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To the Graduate Council:

I am submitting herewith a dissertation written by Christopher Ryan Loga entitled "Extension Theorems on Matrix Weighted Sobolev Spaces." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Michael W. Frazier, Major Professor

We have read this dissertation and recommend its acceptance:

Stefan Richter, Joan R. Lind, Tuoc Phan, Michael W. Guidry

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Extension Theorems on Matrix Weighted Sobolev Spaces

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Christopher Ryan Loga

August 2016

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To Amanda, my wife, the only person who knows just how much this means to me.

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Abstract

Let D a subset of $\mathbb{R}^n$ [R n] be a domain with Lipschitz boundary and $1 \leq p < \infty$ [1 less than or equal to p less than infinity]. Suppose for each x in $\mathbb{R}^n$ that $W(x)$ is an $m \times m$ [m by m] positive definite matrix which satisfies the matrix A_p [A p] condition. For $k = 0, 1, 2, 3, ...$ define the matrix weighted, vector valued, Sobolev space $L^p_k(D, W)$ [L p k of D,W] with

$$\left|\left|\overrightarrow{f}\right|\right|^p_{L^p_k(D,W)} = \sum_{|\alpha| \leq k} \int_D \left|\left|W^{1/p}(x)\left(D^\alpha \overrightarrow{f}(x)\right)\right|\right|^p \mathrm{d}x$$

[the weighted L p k norm of vector valued f over D to the p power equals the sum over all alpha with order less than k of the integral over D of the the pth power of the norm of W to the 1 over p times the alpha order derivative of f] where $\overrightarrow{f} = (f_1, \ldots, f_m) : D \to \mathbb{C}^m$ [vector f takes D into C m]. We then aim to show that for $\overrightarrow{f}$ [vector f] in $L^p_k(D, W)$ there exists an extension $E\left(\overrightarrow{f}\right)$ [E of vector f] in $L^p_k(\mathbb{R}^n, W)$ [L p k of R n, W] such that $E\left(\overrightarrow{f}\right) = \overrightarrow{f}$ [E of vector f equals vector f] on D and

$$\left|\left|E\left(\overrightarrow{f}\right)\right|\right|_{L^p_k(\mathbb{R}^n,W)} \leq C \left|\left|\overrightarrow{f}\right|\right|_{L^p_k(D,W)}$$

[the weighted L p k norm of E of vector f on all of R n is less than or equal to a constant times the weighted L p k norm of f on D] for some constant independent of $\overrightarrow{f}$. This theorem generalizes a known result for scalar A_p weights. To prove such a result, we first consider various cases including that of unweighted smooth and Lipschitz domains. We then proceed to go through some standard results for

scalar A_p weights. The scalar A_p weighted smooth and Lipschitz domain case is then addressed. With such intuition in hand, certain facts about matrix weights must be addressed before we can finally prove both smooth and Lipschitz domain results in this new context.

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Chapter 1

Introduction

When considering partial differential equations (PDE), it is essential to measure the size and smoothness of functions by defining norms and considering the associated spaces of functions with finite norm. One such question is that of extending functions in one of these function spaces from the domain where they are defined to the whole space. Such extension theorems are powerful because one can apply certain tools, such as the Fourier transform, defined on $\mathbb{R}^n$, to the extended functions, and then restrict back to the domain to obtain corresponding information there. It is not immediately clear, however, which functions can be extended. The major complication lies in the function space under consideration.

A primary example of extension theorems occurs in the setting of Sobolev spaces on domains. Denoted $L^p_k(D)$, this space consists of all functions with derivatives of order up to k on a domain D such that each derivative is in L^p. In such a setting extension results are known for a variety of domains D.

1.1 History

Extension theorems for Sobolev spaces on smooth domains have been known for a long time. The method predominantly used to prove such theorems is to first simplify the domain using a change of coordinates, and use density results to restrict attention to very smooth functions. Then one defines some sort of integral operator which extends the function to all of $\mathbb{R}^n$, checking that this operator is bounded in the Sobolev norm.

More complicated domains for this extension problem have been considered in the somewhat recent past. For example, consider a domain whose boundary is not necessarily smooth, say a Lipschitz domain D. A. P. Calderón showed in 1961 that an extension operator can be defined on $L_k^p(D)$ with singular integrals for $1 < p < \infty$ (see [1] for more). E. M. Stein in 1970 constructed a modified extension using integral operators (see [18], pp. 180-192). Stein's approach had the added benefit that it included the endpoints $p = 1, \infty$ and his extension operator is independent of the order of differentiability k.

A further result is that of Peter Jones in 1980 (see [11]). He found a more general class of domains, called $(\epsilon - \delta)$ domains, for which an extension operator will exist. Such a class is of particular interest in that it is the sharp class of domains for the unweighted extension problem in $\mathbb{R}^2$. What makes his approach more intriguing is that it does not use an integral operator. Rather, he defined the extension using a summation of polynomials localized on cubes.

Beyond this, S. K. Chua in his paper of 1992 extended Jones's result to scalar A_p weighted Sobolev space (see [3] and also [4]). The A_p condition for a weight w is a condition where the weight in question and its reciprocal satisfies a certain averaging condition. Such a condition is key in that it allows for the use of important estimates when bounding integral norms. Due to such a condition, Chua was able to extend functions in the weighted Sobolev space $L_k^p(D, w)$ by following Jones' approach.

1.2 Further Motivation

Extension theorems have many applications in PDE. For instance, in Evans [7] Ch.5, p. 268 a $k = 1$ version of the smooth boundary extension theorem for L_k^p is proved. Later, on p. 279 this result is used to prove the Gagliardo-Niremberg-Sobolev inequality which shows that L_k^p embeds into L^q space. The result is first proven on all of $\mathbb{R}^n$ and then the extension is used to get the estimate for a domain. This style of argument is used again to show that certain Sobolev spaces embed into Hölder continuous space (p.283) and to prove compactness of the embedding of certain Sobolev spaces into L^q spaces (p.286).

Weighted integrals have become a standard technique in recent years. For instance, consider the work of Fabes, Kenig, and Serapioni in [8]. Here they obtain local regularity estimates and Poincaré inequalities for an elliptic differential operator which may not be uniformly elliptic, but whose ellipticity constants are controlled by an A_p weight. In current work of Joshua Isralowitz and Kabe Moen these ideas are being extended to establish matrix weighted Poincaré/Sobolev inequalities with applications to degenerate elliptic systems.

Weights have also appeared in numerical analysis work such as that of Nochetto, Otarola, and Salgado. In [14] they develop a constructive approach to piecewise polynomial interpolation in Muckenhoupt weighted Sobolev Spaces. They achieve this result by using the fact that $y^\alpha \in A_p$ for a certain range of α.

1.3 Overview of the Results

It is the purpose of this thesis to generalize Chua's result in [3] to the matrix weighted case when D is a Lipschitz domain.

Theorem 1.1. *Let D be a Lipschitz domain, $1 \le p < \infty$, $k = 0, 1, 2, \ldots,$ and $W \in A_p$. Given $\overrightarrow{f} \in L^p_k(D, W)$ there exists an extension $E\left(\overrightarrow{f}\right) \in L^p_k(\mathbb{R}^n, W)$, that is, there exists $E\left(\overrightarrow{f}\right) : \mathbb{R}^n \to \mathbb{R}$ such that $E\left(\overrightarrow{f}\right)\Big|_D = \overrightarrow{f}$ and*

$$\left|\left|E\left(\overrightarrow{f}\right)\right|\right|_{L^p_k(\mathbb{R}^n, W)} \le C_{p,k,D,W} \left|\left|\overrightarrow{f}\right|\right|_{L^p_k(D,W)}.$$

To understand the problem we will consider several different cases. In this way, we will gain a much greater sense of the obstacles needed in proving such a result.

Starting with several preliminaries in Chapters 2 and 3, we are able to briefly build the needed tools to prove the extension theorem. In Chapter 4 the unweighted smooth domain case is proven in detail. Though this result is well-known, a detailed proof still serves to give a better foundation for the subject and the further results that follow. Other unweighted results are expounded upon at the close of the chapter.

In Chapter 5, a brief overview is given for scalar A_p weights. By a scalar weight we mean a non-negative function $w : \mathbb{R}^n \to [0, \infty]$ that is locally integrable on $\mathbb{R}^n$. Such a function is called an A_p weight if it satisfies an A_p condition for corresponding $1 < p < \infty$. Such a condition comes about in several different contexts. In some situations, by limiting, we can then further discuss the $p = 1$ case. These results are now quite standard. For more, one can read [19], Chapter 5.

Using this knowledge, in Chapter 6 we consider the A_p weighted smooth domain case. However, there is a difference when defining the extension to accommodate A_p weights. Namely, we are required to average over cubes as opposed to the lines used in the unweighted situation. In the one dimensional setting, it becomes clear as to how this averaging approach makes use of the A_p condition. When moving to higher dimensions, a result for taking derivatives of vector valued functions is also required. With these preliminaries in hand we are at last able to both understand and prove the multidimensional scalar weighted result for smooth domains. It is also interesting to note that a section of the proof has 3 different methods of proof.

Throughout Chapter 7, much of the previous case is applied as we move toward the more general Lipschitz domain. The critical extra element here is how to define the extension so as to accommodate the domain. In light of what was done previously, we are forced to modify Stein's approach in [18] so as to gain a method involving estimates on Whitney cubes. Such a dyadic decomposition is at the heart of the proof and so such estimates are crucial. With such results in hand, the proof follows similarly to before.

In Chapter 8, matrix weights are introduced as well as the corresponding A_p condition. The motivation for the matrix A_p condition was to find the right conditions on a weight W so that the Hilbert transform preserves $L^p(W)$. The original formulation of the matrix A_p condition is complicated (see for instance [13], [12], and [20]), but in [16] Svetlana Roudenko showed that the condition is equivalent to the following: Let W be a matrix weight on $\mathbb{R}^n$, $1 < p < \infty$, and $1/p+1/p' = 1$. Then $W \in A_p$ if and only if W and $W^{-p'/p}$ are locally integrable and for some constant $C > 0$

$$\left(\int_Q \left(\int_Q ||W^{1/p}(x)W^{-1/p}(t)||^{p'} \frac{\mathrm{d}t}{|Q|}\right)^{p/p'} \frac{\mathrm{d}x}{|Q|}\right)^{1/p} \leq A_{p,n}$$

for all cubes $Q \subset \mathbb{R}^n$. There is also a similar condition for A_1 weights.

Chapter 9 is dedicated to the matrix weighted smooth domain case. The property of A_p weights that turns out to be key in doing this is the following variable switching property: For a cube Q and a vector valued function $\overrightarrow{f}$

$$\frac{1}{|Q|}\int_Q \left(\frac{1}{|Q|}\int_Q \left|\left|W^{1/p}(x)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\right)^p \mathrm{d}x \leq \frac{[A_p(W)]^p}{|Q|}\int_Q \left|\left|W^{1/p}(t)\overrightarrow{f}(t)\right|\right|^p \mathrm{d}t.$$

In this way, the A_p condition has allowed us to line up the variables, going from $W^{1/p}(x)\overrightarrow{f}(t)$ on the left to $W^{1/p}(t)\overrightarrow{f}(t)$ on the right. This variable switching procedure is required to make certain crucial estimates work in the proofs corresponding to a matrix weighted case.

In Chapter 10, we finally give the proof of Theorem 1.1. Most of it is a consequence of Chapters 7 and 8.

As a concluding thought, several open questions as to how the extension problem can be further investigated are considered in Chapter 11. These include questions regarding domains, weights, and more general function spaces.

Chapter 2

Definitions and Notation

Throughout we write $f \in L^p(E, \mu)$ where $1 \leq p < \infty$ to mean

$$||f||^p_{L^p(E,\mu)} = \int_E |f(x)|^p \mathrm{d}\mu(x) < \infty$$

where (E, μ) is a measure space. Furthermore, when f is continuous, we say $f \in L^\infty(E)$ given that

$$||f||_{L^\infty(E)} = \sup_{x \in E} |f(x)| < \infty.$$

When f is not continuous, we use the convention that $||f||_{L^\infty} = \operatorname{ess\,sup} |f|$.

2.1 Sobolev Space

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ a multiindex. We say $f : \Omega \to \mathbb{R}$ has weak derivative, $D^\alpha f$, if there exists $F \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega F g \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f D^\alpha g \, \mathrm{d}x \tag{2.1}$$

for all $g \in C_c^\infty(\Omega)$. We then write $D^\alpha f = F$. All derivatives considered will be in the weak sense while noting that classically differentiable implies weakly differentiable.

Definition 2.1. *For $1 \leq p \leq \infty$, $k = 0, 1, 2, \ldots$, and μ a measure define the Sobolev space*

$$L_k^p(\Omega, \mu) = \{f : \Omega \to \mathbb{R} \mid D^\alpha f \in L^p(\Omega, \mu) \ \ \forall |\alpha| \leq k\}$$

which is complete with respect to the norm

$$||f||_{L_k^p(\Omega,\mu)} = \sum_{|\alpha| \leq k} ||D^\alpha f||_{L^p(\Omega,\mu)}.$$

Namely, a Sobolev space is a set of functions who share a certain measure of smoothness and finite size.

2.2 Domains

For convenience we will often denote $\mathbb{R}^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y \in \mathbb{R}\}$.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded, open set and denote $\partial\Omega$ as its boundary. We say $\partial\Omega$ is C^k if for each point $x_0 \in \partial\Omega$ there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^n \to \mathbb{R}$ such that $\Omega \cap B_r(x_0) = U \cap B_r(x_0)$ where each $U \subset \mathbb{R}^{n+1}$ is a half space given by $\{x = (x', x_{n+1}) : x_{n+1} > \gamma(x')\}$ or one of its rotations about x_0.*

When considering such a definition, it is common to further "flatten the boundary" by applying a smooth change of coordinates to each half space U with the result that $\gamma \equiv 0$. Once this flattening has been applied, it then suffices to consider each associated half space as one of the following:

$$\mathbb{R}_-^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y < 0\}$$
$$\mathbb{R}_+^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$$

It is standard practice to refer to a domain based on its boundary and boundedness. That is, a smooth domain is any domain with smooth boundary. In more modern times, less smooth domains have become of greater interest.

Definition 2.3. *Let $\Gamma : \mathbb{R}^n \to \mathbb{R}$ be a function which satisfies the Lipschitz condition*

$$|\Gamma(x) - \Gamma(x')| \leq M|x - x'| \tag{2.2}$$

for some $M > 0$ and for all $x, x' \in \mathbb{R}^n$. The special Lipschitz domain that Γ determines is given by

$$D = \{(x, y) \in \mathbb{R}^{n+1} : y > \Gamma(x)\}$$

The smallest M for which (2.2) is satisfied is called the Lipschitz bound for D.

It is apparent from this that any half space is a special Lipschitz domain. In a similar manner to before we could also define a bounded Lipschitz domain as any domain whose boundary is locally that of a special Lipschitz domain. However, the following more general definition allows us to consider unbounded domains.

Definition 2.4. *Let $D \subset \mathbb{R}^{n+1}$ be an open set and denote ∂D as its boundary. We say D is a Lipschitz domain if there exists $\epsilon > 0$, an integer N, $M > 0$, and $U_1, U_2, \ldots, U_n, \ldots$ open in $\mathbb{R}^{n+1}$ such that*

(i) Denoting $U_i^\epsilon = \{(x, y) \in U_i : B((x, y), \epsilon) \subset U_i\}$, we have that $\partial D \subset \bigcup_{i=1}^\infty U_i^\epsilon$

(ii) No point in $\mathbb{R}^{n+1}$ is contained in more than N of the U_i's

(iii) For each $i = 1, 2, \ldots$ there exists a special Lipschitz domain D_i whose bound does not exceed M such that

$$U_i \cap D = U_i \cap D_i$$

Any domain with smooth boundary fits the above definition where the special Lipschitz domains involved are just half spaces with flattened boundary. When the domain is also bounded, it is possible to pick a finite number of U_i's. Also, when working through computations it is useful to keep in mind that ϵ, N, $M > 0$, and the U_i's given in Definition 2.4 are all dependent on the associated domain D.

As far as standardized notation goes, Ω will always denote a smooth domain while D will always denote a Lipschitz domain.

Chapter 3

Preliminary Results

Let us now examine several key results which we will use throughout when proving our various extension results. Here, we use the notation $C_c(A)$ to mean the continuous functions with compact support contained in the set A.

3.1 Existence of Derivatives Across a Boundary

The following theorem allows us to more easily check the differentiability of an extension when the domain is smooth. Namely, it suffices to show that the derivatives must satisfy a trace property.

Lemma 3.1. *Let $k = 0, 1, 2, \ldots$. If $f \in C^k\left(\overline{\mathbb{R}^{n+1}_+}\right)$ and $g \in C^k\left(\mathbb{R}^{n+1}_-\right)$ with the property that given $x_0 \in \mathbb{R}^n$ it follows that $D^\alpha g(x,y) \to D^\alpha f(x,0)$ as $y \to 0$ for all $|\alpha| \le k$, then*

$$E(f) = \begin{cases} f(x,y) & y \ge 0 \\ g(x,y) & y < 0 \end{cases}$$

has weak derivatives up to order k.

Proof. Since $E(f)$ is clearly differentiable in both $C^k\left(\mathbb{R}^{n+1}_+\right)$ and $C^k\left(\mathbb{R}^{n+1}_-\right)$ it suffices to show integration by parts on the boundary, which is $x = 0$. First, consider differentiation with respect to y.

Let $\psi \in C_c^\infty(\mathbb{R}^{n+1})$. Choose a ball $B_r^{n+1} \subset \mathbb{R}^{n+1}$ of radius $r > 0$ and centered at $(0,0)$ such that the support of ψ is contained in B_r^{n+1}. Then, we have that

$$\begin{aligned}\iint_{B_r^{n+1}\cap\mathbb{R}_+^{n+1}} f\psi_y \,\mathrm{d}y\mathrm{d}x &= \int_{B_r^n}\int_0^{\sqrt{r^2-|x|^2}} f\psi_y \,\mathrm{d}y\mathrm{d}x \\ &= \int_{B_r^n} (0 - f(x,0)\psi(x,0))\,\mathrm{d}x - \int_{B_r^n}\int_0^{\sqrt{r^2-|x|^2}} f_y\psi \,\mathrm{d}y\mathrm{d}x \\ &= -\int_{B_r^n} f(x,0)\psi(x,0)\,\mathrm{d}x - \iint_{B_r^{n+1}\cap\mathbb{R}_+^{n+1}} f_y\psi\,\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Using the trace property it similarly follows that

$$\iint_{B_r^{n+1}\cap\mathbb{R}_-^{n+1}} g\psi_y \,\mathrm{d}y\mathrm{d}x = \int_{B_r^n} f(x,0)\psi(x,0)\,\mathrm{d}x - \iint_{B_r^{n+1}\cap\mathbb{R}_-^{n+1}} g_y\psi\,\mathrm{d}y\mathrm{d}x.$$

Adding these two equations shows that (2.1) is satisfied and thus yields the differentiability of the extension in the y component. Also, the argument for the x derivatives is even simpler due to the boundary terms always being zero. In this way all first order derivatives of $E(f)$ exist.
Furthermore, assuming we have a derivative of order $|\alpha|$ we can now repeat this argument in any of the y coordinates involved to gain the conclusion for all the derivatives of order $|\alpha|$. Thus the full conclusion follows by induction on the order of differentiability k.

□

It is interesting to note that the above proof does not work for Lipschitz domains. Due to the smoothness of the domain above, we did not have to worry about differentiability in x. However, a problem arises in the Lipschitz domain case due to the jagged nature of the boundary which causes problems when attempting to verify integration by parts in the x-variables. Namely, the boundary may "wobble" too much.

There is, however, no issue in the Lipschitz domain case as long as we have an unlimited amount of differentiability for the function defined on the domain. This is made explicit when we need it later on with Lemma 4.1.

3.2 Density of $C^{\infty}\left(\overline{D}\right)$ in $L^p_k\left(D\right)$

Definition 3.1. *Let $U \subset \mathbb{R}^n$ be open. A Radon measure μ on U is a measure defined on the Borel sets of U that also has the following properties:*

(a) For every compact $K \subset U$ we have that $\mu(K) < \infty$.

(b) μ is a regular measure, that is, it has both inner and outer regularity.

Such a measure μ is important in that integration with respect to μ gives a fundamental density result: C_c is dense in L^p.

