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Duality of Scales

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To the Graduate Council:

I am submitting herewith a dissertation written by Michael Christopher Holloway entitled "Duality of Scales." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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To my parents, Frank and Ruth. Thank you for your unending support.

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Abstract

We establish an interaction between the large scale and small scale using two types of maps from large scale spaces to small scale spaces. First we use slowly oscillating maps, which can be described as those having arbitrarily small variation at infinity. These lead to a Galois connection between certain collections of large scale structures and small scale structures on a given set. Slowly oscillating functions can also be used to define to the notion of a dual pair of scale structures on a space. A dual pair consists of a large and a small scale structure on a space which is maximal with respect to the identity map being slowly oscillating. Finally, slowly oscillating functions and dual pairs are used to explain several well-known classes of large scale structures. The second type of maps studied are pinch-spacing maps. These are maps which respect the large scale structure of the domain, but only at a fixed scale. We use pinch-spacing to characterize and explain connections between the large scale properties of finite asymptotic dimension, property A, coarse embeddability into Hilbert space, exactness, and large scale paracompactness.

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Chapter 1

Introduction

In this dissertation, we will study the category of coarse spaces and that of uniform spaces using maps between the two types of spaces. The goal is to emphasize the interaction between the large and small scale. Uniform space theory, initiated by André Weil in [30], studies the small scale properties of a space. It takes concepts from metric space theory, such as uniform continuity and uniform convergence and extends them to spaces which need not have a metric. The tools of uniform space theory have proven useful in the study of topological groups, which possess a canonical uniform structure. There is a connection between topology and uniform space theory in that every uniform structure has an associated topology; however, there is a key difference in the two fields: while the open sets in a topology are a local concept, a uniform structure consists of covers of the space, so it is global in nature.

Dual to uniform space theory is coarse geometry, which is concerned with the large scale properties of a space. The theory of coarse structures was developed by Higson and Roe in studying signature theory and index theory [12]. Like uniform structures, coarse structures naturally arise from a metric, but also can be defined in a more general setting for spaces which need not necessarily possess a metric. The original motivation for defining coarse structures was that it provided an approach to the Novikov conjecture, a long-standing and still open conjecture which states that the higher signatures of a smooth manifold are homotopy invariant.

In this dissertation, we will study both uniform spaces and coarse spaces. In particular, we will investigate two types of maps between coarse spaces and uniform spaces and the ways these types of maps can be used to reveal information about the associated to structures. The layout is as follows. First, large and small scale structures are defined and their basic properties are reviewed. Next, the large and small scale categories are introduced and we discuss several categorical constructions such as covariant and contravariant structures. Also, direct and inverse limits of scale structures are created as a way to introduce metric approximations of scale spaces. Next, slowly oscillating functions are introduced as a way to connect the small and large scale categories. The notion of a slowly oscillating function was first considered by Higson in the context of metric spaces; see [26]. The collection of continuous, complex-valued, slowly oscillating maps determines a compactification, called the Higson compactification, of a proper metric space. It has been shown that the covering dimension of the Higson corona of a proper metric space is no larger than the asymptotic dimension of that space [7]. Thus, slowly oscillating functions can be used to detect certain large scale properties of a space. We generalize the notion of a slowly oscillating map to functions between large scale spaces and small scale spaces. We will use slowly oscillating maps to establish a link between the set of large scale structures on a given space and the set of small scale structures on that space and in this way explain several well-known large scale structures in terms of slowly oscillating maps. We introduce the notion of a dual pair of scale structures and devise a scheme to construct such dual pairs. It is shown that an element of a dual pair has desirable properties, for example any large scale structure which is an element of a dual pair is coarsely normal. We will show that this construction gives a Galois connection between the large scale structures on a space and the small scale structures on a space. In the final chapter, we introduce the notion of a pinch-spacing map, which is another type of map between large scale spaces and small scale spaces. Using this notion, we produce a new framework for the connection between the large scale properties of finite asymptotic dimension, property A of Guoliang Yu, and coarsely embeddability in Hilbert space. We also use pinch-spacing as a new way to discuss the large scale properties of exactness and large scale paracompactness. In this way, pinch-spacing maps are a more powerful tool than slowly oscillating maps for the purpose of studying the large scale properties of a space, because for a large class of spaces, all of the above properties can be characterized in terms of pinch-spacing maps.

Chapter 2

Scale Structures

2.1 Scales

In this chapter we will define the notion of a small scale structure and a large scale structure on a space. These structures are collections of covers of the space satisfying certain axioms, which formalize the notion of zooming into and zooming out from a space. First, we establish some basic terminology. Much of the material from this chapter comes from [2] and [3].

Definition 2.1.1. By a **scale** on a space X, we mean any cover $\mathcal{U}$ of X. That is, a scale $\mathcal{U}$ is a collection of subsets of X such that $\bigcup_{U \in \mathcal{U}} U = X$.

In defining large and small scale structures, we need a way to compare scales; that is, we want a way to determine if one scale is smaller or larger than another. To do this, we use the notion of star-refinement.

Definition 2.1.2. Let $\mathcal{U}$ and $\mathcal{V}$ be two collections of subsets of X. We say that $\mathcal{U}$ **refines** $\mathcal{V}$, denoted $\mathcal{U} \prec \mathcal{V}$, if for every $U \in \mathcal{U}$, there is some $V \in \mathcal{V}$ such that $U \subseteq V$. If $\mathcal{U} \prec \mathcal{V}$, then we also say that $\mathcal{V}$ **coarsens** $\mathcal{U}$.

Definition 2.1.3. Let $U \subseteq X$ and $\mathcal{V}$ be a collection of subsets of X. Then the **star of U against $\mathcal{V}$** is defined by

$$st(U, \mathcal{V}) = \bigcup \{V \in \mathcal{V} : U \cap V \neq \emptyset\}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are two collections of subsets of X, then the **star of $\mathcal{U}$ against $\mathcal{V}$** is defined as

$$st(\mathcal{U}, \mathcal{V}) = \{st(U, \mathcal{V}) : U \in \mathcal{U}\}.$$

Informally speaking, the process of starring a set against a cover fattens up the set by that cover. For example, in a geodesic metric space, if $\mathcal{V}$ is the collection of all radius r balls and U is an arbitrary subset, then $st(U, \mathcal{V})$ is equal to the $2r$-neighborhood of U; that is, all points within distance $2r$ of a point of U.

Definition 2.1.4. We say that $\mathcal{U}$ is a **smaller scale** than $\mathcal{V}$ if $st(\mathcal{U}, \mathcal{U}) \prec \mathcal{V}$. In this case, we also say that $\mathcal{V}$ is a **larger scale** than $\mathcal{U}$. In the literature, if $\mathcal{U}$ is a smaller scale than $\mathcal{V}$, then it is said that $\mathcal{U}$ star-refines $\mathcal{V}$.

When comparing scales, we don't just require refinement to have a smaller scale, but demand the stronger condition of star-refinement.

We now prove some basic but useful lemmas.

Lemma 2.1.5. *1) If $U \subseteq X$ and $\mathcal{V}$ is a scale of X, then $U \subseteq st(U, \mathcal{V})$.*

2) If $\mathcal{U}$ is a collection of subsets of X and $\mathcal{V}$ is a scale of X, then $\mathcal{U} \prec st(\mathcal{U}, \mathcal{V})$.

3) If $U \subseteq V$ and $\mathcal{W}$ is any collection of subsets of X, then $st(U, \mathcal{W}) \subseteq st(V, \mathcal{W})$.

4) If $\mathcal{U} \prec \mathcal{V}$ and $\mathcal{W}$ is any collection of subsets of X, then $st(\mathcal{U}, \mathcal{W}) \prec st(\mathcal{V}, \mathcal{W})$.

5) If $U \subseteq X$ and $\mathcal{V}_1, \mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1 \prec \mathcal{V}_2$, then $st(U, \mathcal{V}_1) \subseteq st(U, \mathcal{V}_2)$.

6) If $\mathcal{U}$ is any collection of subsets of X and $\mathcal{V}_1, \mathcal{V}_2$ are collections of subsets of X with $\mathcal{V}_1 \prec \mathcal{V}_2$, then $st(\mathcal{U}, \mathcal{V}_1) \prec st(\mathcal{U}, \mathcal{V}_2)$.

7) If $\mathcal{U}$ is a scale and $\mathcal{V}$ is any collection of subsets of X, then $\mathcal{V} \prec st(\mathcal{U}, \mathcal{V})$.

Proof. 1) Let $x \in U$. There is some $V \in \mathcal{V}$ containing x. Hence $V \cap U \neq \emptyset$. Thus, $x \in V \subseteq st(U, \mathcal{V})$.

2) This follows from 1).

3) Let $x \in st(U, \mathcal{W})$. Then there is some $W \in \mathcal{W}$ containing x such that $W \cap U \neq \emptyset$. Hence, $W \cap V \neq \emptyset$. Thus, $x \in W \subseteq st(V, \mathcal{W})$.

4) This follows from 3).

5) Let $x \in st(U, \mathcal{V}_1)$. Then there is some $V_1 \in \mathcal{V}_1$ containing x such that $V_1 \cap U \neq \emptyset$. Since $\mathcal{V}_1 \prec \mathcal{V}_2$, there is some $V_2 \in \mathcal{V}_2$ such that $V_1 \subseteq V_2$. Then V_2 contains x and $V_2 \cap U \neq \emptyset$. Hence, $x \in V_2 \subseteq st(U, \mathcal{V}_2)$.

6) This follows from 5).

7) Let $V \in \mathcal{V}$ and let $x \in V$. Since $\mathcal{U}$ is a scale of X, then there is some $U \in \mathcal{U}$ containing x. Hence, $V \subseteq st(U, \mathcal{V})$. □

Let $f : X \to Y$ be any function between spaces, let $\mathcal{U}$ be a collection of subsets of X, and let $\mathcal{V}$ be a collection of subsets of Y. Define

$$f(\mathcal{U}) := \{f(U) \mid U \in \mathcal{U}\}$$

and

$$f^{-1}(\mathcal{V}) := \{f^{-1}(V) \mid V \in \mathcal{V}\}.$$

Lemma 2.1.6. *If $f : X \to Y$ is any function and $\mathcal{U}, \mathcal{V}$ are any collections of subsets of X, then $f(st(\mathcal{U}, \mathcal{V})) \prec st(f(\mathcal{U}), f(\mathcal{V}))$.*

Proof. This is obvious since if $U \in \mathcal{U}$, then $f(st(U, \mathcal{V})) \subseteq st(f(U), f(\mathcal{V}))$. □

Lemma 2.1.7. *Let $f : X \to Y$ be a function and $\mathcal{U}, \mathcal{V}$ be collections of subsets of Y.*

1) $st(f^{-1}(\mathcal{U}), f^{-1}(\mathcal{V})) \prec f^{-1}(st(\mathcal{U}, \mathcal{V}))$.

2) If f is surjective, then $st(f^{-1}(\mathcal{U}), f^{-1}(\mathcal{V})) = f^{-1}(st(\mathcal{U}, \mathcal{V}))$.

Proof. 1) Let $U \in \mathcal{U}$. Observe that

$$st(f^{-1}(U), f^{-1}(\mathcal{V})) = \bigcup \{f^{-1}(V) : f^{-1}(V) \cap f^{-1}(U) \neq \emptyset\}$$

$$= \bigcup\{f^{-1}(V) : f^{-1}(V \cap U) \neq \emptyset\}$$
$$= f^{-1}(\bigcup\{V : f^{-1}(V \cap U) \neq \emptyset\})$$
$$\subseteq f^{-1}(\bigcup\{V : V \cap U \neq \emptyset\})$$
$$= f^{-1}(st(U, \mathcal{V})).$$

2) We showed in the proof of 1) that for $U \in \mathcal{U}$, is is the case that $st(f^{-1}(U), f^{-1}(V)) \subseteq f^{-1}(st(U, \mathcal{V}))$. Now let $x \in f^{-1}(st(U, \mathcal{V}))$. Then there is some $V \in \mathcal{V}$ so that $V \cap U \neq \emptyset$ and $f(x) \in V$. Say $p \in V \cap U$. By the surjectivity of f, $p = f(q)$ for some q. Then $q \in f^{-1}(U) \cap f^{-1}(V)$, implying that $x \in st(f^{-1}(U), f^{-1}(V))$. □

2.2 Metric Scale Structures

The most natural scale structures arise in metric spaces or more generally pseudo-metric spaces. We first recall the definition of a metric space.

Definition 2.2.1. Let X be a space. A **pseudo-metric** on X is a function $p : X \times X \to \mathbb{R} \cup \{\infty\}$ such that for all $x, y, z \in X$, the following are satisfied:

1) $p(x, y) \geq 0$;

2) $p(x, y) = p(y, x)$;

3) $p(x, z) \leq p(x, y) + p(y, z)$ (triangle inequality).

If it is true that $p(x, y) > 0$ whenever x and y are distinct, then p is said to be a **metric**. A space equipped with a (pseudo)-metric is called a **(pseudo)-metric space**. In cases where we need to specify the metric on X, we will use the notation (X, p).

Notice that we do not require that our pseudo-metrics take finite values. Hence, all pseudo-metrics considered may be ∞-metrics.

If (X, d) is a (pseudo)-metric space, $x \in X$ and $r > 0$, then define the **r-ball about** x to be the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Definition 2.2.2. If U is a subset of metric space X, then we say the **diameter** of U, denoted $diam(U)$, is defined by

$$diam(U) = sup\{d(x, y) : x, y \in U\},$$

where the sup may take value ∞.

If $\mathcal{U}$ is a collection of subsets of X, then the **mesh** of $\mathcal{U}$, denoted $mesh(\mathcal{U})$, is defined by

$$mesh(\mathcal{U}) = sup\{diam(U) : U \in \mathcal{U}\},$$

where again this value may possibly be ∞.

Define the **Lebesgue number** of $\mathcal{U}$, denoted $Leb(\mathcal{U})$, to be the supremum of the set of nonnegative numbers r so that every subset of X of diameter less that r is contained in some element of $\mathcal{U}$.

In a metric space, there is a natural notion of scale which arises from the metric. For example, we may fix a positive number R and look at the space at scale R. For a point x in the space, the points close to x are scale R are those contained in the R-ball about x.

Definition 2.2.3. Let X be a metric space. The **metric small scale structure on** X is defined to be collection of all scales $\mathcal{U}$ on X such that $Leb(\mathcal{U}) > 0$.

The metric small scale structure consists of all scales which are in some sense thick or fat. In order to be in the metric small scale structure, the elements of a scale must have in some sense large overlap with each other, or at least not have arbitrarily small overlap.

We remark that the Lebesgue number lemma, which states that every open cover of a compact metric space has positive Lebesgue number, can be interpreted as saying that every open cover of a compact metric space is included in the metric large scale structure.

Definition 2.2.4. The **metric large scale structure on** X is defined to be the collection of all scales $\mathcal{U}$ on X such that $mesh(\mathcal{U}) < \infty$.

Notice that the be an element of the metric large scale structure, it is not enough for each element of $\mathcal{U}$ to have finite diameter. Rather, it is necessary that there be some uniform bound on the diameters of all elements of the scale. For this reason, an element of the metric large scale structure is sometimes called a "uniformly bounded cover" of X.

At first glance, it may seem that these structures are defined in completely different manners. However, we can slightly modify the definitions of Lebesgue number and mesh to emphasize that they are really dual concepts of each other.

Let $\mathcal{U}$ be a scale on a metric space X. Define

$$Leb(\mathcal{U}) = \sup\{\lambda > 0 : \{B(x, \lambda) : x \in X\} \prec \mathcal{U}\}$$

and

$$mesh(\mathcal{U}) = \inf\{\lambda < \infty : \mathcal{U} \prec \{B(x, \lambda) : x \in X\}\}.$$

Notice that if we use the above definitions, it is clear that mesh is the dual notion to Lebesgue number. Additionally, we are able to describe the 'size' of a cover by comparing it to a specific collection of scales - those consisting of all balls of a fixed radius. Rather than relying only on the metric to compare sizes of scales, we are using the notion of refinement to determine size.

2.3 General Scale Structures

If $\mathcal{U}$ and $\mathcal{V}$ are scales in the metric small scale structure on the metric space (X, d), then it is always possible to find a scale smaller than both $\mathcal{U}$ and $\mathcal{V}$. Specifically, if we define $\lambda = \min\{Leb(\mathcal{U}), Leb(\mathcal{V})\} > 0$, then the collection $\mathcal{W} := \{B(x, \lambda/6) = \{y \in X : d(x, y) < \lambda/6\}$ is an element of the metric small scale structure and is a smaller scale than both $\mathcal{U}$ and $\mathcal{V}$ since the diameter of an element of $st(\mathcal{W}, \mathcal{W})$ is no larger than λ. Informally, when we change from scales $\mathcal{U}$ and $\mathcal{V}$ to scale $\mathcal{W}$, we have zoomed into the space by finding a smaller scale.

On the other hand, given two uniformly bounded covers $\mathcal{U}$ and $\mathcal{V}$, it is always possible to find a scale larger than both $\mathcal{U}$ and $\mathcal{V}$. Namely, let $\lambda = \max\{mesh(\mathcal{U}), mesh(\mathcal{V})\}$, then $\mathcal{W} := \{B(x, 3\lambda\}$ is a uniformly bounded cover which is larger than both $\mathcal{U}$ and $\mathcal{V}$. Here, by finding a larger scale, we are zooming out from the space.

Put another way, the above paragraphs say that in the metric small scale structure it is always possible to zoom in and in the metric large scale structure it is always possible to zoom out. This is the key property that we wish capture in the definition of small and large scale structures. To define these scale structures, we will use terminology from the theory of partially ordered sets.

Definition 2.3.1. For a set X, a **quasiorder** on X is a relation $<$ on X which is transitive. That is, if $x, y, z \in X$ such that $x < y$ and $y < z$, then $x < z$.

Definition 2.3.2. If $(X, \leq)$ is a set with a quasiorder, then a **filter** is a subset $F \subseteq X$ satisfying:

1) F is nonempty;

2) if $x, y \in F$, then there exists $z \in F$ such that $z < x$ and $z < y$; and

3) if $x \in F$, $y \in X$ and $x < y$, then $y \in F$.

If X is a space, then we can quasiorder the collection of all scales of X by using either star-refinement or reverse star-refinement.

Definition 2.3.3. A **small scale structure** on a space X is a filter on the collection of scales on X, where the quasiorder is given by star-refinement. The elements of a small scale structure are called **small scales**. In this case, we say that X is a **small scale space**.

Put another way, a small scale structure is a collection of scales on a space X such that any scale larger than a small scale is also a small scale and given any two small scales, it is always possible to find a small scale smaller than both.

Definition 2.3.4. A **large scale structure** on a space X is a filter on the collection of scales on X, where the quasiorder is given by reverse star-refinement. The elements of a large scale structure are called **large scales**. In this case, we say that X is a **large scale space**.

Put another way, a large scale structure is a collection of scales on a space X such that any scale smaller than a large scale is also a large scale and given any two large scales, it is always possible to find a small scale larger than both.

The ability to always find a smaller scale in a small scale space and to always find a larger scale in a large scale space can be interpreted as saying that it is always possible to zoom in further into a small scale space and that in a large scale space it is always possible to zoom out from any scale to a larger scale.

We now prove an equivalent definition of a large scale structure which is sometimes more useful.

Proposition 2.3.5. *A nonempty collection L of scales on a space X forms a large scale structure on X if and only if it satisfies*

1) for scales $\mathcal{U}$ and $\mathcal{V}$ on X, if $\mathcal{U} \in L$ and $\mathcal{V} \prec \mathcal{U}$, then $\mathcal{V} \in L$;

2) if $\mathcal{U}, \mathcal{V} \in L$, then $st(\mathcal{U}, \mathcal{V}) \in L$.

Proof. ($\Rightarrow$): First assume that L forms a large scale structure on X. Let $\mathcal{U} \in L$ and suppose that $\mathcal{V}$ is a scale of X such that $\mathcal{V} \prec \mathcal{U}$. Notice that $\mathcal{U} \in L$ implies that there is some $\mathcal{W} \in L$ so that $st(\mathcal{U},\mathcal{U}) \prec \mathcal{W}$. Hence, by Lemma 2.1.5, $st(\mathcal{V},\mathcal{V}) \prec st(\mathcal{U},\mathcal{U}) \prec \mathcal{W}$, which implies that $\mathcal{V} \in L$ since L is a filter under reverse star refiment.

Now suppose that $\mathcal{U}, \mathcal{V} \in L$. We know that there is some $\mathcal{W} \in L$ so that $st(\mathcal{U},\mathcal{U}) \prec \mathcal{W}$ and $st(\mathcal{V},\mathcal{V}) \prec \mathcal{W}$. Also, there is some $\mathcal{W}'$ so that $st(\mathcal{W},\mathcal{W}) \prec \mathcal{W}'$. Hence, applying Lemma 2.1.5 again, we have that $st(\mathcal{U},\mathcal{V}) \prec st(\mathcal{W},\mathcal{W}) \prec \mathcal{W}'$. Then by the above paragraph, $st(\mathcal{U},\mathcal{V}) \in L$, which finishes the proof of the forward direction.

($\Leftarrow$): Now assume that L satisfies the given two properties. Let $\mathcal{U} \in L$ and suppose that $\mathcal{V}$ is a scale on X such that $st(\mathcal{V},\mathcal{V}) \prec \mathcal{U}$. Notice that $\mathcal{V} \prec \mathrm{st}(\mathcal{V},\mathcal{V}) \prec \mathcal{U}$, which implies that $\mathcal{V} \prec \mathcal{U}$, so by assumption, $\mathcal{V} \in L$.

Now assume that $\mathcal{U}, \mathcal{V} \in L$. We need to find an element of L which is larger than both. We know that $st(\mathcal{U},\mathcal{U}), st(\mathcal{V},\mathcal{V}) \in L$. Hence, $st(st(\mathcal{U},\mathcal{U}), st(\mathcal{V},\mathcal{V})) \in L$. Again, applying Lemma 2.1.5, we have that $st(\mathcal{U},\mathcal{U}) \prec st(st(\mathcal{U},\mathcal{U}), st(\mathcal{V},\mathcal{V}))$ and $st(\mathcal{V},\mathcal{V}) \prec st(st(\mathcal{U},\mathcal{U}), st(\mathcal{V},\mathcal{V}))$. □

The following result shows that large scale structures are closed under finite unions.

Lemma 2.3.6. *Let L be a large scale structure. If $\mathcal{U}, \mathcal{V} \in L$, then $\mathcal{U} \cup \mathcal{V} \in L$.*

Proof. By Lemma 2.1.5, $\mathcal{U} \prec st(\mathcal{U},\mathcal{V})$ and $\mathcal{V} \prec st(\mathcal{U},\mathcal{V})$, so $\mathcal{U} \cup \mathcal{V} \prec st(\mathcal{U},\mathcal{V}) \in L$. This implies that $\mathcal{U} \cup \mathcal{V} \in L$. □

Similar to defining a topology on a space, we need not specify all elements of a scale structure but need only give a base for the structure.

Definition 2.3.7. A **base of a small scale structure** is a nonempty collection $\mathcal{B}$ of scales such that if $\mathcal{U} \in \mathcal{B}$ and $\mathcal{V} \in \mathcal{B}$, then there is some $\mathcal{W}$ in $\mathcal{B}$ such that $\mathcal{W}$ is smaller than both $\mathcal{U}$ and $\mathcal{V}$.

A **base of a large scale structure** is a nonempty collection $\mathcal{B}$ of scales such that if $\mathcal{U} \in \mathcal{B}$ and $\mathcal{V} \in \mathcal{B}$, then there is some $\mathcal{W}$ in $\mathcal{B}$ such that $\mathcal{W}$ is larger than both $\mathcal{U}$ and $\mathcal{V}$.

Notice that the above is just stating that a base of a scale structure is a filter base under the appropriate ordering. Given a base for a small (large) scale structure, this base can be used to define a small (large) scale structure. One simply takes all scales which are larger (smaller) than some element of the base.

We point out that in order for $\mathcal{B}$ to be a large scale basis, it suffices that $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ implies that there is some $\mathcal{W} \in \mathcal{B}$ such that $st(\mathcal{U}, \mathcal{V}) \prec \mathcal{W}$.

Traditionally, large scale and small scale structures on a space X have been defined using subsets of the product $X \times X$. These sets are often called entourages. In the appendix, we show that using the entourage approach gives an alternative, but it some sense equivalent way to place such structures on a space. That is, there is a natural way to translate from the scales approach to the entourage approach. The covers approach to large scale structures first appeared in [8].

2.4 Topologies and Collections of Bounded Sets

Associated to a small scale structure is a natural topology. It is defined as follows. If X is a space with a small scale structure, then a subseteq $U \subseteq X$ is open if for each $p \in U$ there is a small scale $\mathcal{V}$ so that $st(\{p\}, \mathcal{V}) \subseteq U$.

A small scale structure is called **Hausdorff** if given points x, y in X, there is some small scale $\mathcal{U}$ so that no element of $\mathcal{U}$ contains both x and y. For example, the metric small scale structure is Hausdorff. On the other hand, if (X, p) is a pseudo-metric space and there are distinct points $x, y \in X$ such that $d(x, y) = 0$, then the associated uniform structure is not Hausdorff since $diam(\{x, y\}) = 0$ so must be contained in some element of every small scale.

A Hausdoff uniform structure generates a Hausdorff topology. An easy way to see this is the following set of lemmas:

Lemma 2.4.1. *If X is a small scale space. If $U \subseteq X$, then the interior of U in the topology generated by the small scale structure is equal to the set*

$$I = \{x \in U \mid \text{ there exists a small scale } \mathcal{V} \text{ so that } st(\{x\}, \mathcal{V}) \subseteq U\}.$$

Proof. First, note that $int(U) \subseteq I$ since if $x \in int(U)$, there is some scale $\mathcal{V}$ so that $st(\{x\},\mathcal{U}) \subseteq int(U) \subseteq U$. So we need only show that I is an open subset of U. Let $x \in I$. So we can choose a small scale $\mathcal{V}$ so that $st(\{x\}, \mathcal{V}) \subseteq U$. Choose a scale $\mathcal{W}$ smaller than $\mathcal{V}$. We claim that $st(\{x\}, \mathcal{W}) \subseteq I$, which will prove that I is open. Say $x, y \in W \in \mathcal{W}$. Then $x \in st(\{y\}, \mathcal{W}) \subseteq st(W, \mathcal{W}) \subseteq V$ for some $V \in \mathcal{V}$. But then $st(\{y\}, \mathcal{W}) \subseteq V \subseteq \mathrm{st}(\{x\}, \mathcal{V}) \subseteq U$, proving that $y \in I$. □

Lemma 2.4.2. *If $\mathcal{U}$ is a small scale then the scale $int(\mathcal{U}) := \{int(U) : U \in \mathcal{U}\}$ is also a small scale, where $int(U)$ denotes the interior of U in the topology generated by the small scale.*

Proof. First we show that $int(\mathcal{U})$ is a scale of X. Indeed, let $x \in X$. Choose a scale $\mathcal{V}$ smaller than $\mathcal{U}$. Say $x \in V \in \mathcal{V}$. Then $st(\{x\}, \mathcal{V}) \subseteq st(V, \mathcal{V}) \subseteq U \in \mathcal{U}$. Thus, $x \in int(U)$, proving that $int(\mathcal{U})$ is a cover. To show that it is a small scale, we need to show that it is larger than some small scale. We claim that it is larger than $\mathcal{V}$. Indeed, let $V \in \mathcal{V}$. Then $st(V, \mathcal{V}) \subseteq U$ for some $U \in \mathcal{U}$. Also, if $x \in st(V, \mathcal{V})$, then $st(\{x\}, \mathcal{V}) \subseteq st(V, \mathcal{V}) \subseteq U$, proving that $x \in int(U)$. Thus, $\mathcal{V}$ is a smaller scale that $int(\mathcal{U})$. □

Lemma 2.4.3. *The topology generated by a Hausdorff small scale structure on a space X is Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$. Then there is some small scale $\mathcal{U}$ so that no element of $\mathcal{U}$ contains both x and y. Let $\mathcal{V}$ be a smaller scale and consider the small scale $int(\mathcal{V})$. We may choose elements $A, B \in int(\mathcal{V})$ so that $x \in A$ and $y \in B$. We claim that $A \cap B = \emptyset$. Suppose not. Then there is some $z \in A \cap B$. But then $x, y \in st(A, B) \subseteq U$ for some $U \in \mathcal{U}$, a contradiction. □

It is actually true that the topology generated by a Hausdorff small scale structure is Tychonoff. Conversely, if X is a Tychonoff space, then there is some small scale structure on X so that the topology on X in generated that small scale structure [13].

A topology is basically a collection of subsets of a space satisfying certain axioms. As the above discussion shows, there is a natural topology associated to any small scale structure. Since large scale structures are a dual structure, there is a dual collection of subsets associated to any large scale structure.

Definition 2.4.4. Let X be a large scale space. A set $B \subseteq X$ is **bounded** if there is some large scale $\mathcal{U}$ so that $B \in \mathcal{U}$.

In order to emphasize the duality between collections of bounded sets and topologies, we can use a slightly modified definition of bounded set. Say that a set $B \subseteq X$ is bounded if there is some $x \in B$ and large scale $\mathcal{U}$ so that $B \subseteq st(\{x\}, \mathcal{U})$. This gives the same collection as above, but is dual to the notion of an open set from a small scale.

A collection of bounded sets coming a large scale structure has certain properties which it must satisfy. The following result shows that any collection of subsets satisfying these properties comes from some (not necessarily unique) large scale structure on the space.

Proposition 2.4.5. *A collection $\mathcal{B}$ of subsets of a space X is the collection of bounded subsets for some large scale structure on X if and only if*
1) $\mathcal{B}$ contains all singleton subsets of X;
2) $\mathcal{B}$ is closed under taking subsets; and
3) if $B \in \mathcal{B}$ and $\mathcal{F} \subseteq \mathcal{B}$ with $|\mathcal{F}| < \infty$, then $st(B, \mathcal{F}) \in \mathcal{B}$.

