\varphi|^2(1-|z|^2)^{|\beta|-1}dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$ by Theorem 7.1. Furthermore, we have that $|R^k\varphi|^2(1-|z|^2)^{2k+|\beta|-1}dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$ for all $k \in \mathbb{N}_0$ by Theorem 2.2.

Thus if $f \in H^2(\mathbb{D})$, then we have

$$\begin{aligned}
\|f(|\beta|+R)^m\varphi\|_{A^2_{2m+|\beta|-1}(\mathbb{D})} &\leq \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} |\beta|^{m-k}\|fR^k\varphi\|_{A^2_{2m+|\beta|-1}(\mathbb{D})} \\
&\lesssim \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} |\beta|^{m-k}\|fR^k\varphi\|_{A^2_{2k+|\beta|-1}(\mathbb{D})} \\
&\lesssim \|f\|_{H^2(\mathbb{D})}
\end{aligned}$$

where we have used the pointwise inequalty $(1-|z|^2)^{2m+|\beta|-1} \leq (1-|z|^2)^{2k+|\beta|-1}$. Therefore $|(|\beta|+R)^m\varphi|(1-|z|^2)^{2m+|\beta|-1}dA(z)$ is Carleson measure for $H^2(\mathbb{D})$. □

Theorem 7.2. *Let $\beta \in \mathcal{S}_{d+1}$ and $\varphi \in Hol(\mathbb{D})$ be such that $z^\beta\varphi \in H^2_{d+1}$. Then $z^\beta\varphi \in \mathcal{X}(H^2_{d+1})$ if and only if $z^\beta\varphi \in \mathcal{C}H^2_{d+1}$.*

Proof. (⇐) We have already mentioned that this direction, regardless of the type of symbol, follows from Theorem 3.2.

(⇒) Let $m \in \mathbb{N}$ be such that $2m-(d+1) > -1$ and suppose that $z^\beta\varphi \in \mathcal{X}(H^2_{d+1})$. Let $f = \sum_{\alpha\in\mathcal{S}_{d+1}} z^\alpha f_\alpha \in H^2_{d+1}$. Using the orthogonality of monomials in $A^2_{2m-(d+1)}(\mathbb{B}_{d+1})$ and the fact that $R^m(z^\beta\varphi)$ is z^β times a function which only depends only z_1, we have by Lemma 7.1.1 that

$$\begin{aligned}
\|fR^m(z^\beta\varphi)\|^2_{A^2_{2m-(d+1)}(\mathbb{B}_d)} &= \sum_{\alpha\in\mathcal{S}_{d+1}} \|z^{\alpha+\beta}f_\alpha(|\beta|+R)^m\varphi\|^2_{A^2_{2m-(d+1)}(\mathbb{B}_d)} \\
&= \sum_{\alpha\in\mathcal{S}_{d+1}} \|z^{\alpha+\beta}\|_{A^2_{2m-(d+1)}(\mathbb{B}_{d+1})}(|\beta|+R)^m\varphi\|^2_{A^2_{2m+|\alpha|+|\beta|-1}(\mathbb{D})} \\
&\lesssim \sum_{\alpha\in\mathcal{S}_{d+1}} \frac{\alpha!}{|\alpha|!}\|f_\alpha(|\beta|+R)^m\varphi\|^2_{A^2_{2m+|\alpha|+|\beta|-1}(\mathbb{D})}.
\end{aligned}$$

where we have used that

$$\|z^{\alpha+\beta}\|_{A^2_{2m-(d+1)}(\mathbb{B}_{d+1})} \leq \|z^\alpha\|_{A^2_{2m-(d+1)}(\mathbb{B}_{d+1})} = \frac{\alpha!\Gamma(2m+1)}{\Gamma(2m+1+|\alpha|)} \lesssim \frac{\alpha!}{|\alpha|!}$$