Theorem 3.1 ([17],p. 69, Theorem 3.14)**.** *Let μ be a Radon measure for $\mathbb{R}^n$ and suppose $1 \leq p < \infty$. If $f \in L^p(\mathbb{R}^n, \mu)$, then for every $\epsilon > 0$ there exists $g \in C_c\left(\mathbb{R}^n\right)$ such that $||f - g||_{L^p(\mathbb{R}^n,\mu)} < \epsilon$.*

To show this classical result, one first considers a set of simple functions. Once such a collection S is shown to be dense in L^p, the result is proved with L^p replaced by S. This classical density allows for a stronger density result for L^p_k. To show this we must first prove the following well known theorem which will act as a lemma for our purposes:

Lemma 3.2 (Continuity of Translations in L^p-norm)**.** *If $f \in L^p\left(\mathbb{R}^n, \mu\right)$ for $1 \leq p < \infty$ where μ is a Radon measure on $\mathbb{R}^n$, then, as $t \to 0$,*

$$\left(\int_{\mathbb{R}^n} |f(x+t) - f(x)|^p \mathrm{d}\mu(x)\right)^{1/p} \to 0$$

Proof. First suppose $f \in C_c\left(\mathbb{R}^n\right)$ with $\mathrm{supp} f \subset \overline{B(0,r)}$ for some $r > 0$. Fix $\epsilon > 0$. Since f is continuous it follows that f is uniformly continuous on $\overline{B(0,r)}$. So there exists $\delta > 0$ such that when $|t| < \delta$,

$$|f(x+t) - f(x)| < \frac{\epsilon}{\mu(B(0,r+1))^{1/p}}$$

for all $x \in \overline{B(0,r+1)}$. We may choose $0 < \delta \leq 1$. Then, if $|t| < \delta$, $|f(x+t) - f(x)|$ is supported on $\overline{B(0,r+1)}$. Hence

$$\left(\int_{\mathbb{R}^n} |f(x+t) - f(x)|^p \mathrm{d}\mu(x)\right)^{1/p} < \epsilon$$

Now, let $f \in L^p(\mathbb{R}^n, \mu)$ and again fix $\epsilon > 0$. By Theorem 3.1 there exists $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_{L^p(\mathbb{R}^n,\mu)} < \epsilon/3$. Because $g \in C_c(\mathbb{R}^n)$, the previous case tells us that there exists $\delta > 0$ such that when $|t| < \delta$,

$$||g(x+t) - g(x)||_{L^p(\mathbb{R}^n,\mu)} < \frac{\epsilon}{3}$$

Thus

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f(x+t) - f(x)|^p \mathrm{d}\mu(x)\right)^{1/p} &\leq ||f(x+t) - g(x+t)||_{L^p(\mathbb{R}^n,\mu)} \\ &\quad + ||g(x+t) - g(x)||_{L^p(\mathbb{R}^n,\mu)} \\ &\quad + ||g(x) - f(x)||_{L^p(\mathbb{R}^n,\mu)} \\ &< \epsilon\end{aligned}$$

□

From this we now gain the density result that is the subject of the section.

Lemma 3.3 (Density of C^∞ in L^k_p)**.** *Suppose $f \in L^1_{loc}(\mathbb{R}^{n+1})$ and that $D \subset \mathbb{R}^{n+1}$ is a special Lipschitz domain with curve Γ and bound M. Let $f \in L^p_k(D, \mu)$ where $1 \leq p < \infty$, $k = 0, 1, 2, \ldots$, and μ is a σ-finite Radon measure. Then there exist*

$\{f_\epsilon\}_{\epsilon>0}$ where each $f_\epsilon \in C^\infty\left(\overline{D}\right)$ such that

$$||f_\epsilon - f||_{L^p_k(D,\mu)} \to 0$$

as $\epsilon \to 0$.

Proof. Fix $\epsilon > 0$. Consider $\phi \in C_c^\infty\left(\mathbb{R}^{n+1}\right)$ such that $\operatorname{supp}\phi \subset \Gamma_-$, that is, the downward opening cone centered at the origin (see (2.2)):

$$\Gamma_- = \{(x', y) \in \mathbb{R}^{n+1} : M|x'| < |y|,\ y < 0\}.$$

We also choose ϕ such that $\int_{\mathbb{R}^{n+1}} \phi(x)\mathrm{d}x = 1$. Denote $\phi_\epsilon = \epsilon^{-(n+1)}\phi(x/\epsilon)$ for any $\epsilon > 0$. For $x = (x', y) \in \overline{D}$, define

$$f_\epsilon(x) = \phi_\epsilon * f_0(x) = \int_{\mathbb{R}^{n+1}} \phi_\epsilon(x - s) f_0(s)\mathrm{d}s$$

with the trivial extension $f_0(s) = \begin{cases} f(s) & s \in D \\ 0 & s \notin D \end{cases}$. Thus clearly $f_\epsilon \in C^\infty\left(\overline{D}\right)$. Furthermore, using the Minkowski Integral Inequality we have that

$$\begin{aligned} ||D^\alpha f_\epsilon - D^\alpha f||_{L^p(D,\mu)} &= \left(\int_D \left|\int_{\mathbb{R}^{n+1}} \phi_\epsilon(s)[D^\alpha f_0(x-s) - D^\alpha f(x)]\mathrm{d}s\right|^p \mathrm{d}\mu(x)\right)^{1/p} \\ &= \left(\int_D \left|\int_{\mathbb{R}^{n+1}} \phi(u)[D^\alpha f_0(x-\epsilon u) - D^\alpha f(x)]\mathrm{d}u\right|^p \mathrm{d}\mu(x)\right)^{1/p} \\ &\leq \int_{\mathbb{R}^{n+1}} |\phi(u)| \left(\int_D |D^\alpha f_0(x-\epsilon u) - D^\alpha f(x)|^p \mathrm{d}\mu(x)\right)^{1/p} \mathrm{d}u \end{aligned}$$

where the last expression makes sense because $D^\alpha f_0(x - \epsilon u)$ exists for $x \in D$ and $u \in \operatorname{supp}\phi$ since we are shifting an element in the domain by an element in an upward facing cone. More precisely, $(x', y) \in D$ and $u = (u', v) \in \operatorname{supp}\phi \subset \Gamma_-$ implies that

$$\Gamma(x' - \epsilon u') \leq |\Gamma(x' - \epsilon u') - \Gamma(x)| + \Gamma(x) \leq M|\epsilon u'| + y < |\epsilon v| + y = y - \epsilon v.$$

Therefore $D^\alpha f_0$ can be replaced with $D^\alpha f$. By passing to the trivial extension of $D^\alpha f$, we can use Lemma 3.2 for $D^\alpha f$ to get that

$$F_\epsilon(u) = \left(\int_D |D^\alpha f(x - \epsilon u) - D^\alpha f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p} \to 0$$

as $\epsilon \to 0$ pointwise for each u. We also know that $|\phi(u)|F_\epsilon(u) \leq 2||D^\alpha f||_{L^p(D,\mu)}|\phi(u)|$ is in $L^1(\mathbb{R}^{n+1})$. Thus, by the Dominated Convergence Theorem $||D^\alpha f_\epsilon - D^\alpha f||_{L^p(D,\mu)} \to 0$ as $\epsilon \to 0$. That is, each term in the sum that defines the Sobolev norm (see Definition 2.1) converges to 0 and thus, since the sum has a finite number of terms, the Sobolev norm itself must converge to zero. □

3.3 Constructing Moment Conditions

For future results we would also like to be able to choose a $\phi \in C_c^\infty$ with certain moment conditions.

Lemma 3.4. *Given any closed interval $I = [a, b]$ and $b_0, \ldots, b_k \in \mathbb{R}$, there exists $\phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \phi \subset I$ such that*

$$\int_{-\infty}^{\infty} \lambda^l \phi(\lambda) \mathrm{d}\lambda = b_l \in \mathbb{R}$$

for all $l = 0, 1, \ldots, k$.

Proof. Consider $f(t) = \begin{cases} e^{1/(a-t)} & t > a \\ 0 & t \leq a \end{cases}$ which is known to be in $C^\infty(-\infty, \infty)$. Similarly, $g(t) = \begin{cases} e^{1/(t-b)} & t < b \\ 0 & t \geq b \end{cases}$ is also in $C^\infty(-\infty, \infty)$. Thus their product is as well, i.e.,

$$h(t) = \begin{cases} e^{\frac{a-b}{(a-t)(t-b)}} & a \leq t \leq b \\ 0 & t < a \text{ or } t > b \end{cases} \in C^\infty(-\infty, \infty).$$

Also, $\operatorname{supp} h = [a, b]$.

Now, for the moment conditions, consider such a $\phi \in C_c^\infty(I)$ with $\int \phi = 1$ (here, we normalize ϕ if necessary). Fix $k \in \mathbb{N}$. Then $\int x^j \phi^{(k)}(x)\mathrm{d}x = 0$ for all $j < k$ and

$$\int x^k \phi^{(k)}(x)\mathrm{d}x = (-1)^k k! \int \phi(x)\mathrm{d}x = c_k$$

Define $g_k = \frac{1}{c_k}\phi^{(k)}$.

Next, denote $a_{k,k-1} = \int x^k \left[\frac{1}{c_{k-1}}\phi^{(k-1)}(x)\right] \mathrm{d}x$. Define $g_{k-1} = \frac{1}{c_{k-1}}\phi^{(k-1)} - a_{k,k-1}\, g_k$. Then for $j < k-1$ we have that

$$\int x^j g_{k-1}(x)\mathrm{d}x = 0, \qquad \int x^{k-1} g_{k-1}(x)\mathrm{d}x = 1, \qquad \text{and} \qquad \int x^k g_{k-1}(x)\mathrm{d}x = 0$$

In general, given $g_i, g_{i+1}, \ldots, g_{k-1}, g_k$ with $1 \leq i \leq k$ such that

$$\int x^j g_i(x)\mathrm{d}x = 0 \qquad \text{and} \qquad \int x^i g_i(x)\mathrm{d}x = 1$$

for $i \neq j$ where $j \in 0, 1, \ldots, k$, we denote $a_{j,i-1} = \int x^j \left[\frac{1}{c_{i-1}}\phi^{(i-1)}(x)\right] \mathrm{d}x$ and define

$$g_{i-1} = \frac{1}{c_{i-1}}\phi^{(i-1)} - \sum_{t=i}^{k} a_{t,i-1}\, g_t.$$

It then follows that we gain the existence of a set of functions $\{g_i\}_{0 \leq i \leq k}$ where each $g_i \in C_c^\infty(-\infty, \infty)$ with the moment conditions

$$\int x^j g_i(x)\mathrm{d}x = 0 \qquad \text{and} \qquad \int x^i g_i(x)\mathrm{d}x = 1$$

for $i \neq j$ where $j \in 0, 1, \ldots, k$. Thus, if we want $h \in C_c^\infty$ with $\int x^l h(x)\mathrm{d}x = b_l$ for $0 \leq l \leq k$ we simply define $h = b_0 g_0 + \cdots + b_k g_k$. $\square$

Chapter 4

The Unweighted Case

In our goal of proving Theorem 1.1 we first aim to show the unweighted case. In doing so we bring attention to the major issues at play. Namely, in each case, there are six main steps we will follow:

(A) Define the extension $E(f)$ after first localizing the domain (see Definition 2.4) and assuming that $f \in C^\infty$ (see Lemma 3.3).

(B) Show that the extension is weakly differentiable up to the assumed order k (Here we use Lemma 3.1).

(C) Show that $E(f)$ is bounded in L^p.

(D) Using what was learned in the last step, perform L^p estimates on the derivatives of $E(f)$.

(E) We use density to define the extension for general f and show that the Sobolev norm estimate is maintained.

(F) We use the definition of the domain to define the general extension by using a partition of unity argument.

In accordance with these, we will label each extension theorem proof as such.

These steps have their origin in past results. For instance, the results of E. M. Stein in [18] follow this program to prove the unweighted Lipschitz domain case. This proof is summarized at the end of the section as it will be useful in demonstrating both the scalar and matrix weighted analogues.

4.1 The Smooth Domain Case

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with smooth boundary. Given $f \in L^p_k(\Omega)$ with $1 \le p < \infty$, $k = 0, 1, 2, \ldots$, there exists an extension $E(f) \in L^p_k(\mathbb{R}^{n+1})$, that is, there exists $E(f)(x, y) : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $E(f)|_\Omega = f$ and*

$$||E(f)||_{L^p_k(\mathbb{R}^{n+1})} \le C_{p,k,\Omega}||f||_{L^p_k(\Omega)} \tag{4.1}$$

Proof. (A) Before providing an explicit definition for the extension operator, we make two standard simplifications: First, we can make a C^∞ change of coordinates to “straighten out the boundary” of Ω. We accomplish this by covering the boundary with neighborhoods, straightening each piece, and then using a partition of unity to “glue” the pieces back together. Thus we only consider $\Omega = \mathbb{R}^{n+1}_+$.
As a second simplification, we note that by density we can assume that $f \in C^\infty\left(\overline{\mathbb{R}^{n+1}_+}\right)$ and then pass through to a limit at the end.
For such f, consider

$$g(x, y) = \int_0^\infty f(x, -\lambda y)\phi(\lambda)\,\mathrm{d}\lambda \tag{4.2}$$

where ϕ is a continuous function with compact support in $(0, \infty)$ that satisfies moment properties:

$$\int_0^\infty \lambda^l \phi(\lambda)\,\mathrm{d}\lambda = (-1)^l \tag{4.3}$$

for all $0 \leq l \leq k$. We then set

$$E(f) = \begin{cases} f(x,y) & y \geq 0 \\ g(x,y) & y < 0 \end{cases}$$

(B) For the weak derivatives, notice first that since we assumed f to be smooth it follows that

$$g_{x_j} = \int_0^\infty f_{x_j}(x, -\lambda y)\phi(\lambda)\,\mathrm{d}\lambda$$

for all $0 \leq j \leq n$ and

$$g_y = -\int_0^\infty f_y(x, -\lambda y)\lambda\phi(\lambda)\,\mathrm{d}\lambda$$

Putting these together we have that any derivative of g of order $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = (\alpha', \alpha_{n+1})$ will have the form

$$D^\alpha g = (-1)^{\alpha_{n+1}} \int_0^\infty D^\alpha f(x, -\lambda y)\lambda^{\alpha_{n+1}}\phi(\lambda)\mathrm{d}\lambda \tag{4.4}$$

Namely, g is smooth on the lower half space. Also, we have the trace property that as $y \to 0^+$, $g(x,y) \to f(x,0)$ because of (4.3). Similarly, for all $|\alpha| \leq k$,

$$\begin{aligned} \lim_{y\to 0} D^\alpha g(x,y) &= (-1)^{\alpha_{n+1}} \int_0^\infty D^\alpha f(x,0)\lambda^{\alpha_{n+1}}\phi(\lambda)\mathrm{d}\lambda \\ &= D^\alpha f(x,0). \end{aligned}$$

Thus it follows by Lemma 3.1 that $E(f)$ has weak derivatives of all orders up to k.

(C) We next show that the L^p norm of g and of all of its derivatives are bounded by the corresponding L^p norms of f and its derivatives. Indeed, using Minkowski's integral inequality we have that

$$\begin{aligned}
||g||_{L^p(\mathbb{R}^{n+1}_-)} &= \left(\int_{\mathbb{R}^n}\int_{-\infty}^{0}\left|\int_0^\infty f(x,-\lambda y)\phi(\lambda)\,\mathrm{d}\lambda\right|^p \mathrm{d}y\,\mathrm{d}x\right)^{1/p} \\
&\leq \int_0^\infty \left(\int_{\mathbb{R}^n}\int_{-\infty}^{0} |f(x,-\lambda y)\phi(\lambda)|^p\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p}\mathrm{d}\lambda \\
&= \int_0^\infty \left(\int_{\mathbb{R}^n}\int_{0}^{\infty} |f(x,u)\phi(\lambda)|^p\,\frac{\mathrm{d}u}{\lambda}\,\mathrm{d}x\right)^{1/p}\mathrm{d}\lambda \\
&= \int_0^\infty \frac{|\phi(\lambda)|}{\lambda^{1/p}}\mathrm{d}\lambda \cdot ||f||_{L^p\left(\mathbb{R}^{n+1}_+\right)}.
\end{aligned}$$

But ϕ has compact support in $(0,\infty)$ and thus we have the required bound.

(D) Similarly, for $D^\alpha g$ as given by (4.4), we have that

$$\begin{aligned}
||D^\alpha g||_{L^p(\mathbb{R}^{n+1}_-)} &= \left(\int_{\mathbb{R}^n}\int_{-\infty}^{0}\left|\int_0^\infty D^\alpha f(x,-\lambda y)\lambda^{\alpha_{n+1}}\phi(\lambda)\,\mathrm{d}\lambda\right|^p \mathrm{d}y\,\mathrm{d}x\right)^{1/p} \\
&\leq \int_0^\infty \left(\int_{\mathbb{R}^n}\int_{-\infty}^{0} |D^\alpha f(x,-\lambda y)\lambda^{\alpha_{n+1}}\phi(\lambda)|^p\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p}\mathrm{d}\lambda \\
&= \int_0^\infty \left(\int_{\mathbb{R}^n}\int_{0}^{\infty} |D^\alpha f(x,u)\lambda^{\alpha_{n+1}}\phi(\lambda)|^p\,\frac{\mathrm{d}u}{\lambda}\,\mathrm{d}x\right)^{1/p}\mathrm{d}\lambda \\
&= \int_0^\infty \lambda^{\alpha_{n+1}-1/p}|\phi(\lambda)|\mathrm{d}\lambda \cdot ||D^\alpha f||_{L^p\left(\mathbb{R}^{n+1}_+\right)} \\
&\leq C_{\alpha,p,n}||D^\alpha f||_{L^p\left(\mathbb{R}^{n+1}_+\right)}
\end{aligned}$$

by again using that ϕ has compact support in $(0,\infty)$. Thus g and all of its derivatives are bounded in the L^p norm. That is, given our assumptions on Ω and f, (4.1) is satisfied.

(E) Consider $f \in L^p_k\left(\mathbb{R}^{n+1}_+\right)$. By Lemma 3.3, there exists $\{f_N\} \subset C^\infty\left(\overline{\mathbb{R}^{n+1}_+}\right)$ such that

$$||f_N - f||_{L^p_k\left(\mathbb{R}^{n+1}_+\right)} \to 0$$

as $N \to \infty$. By noting that the extension map E is a linear map, it then follows that for $N, M > 0$

$$\begin{aligned} ||E(f_N) - E(f_M)||_{L^p_k(\mathbb{R}^{n+1})} &= ||E(f_N - f_M)||_{L^p_k(\mathbb{R}^{n+1})} \\ &\leq C_{p,k,n}||f_N - f_M||_{L^p_k(\mathbb{R}^{n+1}_+)} \end{aligned}$$

(Note that the constant is independent of N, M, and f)

Since f_N converges, it is Cauchy (in $\mathbb{R}^{n+1}_+$). Thus $E(f_N)$ is also Cauchy (but in $\mathbb{R}^{n+1}$) and thus, by the completeness of Sobolev space, $E(f_N)$ converges. So we can define $E(f) = \lim E(f_N)$ in the Sobolev norm, that is,

$$||E(f) - E(f_N)||_{L^p_k(\mathbb{R}^{n+1})} \to 0$$

as $N \to \infty$. In particular, $||E(f) - E(f_N)||_{L^p_k(\mathbb{R}^{n+1}_+)} \to 0$ as $N \to \infty$. So it follows that on $\mathbb{R}^{n+1}_+$

$$E(f) = \lim E(f_N) = \lim f_N = f$$

Thus $E(f)$ is an extension of f. Last of all,

$$\begin{aligned} ||E(f)||_{L^p_k(\mathbb{R}^{n+1})} &= \lim ||E(f_N)||_{L^p_k(\mathbb{R}^{n+1})} \\ &\leq \lim C_{p,k,n}||f_N||_{L^p_k(\mathbb{R}^{n+1}_+)} \\ &= C_{p,k,n} \lim ||f_N||_{L^p_k(\mathbb{R}^{n+1}_+)} \\ &= C_{p,k,n}||f||_{L^p_k(\mathbb{R}^{n+1}_+)} \end{aligned}$$

So the norm inequality is preserved.

(F) As the final step, since we have shown that the result holds on any half space (just shift the coordinate axes as needed), we can now show the result for our original smooth domain.

For $i = 1, 2, \ldots$, let U_i be the open sets associated with the smooth domain Ω and Ω_i be the associated half spaces. Consider $\lambda_i, \lambda_0, \lambda_+, \lambda_- : \mathbb{R}^{n+1} \to \mathbb{R}$ all of which are smooth with bounded derivatives independent of i (and dependent only on the parameters of the extension constant C) such that

- supp $\lambda_i \subset U_i$ and $\lambda_i = 1$ on $U_i^{\epsilon/2}$
- supp $\lambda_o \subset \epsilon/2$ neighborhood of Ω and $\lambda_0 = 1$ on $\overline{\Omega}$
- supp $\lambda_+ \subset \epsilon$ neighborhood of $\partial\Omega$ and $\lambda_+ = 1$ if $\text{dist}(x, \partial\Omega) \leq \epsilon/2$
- supp $\lambda_- \subset \Omega$ and $\lambda_- = 1$ if $\text{dist}(x, \partial\Omega) \geq \epsilon/2$

Then set $\Lambda_+ = \lambda_0 \left(\frac{\lambda_+}{\lambda_+ + \lambda_-}\right)$ and $\Lambda_- = \lambda_0 \left(\frac{\lambda_-}{\lambda_+ + \lambda_-}\right)$ and note

- Since λ_0 is supported where $\lambda_+ + \lambda_- \geq 1$, Λ_+ and Λ_- are also smooth with bounded derivatives
- $\Lambda_+ + \Lambda_- = 1$ on $\overline{\Omega}$
- $\Lambda_+ + \Lambda_- = 0$ outside the $\epsilon/2$ neighborhood of Ω.