Proof. Suppose that $\mathcal{B}$ is the collection of bounded subsets of a large scale structure $\mathcal{L}$ on X. Clearly, $\mathcal{B}$ contains all singleton subsets of X and is closed under taking subsets. Let $B \in \mathcal{B}$ and let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a finite subset of $\mathcal{B}$. We have $\{B\} \in \mathcal{L}$ and $\{F_i\} \in \mathcal{L}$ for each i. By Corollary 2.3.6, a large scale structure is closed under finite unions, so $\mathcal{F} \in \mathcal{L}$. But then $st(\{B\}, \mathcal{F}) \in \mathcal{L}$, implying that $st(B, \mathcal{F})$ is bounded.

Now suppose that $\mathcal{B}$ is a collection of subsets of X satisfying the three conditions. We will define a large scale structure on X having bounded sets $\mathcal{B}$. Define $\mathcal{L}$ by $\mathcal{A} \in \mathcal{L}$ if and only if there are finitely many $B_1, \ldots, B_n \in \mathcal{B}$ such that for each nonsingleton set $S \in \mathcal{A}$ there is some $1 \leq i \leq n$ such that $S \subseteq B_i$. Suppose that $\mathcal{A}_1 \in \mathcal{L}$ and each element of $\mathcal{A}_2$ containing more than one point is contained in some element of $\mathcal{A}_1$. There are finitely many $B_1, \ldots, B_n \in \mathcal{B}$ such that each nonsingleton set of $\mathcal{A}_1$ is contained in some B_i and clearly the set $\mathcal{A}_2$ has the same property with respect to $B_1, \ldots, B_n$, implying that $\mathcal{A}_2 \in \mathcal{L}$. Now suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are in $\mathcal{L}$. So we have a finite collection $B_1, \ldots, B_n \in \mathcal{B}$ associated to $\mathcal{A}_1$ and a finite collection $C_1, \ldots, C_m \in \mathcal{B}$ associted to $\mathcal{A}_2$. Note that each nonsingleton set of $st(\mathcal{A}_1, \mathcal{A}_2)$ is contained in some element of $\{st(B_1, \{C_1, \ldots, C_m\}), \ldots, st(B_n, \{C_1, \ldots, C_m\}), st(C_1, \{C_1, \ldots, C_m\}), \ldots, st(C_m, \{C_1, \ldots, C_m\})\}$ implying that $st(\mathcal{A}_1, \mathcal{A}_2) \in \mathcal{L}$. Therefore, $\mathcal{L}$ is a large scale structure with the desired collection of bounded subsets. □

We will term the large scale structure in the above proof the **minimal large scale structure associated with $\mathcal{B}$**.

It is possible that there are many different small scale structures generating a given topology. However, if a topological space X is compact, then there is only one small scale structure generating the topology, namely the small scale structure generated by the collection of all open covers of X. Notice that there is a dual result for large scale structures and bounded sets: a collection of bounded sets $\mathcal{B}$ for a set X is generated by a unique large scale structure on X iff for every collection of bouded sets $\{B_\alpha\}$, where each B_α has at least two elements, there is some finite collection $\{C_1, \ldots, C_n\}$ of bounded sets such that $\{B_\alpha\} \prec \{C_1, \ldots, C_n\}$. In this case, the minimal large scale structure associated to $\mathcal{B}$ is the only large scale structure on X whose bounded sets are equal to $\mathcal{B}$.

Definition 2.4.6. A large scale space X is **large scale anti-Hausdorff** if for every two point $x, y \in X$, there is a bounded set B containing x and y.

The reasoning for the term anti-Hausdorff is that in an anti-Hausdorff space, any two points become indistinguishable from far enough away.

Lemma 2.4.7. *For the large scale space X, the following are equivalent.*

1) X is large scale anti-Hausdorff;

2) the finite union of bounded sets of X is bounded;

3) any finite subset of X is bounded.

2.5 Examples of Scale Structures

2.5.1 Metric Scale Structures

Our first example, and perhaps the most natural example of a scale structure is the metric scale structure defined on a pseudo-metric space. The base of this scale structure consists of all collections of r-balls as r ranges through the positive reals. That is, for (X, d) a pseudo-metric space, the base is equal to

$$\{\{B(x, r) = \{y \in X : d(x, y) < r\} \mid x \in X\} \mid r > 0\}.$$

In the large scale case, to construct the whole large scale structure, we take all scales smaller than some collection of r-balls; that is, all scales of finite mesh. In the small scale case, we take all covers which are larger than some collection of r-balls. This requirement translates to a scale of positive Lebesgue number. Notice that these scale structures are idential to the metric structures defined before.

In the metric large scale structure, a set is bounded if and only if it has finite diameter. Notice also that the topology generated by the metric small scale structure is identical to the topology generated by the metric.

2.5.2 Group Scale Structures

Another example of a scale structure comes from a group.

Definition 2.5.1. Let G be a topological group (that is, a group with a topology such that the map $(g,h) \mapsto gh^{-1}$ is continuous). Then the collection

$$\{\{g \cdot U : g \in G\} : U \text{ a neighborhood of } 1_G\}$$

is the base of a small scale structure on G, called the **group small scale structure**.

Notice that the group small scale structure generates the original topology on the group.

Definition 2.5.2. Let G be any group. The **group large scale structure** is the large scale structure associated with the base $\{g \cdot F : F \subseteq G \text{ finite}\}$.

Recall that for a finitely generated group G, there is a natural metric on G, the word-length metric. It is defined by choosing a finite, symmetric generating set $\{g_1, g_2, \ldots, g_n\}$ and defining the distance from x to y to be the minimum number of generators necessary to represent $y^{-1} \cdot x$. This is the same as the graph metric coming from the Cayley graph using the same generating set. Notice that this metric is left translation invariant; that is, $d(g_1, g_2) = d(h \cdot g_1, h \cdot g_2)$ for all $g_1, g_2, h \in G$. As it turns out, the group large scale structure for a finitely generated group G is the same as the metric large scale structure coming from the word-length metric, regardless of the finite generating set chosen.

Proposition 2.5.3. *If G is a finitely generated group, then the group large scale structure on G is identical with the metric coarse structure coming from the word-length metric based on any finite generating set.*

Proof. Let $\mathcal{U}$ be a scale in the metric large scale structure. Say $mesh(\mathcal{U}) < R < \infty$. Let $F = B(e_G, R)$, which is finite. Let $U \in \mathcal{U}$ and fix $g \in U$. We claim that $U \subseteq g \cdot F$.

Indeed, if $h \in U$, then $R > d(g,h) = d(e, g^{-1} \cdot h)$, implying that $g^{-1} \cdot h \in F$. Thus, $h = g \cdot (g^{-1} \cdot h) \in g \cdot F$, as desired. Thus, the metric large scale structure is contained in the group large scale structure.

On the other hand, let $\mathcal{U}$ be a scale in the group coarse structure. Say, $\mathcal{U} \prec \{g \cdot F : g \in G\}$, where $F \subseteq G$ is finite.Notice that $diam(g \cdot F) = diam(F)$, which implies that $mesh(\mathcal{U}) \leq mesh(\{g \cdot F\}) < \infty$. Hence, every scale of the group large scale structure is included in the metric large scale structure. Therefore, the metric and the group large scale structures are identical. □

2.6 The Large and Small Scale Categories

If $f : (X, d_x) \to (Y, d_y)$ is a map between metric space, then we say that f is uniformly continuous if for every $\epsilon > 0$ there is a $\delta > 0$ so that $d_x(x_1, x_2) < \delta$ implies that $d_y(f(x_1), f(x_2)) < \epsilon$. This statement can be interpreted as a statement about the metric small scale structures on X and Y. It says that if $\mathcal{V}$ is a small scale on Y, say with $Leb(\mathcal{V}) > \epsilon$, then $f^{-1}(\mathcal{V}) := \{f^{-1}(V) \mid V \in \mathcal{V}\}$ has Lebesgue number at least δ. In other words, if $\mathcal{V}$ is a small scale on Y, then $f^{-1}(\mathcal{V})$ is a small scale on X. Using this, we can generalize the notion of uniform continuity to functions between small scale spaces.

Definition 2.6.1. Let X and Y be small scale spaces. A function $f : X \to Y$ is **small scale continuous** if $f^{-1}(\mathcal{V})$ is a small scale of X for every small scale $\mathcal{V}$ of Y.

Notice that the composition of small scale continuous maps is again small scale continuous. Thus, we can define the category of small scale spaces at the category whose objects are small scale spaces and whose morphisms are small scale continuous functions between these spaces.

Now, we can dualize the above definition to the large scale.

We must now establish a convention when discussing large scale spaces. Sometimes we will consider collections of subsets of a large scale space to be large scales, even though they may not be covers of the space. To rectify this, when considering any collection of subsets in a large scale space, we will assume that it has been trivially extended to be a cover. That is, we will assume that it contains all singleton subsets of the space.

Definition 2.6.2. Let X and Y be large scale spaces. A map $f : X \to Y$ is **large scale continuous** if $f(\mathcal{U})$ is a large scale of Y for every large scale $\mathcal{U}$ of X.

Notice that if f is not surjective, then the image of a cover of X will not be a cover of Y, so we invoke the convention that these images have been trivially extended to be covers of Y.

The composition of large scale continuous maps is clearly large scale continuous. In the large scale, we are only concerned with two things being equal up to a uniform bound. For this reason, we define an equivalence relation on maps between large scale spaces.

Definition 2.6.3. Two maps between large scale spaces $f, g : X \to Y$ are **close** if there is a large scale cover $\mathcal{V}$ of Y so that for every $x \in X$ there is some $V \in \mathcal{V}$ containing both $f(x)$ and $g(x)$.

A large scale continuous map $f : X \to Y$ between large scale spaces is a **coarse equivalence** if there is a large scale continuous map $g : Y \to X$ such that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X. In this case, we say that X and Y are **coarsely equivalent**.

We remark that when determining if a map between large (or small) scale spaces is large (small) scale continuous, it suffices to check it on the elements of a base of the structure.

2.7 Product and Coproduct Scale Structures

In this section we point out that the categories of large and small scale spaces contain products and coproducts. For a family of spaces $\{X_\alpha\}_{\alpha \in I}$ and a collection of scales

$\{\mathcal{U}_\alpha\}_{\alpha\in I}$ where each $\mathcal{U}_\alpha$ is a scale of X_α, define

$$\prod_{\alpha\in I}\mathcal{U}_\alpha := \left\{\prod_{\alpha\in I} U_\alpha \mid U_\alpha \in \mathcal{U}_\alpha\right\}.$$

That is, $\prod\limits_{\alpha\in I}\mathcal{U}_\alpha$ is the collection of all blocks in the product made out of elements of the given scales.

Definition 2.7.1. For a family $\{X_\alpha\}_{\alpha\in I}$ of small scale spaces, the **product small scale structure on** $\prod\limits_{\alpha\in I} X_\alpha$ is the small scale structure with base

$$\left\{\prod_{\alpha\in I}\mathcal{U}_\alpha \mid \mathcal{U}_\alpha \text{ a small scale of } X_\alpha, \mathcal{U}_\alpha = \{X_\alpha\} \text{ for all but finitely many } \alpha\right\}.$$

The product structure is a categorical product in the category of ss spaces. This means that given a collection of small scale continuous maps $f_\alpha : X \to X_\alpha$, there is a unique small scale continuous map $f : X \to \prod\limits_{\alpha\in A} X_\alpha$ such that $f_\alpha = \pi_\alpha \circ f$ for each α, where $\pi_\alpha : \prod\limits_{\alpha\in A} X_\alpha \to X_\alpha$ is the projection map.

The fact that the induced map f is small scale continuous is clear by observing that

$$f^{-1}\left(\mathcal{U}_{\alpha_1} \times \cdots \times \mathcal{U}_{\alpha_n} \times \prod_{\alpha\neq\alpha_i}\{X_\alpha\}\right) = \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(\mathcal{U}_{\alpha_i})$$

is a small scale of X.

Notice that we only take finitely many covers which contain more than one element. The reason for this is to ensure that the induced map f is small scale continuous. For example, if $X_n = \mathbb{R}$ for $n \geq 1$ and $f_n : \mathbb{R} \to X_n$ is the identity map for each n, then this gives a map $f : \mathbb{R} \to \prod\limits_{n\geq 1} X_n$ defined by $f(x) = (x, x, x, \ldots)$. Notice that if we define $B(r)$ to be the collection of balls of radius $\mathbb{R}$, then $f^{-1}(\prod\limits_{n\geq 1} B(1/n))$ is the collection of singletons of $\mathbb{R}$, which is not a small scale.

The category of ss spaces also has coproducts; namely, the disjoint union small scale structure.

Definition 2.7.2. For a family $\{X_\alpha\}_{\alpha\in I}$ of small scale spaces, the **disjoint union small scale structure on** $\coprod_{\alpha\in I} X_\alpha$ is the small scale structure with base

$$\left\{\coprod_{\alpha\in I}\mathcal{U}_\alpha \mid \mathcal{U}_\alpha \text{ a small scale of } X_\alpha\right\}.$$

This is a categorical coproduct, meaning that given a collection of small scale continuous maps $f_\alpha : X_\alpha \to X$, there is a unique small scale continuous map $f : \coprod_{\alpha\in A} X_\alpha \to X$ such that $f_\alpha = f \circ i_\alpha$ for each α, where $i_\alpha : X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ is the inclusion map. Notice that the induced map f is small scale continuous since if $\mathcal{U}$ is a small scale of X, then

$$f^{-1}(\mathcal{U}) = \coprod_{\alpha\in I} f_\alpha^{-1}(\mathcal{U})$$

is a small scale of the coproduct.

The large scale category also has products and coproducts.

Definition 2.7.3. For a family $\{X_\alpha\}_{\alpha\in I}$ of largel scale spaces, the **product small scale structure on** $\prod_{\alpha\in I} X_\alpha$ is the large scale structure with base

$$\left\{\prod_{\alpha\in I}\mathcal{U}_\alpha \mid \mathcal{U}_\alpha \text{ a large scale of } X_\alpha\right\}.$$

This is a categorical product since if $\{f_\alpha : X \to X_\alpha\}_{\alpha\in I}$ is a collection of large scale continuous maps, and $\mathcal{U}$ is a large scale of X, then

$$f(\mathcal{U}) = \prod_{\alpha\in I} f_\alpha(\mathcal{U})$$

is a large scale of $\prod_{\alpha\in I} X_\alpha$.

Definition 2.7.4. For a family $\{X_\alpha\}_{\alpha \in I}$ of largel scale spaces, the **disjoint union large scale structure on** $\coprod_{\alpha \in I} X_\alpha$ is the large scale structure with base

$$\left\{ \coprod_{\alpha \in I} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \text{ a large scale of } X_\alpha, \mathcal{U}_\alpha = \{\{x\} : x \in X_\alpha\} \text{ for all but finitely many } \alpha \right\}.$$

This is a categorical coproduct since if $\{f_\alpha : X_\alpha \to X\}_{\alpha \in I}$ is a collection of large scale continuous maps, then

$$f(\mathcal{U}_{\alpha_1} \sqcup \cdots \sqcup \mathcal{U}_{\alpha_1} \sqcup \coprod_{\alpha \neq \alpha_i} \{\{x\} : x \in X_\alpha\}) = \bigcup_{i=1}^{n} f_{\alpha_i}(\mathcal{U}_{\alpha_i})$$

is a large scale of X.

We cannot simply take the disjoint union of any large scales of the X_α's and obtain a coproduct. For example, if $X_n = \mathbb{R}$ for $n \geq 1$, and $f_n = id_{\mathbb{R}}$ for $n \geq 1$, then

$$f\left(\coprod_{n \geq 1} B(n)\right)$$

does not have a uniform bound in $\mathbb{R}$.

Notice that in the large scale case we did not need to restrict the covers in the product structure in order to obtain a categorical product. On the other hand, to form a coproduct, we cannot mimic the construction for the small scale coproduct, but must put a restriction on the uniformly bounded families. This phenomenon further emphasizes the duality between large and small scale structures.

2.8 Covariant And Contravariant Scale Structures

The philosophy behind the covariant point of view is that to understand a space, one can map known spaces into it. This is the approach taken in homotopy theory, where spheres are mapped into a space, and in homology theory, where simplices are mapped into a space. The dual approach, or the contravariant point of view is to map out of an unknown space in order to understand it. This is the point of view of cohomology, where one maps a space to Eilenberg-MacLane spaces.

Following this philosophy, we define covariant scale structures on a set X by inducing a scale structure on a space X via maps for a collection of scale spaces into X, and we define the contravariant scale structures via maps from X to a collection of scale spaces.

Lemma 2.8.1. *If $\{L_\alpha\}_{\alpha \in I}$ is a collection of large scale structures on a set X, then $\bigcap_{\alpha \in I} L_\alpha$ is a large scale structure on X.*

Proof. Let $\mathcal{U}, \mathcal{V} \in \bigcap_{\alpha \in I} L_\alpha$. The $st(\mathcal{U}, \mathcal{V}) \in L_\alpha$ for each α. Then $st(\mathcal{U}, \mathcal{U}), st(\mathcal{V}, \mathcal{V}) \in L_\alpha$ for all α. Hence, $st(st(\mathcal{U}, \mathcal{U}), st(\mathcal{V}, \mathcal{V})) \in L_\alpha$ for all α. Now notice that

$$st(\mathcal{U}, \mathcal{U}), \mathrm{st}(\mathcal{V}, \mathcal{V}) \prec st(st(\mathcal{U}, \mathcal{U}), st(\mathcal{V}, \mathcal{V})) \in \bigcap_{\alpha \in I} L_\alpha,$$

as desired.

Clearly, if $\mathcal{U} \prec \mathcal{V} \in \bigcap_{\alpha \in I} L_\alpha$, then $\mathcal{U} \in \bigcap_{\alpha \in I} L_\alpha$. □

Definition 2.8.2. Given any collection of scales $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ on a set, define the **large scale structure on X generated by** $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ to be the intersection of all large scale structures on X containing $\{\mathcal{U}\}$. Notice that we are not intersecting an empty collection, since the collection of all scales on a space X is a large scale structure.

The next lemma gives a more constructive approach to the large scale structure generated by a collection of scales. For a collection of scales $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$, define

$$st(\mathcal{U}_1, \ldots, \mathcal{U}_n) := st(st(\ldots st(\mathcal{U}_1, \mathcal{U}_2), \mathcal{U}_3), \ldots, \mathcal{U}_n).$$

Lemma 2.8.3. *The large scale structure generated by $\{\mathcal{U}_\alpha\}$ has basis*

$$\{st(\mathcal{U}_1, \ldots, \mathcal{U}_n) : \mathcal{U}_i \in \{\mathcal{U}_\alpha\}, n \geq 1\}.$$

Proof. Clearly any coarse structure containing $\{\mathcal{U}_\alpha\}$ must contain all sets of the above form, so it suffices to show that the collection forms a base for an ls-structure. But this is true since

$$st(st(\mathcal{U}_1,\ldots,\mathcal{U}_n), st(\mathcal{V}_1,\ldots,\mathcal{V}_m)) \prec st(\mathcal{U}_1,\ldots,\mathcal{U}_n,\mathcal{V}_m,\mathcal{V}_{m-1},\ldots,\mathcal{V}_2,\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_m).$$

□

Definition 2.8.4. Let X be a set and $\{f_\alpha : X_\alpha \to X\}_{\alpha\in I}$ be a collection of functions from large scale spaces X_α to X. The **covariant large scale structure generated by** $\{f_\alpha\}$ is defined to be the large scale structure generated by the collection $\{f_\alpha(\mathcal{U}) : \mathcal{U}$ a large scale of $X_\alpha\}_{\alpha\in I}$, where if necessary we extend $f_\alpha(\mathcal{U})$ to a scale by adding all singletons subsets of X to it.

Notice that the contravariant large scale structure on X is the smallest structure making each map f_α large scale continuous.

To define the contravariant small scale structure, we need to introduce the notion of a decreasing sequence of scales on a space.

Definition 2.8.5. A sequence of scales $\{\mathcal{U}_i\}_{i\geq 0}$ on a space X is called a **decreasing sequence of scales** if $\mathcal{U}_i$ is smaller than $\mathcal{U}_{i-1}$ for $i \geq 1$.

In the literature, a decreasing sequence of scales has traditionally been called a **normal sequence** of covers.

Definition 2.8.6. Let X be a set and $\{f_\alpha : X_\alpha \to X\}_{\alpha\in I}$ be a collection of functions from small scale spaces X_α to X. The **covariant small scale structure generated by** $\{f_\alpha\}$ is defined to be the small scale structure consisting of all scales $\mathcal{U}$ of X so that there is some decreasing sequence of scales $\{\mathcal{U}_i\}_{i\geq 0}$ with $\mathcal{U}_0 = \mathcal{U}$ and $f_\alpha^{-1}(\mathcal{U}_i)$ a small scale of X_α for all $\alpha \in I$ and all $i \geq 0$.

We need to show that this is indeed a small scale structure. For this purpose, we prove two lemmas which are related to decreasing sequences of scales.

Lemma 2.8.7. *If S is a small scale structure, and $\mathcal{U}, \mathcal{V} \in S$, then $\mathcal{U} \cap \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is in S.*

Proof. There is a small scale $\mathcal{W} \in S$ which is smaller than both $\mathcal{U}$ and $\mathcal{V}$. If $W \in \mathcal{W}$, then there are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $st(W, \mathcal{W}) \subseteq U$ and $st(W, \mathcal{W}) \subseteq V$. Then $st(W, \mathcal{W}) \subseteq U \cap V$. Thus, $st(\mathcal{W}, \mathcal{W}) \prec \mathcal{U} \cap \mathcal{V}$, impling that $\mathcal{U} \cap \mathcal{U} \in S$. □

Lemma 2.8.8. *If $\{\mathcal{U}_i\}_{i \geq 0}$ and $\{\mathcal{V}_i\}_{i \geq 0}$ are two decreasing sequences of scales, then $\{\mathcal{U}_i \cap \mathcal{V}_i\}_{i \geq 0}$ is a decreasing sequence of scales.*

Proof. By the previous lemma, each $\mathcal{U}_i \cap \mathcal{V}_i$ is a small scale, so we need only show that for $i \geq 1$, is is the case that $\mathcal{U}_i \cap \mathcal{V}_i$ is a smaller scale that $\mathcal{U}_{i-1} \cap \mathcal{V}_{i-1}$. Indeed, let $U \in \mathcal{U}_i$ and $V \in \mathcal{V}_i$. We know there are $\bar{U} \in \mathcal{U}_{i-1}$ and $\bar{V} \in \mathcal{V}_{i-1}$ so that $st(U, \mathcal{U}_i) \subseteq \bar{U}$ and $st(V, \mathcal{V}_i) \subseteq \bar{V}$. Notice that $st(U \cap V, \mathcal{U}_i \cap \mathcal{V}_i)$ is a subset of both $st(U, \mathcal{U}_i)$ and $st(V, \mathcal{V}_i)$. Thus, $st(U \cap V, \mathcal{U}_i \cap \mathcal{V}_i) \subseteq \bar{U} \cap \bar{V} \in \mathcal{U}_{i-1} \cap \mathcal{V}_{i-1}$. Therefore, $\mathcal{U}_i \cap \mathcal{V}_i$ is a smaller scale that $\mathcal{U}_{i-1} \cap \mathcal{V}_{i-1}$. □

Proposition 2.8.9. *The covariant small scale structure is a small scale structure on X.*

Proof. First we show that if $\mathcal{U}$ is an element of the covariant small scale structure and $\mathcal{V}$ is larger than $\mathcal{U}$, then $\mathcal{V}$ is in the covariant small scale structure. Since $\mathcal{U}$ is a small scale, there is a decreasing sequence of scales $\{\mathcal{U}_i\}_{i \geq 0}$ starting with $\mathcal{U}$ such that the inverse image under f_α of each element is a small scale of X_α for each α. Define a new decreasing sequence of scales $\{\mathcal{V}_i\}_{i \geq 0}$ by letting $\mathcal{V}_0 = \mathcal{V}$ and $\mathcal{V}_i = \mathcal{U}_{i-1}$ for $i \geq 1$. Notice that this sequence satisfies the conditions in the definition of the covariant small scale structure since $f_\alpha^{-1}(\mathcal{U})$ is smaller than $f_\alpha^{-1}(\mathcal{V})$. Thus, $\mathcal{V}$ is a small scale.

Now suppose that $\mathcal{U}$ and $\mathcal{V}$ are elements of the covariant small scale structure. So we can find suitable decreasing sequences of scales $\{\mathcal{U}_i\}_{i \geq 0}$ and $\{\mathcal{V}_i\}_{i \geq 0}$ starting with $\mathcal{U}$ and $\mathcal{V}$, respectively. Consider the decreasing sequence of scales $\{\mathcal{U}_{i+1} \cap \mathcal{V}_{i+1}\}_{i \geq 0}$. By the previous lemma, this is a decreasing sequence of scales. Notice also that $\mathcal{U}_1 \cap \mathcal{V}_1$ is smaller than both $\mathcal{U}$ and $\mathcal{V}$. Finally, the inverse image under f_α of each element of this sequence is a small scale of X_α since it is the intersection of two small scales, hence is a small scale by the first lemma. Therefore, $\mathcal{U}_1 \cap \mathcal{V}_1$ is a small scale smaller than both $\mathcal{U}$ and $\mathcal{V}$, as desired. $\square$

Notice that the covariant small scale structure is the largest small scale structure on X so that each function $f_\alpha : X_\alpha \to X$ is small scale continuous.

Now we consider contravariant scale structures.

First we prove a necessary lemma.

Lemma 2.8.10. *If $f : X \to Y$ is a function where X is a set and Y is a small scale space, then the collection*

$$\{f^{-1}(\mathcal{U}) : \mathcal{U} \text{ a small scale of } Y\}$$

forms the base of a small scale structure on X. Additionally, this is the smallest small scale structure on X making f small scale continuous.

Dually, if Y is a large scale space then the collection

$$\{f^{-1}(\mathcal{U}) : \mathcal{U} \text{ a large scale of } Y\}$$

forms the base of a large scale structure on X, which is the largest large scale structure making f large scale continuous.

Proof. The fact that this is a base is immediate, since if $\mathcal{W}$ is a smaller scale than $\mathcal{U}$ and $\mathcal{V}$, then $f^{-1}(\mathcal{W})$ is a smaller scale than $f^{-1}(\mathcal{U})$ and $f^{-1}(\mathcal{V})$. Also, it is clear that this is the smallest small scale structure making f small scale continuous since any structure on X making this small scale continuous must contain all scales of the form $f^{-1}(\mathcal{U})$ where $\mathcal{U}$ is a small scale of Y.

For the large scale case, the same argument works to show that the collection is a base. To show that it is the largest making f large scale continuous, notice that if $\mathcal{U}$ is a large scale on X, then $f(\mathcal{U})$ is a large scale of Y, and $\mathcal{U} \subseteq f^{-1}(f(\mathcal{U}))$ so is contained in the large scale structure generated by the base. $\square$

Definition 2.8.11. Let X be a set and $\{f_\alpha : X \to X_\alpha\}_{\alpha\in I}$ be a collection of functions from X to small scale spaces X_α. Define the **contravariant small scale structure on X with respect to $\{f_\alpha\}_{\alpha\in I}$** in the following way. Consider the induced map $f : X \to \prod_{\alpha\in I} X_\alpha$ defined by $x \mapsto (f_\alpha(x))_{\alpha\in I}$. Form the contravariant structure on X by taking as a base all scales on X of the form $f^{-1}(\mathcal{U})$ where $\mathcal{U}$ is a small scale on $\prod_{\alpha\in I} X_\alpha$.

Recall that the base of small scales for the product structure consists of all scales of the form

$$\mathcal{U}_{\alpha_1} \times \mathcal{U}_{\alpha_2} \times \cdots \times \mathcal{U}_{\alpha_n} \times \prod_{\alpha\neq\alpha_1,\ldots,\alpha_n} X_\alpha,$$

where $\mathcal{U}_{\alpha_i}$ is a small scale of X_{α_i} for each i.

Lemma 2.8.12. *The contravariant small scale structure on X with respect to $\{f_\alpha : X \to X_\alpha\}_{\alpha\in I}$ is the smallest small scale structure on X making each map f_α small scale continuous.*

Proof. First notice that $f : X \to \prod_{\alpha\in I} X_\alpha$ is small scale continuous, so $f_\alpha = \pi_\alpha \circ f$ is small scale continuous, where π_α is projection $\pi_\alpha : \prod_{\alpha\in I} X_\alpha \to X_\alpha$. To show that this is the smallest structure making each f_α small scale continuous, notice that

$$f^{-1}(\mathcal{U}_1 \times \mathcal{U}_2 \times \prod_{\alpha\neq 1,2} X_\alpha) = f_1^{-1}(\mathcal{U}_1) \cap f^{-1}(\mathcal{U}_2),$$

so by induction, any element of base for the contravariant small scale structure on X is the intersection of finitely many $f_i^{-1}(\mathcal{U}_i)$, where $\mathcal{U}_i$ is a small scale cover of X_i. Each of these intersections must be included in any small scale structure making each f_α small scale continuous, proving the claim. □

Definition 2.8.13. Let X be a set and $\{f_\alpha : X \to X_\alpha\}_{\alpha\in I}$ be a collection of functions from X to large scale spaces X_α. Define the **contravariant large scale structure with respect to $\{f_\alpha\}_{\alpha\in I}$** as follows. First, consider the induced map to the product large scale space $f : X \to \prod_{\alpha \in I} X_\alpha,\ x \mapsto (f_\alpha(x))_{\alpha in I}$. Then take as a base the collection $\{f^{-1}(\mathcal{U}) : \mathcal{U}$ a large scale of $\prod_{\alpha \in I} X_\alpha\}$.

Recall that the large scales of the product of the collection $\{X_\alpha\}$ of large scale spaces are of the form

$$\prod_\alpha \mathcal{U}_\alpha,$$

where $\mathcal{U}_\alpha$ is a large scale of X_α for each α.