For $\alpha = 0$, we have that

$$\|f_0(|\beta| + R)^m \varphi\|_{A^2_{2m+|\beta|-1}(\mathbb{D})} \lesssim \|f_0\|_{H^2(\mathbb{D})}$$

since $|(|\beta| + R)^m \varphi|^2 (1 - |z|^2)^{2m+|\beta|-1} dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$ by Lemma 7.1.2. Furthermore, for $\alpha \neq 0$, we use Lemma 2.1.3, which implies that $C := \sup_{z\in\mathbb{D}} |(|\beta| + R)^m \varphi(z)|^2 (1 - |z|^2)^{2m+|\beta|} < \infty$ and thus we have

$$\|f_\alpha(|\beta| + R)^m \varphi\|^2_{A^2_{2m+|\alpha|+|\beta|-1}} \leq C \|f_\alpha\|^2_{A^2_{|\alpha|-1}(\mathbb{D})}$$

It now follows that $|R^m(z^\beta \varphi)|^2 (1 - |z|^2)^{2m-(d+1)} dV(z)$ is a Carleson measure for H^2_{d+1}.

□

7.5 Concluding Remarks

For a function $f \in \text{Hol}(\mathbb{B}_{d+1})$, write $f(z) = \sum_{k=0}^{\infty} f_k(z)$ for the homogeneous expansion of f. For any real number $t > 0$, define the fractional derivative of f of order t to be

$$D^t f(z) = \sum_{n=0}^{\infty} (n+1)^t f_k(z).$$

Note that $D^0 f = f$ and $D^1 f = f + Rf$. The following result of Miroljub Jevtic provides a characterization of $BMOA(\mathbb{B}_{d+1})$ in terms of Carleson measures for the Hardy space $H^2(\mathbb{B}_{d+1})$ which involves these fractional derivatives.

Theorem 7.3 (from [19]). *For a function $f \in Hol(\mathbb{B}_{d+1})$, the following are equivalent*

1. $f \in BMOA(\mathbb{B}_{d+1})$
2. *There is a $t > 0$ such that*

$$|D^t f(z)|^2 (1 - |z|^2)^{2t-1} dV(z)$$

is a Carleson measure for $H^2(\mathbb{B}_{d+1})$.

3. *For all* $t > 0$ *we have that*

$$|D^t f(z)|^2 (1 - |z|^2)^{2t-1} dV(z)$$

is a Carleson measure for $H^2(\mathbb{B}_{d+1})$.

Using this result, one can obtain a method for constructing functions which satisfy the equivalent conditions of Theorem 7.2.

Theorem 7.4. *Let* $\beta \in \mathcal{S}_{d+1}$ *and* $\varphi \in Hol(\mathbb{D})$ *be such that* $z^\beta \varphi \in H^2_{d+1}$. *Then the following are equivalent*

1. $z^\beta \varphi \in \mathcal{X}(H^2_{d+1})$
2. $z^\beta \varphi \in \mathcal{C}H^2_{d+1}$
3. *there is a function* $b \in BMOA(\mathbb{D})$ *such that* $\varphi = D^{\frac{|\beta|}{2}} b$.

Proof. Writing $\varphi(z) = \sum_{n=0}^{\infty} \widehat{\varphi}(n) z^n$, one easily sees that the function $b(z) = \sum_{n=0}^{\infty} (n+1)^{-\frac{|\beta|}{2}} \widehat{\varphi}(n) z^n$ is holomorphic on $\mathbb{D}$ and satisfies $D^{\frac{|\beta|}{2}} b = \varphi$.

If $\beta = 0$, then $b = \varphi$. Furthermore, Corollary 7.1.1 then implies that φ satisifes (1) and (2) if and only if $|R\varphi(z)|^2(1 - |z|^2) dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$. However, this is equivalent to $\varphi \in BMOA(\mathbb{D})$, as mentioned earlier.