With all of this set up, we finally define the extension operator:

$$E(f) = \Lambda_+ \left\{ \frac{\sum_{i=1}^{\infty} \lambda_i E_i(\lambda_i f)}{\sum_{i=1}^{\infty} \lambda_i^2} \right\} + \Lambda_- f \tag{4.5}$$

From this it is apparent that

(a) When $x \in \text{supp } \Lambda_+$, $x \in U_i^{\epsilon/2}$ for some i and thus $\sum_{i=1}^{\infty} \lambda_i^2 \geq 1$

(b) For each x the sums in (4.5) involve at most $N+1$ non-vanishing terms because of condition (ii) in Definition 2.4

(c) $\Lambda_- f$ is well-defined since the support of Λ_- is contained in Ω

(d) The terms $E_i(\lambda_i f)$ are well-defined since the $\lambda_i f$ are given in the half space Ω_i, that is, f is defined on Ω, thus $\lambda_i f$ is supported on $U_i \cap \Omega = U_i \cap \Omega_i$ (see (iii) in Definition 2.4), and thus $E_i(\lambda_i f)$ is defined on $\mathbb{R}^{n+1}$. (Hence each $\lambda_i E_i(\lambda_i f)$ is supported in U_i)

(e) Based on the smoothness of the lambda functions and of each $E_i(\lambda_i f)$, it follows that $E(f)$ has well-defined derivatives as well.

(f) Since for each i $E_i(\lambda_i f) = \lambda_i f$ on D, it follows that indeed $E(f)(x,y) = f(x,y)$ for all $(x,y) \in D$

Now based on the quotient rule we we write (4.5) in the form

$$E(f) = \sum_{i=1}^{\infty} H_i E_i(\lambda_i f) + \Lambda_- f$$

as a replacement for (10.10). Then

$$\begin{aligned}
||E(f)||_{L_k^p(\mathbb{R}^{n+1})} &= \sum_{|\alpha|\leq k} \left|\left| D^\alpha \left(\sum_{i=1}^{\infty} H_i E_i(\lambda_i f) + \Lambda_- f \right) \right|\right|_{L^p(\mathbb{R}^{n+1})} \\
&\leq \sum_{|\alpha|\leq k} \left(\int_{\mathbb{R}^{n+1}} \left|\left| D^\alpha \left(\sum_{i=1}^{\infty} H_i(x) E_i(\lambda_i f)(x) \right) \right|\right|^p \mathrm{d}x \right)^{1/p} \\
&\quad + \sum_{|\alpha|\leq k} ||D^\alpha (\Lambda_- f)||_{L^p(\mathbb{R}^{n+1})}
\end{aligned}$$

The latter term is clearly bounded by $C\,||f||_{L^p_k(\Omega)}$. Also, by Definition 2.4 (ii), $\sum_{i=1}^{\infty}\chi_{U_i} \leq N$ (also see (b) above). Thus the former is

$$\begin{aligned}
&\sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\left|\left|\sum_{i=1}^{\infty}\sum_{\beta+\gamma=\alpha}D^{\beta}H_i(x)D^{\gamma}E_i\left(\lambda_i f\right)(x)\right|\right|^p \mathrm{d}x\right)^{1/p}\\
&\leq \sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\left(\sum_{i=1}^{\infty}\chi_{U_i}\sum_{\beta+\gamma=\alpha}\left|\left|D^{\beta}H_i(x)D^{\gamma}E_i\left(\lambda_i f\right)(x)\right|\right|\right)^p \mathrm{d}x\right)^{1/p}\\
&\leq \sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}N^{p/p'}\sum_{i=1}^{\infty}\left(\sum_{\beta+\gamma=\alpha}\left|\left|D^{\beta}H_i(x)D^{\gamma}E_i\left(\lambda_i f\right)(x)\right|\right|\right)^p \mathrm{d}x\right)^{1/p}\\
&\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\sum_{i=1}^{\infty}\sum_{\beta+\gamma=\alpha}\left|\left|D^{\beta}H_i(x)D^{\gamma}E_i\left(\lambda_i f\right)(x)\right|\right|^p \mathrm{d}x\right)^{1/p}\\
&= C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^{\infty}\sum_{\beta+\gamma=\alpha}\int_{U_i}\left|D^{\beta}H_i(x)\right|^p\left|\left|D^{\gamma}E_i\left(\lambda_i f\right)(x)\right|\right|^p \mathrm{d}x\right)^{1/p}\\
&\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^{\infty}\sum_{\beta+\gamma=\alpha}\int_{U_i}\left|\left|D^{\gamma}E_i\left(\lambda_i f\right)(x)\right|\right|^p \mathrm{d}x\right)^{1/p}\\
&\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^{\infty}\sum_{\beta+\gamma=\alpha}\sum_{|\kappa|\leq k}\int_{\Omega_i}\left|\left|D^{\kappa}\left(\lambda_i f\right)(x)\right|\right|^p \mathrm{d}x\right)^{1/p}\\
&\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^{\infty}\sum_{\beta+\gamma=\alpha}\sum_{|\kappa|\leq k}\sum_{\epsilon+\delta=\kappa}\int_{U_i\cap\Omega}\left|D^{\epsilon}\lambda_i(x)\right|\left|\left|D^{\delta}f(x)\right|\right|^p \mathrm{d}x\right)^{1/p}\\
&\leq C_{p,k,\Omega}||f||_{L^p_k(\Omega)}
\end{aligned}$$

via several applications of Hölder's inequality. □

It is worth further considering the L^{∞} case. While it is possible to extend it (see [18]), the above procedure will not work. The main reason for this is that the density assumption no longer holds. Thus any proof of such a case would have to be different to accommodate for the loss of density.

It is, however, not too difficult to generalize Theorem 4.1 to vector valued functions. This is discussed in detail in the weighted context in Chapter 9, but the basic principle remains the same. Namely, we define each component $E(f_i)$ of $E\left(\overrightarrow{f}\right)$ as we did in our scalar valued case. In this way, the vector case is reduced to a component-wise argument of the scalar case already considered.

4.2 A Summary of the Lipschitz Domain Case

The results thus far considered are well-known by this point. In fact, the domain can be modified and the above results still hold in essence. For example, consider a Lipschitz domain as given by Definition 2.4. A. P. Calderon showed in [1] that this can be done using an extension operator defined using singular integrals, but only for $1 < p < \infty$. E. M. Stein was able to repeat this but with an extension operator similar to the one used above, albeit more complicated. Such an approach had the added benefit that not only did it cover the endpoints $p = 1, \infty$, but it also was independent of the order of differentiability k.

The procedure which Stein used to show his result was mirrored in our proof for the smooth case. In pages 180-192 of [18], this six step approach is used to prove the following:

Theorem 4.2. *Let $D \subset \mathbb{R}^{n+1}$ be a Lipschitz domain. Given $f \in L^p_k(D)$ with $1 \leq p \leq \infty$, $k = 0, 1, 2, \ldots$, there exists an extension $E(f) \in L^p_k(\mathbb{R}^{n+1})$, that is, there exists $E(f)(x, y) : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $E(f)|_D = f$ and*

$$||E(f)||_{L^p_k(\mathbb{R}^{n+1})} \leq C_{p,k,D}||f||_{L^p_k(D)} \tag{4.6}$$

When proving the weak differentiability of this extension, we would like to have a result similar to Lemma 3.1. It is not clear that such a statement exists, though, given the problems that would arise in the attempt to obtain the weak differentiability in the x-components. However, this can be amended by assuming that the function f is perfectly smooth.

Lemma 4.1 ([18], p. 186)**.** *Let $D \subset \mathbb{R}^{n+1}$ be a special Lipschitz domain with curve Γ. If*

$$E(f)(x,y) = \begin{cases} f(x,y) & (x,y) \in \overline{D} \\ g(x,y) & (x,y) \in \overline{D^c} \end{cases}$$

with $f \in C^\infty\left(\overline{D}\right)$, $g \in C^\infty\left(\overline{D^c}\right)$, and, for any multi-index α , $D^\alpha f = D^\alpha g$ on $\partial D = \{(x, \Gamma(x)) : x \in \mathbb{R}^n\}$, then $E(f) \in C^\infty\left(\mathbb{R}^{n+1}\right)$.

Of course, when $f \in C^\infty$, we can treat Lemma 3.1 as a corollary of Lemma 4.1.

The high points of the proof of Theorem 4.2 are as follows:

(A) It is first assumed that D is a special Lipschitz domain and that $f \in C^\infty(\overline{D})$ The extension is then defined on $\overline{D}^c$ as

$$g(x,y) = \int_1^\infty f(x, y + \lambda\delta^*(x,y))\psi(\lambda)\mathrm{d}\lambda$$

where δ^* is a differentiable function that is equivalent to the distance from the boundary in $\overline{D}^c$. Here $\psi \in C([1,\infty))$ is such that $\psi(\lambda)$ decreases like λ^{-N} as $\lambda \to \infty$ for all N, along with suitable moment conditions.

(B) After proving that such δ^* and ψ exist, their properties are used to show the weak differentiability of the extension through the argument used in proving Lemma 4.1. It is worth noting that the computation of a general derivative here requires both the chain and Leibniz rules. The moment conditions are used extensively in a Taylor estimate whose remainder must consequently go to zero.

(C) The L^p estimate for the extension is obtained by using an involved Minkowski Integral Inequality and Fubini style estimate.

(D) For the derivatives, L^p estimates are obtained using the argument above but first Taylor's theorem is applied. We use moment conditions to get rid of all the sum terms leaving only the integral form of the remainder. The remainder is then controlled using a similar argument to before.

(E) As before, density is used to define the more general extension.

(F) As a final step, the same partition of unity argument we used above is employed to finish off the proof.

For our purposes in the sequel, the above argument will be key but the extension will require substantial revision as will the method for obtaining the L^p estimates. Namely, when considering weighted L^p space an integration on lines no longer suffices. This is a consequence of the known conditions for weights which are formulated in terms of cubes, not lines. Because of this, a new method for extending and performing the L^p estimates must be developed. Such a method requires first requires an understanding about the weights involved.

Chapter 5

Scalar A_p Weights

While there are various conditions one might want to place on a weight, the most standard by far is the A_p condition. For scalar weights, such a condition leads to extension results quite naturally. For example, in [3] this condition is used to modify the proof found in [11] to get the most general extension result thus far known for the scalar weighted context. However, for our purposes this chapter serves as a necessary overview for obtaining the scalar weighted case for both smooth and Lipschitz domains. It is also important for understanding the more complicated topic of matrix A_p weights which will be discussed in Chapter 8.

As far as a reference for this material, the majority of the results from this chapter come from [19], chapter 5.

5.1 Motivation

Let $x \in \mathbb{R}^n$. Consider the Hardy-Littlewood Maximal Function:

$$M(f)(x) = \sup_{Q: x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y$$

where Q is a cube and $|Q|$ denotes the Lebesgue measure of the cube. This is useful for bounding lots of different integral operators, such as the Harmonic Extension $\int_{\mathbb{R}^n} |P_t(x-y)f(y)|\,\mathrm{d}y \leq M(f)(x)$ for all $t > 0$.

Now if $f \in L^p$ for $1 < p \leq \infty$, then $M(f) \in L^p$ with

$$\int |M(f)(x)|^p \mathrm{d}x \leq C_p \int |f(x)|^p \mathrm{d}x$$

It is then natural to ask for what measures μ does the following inequality hold:

$$\int |M(f)(x)|^p \mathrm{d}\mu \leq C_{p,\mu} \int |f(x)|^p \mathrm{d}\mu \tag{5.1}$$

A theorem of Hunt-Muckenhoupt-Wheeden says that this will hold if and only if $\mathrm{d}\mu = w(x)\mathrm{d}x$ where w is nonnegative and absolutely continuous with respect to Lebesgue measure, and w is an A_p weight (see also [19], p. 198).

5.2 Definition

Definition 5.1. *Let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. A weight $w \geq 0$ that is locally integrable is an A_p weight, written $w \in A_p$, if for some constant $A > 0$*

$$\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x \left(\frac{1}{|Q|}\int_Q w^{-p'/p}(x)\mathrm{d}x\right)^{p/p'} \leq A \tag{5.2}$$

for all cubes Q. The smallest such A for which this holds is called the A_p bound of w, denoted $A_p(w)$.

For $p = 2$ this says that the average of the weight times the average of its reciprocal cannot be too big, i. e.

$$\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x \cdot \frac{1}{|Q|}\int_Q w^{-1}(x)\mathrm{d}x \leq A$$

That is, w cannot get too big or too small too much of the time.

As an example, it is easy to see that $w(x) = |x|^\alpha$ is an A_p weight on $\mathbb{R}^n$ if and only if $-n < |\alpha| < n(p-1)$. This demonstrates that an A_p weight might be zero, just not too often, and could also be singular as long as it is not too badly behaved.

As far as notation, we denote

$$w(E) = \int_E w(x)\mathrm{d}x$$

for any measurable set E.

5.3 An Averaging Property and Doubling Weights

It is often easier to think about A_p weights using one of several equivalent definitions. Besides (5.1) and the definition provided by (5.2), we also have the following formulation:

Lemma 5.1 ([19], p. 195)**.** *For $1 < p < \infty$, $w \in A_p$ if and only if there exists a constant $C > 0$*

$$\left(\frac{1}{|Q|}\int_Q f(x)\mathrm{d}x\right)^p \leq \frac{C}{w(Q)}\int f^p(x)w(x)\mathrm{d}x \tag{5.3}$$

for all cubes Q and nonnegative functions f. Furthermore, the smallest such C for which this holds is $A_p(w)$

Unless working with L^p estimates, it is unclear why such conditions should be ideal for a weight. This leads to a more general property of A_p - weights that can be taken as a condition in itself.

Definition 5.2. *A weight $w \geq 0$ that is locally integrable is called doubling if there exists C such that*

$$w(2Q) \leq Cw(Q)$$

for any cube Q (here $2Q$ represents the cube with the same center but twice the side length as Q).

The primary example of a doubling weight is $w(x) = |x|^\alpha$ is a doubling weight on $\mathbb{R}^n$ if and only if $|\alpha| > -n$. Using (5.3) it can be shown that any A_p weight must also be a doubling weight.

5.4 A_1 Weights

There are two other classifications of weights, that is, A_∞ and A_1. Both can be thought of as limits of A_p weights with the first understood from the following useful property:

$$\text{If } 1 < p_1 < p_2 < \infty, \text{ then } A_{p_1} \subset A_{p_2}$$

With this, we say that $w \in A_\infty$ if and only if $w \in A_p$ for some $1 < p < \infty$. That is, notationally, $A_\infty = \bigcup_{1<p<\infty} A_p$.

Turning to the $p = 1$ case, we limit $p \to 1$ in (5.2) to gain that $w \in A_1$ if and only if for every cube Q there exists $A > 0$ such that

$$\frac{1}{|Q|} \int_Q w(u) \mathrm{d}u \leq A w(x) \tag{5.4}$$

for almost every $x \in Q$. Taking the sup over all cubes Q, this is equivalent to

$$(Mw)(x) \leq A' w(x) \tag{5.5}$$

for almost every $x \in Q$.

It is then possible to show that $w \in A_1$ implies that $w \in A_p$ for all $1 < p < \infty$. Combining this with some of our prior results yields a more complete statement for containment of weights:

$$\text{If } 1 \leq p_1 < p_2 \leq \infty, \text{ then } A_{p_1} \subset A_{p_2}.$$

Also, as noted earlier, any of the above weights will be doubling, that is, A_∞ is a subset of doubling. Such containments, while useful for intuitive understanding, will be used minimally in the sequel.

However, it is useful to note that we regain a result similar to (5.3), that is, $w \in A_1$ if and only if

$$\frac{1}{|Q|} \int_Q f(x) \mathrm{d}x \leq \frac{C}{w(Q)} \int f(x) w(x) \mathrm{d}x \tag{5.6}$$

for all cubes Q and nonnegative functions f (see [19], p. 197).

5.5 Density of $C^\infty \left(\overline{D}\right)$ in $L^p_k (D, w)$

Definition 5.3. *For $1 \leq p < \infty$ we define the scalar weighted L^p norm of f to be*

$$||f||_{L^p(E,w)} = \left(\int_E |f(x)|^p w(x) \mathrm{d}x \right)^{1/p} \tag{5.7}$$

for any $E \subset \mathbb{R}^n$ measurable.

Using this and Definition 2.1 we make a clear and intuitive definition for scalar weighted Sobolev space.

Definition 5.4. *For $1 \leq p \leq \infty$, $k = 0, 1, 2, \ldots$, and w a scalar weight, the Sobolev space*

$$L^p_k (D, w) = \{ f : D \to \mathbb{R} \mid D^\alpha f \in L^p (D, w) \ \forall |\alpha| \leq k \}$$

which is complete with respect to the norm

$$||f||_{L^p_k(D,w)} = \sum_{|\alpha| \leq k} ||D^\alpha f||_{L^p(D,w)}.$$

With such a definition, it is a simple matter to reclaim the density of $C^\infty\left(\overline{D}\right)$ in $L_k^p(D,w)$. As long as w is a doubling weight, the measure induced by w is guaranteed to be a Radon measure and thus we reclaim the following result:

Corollary 5.1 (Density of C^∞ in weighted L_p^k)**.** *Suppose $f \in L^1_{loc}\left(\mathbb{R}^{n+1}\right)$ and that $D \subset \mathbb{R}^{n+1}$ is a special Lipschitz domain with curve Γ and bound M. Let $f \in L_k^p\left(D,w\right)$ where $1 \leq p < \infty$, $k = 0,1,2,...$, and w is a doubling weight. Then there exist $\{f_\epsilon\}_{\epsilon>0}$ where each $f_\epsilon \in C^\infty\left(\overline{D}\right)$ such that, as $\epsilon \to 0$,*

$$||f_\epsilon - f||_{L_k^p(D,w)} \to 0.$$

Chapter 6

The Scalar A_p Weighted Smooth Domain Case

With these results we are now able to prove the scalar weighted, smooth domain version of Theorem 1.1. The extension operator which we will use is given as an integration over a cube whose size is roughly the same as its distance from the boundary. Another interesting note is that it is actually possible to give multiple proofs of the L^p bounds using the three different characterizations of A_p - weights when $1 < p < \infty$ (and two proof for the $p = 1$ case). These will be presented along with the rest of the argument in the same pattern as that of its unweighted counterpart, Theorem 4.1.

6.1 Whitney Cubes and the One Dimensional Result

To aid in understanding, we first consider the one dimensional case where the domain we start with is $\mathbb{R}_+$. This helps to introduce the central issues that come about by adding in an A_p weight. Such issues include a finite interval of integration, doubling as a means of doing weighted estimates, and, most notably, the pivotal role that the

A_p condition plays in the proof. There is also a final issue of over counting cubes that is much simpler in the one-dimensional case where cubes are really just intervals.

In all further cases, what follows depends heavily on the concept of Whitney cube decompositions for the upper and lower half spaces, or, more generally, any complement of a nonempty closed region contained in $\mathbb{R}^n$.

Lemma 6.1 ([18], p.167)**.** *For any closed set $F \subset \mathbb{R}^n$, there exists a collection of closed cubes $\{Q_k\}_{k=1}^{\infty}$ such that $\text{int}(Q_k) \cap \text{int}(Q_\ell) = \emptyset$ for $k \neq \ell$, $\cup_{k=1}^{\infty} = F^c$, and $\text{dist}(F, Q_k) \approx \text{side}(Q_k)$ where the equivalence constants depend only on n. Such a construction is often called the Whitney cube decomposition for $U = F^c$.*

To understand (and prove) the existence of such a decomposition, it is helpful to first consider dyadic intervals. For the positive real line, this consists of the set of intervals$\{[2^q, 2^{q+1}] : q \in \mathbb{Z}\}$ (a similar result clearly holds for the lower half plane). Using this, we can obtain a dyadic cube decomposition for the upper (and lower) half planes or the various quadrants of $\mathbb{R}^n$. This decomposition works as a Whitney decomposition since it has the further desired property that the side length of a cube is the same as its distance from the boundary.

To obtain the construction for a more general domain, we start with the dyadic decomposition. Then, by removing cubes and dyadically decomposing the cubes that remain, we obtain a cube decomposition for the domain in question. To better understand this and for other details on the Whitney cube decomposition the reader may reference [18], p.167-170.

Theorem 6.1. *(1-Dimensional Case) Given $f \in L_k^p(\mathbb{R}_+, w)$ with $1 \leq p < \infty$, $k = 0, 1, 2, \ldots$, and $w \in A_p$, there exists an extension $E(f) \in L_k^p(\mathbb{R}, w)$, that is, there exists $E(f) : \mathbb{R} \to \mathbb{R}$ such that $E(f)|_{\mathbb{R}_+} = f$ and*

$$||E(f)||_{L_k^p(\mathbb{R},w)} \leq C_{p,k,w} ||f||_{L_k^p(\mathbb{R}_+,w)} \tag{6.1}$$

It is worth noting that in this sub-case we are extending a domain that is the one dimensional equivalent of the upper half space. We are thus able to skip part (F) in the proof since we will no longer require a partition of unity argument to go from the upper half space to the general smooth domain (see the proof of Theorem 4.1).