Lemma 2.8.14. *The contravariant large scale structure on X with respect to $\{f_\alpha\}_{\alpha \in I}$ is the largest large scale structure making each map $f_\alpha : X \to X_\alpha$ large scale continuous.*

Proof. Let $\mathcal{U}$ be an element of the base of the contravariant large scale structure; say $\mathcal{U} = f^{-1}(\prod_{\alpha \in I}(\mathcal{U}_\alpha))$, where $\mathcal{U}_\alpha$ is a large scale of X_α. Then $f_\alpha(\mathcal{U}) = f_\alpha(f^{-1}(\prod_{\alpha \in I}(\mathcal{U}_\alpha))) = \pi_\alpha \circ f \circ f^{-1}(\prod_{\alpha \in I} \mathcal{U}_\alpha) \subseteq \pi_\alpha(\prod_{\alpha \in I} \mathcal{U}_\alpha) = \mathcal{U}_\alpha$ is a large scale of X_α. Hence each f_α is large scale continuous.

To show that this is the largest large scale structure making each f_α large scale continuous, notice that if $\mathcal{U}$ is a large scale of X, then $\mathcal{U} \prec f^{-1}(\prod_{\alpha \in I} f_\alpha(\mathcal{U}))$ is an element of the contravariant structure. $\square$

2.9 Direct and Inverse Limits

As an application of the covariant and contravariant scale structures, we define the define direct and inverse limits of scale structures.

Definition 2.9.1. A **directed set** is a nonempty set I with a relation $\leq$ satisfying:

1) $i \leq i$ for all $i \in I$;

2) $i \leq j$ and $j \leq k$ implies that $i \leq k$; and

3) if $i, j \in I$, then there is some $k \in I$ satisfying $i \leq k$ and $j \leq k$.

Definition 2.9.2. Let $\mathcal{C}$ be a category. Let $(I, \leq)$ be a directed set, $\{A_i \mid i \in I\}$ be a collection of objects in $\mathcal{C}$ and let $\{\phi_i^j : A_i \to A_j\}_{i \leq j}$ be a collection of morphisms satisfying

1) $\phi_i^i = id_{A_i}$ for all i;

2) $\phi_i^k = \phi_j^k \circ \phi_i^j$ for all $i \leq j \leq k$.

The pair $(\{A_i\}, \{\phi_i^j\})$ is called a **direct system** over I.

Definition 2.9.3. The **direct limit** of the direct system $(\{A_i\}, \{\phi_i^j\})$ is an object $\varinjlim A_i$ together with a collection of morphisms $\{\alpha_i : X_i \to \varinjlim A_i\}_{i \in I}$ satisfying $\alpha_i = \alpha_j \circ \phi_i^j$ for all $i \leq j$ which is universal in the sense that if B is any other object such that there are morphisms $\beta_i : A_i \to B$ satisfying $\beta_i = \beta_j \circ \phi_i^j$ for all $i \leq j$, then there is a unique morphism $u : \varinjlim A_i \to B$ such that $\beta_i = u \circ \alpha_i$ for all $i \in I$.

Example 2.9.4. *Direct limits exist in the category of sets. Indeed, let $(\{A_i\}, \{\phi_i^j\})$ be a directed system of sets. Then*

$$\varinjlim A_i = \bigsqcup_{i \in I} A_i / \sim,$$

where $\sim$ is the equivalence relation on $\bigsqcup_{i \in I} A_i$ defined by $a \in A_i \sim b \in A_j$ iff there is some k so that $\phi_i^k(a) = \phi_j^k(b)$.

Proposition 2.9.5. *If $(\{A_i\}, \{\phi_i^j\})$ is a direct system of large scale space, then the direct limit $\varinjlim A_i$ exists in the category of large scale spaces.*

Proof. Let the underlying set of $\varinjlim A_i$ be the direct limit in the category of sets of the system $(\{A_i\}, \{\phi_i^j\})$. To define the scale structure on $\varinjlim A_i$, take the covariant scale structure induced by the collection of maps $\{\alpha_i : A_i \to \varinjlim A_i\}_{i \in I}$. Now if B is any other large scale space such that there are large scale continuous maps $\beta_i : A_i \to A$ satisfying $\beta_i = \beta_j \circ \phi_i^j$ for all $i \leq j$, then we get a function $u : \varinjlim A_i \to B$ such that $\beta_i = u \circ \alpha_i$ for all $i \in I$. To finish the proof, we need to show that this map is large scale continuous. Let's check that the image of a basis element of the structure on $\varinjlim A_i$ is a large scale of A. Indeed, by Lemma 2.8.3 a basis element is of the form $st(f_{\alpha_1}(\mathcal{U}_1), \ldots, f_{\alpha_n}(\mathcal{U}_n))$, where each $\mathcal{U}_i$ is a large scale of X_i. Applying Lemma 2.1.6 , we get that $u(st(f_{\alpha_1}(\mathcal{U}_1), \ldots, f_{\alpha_n}(\mathcal{U}_n))) \prec st(u(f_{\alpha_1}(\mathcal{U}_1), \ldots, u(f_{\alpha_n}(\mathcal{U}_n)))) = st(g_{\alpha_1}(\mathcal{U}_1), \ldots, g_{\alpha_n}(\mathcal{U}_n))$ is a large scale of B since each $g_{\alpha_i}(\mathcal{U}_i)$ is a large scale of A. □

Proposition 2.9.6. *If* $(\{A_i\}, \{\phi_i^j\})$ *is a direct system of small scale space, then the direct limit* $\varinjlim A_i$ *exists in the category of small scale spaces.*

Proof. Let the underlying set of $\varinjlim A_i$ be the direct limit in the category of sets of the system $(\{A_i\}, \{\phi_i^j\})$. To define the scale structure on $\varinjlim A_i$, take the covariant scale structure induced by the collection of maps $\{\alpha_i : A_i \to \varinjlim A_i\}$. Now if B is any other small scale space such that there are small scale continuous maps $\beta_i : A_i \to B$ satisfying $\beta_i = \beta_j \circ \phi_i^j$ for all $i \leq j$, then we get a function $u : \varinjlim A_i \to B$ such that $\beta_i = u \circ \alpha_i$ for all $i \in I$. To finish the proof, we need to show that this map is small scale continuous.

Let $\mathcal{U}$ be a small scale of B. We need to show that there is a decreasing sequence of scales of $\varinjlim A_i$ so that for each term in the sequence, the inverse image under each f_α is a small scale of X_α. To do this, choose a decreasing sequence of scales $\{\mathcal{U}_i\}_{i\geq 0}$ of B starting with $\mathcal{U} = \mathcal{U}_0$. We claim that the decreasing sequence $\{u^{-1}(\mathcal{U}_i)\}_{i\geq 0}$ is the desired sequence. Indeed, for each α, we have that $f_\alpha^{-1}(u^{-1}(\mathcal{U}_i)) = g_\alpha^{-1}(\mathcal{U}_i)$ is a small scale of X_α since we assumed that each map g_α is small scale continuous. □

Definition 2.9.7. Let $\mathcal{C}$ be a category. Let $(I, \leq)$ be a directed set, $\{A_i \mid i \in I\}$ be a collection of objects in $\mathcal{C}$ and let $\{\phi_i^j : A_j \to A_i\}_{i\leq j}$ be a collection of morphisms satisfying

1) $\phi_i^i = id_{A_i}$ for all i;

2) $\phi_i^k = \phi_i^j \circ \phi_j^k \circ \phi_i^j$ for all $i \leq j \leq k$.

The pair $(\{A_i\}, \{\phi_i^j\})$ is called a **inverse system** over I.

Definition 2.9.8. The **inverse limit** of the inverse system $(\{A_i\}, \{\phi_i^j\})$ is an object $\varprojlim A_i$ together with a collection of morphisms $\{\alpha_i : \varprojlim A_i \to A_i\}_{i\in I}$ satisfying $\alpha_i =$

$\phi_i^j \circ \alpha_j$ for all $i \leq j$ which is universal in the sense that if B is any other object such that there are morphisms $\beta_i : B \to A_i$ satisfying $\beta_i = \phi_i^j \circ \beta_j$ for all $i \leq j$, then there is a unique morphism $u : B \to \varprojlim A_i$ such that $\beta_i = \alpha_i \circ u$ for all $i \in I$.

Example 2.9.9. *Inverse limits exist in the category of sets. Indeed, let $(\{A_i\}, \{\phi_i^j\})$ be an inverse system of sets. Then*

$$\varprojlim A_i = \{(a_i) \in \prod_{i \in I} A_i \mid a_i = f_{ij}(a_j) \textit{ for all } i \leq j\}.$$

Proposition 2.9.10. *If $(\{A_i\}, \{\phi_i^j\})$ is an inverse system of large scale space, then the inverse limit $\varprojlim A_i$ exists in the category of large scale spaces.*

Proof. Let the underlying set of $\varprojlim A_i$ be the inverse limit in the category of sets of the system $(\{A_i\}, \{\phi_i^j\})$. To define the scale structure on $\varprojlim A_i$, take the contravariant scale structure induced by the collection of maps $\{\alpha_i : \varinjlim A_i \to A_i\}$. Now if B is any other large scale space such that there are large scale continuous maps $\beta_i : B \to A_i$ satisfying $\beta_i = \phi_i^j \circ \beta_j$ for all $i \leq j$, then we get a function $u : B \to \varinjlim A_i$ such that $\beta_i = \alpha_i \circ u$ for all $i \in I$. To finish the proof, we need to show that this map is large scale continuous. This is true since $u(\mathcal{U}) \prec f^{-1}(\prod_{\alpha \in A} g_\alpha(\mathcal{U}))$, where f is the induced map to the product. Hence if $\mathcal{U}$ is a large scale of B, then $u(\mathcal{U})$ refines a large scale of $\varprojlim A_i$. □

Proposition 2.9.11. *If $(\{A_i\}, \{\phi_i^j\})$ is an inverse system of small scale space, then the inverse limit $\varprojlim A_i$ exists in the category of small scale spaces.*

Proof. Let the underlying set of $\varprojlim A_i$ be the inverse limit in the category of sets of the system $(\{A_i\}, \{\phi_i^j\})$. To define the scale structure on $\varprojlim A_i$, take the contravariant scale structure induced by the collection of maps $\{\alpha_i : \varinjlim A_i \to A_i\}$. Now if B is any other large scale space such that there are small scale continuous maps $\beta_i : B \to A_i$ satisfying $\beta_i = \phi_i^j \circ \beta_j$ for all $i \leq j$, then we get a function $u : B \to \varinjlim A_i$ such that $\beta_i = \alpha_i \circ u$ for all $i \in I$. To finish the proof, we need to show that this map is small scale continuous. This is true since

$$u^{-1}(f^{-1}(\mathcal{U}_{\alpha_1} \times \cdots \times \mathcal{U}_{\alpha_n} \times \prod_{\alpha \neq \alpha_i} \{X_\alpha\})) = \bigcap_{i=1}^{n} \beta_{\alpha_i}^{-1}(\mathcal{U}_{\alpha_i}),$$

which is a small scale of B since each β_α is small scale continuous. □

2.9.1 Metric Approximations

Definition 2.9.12. A sequence of scales $\{\mathcal{U}_i\}_{i\geq 0}$ on a space X is called a **decreasing sequence of scales** if $\mathcal{U}_i$ is smaller than $\mathcal{U}_{i-1}$ for $i \geq 1$. The sequence is an **increasing sequence of scales** if $\mathcal{U}_{i-1}$ is smaller than $\mathcal{U}_i$ for $i \geq 1$.

The proof of the following Proposition is adapted from Theorem 1.14 of [13].

Proposition 2.9.13. *If X is any set and $\{\mathcal{U}_i\}_{i\geq 0}$ is a decreasing sequence of scales on X, then there is a pseudometric d on X so that $\{\mathcal{U}_i\}_{i\geq 0}$ is a base for the metric small scale structure induced by d.*

Proof. Define $t(x, y) = 0$ if for every n, there is some $u \in \mathcal{U}_n$ containing both x and $y, t(x, y) = 2$ if no element of $\mathcal{U}_0$ contains both x and y. Otherwise, define $t(x, y) = 2^{1-n}$, where $n = \max\{i : \exists U \in \mathcal{U}_i$ containing both x and $y\}$. Now define $d(x, y) = \inf\{\sum_t(x_i, x_{i+1})\}$, where the infimum is taken over all finite chains $x = x_1, x_2, \ldots, x_n = y$. Then d is clearly a pseudo-metric.

Now we show that $\{\mathcal{U}_i\}_{i\geq 0}$ is a base of the metric small scale structure.

First, we show that each $\mathcal{U}_i$ is a small scale. We claim that $B(2^{-i}) \prec \mathcal{U}_i$ for each i. To show this, it suffices to show that if $2^{-i} \leq d(x, y) < 2^{1-i}$, then there is some element of $\mathcal{U}_i$ containing both x and y. Since $d(x, y) < 2^{1-i}$, there is some chain $x = x_1, x_2, \ldots, x_n = y$ such that $t(x_1, x_2) + \cdots + t(x_{n-1}, x_n) < 2^{1-i}$. We proceed by induction on n. If $n = 2$, then x and y are contained in some element of $\mathcal{U}_i$, so we are done. So assume that the result holds for all chains of length less than n. We may choose i and j maximal so that $t(x_1, x_2) + \cdots + t(x_{i-1}, x_i) < 2^{-i}$ and

$t(x_i, x_{i+1}) + \cdots + t(x_{j-1}, x_j) < 2^{-i}$. It follows that $t(x_j, x_{j+1}) + \cdots + t(x_{n-1}, x_n) < 2^{-1}$. Then by induction, there are elements of $\mathcal{U}_{i+1}$ containing the pairs $(x, x_i), (x_i, x_j)$, and (x_j, y). Since $\mathcal{U}_{i+1}$ is smaller than $\mathcal{U}_i$, it follows that some element of $\mathcal{U}_i$ contains x and y, as desired.

To show that the $\mathcal{U}_i$ form a basis, we need to show that every metric small scale is larger than some $\mathcal{U}_i$. Notice that $mesh(\mathcal{U}_i) < 2^{1-i}$ so since any metric small scale has positive Lebesgue number, it follows that it must contain some $\mathcal{U}_i$. □

Corollary 2.9.14. *If X is a small scale space, then the small scale structure on X is induced by a pseudo-metric if and only if the scale structure has a countable base.*

Proposition 2.9.15. *[27] If X is any set and $\{\mathcal{U}_i\}_{i\geq 1}$ is an increasing sequence of scales on X, then there is a metric d on X so that $\{\mathcal{U}_i\}_{i\geq 1}$ is a base for the metric large scale structure induced by d.*

Proof. For distinct $x, y \in X$, define $d(x, y) = i$, where i is the minimum so that some element of $\mathcal{U}_i$ contains both x and y, and ∞ if there is no such i, and define $d(x, x) = 0$ for all $x \in X$. Now if $d(x, y) = i$ and $d(y, z) = j$ with $j \geq i$, then this means there are elements of $\mathcal{U}_j$ containing the pairs (x, y) and (y, z). Then since $\mathcal{U}_{j+1}$ is larger than $\mathcal{U}_j$, there is some element of $\mathcal{U}_j$ containing both x and z. Hence, $d(x, z) \leq 1 + j \leq d(x, y) + d(y, z)$, which proves that d is a metric. Each $\mathcal{U}_i$ is a large scale since $mesh(\mathcal{U}_i) \leq i$ and they form a base since if $mesh(\mathcal{U}) < M$, then $\mathcal{U} \prec \mathcal{U}_{M+1}$. □

Corollary 2.9.16. *If X is a large scale space, then the large scale structure on X is induced by a metric if and only if the scale structure has a countable base.*

As an application of the direct limit construction, we will show that every scale space can, in some sense, be built out of metric scale spaces.

Theorem 2.9.17. *Let X be a large scale space. Then X is the direct limit of a collection of metric large scale spaces.*

Proof. Let X be a large scale space. Let

$$I = \{\{\mathcal{U}_i\}_{i\geq 0} \mid \{\mathcal{U}_i\}_{i\geq 0} \text{ an increasing sequence of scales of } X\}.$$

We can make I into a directed set by ordering elements in the following way: To each element $\{\mathcal{U}_i\}_{i\geq 0}$ of I, there is an associated metric $d_{\{\mathcal{U}_i\}}$ on X such that the metric structure defined by $d_{\{\mathcal{U}_i\}}$ is generated by $\{\mathcal{U}_i\}_{i\geq 0}$. Order the elements of I by $\{\mathcal{U}_i\}_{i\geq 0} \leq \{\mathcal{V}_i\}_{i\geq}$ if $id_X : (X, d_{\{\mathcal{U}_i\}_{i\geq 0}}) \to (X, d_{\{\mathcal{V}_i\}_{i\geq 0}})$ is large scale continuous.

We claim that $X = \varinjlim (X, d_{\{\mathcal{U}_i\}_{i\geq 0}})$. That is, the large scale structure on X is the smallest one containing all large scales from each $(X, d_{\{\mathcal{U}_i\}_{i\geq 0}})$. Indeed, X clearly contains all such large scales, and given any large scale $\mathcal{U}$ of X, there is an increasing sequence of large scales of X, $\{\mathcal{U}_i\}_{i\geq 0}$ with $\mathcal{U}_i = \mathcal{U}$, proving that X is the smallest containing all such scales. □

Theorem 2.9.18. *Let X be a small scale space. Then X is the inverse limit of a collection of metric small scale spaces.*

Proof. Let X be a small scale space. Let

$$I = \{\{\mathcal{U}_i\}_{i\geq 0} \mid \{\mathcal{U}_i\}_{i\geq 0} \text{ a decreasing sequence of scales of } X\}.$$

We can make I into a directed set by ordering elements in the following way. To each element $\{\mathcal{U}_i\}_{i\geq 0}$ of I, there is an associated pseudometric $d_{\{\mathcal{U}_i\}}$ on X such that the metric structure defined by $d_{\{\mathcal{U}_i\}}$ is generated by $\{\mathcal{U}_i\}_{i\geq 0}$. Order the elements of I by $\{\mathcal{U}_i\}_{i\geq 0} \leq \{\mathcal{V}_i\}_{i\geq}$ if $id_X : (X, d_{\{\mathcal{V}_i\}_{i\geq 0}}) \to (X, d_{\{\mathcal{U}_i\}_{i\geq 0}})$ is small scale continuous.

We claim that $X = \varprojlim (X, d_{\{\mathcal{U}_i\}_{i\geq 0}})$. We know that $id_X : X \to (X, d_{\{\mathcal{U}_i\}})$ is small scale continuous for every decreasing sequence of scales $\{\mathcal{U}_i\}$ of X, so we need only show that every small scale of X is contained in the inverse limit small scale structure. Indeed, let $\mathcal{U}$ be a small scale of X. We can choose a decreasing sequence of scales $\{\mathcal{U}_i\}_{i\geq 0}$ starting with $\mathcal{U}$. So $\mathcal{U}$ is a small scale of $(X, d_{\{\mathcal{U}_i\}})$. Let f be the induced map to the product of all spaces $(X, d_{\{\mathcal{V}_i\}})$. Then

$$\mathcal{U} = f^{-1}\left(\mathcal{U} \times \prod_{\{\mathcal{V}_i\} \neq \{\mathcal{U}_i\}} \{X\}\right),$$

which is an element of the inverse limit. □

Chapter 3

Slowly Oscillating Functions

3.1 Interaction Between Topology and Large Scale Structures: Proper Coarse Structures

The purpose of this section is make the appropriate setting for compactifications induced by large scale structures. We begin by translating Roe's notion of a proper coarse structure to that of a proper large scale structure and develop some relevant properties. For example, one can easily prove that a proper large scale space is paracompact.

Definition 3.1.1. A large scale structure on a Hausdorff topological space X is **proper** if all its bounded sets are precompact, and it contains an open, uniformly bounded cover.

For example, the metric large scale structure on a proper metric space is proper, since in a proper metric space, a set is compact if and only if it is closed and bounded, and the cover by open balls of radius 1 is a uniformly bounded open cover.

Lemma 3.1.2. *Let X be a locally compact Hausdorff space with a proper large scale structure. Every large scale cover can be coarsened to an open uniformly bounded cover.*

Proof. Let $\mathcal{U}$ be a large scale cover of X and $\mathcal{V}$ be an open large scale cover. Notice that $st(\mathcal{U}, \mathcal{V})$ is an open coarsening of $\mathcal{U}$. □

In the appendix, we show that our notion of properness is equivalent to the notion of proper coarse structure as defined by John Roe (Definition 2.22 [27]).

For the next result, recall that a locally compact Hausdorff space is paracompact if and only if it is a disjoint union of open σ-compact sets; see Theorem 12.11 of [4].

Proposition 3.1.3. *If X is has a proper large scale structure, then X is paracompact.*

Proof. We shall show that X is the disjoint union of σ-compact open sets. The result will then follow since X is locally compact. Let $\mathcal{U}$ be an open large scale of X. Define $st^0(\mathcal{U}) = \mathcal{U}$ and for $i \geq 1$, define $st^i(\mathcal{U}) = st(st^{i-1}(\mathcal{U}), \mathcal{U})$. Define an equivalence relation on X by $x \sim y$ if and only if there is an n such that $x, y \in U$ for some $U \in st^n(\mathcal{U})$. We claim that the equivalence classes of this relation are open and σ-compact. Indeed, let $[x]$ be an equivalence class with representative x. Choose $U \in \mathcal{U}$ with $x \in U$. Let $st^0(U) = st(U, \mathcal{U})$ and $st^{i+1}(U) = st(st^i(U), \mathcal{U})$ for $i \geq 0$. Note that for every $y \in [x]$ there is an n such that $y \in st^n(U)$. Thus, $[x] = \bigcup_{n=0}^{\infty} st^n(U)$. Let $z \in cl(st^n(U))$, and choose $V \in \mathcal{U}$ containing z. Then $V \cap st^n(U) \neq \emptyset$, implying that $z \in st^{n+1}(U)$. Thus, $cl(st^n(U)) \subseteq st^{n+1}(U)$, implying that $[x] = \bigcup_{n=0}^{\infty} st^n(U) = \bigcup_{n=0}^{\infty} cl(st^n(U))$, where each $cl(st^n(U))$ is compact since X is proper. Hence each equivalence class is a countable union of compact sets, as desired. □

3.2 Interaction between Large and Small Scales: Slowly Oscillating Functions

We introduce slowly oscillating functions for metric space in such a way that they can be viewed as a dualization of small scale continuous functions. We will later generalize slowly oscillating functions as maps between large scale spaces and small scale spaces.

A sequence of points $\{x_n\}_{n\geq 1}$ in a large scale space X is said to **diverge to infinity** if each bounded subset of X only contains finitely many terms of the sequence. A function $f : X \to \mathbb{C}$ from a large scale space X is C_0 if for any sequence of points $\{x_n\}$ diverging to infinity, $\lim\limits_{n\to\infty} f(x_n) = 0$. We will study classes of functions which may not be C_0 but which will have C_0-variation. Notice that the elements of scales are thought of like points, e.g. the cover by R-balls in a metric space look like a cover by points when viewed from far away. Slowly oscillating functions are functions whose variation is C_0 at all scales. Dually, uniformly continuous functions can be defined in terms of variation at different scales.

Definition 3.2.1. A set $B \subseteq X$ is called **weakly bounded** if for every coarse component C of X, the set $B \cap C$ is bounded.

Notice that the union of finitely many weakly bounded sets is again weakly bounded.

Let X be any space which has a notion of bounded sets, $\mathcal{U}$ be a scale of X, and $\epsilon > 0$. We say that $f : X \to \mathbb{C}$ has **C_ϵ-variation at scale $\mathcal{U}$** if there is a weakly bounded set B such that $\text{diam}(f(U \setminus B)) \leq \epsilon$ for all $U \in \mathcal{U}$ such that $U \nsubseteq B$. The function f has **C_0-variation at scale $\mathcal{U}$** if it has C_ϵ-variation at scale $\mathcal{U}$ for all $\epsilon > 0$. Here are some important observations:

- Constant functions have C_0-variation at any scale $\mathcal{U}$.
- f has C_0-variation at scale $\{X\}$ if and only if f is a C_0 function on X.
- f has C_0-variation at scale $\{\{x\} : x \in X\}$ for any function f.

We can use the notion of C_ϵ variation as an alternative means of defining small scale continuity. Here the bounded sets of X are taken to be the sets which are precompact in the topology induced by the small scale structure. Notice that this means that the space is coarsely anti-Hausdorff.

Proposition 3.2.2. *Let X be a ss-space. A continuous function $f : X \to \mathbb{C}$ is small scale continuous if and only if for every $\epsilon > 0$, there exists a small scale $\mathcal{U}$ such that f has C_ϵ-variation at scale $\mathcal{U}$.*

Proof. The forward direction is clear. Assume that for every $\epsilon > 0$, there exists a small scale $\mathcal{U}$ such that f has C_ϵ-variation at scale $\mathcal{U}$. Fix $\epsilon > 0$. Choose a small scale $\mathcal{U}$ and a precompact set B such that $diam(f(U \setminus B)) < \epsilon$ for all $U \in \mathcal{U}$ not totally contained in B. Since B is precompact, there is a cover $\mathcal{V}$ of X so that $diam(f(V \cap B)) < \epsilon$ for all $V \in \mathcal{V}$. By choosing a star refinement of both $\mathcal{U}$ and $\mathcal{V}$ we obtain a cover $\mathcal{W}$ so that $diam(f(W)) < \epsilon$ for all $W \in \mathcal{W}$. □

We can now dualize this notion to the large scale, and this will give a way to define compactifications of a space. First, we remind the reader some facts about compactifications of Tychonoff spaces.

A **compactification** of a Tychonoff space X is a compact space Y containing X such that $\overline{X} = Y$. An easy way to obtain a compactification of X, we can embed X into a compact space and take the closure of X.

It turns out that all compactifications of X are obtained via embeddings

$$X \hookrightarrow \prod_{f \in A \subseteq C_b(X)} \overline{f(X)}$$
$$x \mapsto (f(x))_{f \in A}$$

where $C_b(X)$ is the set of bounded, continuous functions from X to $\mathbb{C}$ and A is a unital C*-subalgebra which separates the points of X where the operations are pointwise and involution is complex conjugation. The reason this is true is that given a compactification Y and an embedding $i : X \hookrightarrow Y$, we get a unital C*-subalgebra

$$A = \{f \in C_b(X) \mid f \text{ extends continuously to } Y\}.$$

Equivalently, there is an induced map $i^* : C(Y) \hookrightarrow C_b(X)$ defined by $i^*(f) = f \circ i$, and $i^*(C(Y))$ is the desired C*-subalgebra.

On the other hand, given a unital C* subalgebra $A \subseteq C_b(X)$ which separates the points of X, consider $\overline{i(X)}$, where i is the embedding

$$X \hookrightarrow \prod_{f \in A \subseteq C_b(X)} \overline{f(X)}$$
$$x \mapsto (f(x))_{f \in A}$$

Observe that a bounded, continuous function $f : X \to \mathbb{C}$ extends to $\overline{i(X)}$ if and only if $f \in A$.

Here are some examples of compactifications obtained from subalgebras.

1. The Stone-Cech Compactification is the compactification associated to $C_b(X)$.

2. The one-point compactification is the compactification associated to the subring of complex-valued functions whose values tend to a constant value at ∞.

3. The Smirnov (or Samuel) compactification is the compactification associated to the subring of uniformly continuous functions from X to $\mathbb{C}$.

The concept of C_0-variation at a scale provides a connection between scales of a proper metric space and compactifications of that space. For each cover $\mathcal{U}$ of X and $\epsilon \geq 0$, we can define a collection of functions

$$C_\epsilon(\mathcal{U}) = \{f : X \to \mathbb{C} \mid f \text{ is bounded, continuous and has } C_\epsilon\text{-variation at scale } \mathcal{U}\ \}.$$

If X is a Tychonoff topological space with proper large scale structure, then the collections $C_0(\mathcal{U})$ are unital C*-subalgebras of $C_b(X)$ which separate the points of X and therefore determine compactifications of X. For example, the one-point compactification corresponds $C_0(\{X\})$, and the Stone-Čech compactification corresponds $C_0(\{\{x\} : x \in X\})$.

For a proper large scale space X, define $H(X)$ to the the collection of all bounded, continuous function $f : X \to \mathbb{C}$ such that for all large scales $\mathcal{U}$, f has C_ϵ-variation at scale $\mathcal{U}$. We shall call elements of $H(X)$ the Higson functions of X. Notice that $H(X)$ is the intersection of all $C_\epsilon(\mathcal{U})$ as the $\mathcal{U}$'s range through all large scales of X. Hence, $H(X)$ is a unital, C*-subalgebra of $C_b(X)$ and it clearly separates the points of X. Hence, $H(X)$ determines a compactification hX, called the **Higson compactification** of X. This compactification can be characterized by the property that a bounded continuous function $f : X \to \mathbb{C}$ extends continuously to hX if and only if f is Higson.

The Higson compactification was first defined by Nigel Higson in [11] and was used as a tool for doing index theory on non-compact manifolds. The Higson compactification was studied by Keesling in [16], who showed that under very general conditions, the one-dimensional Čech cohomology of the Higson compactification of a space contains a subgroup isomorphic the the additive reals.

Notice that when measuring the variation of a function at a scale, we did not really need the fact that the codomain Y was a metric space, but really only used the small scale structure on Y. Hence, we can generalize the notion of a slowly oscillating functions to maps between large scale spaces and small scale spaces.

Definition 3.2.3. A function $f : X_L \to Y_S$ between an ls-space X_S and an ss-space Y_S is said to be **slowly oscillating** if for every large scale $\mathcal{U}$ of X and every small scale $\mathcal{V}$ of Y_S there exists a weakly bounded set $B \subseteq X$ such that $f(\mathcal{U} \setminus B) = \{f(U \setminus B) : U \in \mathcal{U}\} \prec \mathcal{V}$.

The next lemma gives an equivalent definition of slowly oscillating which will in some cases be easier to work with.

Lemma 3.2.4. *Let X be a large scale space and Y be a small scale space. The following are equivalent:*

1) for every large scale $\mathcal{U}$ of X and every small scale $\mathcal{V}$ of Y, there is a bounded set $B \subseteq X$ so that $\{f(U \setminus B) \mid U \in \mathcal{U}\} \prec \mathcal{V}$;

2) for every large scale $\mathcal{U}$ of X and every small scale $\mathcal{V}$ of Y, there is a bounded set $B \subseteq X$ so that $\{f(U) \mid U \in \mathcal{U}, U \cap B = \emptyset\} \prec \mathcal{V}$.