If $|\beta| > 0$, then Theorem 7.1 implies that φ satisfies (1) or (2) if and only if $|\varphi(z)|^2 (1 - |z|^2)^{|\beta|-1} dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$. Therefore, we have that $|D^{\frac{|\beta|}{2}} b(z)|^2 (1 - |z|^2)^{|\beta|-1} dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$. By Jevtic's result, this is equivalent to $b \in BMOA(\mathbb{D})$. □

We conclude the paper with a generalization of the main theorem which easily follows from the results of the previous sections. For $i \in \{1, 2, ..., d+1\}$, define

$$S^i_{d+1} = \left\{ \alpha = (\alpha_1, ..., \alpha_{d+1}) \in \mathbb{N}_0^{d+1} \mid \alpha_i = 0 \right\}.$$

It is clear that all of the previous results and their proofs remain valid if $\mathcal{S}_{d+1}$ is replaced with $\mathcal{S}_{d+1}^i$ for any $i \in \{1, 2, ..., d+1\}$.

Theorem 7.5. *Let $N \in \mathbb{N}$ be fixed, and let*

$$\varphi = \sum_{i=1}^{d+1} \sum_{\beta \in \mathcal{S}_{d+1}^i, |\beta| \leq N} z^\beta \varphi_{\beta,i} \in H_{d+1}^2$$

where each $\varphi_{\beta,i} \in Hol(\mathbb{D})$ is a function of the variable z_i. Then the following are equivalent

1. *$\varphi \in \mathcal{X}(H_{d+1}^2)$*

2. *$z^\beta \varphi_{\beta,i} \in \mathcal{X}(H_{d+1}^2)$ for all $\beta \in \mathcal{S}_{d+1}^i$ with $|\beta| \leq N$ and for all $i \in \{1, ..., d+1\}$.*

3. *The measure*

$$|R^m(z^\beta \varphi_{\beta,i})|^2 (1 - |z|^2)^{2m-(d+1)} dV(z)$$

   is a Carleson measure for H_{d+1}^2 for all $\beta \in \mathcal{S}_{d+1}^i$ with $|\beta| \leq N$, any $i \in \{1, ..., d+1\}$, and any $m \in \mathbb{N}$ with $2m - (d+1) > -1$.

4. *$\varphi \in \mathcal{C}H_{d+1}^2$.*

Proof. $(i) \Rightarrow (ii)$. For any $f, g \in \mathrm{Hol}(\overline{\mathbb{D}})$, we see that if (i) holds, then for any $\beta \in S_{d+1}^i$, we have

$$|\langle fg, \varphi_{\beta,i} \rangle_{A_{|\beta|-1}^2}| = \frac{|\beta|!}{\beta!} |\langle fgz^\beta, \varphi \rangle_{H_{d+1}^2}| \leq \|\varphi\|_{\mathcal{X}(H_{d+1}^2)} \|f\|_{H^2(\mathbb{D})} \|g\|_{A_{|\beta|-1}^2}$$

so that (ii) holds by Theorem 7.1.

$(ii) \Rightarrow (iii)$. This is immediate from Theorem 7.2.

$(iii) \Rightarrow (iv)$. This follows by the linearity of R^m and the convexity of the real-valued map $x \mapsto x^2$.

$(iv) \Rightarrow (i)$. This follows by Theorem 3.2. $\square$

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Vita

James Allen Sunkes, III was born in Huntsville, Alabama to Jim and Mary Sunkes in 1987. After graduating from Madison County High School in 2005, James continued his education at Tennessee Technological University in Cookeville, TN. During his sophomore year there, he changed his major from Physics to Mathematics, after completing the introductory Calculus sequence. After completing his mathematics degree at Tennessee Tech, James continued his studies at the University of Tennessee pursuing the fields of complex analysis and operator theory. James is now pursuing a job in academia, where he is hoping he can teach at the collegiate level. James lives with his wife, Kimberly, and is expecting a child in May of 2016.