Proof. (A) In light of 5.1, we first assume that $f \in C^{\infty}(\mathbb{R}_+)$. For such f, consider

$$g(y) = \int_{1/2}^{2} f(-\lambda y)\phi(\lambda)\mathrm{d}\lambda \tag{6.2}$$

where $y < 0$. Here $\phi \in C_c^{\infty}(\mathbb{R}_-)$ with $\operatorname{supp}\phi \subset \left[\frac{1}{2}, 2\right]$ and the following moment properties:

$$\int \phi(\lambda)\mathrm{d}\lambda = 1 \quad \text{and} \quad \int \lambda^l \phi(\lambda)\mathrm{d}\lambda = (-1)^l \tag{6.3}$$

for $1 \leq l \leq k$. We then set

$$E(f) = \begin{cases} f(y) & y \geq 0 \\ g(y) & y < 0 \end{cases}$$

(B) Since we assumed f to be smooth it follows that g is smooth with derivatives

$$\frac{\partial^l g}{\partial y^l}(y) = \int_{1/2}^{2} \frac{\partial^l f}{\partial y^l}(-\lambda y)(-\lambda)^l \phi(\lambda)\mathrm{d}\lambda \tag{6.4}$$

for $0 \leq l \leq k$. Using the moment conditions in (6.3) we then have that $\frac{\partial^l g}{\partial y^l}(y) \to \frac{\partial^l f}{\partial y^l}(0)$ as $y \to 0$ for each $0 \leq l \leq k$. That is, g is smooth on the lower half space with the necessary trace conditions and thus we can use the existence of derivatives across the boundary (Lemma 3.1) to gain that $E(f)$ is smooth.

(C) There are three different methods for obtaining the L^p bound for $E(f)$, each of which begins in the same way. As seen before in the proof of Theorem 4.1, we need only bound the norm of g by the norm of f. We can also make the immediate pointwise estimate using standard estimates and applying a change of variables from λ to v:

$$|g(y)| \leq \frac{C}{|y|} \int_{-(1/2)y}^{-2y} |f(v)| \mathrm{d}v \tag{6.5}$$

while keeping in mind that here $y < 0$ (so the expression is well-defined).

The key observation here is that this expression is an average of f over an interval which is similar in style to a dyadic Whitney interval. In the scalar A_p weighted, smooth domain context there are three different ways to bound it each of which relies on a different (but equivalent) definition of the A_p condition.

(Averaging Method) The averaging property formulation of the A_p condition comes up very naturally in the calculations. Namely, we want to just consider the extension on a Whitney interval in $\mathbb{R}_+$ and then sum up the L^p norms over all such cubes. To this end, we will denote $\mathcal{W}_- = \{I_q\}_{q=-\infty}^{\infty}$ to be a Whitney decomposition for $\mathbb{R}_-$ and similarly $\mathcal{W}_+ = \{J_q\}_{q=-\infty}^{\infty}$ as its $\mathbb{R}_+$ equivalent.

Fix $y \in I_q$ where $I_q \in \mathcal{W}_-$, that is, let $I_q = [-2^{q+1}, -2^q]$ for $q \in \mathbb{Z}$. Then $2^q \leq -y \leq 2^{q+1}$. Thus

$$2^{q-1} \leq -\frac{1}{2}y \leq -2y \leq 2^{q+2}$$

which yields an even larger interval $I_q^* = [2^{q-1}, 2^{q+2} + 2^q + 2^{q-1}]$ whose total length is $2^{q+2} + 2^q = 5 \cdot 2^q$. Hence $|y| \geq 2^q = \frac{1}{5}|I_q^*|$.

For such $y \in I_q$ we can then replace (6.5) by

$$|g(y)| \leq \frac{C'}{|I_q^*|} \int_{I_q^*} |f(v)| \mathrm{d}v$$

Next, we can use the averaging equivalence (5.3) for A_p weights to get that

$$\begin{aligned}|g(y)|^p &\leq C_p \left(\frac{1}{|I_q^*|} \int_{I_q^*} |f(v)| \mathrm{d}v \right)^p \\ &\leq \frac{C_{p,w}}{w(I_q^*)} \int_{I_q^*} |f(v)|^p w(v) \mathrm{d}v\end{aligned}$$

for all $y \in I_q$. We then calculate the L^p norm by summing over all such intervals. Here we recall that w must also be a doubling weight. By doubling I_q^* just once we get $2I_q^* = [-2^{q+1}, 2^{q+3}]$ which contains $I_q = [-2^{q+1}, -2^q]$. Thus, since w must be doubling, $w(I_q) \leq w(2I_q^*) \leq Cw(I_q^*)$ for some constant $C > 0$ as in Definition 5.2. Also, note that I_q^* is contained in exactly 4 intervals $J_q \in \mathcal{W}_+$. Thus

$$\begin{aligned}||g||^p_{L^p(\mathbb{R}_-,w)} &= \sum_{I_q \in \mathcal{W}_-} \int_{I_q} |g(y)|^p w(y) \mathrm{d}y \\ &\leq C_{p,w} \sum_{I_q \in \mathcal{W}_-} \int_{I_q} \left(\frac{1}{w(I_q^*)} \int_{I_q^*} |f(v)|^p w(v) \mathrm{d}v \right) w(y) \mathrm{d}y \\ &= C_{p,w} \sum_{I_q \in \mathcal{W}_-} \frac{w(I_q)}{w(I_q^*)} \int_{I_q*} |f(v)|^p w(v) \mathrm{d}v \\ &\leq C_{p,w} \sum_{I_q \in \mathcal{W}_-} C \int_{I_q*} |f(v)|^p w(v) \mathrm{d}v \\ &\leq C_{p,w} \cdot 4 \sum_{J_q \in \mathcal{W}_+} \int_{J_q} |f(v)|^p w(v) \mathrm{d}v \\ &= C_{p,w} ||f||^p_{L^p(\mathbb{R}_+,w)}\end{aligned}$$

As a final remark, we can use this method to show that the $p = 1$ case must also hold because of the analogous averaging condition (5.6).

(Maximal Function Method) This is perhaps the most natural proof to think of at the start but requires a bit of trickery as far as where the functions are actually defined. Consider the trivial extension of f to all of $\mathbb{R}$:

$$f_0(x,y) = \begin{cases} f(y) & y > 0 \\ 0 & y \leq 0 \end{cases}$$

We can then rewrite (6.5) using the maximal function over intervals $I \subset \mathbb{R}$:

$$\begin{aligned} |g(y)| &\leq \frac{C}{|y|} \int_{-(1/2)y}^{-2y} |f(v)| \mathrm{d}v \\ &\leq \frac{4C}{|4y|} \int_{2y}^{-2y} |f_0(v)| \mathrm{d}v \\ &\leq C \sup_{I \,:\, y \in I} \frac{1}{|I|} \int_I |f_0(v)| \mathrm{d}v \\ &= C\, M(f_0)(y) \end{aligned}$$

Using (5.1) we thus have that

$$\begin{aligned} ||g||^p_{L^p(\mathbb{R}_-,w)} &= \int_{-\infty}^{0} |g(y)|^p \mathrm{d}y \\ &\leq C_p \int_{-\infty}^{0} |M(f_0)(x,y)|^p \mathrm{d}y \\ &\leq C_p \int_{-\infty}^{\infty} |M(f_0)(y)|^p \mathrm{d}y \\ &\leq C_{p,w} \int_{-\infty}^{\infty} |f_0(y)|^p w(y) \mathrm{d}y \\ &= C_{p,w} \int_{0}^{\infty} |f(,y)|^p w(x,y) \mathrm{d}y \\ &= C_{p,w} ||f||^p_{L^p(\mathbb{R}_+,w)}. \end{aligned}$$

It is worth noting that this proof cannot be used in the case $p = 1$ due to the fact that the maximal function is not bounded in L^1 (for example, see [18], p.5).

(Direct Method) This proof is only difficult in that it requires a slight reformulation of the A_p condition and is thus not as immediately obvious. Namely, for $Q \subset \mathbb{R}^n$ a cube, we have that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q[w^{1/p}(x)w^{-1/p}(t)]^{p'}\mathrm{d}t\right)^{p/p'}\mathrm{d}x \\
&\qquad = \frac{1}{|Q|}\int_Q\left(w^{p'/p}(x)\right)^{p/p'}\left(\frac{1}{|Q|}\int_Q w^{-p'/p}(t)\mathrm{d}t\right)^{p/p'}\mathrm{d}x \\
&\qquad = \frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\left(\frac{1}{|Q|}\int_Q w^{-p'/p}(t)\mathrm{d}t\right)^{p/p'} \\
&\qquad \leq A_p(w) \qquad (6.6)
\end{aligned}$$

Next, we proceed as we did in the "Averaging Method" by fixing an interval in the Whitney decomposition of $\mathbb{R}_-$ and doing size estimates. Using that same notation, we then have that (by using Hölder's inequality on the inside integral)

$$\begin{aligned}
||g||^p_{L^p(\mathbb{R}_-,w)} &\leq C_p\sum_{I_q\in\mathcal{W}_-}\int_{I_q}\left(\frac{1}{|I_q^*|}\int_{I_q^*}|f(v)|\mathrm{d}v\right)^p w(y)\mathrm{d}y \\
&= C_p\sum_{I_q\in\mathcal{W}_-}\int_{I_q}\left(\frac{1}{|I_q^*|}\int_{I_q^*}|f(v)|w^{1/p}(y)\mathrm{d}v\right)^p\mathrm{d}y \\
&= C_p\sum_{I_q\in\mathcal{W}_-}\int_{I_q}\left(\frac{1}{|I_q^*|}\int_{I_q^*}w^{1/p}(y)w^{-1/p}(v)|f(v)|w^{1/p}(v)\mathrm{d}v\right)^p\mathrm{d}y \\
&\leq C_p\sum_{I_q\in\mathcal{W}_-}\int_{I_q}\left\{\begin{array}{l}\left(\frac{1}{|I_q^*|}\int_{I_q^*}[w^{1/p}(y)w^{-1/p}(v)]^{p'}\mathrm{d}v\right)^{p/p'} \\ \cdot\left(\frac{1}{|I_q^*|}\int_{I_q^*}|f(v)|^p w(v)\mathrm{d}v\right)\end{array}\right\}\mathrm{d}y \\
&= C_p\sum_{I_q\in\mathcal{W}_-}\left\{\begin{array}{l}\int_{I_q}\left(\frac{1}{|I_q^*|}\int_{I_q^*}[w^{1/p}(y)w^{-1/p}(v)]^{p'}\mathrm{d}v\right)^{p/p'}\mathrm{d}y \\ \cdot\left(\frac{1}{|I_q^*|}\int_{I_q^*}|f(v)|^p w(v)\mathrm{d}v\right)\end{array}\right\}
\end{aligned}$$

where for each $I_q\in\mathcal{W}_-$ (in $\mathbb{R}_-$) the corresponding I_q^* (in $\mathbb{R}_+$) can be doubled to give that $I_q\subset 2I_q^*$. This comes into play because of the difficulty in that both averages in the A_p condition must be over the same cube. However, based on what has been said this is easily rectified as we can now bound the first integral by

$$\int_{2I_q^*}\left(\frac{2}{|2I_q^*|}\int_{2I_q^*}[w^{1/p}(y)w^{-1/p}(v)]^{p'}\mathrm{d}v\right)^{p/p'}\mathrm{d}y \le C_pA_p(w)|2I_q^*|$$

since the weight is defined everywhere.

For the $p = 1$ case, we notice that by Fubini

$$\begin{aligned}\int_{I_q^*}\left(\frac{1}{|I_q^*|}\int_{I_q^*}|f(v)|\mathrm{d}v\right)w(y)\mathrm{d}y &\le \frac{|2I_q^*|}{|I_q^*|}\int_{I_q^*}|f(v)|\frac{1}{|2I_q^*|}\int_{2I_q^*}w(y)\mathrm{d}y\mathrm{d}v\\ &\le 2A_1(w)\int_{I_q^*}|f(v)|w(v)\mathrm{d}v\end{aligned}$$

using (5.4).

In either case, each I_q^* is contained in 4 cubes $J_q \in \mathcal{W}_+$ (as before, in the "Averaging Method“). Therefore we at last have that

$$\begin{aligned}||g||^p_{L^p(\mathbb{R}_-,w)} &\le C_p\sum_{I_q-\in\mathcal{W}_-}C_pA_p(w)|2I_q^*|\frac{1}{|I_q^*|}\int_{I_q^*}|f(v)|^pw(v)\mathrm{d}v\\ &= C_{p,w}\sum_{I_q\in\mathcal{W}_-}2\int_{I_q^*}|f(v)|^pw(v)\mathrm{d}v\\ &\le C_{p,w}4\sum_{J_q\in\mathcal{W}_+}\int_{J_q}|f(v)|^pw(v)\mathrm{d}v\\ &= C_{p,w}||f||^p_{L^p(\mathbb{R}_+,w)}\end{aligned}$$

(D) Based on the derivatives of g as given in (6.4) we see that

$$\begin{aligned}\left|\frac{\partial^l g}{\partial y^l}(y)\right| &\le \int_{1/2}^{2}\left|\frac{\partial^l f}{\partial y^l}(-\lambda y)\right||(-\lambda)|^l\,|\phi(\lambda)|\,\mathrm{d}\lambda\\ &\le \frac{C_l}{|y|}\int_{-\frac{1}{2}y}^{-2y}\left|D^lf(v)\right|\mathrm{d}v\end{aligned}$$

for each $0 \leq l \leq k$. This is the derivative form of the average expression in (6.5). Because of this, we can now repeat any of the three arguments in part (C) to bound the derivatives in L^p norm.

(E) It then further follows that we can extend $f \in L^p_k(\mathbb{R}_+, w)$ given Corollary 5.1 and the argument provided in part (E) of the proof of Theorem 4.1. $\square$

6.2 Derivatives of Vector Valued Compositions

For the general dimensional case we will require a formula for taking higher order derivatives of a generic vector valued composition.

Lemma 6.2. *Let $F(x) = f(u(x))$ where $u = (u_1, \dots, u_n), x = (x_1, \dots, x_m)$. For α a multi-index and ℓ a positive integer, consider all length ℓ lists of nonzero multi-indices which sum to α:*

$$\mathcal{B}_{\alpha,\ell} = \{\{\beta_1, \dots, \beta_\ell\} : \beta_1 + \cdots + \beta_\ell = \alpha \text{ where } |\beta_j| > 0 \ \forall j = 1, \dots, \ell\}.$$

Then

$$D^\alpha F = \sum_{\ell=1}^{|\alpha|} \sum_{\{\beta_1,\dots,\beta_\ell\} \in \mathcal{B}_{\alpha,\ell}} \sum_{n_1=1}^{n} \cdots \sum_{n_\ell=1}^{n} \left(\prod_{r=1}^{\ell} D^{\beta_r} u_{n_r} \right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f \right). \tag{6.7}$$

Here e_i denotes the ith component unit vector. It is worth noting that because of the chain rules involved, the derivatives of f in a sense count the number of u_{n_r}'s in any given term.

Proof. We proceed by induction on $|\alpha|$. For $|\alpha| = 1$, $D^\alpha = \frac{\partial}{\partial x_i}$ for some $1 \leq i \leq m$. Thus

$$D^\alpha F = \sum_{j=1}^{n} D^{e_j} f D^{e_i} u_j.$$

Here, since $|\alpha| = 1$, the two outermost sums are absent and thus the product will be also. The only sum here corresponds to $j = n_1$ and thus the theorem holds in such a case.

Assume that (6.7) holds for some α. Consider a derivative of order $|\alpha| + 1$, that is, $\alpha' = \alpha + e_i$ for some $1 \leq i \leq m$. Then, by product rule,

$$\begin{aligned}
D^{\alpha'} F &= D^{e_i} \left[\sum \left(\prod_{r=1}^{\ell} D^{\beta_r} u_{n_r} \right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f \right) \right] \\
&= \sum \left(\sum_{r=1}^{\ell} D^{e_i + \beta_r} u_{n_r} \prod_{t \neq r} D^{\beta_t} u_{n_t} \right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f \right) \\
&\quad + \sum \left(\prod_{r=1}^{\ell} D^{\beta_r} u_{n_r} \right) \left(\sum_{n_{\ell+1}=1}^{n} D^{e_{n_{\ell+1}} + \sum_{s=1}^{\ell} e_{n_s}} f D^{e_i} u_{n_{\ell+1}} \right) \\
&= \sum_{\ell=1}^{|\alpha|} \sum_{\beta'_1 + \cdots + \beta'_\ell = \alpha'} \sum_{n_1=1}^{n} \cdots \sum_{n_\ell=1}^{n} \left(\prod_{r=1}^{\ell} D^{\beta'_r} u_{n_r} \right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f \right) \\
&\quad + \sum_{\ell=1}^{|\alpha|} \sum_{\beta_1 + \cdots + \beta_\ell = \alpha} \sum_{n_1=1}^{n} \cdots \sum_{n_\ell=1}^{n} \sum_{n_{\ell+1}=1}^{n} \left(D^{e_i} u_{n_{\ell+1}} \prod_{r=1}^{\ell} D^{\beta_r} u_{n_r} \right) \left(D^{\sum_{s=1}^{\ell+1} e_{n_s}} f \right) \\
&= \sum_{\ell=1}^{|\alpha'|} \sum_{\beta_1 + \cdots + \beta_\ell = \alpha'} \sum_{n_1=1}^{n} \cdots \sum_{n_\ell=1}^{n} \left(\prod_{r=1}^{\ell} D^{\beta_r} u_{n_r} \right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f \right)
\end{aligned}$$

by noting that we can go from $\beta_1 + \cdots + \beta_\ell = \alpha$ to

$$\beta_1 + \cdots + \beta_{j-1} + (\beta_j + e_i) + \beta_{j+1} + \cdots + \beta_\ell = \alpha + e_i = \alpha'$$

in the first sum and from $\beta_1 + \cdots + \beta_\ell = \alpha$ to

$$\beta_1 + \cdots + \beta_\ell + \beta_{\ell+1} = \alpha + e_i = \alpha'$$

where $\beta_{\ell+1} = e_i$ in the second sum. So the α' case holds. □

6.3 The General Result

We now can prove the general version of Theorem 6.1. Here the biggest transitions are from intervals to cubes and the additional complications related to the derivatives. It may also be useful to the reader to first consider the $n = 1$ case.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with smooth boundary. Given $f \in L_k^p(\Omega, w)$ with $1 \leq p < \infty$, $k = 0, 1, 2, \ldots$, and $w \in A_p$, there exists an extension $E(f) \in L_k^p(\mathbb{R}^{n+1}, w)$, that is, there exists $E(f)(x, y) : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $E(f)|_\Omega = f$ and*

$$||E(f)||_{L_k^p(\mathbb{R}^{n+1},w)} \leq C_{p,k,n,w}||f||_{L_k^p(\Omega,w)} \tag{6.8}$$

Proof. (A) As before we assume that $f \in C^\infty\left(\overline{\mathbb{R}_+^{n+1}}\right)$, that is, $\partial\Omega = \mathbb{R}^n$ (i.e. $\Gamma \equiv 0$). For such f, consider

$$g(x, y) = \int_{[-1,1]^n} \int_{1/2}^{2} f(x + \mu y, -\lambda y)\phi(\lambda)\mathrm{d}\lambda\, \psi(\mu)\mathrm{d}\mu \tag{6.9}$$

where $x \in \mathbb{R}^n$ and $y < 0$. Here $\mu = (\mu_1, \ldots, \mu_n)$ (so $x + \mu y = (x_1 + \mu_1 y, \ldots, x_n + \mu_n y)$ and $\mathrm{d}\mu = \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_n$) and $\psi(\mu) = \psi_1(\mu_1)\psi_2(\mu_2)\cdots\psi_n(\mu_n)$ where both ϕ and each ψ_j are smooth functions with compact supports within their limits of integration and the following moment properties:

$$\begin{array}{cc} \int \phi(\lambda)\mathrm{d}\lambda = 1, & \int \lambda^l \phi(\lambda)\mathrm{d}\lambda = (-1)^l \\ \int \psi_j(\mu_j)\mathrm{d}\mu_j = 1, & \int \mu_j^l \psi_j(\mu_j)\mathrm{d}\mu_j = 0 \end{array} \tag{6.10}$$

for $1 \leq l \leq k$ and $1 \leq j \leq n$. We then set

$$E(f) = \begin{cases} f(x, y) & y \geq 0 \\ g(x, y) & y < 0 \end{cases}.$$

(B) Here we must verify trace properties so as to use Lemma 3.1 to gain the differentiability of $E(f)$. First off, it is clear that by using (6.10) we have that $g(x,y) \to f(x,0)$ as $y \to 0$.