Proof. Clearly 1) imples 2) since if $U \cap B = \emptyset$, then $U \setminus B = U$.

Now assume 2). For a large scale $\mathcal{U}$ of X and a small scale $\mathcal{V}$ of Y, choose a bounded $B \subseteq X$ so that $\{f(U) \mid U \in \mathcal{U}, U \cap B = \emptyset\} \prec \mathcal{V}$. Now B is bounded, so $B' := st(B, \mathcal{U})$ is also. Notice that if $U \in \mathcal{U}$ and $U \cap B = \emptyset$, then $f(U \setminus B') \subseteq f(U) \subseteq V$ for some $V \in \mathcal{V}$, and if $U \cap B \neq \emptyset$, then $U \subseteq B'$, so $f(U \setminus B') = f(\emptyset) \subseteq V$ for any $V \in \mathcal{V}$. Hence, 2) implies 1). □

3.3 The Higson Corona

Given a proper large scale space, we define the **Higson corona** of X to be $\nu X := hX \setminus X$. Associating to a proper large scale space its Higson corona is functorial (Proposition 2.41 of [27]). Further, for a proper metric space X the covering dimension of the corona is no larger than the asymptotic dimension of X and that they agree if X has finite asymptotic dimension [7]. Recently, it was shown that the Higson corona functor gives an equivalence of categories between the category of totally bounded metric spaces with C_0 coarse structures and the category of compact metric spaces [20].

In this section, we will consider when it is the case that a subset of a large scale space determines the corona. That is, if $A \subseteq X$, then when is the closure of A in the Higson compactification of X equal to $\overline{A} \cup \nu X$? We define a subset $A \subset X$ of the large scale structure X to be **cobounded** if the inclusion $i : A \to X$ is a coarse equivalence. We will show that a sufficient condition on A for the conclusion of the question to be true is coboundedness. The following theorem is of a more general nature and will be useful for our proof.

Theorem 3.3.1. *Let X be a space with Hausdorff compactification μX and $A \subset X$. If A has the property that continuous functions $f, g : \mu X \to [0, \infty)$ must be equal if they agree on A, then the closure of A in μX contains $\mu X \setminus X$.*

Proof. Let $x \in \mu X \setminus X$ and $\epsilon > 0$ and suppose further that there is some $g \in C(\mu X)$ such that $|g(a) - g(x)| \geq \epsilon$ for all $a \in A$. Any extension $\tilde{g}$ of $g|A$ has to have the property that $g - \tilde{g}$ is 0 at x. We will manufacture an extension that does not have this property. Let $\alpha : \mu X \to [0, 1]$ be a continuous function that is 1 at x and 0 on the complement of $g^{-1}((g(x) - \frac{\epsilon}{2}, g(x) + \frac{\epsilon}{2}))$. Notice that $h = g + \alpha$ is a continuous extension of g which does not have the property that $g - h$ is 0 at x. It follows that $x \in \overline{A}^{\mu X}$. □

The philosophy behind this section is to prove an analogue of the well known, see [19], result that states that every metric space is coarsely equivalent to a uniformly discrete metric space. This uniformly discrete subspace will have properties that we will need in order to apply theorem 3.3.1.

Definition 3.3.2. Let X be a space with a proper coarse structure. A **discrete core** D of X is a topologically discrete subset such that the inclusion $D \hookrightarrow X$ is a coarse equivalence.

Definition 3.3.3. Let X be a space with a proper coarse structure. A continuous function $f : X \to \mathbb{C}$ is $\mathbf{C_0}$ if for every $\epsilon > 0$ there is a bounded set B in X such that $|f(x)| < \epsilon$ for $x \in X \setminus B$.

Note that a function $f : X \to \mathbb{C}$ on a proper large scale structure X is C_0 if and only if it extends to be 0 on the Higson Corona νX.

Proposition 3.3.4. *If X has a proper coarse structure, then X has a discrete core D. Furthermore, if $f : D \to [0, 1]$ is Higson, then f extends to a Higson function F on X and any other extension g of f has the property that $g - F$ is C_0 on X.*

Proof. Let $\mathcal{V}$ be an open uniformly bounded cover of X. By Proposition 3.1.3, X is paracompact, so we can refine $\mathcal{V}$ to a locally finite, uniformly bounded open cover $\mathcal{U} = \{U_s\}_{s\in S}$. For each $s \in S$ choose $x_s \in U_s$ and put $D = \{x_s\}_{s\in S}$. We claim that D is a discrete core of X. First we show that the inclusion map $i : D \to X$ is coarse. The subset D is discrete since the cover $\mathcal{U}$ is locally finite. The large scale structure on D is the inherited structure from X; i.e., a family $\mathcal{B}$ of subsets of D is uniformly bounded if and only if $\mathcal{B}$ is uniformly bounded in X. Thus if $\mathcal{B}$ is uniformly bounded in D, then $i(\mathcal{B}) = \mathcal{B}$ is uniformly bounded in X, so i is bornologous. Also, if $B \subset X$ is bounded, then $i^{-1}(B) \subseteq B$ is bounded in D. Hence, i is coarse. Now define $p : X \to D$ by sending each $x \in X$ to some $x_s \in D$ where $U_s \in \mathcal{U}$ is some set containing x. Let $\mathcal{B}$ be uniformly bounded in X. Then $p(\mathcal{B})$ refines $\mathrm{St}(\mathcal{B},\mathcal{U})$, so $p(\mathcal{B})$ is uniformly bounded in D, implying that p is bornologous. If $B \subseteq D$ is bounded, then $p^{-1}(B) \subseteq \mathrm{St}(B,\mathcal{U})$, so is bounded in X. Thus p is coarse. Note that both $\{x, (p \circ i)(x)\}_{x\in D}$ and $\{x, (i \circ p)(x)\}_{x\in X}$ refine $\mathcal{U}$, so that $p \circ i$ and $i \circ p$ are close to the identity maps. Thus, D and X are coarsely equivalent.

Now assume that $f : D \to [0, 1]$ is Higson. Since $\mathcal{U}$ is locally finite, we may assume that for each $U_s \in \mathcal{U}$, it is the case that $U_s \cap D = \{x_s\}$ for if not we may replace U_s by $U_s \setminus (D \setminus \{x_s\})$. Choose a partition of unity $\{\phi_s\}_{s\in S}$ with $\mathrm{supp}(\phi_s) \subseteq U_s$ for each s. Define $F(x) = \sum_{s\in S} f(x_s) \cdot \phi_s(x)$. Then F is continuous and since $\phi_t(x_s) = 1$ for $t = s$ and $\phi_t(x_s) = 0$ for $t \neq s$, then F extends f. We now show that F is slowly oscillating. Fix $\epsilon > 0$. Let $\mathcal{B}$ be the cover of $[0, 1]$ by ϵ balls, and let $\mathcal{V}$ be a uniformly bounded cover of X. Let $\mathcal{W} = \{D \cap st(V,\mathcal{U}) : V \in \mathcal{V}\}$, which is a uniformly bounded cover of D. Then there is a bounded set B of D such that $f(\mathcal{W})|_{D\setminus B}$ refines $\mathcal{B}$. Put $\hat{B} = st(B,\mathcal{U})$, which is bounded in X. Let $x, y \in V \setminus \hat{B}$ for some $V \in \mathcal{V}$. There are finitely many s such that $x \in U_s$. Let x_{s_1} be such that $x \in U_{s_1}$ and $f(x_{s_1}) = \max\{f(x_s) : x \in U_s\}$. Similarly, choose x_{s_2} so that $f(x_{s_2}) = \min\{f(x_s) : y \in U_s\}$. Then $F(x) - F(y) = \sum_{s\in S} f(x_s) \cdot \phi_s(x) - \sum_{s\in S} f(x_s) \cdot \phi_s(y) \leq f(x_{s_1}) - f(x_{s_2}) < \epsilon$ since $x_{s_1}, x_{s_2} \in D \cap st(V,\mathcal{U}) \setminus B$. Similarly, $F(y) - F(x) < \epsilon$. Hence $F(\mathcal{V})|_{X\setminus\hat{B}}$ refines $\mathcal{B}$, so F is slowly oscillating.

Claim: Any slowly oscillating extension g of f will have the property that $F - g$ is C_0 on X.

Notice that $F - g$ is Higson and is 0 on A. Let $\mathcal{V}$ be a uniformly bounded cover of X. Then for any $\epsilon > 0$, there is a weakly bounded $B \subseteq X$ such that $mesh(\{f(U \setminus B) : U \in st(\mathcal{V}, \mathcal{U})\}) < \epsilon$. So then $mesh(\{f(U \setminus st(B, st(\mathcal{U}, \mathcal{V}))\}) < \epsilon$. Each set $U \setminus st(B, st(\mathcal{U}, \mathcal{V}))$ contains an element x of D, and $F - g(x) = 0$. This means that, outside of weakly bounded sets, $F - g$ is arbitrarily close to 0 and hence is C_0. This proves the claim. □

Theorem 3.3.5. *Let X be a proper large scale structure and $A \subset X$ be a closed cobounded subspace. The closure of A in the Higson compactification $h(X)$ is $A \cup \nu X$.*

Proof. Since A is closed, its inherited coarse structure is proper, so we may choose a discrete core D of A. Notice that this discrete core D of A will also be a discrete core of the entire space. Let $f : A \to [0, \infty)$ be a map which extends to a continuous function $\tilde{f} : h(X) \to [0, \infty)$. Since f extends to $h(X)$, it must be slowly oscillating. Let g be another extension of f. Then both g and $\tilde{f}$ are continuous extensions of $f \mid D$, so by Proposition 3.3.4, they must agree on $h(X) \setminus X$. The result then follows from Theorem 3.3.1. □

Chapter 4

Duality

4.1 Contravariant and Covariant Mixed Structures

Suppose that we have a slowly oscillating function $f : X \to Y$ where X has large scale structure $\mathcal{L}$ and Y is some small scale space. We may replace the large scale structure on X with any other large scale structure $\mathcal{M}$ having the same bounded sets as $\mathcal{L}$ such that $id : X_{\mathcal{M}} \to X_{\mathcal{L}}$ is large scale continuous, and the map $f : X_{\mathcal{M}} \to Y$ will still be slowly oscillating. On the other hand, creating a new large scale structure $\mathcal{N}$ on X by adding large scales to $\mathcal{L}$ may result in $f : X_{\mathcal{N}} \to Y$ no longer being slowly oscillating. Proposition 4.1.7 will show that there is a maximum structure on X having the same bounded sets as $\mathcal{L}$ such that f is slowly oscillating with respect to this structure.

Alternatively, one could start with a function $f : X \to Y$ where Y is a small scale space and a collection $\mathcal{B}$ of subsets of X satisfying the conditions of Proposition 2.4.5, that is, a collection of bounded sets. Our next proposition shows that there will always be a large scale structure $\mathcal{L}$ on X such that $f : X_{\mathcal{L}} \to Y$ is slowly oscillating, namely the minimal coarse structure associated with $\mathcal{B}$. Hence, we need only specify the bounded sets on the domain in order to create this maximal large scale structure.

Proposition 4.1.1. *Let Y be a small scale space. Given a space X and a collection $\mathcal{B}$ of bounded sets on X, any function $f : X \to Y$ is slowly oscillating if X is equipped with the minimal large scale structure associated with $\mathcal{B}$.*

Proof. Let $\mathcal{U}$ be a large scale of X. Then there are bounded sets $B_1, \ldots, B_n$ such that for each nonsingleton set $S \in \mathcal{U}$, there is an i so that $S \subseteq B_i$. Then B_i's are bounded so in particular they are weakly bounded, which implies that their union $B = B_1 \cup \cdots \cup B_n$ is weakly bounded. Then the collection $\{f(U \setminus B) : U \in \mathcal{U}\}$ will be a collection of singleton subsets of Y, which clearly refines any uniform cover of Y. Thus, f is slowly oscillating. □

There is a natural duality between the topology induced by an ss-structure and the bounded sets induced by an ls-structure:

- U is open if for each $x \in U$ there is an small scale $\mathcal{U}$ such that $st(x,\mathcal{U}) \subseteq U$.
- B is bounded if for each $x \in B$ there is an large scale $\mathcal{U}$ such that $B \subseteq st(x,\mathcal{U})$.

As naturally as it is to look for small scale structures that induce a given topology, one may also look for large scale structures that have a prescribed collection of bounded sets.

Definition 4.1.2. Let X be a set with bounded sets $\mathcal{B}$ and let Δ be a collection of pairs (Y, f) where Y is a small scale space and $f : X \to Y$ is a function. We say that X has the **contravariant mixed large scale structure** with respect to Δ and $\mathcal{B}$ if X has the maximal large scale structure on X such that each function $f : Y \to X$ for $(Y, f) \in \Delta$ slowly oscillating and the bounded subsets of X are exactly equal to $\mathcal{B}$.

Definition 4.1.3. Let Y be a uniformizable topological space and let Δ be a collection of pairs (X, f) where X is a large scale space and $f : X \to Y$ is a function. The **covariant mixed small scale structure** with respect to Δ is the maximal small scale structure on Y such that each function $f : X \to Y$ is slowly oscillating and such that this small scale structure generates a topology on Y which is no larger than the original one.

Corollary 4.1.4. *If X is a set having a collection of bounded sets which has only finitely many coarse components with more than one element, then the minimal large scale structure is the contravariant mixed large scale structure with respect to all functions to all uniform spaces of cardinality no greater than the cardinality of X.*

Proof. Let $\mathcal{B}$ be the collection of bounded sets on X and suppose that X has a large scale structure strictly larger than the minimal large scale structure. So there is some large scale $\mathcal{U}$ such that for every finite collection $B_1, \ldots, B_n \in \mathcal{B}$, there is some $U \in \mathcal{U}$ with $|U| > 1$ and $U \setminus B_i \neq \emptyset$ for each $i = 1, \ldots, n$. Consider the space X with the maximal small scale structure; that is, every cover of X is a small scale. We claim that $id_X : X \to X$ is not slowly oscillating. Indeed, let $\mathcal{V}$ be the cover of X by singletons. Let B be a weakly bounded subset of X. We claim that there is some $U \in \mathcal{U}$ with $|U| > 1$ and $U \cap B = \emptyset$. Indeed, if not, then $st(B,\mathcal{U})$ is weakly bounded. Let $C_1, \ldots, C_n$ be the coarse components of X. For $i = 1, \ldots, n$, let $B_i = C_i \cap st(B,\mathcal{U})$. But then every nonsingleton element of $\mathcal{U}$ is contained in some B_i, a contradiction. Thus, id_X is not slowly oscillating. □

Note that we need the assumption of only finitely many coarse components. Consider the coarse disjoint union of infinitely many copies of $\{0, 1\}$ where each copy of $\{0, 1\}$ has maximal coarse structure. This structure is strictly larger than the minimal structure, but every function from it is slowly oscillating.

Lemma 4.1.5. *Let X be a set with a collection of bounded sets $\mathcal{B}$. Given any function $f : X \to Y$, where Y is a small scale space, there is a maximal large scale structure on X with bounded sets exactly $\mathcal{B}$ making f slowly oscillating.*

Proof. Define a large scale structure on X by making $\mathcal{U}$ a large scale if and only if

1) for each small scale $\mathcal{V}$ of Y there is a weakly bounded set $B \subseteq X$ such that $\{f(U \setminus B) : U \in \mathcal{U}\} \prec \mathcal{V}$; and

2) $st(B,\mathcal{U})$ is bounded for each bounded set B.

Notice that the second condition implies that $st(B,\mathcal{U})$ is weakly bounded for each weakly bounded set B.

Clearly this is closed under refinements so we need only show that if $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfy the two conditions, then $st(\mathcal{U}_1,\mathcal{U}_2)$ do, too. Let $\mathcal{V}$ be a small scale of Y. Choose a star-refinement $\mathcal{W}$ of $\mathcal{V}$. We may choose weakly bounded sets B_1 and B_2 so that $\{f(U \setminus B_i) : U \in \mathcal{U}_i\} \prec \mathcal{W}$ for $i = 1, 2$. Put $B = st(B_1 \cup B_2, \mathcal{U}_2)$, which is weakly bounded. Let $U \in \mathcal{U}_1$. We will show that $f(st(U,\mathcal{U}_2)) \subseteq V$ for some $V \in \mathcal{V}$. Suppose that $A \in \mathcal{U}_2$, $A \cap U \neq \emptyset$, and $A \setminus B \neq \emptyset$. Say $p \in A \cap U$. Notice that $p \notin B_1 \cup B_2$ since then $U \subseteq B$. There are $W, W' \in \mathcal{W}$ so that $f(U \setminus B_1) \subseteq W$ and $f(A \setminus B) \subseteq W'$. So $f(p) \in W \cap W'$. Thus, $f(st(U,\mathcal{U}_2)) \subseteq st(W,\mathcal{W}) \subseteq V$ for some $V \in \mathcal{V}$. It is clear that $st(\mathcal{U}_1,\mathcal{U}_2)$ satisfies the second condition. Hence, this collection forms a coarse structure. Also, it is clear that this collection is the maximal so that f is slowly oscillating. □

Theorem 4.1.6. *Let X be a set with bounded sets $\mathcal{B}$ and Δ a collection of pairs (Y, f), where Y is a small scale space and $f : X \to Y$ is any function. Then there exists a maximal large scale structure on X with bounded sets $\mathcal{B}$ so that f is slowly oscillating for every (Y, f) in Δ; i.e., the contravariant mixed large scale structure on X with respect to Δ exists.*

Proof. For each pair (Y, f) in Δ, we can form the maximal large scale structure on X so that f is slowly oscillating. The intersection of large scale structures is again a large scale structure, so to form the contravariant large scale structure with respect to Δ, intersect the maximal coarse structure with respect to each function f in Δ. □

Corollary 4.1.7. *Let X be a large scale space, Y be a small scale space, and $f : X \to Y$ be slowly oscillating. There exists a maximum large scale structure X_{L_0} on X with the same bounded sets as X such that $id_X : X \to X_{L_0}$ is large scale continuous and $f : X_{L_0} \to Y$ is slowly oscillating.*

Remark 4.1.8. Let $\mathcal{B}$ be a collection of bounded sets for a space X. As stated above, there is a minimum large scale structure on X with bounded sets equal to $\mathcal{B}$. Also, there is a maximum structure with bounded sets $\mathcal{B}$ which is formed by taking all covers $\mathcal{U}$ of X with the property that $st(B,\mathcal{U}) \in \mathcal{B}$ for all $B \in \mathcal{B}$.

We now consider the codomain of a slowly oscillating map. The small scale structure on the codomain induces a topology on Y, but there may be other uniform structures which induce the same topology. It is well known (see [13] Theorem 20 on page 10) that there is a largest uniformity on a uniformizable space X which induces the topology on X. On the other hand, there need not be a minimal small scale structure generating the topology of a space. In fact, a Tychonoff space has a minimal uniformity generating its topology if and only if it is locally compact [29]. The maximal uniformity generating a given topology is built by considering all decreasing sequences of *open* scales. In particular, for a paracompact space, the collection of all open covers is the largest small scale structure generating the topology. It is not difficult to show that the collection of all scales which are in some sequence forms a base for a uniformity on X which induce the same topology on X and is the maximal such uniformity. We adapt this technique to obtain the following result.

Theorem 4.1.9. *Let X be a uniformizable topological space with topology Γ and Δ a collection of pairs (Y, f) where Y is a large scale structure and $f : Y \to X$ is a function. There exists a maximal small scale structure such that*

1. The small scale structure generates a topology τ on X such that $\tau \subseteq \Gamma$.

2. $f : Y \to X$ is slowly oscillating for each pair $(Y, f) \in \Delta$; i.e the covariant mixed small scale structure on X with respect to Δ exists.

Additionally, if for each $(Y, f) \in \Delta$, the map f is slowly oscillating (where X is considered with its original small scale structure), then $\tau = \Gamma$.

Proof. Let M be the maximal small scale structure generating the topology on X. Let $\mathcal{F}$ be the collection of all decreasing sequences of scales $\{\mathcal{U}_n : n \geq 1\}$ of uniform covers from M that satisfy the following slowly oscillating condition: For each uniformly bounded cover $\mathcal{V}$ of X, $n \geq 1$, and $(Y, f) \in \Delta$ there is a weakly bounded set B_n such that $\{f(V \setminus B_n) : V \in \mathcal{V}\}$ refines $\mathcal{U}_n$. Notice that this collection is nonempty since we at least have $\{\mathcal{U}_i = \{X\} : i \geq 1\}$ as such a normal sequence.

Let $\{\mathcal{U}_n : n \geq 1\}$ and $\{\mathcal{V}_n : n \geq 1\}$ be in $\mathcal{F}$ and $(f, Y) \in \Delta$. Their intersection is the normal sequence $\{\mathcal{U}_n \cap \mathcal{V}_n : n \geq 1\}$. We will show that the intersection $\{\mathcal{U}_n \cap \mathcal{V}_n : n \geq 1\}$ also satisfies the slowly oscillating property. Let $\mathcal{W}$ be a uniformly bounded cover of X and $n \geq 1$. Let B_n and C_n be weakly bounded subsets of X such that $\{f(W \setminus B_n) : W \in \mathcal{W}\} \prec \mathcal{U}_n$ and $\{f(W \setminus C_n) : W \in \mathcal{W}\} \prec \mathcal{V}_n$. Notice then that $B_n \cup C_n$ is weakly bounded and that each element of $\{f(W \setminus (B_n \cup C_n)) : W \in \mathcal{W}\}$ lies in an element of $\mathcal{U}_n \cap \mathcal{V}_n$ and thus the sequence $\{\mathcal{U}_n \cap \mathcal{V}_n : n \geq 1\}$ satisfies the slowly oscillating property.

The union of all normal sequences in $\mathcal{F}$ is a uniformity on Y. It is, by design, the largest uniformity on Y for which f is slowly oscillating. Since the decreasing sequences come from a uniformity generating the original topology, the topology induced must be no larger than the original topology.

Now assume that for each $(Y, f) \in \Delta$, the function f is slowly oscillating. In this case, all scales from the original structure are elements of some decreasing sequence of scales with the slowly oscillaing condition. Hence, the topology generated by the new small scale structure is no smaller than the original topology. Therefore, in this case the topology generated by the maximal small scale structure is equal to the original topology on X. □

Corollary 4.1.10. *Let X be a large scale space, Y be a small scale space, and $f : X \to Y$ be slolwy oscillating. There exists a maximal small scale structure Y_{S_0} inducing the same topology on Y such that $id_Y : Y_{S_0} \to Y$ is uniformly continuous and $f : X \to Y_{S_0}$ is slowly oscillating.*

Remark 4.1.11. Notice that when maximizing a small scale structure as in the above proposition, if one wishes to preserve the topology, then it is necessary to start with a function which is already slowly oscillating. In the case of maximizing a large scale structure one need only specify the bounded sets and the resulting maximal large scale structure will still have these bounded sets. The reason for this is that for a fixed large scale structure and function, it may be the case that no uniformity on the codomain which generates the topology will make the function slowly oscillating. For example, if f is a function from some large scale space to a compact space which is not slowly oscillating, then since there is only one small scale structure which generates the topology on the codomain, it is not possible to find any small scale structure generating the topology which will make the function slowly oscillating. However, we can find a small scale structure generating a smaller topology.

4.2 The C_0 Large Scale Structure

The remainder of this section will be devoted to an extensively studied proper large scale structure, which is equivalent to the so-called C_0 coarse structure which was introduced by Nick Wright in [31]. We define a C_0 large scale structure and in the appendix we show that it is the correct translation of C_0 coarse structures in the setting of large scale structures.

Definition 4.2.1. Let (X, d) be a metric space. Define the C_0 **large scale structure with respect to** d to be the covariant mixed large scale structure with respect to $\{id_X : X \to X\}$, where the small scale structure on X is equal to the metric small scale structure.

Definition 4.2.2. Let (X, d) be a metric space. We call a collection $\mathcal{U}$ **metrically** d**-bounded** if there is an $M > 0$ such that $mesh(\mathcal{U}) < M$.

Proposition 4.2.3. *The C_0 large scale structure on a proper metric space (X, d) is equal to the collection of all metrically d-bounded scales $\mathcal{U}$ of X such that for every $\epsilon > 0$ there is a precompact $K \subseteq X$ such that $mesh(\{U \setminus K\}_{U \in \mathcal{U}}) < \epsilon$.*

We remark that if the metric on X is proper, then in the C_0 large scale structure, a set $B \subseteq X$ is bounded if and only if it is pre-compact.

One interesting aspect of the C_0 structure is that it is often nonmetrizable, as the following proposition shows.

Proposition 4.2.4. *If (X, d) is a proper, unbounded, geodesic metric space, then $C_0(X)$ is not metrizable.*

Proof. Suppose that $\hat{d}$ is a metric whose metric large scale structure is equal to the C_0 structure on (X, d). For $n = 1, 2, 3, \ldots$, let $\mathcal{B}_n = \{B_{\hat{d}}(x, n) : x \in X\}$, the collection of all n-balls around points of X. Each of these collections is uniformly bounded in the C_0 structure induced by d, so for each n, we may choose a compact K_n such that $mesh_d\{B \setminus K_n \mid B \in \mathcal{B}\} < \frac{1}{n^2}$. Also, we may choose M_n such that $mesh_d(\mathcal{B}_\setminus) < M_n$. Now for each n, choose $x_n \in X \setminus (N_{n+M_n}(K_n))$, where $N_{n+M_n}(K_n)$ denotes the $n + M_n$ neighborhood of K_n. Then the collection $\{B_d(x_n, \frac{1}{n}) : n \geq 1\}$ is uniformly bounded in the C_0 structure of (X, d) so it must be bounded in the $\hat{d}$ metric. But since X is geodesic, we may choose for each n a point y_n such that $\frac{1}{n^2} < d(x_n, y_n) < \frac{1}{n}$. But $diam_d(B_{\hat{d}}(x_n, n)) < \frac{1}{n^2}$, so $\hat{d}(x_n, y_n) > n$. Thus, $mesh_{\hat{d}}\{B_d(x_n, \frac{1}{n}) : n \geq 1\}$ is unbounded, a contradition. Therefore, no metric generates the C_0 coarse structure. □

On the other hand, for uniformly discrete metric spaces, the C_0 structure is metrizable and is generated by an ultrametric. In particular, this implies that the C_0 structure has asymptotic dimension zero.

Proposition 4.2.5. *If (X, d) is a proper metric space which is uniformly discrete (that is, there is an $r > 0$ such that $d(x, y) > r$ for all $x, y \in X$), then $C_0(X, d)$ is metrizable.*

Proof. Fix a basepoint $x_0 \in X$. Define a metric $\hat{d}$ on X by $\hat{d}(x, y) = 0$ if $x = y$ and for $x \neq y$, define $\hat{d}(x, y) = 3^n$, where $n = \min\{k : x, y \in B_d(x_0, k)\}$. We claim that $B_{\hat{d}}$, the bounded coarse structure associated with $\hat{d}$ is equal to $C_0(X, d)$.

Let $\mathcal{B}$ be uniformly bounded in $(X, \hat{d})$. Then $mesh_{\hat{d}}(\mathcal{B}) = M < \infty$. Choose n so that $3^n > M$. Then outside of $B(x_0, n)$, the collection $\mathcal{B}$ consists only of singletons. Thus, $mesh_d(\mathcal{B}) < 2n < \infty$. Also, $mesh_d(\{B \setminus B_d(x_0, n) : B \in \mathcal{B}\}) = 0$. Thus, $\mathcal{B} \in C_0(X, d)$.

Now suppose that $\mathcal{B} \in C_0(X, d)$. Choose K compact so that $mesh_d(\{B \setminus K : B \in \mathcal{B}\}) < r/2$. Then outside of $K, \mathcal{B}$ consists of only singletons. Choose n so that $K \subseteq B_d(x_0, n)$. Then $mesh_{\hat{d}}(\mathcal{B}) \leq 3^n < \infty$. So $\mathcal{B}$ is uniformly bounded in $(X, \hat{d})$. □

4.3 Galois Connections

Definition 4.3.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be two posets. A **Galois connection** between $\mathcal{P}$ and $\mathcal{Q}$ is a pair of fuctions (ϕ^*, ϕ_*) with $\phi^* : \mathcal{P} \to \mathcal{Q}$ and $\phi_* : \mathcal{Q} \to \mathcal{P}$ such that

1) both functions are monotone;

2) $x \leq \phi_*\phi^*(x)$ and $\phi^*\phi_*(y) \leq y$ for every $x \in \mathcal{P}$ and $y \in \mathcal{Q}$.

The notion of a Galois connection is a generalization of the correspondence between subgroups and subfields which arises in Galois theory of fields. See [25] for more information on Galois connections.

The concept of Galois connection can explain certain results from coarse geometry. Given a proper large scale structure on a space there is an associated compactification, the Higson compactification with respect to that structure. Conversely, given a compactification, there is an associated large scale structure, namely the continuously controlled large scale structure. Proposition 2.45 of [27] can be reinterpreted as saying that the operations of taking the Higson compactification with respect to a large scale structure and taking the continuously controlled coarse structure associated to a compactification forms a Galois connection between the compactifications of a space and the large scale structures on that space.

We will now show that the process described above of maximizing large scale structure and maximizing small scale structures with respect to a slowly oscillating function is a Galois connection between a collection of large scale structures on the domain and a collection of small scale structures on the codomain.

Suppose we have a function $f : X \to Y$ where Y is a uniformizable space. Let $\mathcal{B}$ be a collection of subsets of X which are the bounded sets for some large scale structure on X. Let $\mathcal{P}$ be the poset, ordered by inclusion, of all large scale structures on X

with bounded set $\mathcal{B}$ for which f is slowly oscillating for some small scale structure inducing the topology on Y. Let $\mathcal{Q}$ be the poset, ordered by reverse inclusion, of small scale structures on Y inducing the given topology on Y for which f is slowly oscillating for some large scale structure in $\mathcal{P}$. Define $\phi^* : \mathcal{P} \to \mathcal{Q}$ by mapping a large scale structure L to the maximal element of $\mathcal{Q}$ for which f is slowly oscillating when X has large scale structure L, and define $\phi_* : \mathcal{Q} \to \mathcal{P}$ by mapping a small scale structure S to the maximal element of $\mathcal{P}$ for which f is slowly oscillating when Y has small scale structure S.

Theorem 4.3.2. *The functions ϕ^* and ϕ_* form a Galois connection from $\mathcal{P}$ to $\mathcal{Q}$.*

Proof. All conditions are easy to verify. For example, let $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{P}$ with $\mathcal{L}_1 \subseteq \mathcal{L}_2$. We show that $\phi^*\mathcal{L}_2 \subseteq \phi^*\mathcal{L}_1$. Let $\mathcal{U}$ be a small scale in $\phi^*\mathcal{L}_2$ and let $\mathcal{B}$ be a large scale in $\mathcal{L}_1$. But then $\mathcal{B}$ is an element of $\mathcal{L}_2$, so there is a weakly bounded set $K \subseteq X$ such that $\{B \setminus K : B \in \mathcal{B}\} \prec \mathcal{U}$. Hence $\mathcal{U} \in \phi^*\mathcal{L}_1$, implying that $\phi^*\mathcal{L}_2 \subseteq \phi^*\mathcal{L}_1$. □

Corollary 4.3.3. *For the functions ϕ^* and ϕ_* defined above, it is the case that $\phi^* \circ \phi_* \circ \phi^* = \phi^*$ and $\phi_* \circ \phi^* \circ \phi_* = \phi_*$.*

Proof. This is a consequence of being a Galois connection (see [25][Theorem 9]), but we include a proof. Let $\mathcal{L} \in \mathcal{A}$. By the definition of a Galois connection we have that $\mathcal{L} \subseteq \phi_*\phi^*\mathcal{L}$. Then by the monotonicity of ϕ^* we have $\phi^*\phi_*\phi^*\mathcal{L} \subseteq \phi^*\mathcal{L}$. At the same time we have $\phi^*\mathcal{L} \subseteq \phi^*\phi_*\phi^*\mathcal{L}$. Thus $\phi^*\mathcal{L} = \phi^*\phi_*\phi^*\mathcal{L}$. The proof that $\phi_* \circ \phi^* \circ \phi_* = \phi_*$ is similar. □

Corollary 4.3.4. *Let X be a large scale space, Y be a small scale space, and $f : X \to Y$ be a slowly oscillating function. Then there is a large scale structure L on X and a small scale structure S on Y so that $f : X_L \to Y_S$ is slowly oscillating and both L and S are maximal with respect to this property.*

Notice that the Galois connection technique shows that one reaches this maximal state only after alternatingly maximizing 3 times. Intuitively, this makes sense.

Suppose one starts with a slowly oscillating function $f : X \to Y$ and first maximizes the large scale structure on X. Once one has maximized the small scale structure on Y, it only makes it more difficult to add large scales to the existing large scale structure on X.

Example 4.3.5.

Let $(X, d) = (\mathbb{N}, d)$ with standard metric. Notice that $id_X : X \to X$ is not slowly oscillating when X is given the metric large and small scale structures. Indeed, the cover by singletons is a small scale of X, but clearly there is no bounded set B so that outside of B the cover by radius 5 balls refines the cover by singletons. Hence, when we maximize the large scale structure or small scale structure with respect to the opposite metric scale structure, we will get a smaller scale structure than we started with.

First, we consider maximizing the large scale structure. It is easy to see that the maximal large scale structure is the metric structure generated by the metric $\tilde{d}$ defined by $\tilde{d}(x, y) := d(x^2, y^2)$ since in this metric, for any R, the cover by R-balls consists of singletons outside of some finite set.

Now, construct the maximal small scale structure as follows: For every increasing sequence of integers $1 = n_1 < n_2 < n_3 < \ldots$, let $\mathcal{U}_{\{n_i\}} := \bigcup_{i \geq 1} \{B(x, i) : x \in [n_i, n_{i+1})\}$. Give $\mathbb{N}$ the small scale structure generated by the collection of all $\mathcal{U}_{\{n_i\}}$. For any cover of $\mathbb{N}$ by R-balls, and any $\mathcal{U}\{n_i\}$, notice that the cover by R-balls refines $\mathcal{U}_i$ outside of $\{1, \ldots, n_R\}$, hence the identity map is slowly oscillating when $\mathbb{N}$ has the given small scale structure. Also, it is the maximal small scale structure making the identity map slowly oscillating. Indeed, if $\mathcal{V}$ is a small scale in the maximal small scale structure, then for $i \geq 1$, we can choose an increasing sequence n_i so that outside of $\{1, \ldots, n_i\}$, the cover by i-balls refines $\mathcal{V}$. Hence, $\mathcal{U}_{\{n_i\}} \prec \mathcal{V}$. Notice that this small scale structure generates the same topology on $\mathbb{N}$ since for any $n \in \mathbb{N}$, if we let $\{n_i\}$ be an increasing sequence of integers with $n_2 > n$, then $st(\{n\}, \mathcal{U}_{\{n_i\}}) = \{n\}$, hence this small scale structure gives $\mathbb{N}$ the discrete topology. However, this small scale structure is strictly smaller than the original one.

Notice that if we first maximize the large scale structure with respect to the metric structure induced by d, then maximizing the small scale structure with respect to the maximal large scale structure gives the same small scale structure on $\mathbb{N}$. This is since the small scale structure induced by d is already the maximal structure on $\mathbb{N}$. On the other hand, if we first maximize the small scale structure and then maximize the large scale structure with respect to the new small scale structure, then we get back the metric large scale structure. Indeed, suppose the resulting structure is any larger. Then it contains some scale $\mathcal{U}$ with infinite mesh. This means that for each i, we can choose an integer n_i so that there is a set in $\mathcal{U}$, $U_i \subseteq \{1, \ldots, n_i\}$ and $diam(U_i) > i$. But then $\mathcal{U}$ does not refine $\mathcal{V}_{\{n_i\}}$ outside of any bounded set.

Example 4.3.6.

Let (X, d) be a metric space. We can generalize the above example to arbitrary metric spaces. Fix a point $x \in X$. For each increasing sequence of positive real numbers $0 < n_1 < n_2 < n_3 < \ldots$ and each open cover $\mathcal{V}$ of X, let

$$\mathcal{U}_{\{n_i\},\mathcal{V}} := \{V \in \mathcal{V} : V \cap B(x, n_1) \neq \emptyset\} \cup \bigcup_{i \geq 2} \{B(y, i) \mid y \in B(x, n_{i+1}) \setminus B(x, n_i)\}.$$

Each of these covers is open so is in the maximal small scale structure generating the topology on X. Also, each is clearly in the maximal small scale structure making the identity map slowly oscillating. Now, assume that $\mathcal{U}$ is a small scale cover in the maximal structure. For each $i \geq 1$ we can find n_i so that outside of $B(x, n_i)$, $\mathcal{U}$ coarsens the cover of X by radius i balls. Then $\mathcal{U}$ coarsens $\mathcal{V}_{\{n_i\},\int(\mathcal{U})}$. This proves that the collection of $\mathcal{U}_{\{n_i\},\mathcal{V}}$ generate the largest small scale structure on X so that the identity map is slowly oscillating. Call the resulting structure $D_0(X)$.

Once we have maximized the small scale structure, we can maximize the large scale structure. However, we will gain no new covers. Indeed, if $\mathcal{U}$ is a cover not in the original large scale structure, then $mesh(\mathcal{U}) = \infty$. So we can choose a sequence of elements U_i of $\mathcal{U}$ and an increasing sequence of real numbers $\{n_i\}$ so that $U_i \subseteq B(x, n_{i+1}) \setminus B(X, n_i)$ and $diam(U_i) > i$. But then there is no bounded set B so that outside of $B, \mathcal{U}$ refines the cover $\mathcal{V}_{\{n_i\},\{B(x,1)\}}$. Thus, the maximal large scale structure on X is the original metric one.

Now let's consider what happens if we maximize in the opposite order. In this case, we add the assumption that the metric is proper; that is, a set is compact if and only if it is closed and bounded. First, we hold the metric small scale structure fixed and maximize the large scale structure. This results in $C_0(X)$. Now we can maximize the small scale structure with respect to $C_0(X)$. We claim that the resulting small scale structure is equal to the original metric small scale structure. If not, then it includes some small scale $\mathcal{V}$ such that $Leb(\mathcal{V}) = 0$. This means that for each $i \geq 1$, there is some $x_i \in X$ such that $B(x_i, \frac{1}{i})$ is not contained in any element of $\mathcal{V}$. Notice that the x_i cannot all be contained in some bounded set. Indeed, suppose that B is a bounded set containing all $\{x_i\}$. Then the closure of B is compact, so if we consider the cover $int(\mathcal{V})$ of the closure of B, this cover must have positive Lebesgue number, a contradiction. Now we build a C_0 large scale $\mathcal{U}$ so that there is no bounded set outside of which $\mathcal{U}$ refines $\mathcal{V}$. We will build an increasing sequence $\{r_i\}$ by first setting $r_1 = d(x_1, x_2)$. Once r_i has been chosen, pick some x_{n_i} not contained in $B(x_1, r_i)$ and put $r_{i+1} = d(x_1, x_{n_i})$. Define a C_0 cover

$$\mathcal{U} = \bigcup_{i \geq 1} \{B(x, \frac{1}{n_i^2}) : x \in B(x_1, r_{i+1}) \setminus B(x_1, r_i)\}.$$

4.4 Dual Pairs

Definition 4.4.1. Let X be a large scale space with large scale structure L, Y be a small scale space with small scale structure S, and $f : X \to Y$ be slowly oscillating. We say that (X_L, Y_S) are **dual with respect to** f if the structures L and S are maximal with respect to f being slowly oscillating. This means that first, L cannot be enlarged to a large scale structure with the same bounded sets having f still slowly oscillating, and S cannot be enlarged to a small scale structure generating the same topology but with f still slowly oscillating. If the function f is understood, we will simply say that (X_L, Y_S) is a **dual pair**.

An important case to consider is when $X = Y$ as sets and $f = id_X$.

Example 4.3.6 shows that for large scale spaces, metric large scale structures are always elements of dual pairs, and that for small scale spaces, proper metric spaces are always elements of dual pairs. For the case of a proper metric space (X, d), then $(C_0(X), (X, d))$ is a dual pair. In general, Proposition 4.3.4 says that given a slowly oscillating function between a large scale space and a small scale space, the structures on the domain and codomain can always be increased in just two steps to get a dual pair. More generally, given a function between a large scale space and a small scale space, we can always find a dual pair in two steps, although the structures may be smaller than the structures we started with.

Lemma 4.4.2. *If $f : X \to Y$ is a function between large scale spaces with the property that f maps bounded sets of X to bounded sets of Y and the inverse image of a bounded set of Y is bounded in X, then f also has the property that the image of a weakly bounded set is bounded and the inverse image of a weakly bounded set is weakly bounded.*

Proof. We claim that each coarse component of X maps to the intersection of $f(X)$ with a coarse component of Y. Let $\sim$ be the relation defined by $x \sim y$ if $\{x, y\}$ is bounded. Observe that for $x, y \in X$,

$$x \sim y \Rightarrow \{x, y\} \text{ is bounded } \Rightarrow \{f(x), f(y)\} \text{ is bounded} \Rightarrow f(x) \sim f(y),$$

which shows that each coarse component of the domain is mapped into a single coarse component of the codomain. At the same time,

$$f(a) \sim f(b) \Rightarrow \{f(a), f(b)\} \text{ is bounded } \Rightarrow f^{-1}(\{f(a), f(b)\}) \text{ is bounded } \Rightarrow a \sim b,$$

which shows that the inverse image of a coarse component is contained in a single coarse component.

The statement about weakly bounded sets follows since to determine is a set is weakly bounded, we consider its intersection with each coarse component. So if B is weakly bounded in the domain, then $f(B)$ intersected with a coarse component of the domain is equal to the image of the intersection of B with the inverse image of that coarse component, hence is bounded. Similarly, the result holds for the inverse image of a weakly bounded set. □

Theorem 4.4.3. *Let X_L be a large scale space and Y_S be a small scale space such that $f : X_L \to Y_S$ is slowly oscillating and the structure on X_L is the maximal such that f is slowly oscillating (in particular, if X_L and Y_S are dual with respect to f.) Let $g : Z \to X_L$ be a map from a large scale space such that g maps bounded sets to bounded sets and such that the inverse image of a bounded set is bounded. Then $f \circ g$ is slowly oscillating if and only if g is large scale continuous.*

Proof. First suppose that g is large scale continuous. Let $\mathcal{U}$ be a large scale of X and $\mathcal{V}$ a small scale of Y. Then $g(\mathcal{U})$ is a large scale of X, so there is some weakly bounded set $B \subseteq X$ so that $\{f(g(U) \setminus B) \mid U \in \mathcal{U}\} \prec \mathcal{V}$. By assumption, $g^{-1}(B)$ is weakly bounded. Notice that $\{(f \circ g)(U \setminus g^{-1}(B)) : U \in \mathcal{U}\} \prec \{f(g(U) \setminus B) : U \in \mathcal{U}\} \prec \mathcal{V}$, proving that $f \circ g$ is slowly oscillating.

Now suppose that $f \circ g$ is slowly oscillating. Let $\mathcal{U}$ be a large scale of Z. We need to show that for every bounded $B \subseteq X$, it is the case that $st(B, g(\mathcal{U}))$ is bounded. Notice that $st(B, g(\mathcal{U})) \subseteq g(st(g^{-1}(B), \mathcal{U}))$ and $g(st(g^{-1}(B), \mathcal{U}))$ is bounded since $g^{-1}(B)$ is bounded, hence $st(g^{-1}(B), \mathcal{U}))$ is bounded. Now, let $\mathcal{V}$ be a small scale of Y. We know that there is a weakly bounded $B \subseteq Z$ so that $\{(f \circ g)(U \setminus B) \mid U \in \mathcal{U}\} \prec \mathcal{V}$. By assumption, $g(B)$ is weakly bounded in X. Notice that $\{f(g(U) \setminus g(B)) \mid U \in \mathcal{U}\} \prec \{(f \circ g)(U \setminus B) \mid U \in \mathcal{U}\} \prec \mathcal{V}$. Hence, by the definition of the large scale structure on X, it is the case that $g(\mathcal{U})$ is a large scale of X, implying that g is large scale continuous. □

Theorem 4.4.4. *Let X_L be a large scale space and Y_S be a small scale space such that $f : X_L \to Y_S$ is slowly oscillating and the structure on Y_S is the maximal such that f is slowly oscillating (in particular, if X_L and Y_S are dual with respect to f.) Let $g : Y \to K$ be continuous. Then $g \circ f$ is slowly oscillating if and only if g is small scale continuous.*

Proof. First suppose that g is small scale continuous. Let $\mathcal{U}$ be a large scale of X and $\mathcal{V}$ be a small scale of K. Then $g^{-1}(\mathcal{V})$ is a small scale of Y. Hence, there is a weakly bounded set $B \subseteq X$ such that outside of B, the image of $\mathcal{U}$ under f refines $g^{-1}(\mathcal{V})$. Then outside of B, the image of $\mathcal{U}$ under $g \circ f$ refines $\mathcal{V}$.

Now suppose that $g \circ f$ is slowly oscillating. Let $\mathcal{V}$ be a small scale of K. We can choose a decreasing sequence of small scales of K, $\{\mathcal{V}_i\}_{i \geq 0}$, with $\mathcal{V}_0 = \mathcal{V}$. Then $\{g^{-1}(\mathcal{V}_i)\}_{i \geq 0}$ is a decreasing sequence of scales of X and it satisfies the slowly oscillating condition since $g \circ f$ is slowly oscillating. Also, by the continuity of g, $\{g^{-1}(int(\mathcal{V}_i))\}_{i \geq 0}$ is a decreasing sequence of open scales of Y, and $g^{-1}(int(\mathcal{V}_i)) \prec g^{-1}(\mathcal{V}_i)$ for each i. Thus, by the definition of the small scale structure on Y, $g^{-1}(\mathcal{V})$ is a small scale of Y. $\square$

We remark that it is always true that the composition of a small scale continuous and a slowly oscillating function is slowly oscillating.

Corollary 4.4.5. *If X_L is a proper large scale space and X_S is a small scale space with the same topology and X_S is maximal with repsect to the identity being slowly oscillating (in particular, if X_L and X_S are dual with respect to id_X, and X_L and X_S have the same topology), then the bounded Higson functions on X_L are exactly the bounded, complex-valued, uniformly continuous continuous functions on X_S.*

Corollary 4.4.6. *If X_L and X_S are dual with respect to id_X and X_L and X_S have the same topology, then the Higson compactification of X_L and the Smirnov compactification of X_S are homeomorphic.*

Recall that for a metric space X, the small scale structure $D_0(X)$ is the maximal small scale structure on X generating the metric topology such that the identity map of X is slowly oscillating, where the large scale structure on X is the metric large scale structure.

Corollary 4.4.7. *If X is a proper metric space, then the Higson compactification of $C_0(X)$ and the Smirnov compactification of X are homeomorphic.*

If X is a metric space, then the Higson compactification of X and the Samuel compactification of $D_0(X)$ are homeomorphic.

Proof. The first statement follows since $C_0(X)$ is proper with respect to the metric topology. The second statement follows since the topology on $D_0(X)$ is the same as the metric topology by construction. □

As an application of the above machinery, we provide a new proof of the well-known fact that the Higson compactification of an unbounded metric space is never metrizable.

First, suppose that X is an unbounded metric space. Select a subset A of X as follows. For $i = 1, 2, 3, \ldots$, choose points $x_i \in X$ as follows. First, let x_1 be any point in X. Once x_i has been chosen, pick x_{i+1} so that $d(x_{i+1}, \{x_1, x_2, \ldots, x_i\}) \geq i + 1$. Notice that every function on A is slowly oscillating (where A has the inherited metric large scale structure). Thus, the Higson compactification of A is equal to the Stone-Cech compactification of A, which is nonmetrizable. We will show that every function on A extends to a continuous and slowly oscillating function on X, and the desired result will be a consequence of the following lemma.

Lemma 4.4.8. *If $A \subseteq X$ such that the inherited large scale structure on A is proper with respect to the inherited topology, and every continuous and slowly oscillating function $f : A \to [0, 1]$ extends to a continuous and slowly oscillating function $g : X \to [0, 1]$, then the Higson compactification of A is equal to the closure of A taken in the Higson compactification of X.*

Proof. We will show that a continuous $f : A \to [0, 1]$ extends to a continuous $\tilde{f} : \overline{A} \to [0, 1]$ if and only if f is slowly oscillating.

First suppose that $f : A \to [0,1]$ extends to a continuous map $\tilde{f} : \overline{A} \to [0,1]$. Using the Tietze extension theorem, extend $\tilde{f}$ to a continuous map $g : hX \to [0,1]$. Notice that $g\mid_X : X \to [0,1]$ is continuous and has a continuous extension to hX, hence is slowly oscillating. Thus, $f = g\mid_A$ is slowly oscillating.

Now suppose that $f : A \to [0,1]$ is continuous and slowly oscillating. Let $g : X \to [0,1]$ be a continuous and slowly oscillating extension. Then g has a continuous extension $\tilde{g} : hX \to [0,1]$ and so $\tilde{g}\mid_{\overline{A}} : \overline{A} \to [0,1]$ is a continuous extension of f, proving the result. □

We will also make use of Katetov's Theorem.

Theorem 4.4.9. *(Katetov's Theorem)[15]: If Y is a small scale space and A is a subset of Y, then any small scale continuous function $f : A \to [0,1]$ extends to a small scale continuous function $\tilde{f} : X \to [0,1]$.*

Notice that in the inherited small scale structure, A is uniformly discrete; that is, every cover of A is uniform. Also, with the inherited scale structures, $id_A : A \to A$ is slowly oscillating. This is because every large scale of A consists of only singletons outside of a bounded set. Hence, the inherited small scale structure on A is maximal so that the identity map is slowly oscillating. Let $f : A \to [0,1]$ be continuous and slowly oscillating. Then f is also small scale continuous. By Katetov's Theorem, f extends to a small scale continuous function $\tilde{f} : X \to [0,1]$. Then $\tilde{f} : X \to [0,1]$ is continuous and slowly oscillating. Thus, every continuous and slowly oscillating function $f : A \to [0,1]$ extends to a continuous and slowly oscillating function $\tilde{f} : X \to [0,1]$ and the desired result holds.

4.5 Continuous and Uniform Control

The philosophy behind this section is to take a topological space or ss-structure and define a large scale structure on a portion of it by declaring a certain subspace to be infinitely far away. We aim to make this vague point of view more concrete and to discuss some well known and some new ways of achieving this goal.

Recall the following Theorem/definition from Roe [27](Theorem 2.27 on page 26)

Theorem 4.5.1. *Let X be a locally compact paracompact space with a compactification μX. Let $E \subseteq \mu X \times \mu X$. The following conditions are equivalent:*

1) The closure $\overline{E}$ of E meets the compliment of $X \times X$ only in the diagonal.

2) E is proper and, for every net (x_λ, y_λ) in E, if the net x_λ converges to a point x in $\mu X \setminus X$ then the net y_λ converges to x.

3) E is proper, and for every point $x \in \mu X \setminus X$ and every neighborhood V of x in μX there is a neighborhood $U \subseteq V$ of x such that $E \cap (U \times X \setminus V) = \emptyset$.

The **continuously μX controlled coarse structure** X_{CC} is the large scale structure on X that consists of all sets E satisfying any of the above conditions. It is worth pointing out that this large scale structure need not always be proper, see [20] for further details.

We develop two points of view for translating the above definition of continuous control from coarse structures to large scale structures. One of the points of view comes from the notion of duality while the other comes from ideas developed by Damas in [22] about Dugundji covers.

4.5.1 An Exterior Approach

Definition 4.5.2. Let X be a locally compact Hausdorff space and μX a compactification of X. Let $S_\mu(X)$ be the uniform structure on X induced by μX. The large structure $L_\mu(X)$ is the maximal large scale structure on X such that the identity map $(X, L_\mu(X)) \to S_\mu(X)$ is slowly oscillating. We call $L_\mu(X)$ the **continuously μX controlled large scale structure**.

Theorem 4.5.3. *Let X be a locally compact paracompact space with a compactification μX. $L_\mu(X)$ is the large scale structure corresponding to the continuously μX controlled coarse structure X_{CC} on X.*

Proof. Let $\mathcal{U}$ be a uniform cover of X and let $\mathcal{B}$ be a uniformly bounded family in X. Say $\bigcup_{B\in\mathcal{B}}(B \times B) \subseteq E$ where E is continuously controlled. Extend $\mathcal{U}$ to a uniform cover $\mathcal{U}'$ of μX. Then we may refine $\mathcal{U}'$ to an open cover $\mathcal{V}$ of μX since the uniformity inducing a compact topology consists of all covers normal with respect to the family of open covers. For each $p \in \mu X \setminus X$ choose $V_p \in \mathcal{V}$ with $p \in V_p$. By the definition of the continuously controlled coarse structure we may choose $U_p \subseteq V_p$ with U_p open, $p \in U_p$, and $E \cap (U \times X \setminus V) = \emptyset$. Put $U = \bigcup_{p\in\mu X\setminus X} U_p$. Then $(X \setminus U) \times (X \setminus U)$ is continuously controlled, so $X \setminus U$ is bounded. We claim that $\{B \setminus (X \setminus U)\}_{B\in\mathcal{B}} \prec \mathcal{U}$. Let $b_1 \in B \cap U$. Then there is some $p \in \mu X \setminus X$ such that $b_1 \in U_p$. We'll show that any other point of $B \cap U$ is contained in V_p. Let $b_2 \in B \cap U$. Then b_2 must be in V_p since $(b_1, b_2) \in E$ and $E \cap (U_p \times (X \setminus V_p)) = \emptyset$. Thus $B \cap U \subseteq V_p \cap X$ which is contained in some element of $\mathcal{U}$, as desired. Therefore, the identity map id: $X \to X$ is slowly oscillating. □

Theorem 4.5.4. *Let X be a locally compact paracompact space with a compactification μX. Then $L_\mu(X)$ and $S_\mu(X)$ form a dual pair.*

Proof. One need only observe that the small scale structure on $S_\mu(X)$ is maximal as it uniquely extends to a small scale structure on μX and that small scale structure is unique. □

The following corollary is well known.

Corollary 4.5.5. *[20] [22] The C_0 large scale structure on a locally compact metric X with metrizable compactification μX is the continuously controlled coarse structure induced by μX*

Definition 4.5.6. Let X be a set and X'_S be a small structure on X. Let X'_L be the minimal large scale structure on X with bounded sets being pre-compact in the topology induced by the small scale structure. The **uniformly controlled large scale structure on** X is the dual pair (X_S, X_L) associated to the identity $X'_L \to X'_S$ which is slowly oscillating.

Proposition 4.5.7. *Let X be topological space with a compactification μX and let X'_S be a uniform structure on X induced by μX. Then the uniformly controlled large scale structure on X induced by X'_S is the continuously μX controlled large scale structure.*

4.5.2 An Interior Approach

We generalize the C_0 coarse structure in this section using the ideas from [1]. The idea behind the C_0 coarse structure of a metric space X with compactification μX is that one wants a coarse structure on X so that the corona $\mu X \setminus X$ is declared infinitely far away. We consider the more general case in which one has a topological space (or uniform space) X and declares a subset $A \subseteq X$ to be infinitely far away by defining a coarse structure that reflects this behavior.

Let X be a space and $A \subseteq X$. A family $\mathcal{U}$ of $X \setminus A$ is **(X,A) topologically controlled** ($(X, A)-$controlled for short) if for each $x \in A$ and each neighborhood V of x there is a neighborhood $W \subseteq V$ of x such that $st(W, \mathcal{U}) \subseteq V$. Damas in [22] similarly defines canonical covers for compact spaces X with nowhere dense subsets A (coronas of compactifications) which are topologically controlled families with the added condition that the covers be locally finite and open.

Theorem 4.5.8. *Let X be a space and $A \subseteq X$. The collection of all (X, A) topologically controlled families is a ls-structure on $X \setminus A$.*

Proof. The only property of being a ls-structure that is not easy to check is whether the star of an (X, A)-controlled cover against another (X, A)-controlled cover is (X, A)-controlled.

Suppose $\mathcal{U}_i$, $i = 1, 2$ are (X, A)-controlled. We need to show that $st(\mathcal{U}_2, \mathcal{U}_1)$ is an (X, A)-controlled cover. Let V be a neighborhood of some x in $X \setminus A$.

- Pick a neighborhood W_0 of x such that $st(W_0, \mathcal{U}_1) \subseteq V$
- Pick a neighborhood W_1 of x such that $st(W_1, \mathcal{U}_2) \subseteq W_0$

- Pick a neighborhood W_2 of x such that $st(W_2,\mathcal{U}_1) \subseteq W_1$

Suppose $T \in \mathcal{U}_2$ and $st(T,\mathcal{U}_1) \cap W_2 \neq \emptyset$. Notice then that

$$st(T,\mathcal{U}_1) \subseteq st(st(st(W_2,\mathcal{U}_1),\mathcal{U}_2),\mathcal{U}_1) \subseteq st(st(W_1,\mathcal{U}_2),\mathcal{U}_1) \subseteq st(W_0,\mathcal{U}_1) \subseteq V.$$

This means that $st(\mathcal{U}_2,\mathcal{U}_1)$ is (X, A)-controlled.

□

The following proposition, proven by Damas, shows that these covers come from continuously controlled coarse structures.

Proposition 4.5.9. *[Corollary 7 from [22]] Let X be a locally compact Hausdorff space with compactification μX and corona A. A family of subsets of $X = \mu X \setminus A$ is $(\mu X, A)$ topologically controlled if and only if it is continuously μX controlled.*

4.6 Comparison of Classes of Coarse Structures

In [21], Mine, Yamashita, and Yamauchi defined the above coarse structure as the C_0 coarse structure associated to $\mathcal{U}$ and showed that it is equal to the continuously controlled coarse structure induced by the Samuel compactification of $\mathcal{U}$. Since the continuously controlled large scale structure associated to a compactification is equal to the uniformly controlled large scale structure associated to the unique uniformity on that compactification, it follows that the class of uniformly controlled and the class of continuously controlled coarse structures are one and the same.

For a metric space, we may recover the C_0 structure of Wright by taking the metric uniformity, forming its Samuel compactification, and then taking the uniformly controlled large scale structure induced by the unique uniformity inherited from the Samuel compactification.

Roe shows that every metric coarse structure is equal to the continuously $h(X)$ controlled coarse structure associated to the Higson compactification with respect to the metric. On the other hand, we have shown that not every C_0 structure is metrizable. Thus, the class of continuously controlled and uniformly controlled large scale structures properly contains the class of metrizable coarse structures.

Chapter 5

Pinch-Spacing

5.1 Preliminaries

Definition 5.1.1. Let X be any set and $1 \leq p < \infty$. Define

$$\ell_p(X) = \{f : X \to \mathbb{R} : \sum_{x \in X} |f(x)|^p < \infty\};$$

$$\ell_{p,+}(X) = \{f : X \to [0,1] : \sum_{x \in X} |f(x)|^p = 1\};$$

$$\Delta_p(X) = \{f : X \to [0,1] : f \text{ has finite support, } \sum_{x \in X} |f(x)|^p = 1\};$$

$$\Delta_p^{(n)}(X) := \{f \in \Delta_p(X) : |supp(f)| \leq n+1\}.$$

Notice that

$$\Delta_p^{(n)}(X) \subseteq \Delta_p(X) \subseteq \ell_{p,+}(X) \subseteq \ell_p(X).$$

For each p, it is the case that $\ell_p(X)$ is a Banach space with norm $||f||_p := (\sum_{x \in X} |f(x)|^p)^{1/p}$.

Remark 5.1.2. For $p = 2$, we get not just a Banach space but a Hilbert space: $\ell_2(X)$ is a Hilbert space with inner product $\langle f, g\rangle := \sum_{x\in X} f(x)g(x)$.