Now, let α be a multi-index. Since we assumed f to be smooth it follows that by considering $\alpha = (\alpha', \alpha_{n+1})$ we have that

$$D^{\alpha'} g(x,y) = \int_{[-1,1]^n} \int_{1/2}^{2} D^{\alpha'} f(x+\mu y, -\lambda y)\phi(\lambda)\mathrm{d}\lambda\, \psi(\mu)\mathrm{d}\mu.$$

For the derivatives in y we must use (6.7). Combining this with the above equation gives that

$$D^{\alpha} g(x,y) = \sum \int_{[-1,1]^n} \int_{1/2}^{2} \mu_1^{l_1} \cdots \mu_n^{l_n} (-\lambda)^{l_{n+1}} D^{(\alpha_1+l_1,\dots,\alpha_n+l_n,l_{n+1})} f(x+\mu y, -\lambda y)\mathrm{d}\lambda\mathrm{d}\mu \tag{6.11}$$

where the sum has a finite number of terms and $l_1 + \cdots + l_{n+1} = \alpha_{n+1}$ with each $l_i \geq 0$. Hence, there is a single integral term with $l_i = 0$ for all $i = 1, \dots, n$ so that $l_{n+1} = \alpha_{n+1}$. Consequently, this particular terms satisfies

$$\iint (-\lambda)^{\alpha_{n+1}} D^{(\alpha', \alpha_{n+1})} f(x+\mu y, -\lambda y)\mathrm{d}\lambda\mathrm{d}\mu \to D^{\alpha} f(x,0)$$

as $y \to 0$ by using (6.10). Otherwise, there exists $l_i > 0$ for some $i = 1, \dots, n$ and thus, by using (6.10), all other terms in the sum for (6.11) go to zero as $y \to 0$.

In summary, since f is assumed to be smooth we can calculate the derivatives of g which demonstrates that g is smooth on the lower half space with the necessary trace conditions. Thus we can use the existence of derivatives across the boundary to gain the necessary differentiability.

(C) As before, each of the three methods begins in the same way. We can make the immediate pointwise estimate using standard estimates and applying a change of variables from μ and λ to u and v, respectively:

$$|g(x,y)| \leq \frac{C}{|y|^{n+1}} \int_{x_n+y}^{x_n-y} \cdots \int_{x_1+y}^{x_1-y} \int_{-(1/2)y}^{-2y} |f(u_1, \ldots, u_n, v)| \mathrm{d}v \, \mathrm{d}u_1 \cdots \mathrm{d}u_n \tag{6.12}$$

From here we proceed to give the three different methods of proof.

(Averaging Method) Let $\mathcal{W}_-$ be the dyadic Whitney decomposition for the lower half space given by cubes of the form

$$Q_- = [q_1 2^q, (q_1+1)2^q] \times \cdots \times [q_n 2^q, (q_n+1)2^q] \times \left[-2^{q+1}, -2^q\right]$$

and further let $\mathcal{W}_+$ be its upper half space equivalent.

Fix $(x,y) \in Q_-$ where $Q_- \in \mathcal{W}_-$. Then there exists $q \in \mathbb{Z}$ and for each $1 \leq j \leq n$ there exists $q_j \in \mathbb{Z}$ such that $2^q \leq -y \leq 2^{q+1}$ and $q_j 2^q \leq x_j \leq (q_j+1)2^q$. Then

$$q_j 2^q - 2^{q+1} \leq x_j + y \leq x_j - y \leq q_j 2^q + 2^q + 2^{q+1}$$

which represents an interval of total length $2^q + 2^{q+2}$ and

$$2^{q-1} \leq -\frac{1}{2}y \leq -2y \leq 2^{q+2}$$

which represents an interval of total length $2^{q+2} - 2^{q-1}$. Adding $2^{q-1} + 2^q$ to the top limit then yields a new cube $Q^* \subset \mathbb{R}^{n+1}_+$ with sides of length $5(2^q)$. Explicitly,

$$Q^* = \left[q_1 2^q - 2^{q+1}, q_1 2^q + 2^q + 2^{q+1}\right] \times \cdots \times \left[q_n 2^q - 2^{q+1}, q_n 2^q + 2^q + 2^{q+1}\right]$$
$$\times \left[2^{q-1}, 2^{q+2} + 2^q + 2^{q-1}\right]$$

Also, $|y|^{n+1} \geq (2^q)^{n+1}$.

For such (x, y) we can then replace (6.12) by

$$
\begin{aligned}
|g(x,y)| &\leq \frac{C}{(2^q)^{n+1}} \int_{q_n 2^q - 2^{q+1}}^{q_n 2^q + 2^q + 2^{q+1}} \cdots \int_{q_1 2^q - 2^{q+1}}^{q_1 2^q + 2^q + 2^{q+1}} \int_{2^{q-1}}^{2^{q+2} + 2^q + 2^{q-1}} |f(u_1, \ldots, u_n, v)| \mathrm{d}v \mathrm{d}u \\
&= \frac{5^{n+1} C}{|Q^*|} \iint_{Q^*} |f(u,v)| \mathrm{d}u \mathrm{d}v
\end{aligned}
$$

Next, we can use the averaging equivalence (5.3) for A_p weights to get that

$$
\begin{aligned}
|g(x,y)|^p &\leq C_{p,n} \left(\frac{1}{|Q^*|} \iint_{Q^*} |f(u,v)| \mathrm{d}u \mathrm{d}v \right)^p \\
&\leq C_{p,n,w} \frac{1}{w(Q^*)} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v
\end{aligned}
$$

for all $(x, y) \in Q_-$. We then calculate the L^p norm by summing over all such cubes. Here we recall that w is also a doubling weight and that by doubling Q^* just once we can contain Q_-. Lastly, note that Q^* is contained in at most $5 \cdot 10^n$ cubes $Q_+ \in \mathcal{W}_+$. Thus

$$
\begin{aligned}
||g||^p_{L^p(\mathbb{R}^{n+1}_-, w)} &= \sum_{Q_- \in \mathcal{W}_-} \iint_{Q_-} |g(x,y)|^p w(x,y) \mathrm{d}y \mathrm{d}x \\
&\leq C_{p,n,w} \sum_{Q_- \in \mathcal{W}_-} \iint_{Q_-} \left(\frac{1}{w(Q^*)} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v \right) w(x,y) \mathrm{d}y \mathrm{d}x \\
&= C_{p,n,w} \sum_{Q_- \in \mathcal{W}_-} \frac{w(Q_-)}{w(Q^*)} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v \\
&\leq C_{p,n,w} \sum_{Q_- \in \mathcal{W}_-} C \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v \\
&\leq C_{p,n,w} \cdot 5 \cdot 10^n \sum_{Q_+ \in \mathcal{W}_+} \iint_{Q_+} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v \\
&= C_{p,n,w} ||f||^p_{L^p\left(\mathbb{R}^{n+1}_+, w\right)}
\end{aligned}
$$

We can further use this method to show that the $p = 1$ case must also hold because of the analogous averaging condition (5.6).

(Maximal Function Method) Consider the trivial extension of f to all of $\mathbb{R}^{n+1}$:

$$f_0(x,y) = \begin{cases} f(x,y) & y > 0 \\ 0 & y \leq 0 \end{cases}$$

We can then rewrite (6.5) using the maximal function over cubes $Q \subset \mathbb{R}^{n+1}$:

$$\begin{aligned}
|g(x,y)| &\leq \frac{C}{|y|^{n+1}} \int_{x_n+y}^{x_n-y} \cdots \int_{x_1+y}^{x_1-y} \int_{-(1/2)y}^{-2y} |f_0(u_1,\ldots,u_n,v)| \mathrm{d}v \, \mathrm{d}u_1 \cdots \mathrm{d}u_n \\
&\leq \frac{4^{n+1}C}{|4y|^{n+1}} \int_{x_n+2y}^{x_n-2y} \cdots \int_{x_1+2y}^{x_1-2y} \int_{2y}^{-2y} |f_0(u_1,\ldots,u_n,v)| \mathrm{d}v \, \mathrm{d}u_1 \cdots \mathrm{d}u_n \\
&\leq C_n \sup_{Q:(x,y)\in Q} \frac{1}{|Q|} \iint_Q |f_0(u,v)| \mathrm{d}v \mathrm{d}v \\
&= C_n M(f_0)(x,y)
\end{aligned}$$

Using (5.1) we thus have that

$$\begin{aligned}
||g||^p_{L^p\left(\mathbb{R}^{n+1}_-,w\right)} &= \int_{\mathbb{R}^n} \int_{-\infty}^0 |g(x,y)|^p \mathrm{d}y \mathrm{d}x \\
&\leq C_{p,n} \int_{\mathbb{R}^n} \int_{-\infty}^0 |M(f_0)(x,y)|^p \mathrm{d}y \mathrm{d}x \\
&\leq C_{p,n} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |M(f_0)(x,y)|^p \mathrm{d}y \mathrm{d}x \\
&\leq C_{p,n,w} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |f_0(x,y)|^p w(x,y) \mathrm{d}y \mathrm{d}x \\
&= C_{p,n,w} \int_{\mathbb{R}^n} \int_0^{\infty} |f(x,y)|^p w(x,y) \mathrm{d}y \mathrm{d}x \\
&= C_{p,n,w} ||f||^p_{L^p\left(\mathbb{R}^{n+1}_+,w\right)}.
\end{aligned}$$

(Direct Method) Recall the rearranged A_p condition which was given in (6.6):

$$\frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q [w^{1/p}(x) w^{-1/p}(t)]^{p'} \mathrm{d}t \right)^{p/p'} \mathrm{d}x \leq A_p(w) \tag{6.13}$$

where $Q \subset \mathbb{R}^{n+1}$ is a cube. Proceeding as we did in the "Averaging Method," fix a cube in a Whitney decomposition of the lower half space. Using that same notation, we then have that (by using Hölder's inequality on the inside integral)

$$\begin{aligned}
&||g||^p_{L^p\left(\mathbb{R}^{n+1}_-,w\right)} \\
&\leq C_{p,n} \sum_{Q_- \in \mathcal{W}_-} \iint_{Q_-} \left(\frac{1}{|Q^*|} \iint_{Q^*} |f(u,v)| \mathrm{d}u \mathrm{d}v \right)^p w(x,y) \mathrm{d}y \mathrm{d}x \\
&= C_{p,n} \sum_{Q_- \in \mathcal{W}_-} \iint_{Q_-} \left(\frac{1}{|Q^*|} \iint_{Q^*} |f(u,v)| w^{1/p}(x,y) \mathrm{d}u \mathrm{d}v \right)^p \mathrm{d}y \mathrm{d}x \\
&= C_{p,n} \sum_{Q_- \in \mathcal{W}_-} \iint_{Q_-} \left(\frac{1}{|Q^*|} \iint_{Q^*} w^{1/p}(x,y) w^{-1/p}(u,v) |f(u,v)| w^{1/p}(u,v) \mathrm{d}u \mathrm{d}v \right)^p \mathrm{d}y \mathrm{d}x \\
&\leq C_{p,n} \sum_{Q_- \in \mathcal{W}_-} \iint_{Q_-} \left\{ \begin{array}{l} \left(\frac{1}{|Q^*|} \iint_{Q^*} [w^{1/p}(x,y) w^{-1/p}(u,v)]^{p'} \mathrm{d}u \mathrm{d}v \right)^{p/p'} \\ \cdot \left(\frac{1}{|Q^*|} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v \right) \end{array} \right\} \mathrm{d}y \mathrm{d}x \\
&= C_{p,n} \sum_{Q_- \in \mathcal{W}_-} \left\{ \begin{array}{l} \iint_{Q_-} \left(\frac{1}{|Q^*|} \iint_{Q^*} [w^{1/p}(x,y) w^{-1/p}(u,v)]^{p'} \mathrm{d}u \mathrm{d}v \right)^{p/p'} \mathrm{d}y \mathrm{d}x \\ \cdot \left(\frac{1}{|Q^*|} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u \mathrm{d}v \right) \end{array} \right\}
\end{aligned}$$

where for each $Q_- \in \mathcal{W}_-$ (in the lower half space) the corresponding Q^* (in the upper half space) can be doubled to give that $Q_- \subset 2Q^*$. This comes into play because of the difficulty in that both averages in the A_p condition must be over the same cube! However, based on what has been said this is easily rectified as we can now bound the first integral by

$$\int_{2Q^*} \int_{2Q^*} \left(\frac{2^{n+1}}{|2Q^*|} \int_{2Q^*} [w^{1/p}(x,y) w^{-1/p}(u,v)]^{p'} \mathrm{d}v \mathrm{d}u \right)^{p/p'} \mathrm{d}y \mathrm{d}x \leq C_{p,n} A_p(w) |2Q^*|$$

since the weight is defined everywhere. Also, each Q^* is contained in at most $5 \cdot 10^n$ cubes $Q_+ \in \mathcal{W}_+$. Therefore we at last have that

$$\begin{aligned}
||g||^p_{L^p\left(\mathbb{R}^{n+1}_-,w\right)} &\leq C_{p,n} \sum_{Q_- \in \mathcal{W}_-} C_{p,n} A_p(w) |2Q^*| \frac{1}{|Q^*|} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u\mathrm{d}v \\
&= C_{p,n,w} \sum_{Q_- \in \mathcal{W}_-} 2^{n+1} \iint_{Q^*} |f(u,v)|^p w(u,v) \mathrm{d}u\mathrm{d}v \\
&\leq C_{p,n,w} 5 \cdot 10^n \sum_{Q_+ \in \mathcal{W}_+} \iint_{Q_+} |f(u,v)|^p w(u,v) \mathrm{d}u\mathrm{d}v \\
&= C_{p,n,w} ||f||^p_{L^p\left(\mathbb{R}^{n+1}_+,w\right)}.
\end{aligned}$$

Here the $p = 1$ case also follows as it did in one dimension.

(D) To bound $D^\alpha g$, we again consider (6.11):

$$D^\alpha g(x,y) = \sum \int_{[-1,1]^n} \int_{1/2}^2 \mu_1^{l_1} \cdots \mu_n^{l_n} (-\lambda)^{k_{n+1}} D^\beta f(x + \mu y, -\lambda y) \mathrm{d}\lambda \mathrm{d}\mu$$

where $l_i \geq 0$ for each $1 \leq i \leq n+1$ and $|\beta| = |\alpha|$. For each term of this sum, a similar estimate to (6.12) then follows:

$$\begin{aligned}
|D^\alpha g(x,y)| &\leq C_{\alpha,n} \int_{[-1,1]^n} \int_{1/2}^2 |D^\beta f(x + \mu y, -\lambda y)| \mathrm{d}\lambda \mathrm{d}\mu \\
&\leq \frac{C_{\alpha,n}}{|y|^{n+1}} \int_{x_n+y}^{x_n-y} \cdots \int_{x_1+y}^{x_1-y} \int_{-(1/2)y}^{-2y} |D^\beta f(u_1, \ldots, u_n, v)| \mathrm{d}v \, \mathrm{d}u_1 \cdots \mathrm{d}u_n
\end{aligned}$$

We can thus follow the approach of any of the three methods to finish bounding each term of the sum and hence the entire derivative. Thus g and all of its derivatives are bounded in the L^p norm. That is, given our assumptions on Ω and f, (6.8) is satisfied.

(E) As in the proof of Theorem 4.1, the details for the density argument, while omitted, will follow using Corollary 5.1.

(F) As the final step, since we have shown that the result holds on any half space (just shift the coordinate axes as needed), we can now show the result for our original smooth domain. To do so, we proceed as we did in part (F) of the proof of Theorem 4.1.

Define the extension operator:

$$E(f) = \Lambda_+ \left\{ \frac{\sum\limits_{i=1}^{\infty} \lambda_i E_i(\lambda_i f)}{\sum\limits_{i=1}^{\infty} \lambda_i^2} \right\} + \Lambda_- f \tag{6.14}$$

where, for $i = 1, 2, \ldots$, U_i are the open sets associated with the smooth domain Ω and Ω_i are the associated half spaces. Also $\lambda_i, \lambda_0, \lambda_+, \lambda_- : \mathbb{R}^{n+1} \to \mathbb{R}$ are all smooth with bounded derivatives independent of i such that

- supp $\lambda_i \subset U_i$ and $\lambda_i = 1$ on $U_i^{\epsilon/2}$
- supp $\lambda_o \subset \epsilon/2$ neighborhood of Ω and $\lambda_0 = 1$ on $\overline{\Omega}$
- supp $\lambda_+ \subset \epsilon$ neighborhood of $\partial\Omega$ and $\lambda_+ = 1$ if $\operatorname{dist}(x, \partial\Omega) \leq \epsilon/2$
- supp $\lambda_- \subset \Omega$ and $\lambda_- = 1$ if $\operatorname{dist}(x, \partial\Omega) \geq \epsilon/2$

and, with $\Lambda_+ = \lambda_0 \left(\frac{\lambda_+}{\lambda_+ + \lambda_-} \right)$ and $\Lambda_- = \lambda_0 \left(\frac{\lambda_-}{\lambda_+ + \lambda_-} \right)$,

- Λ_+ and Λ_- are smooth with bounded derivatives
- $\Lambda_+ + \Lambda_- = 1$ on $\overline{\Omega}$
- $\Lambda_+ + \Lambda_- = 0$ outside the $\epsilon/2$ neighborhood of Ω.

Based on the quotient rule we can write (6.14) in the form

$$E\left(f\right) = \sum_{i=1}^{\infty} H_i E_i \left(\lambda_i f\right) + \Lambda_- f.$$

Then

$$||E(f)||_{L_k^p(\mathbb{R}^{n+1},w)} \leq \sum_{|\alpha|\leq k} \left(\int_{\mathbb{R}^{n+1}} \left| D^\alpha \left(\sum_{i=1}^{\infty} H_i(x) E_i \left(\lambda_i f\right)(x) \right) \right|^p w(x)\mathrm{d}x \right)^{1/p} + \sum_{|\alpha|\leq k} ||D^\alpha \left(\Lambda_- f\right)||_{L^p(\mathbb{R}^{n+1},w)}$$

The latter term is clearly bounded by $C\,||f||_{L_k^p(\Omega,w)}$. Also, by Definition 2.4 (ii), $\sum_{i=1}^{\infty} \chi_{U_i} \leq N$ (also see (b) above). Thus the former is

$$\begin{aligned}
&\sum_{|\alpha|\leq k} \left(\int_{\mathbb{R}^{n+1}} \left| \sum_{i=1}^{\infty} \sum_{\beta+\gamma=\alpha} D^\beta H_i(x) D^\gamma E_i \left(\lambda_i f\right)(x) \right|^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq \sum_{|\alpha|\leq k} \left(\int_{\mathbb{R}^{n+1}} \left(\sum_{i=1}^{\infty} \chi_{U_i} \sum_{\beta+\gamma=\alpha} \left| D^\beta H_i(x) D^\gamma E_i \left(\lambda_i f\right)(x) \right| \right)^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq \sum_{|\alpha|\leq k} \left(\int_{\mathbb{R}^{n+1}} N^{p/p'} \sum_{i=1}^{\infty} \left(\sum_{\beta+\gamma=\alpha} \left| D^\beta H_i(x) D^\gamma E_i \left(\lambda_i f\right)(x) \right| \right)^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq C_{p,\Omega} \sum_{|\alpha|\leq k} \left(\int_{\mathbb{R}^{n+1}} \sum_{i=1}^{\infty} \sum_{\beta+\gamma=\alpha} \left| D^\beta H_i(x) D^\gamma E_i \left(\lambda_i f\right)(x) \right|^p w(x)\mathrm{d}x \right)^{1/p} \\
&= C_{p,k,\Omega} \sum_{|\alpha|\leq k} \left(\sum_{i=1}^{\infty} \sum_{\beta+\gamma=\alpha} \int_{U_i} |D^\beta H_i(x)|^p \, |D^\gamma E_i \left(\lambda_i f\right)(x)|^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq C_{p,k,\Omega} \sum_{|\alpha|\leq k} \left(\sum_{i=1}^{\infty} \sum_{\beta+\gamma=\alpha} \int_{U_i} |D^\gamma E_i \left(\lambda_i f\right)(x)|^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq C_{p,k,\Omega} \sum_{|\alpha|\leq k} \left(\sum_{i=1}^{\infty} \sum_{\beta+\gamma=\alpha} \sum_{|\kappa|\leq k} \int_{\Omega_i} |D^\kappa \left(\lambda_i f\right)(x)|^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq C_{p,k,\Omega} \sum_{|\alpha|\leq k} \left(\sum_{i=1}^{\infty} \sum_{\beta+\gamma=\alpha} \sum_{|\kappa|\leq k} \sum_{\epsilon+\delta=\kappa} \int_{U_i\cap\Omega} |D^\epsilon \lambda_i(x)| \left| D^\delta f(x) \right|^p w(x)\mathrm{d}x \right)^{1/p} \\
&\leq C_{p,k,\Omega} ||f||_{L_k^p(\Omega,w)}
\end{aligned}$$

via several applications of Hölder's inequality. □

Chapter 7

The Scalar A_p Weighted Lipschitz Domain Case

The proof of this extension result follows using the method developed in [18]. This was summarized near the end of Chapter 4. However, significant modification is now necessary due to the involvement of weights. The most technical of these issues is handled by the following results.

7.1 Estimates on Whitney Cubes

To define the extension operator, it is necessary to take derivatives of a distance function. The following lemma aids in this.