Remark 5.1.3. Recall that for a space X, a partition of unity on X is a collection of functions $\{\phi_s : X \to [0,1]\}_{s\in S}$ such that for each $x \in X$, it is the case that $\sum_{s\in S} \phi_s(x) = 1$.

For a set S and a space X, there is a one-to-one correspondence between partitions of unity on X indexed by S and maps $f : X \to \ell_{1,+}(S)$. This correspondence sends the partition of unity $\{\phi_s\}_{s\in S}$ to the function $\psi : X \to \ell_{1,+}(S)$ defined by $\psi(x)(s) = \phi_s(x)$.

A partition of unity $\{\phi_s\}_{s\in S}$ is **simplicial** if for each $x \in X$, there are only finitely many $s \in S$ such that $\phi_s(x) > 0$. Restricting the above correspondence to the set of simplicial partitions of unity gives a correspondence between the set of simplicial partitions of unity and functions $\psi : X \to \Delta_1(X)$.

Definition 5.1.4. For a set X, define

$$\ell_\infty(X) = \{f : X \to \mathbb{R} : \sup_{x\in X} |f(x)| < \infty\}.$$

This is a Banach space with $||f||_\infty = \sup_{x\in X} |f(x)|$.

5.2 Pinch-Spacing

Definition 5.2.1. A function $f : X \to Y$ between metric spaces is said to have (R, ϵ) **variation** if $d(x, y) \leq R$ implies that $d(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

More generally, consider a function $f : X \to Y$, where Y is a metric space. Let $\mathcal{U}$ be a collection of subsets of X and $\epsilon > 0$. We say that f has $(\mathcal{U}, \epsilon)$ **variation** if $mesh(f(\mathcal{U})) < \epsilon$.

Even more generally, consider a function $f : X \to Y$. For a collection $\mathcal{U}$ of subsets of X and a collection $\mathcal{V}$ of subsets of Y, we say that f has $(\mathcal{U}, \mathcal{V})$**-variation** if $f(\mathcal{U}) \prec \mathcal{V}$.

Definition 5.2.2. Let X be a large scale space and K be a metric space. Let $c > 0$. Then X **c-pinch-spaces** to K if for every large scale $\mathcal{U}$ of X and every $\epsilon > 0$, there is a large scale $\mathcal{V}$ of X and a function $f : X \to K$ such that

(1) f has $(\mathcal{U}, \epsilon)$-variation;

(2) if $x, y \in X$ and no element of $\mathcal{V}$ contains both x and y, then $d(f(x), f(y)) \geq c$.

More generally, let X be a large scale space and Y be a small scale space. Fix a small scale $\mathcal{C}$ of Y. Then X **$\mathcal{C}$-pinch-spaces** to Y if for every large scale $\mathcal{U}$ of X and every small scale $\mathcal{V}$ of Y there is a large scale $\mathcal{W}$ of X and a function $f : X \to Y$ such that

(1) f has $(\mathcal{U}, \mathcal{V})$-variation;

(2) $f^{-1}(\mathcal{C}) \prec \mathcal{W}$.

Notice that it is sufficient to consider only scales which are part of a basis for the scale structures on either the domain or the codomain.

We remark that if X and Y are metric spaces with the corresponding metric scale structures, then X c-pinch-spaces to Y if and only if for every $R, \epsilon > 0$, there is some $S > 0$ and $f : X \to Y$ so that 1) $d(x, y) < R$ implies that $d(f(x), f(y)) < \epsilon$; and 2) $d(x, y) > S$ implies that $d(f_x, f_y) \geq c$.

A pinch-spacing map is similar to a coarse map, but only at a fixed scale. That is, a pinch-spacing map must map close points to close points and must send points which are sufficiently far away in the domain to distant points in the codomain. However, it doesn't have to do it uniformly for all scales like a coarse map, but only at a one chosen scale at a time. The ability of X to pinch-space to Y says that at any scale of X, there is a map which respects that scale.

Notice that a bounded metric space X will c-pinch-spaces to any other space, since a constant map with satisfy the pinching and spacing condition for $S > diam(X)$.

The property of pinch-spacing to a particular small scale space is a coarse invariant.

Proposition 5.2.3. *Let X and Y be coarsely equivalent large scale spaces and let K be a small scale space. Let $\mathcal{C}$ be a small scale of K. Then X $\mathcal{C}$ pinch-spaces to K if and only if Y $\mathcal{C}$ pinch-spaces to K.*

Proof. Suppose that X $\mathcal{C}$ pinch-spaces to K. Let $\mathcal{U}$ be a large scale of Y. Since X and Y are coarsely equivalent, there are large scale continuous maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f$ is close to the id_X and $f \circ g$ is close to id_Y. That is, there is a large scale $\mathcal{B}$ of Y so that for all $p \in Y$, there is some $B \in \mathcal{B}$ so that $p, (f \circ g)(p) \in B$. Since $g(\mathcal{U})$ is a large scale of X. Hence, there is a large scale $\mathcal{W}$ of X and a function $h : X \to K$ so that $h(g(\mathcal{U})) \prec \mathcal{V}$ and $h^{-1}(\mathcal{C}) \prec \mathcal{W}$. Then $(h \circ g) : Y \to K$ has the property that $(h \circ g)(\mathcal{U}) \prec \mathcal{V}$ and $(h \circ g)^{-1}(\mathcal{C}) \prec st(f(\mathcal{W}, \mathcal{B}))$. That is, Y $\mathcal{C}$-pinch-spaces to K. □

More trivially, the property of being a pinch-space of a particular large scale space is a small scale invariant.

Proposition 5.2.4. *Let X be a large scale space and Y_1, Y_2 small scale spaces. Assume that Y_1 and Y_2 are small scale equivalent. That is, there are small scale continuous maps $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$ so that $g \circ f = id_{Y_2}$ and $f \circ g = id_{Y_1}$. If $\mathcal{C}$ is a small scale of Y_1, then X $\mathcal{C}$ pinch-spaces to Y_1 if and only if X $f(\mathcal{C})$ pinch-spaces to Y_2.*

Proof. Straightforward. □

Proposition 5.2.5. *If $\mathcal{B}$ is a Banach space, then $\mathcal{B}$ c-pinch-spaces to itself for any $c > 0$. In particular, if $\mathcal{H}$ is a Hilbert space, then $\mathcal{H}$ c-pinch-spaces to itself for any $c > 0$.*

Proof. Fix $c > 0$. For $R, \epsilon > 0$, define $f : \mathcal{B} \to \mathcal{B}$ by $f(x) = \frac{\epsilon}{R} \cdot x$. Choose $S > \frac{cR}{\epsilon}$. Notice that $d(f(x), f(y)) = ||f(x) - f(y)|| = ||\frac{\epsilon}{R} \cdot x - \frac{\epsilon}{R} \cdot y|| = \frac{\epsilon}{R}||x - y|| = \frac{\epsilon}{R} d(x, y)$. Thus, if $d(x, y) < R$, then $d(f(x), f(y)) = \frac{\epsilon}{R} \cdot d(x, y) < \frac{\epsilon}{R} \cdot R$. Also, if $d(x, y) > S$, then $d(f(x), f(y)) > \frac{\epsilon}{R} \cdot S > c$, as desired. □

Notice that in general, a metric space need not pinch-space to itself. Consider, for example, $\mathbb{N}$. It is the case that $\mathbb{N}$ does not c-pinch-space to itself for any $c > 0$. Indeed, suppose we had a pinch-spacing map $f : \mathbb{N} \to \mathbb{N}$ for $R = 2$ and $\epsilon = \frac{1}{2}$. It follows from the pinching property that f is a constant map. However, this violates the spacing property.

It is not even true that every geodesic metric space c-pinch-spaces to itself for some $c > 0$.

Theorem 5.2.6. *If $d \geq 3$, then the d-regular tree T_d does not c-pinch-space to itself for any $c > 0$.*

Proof. Let $c > 0$. Choose $0 < \epsilon < \min\{\frac{c}{2}, \frac{1}{2}\}$. Let f be a pinching map for $R = 1$ and ϵ. Let $S > 0$. We will show that f cannot be spacing. To arrive at a contradiction, we will build an S-separted set which must map into a set which contains a $\frac{c}{2}$ net of smaller cardinality. Thus, at least two points which are distance greater than S must be mapped to points within distance c. Fix a vertex v. For each $n = 1, 2, 3, \ldots$, let $S(n)$ be the sphere of radius n centered at v, which is $2n$-separated. Then $|S(v, n)| = d(d-1)^{n-1}$. Notice that $f(S(n)) \subseteq B(f(v), n\epsilon)$. Within $B(f(v), n\epsilon)$ we can build a $\frac{c}{2}$ net, $N(n)$ by starting with $f(v)$ and then choosing all points which are distance ϵ apart. Then $|N(2k)| \leq 2d \sum_{i=0}^{k-1} (d-1)^i$. Note that $\lim_{k\to\infty} \frac{|S(2k)|}{|N(2k)|} = \infty$. Thus we can choose $k > S$ so that $|S(2k)| > |N(2k)|$, which shows that f cannot be spacing.

□

Corollary 5.2.7. *The Cayley graph of F_n does not pinch-space to itself for $n \geq 2$.*

Example 5.2.8. *If $n \geq 2$, then hyperbolic n space H^n does not c-pinch-space to itself for any $c > 0$.*

Example 5.2.8 can be proved using the same technique as for the d-regular tree using that fact that the surface area of a sphere in hyperbolic space grows exponentially with respect to radius.

5.3 Pinch Spacing and Dimension

Definition 5.3.1. If Y is a small scale space, then we say that the **small scale dimension** of Y is no larger than n, denoted $ssdim(Y) \leq n$ if for every small scale $\mathcal{U}$ of Y, there is a small scale refinement $\mathcal{V}$ of $\mathcal{U}$ so that $mult(\mathcal{V}) \leq n+1$.

Remark: The above definition is what Isbell calls large dimension of a uniform space.

Theorem 5.3.2. *(Theorem 3.12 of [13]) If K is an n-dimensional simplicial complex, then $ssdim(K) = n$.*

Definition 5.3.3. If X is a large scale space, then we say that the **asymptotic dimension** of X is no larger than n, denoted $asdim(X) \leq n$ if for every large scale $\mathcal{U}$ of X, there is a large scale coarsening $\mathcal{V}$ of $\mathcal{U}$ so that $mult(\mathcal{U}) \leq n+1$.

It turns out that the class of zero asymptotic dimension metric spaces c-pinch-space to themselves for all $c > 0$.

Proposition 5.3.4. *If X is a metric space and $asdim(X) = 0$, then X c-pinch-spaces to itself for any $c > 0$.*

Proof. Let $c, R, \epsilon > 0$. Put $T = \max R, c$. Since $asdim(X) = 0$, there is a T-disjoint uniformly bounded cover $\mathcal{U}$ of X. That is, $\inf_{x \in U, x' \in U'} d(x, x') > T$ for every $U \neq U' \in \mathcal{U}$. Notice that this implies that each $x \in X$ is contained in only one element of $\mathcal{U}$. For each $U \in \mathcal{U}$, choose an element $x_u \in U$. Define $f : X \to X$ by defining $f(y) = x_U$, where $y \in U$. Put $S = mesh(\mathcal{U}) < \infty$. If $d(x, y) < R \leq T$, then x and y are contained in the same element U of $\mathcal{U}$, so $d(f(x), f(y)) = d(x_U, x_U) = 0 < \epsilon$. If $d(x, y) > S$, then x and y are contained in distinct elements of $\mathcal{U}$, so $d(f(x), f(y)) > T \geq c$. □

Theorem 5.3.5. *If X is a large scale space and Y is a small scale space and X $\mathcal{C}$-pinch-spaces to Y for some small scale $\mathcal{C}$, then $asdim(X) \leq ssdim(Y)$.*

Proof. Let $n = sdim(Y)$. Let $\mathcal{U}$ be a large scale of X. We know that there is a uniform refinement $\mathcal{D}$ of $\mathcal{C}$ with multiplicity $\leq n+1$. Choose a pinch-spacing map $f : X \to Y$ with $(\mathcal{U}, \mathcal{D})$-variation and a large scale $\mathcal{W}$ of X so that $f^{-1}(\mathcal{C}) \prec \mathcal{W}$. Put $\mathcal{V} = f^{-1}(\mathcal{D})$. Notice that $\mathcal{D} \prec \mathcal{C}$, so $\mathcal{V} = f^{-1}(\mathcal{D}) \prec f^{-1}(\mathcal{C}) \prec \mathcal{W}$, hence $\mathcal{V}$ is uniformly bounded. Also, $mult(\mathcal{V}) \leq mult(\mathcal{D}) \leq n+1$. Finally, notice that $\mathcal{U} \prec \mathcal{V}$. Indeed, if $U \in \mathcal{U}$, then there is some $D \in \mathcal{D}$ so that $f(U) \subseteq D$. Hence, $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(D) \in \mathcal{V}$. Thus, we have found a uniformly bounded coarsening with multiplicity $\leq ssdim(X)+1$. □

Corollary 5.3.6. *If X c-pinch-spaces to $\mathbb{R}^n$ for some $c > 0$ and some n, then $asdim(X) \leq n$.*

Corollary 5.3.7. *Let $c > 0$. Then $\mathbb{R}^n$ c-pinch-spaces to $\mathbb{R}^m$ if and only if $n \leq m$.*

Proof. The if direction follows since for $n \leq m$, we can isometrically embed $\mathbb{R}^n$ into $\mathbb{R}^m$ and then scale $\mathbb{R}^m$ to get the desired pinch-spacing map. The only if direction follows since $asdim(\mathbb{R}^n) = n = ssdim(\mathbb{R}^n)$. □

Corollary 5.3.8. *The Cayley graph of $\mathbb{Z}^2$ does not c-pinch-space to itself for any $c > 0$.*

Proof. The follows since the Cayley graph of $\mathbb{Z}^2$ has asymptotic dimension 2 but small scale dimension 1. □

5.4 Pinch-Spacing to Metric Spaces and Banach Spaces

Proposition 5.4.1. *If a large scale space X coarsely embeds in a metric space, then X is metrizable.*

Proof. Let $f : X \to Y$ be a coarse embedding, where Y is a metric space. Since a subspace of a metric space is a metric space, we may, without loss of generality, assume that f is onto. So there is a large scale continuous map $g : Y \to X$ so that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.

To show that X is metrizable, it is sufficient to show that it has a countable basis. For each natural number n, let $\mathcal{V}_n := \{B(y, n) : y \in Y\}$. Then $\{\mathcal{V}_n : n \in \ltimes\}$ is a large scale basis for the large scale structure on Y.

Say $g \circ f$ is $\mathcal{W}$-close to id_X. That is, $\mathcal{W}$ is uniformly bounded and for every $x \in X$, there is some $W \in \mathcal{W}$ so that $x, gf(x) \in W$. We will show that the collection $\{st(g(\mathcal{V}_n), \mathcal{W}) : n \in \mathbb{N}\}$ is a large scale basis for the structure on X.

Let $\mathcal{U}$ be a large scale of X. Then $f(\mathcal{U})$ is uniformly bounded in Y, so there is some $n \in \mathbb{N}$ so that $f(\mathcal{U}) \prec \mathcal{V}_n$. We claim that $\mathcal{U} \prec st(g(\mathcal{V}_n), \mathcal{W})$. Indeed, let $U \in \mathcal{U}$. Then there is some $V \in \mathcal{V}_n$ so that $U \subseteq V$. Let $x \in U$. Choose $W \in \mathcal{W}$ so that $x, gf(x) \in W$. Notice that $gf(x) \in g(V) \cap W$. Thus, $x \in st(g(V), \mathcal{W})$. Hence, $U \subseteq st(g(V), \mathcal{W}) \in st(g(\mathcal{V}_n), \mathcal{W})$, as claimed. □

Lemma 5.4.2. *If X and Y are metric spaces, then a map $f : X \to Y$ is a coarse embedding if and only if there are functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ so that $\lim_{t\to\infty} \rho_-(t) = \infty$ and for all $x, y \in X$, it is the case that*

$$\rho_-(d(x, y)) < d(f(x), f(y)) < \rho_+(d(x, y)).$$

Proof. ($\Rightarrow$): Suppose that $f : X \to Y$ is a coarse embedding. Without loss of generality, we may assume that f is onto. Then there is a map $g : Y \to X$ which is large scale continuous, $g \circ f$ is closed to id_X, and $f \circ g$ is closed to id_Y. For $t \geq 0$, let $B(t) = \{\overline{B}(x, t) = \{y \in X : d(x, y) \leq t\} : x \in X\}$. Since f is large scale continuous, then for each t, there is $M_t < 0$ so that $mesh(f(B(t))) < M_t$. Put $\rho_+(t) = M_t$. Let $x, y \in X$ and say $d(x, y) = t$. Then $x, y \in \overline{B}(x, t)$. Thus, $f(x), f(y) \in f(\overline{B}(x, y))$, which implies that $d(f(x), f(y)) < M_t = \rho_+(t) = \rho_+(d(x, y))$.

For each $t \geq 0$, put $\rho_-(t) = \inf\limits_{x,y\in X, d(x,y)=t} d(f(x), f(y))$. Clearly, $\rho_-(d(x, y)) \leq d(f(x), f(y))$ for all $x, y \in X$. We claim that $\rho_-(t) \to \infty$ as $t \to \infty$. If not, then there is $M > 0$ such that for every t, there is $s > t$ such that $\rho_-(s) \leq M$. Put

$\mathcal{V} = \{B(x, M) : x \in M\}$, which is uniformly bounded in Y. Hence, $g(V)$ is uniformly bounded in X; say $mesh(g(\mathcal{V})) = t < \infty$. Notice that $g \circ f$ is close to id_X, so there is $N < \infty$ such that $d(g \circ f(p), p) < N$ for all $p \in X$. We may choose $s > t + 2N$ so that there are points $x, y \in X$ with $d(x, y) = S$, but $d(f(x), f(y)) < M$. But then $d(g(f(x), g(f(y)))) \geq d(x, y) - d(x, g(f(x))) - d(y, g(f(y))) > S - 2M > t$, a contradiction.

($\Leftarrow$): First, we show that f is large scale continuous. For each $r > 0$, put $B(r) = \{B(x, r) : x \in X\}$. To show the large scale continuity of f, it suffices to show that the image of each $B(r)$ is uniformly bounded. We claim that $mesh(f(B(r))) < 2 \cdot \rho_+(r)$. Indeed, if $y, z \in B(x, r),$, then $d(f(y), f(z)) \leq d(f(y), f(x)) + d(f(x), f(z)) < \rho_+(r) + \rho_+(r) = 2 \cdot \rho_+(r)$.

Now, define $g : f(X) \to X$ by choosing for each $y \in f(X)$ some $x \in X$ such that $f(x) = y$. First, we show that g is large scale continuous by showing that $g(B(r))$ is uniformly bounded for each $r > 0$. Indeed, fix $r > 0$ and choose t so that $\rho_-(t_0) > r$ for all $t_0 \geq t$. Notice that if $y, z \in B(x, r)$, then $d(g(y), g(z)) \leq d(g(y), g(x)) + d(g(x), g(z)) \leq 2t$.

Finally, notice that $f \circ g = id_{g(X)}$ and that if we choose M so that $\rho_-(t) > 0$ for all $t \geq M$, then $d(x, g(f(x))) \leq M$ for all $x \in X$, so $g \circ f$ is close to id_X. □

Lemma 5.4.3. *If X coarsely embeds in a metric space Y, and Y c-pinch-spaces to itself, then X c-pinch-spaces to Y.*

Proof. Say $f : X \to Y$ is a coarse embedding. So there are function $\rho_1, \rho_+ : [0, \infty) \to [0, \infty)$ such that $lim_{t \to \infty} \rho_-(t) = \infty$ and for all $x, y \in X$,

$$\rho_-(d(x, y)) < d(f(x), f(y)) < \rho_+(d(x, y)).$$

Fix $R, \epsilon > 0$. Choose a c-pinch-spacing map $g : T \to T$ for $\rho_+(R)$ and ϵ and let T be the associated constant. Choose S so that $\rho_(S) > T$. We claim that $g \circ f$ is a c-pinch-spacing map with constant S for R and ϵ. Indeed, if $d(x, y) < R$, then

$d(f(x), f(y)) < \rho_+(R)$, which implies that $d(g(f(x)), g(f(y))) < \epsilon$. Also, if $d(x, y) > S$,then $d(f(x), f(y)) > \rho_-(S) > T$, which implies that $d(g(f(x)), g(f(y))) > c$. □

While Theorem 5.4.1 shows that only metric spaces can coarsely embed in metric spaces, it is the case that any space can and does pinch-space to a metric (actually, Banach) space.

Theorem 5.4.4. *(Kuratowski)[17] If X is a metric space, then X isometrically embeds in $\ell_\infty(X)$.*

Proof. Consider the map $x \mapsto f_x$, where $f_x(y) = d(x, y) - d(y, x_0)$ for some fixed $x_0 \in Y$. It is easy to check this this map is an isometry. □

Corollary 5.4.5. *If X is a metric space, then X c-pinch-spaces to the Banach space $\ell_\infty(X)$ for any $c \geq 0$.*

Proof. Theorem 5.4.4 implies that X isometrically embeds in $\ell_\infty(X)$. The space $\ell_\infty(X)$ c-pinch-spaces to itself for any $c > 0$ by 5.2.5, so the result follows from Lemma 5.4.3. □

Remark: By a result of Fréchet [9], ever separable metric space isometrically embeds in $\ell^\infty(\mathbb{N})$, so it follows that every separable metric space c-pinch-spaces to $\ell^\infty(\mathbb{N})$ for any c.

Proposition 5.4.6. *If X is a large scale space, then X c-pinche-spaces to the Banach space $\ell_\infty(X)$ for any $c > 0$.*

Proof. Let $c > 0$. Fix a large scale $\mathcal{U}$ of X and $\epsilon > 0$. We may choose an increasing sequence of scales $\{\mathcal{U}_i\}_{i \geq 1}$ starting with $\mathcal{U}_1 = \mathcal{U}$. Consider the metric from Lemma 2.9.15, defined using this increasing sequence of scales. This metric may take value ∞, so say that $\{X_\alpha\}_{\alpha \in I}$ are the coarse components from this metric; that is the X_α are equivalence classes of points of defined by the relation that two points are related if they are finite distance apart. For each $\alpha \in I$, choose $x_\alpha \in X_\alpha$. Define

$f : X \to \ell_\infty(X)$, $x \mapsto f_x$ by $f_x(y) = \frac{\epsilon}{2}(d(x,y) - d(x_\alpha, y))$, if $x, y \in X_\alpha$ and $f_x(y) = c$ if x and y are in different coarse components of X. Notice that if $x \in X_\alpha$, then $sup_{y \in X}|f_x(y)| = \max \frac{\epsilon}{2} d(x, x_\alpha), c$, so $f_x \in \ell_\infty(X)$.

Now we show that f satisfies the pinch-spacing requirements. Suppose that $x \neq y \in U$ for some $U \in \mathcal{U}$. Then $d(x,y) = 1$. Hence, there is a coarse component, say X_α containing both x and y. Then

$$d(f_x, f_y) = \sup_{z \in X} |f_x(z) - f_y(z)| = \sup_{z \in X_\alpha} |f_x(z) - f_y(z)| = \frac{\epsilon}{2} d(x,y) = \frac{\epsilon}{2} < \epsilon.$$

Choose i so that $i \geq \frac{c}{\epsilon}$. Suppose $x, y \in X$ and no element of $\mathcal{U}_i$ contains both x and y. Then $d(x,y) > i \geq \frac{c}{\epsilon}$. First assume that x and y are contained in the same coarse component, say X_α. Then

$$d(f_x, f_y) = \sup_{z \in X} |f_x(z) - f_y(z)| = \sup_{z \in X_\alpha} |f_x(z) - f_x(z)| = \epsilon d(x,y) > \epsilon \frac{c}{\epsilon} = c.$$

Now assume that no coarse component contains both x and y. Then $d(f_x, f_y) \geq |f_x(y) - f_y(y)| = f_x(y) = c$. □

5.5 Asymptotic Dimension

Asymptotic dimension, the large scale analogue of topological covering dimension, is an important large scale invariant. Originally defined by Gromov for metric spaces in [10], asymptotic dimension has been extensively studied since Guoliang Yu showed that the Novikov conjecture holds for groups of finite asymptotic dimension [32]. In this section, we characterize asymptotic dimension in terms of pinch-spacing maps to finite dimensional simplicial complexes.

Definition 5.5.1. If Δ is a simplicial complex with vertex set V and $v \in V$, then the star of v is defined by $st(v) := \{\phi \in \Delta : \phi(v) \neq 0\}$.

Theorem 5.5.2. *Let X be a large scale space. The following are equivalent:*

1) $asdim(X) \leq n$;

2) X 2-pinch-spaces to a simplicial complex of dimension $\leq n$ (with ℓ_1 metric);

3) X 2-pinch-spaces to $\Delta_1^{(n)}(X)$;

4) X $2^{1/p}$-pinch-spaces to $\Delta_p^{(n)}(X)$ for some $1 \leq p < \infty$;

5) X $2^{1/p}$-pinch-spaces to $\Delta_p^{(n)}(X)$ for all $1 \leq p < \infty$.

Proof. Clearly 3) implies 2).

2) $\Rightarrow$ 1): This follows from Theorem 5.3.2 and 5.3.5. Alternatively, suppose that X 2-pinch-spaces to Δ, a simplicial complex of dimension $\leq n$. Let $\mathcal{U}$ be a uniformly bounded cover of X and $0 < \epsilon < \frac{1}{n+1}$. Choose a uniformly bounded cover $\mathcal{V}$ of X and $f : X \to \Delta$, $x \mapsto f_x$ so that $mesh(f(\mathcal{U})) < \epsilon$ and with the property that if no element of $\mathcal{V}$ contains both x and y, then $d(f_x, f_y) \geq 2$.

Let I be the vertex set of Δ. Put $\mathcal{W} = f^{-1}(\{st(i) : i \in I\})$. We claim that $\mathcal{U} \prec \mathcal{W} \prec st(\mathcal{V}, \mathcal{V})$ and $\mathcal{W}$ has multiplicity $\leq n+1$.

Let $j \in I$. Fix $x \in f^{-1}(st(j))$ and some $V \in \mathcal{V}$ containing x. Let $y \in f^{-1}(st(j))$. Suppose by way of contradiction that no element of $\mathcal{V}$ contains both x and y. Then $d(f_x, f_y) \geq 2$. But $f_x(j), f_y(j) > 0$, so $d(f_x, f_y) = \sum_{i \in I} |f_x(i) - f_y(i)| < \sum_{i \in I} f_x(i) + \sum_{i \in I} f_y(i) = 2$, a contradiction. Hence, there is some element of $\mathcal{V}$ containing both x and y. Thus, $f^{-1}(st(j)) \subseteq st(\{x\}, \mathcal{V}) \subseteq st(V, \mathcal{V}) \in st(\mathcal{V}, \mathcal{V})$. Thus, $\mathcal{W} \prec st(\mathcal{V}, \mathcal{V})$ is uniformly bounded.

Now, let $U \in \mathcal{U}$ and fix $x \in U$. We claim that there is some $i \in I$ so that i is in the support of f_y for all $y \in U$. Suppose not. Say $support(f_x) = \{i_0, \ldots, i_n\}$. Then for each $j = 0, \ldots, n$, we may choose $y_j \in U$ with $i_j \notin support(f_{y_j})$. But $d(f_x, f_{y_j}) < \frac{1}{n+1}$, implying that $f_x(p_j) < \frac{1}{n+1}$. Then $\|f_x\| = \sum_{i \in I} f_x(i) < (n+1)\frac{1}{n+1} = 1$, a contradiction. Thus, there is some $i \in I$, which is in the support of f_y for all $y \in U$. Thus, $U \subseteq f^{-1}(st(i))$. Thus, $\mathcal{U} \prec \mathcal{W}$.

Notice that for each $x \in X$, it is the case that f_x is supported on at most $n+1$ vertices. Hence, $\mathcal{W}$ has multiplicity no larger than $n+1$. Therefore, $asdim(X) \leq n$.

1) $\Rightarrow$ 3): Assume that $asdim(X) \leq n$. We claim that X 2-pinch-spaces to $\Delta_1^{(n)}(X)$. Indeed, fix a uniformly bounded cover $\mathcal{U}$ of X and $\epsilon > 0$. We can choose an increasing sequence of large scales $\{\mathcal{U}_i\}_{i\geq 1}$ as follows. Put $\mathcal{U}_1 = \mathcal{U}$ and then for each i, choose a cover $\mathcal{U}_{i+1}$ of multiplicity $\leq n+1$ such that $st(\{x : x \in X\}, st(\mathcal{U}_i,\mathcal{U}_i)) \prec \mathcal{U}_{i+1}$. Additionally, we may stipulate that there is an injective map from each $\mathcal{U}_i$ to X by choosing for each $x \in X$ one element U of $\mathcal{U}_{i+1}$ so that $st(x, st(\mathcal{U}_i,\mathcal{U}_i)) \subseteq U$ and restricting the cover to this collection.

Now, we can define an ∞-metric d using the $\mathcal{U}_i$ as in Lemma 2.9.15.

We first claim that $\max\limits_{U\in\mathcal{U}_i} d(x, X\setminus U) \geq i$. Suppose not. Then for each $U \in \mathcal{U}_i$, there is some $y \in X\setminus U$ so that $d(x,y) \leq i-1$. Then we can find $V_{y_U} \in \mathcal{U}_{i-1}$ containing both x and y. Suppose that $x \in V_x \in \mathcal{U}_{i-1}$. Then $\{y_U\}_{U\in\mathcal{U}} \subseteq st(\{x\},\mathcal{U}_{i-1}) \subseteq st(U_x,\mathcal{U}_{i-1}) \in st(\mathcal{U}_{i-1},\mathcal{U}_{i-1})$. Hence, $\{y_U\}_{U\in\mathcal{U}} \subseteq V$ for some $V \in \mathcal{U}_i$. But then $y_V \in V \cap (X \setminus V)$, a contradiction.

Choose an integer M so that $\frac{2n+3}{M} \leq \frac{\epsilon}{2n+2}$. For each $U \in \mathcal{U}_M$, define $t_U : X \to \mathbb{R}$ by $t_U(x) = d(x, X\setminus U)$, if $d(x, X\setminus U) < \infty$ and $t_U(x) = M$, if $d(x, X\setminus U) = \infty$. Notice that $|t_U(x) - t_U(y)| \leq d(x,y)$ for each $x, y \in X$. Now define $\phi_U(x) = \frac{t_U(x)}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)}$. This gives a map $f : X \to \Delta^{(n)}(\mathcal{U}_M)$ by $f_x(U) := \phi_U(x)$. Notice that

$$\begin{aligned}
|\phi_U(x) - \phi_U(y)| &= \left| \frac{t_U(x)}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)} - \frac{t_U(y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(y)} \right| \\
&= \left| \frac{t_U(x) - t_U(y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)} + \frac{t_U(y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)} - \frac{t_U(y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(y)} \right| \\
&\leq \frac{d(x,y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)} + t_U(y) \left| \frac{1}{\sum\limits_{V\in\mathcal{U}_M} t_v(x)} - \frac{1}{\sum\limits_{V\in\mathcal{U}_M} t_V(y)} \right| \\
&= \frac{d(x,y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)} + \frac{t_U(y)}{\sum\limits_{V\in\mathcal{U}_M} t_V(y)} \frac{1}{\sum\limits_{V\in\mathcal{U}_M} t_V(x)} \sum_{V\in\mathcal{U}_M} |t_V(x) - t_V(y)| \\
&\leq \frac{d(x,y)}{M} + \frac{(2k+2)d(x,y)}{M}
\end{aligned}$$

$$= \frac{2k+3}{M} d(x,y) \leq \frac{\epsilon}{2k+2} d(x,y).$$

We now show that the map $f : X \to \Delta(\mathcal{U}_M)$ has $(\mathcal{U}, \epsilon)$-variation. Indeed, let $U \in \mathcal{U}$. If $x, y \in U$, then $d(x,y) \leq 1$. Hence,

$$\begin{aligned} d(f_x, f_y) &= \sum_{U \in \mathcal{U}_M} |f_x(U) - f_y(U)| \\ &= \sum_{U \in \mathcal{U}_M} |\phi_U(x) - \phi_U(y)| \\ &\leq (2k+2) \frac{\epsilon}{2k+2} d(x,y) \\ &= \epsilon d(x,y) \leq \epsilon. \end{aligned}$$

Now we show the spacing condition. Suppose no element of $\mathcal{U}_M$ contains both x and y. Then for each $U \in \mathcal{U}_M$, it is the case that $\phi_U(x) = 0$ or $\phi_U(y) = 0$. Hence, $d(f_x, f_y) = \sum_{U \in \mathcal{U}_M} |\phi_U(x) - \phi_U(y)| = \sum_{U \in \mathcal{U}_M} \phi_U(x) + \sum_{U \in \mathcal{U}_M} \phi_U(y) = 2$.

Finally, notice that since we chose $\mathcal{U}_M$ so that there is an injective map from $\mathcal{U}_M$ to X, then we get a distance-preserving map $i : \Delta^{(n)}(\mathcal{U}_M) \to \Delta^{(n)}(X)$. So we get the pinch-spacing map to $\Delta^{(n)}(X)$ by setting $g = i \circ f$.

3) $\Rightarrow$ 5): Let $1 \leq p < \infty$. Given a map $f : X \to \Delta_1^{(n)}(X)$, define $g : X \to \Delta_p^{(n)}(X)$ by $g_x(z) := (f_x(z))^{1/p}$. Notice that $\sum_{z \in X} |g_x(z)|^p = \sum_{z \in X} |f_x(z)| = 1$, so the map is well-defined. Also, for $p \geq 1$ and $a, b \geq 0$, it is the case that $|a - b|^p \leq |a^p - b^p|$, so we get that

$$\begin{aligned} ||g_x - g_y||_p^p &= \sum_{z \in X} |g_x(z) - g_y(z)|^p \\ &\leq \sum_{z \in X} |g_x(z)^p - g_y(z)^p| \\ &= \sum_{z \in X} |f_x(z) - f_y(z)| = ||f_x - f_y||_1. \end{aligned}$$

Thus, we can choose g to be a pinching map. Further, $||f_x - f_y||_1 = 2$ implies that $supp(f_x) \cap supp(f_y) = \emptyset$, which implies that $||g_x - g_y||_p = 2^{1/p}$. Thus, g will satisfy the spacing condition.

5) $\Rightarrow$ 4): Immediate

4) $\Rightarrow$ 3): Given a map $f : X \to \Delta_p^{(n)}(X)$, define $g : X \to \Delta_1^{(n)}(X)$ by $g_x(z) := f_x(z)^p$. Following the proof of Proposition 3.3 from [6], we get that $||g_x - g_y||_1 \leq K||f_x - f_y||_p$ for some constant K. Thus, we can built a pinch-spacing map $g : X \to \Delta_1(X)$ by choosing the correct parameters for a pinch-spacing map $f : X \to \Delta_p(X)$. □

5.6 Property A

Property A was introduced by Guoliang Yu in [33] as a sufficient condition for a metric space to coarsely embed in Hilbert space. It can by viewed as a nonequivariant version of amenability.

Definition 5.6.1. A discrete metric space X has **property A** if for all $R, \epsilon > 0$, there exists a family $\{A_x\}_{x \in X}$ of finite, non-empty subsets of $X \times \mathbb{N}$ such that

1) for all $x, y \in X$ with $d(x, y) \leq R$, we have $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \epsilon$;

2) there exists S such that for each $x \in X$, if $(y, n) \in A_x$, then $d(x, y) \leq S$.

Lemma 5.6.2. *If A_x and A_y are finite sets and $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \epsilon$, then $\frac{|A_y|}{|A_x|} < 1 + \epsilon$.*

Proof. Observe that

$$\begin{aligned}
\epsilon &> \frac{|A_x \triangle A_y|}{|A_x \cap A_y|} \\
&= \frac{|A_x| + |A_y| - 2|A_x \cap A_y|}{|A_x \cap A_y|} \\
&= \frac{|A_x|}{|A_x \cap A_y|} + \frac{|A_y|}{|A_x \cap A_y|} - 2 \geq \frac{|A_x|}{|A_y|} + \frac{|A_y|}{|A_y|} - 2 \\
&= \frac{|A_x|}{|A_y|} - 1.
\end{aligned}$$

Hence, $1 + \epsilon > \frac{|A_x|}{|A_y|}$. $\square$

Lemma 5.6.3. *Given a partition of unity $\phi : X \to \ell_1(S)$ and $\epsilon > 0$, there exists a partition of unity $\psi : X \to \Delta(S)$ such that $d(\phi(x) - \psi(x)) < \epsilon$ for all $x \in X$ and $supp(\psi_x) \subseteq supp(\phi_x)$ for each $x \in X$.*

Proof. For each $x \in X$, we can find $s_1, \ldots, s_n \in S$ such that $(1 - \sum_{i=1}^n \phi_x(s_i)) < \frac{\epsilon}{2}$. Define $\psi_x(s_1) = \phi_x(s_1) + 1 - \sum_{i=1}^n \phi_x(s_i)$, $\psi_x(s_i) = \phi_x(s_i)$ for $i = 2, \ldots, n$, and $\psi_x(s) = 0$ for $s \neq s_1, \ldots, s_n$. $\square$

Proposition 5.6.4. *If $f : X \to \Delta_1(X)$ has (R, ϵ)-variation, and $d(x, y) > S$ implies that $d(f_x, f_y) \geq 2$, then there exists $g : X \to \Delta_1(X)$ with (R, ϵ) variation such that $supp(g_x) \subseteq B(x, S)$ for all $x \in X$. In particular, $d(x, y) > 2S$ implies that $d(g_x, g_y) = 2$.*

Proof. Note that give a map $\phi : X \to X$, there is an induced map $\tilde{\phi} : \Delta_1(X) \to \Delta_1(X)$ given by $\tilde{\phi}(f)(x) = \sum_{z \in \phi^{-1}(\{x\})} f(z)$.

For each $z \in X$, consider $S_z = \{x \in X : z \in supp(f_x)\}$. Notice that $diam(S_z) \leq S$ since if $x, y \in S_z$, then $f_x(z), f_y(z) \neq 0$, which implies that $d(f_x, f_y) < 2$. Then by assumption, $d(x, y) \leq S$.

For each $z \in X$, choose $z' \in S_z$ and define $\phi(z) = z'$. We claim that $g : X \to \Delta_1(X)$ given by $g_x = \tilde{\phi}(f_x)$ has (R, ϵ)-variation and $supp(g_x) \subseteq B(x, S)$ for all $x \in X$.

Indeed, suppose that $d(x, y) < R$. Then

$$
\begin{aligned}
d(g_x, g_y) &= \sum_{z \in Z} |g_x(z) - g_y(z)| \\
&= \sum_{z \in Z} |\sum_{z' \in S_z} f_x(z') - \sum_{z'' \in Z} f_y(z'')| \\
&= \sum_{z \in Z} |\sum_{z' \in S_z} (f_x(z') - f_y(z'))| \\
&= \sum_{z' \in Z} |f_x(z') - f_y(z')| \\
&= d(f_x, f_y) < \epsilon.
\end{aligned}
$$

Now, suppose that $z \in supp(g_x)$. Then there is some $z' \in X$ such that $\phi(z') = z$ and $z \in supp(f_x)$. But by the way that ϕ is defined, this implies that $z' \in supp(f_z)$. Thus, $d(x, z) \leq S$, as desired. □

The above result shows that given an (R, ϵ)-pinching map to $\Delta_1(X)$, we may assume that the supports of the images are uniformly bounded.

Theorem 5.6.5. *Let X be a discrete metric space. If X has property A, then X 2-pinch-spaces to $\ell_{1,+}(X)$.*

Proof. Suppose that X has property A. Fix $R, \epsilon > 0$. Choose a collection $\{A_x\}_{x \in X}$ of subsets of $X \times \mathbb{N}$ and $S > 0$ from the definition of property A for R and $\frac{\epsilon}{2}$. For each $x \in X$, let $\pi_x : A_x \to X$ be the projection map restricted to A_x. Define $t_x : X \to \mathbb{N}$ by $t_x(z) = |\pi_x^{-1}(\{z\})|$. Now define $f : X \to \ell_{1,+}(X)$ by $f_x(z) = \frac{t_x(z)}{\sum_{z' \in X} t_x(z')} = \frac{t_x(z)}{\sum_{z' \in X} t_x(z')}$.

Let $x, y \in X$ with $d(x, y) < R$. Assume that $|A_y| \geq |A_x|$. Then

$$\begin{aligned}
d(f_x, f_y) &= \sum_{z \in Z} |f_x(z) - f_y(z)| \\
&\leq \sum_{z \in Z} (|f_x(z) - \frac{|A_y|}{|A_x|} f_y(z)| + |\frac{|A_y|}{|A_x|} f_y(z) - f_y(z)|) \\
&= \sum_{z \in X} (\frac{1}{|A_x|} ||A_x| f_x(z) - |A_y| f_y(z)| + |\frac{|A_y|}{|A_x|} - 1| f_y(z)) \\
&\leq \frac{|A_x \triangle A_y|}{|A_x|} + \frac{\epsilon}{2} \\
&\leq \frac{|A_x \triangle A_y|}{|A_x \cap A_y|} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Also, if $d(x, y) > 2S$, then $d(f_x, f_y) = \sum_{z \in X} |f_x(z) - f_y(z)| = \sum_{z \in X} f_x(z) + \sum_{z \in X} f_y(z) = 2$. □

Lemma 5.6.6. *Given any map $\phi : S \to X$, there is an induced map $\tilde{\phi} : \ell_1(S) \to \ell_1(X)$. Additionally, if f and g are positive functions, then $d(\tilde{\phi}(f), \tilde{\phi}(g)) \leq d(f, g)$.*

Proof. Define $\tilde{\phi}(f) \in \ell_1(X)$ by $\tilde{\phi}(f)(x) = \sum_{s \in \phi^{-1}(\{x\})} f(s)$.

For f and g taking only positive values,

$$\begin{aligned} d(\tilde{\phi}(f), \tilde{\phi}(g)) &= \sum_{x \in X} |\tilde{\phi}(f)(s) - \tilde{\phi}(g)(s)| \\ &= \sum_{x \in X} | \sum_{s \in \phi^{-1}(\{x\})} f(s) - g(s)| \\ &\leq \sum_{s \in S} |f(s) - g(s) = d(f, g). \end{aligned}$$

□

Theorem 5.6.7. *For a space large scale space X, the following are equivalent:*

1) X 2-pinch-spaces to $\ell_{1,+}(X)$;

2) X 2-pinch-spaces to $\Delta_1(X)$;

3) X $\sqrt{2}$-pinch-spaces to $\Delta_2(X)$;

4) X 2-pinch-spaces to some simplicial complex with ℓ_1 metric;

5) X $2^{1/p}$-pinch-spaces to $\Delta_p(X)$ for each $1 \leq p < \infty$;

6) X $2^{1/p}$-pinch-spaces to $\Delta_p(X)$ for some $1 \leq p < \infty$.

Proof. 1) ⇒ 2): Let $\mathcal{U}$ be a large scale of X,and fix $\epsilon > 0$. Choose a large scale $\mathcal{V}$ a pinch-spacing map $f : X \to \ell_{1,+}(X)$ for R and $\frac{\epsilon}{3}$. Using Lemma 5.6.3, find $g : X \to \Delta_1(X)$ so that $supp(g_x) \subseteq supp(f_x)$ and $d(f_x, g_x) < \frac{\epsilon}{3}$ for each $x \in X$. Observe that if $x, y \in U$ for some $U \in \mathcal{U}$, then $d(g_x, g_y) \leq d(g_x, f_x) + d(f_x, f_y) + d(f_y, g_y) < \epsilon$. Also, if x and y are contained in no element of $\mathcal{V}$, then $supp(f_x) \cap supp(f_y) = \emptyset$, so $supp(g_x) \cap supp(g_y) = \emptyset$. Thus, $d(g_x, g_y) = 2$.

2) ⇒ 1): Clear, since $\Delta_1(X) \subseteq \ell_{1,+}(X)$.

2) ⇒ 4): Clear.

2) ⇒ 3): Suppose that X 2-pinch-spaces to $\Delta_1(X)$ Let $\epsilon > 0$ and fix a large scale $\mathcal{U}$ of X. Choose a pinch-spacing map f to $\Delta_1(X)$ for $\mathcal{U}$ and ϵ^2. Define $g : X \to \Delta_2(X)$ by $g_x(U) = f_x(U)^{1/2}$. Notice that $\left(\sum_{z \in X} (g_x(z))^2\right)^{1/2} = \left(\sum_{z \in X} f_x(z)\right)^{1/2} = 1$. Also, if

x, y are contained in some element of U, then

$$
\begin{aligned}
d(g_x, g_y) &= \left(\sum_{z \in X} (g_x(z) - g_y(z))^2 \right)^{1/2} \\
&\le \left(\sum_{z \in X} (f_x(z) - f_y(z)) \right)^{1/2} \\
&< \sqrt{\epsilon^2} = \epsilon.
\end{aligned}
$$

3) $\Rightarrow$ 2): Suppose that X 2-pinch-spaces to $\Delta_2(X)$. Fix a large scale $\mathcal{U}$ of X and $\epsilon > 0$. Choose a pinch-spacing map f to $\Delta_2(X)$ for $\mathcal{U}$ and $\sqrt{\epsilon}$. Define $g : X \to \Delta_1(X)$ by $g_x(z) = (f_x(z))^2$. It is easy to see that this is a 2-pinch spacing map for $\mathcal{U}$ and ϵ.

4) $\Rightarrow$ 2): Fix a large scale $\mathcal{U}$ and $\epsilon > 0$. Choose a large scale $\mathcal{V}$ and a 2-pinch-spacing map $f : X \to \Delta_1(S)$ for some vertex set S. For each $s \in S$, let $T_x = \{x \in X : s \in supp(f_x)\}$. Define a map $\phi : S \to X$ by choosing, for each $s \in S$, some $\phi(s) = x \in T_s$ if $T_s \neq \emptyset$ and letting $\phi(s)$ be any element of X if $T_s = \emptyset$.

We claim that the map $g : X \to \Delta_1(X)$ given by $g_x = \tilde{\phi}(f_x)$ is a 2-pinch-spacing map with respect to $\mathcal{V}$. By Lemma 5.6.6, if x and y are contained in some element of $\mathcal{U}$, then $d(g_x, g_y) = d(\tilde{\phi}(f), \tilde{\phi}(g)) \le d(f_x, f_y) < \epsilon$. Now suppose that no element of $st(\mathcal{V}, \mathcal{V})$ contains both x and y. We need to show that $supp(g_x) \cap supp(g_y) = \emptyset$. Suppose $z \in supp(g_x) \cap supp(g_y)$. Then there are $s_x, s_y \in S$ so that $x, z \in T_{s_x}$ and $y, z \in T_{s_y}$. Thus, there are V_x, V_y in $\mathcal{V}$ so that $x, z \in V_x$ and $y, z \in V_y$. Hence, $x, y \in st(\{z\}, \mathcal{V})$, which is contained in some element of $st(\mathcal{V}, \mathcal{V})$, a contradiction. □

There is a partial converse for Theorem 5.6.5.

Lemma 5.6.8. *Given $\epsilon > 0$ and $f : X \to \Delta_1(X)$ so that $|supp(f_x)| < M$ for all $x \in X$, there is $g : X \to \Delta_1(X)$ so that 1) $g_x(z)$ is rational for each $x \in X$;*

2) $g_x(z)$ can be written with denominator $K < \frac{M(M-1)}{2\epsilon}$;

3) $d(f_x, g_x) < \epsilon$ for each $x \in X$;

4) $supp(g_x) \subseteq supp(f_x)$.

Proof. Fix $\epsilon > 0$. Choose an integer $K < \frac{M(M-1)}{2\epsilon}$. Let $x \in X$. Say $supp(f_x) = \{z_1, \ldots, z_n\}$. For $i = 1, \ldots, n-1$, choose a nonnegative rational number $q_i \in [f_x(z_i) - \frac{\epsilon}{M(M-1)}, f_x(z_i)]$ such that q_i has denominator K. For $i = 1, \ldots, n-1$, define $g_x(z_i) = q_i$, define $g_x(z_n) = 1 - \sum_{i=1}^{n-1} q_i$, and define $g_x(z) = 0$ for $z \notin supp(f_x)$. Clearly conditions 1), 2), and 4) hold. For condition 3), first note that for $i = 1, \ldots, n-1$, we have $|g_x(z_i) - f_x(z_i)| < \frac{\epsilon}{M(M-1)}$, and that

$$\begin{aligned}|g_x(z_n) - f_x(z_n)| &= g_x(z_n) - f_x(z_n) \\ &= (1 - \sum_{i=1}^{n-1} q_i) - (1 - \sum_{i=1}^{n-1} f_x(z_i)) \\ &= \sum_{i=1}^{n-1} (f_x(z_i) - q_i) \\ &\leq \sum_{i=1}^{n-1} \frac{\epsilon}{M(M-1)} = \frac{\epsilon}{M}.\end{aligned}$$

Hence,

$$\begin{aligned}d(f_x, g_x) &= \sum_{z \in Z} |f_x(z) - g_x(z)| \\ &= \sum_{i=1}^{n} |f_x(z_i) - g_x(z_i)| \\ &\leq \sum_{i=1}^{n} \frac{\epsilon}{M} = \epsilon.\end{aligned}$$

□

Theorem 5.6.9. *Suppose that X 2-pinch-spaces to $\Delta_1(X)$, and for a given large scale $\mathcal{U}$ and $\epsilon > 0$, there are $S, M < \infty$ so that for the associated map $f : X \to \Delta_1(X)$, it is the case that $supp(f_x) \subseteq B(x, S)$ and $|supp(f_x)| < M$ for every $x \in X$. Then X has property A.*

Proof. Fix $R, \epsilon > 0$. Choose $\epsilon' > 0$ such that $\epsilon' < \frac{2\epsilon}{2-\epsilon}$. This implies that $\frac{\epsilon'}{1-\frac{\epsilon'}{2}} < \epsilon$.

Choose a 2-pinch-spacing map $g : X \to \Delta_1(X)$ for R and $\frac{\epsilon}{3}$. Using Lemma 5.6.8, we can get a 2-pinch-spacing map $f : X \to \Delta_1(X)$ so that $d(f_x, g_x) < \frac{\epsilon'}{3}$ for all x and $f_x(z) = \frac{n}{K}$ for some integer n and fixed integer K. Then for $x, y \in X$, if $d(x, y) < R$, then $d(f_x, f_y) \leq d(f_x, g_x) + d(g_x, g_y) + d(g_y, g_x) < \epsilon'$.

For each $x \in X$, define $A_x = \{(z, n) : 0 < f_x(z) \leq \frac{n}{K}\}$. Then clearly $A_x \subseteq B(x, S)$ for all $x \in X$. Notice that $|A_x| = K$ for all $x \in X$.

Suppose that $d(x, y) < R$. Then

$$|A_x \triangle A_y| = \sum_{z \in Z} K|f_x(z) - f_y(z)| = Kd(f_x, f_y) < K\epsilon'.$$

Hence, $K\epsilon' > |A_x \triangle A_y| = |A_x| + |A_y| - 2|A_x \cap A_y| = 2K - 2|A_x \cap A_y|$. Thus, $|A_x \cap A_y| > K(1 - \frac{\epsilon'}{2})$.

Thus,

$$\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \frac{K\epsilon'}{K(1 - \frac{\epsilon'}{2})} = \frac{\epsilon'}{1 - \frac{\epsilon'}{2}} < \epsilon.$$

□

Corollary 5.6.10. *If X is a bounded geometry metric space, then X has property A if and only if X 2-pinch-spaces to $\Delta_1(X)$.*

Proof. The forward direction follows from Theorem 5.6.5 and Theorem 5.6.7.

For the reverse direction, we may use Proposition 5.6.4 and the bounded geometry of X to satisfy the hypothesis of Theorem 5.6.9. □

Proposition 5.6.11. *If X has finite asymptotic dimension, then X 2-pinch-spaces to $\Delta_1(X)$, and for a given large scale $\mathcal{U}$ and $\epsilon > 0$, there are $S, M < \infty$ so that for the associated map $f : X \to \Delta_1(X)$, it is the case that supp$(f_x) \subseteq B(x, S)$ and $|$supp$(f_x)| < M$ for every $x \in X$.*

Notice that the pinch-spacing maps from the proof of Theorem 5.5.2 satisfy the conditions of the hypothesis.

Corollary 5.6.12. *If X has finite asymptotic dimension, then X has property A.*

Following Sako [28], we define property A for more general large scale space. First, we need the analogue for bounded geometry spaces.

Definition 5.6.13. A large scale space is **uniformly locally finite** if for every large scale $\mathcal{U}$ of X, it is the case that $\sup_{U\in\mathcal{U}} |U| < \infty$.

Definition 5.6.14. A uniformly locally finite large scale space X is said to have **property A** if for every $\epsilon > 0$ and every large scale $\mathcal{U}$ of X there exists a large scale $\mathcal{V}$ of X and a family of subsets $\{A_x \subseteq X \times \mathbb{N} : x \in X\}$ such that

1) $\frac{|A_x \triangle A_y|}{A_x \cap A_y|} < \epsilon$ if there is some $U \in \mathcal{U}$ containing both x and y;

2) $\{\pi_X(A_x) \mid x \in X\} \prec \mathcal{V}$, where $\pi_X : X \times \mathbb{N}$ is the projection map.

Using the same proof as in the metric case, we obtain the following result.

Theorem 5.6.15. *If X is a uniformly locally finite large scale space, then X has property A if and only if X 2-pinch-spaces to $\Delta(X)$.*

The above can be used to define property A for general large scale spaces.

Definition 5.6.16. A large scale space has **property A** if it 2-pinch-spaces to $\Delta(X)$.

In [18], the following definition is given:

Definition 5.6.17. A metric space X has **property A** if for all $R, \epsilon > 0$, there is $M > 0$ and a partition of unity $\{\phi_s\}_{s\in S}$ so that

1) $d(x, y) \leq R$ implies that $\sum_{x\in S} |\phi_s(x) - \phi_s(y)| < \epsilon$,

2) the diameter of the support of each ϕ_s is at most M.

Notice that in the case of metric large scale space, this definition coincides with our definition.

5.7 Embedding in Hilbert Space

Recall that a map $f : X \to Y$ between metric spaces is a **coarse embedding** if there are non-decreasing functions $\rho_-, \rho_+ : [0, \infty), [0, \infty)$ so that $\rho_-(d(x, y)) \leq$

$d(f(x), f(y)) \leq \rho_+(d(x,y))$ for all $x, y \in X$. Coarse embeddings into Hilbert space have been extensively studied since Guoliang Yu showed that a space which coarsely embeds in Hilbert space satisfies the coarse Baum-Connes conjecture [33].

Theorem 5.7.1. *Let X be a metric space. If X coarsely embeds in Hilbert space, then X c-pinch-spaces to the unit sphere of some Hilbert space for every* $0 < c < \sqrt{2}$.

Proof. We mimic the proof of Theorem 5.2.8 from [24]. For a Hilbert space H, consider the Fock space

$$Exp(H) := \mathbb{R} \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus \cdots .$$

There is a map $exp : H \to Exp(H)$ given by

$$exp(v) = 1 \oplus v \oplus (\frac{1}{\sqrt{2!}} v \otimes v) \oplus (\frac{1}{\sqrt{3!}} v \otimes v \otimes v) \oplus \cdots .$$

This map has the property that $\langle exp(v), exp(w) \rangle = e^{\langle v,w \rangle}$ for all $v, w \in H$.

Say $f : X \to H$ is a coarse embedding of X into a Hilbert space with associated functions ρ_- and ρ_+.

Fix $R, \epsilon > 0$. For $t > 0$, define $g : X \to Exp(H)$ by $g(x) = e^{-t||f(x)||^2} exp(\sqrt{2t} f(x))$. Then $\langle g_x, g_y \rangle = e^{-t||f(x)-f(y)||^2}$, so that $e^{-t\rho_+(d(x,y))^2} \leq \langle g_x, g_y \rangle \leq e^{-t\rho_-(d(x,y))^2}$.

Notice that if $d(x, y) < R$, then

$$\begin{aligned} d(g(x), g(y)) &= \sqrt{2||g(x)||^2 + 2||g(y)||^2 - 2\langle g(x), g(y) \rangle} \\ &\leq \sqrt{2 - 2e^{-t\rho_+(R)^2}}, \end{aligned}$$

so if we choose t sufficiently small so that $\sqrt{2 - 2e^{-t\rho_+(R)^2}} < \epsilon$ it follows that g satisfies the pinching condition for R and ϵ.

For the spacing condition, notice that $d(g(x), g(y)) \geq \sqrt{2 - 2e^{-t\rho_-(d(x,y))^2}}$, so to satisfy the spacing condition, choose $S > 0$ so that $\sqrt{2 - 2e^{-t\rho_-(S)^2}} > c$, which is possible since $\rho_-(x) \to \infty$ as $x \to \infty$. ∎

There is a converse to the previous result, but instead of pinch-spacing for all $0 < c < \sqrt{2}$, it suffices that X c-pinch-spaces for some $c > 0$.

Theorem 5.7.2. *If X is a metric space and there is a Hilbert space H so that for some $0 < c$, it is the case that X c-pinch-spaces to the unit sphere of H, then X coarsely embeds in a Hilbert space.*

Proof. For $i = 1, 2, \ldots$, choose c-pinch-spacing maps to the unit sphere of H, $f_i : X \to H$ and $S_i > 0$ for $R_i = i$ and $\epsilon_i = \frac{1}{2^i}$. We may further assume that the S_i form an increasing sequence.

Fix a point $z \in X$. Define $f : X \to \bigoplus_{i \geq 1} H$ by $f(x) = (f_i(x) - f_i(z))_{i \geq 1}$. Say $\lceil d(x, z) \rceil = N$. Then

$$\sum_{i \geq 1} ||f_i(x) - f_i(z)||^2 < \sum_{i=1}^{N} 2 + \sum_{i=N+1}^{\infty} \left(\frac{1}{2^i}\right)^2 < \infty,$$

so f is well-defined.

Notice that if $\lceil d(x, y) \rceil = N$, then $d(f(x), f(y))^2 \leq \sum_{i=1}^{N} 2 + \sum_{i=N+1}^{\infty} \left(\frac{1}{2^N}\right)^2$. So we may define $\rho_+(N) = \sqrt{\sum_{i=1}^{N} 2 + \sum_{i=N+1}^{\infty} \left(\frac{1}{2^N}\right)^2}$ for each integer N and then extending to all of $[0, \infty)$ by $\rho_+(k) := \rho_+(\lceil k \rceil)$.