Lemma 7.1 ([18], p. 171, 182)**.** *Let $D \subset \mathbb{R}^{n+1}$ be a special Lipschitz domain with curve Γ and bound M. Denote $\delta(x,y) = dist\left((x,y), \overline{D}\right)$. Then there exists a function $\Delta(x,y) = \Delta\left((x,y), \overline{D}\right)$ defined in $\overline{D}^c$ such that the following hold:*

(a) There exist constants $c_1, c_2 > 0$ such that

$$c_1\delta(x,y) \leq \Delta(x,y) \leq c_2\delta(x,y).$$

(b) $\Delta \in C^\infty\left(\overline{D}^c\right)$ *where for some constants* $B_\alpha > 0$

$$|D^\alpha \Delta(x,y)| \leq B_\alpha\, \delta(x,y)^{1-|\alpha|}.$$

(c) *There exists* $c_M > 0$ *such that*

$$c_M \Delta(x,y) \geq \Gamma(x) - y.$$

We call such a function Δ the regularized distance. It is also useful to note that $\delta(x,y) \leq \Gamma(x) - y$. Namely, this all amounts to saying that $\Delta(x,y) \approx \delta(x,y) \approx \Gamma(x) - y$ for all $(x,y) \in \overline{D}^c$ where the constants for the second equivalence depend on M. It is useful to note that c_2 depends on n and explicitly $c_M = 5\sqrt{1+M^2}$.

One use of Lemma 3 is to acquire some useful facts when performing estimates involving Whitney cubes.

Lemma 7.2. *Let* $D \subset \mathbb{R}^{n+1}$ *be a special Lipschitz domain with Lipschitz graph* Γ *and bound* M*. Define for* $x \in \mathbb{R}^n$ *the upper cone with vertex* $(x, \Gamma(x))$ *and aperture* $A > 0$*:*

$$\Gamma_+^A(x) = \{(x', y') : y' - \Gamma(x) \geq A|x' - x|\}.$$

If $A > M$*, then* $\operatorname{int}(\Gamma_+^A(x)) \subset D$ *for all* $x \in \mathbb{R}^n$*. Furthermore, the following hold:*

(a) *For* $N \geq 2$ *and* c_1, c_2, c_M *be as in Lemma 3, let*

$$c_1' = \frac{1}{2c_2\sqrt{n}M}, \quad c_2' = Nc_M, \quad \text{and} \quad L = 2\frac{c_1'}{c_2'} + 1.$$

Then, for any $(x,y) \in \overline{D}^c$*,*

$$R(x,y) = [x_1 - c_1'\Delta, x_1 + c_1'\Delta] \times \cdots \times [x_n - c_1'\Delta, x_n + c_1'\Delta] \times [y + c_2'\Delta, y + Lc_2'\Delta]$$

is a cube contained in $\Gamma_+^{2M}(x,y)$ *where* $\Delta = \Delta(x,y)$*.*

(b) For

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \times [c, d]$$

a Whitney cube of $\overline{D}^c$, *we define*

$$Q^* = \left[a_1 - c_1'\overline{\Delta}(Q), b_1 + c_1'\overline{\Delta}(Q)\right] \times \cdots \times \left[a_n - c_1'\overline{\Delta}(Q), b_n + c_1'\overline{\Delta}(Q)\right]$$
$$\times \left[c + c_2'\underline{\Delta}(Q), d + c_2' L\overline{\Delta}(Q)\right]$$

where $\overline{\Delta}(Q) = \max_{(x,y)\in Q} \Delta(x, y)$ *and* $\underline{\Delta}(Q) = \min_{(x,y)\in Q} \Delta(x, y)$.
Then $\bigcup_{(x,y)\in Q} R(x, y) \subset Q^*$. *Moreover, if* $\{Q_k\}_{k=1}^{\infty}$ *is the Whitney decomposition for* $\overline{D}^c$, *then there exists sufficiently large* $N \geq 2$ *and a constant* $C_{n,M} > 0$ *such that* $\operatorname{card}(\{k \in \mathbb{N} : (x', y') \in Q_k^*\}) \leq C_{n,m}$ *for all* $(x', y') \in D$.

(c) We can further choose $N \geq 2$ *such that for each Whitney cube* Q *of* $\overline{D}^c$ *we have that* $Q^* \subset \Gamma_+^{2M}(x)$ *for all* $(x, y) \in Q$.

(d) For each Whitney cube Q *of* $\overline{D}^c$ *there exists a cube* $Q^{**} \supset Q \cup Q^*$ *such that* $\frac{|Q^{**}|}{|Q^*|} \leq C_{n,M}$ *for some constant* $C_{n,M} > 0$.

Proof. (a) Clearly, $R(x, y)$ is a cube. To see that it is indeed contained in $\Gamma_+^{2M}(x)$, consider $(x', y') \in R(x, y)$. Then via Lemma 3 (c),

$$y' \geq y + c_2'\Delta = y + Nc_M\Delta \geq y + N(\Gamma(x) - y) \geq y + 2(\Gamma(x) - y) = \Gamma(x) + \Gamma(x) - y,$$

which guarantees that $y' - \Gamma(x) \geq \Gamma(x) - y$ and hence $y' > \Gamma(x)$. Thus we also have that for each component x_j' of x'

$$|x_j - x_j'| \leq c_1'\Delta = \frac{\Delta}{c_2\sqrt{n}2M} \leq \frac{\delta}{\sqrt{n}2M} \leq \frac{\Gamma(x) - y}{\sqrt{n}2M} \leq \frac{y' - \Gamma(x)}{\sqrt{n}2M}.$$

So $2M|x - x'| \leq y' - \Gamma(x)$ and hence $(x', y') \in \Gamma_+^{2M}(x)$.
(b) For $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \times [c, d]$ we have that $\underline{\Delta}(Q) \leq \Delta(x, y) \leq \overline{\Delta}(Q)$ for all $(x, y) \in Q$. Based on the definition of $R(x, y)$ in part (a) and the definition of Q^*

it then follows that $\bigcup_{(x,y)\in Q} R(x,y) \subset Q^*$.

Now, let $(x',y') \in D$. Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \times [c,d]$ be a Whitney cube for $\overline{D}^c$ with side length $l = d - c = b_j - a_j$ for $j = 1, \ldots, n$ such that $(x', y') \in Q^*$. Let $(x,y) \in Q$. Then, by Lemma 3,

$$l \approx \delta(x,y) \approx \Gamma(x) - y \approx \Delta(x,y) \approx \overline{\Delta}(Q) \approx \underline{\Delta}(Q).$$

Namely, in addition to c_1 and c_2 in Lemma 3, there exists $c_3, c_4 > 0$ depending only on n and M such that

$$c_3 l \leq \underline{\Delta}(Q) \leq \Delta(x,y) \leq \overline{\Delta}(Q) \leq c_4 l.$$

It also follows that for $j = 1, \ldots, n$,

$$\begin{aligned} |x_j - x_j'| &\leq b_j - a_j + c_1' \overline{\Delta}(Q) \\ &= l + c_1' \overline{\Delta}(Q) \\ &\leq (1 + c_1' c_4) l. \end{aligned}$$

So $|x - x'| \leq c_5 l$ where $c_5 = (1 + c_1' c_4)\sqrt{n}$.

It also follows that

$$\begin{aligned} y' - \Gamma(x') &\leq d + c_2' L \overline{\Delta}(Q) - \Gamma(x') \\ &\leq d - y + y - \Gamma(x) + |\Gamma(x) - \Gamma(x')| + c_2' L \overline{\Delta}(Q) \\ &\leq l + M|x - x'| + c_2' c_4 L l \\ &\leq c_6 l \end{aligned}$$

since $y < \Gamma(x)$. Here $c_6 = c_5 M + c_2' c_4 L + 1$. Also,

$$y' - \Gamma(x') \geq c + c_2' \underline{\Delta}(Q) - \Gamma(x') \geq N c_3 c_M l - (\Gamma(x') - c).$$

But, using Lemma 3(c),

$$\begin{aligned}\Gamma(x') - c &\leq |\Gamma(x') - \Gamma(x)| + \Gamma(x) - y + y - c \\ &\leq Mc_5 l + c_M \Delta(x,y) + l \\ &\leq c_7 l\end{aligned}$$

where $c_7 = c_5 M + c_4 c_M + 1$. So $y' - \Gamma(x') \geq (Nc_3 c_M - c_7)l$.
Choosing $N \geq \frac{c_7+1}{c_3 c_M}$ (where $c_7+1 = \sqrt{n}M + \frac{c_4}{2c_2} + c_4 c_M + 2$) then gives that $y' - \Gamma(x') \geq l$.
Hence

$$\begin{aligned}|y' - y| &\leq y' - \Gamma(x') + |\Gamma(x') - \Gamma(x)| + \Gamma(x) - y \\ &\leq y' - \Gamma(x') + c_5 Ml + c_4 c_M l \\ &\leq c_7 (y' - \Gamma(x'))\end{aligned}$$

and

$$|x - x'| \leq c_5 (y' - \Gamma(x')).$$

So

$$(x,y) \in \begin{array}{c}[x_1' - c_5(y' - \Gamma(x')), x_1' + c_5(y' - \Gamma(x'))] \times \cdots \\ \times [x_n' - c_5(y' - \Gamma(x')), x_n' + c_5(y' - \Gamma(x'))] \\ \times [y' - c_7(y' - \Gamma(x')), y' + c_7(y' - \Gamma(x'))]\end{array} = S.$$

Now let

$$\mathcal{Q} = \{Q : Q \text{ is a Whitney cube for } \overline{D}^c \text{ and } (x', y') \in Q^*\}.$$

Then $\bigcup_{Q \in \mathcal{Q}} Q \subset S$. So

$$\left| \bigcup_{Q \in \mathcal{Q}} Q \right| \leq |S| = 2^{n+1} c_5^n c_7 (y' - \Gamma(x'))^{n+1}$$

Also, $|Q| = l^{n+1} \geq \frac{1}{c_6^{n+1}}(y' - \Gamma(x'))^{n+1}$ for all $Q \in \mathcal{Q}$. Thus

$$\begin{aligned} 2^{n+1}c_5^n c_7(y' - \Gamma(x'))^{n+1} &\geq \left| \bigcup_{Q \in \mathcal{Q}} Q \right| \\ &= \sum_{Q \in \mathcal{Q}} |Q| \\ &\geq \operatorname{card}(\mathcal{Q}) \cdot \min_{Q \in \mathcal{Q}} |Q| \\ &\geq \operatorname{card}(\mathcal{Q}) \cdot \frac{1}{c_6^{n+1}}(y' - \Gamma(x'))^{n+1}. \end{aligned}$$

Hence $\operatorname{card}(\mathcal{Q}) \leq 2^{n+1}c_5^n c_6^{n+1} c_7$. Therefore the number of Whitney cubes Q with (x', y') contained in Q^* is bounded independent of (x', y').

(c) For $(x, y) \in Q$ and $(x', y') \in Q^*$, with Q and Q^* as in part (b), we have that

$$\begin{aligned} \frac{y' - \Gamma(x)}{|x' - x|} &\geq \frac{c + c_2' \underline{\Delta}(Q) - \Gamma(x)}{c_5 l} \\ &\geq \frac{N c_3 c_M}{c_5} - \frac{\Gamma(x) - c}{c_5 l}. \end{aligned}$$

But

$$\Gamma(x) - c \leq \Gamma(x) - y + y - c \leq (c_4 c_M + 1)l.$$

So

$$\frac{y' - \Gamma(x)}{|x' - x|} \geq \frac{N c_3 c_M - c_4 c_M - 1}{c_5}.$$

Choosing N large such that the above is greater than $2M$ then gives that $N \geq \frac{2c_5 M + c_4 c_M + 1}{c_3 c_M}$ where the numerator is explicitly $2\sqrt{n}M + \frac{c_4}{c_2} + c_4 c_M + 1$.

(d) Let Q and Q^* be as in (b). Define

$$\begin{aligned} Q^{**} = &\left[\frac{a_1 + b_1}{2} - \frac{d - c + c_2' L \overline{\Delta}(Q)}{2}, \frac{a_1 + b_1}{2} + \frac{d - c + c_2' L \overline{\Delta}(Q)}{2} \right] \times \cdots \\ &\times \left[\frac{a_n + b_n}{2} - \frac{d - c + c_2' L \overline{\Delta}(Q)}{2}, \frac{a_n + b_n}{2} + \frac{d - c + c_2' L \overline{\Delta}(Q)}{2} \right] \\ &\times \left[c, d + c_2' L \overline{\Delta}(Q) \right]. \end{aligned}$$

Then Q^{**} is clearly a cube with $[c,d]\cup[c+c_2'\underline{\Delta}(Q), d+c_2'L\overline{\Delta}(Q)]\subset\big[c, d+c_2'L\overline{\Delta}(Q)\big]$. Also, for each $j=1,\dots,n$, the conditions

$$\left|x_j-\frac{a_j+b_j}{2}\right|\le\frac{b_j-a_j}{2},\qquad\left|x_j-\frac{a_j+b_j}{2}\right|\le\frac{b_j-a_j+2c_1'\overline{\Delta}(Q)}{2},$$
$$\text{and}\qquad\left|x_j-\frac{a_j+b_j}{2}\right|\le\frac{d-c+c_2'L\overline{\Delta}(Q)}{2},$$

characterize the x_j-coordinate of x for $x\in Q$, $x\in Q^*$, and $x\in Q^{**}$, respectively. Furthermore,

$$b_j-a_j\le b_j-a_j+2c_1'\overline{\Delta}(Q)=d-c+c_2'(L-1)\overline{\Delta}(Q)\le d-c+c_2'L\overline{\Delta}(Q).$$

Thus $Q\cup Q^*\subset Q^{**}$. Moreover, since

$$c_2'L\overline{\Delta}(Q)-c_2'\underline{\Delta}(Q)\ge c_2'(L-1)\overline{\Delta}(Q)=2c_1'\overline{\Delta}(Q)\ge 2c_1'c_3l,$$

it follows that $|Q^*|\ge(l+2c_1'c_3l)^{n+1}$. Also $|Q^{**}|\le(l+c_2'Lc_4l)^{n+1}$. So

$$\frac{|Q^{**}|}{|Q^*|}\le\frac{(1+c_2'Lc_4)^{n+1}}{(1+2c_1'c_3)^{n+1}}.$$

□

It is worth noting that by letting N_1 be as in part (b) and N_2 as in part (c), we have that $N_1-N_2=\frac{2c_2-c_4-2\sqrt{n}M}{2c_2c_3c_M}$. Thus, if $2c_2+2\sqrt{n}M\ge c_4$, then N_1 will suffice for both estimates and N_2 will suffice when $c_4\ge 2c_2+2\sqrt{n}M$. From now on, however, consider N as the maximum of the $N>0$ chosen in parts (b) and (c).

Also, for what follows, it is important to note that

$$|Q^*|=(l+2c_1'c_2'L\overline{\Delta})^n(l+c_2'(L\overline{\Delta}-\underline{\Delta}))\le\left(\frac{1}{c_3}+2c_1'\right)^n\left(\frac{1}{c_3}+c_2'L\right)\overline{\Delta}^{n+1}.\qquad(7.1)$$

7.2 The Result

Theorem 7.1. *Let D be a Lipschitz domain, $1 \leq p < \infty$, $k = 0, 1, 2, \ldots$, and $w \in A_p$. Given $f \in L_k^p(D, W)$ there exists an extension $E(f) \in L_k^p(\mathbb{R}^n, w)$, that is, there exists $E(f) : \mathbb{R}^n \to \mathbb{R}$ such that $E(f)|_D = f$ and*

$$||E(f)||_{L_k^p(\mathbb{R}^n, w)} \leq C_{p,k,D,w} ||f||_{L_k^p(D,w)} . \tag{7.2}$$

To demonstrate this result, we must still modify the approach laid out in [18] which was summarized while proving Theorem 4.2. These further modifications result from the fact that the original approach was represented as an integral over a line while the A_p condition corresponds to cubes. Thus we define our extension operator as an integration over a specific region which yields different estimates.

Proof. (A) Assume (as in all cases) that $f \in C_c^\infty\left(\overline{D}\right)$ where $D \subset \mathbb{R}^{n+1}$ is a special Lipschitz domain with Lipschitz graph Γ and bound M.
Now we can explicitly define our extension. Let $(x, y) \in \overline{D}^c$. Consider the function

$$g(x, y) = \int_{[-1,1]^n} \int_1^L f(x + c_1' \mu \Delta(x, y), y + c_2' \lambda \Delta(x, y))\, \phi(\lambda) \mathrm{d}\lambda\, \psi(\mu) \mathrm{d}\mu \tag{7.3}$$

with $x = (x_1, \ldots, x_n)$, $\mu = (\mu_1, \ldots, \mu_n)$, $\mathrm{d}\mu = \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_n$, and $\psi(\mu) = \psi_1(\mu_1) \cdots \psi_n(\mu_n)$. Here the $\phi, \psi_j \in C^\infty$ are compactly supported in $[1, L]$ and $[-1, 1]$, respectively, and satisfy the moment conditions

$$\begin{aligned} \textstyle\int \phi(\lambda) \mathrm{d}\lambda = 1, &\quad \textstyle\int \lambda^l \phi(\lambda) \mathrm{d}\lambda = 0 \\ \textstyle\int \psi_j(\mu_j) \mathrm{d}\mu_j = 1, &\quad \textstyle\int \mu^l \psi_j(\mu_j) \mathrm{d}\mu_j = 0 \end{aligned} \tag{7.4}$$

for all $1 \leq j \leq n$ and $1 \leq l \leq k$. Also, the constants c_1', c_2', L are chosen as in Lemma 7.2 (a) making g well-defined. We then define

$$E(f) = \begin{cases} f(x,y) & (x,y) \in \overline{D} \\ g & (x,y) \in \overline{D}^c \end{cases} \tag{7.5}$$

(B) In computing the derivatives, we use (6.7) to get that

$$D^\alpha g = \sum_{\ell=1}^{|\alpha|} \sum_{\{\beta_1,\dots,\beta_\ell\}\in\mathcal{B}_{\alpha,\ell}} \sum_{n_1=1}^{n+1} \cdots \sum_{n_\ell=1}^{n+1} \iint \prod_{r=1}^{\ell} \left(D^{\beta_r} x_{n_r} + c' \mu_{n_r} D^{\beta_r} \Delta\right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f\right) \phi\psi \, \mathrm{d}\mu \tag{7.6}$$

where $\mu_{n+1} = \lambda$ and $c' = c_1'$ or c_2' as the case may be.

By assuming the smoothness of f, we thus know that g will also be smooth. To gain the weak differentiability of $E\left(\overrightarrow{f}\right)$ in then follows by Lemma 4.1 that we need only show that the following trace property is satisfied for all derivatives α up to order k:

$$D^\alpha g(x,y) \to D^\alpha f(x_0,y_0) \text{ as } (x,y) \to (x_0,y_0) \text{ for any } (x_0,y_0) = (x_0,\Gamma(x_0)) \tag{7.7}$$

Since $\delta \approx \Delta$ and using (7.4) it follows that (7.7) holds for $\alpha = 0$. Assume $1 \leq |\alpha| \leq k$. Then, in a similar way, each derivative $D^\alpha g$ will have a term of the form $\iint D^\alpha f \phi\psi$ which will satisfy (7.7). Explicitly, if $\sum_{s=1}^{\ell} e_{n_s} = \alpha$, then $\ell = |\alpha|$. Thus $\alpha = \sum_{r=1}^{\ell} \beta_r$ and thus for each r there exists a t such that $\beta_r = e_{n_t}$. Also, exactly one set of n_r will be such that $n_r = n_t$. When this occurs, $D^{\beta_r} x_{n_r} = 1$ for all r and thus the leading term in the product in (7.6) will yield a single term of 1.

So it remains to show that all other terms must go to zero. By fixing α, ℓ, $\beta_1, \dots, \beta_\ell$, and $n_1, \dots, n_\ell$ we have that the terms under the integral in (7.6) all have the form

$$D^\gamma f \prod_{r=1}^{\ell} D^{\beta_r}[x_{n_r} + c' \mu_{n_r} \Delta].$$

We need only examine the highest order derivatives of $\mu_{n_r}\Delta$ since bounding the lower order terms follows similarly. Thus, by writing $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = (\alpha', \alpha_{n+1})$, and, similarly, $\beta_r = ((\beta_r)', (\beta_r)_{n+1})$ for each $1 \le r \le \ell$, and $\gamma = (\gamma', \gamma_{n+1})$ it suffices to consider

$$D^\gamma f \prod_{r=1}^{\ell} c' \mu_{n_r} D^{\beta_r} \Delta = C\, D^{(\gamma', \gamma_{n+1})} f\, \mu^{\gamma'} \lambda^{\gamma_{n+1}} D^{\beta_1}\Delta \cdots D^{\beta_\ell}\Delta \tag{7.8}$$

where $|\gamma| = \ell \ge 1$ and $\sum_{r=1}^{\ell} (\beta_r)_i = \alpha^i$ for each $1 \le i \le n+1$ because of the product and chain rules involved in calculating (7.6). (See Lemma 6.2) For simplicity, denote

$$\mathcal{R}_\alpha = D^{\beta_1}\Delta \cdots D^{\beta_l}\Delta$$

for this combination of regularized distance derivatives. It is important to note here that by Lemma 7.1 (b)

$$|\mathcal{R}_\alpha| \le B_\alpha \delta^{1-|\beta_1|} \cdots \delta^{1-|\beta_l|} = B_\alpha \delta^{|\gamma| - |\alpha|} \tag{7.9}$$

which is singular for exponents $|\gamma| \le |\alpha| - 1$ as we approach the boundary.