Notice that if $d(x, y) > S_i$, then $d(f(x), f(y))^2 \geq \sum_{i=1}^{\lfloor S_i \rfloor - 1} 1 = \lfloor S_i \rfloor - 1$. Hence, we can define ρ_- by setting $\rho_-(S_i) = \sqrt{\lfloor S_i \rfloor - 1}$ for each S_i and extending to all of $[0, \infty)$ as before. Since the S_i form an increasing sequence, we get that $\rho_-(t) \to \infty$ as $t \to \infty$. □

Corollary 5.7.3. *If a metric space X has property A, then X coarsely embeds in Hilbert space.*

Lemma 5.7.4. *Let X be a large scale space and B a Banach space. If X c_0-pinch-spaces to B for some $c_0 > 0$, then X c-pinch-spaces to B for all $c > 0$.*

Proof. Fix $c > 0$. Notice that if f is a $(\mathcal{U}, \frac{c_0}{c} \cdot \epsilon)$ pinch-spacing map with constant S, then $g := \frac{c}{c_0} \cdot f$ is a $(\mathcal{U}, \epsilon)$ pinch-spacing map with the same constant. □

Theorem 5.7.5. *If X is c_0-pinch-spaces to Hilbert space for some $c_0 > 0$, then X d-pinch-spaces to the unit sphere of Hilbert space for all $0 < d < \sqrt{2}$.*

Proof. Say X c_0-pinch-spaces to the Hilbert H.By Lemma 5.7.4, X c-pinch-spaces to H for all $c > 0$. Let $0 < d < \sqrt{2}$. Fix a large scale $\mathcal{U}$ of X and $\epsilon > 0$. By Theorem 5.7.1, H d-pinch-spaces to the unit sphere of a Hilbert space G. Let g be an (ϵ, ϵ)-pinching map with d-spacing constant $c > 0$. Now choose an (R, ϵ)-pinching, c-spacing map f with constant $S > 0$. Notice that $g \circ f : X \to G$ is (R, ϵ)-pinching and d-spacing with constant S, proving the X d-pinch-spaces to the unit sphere of G. □

Theorem 5.7.6. *If a metric space X c-pinch-spaces to a Hilbert space H for some $c > 0$, then X coarsely embeds in Hilbert space.*

Proof. By the previous theorem, X d-pinch-spaces to the unit sphere of Hilbert space for any $0 < d < \sqrt{2}$. So by Theorem 5.7.2, X coarsely embeds in Hilbert space. □

Corollary 5.7.7. *If X is a metric space, then the following are equivalent:*

1) X coarsely embeds in Hilbert space;

2) X c-pinch-spaces to Hilbert space for all $c > 0$;

3) X c-pinch-spaces to Hilbert space for some $c > 0$.

Corollary 5.7.8. *There exist metric spaces which do not c-pinch-space to Hilbert space for any $c > 0$.*

This follows since there are examples of metric spaces which do not coarsely embed in Hilbert space. For example, a sequence of expander graphs does not coarsely embed. This shows that pinch-spacing to Hilbert space is a strong condition, unlike pinch-spacing to Banach space since its true that every metric, and in fact every large scale space, pinch-spaces to Banach space.

We will now consider pinch-spacing to ℓ_p for other values of p. To accomplish this, we will make use of the Mazur map. Given a set A and $1 \leq p, q < \infty$, define the **Mazur map** $M_{p,q} : \ell_p(A) \to \ell_q(A)$ by

$$M_{p,q}(f)(a) := |f(a)|^{p/q} sign(f(x))$$

for all $f \in \ell_p(A)$. Notice that the Mazur map preserves norm and $M_{p,q} = M_{q,p}^{-1}$.

Proposition 5.7.9. *(Proposition 5.4.6 of [24]) Let $1 \le p < q < \infty$. Then there exists a constant $C > 0$, which depends only on p and q, such that*

$$\frac{p}{q}||f-g||_p \le ||M_{p,q}(f) - M_{p,q}(g)||_q \le C||f-g||_p^{p/q}$$

for every $f, g \in \ell_p(A)$ satisfying $||f||_p = ||g||_p = 1$.

Notice that if $q < p$, then by using $M_{q,p}$ we also get that

$$C||f-g||_p^{q/p} \le ||M_{p,q}(f) - M_{p,q}(g)||_q \le \frac{q}{p}||f-g||_p$$

for a constant C depending only on p and q.

By mimicking the construction for ℓ_2, we get the following result.

Theorem 5.7.10. *If a metric space X c-pinch-spaces to the unit sphere of $\ell_p(A)$ for some $c > 0$ and some $1 \le p < \infty$, then X coarsely embeds in ℓ_p.*

Corollary 5.7.11. *If a metric space X c-pinch-spaces to the unit sphere of $\ell_p(A)$ for some $c > 0$ and some $1 \le p < \infty$, then X coarsely embeds in ℓ_q for all $1 \le q < \infty$. In particular, X coarsely embeds in Hilbert space.*

Proof. Using Proposition 5.7.9, d-pinch-spacing maps from X to the unit sphere of $\ell_q(X)$ by composing c-pinch-spacing maps to the unit sphere of $\ell_p(X)$ with the Mazur map $M_{p,q}$. Here, the constant d depends only on p and q, not on the choice of pinch spacing map to $\ell_p(X)$. To get the coarse embedding, we simply apply Theorem 5.7.10. □

Corollary 5.7.12. *For any $1 \le p < \infty$, it is the case that ℓ_2 coarsely embeds in ℓ_p.*

Proof. By Theorem 5.7.1, ℓ_2 pinch-spaces to the unit sphere of Hilbert space for any $0 < c < \sqrt{2}$. By composing the associated pinch-spacing maps with the Mazur map, we get a d-pinch-spacing maps to the unit sphere of ℓ_p for some d depending only on

p. That is, ℓ_2 d-pinch-spaces to the unit sphere of ℓ_p. Thus, by Theorem 5.7.10, ℓ_2 coarsely embeds in ℓ_p. □

Theorem 5.7.13. *(Proposition 4.1 of [23])* ℓ_p *coarsely embeds in* ℓ_2 *for all* $1 \leq p \leq 2$.

Theorem 5.7.14. *The following are equivalent for a metric space* X*:*

1) X *c-pinch-spaces to* ℓ_p *for some* $1 \leq p \leq 2$ *and some* $c > 0$*;*

2) X *c-pinch-spaces to* ℓ_p *for all* $1 \leq p \leq 2$ *and some* $c > 0$*;*

3) X *c-pinch-spaces to* ℓ_p *for some* $1 \leq p \leq 2$ *and all* $c > 0$*;*

4) X *c-pinch-spaces to* ℓ_p *for all* $1 \leq p \leq 2$ *and all* $c > 0$*;*

5) X *d-pinch-spaces to the unit sphere in* ℓ_p *for some* $d > 0$ *and some* $1 \leq p < \infty$;

6) X *d-pinch-spaces to the unit sphere in* ℓ_p *for some* $d > 0$ *and all* $1 \leq p < \infty$;

7) X *coarsely embeds in* ℓ_p *for some* $1 \leq p \leq 2$*;*

8) X *coarsely embeds in* ℓ_p *for all* $1 \leq p \leq 2$.

Moreover, any of the above conditions implies that X *coarsely embeds in* ℓ_p *for all* $1 \leq p < \infty$.

Proof. 1) ⇒ 8): By Theorem 5.7.13, ℓ_p coarsely embeds in ℓ_2. Composing a pinch-spacing map with this coarse embedding gives a pinch-spacing map to ℓ_2. That is X pinch-spaces to ℓ_2, so by Corollary 5.7.7, it follows that X coarsely embeds in ℓ_2. But by Corollary 5.7.12, ℓ_2 coarsely embeds in ℓ_q for all $1 \leq q < \infty$, so it follows that X also coarsely embeds in ℓ_q for all $1 \leq q < \infty$.

7) ⇒ 8): This follows since ℓ_p coarsely embeds in ℓ_2 for $1 \leq p \leq 2$ and ℓ_2 coarsely embeds in ℓ_q for all $1 \leq q < \infty$.

8) ⇒ 6): From Theorem 5.7.1, ℓ_2 c-pinch-spaces to the unit sphere of ℓ_2 for some $0 < c < \sqrt{2}$. Composing the pinch-spacing map with the Mazur map gives a d-pinch-spacing map to the unit sphere of ℓ_p.

5) ⇒ 6): Follows from using the Mazur map.

6) ⇒ 4): Follows from Lemma 5.7.4.

4) ⇒ 7): Follows since pinch-spacing to ℓ_2 imples coarse embeddability into ℓ_2.

3) ⇒ 1): Clear.

2) $\Rightarrow$ 1): Clear.

8) $\Rightarrow$ 5): Follows from Theorem 5.7.1.

8) $\Rightarrow$ 3): Clear.

8) $\Rightarrow$ 2): Clear. □

We remark that while it is true by the previous Theorem that pinch-spacing to ℓ_p is equivalent to pinch-spacing to the unit sphere ℓ_p for $1 \leq p \leq 2$, the same does not hold for $2 < p < \infty$. Indeed, pinch-spacing to the unit sphere of any ℓ_p implies coarse embeddability into Hilbert space. Notice that if $p > 2$, the ℓ_p pinch-spaces to itself but does not coarsely embed in Hilbert space by a result of Johnson [14]. Another observation is that for $1 \leq p \leq 2$ pinch-spacing to ℓ_p is equivalent to coarsely embedding into ℓ_p. One can ask is the same is true for $p > 2$.

We end this section with a remark about non-metrizable large scale spaces. Since pinch-spacing is done one scale at a time, it is possible for a non-metrizable large scale space to pinch-space to Hilbert space. For example, if the space has a metric approximation where all approximating spaces pinch-space to Hilbert space. However, a non-metrizable large scale space can never coarsely embed in Hilbert space, since coarsely embedding in a metric space implies that the embedded space is metric.

5.8 Coarse Embeddability Versus Property A

We know if a bounded geometry space X $\sqrt{2}$-pinch-spaces to $\Delta_2(X)$, then X has property A. It is also true that if X c-pinch-spaces to the unit sphere in Hilbert space for some $c > 0$, then X coarsely embeds in Hilbert space. If the following conjecture is true, then we get a way to use pinch-spacing to the unit sphere of Hilbert space as a sufficient condition for property A.

Conjecture 5.8.1. *If X $\sqrt{2}$-pinch-spaces to the unit sphere of $\ell_2(I)$, then there is some index set J so that X $\sqrt{2}$-pinch-spaces to $\Delta_2(J)$.*

We offer the following finite-dimensional example as evidence that the conjecture may be true.

Example 5.8.2. *There is a* π*-Lipschitz map* $f : S^1 \to S^8$ *so* $f(S^1)$ *is contained in the first orthant and if* $d(x,y) \geq \sqrt{2}$*, then* $d(f(x), f(y)) = \sqrt{2}$*. This map is given by*

$$f((\cos\theta, \sin\theta)) = \begin{cases} (\cos 2\theta, \sin 2\theta, 0,0,0,0,0,0) & \text{if } 0 \leq \theta \leq \frac{\theta}{4} \\ (0, \cos(2(\theta - \frac{\pi}{4})), \sin(2(\theta - \frac{\pi}{4})), 0,0,0,0,0) & \text{if } \frac{\pi}{4} \leq \theta \leq \frac{2\pi}{4} \\ (0,0, \cos(2(\theta - \frac{2\pi}{4})), \sin(2(\theta - \frac{2\pi}{4})), 0,0,0,0) & \text{if } \frac{2\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\ (0,0,0, \cos(2(\theta - \frac{3\pi}{4})), \sin(2(\theta - \frac{3\pi}{4})), 0,0,0) & \text{if } \frac{3\pi}{4} \leq \theta \leq \frac{4\pi}{4} \\ (0,0,0,0, \cos(2(\theta - \frac{4\pi}{4})), \sin(2(\theta - \frac{4\pi}{4})), 0,0) & \text{if } \frac{4\pi}{4} \leq \theta \leq \frac{5\pi}{4} \\ (0,0,0,0,0, \cos(2(\theta - \frac{5\pi}{4})), \sin(2(\theta - \frac{5\pi}{4})), 0) & \text{if } \frac{5\pi}{4} \leq \theta \leq \frac{6\pi}{4} \\ (0,0,0,0,0,0, \cos(2(\theta - \frac{6\pi}{4})), \sin(2(\theta - \frac{6\pi}{4}))) & \text{if } \frac{6\pi}{4} \leq \theta \leq \frac{7\pi}{4} \\ (\sin(2(\theta - \frac{7\pi}{4})), 0,0,0,0,0,0, \cos(2(\theta - \frac{7\pi}{4}))) & \text{if } \frac{7\pi}{4} \leq \theta \leq \frac{8\pi}{4} \end{cases}$$

Thus, is $g : X \to S^1$ is an (R, ϵ)-pinching, $(S, \sqrt{2})$-spacing map, then $f \circ g : X \to S^8$ is an $(R, \pi \cdot \epsilon)$-pinching, $(S, \sqrt{2})$-spacing map to the first orthant of S^8.

If the above conjecture is true, then we get a positive answer to the following conjecture.

Conjecture 5.8.3. *If* X $\sqrt{2}$*-pinch-spaces to the unit sphere in Hilbert space, then* X *has property A.*

5.9 Connection to Exact Spaces

We extend the definition of an exact metric space [5] to the setting of large scale spaces.

Definition 5.9.1. A large scale space X is **exact** if for every large scale $\mathcal{U}$ of X and $\epsilon > 0$ there is a partition of unity $\{\phi_i\}_{i\in I}$ on X and a large scale $\mathcal{V}$ of X so that

1) the function $\phi : X \to \ell_{1,+}(I)$ defined by $\phi(x)(i) := \phi_i(x)$ has $(\mathcal{U}, \epsilon)$-variation;

2) $\{\phi_i\}_{i\in I}$ is subordinate to $\mathcal{V}$.

This can be interpreted as a statement about pinch-spacing.

Theorem 5.9.2. *For a large scale space X, the following are equivalent:*

1) X is exact;

2) X 2-pinch-spaces to $\ell_{1,+}(I)$ for some set I;

3) X 2-pinch-spaces to $\ell_{1,+}(X)$;

4) X 2-pinch-spaces to $\Delta_1(X)$.

Proof. Clearly, 3) $\Rightarrow$ 2).

2) $\Rightarrow$ 1): Let $\mathcal{U}$ be a large scale of X and $\epsilon > 0$. Choose a pinch-spacing map $\phi : X \to \ell_{1,+}(I)$ with corresponding large scale $\mathcal{V}$. For each $i \in I$, choose some $x_i \in supp(\phi_i)$. Notice that $supp(\phi_i) \subseteq st(\{x_i\}, \mathcal{V})$. Thus, $\{\phi_i\}_{i\in I}$ is subordinate to the large scale $st(\{\{x\} : x \in X\}, \mathcal{V})$.

1) $\Rightarrow$ 3): Let $\mathcal{U}$ be a large scale of X and $\epsilon > 0$. Choose a partition of unity $\phi : X \to \ell_{1,+}(I)$ having $(\mathcal{U}, \epsilon)$-variation. Define a map $f : I \to X$ as follows. For each $i \in I$, choose some $x_i \in supp(\phi_i)$. This induces a map $\tilde{f} : \ell_{1,+}(I) \to \ell_{1,+}(X)$ which decreases distances. So a pinching map ϕ gives rise to a pinching map $\tilde{f} \circ \phi$. We claim that if $\{\phi_i\}_{i\in I}$ is subordinate to $\mathcal{V}$, then $\tilde{f} \circ \phi$ is spacing with respect to $st(\{x : x \in X\}, \mathcal{V})$. Indeed, suppose that $d(\tilde{f}(\phi_x), \tilde{f}(\phi_y)) < 2$. Then there is some $z \in X$ such that $\tilde{f}(\phi_x(z)), \tilde{f}(\phi_y(z)) > 0$. Then there are $i_x, i_y \in I$ such that $x, z \in supp(\phi_{i_x})$ and $y, z \in supp(\phi_{i_y})$ Thus, $x, y \in st(\{z\}, \mathcal{V})$.

Clearly 4) $\Rightarrow$ 3).

3) $\Rightarrow$ 4) follows from Lemma 5.6.3. □

Corollary 5.9.3. *If X has finite asymptotic dimension or X has property A, then X is exact.*

Corollary 5.9.4. *If X is uniformly locally finite and X is exact, then X has property A.*

Corollary 5.9.5. *If X is metric and X is exact, then X coarsely embeds in Hilbert space.*

5.10 Large Scale Paracompactness

Recall the definition of large scale paracompactness [18]:

Definition 5.10.1. A metric space X is **large scale paracompact** if there is a simplicial complex K so that for each uniformly bounded cover $\mathcal{U}$ of X and for all $\lambda, C > 0$, there is a (λ, C)-Lipschitz function $f : X \to K$ such that $\mathcal{V} := \{f^{-1}(st(v))\}_{v \in K^{(0)}}$ is uniformly bounded and $\mathcal{U}$ refines $\mathcal{V}$.

Proposition 5.10.2. *If X is large scale paracompact, then X is exact.*

Proof. We'll show that X 2-pinch-spaces to K and the result will follow from Theorem 5.9.2. Let $R, \epsilon > 0$. Let $\mathcal{U}$ be the cover of X consisting of all sets of diameter no larger than R. Choose an $(\frac{\epsilon}{2R}, \frac{\epsilon}{2})$-Lipschitz map $f : X \to K$ so that $\mathcal{V} = \{f^{-1}(st(v))\}_{v \in K^{(0)}}$ is uniformly bounded and $\mathcal{U} \prec \mathcal{V}$. Notice that if $d(x, y) < R$, then $d(f(x), f(y)) \leq \frac{\epsilon}{2R} \cdot R + \frac{\epsilon}{2} = \epsilon$. Put $S = mesh(\mathcal{V})$. Then if $d(x, y) > S$, it follows that for each vertex $v \in K^{(0)}$, it is not the case that both $f(x)(v) > 0$ and $f(y)(v) > 0$. Thus, $d(f(x), f(y)) = 2$. That is f is a pinch-spacing map. □

Corollary 5.10.3. *If X is large scale paracompact, then X coarsely embeds in Hilbert space.*

Proposition 5.10.4. *If X has bounded geometry and is exact, then X is large scale paracompact.*

Proof. Let $\mathcal{U}$ be a uniformly bounded cover of X, and fix $\lambda, C > 0$. Choose $\epsilon < \min\{2, C\}$ and $R > mesh(\mathcal{U})$ so that $\frac{2-\epsilon}{R} \leq \lambda$. Using the exactness of X and

Proposition 5.6.4, we can choose an $(R, \frac{\epsilon}{2})$-pinch spacing map $f : X \to \Delta_1(X)$ and $S > R$ so that $supp(f(x)) \subseteq B(x, S)$ for each $x \in X$. We need to slightly modify f so that for each $U \in \mathcal{U}$,there is some $y \in X$ so that $f(x)(y) \neq 0$ for all $x \in U$. To do this, first choose for each $U \in \mathcal{U}$ some $x_U \in U$. Then for each $x \in X$, add a total weight of $\frac{\epsilon}{2}$ to the values of $f(x)$ by evenly distributing the extra weight to all x_U such that $x \in U$. Notice that we are using the bounded geometry assumption here since it implies that there are only finitely many U containing any given x. Finally, renormalize by dividing by $1 + \frac{\epsilon}{2}$. Notice that the resulting map has (R, ϵ)-variation and still has the property that $supp(f(x)) \subseteq B(x, S)$ for each $x \in X$. By Proposition 2.4 of [18], this map is (λ, C)-Lipschitz and by construction, each $U \in \mathcal{U}$ is contained in $f^{-1}(st(v))$ for some $v \in X$. □

Corollary 5.10.5. *If X has bounded geometry, then the following are equivalent:*

1) X is exact;

2) X is large scale paracompact;

3) X has property A.

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Appendix

Appendix A

Uniform and Coarse Spaces Versus Large and Small Scale Spaces

Small scale structures and coarse structures on a space X are sometimes defined using subsets of the product $X \times X$ rather than using covers of X. In this approach, elements of the scale structure are often called **entourages** rather than scales. This appendix serves two purposes. First, we will show that the entourages approach to scale structures equivalent to our use of covers. Second, we will show that several of the constructions from the main body are equivalent to some known coarse structures.

The author remarks that when defining scale structures using entourages, we will use the terms uniform structure and coarse structure in place of small scale structure and coarse structure.

For a set X, the **diagonal** of X is defined to be $\Delta = \{(x, x) : x \in X\}$. For a subset $U \subseteq X \times X$, the **inverse** of U is defined to be $U^{-1} = \{(y, x) : (x, y) \in U\}$. For two sets $U, V \subseteq X \times X$, the **composition** (or the **product**) of U and V is defined as $U \circ V = \{(x, z) \mid (x, y) \in U \text{ and } (y, z) \in V \text{ for some } y \in X\}$. For a set $E \subseteq X \times X$ and $x \in X$, let $E[x] = \{y \in X : (y, x) \in E\}$.

Definition A.0.6. A **uniform structure** on a set X is a collection $\mathcal{U}$ of subsets of $X \times X$ satisfying

1) $\Delta \subseteq U$ for all $U \in \mathcal{U}$;

2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

3) for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;

4) if $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$;

5) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

Definition A.0.7. If X and Y are uniform spaces, then a function $f : X \to Y$ is **uniformly continuous** if $f^{-1}(E)$ is an entourage of Y for every entourage E of X.

Definition A.0.8. [27] A **coarse structure** on a set X is a collection $\mathcal{U}$ of subsets of $X \times X$ satisfying

1) $\Delta \in \mathcal{U}$;

2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

3) if $U, V \in \mathcal{U}$, then $U \circ V \in \mathcal{U}$;

4) if $U \in \mathcal{U}$ and $V \subseteq U$, then $V \in \mathcal{U}$;

5) if $U, V \in \mathcal{U}$, then $U \cup V \in \mathcal{U}$.

Definition A.0.9. If X and Y are coarse spaces, then a function $f : X \to Y$ is **bornologous** if $f(E)$ is an entourage of Y for every entourage E of X.

In terms of entourages, we define a small scale entourage base as a collection $\mathcal{B}$ of symmetric entourages such that if $E, F \in \mathcal{B}$, then there exists $G \in \mathcal{B}$ such that $G \circ G \subseteq E \cap F$. To make a small scale entourage base into a uniform structure, add all supersets of elements of $\mathcal{B}$. Similarly, a large scale entrourage base is a collection $\mathcal{B}$ of symmetric entourages such that if $E, F \in \mathcal{B}$, then there exists $G \in \mathcal{B}$ such that $E \circ F \subseteq G$. To make a large scale entourage base into a coarse structure, add all subsets of elements of $\mathcal{B}$.

Fix a set X. Let $\mathcal{C}(X)$ denote the set of coarse structures on X and $\mathcal{L}(X)$ denote the set of small scale structures on X. Our goal is to show that $\mathcal{C}(X)$ is equivalent to $\mathcal{L}(X)$ in the following sense. The sets $\mathcal{C}(X)$ can be made into a poset by defining $U \leq V$ if $id_X : X_U \to X_V$ is bornologous, where X_U denotes X with the uniform structure U. Similary, $\mathcal{L}(X)$ is a poset by declaring $U \leq V$ if $id_X : X_U \to X_V$ is large scale continuous. We will show that there are bijective, order-preserving maps $\ell : \mathcal{C}(X) \to \mathcal{L}(X)$ and $c : \mathcal{L}(X) \to \mathcal{C}(X)$. We comment that the analogous result holds for uniform structures and small scale structures on a set X, and is proved by dualizing the arguments.

First, we define $\ell : \mathcal{C}(X) \to \mathcal{L}(X)$. Let C be a coarse structure on X. Define $\ell(C)$ by taking all scales $\mathcal{U}$ on X for which there is some entourage $E \in C$ such that $\bigcup\limits_{U \in \mathcal{U}} (U \times U) \subseteq E$.

Now we define $c : \mathcal{L}(X) \to \mathcal{C}(X)$. Given a large scale structure L, define $c(L)$ by taking all subsets $E \subseteq X \times X$ for which there is some scale $\mathcal{U} \in L$ such that $E \subseteq \bigcup\limits_{U \in \mathcal{U}} (U \times U)$.

Proposition A.0.10. *The functions ℓ and c are bijective and order-preserving.*

Proof. We claim that $\ell \circ c = id_{\mathcal{L}(X)}$, the identity map on $\mathcal{L}(X)$ and $c \circ \ell = id_{\mathcal{C}(X)}$, the identity map on $\mathcal{C}(X)$.

First, let $L \in \mathcal{L}(X)$. Let $\mathcal{B}$ be a large scale in $\mathcal{L}$. Put $E = \bigcup\limits_{B \in \mathcal{B}} B \times B$. Then $E \in c(L)$. Also, $\bigcup\limits_{B \in \mathcal{B}} \subseteq E$, so $\mathcal{B} \in (\ell \circ c)(L)$. Thus, $L \subseteq (\ell \circ c)(L)$. Now let $\mathcal{B} \in (\ell \circ c)(L)$. Then there is some $E \in c(L)$ such that $\bigcup\limits_{B \in \mathcal{B}} \subseteq E$. Since $E \in c(L)$, there is some $\mathcal{D} \in L$ such that $E \subseteq \bigcup\limits_{D \in \mathcal{D}} D \times D$. Hence $\bigcup\limits_{B \in \mathcal{B}} B \times B \subseteq \bigcup_{D \in \mathcal{D}} D \times D$. It follows that $\mathcal{B} \prec \mathcal{D}$, and so $\mathcal{B} \in L$. Thus $L = (\ell \circ c)(L)$. Therefore, $\ell \circ c = id_{\mathcal{L}(X)}$.

For $E \subseteq X \times X$ and $p \in E$, define $E[x] = \{y \in X : (y, x) \in E\}$. Now let $C \in \mathcal{C}(X)$ and let $E \in C$. Put $F = E \cup \Delta$. Put $\mathcal{B} = \{F[a] : a \in X\}$. Let $(x, y) \in \bigcup\limits_{B \in \mathcal{B}} (B \times B)$. Then there is some $a \in X$ such that $(x, y) \in F[a] \times F[a]$. Thus, $(x, a) \in F$ and $(a, y) \in F^{-1}$, implying that $(x, y) \in F \circ F^{-1}$. Thus, $\bigcup\limits_{B \in \mathcal{B}} B \times B \subseteq F \circ F^{-1} \in C$. That is, $\mathcal{B} \in \ell(C)$. Now let $(x, y) \in E$. Then $x, y \in F[y]$, so $(x, y) \in F[y] \times F[y]$. Thus, $E \subseteq \bigcup\limits_{B \in \mathcal{B}} B \times B$. Thus, $E \in (c \circ \ell)(C)$. Hence $\mathcal{C} \subseteq (c \circ \ell)(C)$. Now let $E' \in (c \circ \ell)(C)$. So there is a $\mathcal{D} \in \ell(C)$ such that $E' \subseteq \bigcup\limits_{D \in \mathcal{D}} D \times D$. But $\mathcal{D} \in \ell(C)$, so $\bigcup\limits_{D \in \mathcal{D}} D \times D \subseteq F$ for some $F \in \mathcal{C}$. Then $E' \subseteq F$, implying that $E' \in \mathcal{C}$ since coarse structures are closed under subsets. Thus $(c \circ \ell)(C) \subseteq C$. Therefore, $c \circ \ell = id_{\mathcal{C}(X)}$. Hence, we have shown that there is a 1-1 correspondence between $\mathcal{C}(X)$ and $\mathcal{L}(X)$.

Now, we show that this correspondence is order-preserving. Suppose that $C_1, C_2 \in \mathcal{C}(X)$ with $C_1 \leq C_2$. Let $\mathcal{B} \in \ell(C_1)$. Then there is some entourage $E \in C_1$ so that $\bigcup_{B \in \mathcal{B}} (B \times B) \subseteq E$. We assumed that $C_1 \leq C_2$, so $E \in C_2$, implying that $\mathcal{B} \in \ell(C_2)$. Hence, $\ell(C_1) \leq \ell(C_2)$. That is, ℓ preserves order. By a similar argument, c preserves order. □

A.1 Proper Large Scale Spaces

Now we recall the original definition of proper coarse structure as given by Roe. This definition is given in terms of the entourage approach to coarse structures. We will show that Roe's definition in terms of entourages is equivalent to our definition of properness in terms of covers.

Definition A.1.1. [27] A coarse structure on a Hausdorff space X is said to be **proper** if there exists a controlled neighborhood of the diagonal and every bounded subset of X is pre-compact.

Notice that both definitions of properness imply that the underlying topological space X is locally compact. First we will show that these definitions are equivalent; that is, a proper coarse structure induces a proper large scale structure. Also, in Proposition 3.1.3 we show that a space with a proper structure must also be paracompact.

Proposition A.1.2. *A proper large scale structure induces a proper coarse structure. Conversely, a proper coarse structure induces a proper large scale structure.*

Proof. ($\Rightarrow$) This direction is clear as one can take the open uniformly bounded cover $\mathcal{U}$ of X and notice that $\bigcup_{U \in \mathcal{U}} U \times U$ is a controlled neighborhood of the diagonal.

($\Leftarrow$) Take a controlled neighborhood $E \subseteq X \times X$ of the diagonal. Notice that for every $x \in X$ there exists some open pre-compact U_x containing x such that $U_x \times U_x \subseteq E$. Take $\mathcal{U} = \{U_x : x \in X\}$ as an open uniformly bounded cover. □

Proposition A.1.3. *The C_0 large scale structure is the large scale structure induced by the C_0 coarse structure.*

Proof. First, let $\mathcal{B}$ be a uniformly bounded family in the induced large scale structure. By the definition of the induced large scale structure, there is a C_0 controlled set E such that $\bigcup_{B \in \mathcal{B}}(B \times B) \subseteq E$. First note that we may choose a compact $K \subseteq X$ such that $(x, y) \in E \setminus (K \times K)$ implies that $d(x, y) < 1$. Then it follows that $mesh(\mathcal{B} < \max\{diam(K), 1\}$. Now fix $\epsilon > 0$. Choose a compact $K \subseteq X$ such that $(x, y) \in E \setminus (K \times K)$ implies that $d(x, y) < \epsilon$. Let $x, y \in B \setminus K$ for some $B \in \mathcal{B}$. Then $(x, y) \in (B \setminus K) \times (B \setminus K) \subseteq (B \times B) \setminus (K \times K) \subseteq E \setminus (K \times K)$. Hence $d(x, y) < \epsilon$. That is, $\mathcal{B}$ is uniformly bounded in the C_0 large scale structure.

Now let $\mathcal{B}$ be uniformly bounded in the C_0 coarse structure. Put $E = \bigcup_{B \in \mathcal{B}}(B \times B)$. We will show that E is controlled in the C_0 coarse structure, which will imply that $\mathcal{B}$ is uniformly bounded in the induced large scale structure. Let $M > 0$ such that $mesh(\mathcal{B}) < M$. Fix $\epsilon > 0$ and choose K compact such that $mesh(\{B \setminus K\}_{B \in \mathcal{B}}) < \epsilon$. Let J be the closure of the M-neighborhood of K, which is compact since X is proper. If $(x, y) \in E \setminus J$, then neither x nor y is in K. Thus $d(x, y) < \epsilon$. Hence, E is controlled in the C_0 coarse structure. Therefore, the C_0 large scale structure is equal to the large scale structure induced by the C_0 coarse structure. □

Vita

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