Since the term $D^\alpha f$ occurs only in the lower order terms, we can perform a Taylor expansion about $(\mu, \lambda) = (0, 1)$:

$$\begin{aligned} &D^\gamma f(x + c_1'\mu\Delta, y + c_2'\lambda\Delta) \\ &\quad = \sum_{|\zeta| \le |\alpha| - |\gamma|} \frac{D^{\zeta+\gamma} f(x, y + c_2'\Delta)(c_1'\Delta)^{|\zeta'|}(c_2'\Delta)^{\zeta_{n+1}}}{\zeta!} \mu^{\zeta'} (\lambda - 1)^{\zeta_{n+1}} \\ &\qquad + \sum_{|\zeta| = |\alpha| - |\gamma| + 1} \frac{D^{\zeta+\gamma} f\left(\theta(x, y, \mu, \lambda)\right)(c_1'\Delta)^{|\zeta'|}(c_2'\Delta)^{\zeta_{n+1}}}{\zeta!} \mu^{\zeta'} (\lambda - 1)^{\zeta_{n+1}}. \end{aligned}$$

for some θ on the the line segment between $(x, y + c_2'\Delta)$ and $(x + c_1'\mu\Delta, y + c_2'\lambda\Delta)$. Thus integrating (7.8) yields

$$\iint D^\gamma f\, \mu^{\gamma'} \lambda^{\gamma_{n+1}} \mathcal{R}_\alpha \psi(\mu)\phi(\lambda)\mathrm{d}\mu\mathrm{d}\lambda$$
$$= \sum_{|\zeta|\le|\alpha|-|\gamma|} \left\{ \begin{array}{l} \frac{(c_1'\Delta)^{|\zeta'|}(c_2'\Delta)^{\zeta_{n+1}}\mathcal{R}_\alpha}{\zeta!} D^{\zeta+\gamma} f\,(x, y + c_2'\Delta) \\ \cdot \int \mu^{\gamma'+\zeta'}\psi(\mu)\mathrm{d}\mu \int \lambda^{\gamma_{n+1}}\,(\lambda-1)^{\zeta_{n+1}}\,\phi(\lambda)\mathrm{d}\lambda \end{array} \right\} + O(\Delta^{|\alpha|-|\gamma|+1}\mathcal{R}_\alpha)$$

since ψ and ϕ are smooth with compact support and f is bounded on the boundary Γ. Due to the moment conditions in (7.4) all of the sum terms are identically zero. The remainder also goes to zero as $(x, y) \to (x_0, y_0)$ using (7.9) since $|\Delta|^{|\alpha|-|\gamma|+1} |\mathcal{R}_\alpha| \le B_\alpha\delta$.

(C) Now that we have the derivatives, we must bound each one in the L^p norm. First let $1 < p < \infty$ and consider bounding g pointwise. Fix $(x, y) \in \overline{D}^c$. Then, by applying the change of variables $u = x + c_1'\mu\Delta$, $v = y + c_2'\lambda\Delta$ we have that $\mathrm{d}u = (c_1'\Delta)^n\mathrm{d}\mu$, $\mathrm{d}v = c_2'\Delta\mathrm{d}\lambda$ and thus

$$\begin{aligned} |g(x, y)| &= \left| \iint f(u, v)\phi\left(\frac{v-y}{c_2'\Delta}\right)\frac{\mathrm{d}v}{c_2'\Delta}\,\psi\left(\frac{u-x}{c_1'\Delta}\right)\frac{\mathrm{d}u}{(c_1'\Delta)^n}\right| \\ &\le \frac{C_{n,M}}{|\Delta|^{n+1}}\iint_{R(x,y)} |f(u, v)|\,\mathrm{d}v\mathrm{d}u \end{aligned} \tag{7.10}$$

where $R(x, y)$ is the cube defined in Lemma 7.2(a). Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \times [c, d]$ be a Whitney cube for $\overline{D}^c$. Fix $(x, y) \in Q$. Letting $l = b_j - a_j = d - c$ for all $1 \le j \le n$ it follows that $l \approx \delta^*(x, y) \approx \Delta(x, y) \approx \overline{\Delta}(Q) \approx \underline{\Delta}(Q)$. Thus, using (7.1), we have that

$$|g(x, y)|^p \le C_{p,n,M}\left(\frac{1}{|Q^*|}\iint_{Q^*} |f(u, v)|\,\mathrm{d}v\mathrm{d}u\right)^p$$

where $\bigcup_{(x,y)\in Q} R(x, y) \subset Q^* \subset D$ is given by Lemma 7.2 (b) and 7.2 (c).

By Lemma 7.1 (d), Q^* can be enlarged to a cube Q^{**} containing Q and whose size is equivalent to that of Q^*. Thus

$$\begin{aligned}\iint_Q \left(\frac{1}{|Q^*|}\iint_{Q^*}\left[w^{\frac{1}{p}}(x,y)w^{-\frac{1}{p}}(u,v)\right]^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\mathrm{d}y\mathrm{d}x \\ \leq \iint_{Q^{**}}\left(\frac{C_{n,M}}{|Q^{**}|}\iint_{Q^{**}}\left[w^{\frac{1}{p}}(x,y)w^{-\frac{1}{p}}(u,v)\right]^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\mathrm{d}y\mathrm{d}x \\ \leq C_{p,n,M}A_p(w)|Q^{**}|\end{aligned}$$

by applying the re-imagined A_p condition (6.13). Thus, we have a "variable switching property:"

$$\begin{aligned}&\iint_Q \left(\frac{1}{|Q^*|}\iint_{Q^*}|f(u,v)|\,\mathrm{d}v\mathrm{d}u\right)^p w(x,y)\mathrm{d}y\mathrm{d}x \\ &\leq \iint_Q\left(\frac{1}{|Q^*|}\iint_{Q^*}\left[w^{\frac{1}{p}}(x,y)w^{-\frac{1}{p}}(u,v)\right]|f(u,v)|\,w^{\frac{1}{p}}(u,v)\mathrm{d}v\mathrm{d}u\right)^p\mathrm{d}y\mathrm{d}x \\ &\leq \iint_Q\left(\frac{1}{|Q^*|}\iint_{Q^*}\left|\left[w^{\frac{1}{p}}(x,y)w^{-\frac{1}{p}}(u,v)\right]^{p'}\mathrm{d}v\mathrm{d}u\right)^{p/p'}\frac{1}{|Q^*|}\,||f||^p_{L^p(Q^*,w)}\,\mathrm{d}y\mathrm{d}x \\ &\leq C_{n,M}A_p(w)\,||f||^p_{L^p(Q^*,w)}\end{aligned}$$

By Lemma 7.1 (b), each point of D is contained in at most $C_{n,M} < \infty$ regions Q^*. Letting $\{Q_i\}_{i=1}^{\infty}$ be the Whitney decomposition for $\overline{D}^c$, it then follows that

$$\begin{aligned}
||g||^p_{L^p\left(\overline{D}^c, w\right)} &= \sum_{i=1}^{\infty} ||g||^p_{L^p(Q_i,w)} \\
&\leq C_{p,n,M} \sum_{i=1}^{\infty} \iint_{Q_i} \left(\frac{1}{|Q_i^*|} \iint_{Q_i^*} |f(u,v)| \, \mathrm{d}v\mathrm{d}u \right)^p w(x,y)\mathrm{d}x\mathrm{d}y \\
&\leq C_{p,n,M,w} \sum_{i=1}^{\infty} \iint_{Q_i^*} |f(u,v)|^p \, w(u,v)\mathrm{d}u\mathrm{d}v \\
&= C_{p,n,M,w} \sum_{i=1}^{\infty} \iint_D \chi_{Q_i^*} \, |f(u,v)|^p \, w(u,v)\mathrm{d}u\mathrm{d}v \\
&\leq C_{p,n,M,w} \iint_D |f(u,v)|^p \, w(u,v)\mathrm{d}u\mathrm{d}v \\
&= C_{p,n,M,w} ||f||^p_{L^p(D,w)}
\end{aligned}$$

The variable switching property given above also works when $p = 1$:
By an application of Fubini's Theorem (and using the same notation)

$$\int_Q \left(\frac{1}{|Q^*|} \int_{Q^*} |f(t)| \, \mathrm{d}t \right) w(x)\mathrm{d}x \leq \frac{|Q^{**}|}{|Q^*|} \int_{Q^*} |f(t)| \left(\frac{1}{|Q^{**}|} \int_{Q^{**}} w(x)\mathrm{d}x \right) \mathrm{d}t$$

where Q is a cube in $\mathbb{R}^{n+1}$. Thus using (5.4)

$$\int_Q \left(\frac{1}{|Q|} \int_Q |f(t)| \, \mathrm{d}t \right) w(x)\mathrm{d}x \leq C_{n,M} A_1(w) \, ||f||_{L^1(Q^*,w)} \, .$$

So we can bound g in the L^p norm for all $1 \leq p < \infty$.
(D) Next we aim to bound the derivatives in a like manner. In doing so we can again assume that our derivative has the form of (7.8), that is,

$$\mathcal{R}_\alpha(x,y) \int_{[-1,1]^n} \int_1^L D^{(\gamma', \gamma_{n+1})} f(x + c_1' \mu \Delta(x,y), y + c_2' \lambda \Delta(x,y)) \mu^{\gamma'} \lambda^{\gamma_{n+1}} \phi(\lambda) \mathrm{d}\lambda \psi(\mu) \mathrm{d}\mu \tag{7.11}$$

where $1 \leq |\gamma| \leq |\alpha|$. This follows since the lower order derivative terms will be bounded by similar arguments and we have already bounded the $\gamma = 0$ term.
It is important to note that $|\mathcal{R}_\alpha| \leq B_\alpha \delta^{1-|\beta_1|} \cdots \delta^{1-|\beta_l|} = B_\alpha \delta^{l-|\alpha|}$. Thus, if $|\gamma| = |\alpha|$, then $|\mathcal{R}_\alpha| = B_\alpha$. Hence, we recover a pointwise bound similar to (7.10):

$$\begin{aligned}
&\left|\mathcal{R}_\alpha \iint D^\gamma f(\cdot,\cdot)\mu^{\gamma'}\lambda^{\gamma_{n+1}}\phi\psi \mathrm{d}\lambda \mathrm{d}\mu\right| \\
&\qquad \leq |\mathcal{R}_\alpha| \iint |D^\gamma f(x + c_1' u\Delta, y + c_2' v\Delta)| \, |\mu|^{\gamma'} \, |\lambda|^{\gamma_{n+1}} \, |\psi(\mu)| \, |\phi(\lambda)| \, \mathrm{d}\lambda \mathrm{d}\mu \\
&\qquad \leq C_{\alpha,n,M} \frac{1}{|\Delta|^{n+1}} \iint_{R(x,y)} |D^\gamma f(u,v)| \, \mathrm{d}v \mathrm{d}u
\end{aligned}$$

since $\psi, \phi \in C_c^\infty$. Thus we have an estimate similar to (7.10) with $\overrightarrow{f}$ replaced by $D^\gamma \overrightarrow{f}$ and thus we can apply the Whitney cube estimates to finish bounding this particular derivative.
If $|\gamma| \leq |\alpha|$, we need $\alpha| - |\gamma|$ powers of Δ to cancel the powers in the estimate (7.9) for $\mathcal{R}_\alpha$. Since $|\gamma| \geq 1$, there exists $j \in \{1, \ldots, n+1\}$ such that $\gamma_j \geq 1$. For such j, we then do a Taylor expansion while employing the integral form of the remainder which produces the powers of Δ necessary for bounding $\mathcal{R}_\alpha$ as follows:
If $j \in \{1, \ldots, n\}$, then by expanding about $\mu_j = -1$ we have that

$$\begin{aligned}
&D^\gamma f(\cdot, x_j + c_2' \mu_j \Delta, \cdot) \\
&\quad = \sum_{\ell=0}^{|\alpha|-|\gamma|-1} \frac{(c_1'\Delta)^\ell (\mu_j + 1)^\ell}{\ell!} D^{\gamma + \ell e_j} f(\cdot, x_j - c_1'\Delta, \cdot) \\
&\qquad + \frac{(c_1'\Delta)^{|\alpha|-|\gamma|}}{(|\alpha| - |\gamma| - 1)!} \int_{-1}^{\mu_j} D^{\gamma + (|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1' u\Delta, \cdot)(\mu_j - u)^{|\alpha|-|\gamma|-1} \mathrm{d}u
\end{aligned}$$

Thus the integration with respect to μ_j in (7.11) becomes

$$\begin{aligned}
&\int_{-1}^{1} D^{\gamma} f(\cdot, x_j + c_1'\mu_j\Delta, \cdot)\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j \\
&= \sum_{\ell=0}^{|\alpha|-|\gamma|-1} \frac{(c_1'\Delta)^{\ell}}{\ell!} D^{\gamma+\ell e_j} f(\cdot, x_j - c_1'\Delta, \cdot) \int_{-1}^{1} (\mu_j + 1)^{\ell}\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j \\
&+ \frac{(c_1'\Delta)^{|\alpha|-|\gamma|}}{(|\alpha| - |\gamma| - 1)!} \int_{-1}^{1}\int_{-1}^{\mu_j} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1' u\Delta, \cdot)(\mu_j - u)^{|\alpha|-|\gamma|-1}\mathrm{d}u\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j
\end{aligned}$$

Here the sum term is identically zero using the moment conditions in (7.4) since $\gamma_j \geq 1$. Moreover, we can now apply Fubini's Theorem to get that

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{\mu_j} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1' u\Delta, \cdot)(\mu_j - u)^{|\alpha|-|\gamma|-1}\mathrm{d}u\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j \\
&\qquad = \int_{-1}^{1} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1' u\Delta, \cdot) \int_{u}^{1} (\mu_j - u)^{|\alpha|-|\gamma|-1}\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j\mathrm{d}u
\end{aligned}$$

where the inside integral will be bounded by a constant depending only on α since $\psi, \phi \in C_c^{\infty}$. Therefore, by denoting $\mu' = (\mu_1, \cdots, \mu_{j-1}, \mu_{j+1}, \cdots, \mu_n)$, we have that

$$\begin{aligned}
&\left|\mathcal{R}_{\alpha} \iint D^{\gamma} f(\cdot, \cdot)\mu^{\gamma'}\lambda^{\gamma_{n+1}}\phi\psi\mathrm{d}\lambda\mathrm{d}\mu\right| \\
&\quad \leq C_{\alpha,n,M}|\Delta|^{|\alpha|-|\gamma|}|\mathcal{R}_{\alpha}| \int_{[-1,1]^{n-1}}\int_{1}^{L}\int_{-1}^{1} \left|D^{\beta} f(\cdot, x_j + c_1' u\Delta, \cdot)\right| \\
&\qquad\qquad \cdot \int_{-1}^{1} |\mu_j - u|^{|\alpha|-|\gamma|-1}|\mu_j|^{\gamma_j}|\psi_j(\mu_j)|\mathrm{d}\mu_j\mathrm{d}u\,\mathrm{d}\lambda\,\mathrm{d}\mu' \\
&\quad \leq C_{\alpha,n,M}\mathcal{B}_{\alpha} \int_{[-1,1]^{n-1}}\int_{-1}^{1}\int_{1}^{L} \left|D^{\beta} f(\cdot, x_j + c_1' u\Delta, \cdot)\right| \mathrm{d}\lambda\,\mathrm{d}u\,\mathrm{d}\mu' \\
&\quad \leq \frac{C_{\alpha,n,M}}{|\Delta|^{n+1}} \iint_{R(x,y)} \left|D^{\beta} f(u, v)\right| \mathrm{d}v\mathrm{d}u
\end{aligned}$$

where $|\beta| = |\alpha|$ and the next to last inequality follows using (7.9). Thus we again have an estimate similar to (7.10) with $\overrightarrow{f}$ replaced by $D^{\beta}\overrightarrow{f}$. As a final case, if $j = n + 1$, then we similarly expand about $\lambda = 1$ to again gain the above result.

In a similar manner to the above, all lesser order derivatives will be bounded, i.e.,

$$||D^{\alpha} g||_{L^p\left(\overline{D}^c, w\right)} \leq C_{p,k,D,w} \, ||f||_{L^p_k(D,W)} \, .$$

Therefore, based on our initial assumptions, the theorem holds.

(E) It then further follows that we can extend any $f \in L^p_k(D, w)$ given Corollary 5.1 and the argument provided in the proof of Theorem 4.1.

(F) We can also finally relax the special Lipschitz domain assumption by repeating the argument given in part (F) of the proof of Theorem 6.2. (Here, one need only replace each symbol Ω with a D) □

Chapter 8

Matrix A_p Weights

As alluded to already, the results thus far are a compilation of known results presented in a context different from their original sources with new methods of proof considered in the weighted cases. Though not all of these results can be found explicitly in the literature, the results themselves are implied by the work of E. M. Stein (see [18] and [19]), Peter Jones ([11]), and S.K. Chua ([3], [4], and [5]). However, such a treatment is useful in understanding the matrix weighted case. Namely, we can proceed by modifying as needed the prior results to gain new insights into the the extension problem for matrix weights.

8.1 Matrix Weights

Let us begin with some definitions. In the following, we will always use $||A||$ for the operator norm of a matrix A and $\left|\left|\overrightarrow{f}\right|\right|$ as the Euclidean norm of a vector valued function $\overrightarrow{f}$.

Definition 8.1. *Let $\mathcal{M}$ be the cone of non-negative positive definite $m \times m$ matrices. A matrix weight on $\mathbb{R}^n$ is a map $W : \mathbb{R}^n \to \mathcal{M}$ such that $W(x)$ is invertible for almost every x.*

It is known that any positive definite operator (often called just positive or definite) must also be self-adjoint. This can be seen, for example, using the following result:

Lemma 8.1 ([6], p. 33)**.** *Let M be an operator on a Hilbert space $\mathcal{H}$.*

(a) M is self adjoint if and only if $\langle Mx, x\rangle_{\mathcal{H}} \in \mathbb{R}$ for all $x \in \mathcal{H}$.

(b) If M is positive definite, then M is self-adjoint.

With this in hand, we also note that self-adjoint operators can be diagonalized (see [6] p.46). In the case of positive operators, the eigenvalues are all positive scalars as well. In this way, we can define roots of any positive operator with the resulting matrix being positive (and thus also self adjoint).

Definition 8.2. *Let $1 \le p < \infty$ and W be a matrix weight. For a measurable $\overrightarrow{f} = (f_i)_{1\le i\le m} : \mathbb{R}^n \to \mathbb{R}^m$, let*

$$\left|\left|\overrightarrow{f}\right|\right|_{L^p(E,W)} = \left(\int_E \left|\left|W^{1/p}(t)\overrightarrow{f}(t)\right|\right|^p \mathrm{d}t\right)^{1/p} \tag{8.1}$$

where $E \subset \mathbb{R}^n$ is measurable.

It is worth noting here that when $m = 1$, it follows that

$$\begin{aligned} ||f||_{L^p(\mathbb{R}^n,w)} &= \left(\int_{\mathbb{R}^n} |w^{1/p}(t)f(t)|^p \mathrm{d}t\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} |f(t)|^p \mathrm{d}w(t)\right)^{1/p} \end{aligned}$$

which agrees with Definition 5.3.

The intuition behind the above definition is that for $p = 2$ one would want:

$$\begin{aligned}
\left|\left|\vec{f}\right|\right|^2_{L^2(\mathbb{R}^n,W)} &= \int_{\mathbb{R}^n} \left|\left\langle W(t)\vec{f}(t), \vec{f}(t)\right\rangle\right| \mathrm{d}t \\
&= \int_{\mathbb{R}^n} \left|\left\langle W^{1/2}(t)W^{1/2}(t)\vec{f}(t), \vec{f}(t)\right\rangle\right| \mathrm{d}t \\
&= \int_{\mathbb{R}^n} \left|\left\langle W^{1/2}(t)\vec{f}(t), W^{1/2}(t)\vec{f}(t)\right\rangle\right| \mathrm{d}t \\
&= \int_{\mathbb{R}^n} \left|\left|W^{1/2}(t)\vec{f}(t)\right|\right|^2 \mathrm{d}t.
\end{aligned}$$

So we define the general norm in such a way for all $1 \leq p < \infty$.

8.2 The A_p Condition

The motivation for the matrix A_p condition was to find a way to preserve the Hilbert transform:

$$\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \mapsto H\vec{f} = \begin{pmatrix} Hf_1 \\ \vdots \\ Hf_m \end{pmatrix}$$

Explicitly, one then wants to define the A_p condition such that $\left|\left|H\vec{f}\right|\right|_{L^p(W)} \leq C\left|\left|\vec{f}\right|\right|_{L^p(W)}$ if and only if $W \in A_p$ for $1 < p < \infty$. Formulating this directly is a bit of a pain (see the work of Nazarov, Treil, and Volberg in [12], [13], and [20]), but Svetlana Roudenko showed the following more direct (for our purposes, at least) equivalence:

Lemma 8.2 ([16], Cor. 3.3)**.** *Let W be a matrix weight on $\mathbb{R}^n$, $1 < p < \infty$, and $1/p + 1/p' = 1$. Then $W \in A_p$ if and only if W and $W^{-p'/p}$ are locally integrable and* *for some constant* $A_{p,n} > 0$

$$\left(\frac{1}{|Q|}\int_Q \left(\frac{1}{|Q|}\int_Q ||W^{1/p}(x)W^{-1/p}(t)||^{p'}\mathrm{d}t\right)^{p/p'}\mathrm{d}x\right)^{1/p} \leq A_{p,n} \tag{8.2}$$

for all cubes $Q \subset \mathbb{R}^n$.

We then denote the minimal constant $A_{p,n}$ as the matrix A_p bound for W, $A_p(W)$.

When $p = 1$ we say $W \in A_1$ if W and W^{-1} are locally integrable and there exists a constant $C > 0$ such that for all cubes $Q \subset \mathbb{R}^n$,

$$\operatorname*{ess\,sup}_{u\in Q} \frac{1}{|Q|}\int_Q ||W(t)W^{-1}(u)||\mathrm{d}t \leq C \tag{8.3}$$

(see [9] p. 1227). Here the smallest C is the A_1 bound, $A_1(W)$.

An important example occurs when $m = 1$. In this case, we have that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\left(\frac{1}{|Q|}\int_Q w^{-p'/p}(t)\mathrm{d}t\right)^{p/p'} \\
&\qquad = \frac{1}{|Q|}\int_Q \left(w^{p'/p}(x)\right)^{p/p'}\left(\frac{1}{|Q|}\int_Q w^{-p'/p}(t)\mathrm{d}t\right)^{p/p'}\mathrm{d}x \\
&\qquad = \frac{1}{|Q|}\int_Q \left(\frac{1}{|Q|}\int_Q |w^{1/p}(x)w^{-1/p}(t)|^{p'}\mathrm{d}t\right)^{p/p'}\mathrm{d}x
\end{aligned}$$

Indeed, we previously used this property of scalar weights in the proofs of Theorems 6.1 and 6.2 (also see (6.6) and (6.13)) to gain an associated "variable switching property." Such a property has an important generalization in the matrix case.

Fix $x \in Q$ for a cube $Q \subset \mathbb{R}^n$. Then we have the pointwise result that

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\left|W^{1/p}(x)\vec{f}(t)\right|\right|\mathrm{d}t\right)^p \\
&\qquad \leq \left(\frac{1}{|Q|}\int_Q ||W^{1/p}(x)W^{-1/p}(t)||\;\left|\left|W^{1/p}(t)\vec{f}(t)\right|\right|\mathrm{d}t\right)^p \\
&\qquad \leq \left(\frac{1}{|Q|}\int_Q ||W^{1/p}(x)W^{-1/p}(t)||^{p'}\mathrm{d}t\right)^{p/p'}\frac{1}{|Q|}\int_Q \left|\left|W^{1/p}(t)\vec{f}(t)\right|\right|^p\mathrm{d}t
\end{aligned}$$

using Hölder's inequality. We then gain the full result by taking the average over Q in the x variable:

$$\left(\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q\left|\left|W^{1/p}(x)\overrightarrow{f}(t)\right|\right|\mathrm{d}t\right)^p\mathrm{d}x\right)^{1/p}\le\frac{A_p(W)}{|Q|^{1/p}}\left|\left|\overrightarrow{f}\right|\right|_{L^p(Q,W)}.$$

To prove an extension theorem we would naturally want to use the matrix equivalent of the scalar case. This is indeed a good idea except for one slight issue. Based on the way the L^p norm given in (8.1) is designed it is not immediately apparent that we can proceed as before. There are, however, two key techniques that prove useful. The first is that we can use the features of the A_p inequality to get what we want as we did in the scalar case; the key feature is the variable switching property that we noticed above. The second is to try to put as much as we can into a scalar setting to avoid issues with vectors and matrices. This is the key theme that moves one from the scalar case to the vector case. The following lemma is a key step in this.

Lemma 8.3. *Let $1\le p<\infty$. If $W\in A_p$ (as a matrix weight), then $w=||W^{1/p}||^p\in A_p$ (as a scalar weight).*

Proof. Suppose $W\in A_p$ for $1<p<\infty$. Then for all cubes $Q\subset\mathbb{R}^n$,

$$\begin{aligned}&\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\left(\frac{1}{|Q|}\int_Q w^{-p'/p}(x)\mathrm{d}x\right)^{p/p'}\\&\qquad=\frac{1}{|Q|}\int_Q||W^{1/p}(x)||^p\mathrm{d}x\left(\frac{1}{|Q|}\int_Q||W^{1/p}(t)||^{-p'}\mathrm{d}t\right)^{p/p'}\\&\qquad=\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q||W^{1/p}(x)||^{p'}||W^{1/p}(t)||^{-p'}\mathrm{d}t\right)^{p/p'}\mathrm{d}x\end{aligned}$$

But

$$\begin{aligned}||W^{1/p}(x)||^{p'}||W^{1/p}(t)||^{-p'}&=||W^{1/p}(x)W^{-1/p}(t)W^{1/p}(t)||^{p'}||W^{1/p}(t)||^{-p'}\\&\le||W^{1/p}(x)W^{-1/p}(t)||^{p'}||W^{1/p}(t)||^{p'}||W^{1/p}(t)||^{-p'}\\&=||W^{1/p}(x)W^{-1/p}(t)||^{p'}\end{aligned}$$

since W is invertible almost everywhere.

Thus $||W^{1/p}||^p \in A_p$. Furthermore, when $p = 1$,

$$\begin{aligned} \frac{1}{|Q|}\int_Q ||W(u)||\,\mathrm{d}u &\leq \frac{1}{|Q|}\int_Q \left|\left|W(u)W^{-1}(x)\right|\right|\,||W(x)||\,\mathrm{d}u \\ &\leq A_1(W)\,||W(x)|| \end{aligned}$$

for almost every $x \in Q$. In this way (5.4) implies (8.3), that is, $||W|| \in A_1$. □

The expression for w above is also independent of p since w will just be the eigenvalue of largest magnitude for the matrix W.

8.3 Density of $C^\infty\left(\overline{D}\right)$ in $L^p_k(D, W)$

Lastly, we need to recover an analogous density result. From the scalar setting it becomes apparent that we must repeat the arguments of Theorem 3.1 and Lemmas 3.2 and 3.3 (see also Corollary 5.1). In doing so the key tool is the fundamental measure theoretic result:

Theorem 8.1 (Lusin's Theorem, [17], p.55)**.** *Let μ be a Radon measure on a locally compact Hausdorff space X. Suppose f is a complex measurable function on X, $|E| < \infty$, $f(x) = 0$ if $x \notin E$, and $\epsilon > 0$. Then there exists a $g \in C_c(X)$ such that*

$$\mu\left(\{x : f(x) \neq g(x)\}\right) < \epsilon.$$

Furthermore, it is possible to find g such that

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|.$$

Using this result, we can now walk back through the density proofs.

Lemma 8.4 (Density of $C^\infty\left(\overline{D}\right)$ in $L^p_k(D, W)$)**.** *Suppose $D \subset \mathbb{R}^{n+1}$ is a special Lipschitz domain with Lipschitz graph Γ and bound M. Let $\overrightarrow{f} \in L^p_k(D, W)$ where*

$1 \leq p < \infty$, $k = 0, 1, 2, ...,$ and $W \in A_p$. Then there exist $\left\{\overrightarrow{f_\epsilon}\right\}_{\epsilon>0}$ where each $\overrightarrow{f_\epsilon} \in C_c^\infty\left(\overline{D}\right)$ such that

$$\left|\left|\overrightarrow{f_\epsilon} - \overrightarrow{f}\right|\right|_{L_k^p(D,W)} \to 0$$

as $\epsilon \to 0$.

Proof. We start by modifying the classical proof to regain the density of $C_c(\mathbb{R}^{n+1})$ in $L^p(\mathbb{R}^{n+1}, W)$ (the classical proof can be found in [17] p. 69). Let $\overrightarrow{f} \in L^p(\mathbb{R}^{n+1}, W)$. Consider the set of functions

$$S = \left\{\overrightarrow{s}(x) : \exists r > 0 \text{ such that } ||\overrightarrow{s}(x)|| = 0 \ \forall |x| \geq r \text{ and } \sup_{x\in\mathbb{R}^n} ||\overrightarrow{s}(x)|| < \infty\right\}.$$

Define $\overrightarrow{f_N}(x) = \chi_{\{|x|\leq N\}\cap\left\{||\overrightarrow{f}(x)||\leq N\right\}}\overrightarrow{f}(x)$. It then follows by the dominated convergence theorem that $\overrightarrow{f_N}(x) \to \overrightarrow{f}(x)$ in $L^p(\mathbb{R}^{n+1}, W)$ as $N \to \infty$. Hence S is dense in $L^p(\mathbb{R}^{n+1}, W)$.

Now, let $\overrightarrow{s}(x) \in S$. By Lemma 8.3, $w = ||W^{1/p}||^p$ is an A_p weight. Thus for all $r > 0$ $w(B_r) < \infty$. Let $r > 0$ be such that $||\overrightarrow{s}(x)|| = 0$ for all $x \notin B_r$. We also know that $\sup_{x\in\mathbb{R}^{n+1}} |s_i(x)| \leq C$ for some constant $C > 0$.

Fix $\epsilon > 0$. By Lusin's Theorem (Theorem 8.1), for each $1 \leq i \leq m$ there exists $g_i \in C_c(\mathbb{R}^m)$ such that, setting $E_i = \{x : s_i(x) \neq g_i(x)\}$, we have that

$$w(E_i) < \frac{\epsilon}{m^{3p/2}2^pC^p}$$

and

$$\sup_{x\in\mathbb{R}^n} |g_i(x)| \leq \sup_{x\in\mathbb{R}^n} |s_i(x)| \leq C.$$

Then

$$\int_{\mathbb{R}^{n+1}} \left|\left|W^{1/p}(x)\left[\overrightarrow{g}(x) - \overrightarrow{s}(x)\right]\right|\right|^p \mathrm{d}x < \epsilon.$$

So $C_c(\mathbb{R}^{n+1})$ is dense in S and thus dense in $L^p(\mathbb{R}^{n+1}, W)$. By a standard argument (compare to Lemma 3.2) we see the continuity of translations in $L^p(\mathbb{R}^{n+1}, W)$ for all $\overrightarrow{f} \in C_c(\mathbb{R}^{n+1})$:

$$\left(\int_{\mathbb{R}^{n+1}} \left|\left|W^{1/p}(x)\left[\overrightarrow{f}(x+t) - \overrightarrow{f}(x)\right]\right|\right|^p \mathrm{d}x\right)^{1/p} \to 0$$

as $t \to 0$. Then, using the density of $C_c(\mathbb{R}^{n+1})$ in $L^p(\mathbb{R}^{n+1}, W)$, we attain such continuity for all $\overrightarrow{f} \in L^p(\mathbb{R}^{n+1}, W)$.

For the remainder of the proof, we use the notation $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$. Fix $\epsilon > 0$. Consider $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ where $\operatorname{supp}\phi \subset \Gamma_-$, that is, the downward opening cone centered at the origin:

$$\Gamma_- = \{x \in \mathbb{R}^{n+1} : M|x'| < |x_{n+1}|,\ x_{n+1} < 0\}. \tag{8.4}$$

We also choose ϕ such that $\int_{\mathbb{R}^{n+1}} \phi(x)\mathrm{d}x = 1$. Denote $\phi_\epsilon = \epsilon^{-n}\phi(x/\epsilon)$ for any $\epsilon > 0$. Define

$$\overrightarrow{f_\epsilon}(x) = \phi_\epsilon * \overrightarrow{f_0}(x) = \int_{\mathbb{R}^{n+1}} \phi_\epsilon(x-t)\overrightarrow{f_0}(t)\mathrm{d}t$$

with the trivial extension $\overrightarrow{f_0}(t) = \begin{cases} \overrightarrow{f}(t) & t \in D \\ 0 & t \notin D \end{cases}$. Then $\overrightarrow{f_\epsilon} \in C^\infty\left(\overline{D}\right)$.

By the Minkowski integral inequality we have that

$$||D^\alpha \overrightarrow{f_\epsilon} - D^\alpha \overrightarrow{f}||_{L^p(D,W)} \leq \int_{\mathbb{R}^{n+1}} |\phi(u)| \left|\left|D^\alpha \overrightarrow{f_0}(x - \epsilon u) - D^\alpha \overrightarrow{f}(x)\right|\right|_{L_p(D,W)} \mathrm{d}u$$

where the last expression makes sense because $D^\alpha f_0(x - \epsilon u)$ exists for $x \in D$ and $u \in \operatorname{supp}\phi$ since we are shifting an element in the domain by an element in an upward facing cone.

Therefore $D^\alpha f_0$ can be replaced with $D^\alpha f$. By passing to the trivial extension of

$D^\alpha f$, we can use the continuity of translations for $D^\alpha f$ to get that

$$
\begin{aligned}
F_\epsilon(u) &= \left|\left| D^\alpha \overrightarrow{f_0}(x-\epsilon u) - D^\alpha \overrightarrow{f}(x) \right|\right|_{L_p(D,W)} \\
&= \left(\int_D \left|\left| W^{1/p}(x) \left[D^\alpha \overrightarrow{f}(x-\epsilon u) - D^\alpha \overrightarrow{f}(x) \right] \right|\right|^p \mathrm{d}x \right)^{1/p} \\
&\to 0
\end{aligned}
$$

as $\epsilon \to 0$ pointwise for each u. Also

$$
|\phi(u)| F_\epsilon(u) \leq 2 || D^\alpha \overrightarrow{f} ||_{L^p(D,W)} |\phi(u)|
$$

which is in $L^1(\mathbb{R}^{n+1})$ since the Sobolev norm is finite. By the dominated convergence theorem we thus get the convergence of each term in the Sobolev norm and thus the entire norm after summing. $\square$

Chapter 9

The Matrix A_p Weighted Smooth Domain Case

In light of the work accomplished in Chapter 6 (which covered the scalar weighted smooth case) and Chapter 8 (which pertained to results for matrix weights), we can now finally prove the first of two new extension results. Here we consider the smooth domain case. For the Lipschitz domain case (which is, in effect, a generalization of the following result), the reader is referred to Chapter 10.

Theorem 9.1 (Smooth Matrix Weighted Extension)**.** *Let $1 < p < \infty$, $k = 0, 1, 2, \ldots$, $W \in A_p$, and $\Omega \subset \mathbb{R}^{n+1}$ a domain with smooth boundary. If $\overrightarrow{f} \in L^p_k(\Omega, W)$, then there exists an extension $E\left(\overrightarrow{f}\right) \in L^p_k(\mathbb{R}^{n+1}, W)$, that is, there exists $E\left(\overrightarrow{f}\right)(x, y) : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $E\left(\overrightarrow{f}\right)|_\Omega = \overrightarrow{f}$ and*

$$\left|\left|E\left(\overrightarrow{f}\right)\right|\right|_{L^p_k(\mathbb{R}^{n+1},W)} \leq C_{p,k,n,W} \left|\left|\overrightarrow{f}\right|\right|_{L^p_k(\Omega,W)} \tag{9.1}$$

To prove this, we will repeat the arguments of Theorem 6.2 using the vector valued generalization of the "Direct Method." Indeed, that is really just the $m = 1$ case of this theorem.

Proof. (A) Assume that $\overrightarrow{f} \in C^\infty\left(\overline{\mathbb{R}^{n+1}_+}\right)$. Consider

$$\begin{aligned}\overrightarrow{g}(x,y) &= \left(\begin{array}{l}\displaystyle\int_{-1}^{1}\cdots\int_{-1}^{1}\int_{1/2}^{2} \quad f_i(x_1+\mu_1 y,\ldots,x_n+\mu_n y,-\lambda y) \\ \qquad\qquad\qquad\qquad\cdot\phi(\lambda)\mathrm{d}\lambda\,\psi_1(\mu_1)\mathrm{d}\mu_1\cdots\psi_n(\mu_n)\mathrm{d}\mu_n\end{array}\right)_{1\le i\le m} \\ &= \int_{[-1,1]^n}\int_{1/2}^{2}\overrightarrow{f}(x+\mu y,-\lambda y)\phi(\lambda)\mathrm{d}\lambda\,\psi(\mu)\mathrm{d}\mu\end{aligned}$$

for $y<0$. Here ϕ and each ψ_j are smooth functions with compact supports within their limits of integration and the following moment properties:

$$\begin{aligned}\int\phi(\lambda)\mathrm{d}\lambda = 1, &\quad \int\lambda^l\phi(\lambda)\mathrm{d}\lambda = (-1)^l \\ \int\psi_j(\mu_j)\mathrm{d}\mu_j = 1, &\quad \int\mu^l\psi_j(\mu_j)\mathrm{d}\mu_j = 0\end{aligned} \tag{9.2}$$

for $1\le l\le k$ and $1\le j\le n$. We then set

$$E\left(\overrightarrow{f}\right) = \begin{cases}\overrightarrow{f}(x,y) & y\ge 0 \\ \overrightarrow{g}(x,y) & y<0\end{cases}.$$

(B) Considering that both $\overrightarrow{f}$ and $\overrightarrow{g}$, as well as all of their derivatives, are defined component wise, let us fix i and omit the index by writing f and g, respectively. It is clear then that by using (9.2) we have that $g(x,y)\to f(x,0)$ as $y\to 0$.
Next, let α be a multi-index. Since we assumed f to be smooth it follows that by considering $\alpha=(\alpha',\alpha_{n+1})$ we have that

$$D^{\alpha'}g(x,y) = \int_{[-1,1]^n}\int_{1/2}^{2} D^{\alpha'}f(x+\mu y,-\lambda y)\phi(\lambda)\mathrm{d}\lambda\,\psi(\mu)\mathrm{d}\mu.$$

For the derivatives in y we must use (6.7). Combining this with the above equation gives that

$$D^\alpha g(x,y) = \sum \int_{[-1,1]^n} \int_{1/2}^{2} \mu_1^{l_1} \cdots \mu_n^{l_n} (-\lambda)^{l_{n+1}} D^{(\alpha_1+l_1,\ldots,\alpha_n+l_n,l_{n+1})} f(x+\mu y, -\lambda y) \mathrm{d}\lambda \mathrm{d}\mu \tag{9.3}$$

where the sum has a finite number of terms and $l_1 + \cdots + l_{n+1} = \alpha_{n+1}$ with each $l_i \geq 0$. Hence, there is a single integral term with $l_i = 0$ for all $i = 1, \ldots, n$ so that $l_{n+1} = \alpha_{n+1}$. Consequently, this particular terms satisfies

$$\iint (-\lambda)^{\alpha_{n+1}} D^{(\alpha',\alpha_{n+1})} f(x+\mu y, -\lambda y) \mathrm{d}\lambda \mathrm{d}\mu \to D^\alpha f(x,0)$$

as $y \to 0$ by using (9.2). Otherwise, there exists $l_i > 0$ for some $i = 1, \ldots, n$ and thus, by using (9.2), all other terms in the sum for (9.3) go to zero as $y \to 0$.
In summary, since $\overrightarrow{f}$ is assumed to be smooth we can calculate the derivatives of $\overrightarrow{g}$ which demonstrates that $\overrightarrow{g}$ is smooth on the lower half space with the necessary trace conditions. Thus we can use the existence of derivatives across the boundary (Lemma 3.1) to gain the necessary differentiability.
(C) Here we use the matrix analog of the Direct Method used when proving Theorems 6.1 and 6.2. In doing so, the key result is the "variable switching property" gained as a consequence of (8.2) and (8.3). As before, we need only bound the Sobolev norm of $\overrightarrow{g}$ on the lower half space by the Sobolev norm of $\overrightarrow{f}$ on the upper half space.
Fix $y < 0$. Then, by applying a change of variables from μ and λ to u and v, respectively, we have that

$$\begin{aligned} \left\| W^{\frac{1}{p}}(x,y) \overrightarrow{g}(x,y) \right\| &\leq C_{\beta,n} \iint \left\| W^{\frac{1}{p}}(x,y) \overrightarrow{f}(u,v) \right\| \frac{\mathrm{d}v}{|y|} \frac{\mathrm{d}u}{|y|^n} \\ &= \frac{C_{\beta,n}}{|y|^{n+1}} \int_{x_n-y}^{x_n+y} \cdots \int_{x_1-y}^{x_1+y} \int_{-(1/2)y}^{-2y} \left\| W^{\frac{1}{p}}(x,y) \overrightarrow{f}(u,v) \right\| \mathrm{d}v \mathrm{d}u \end{aligned} \tag{9.4}$$

Fix $(x,y) \in Q_-$ where $Q_- \in \mathcal{W}_-$. In particular, there exists $q \in \mathbb{Z}$ and for each $1 \leq j \leq n$ there exists $q_j \in \mathbb{Z}$ such that $2^q \leq -y \leq 2^{q+1}$ and $q_j 2^q \leq x_j \leq (q_j+1)2^q$. Then

$$q_j 2^q - 2^{q+1} \leq x_j + y \leq x_j - y \leq q_j 2^q + 2^q + 2^{q+1}$$

which is an interval of length $2^q + 2^{q+2}$ and

$$2^{q-1} \leq -\frac{1}{2}y \leq -2y \leq 2^{q+2}$$

which is an interval of length $2^{q+2} - 2^{q-1}$. Adding $2^{q-1} + 2^q$ to the top limit then yields a new cube $Q^* \subset \mathbb{R}^{n+1}_+$ of size $|Q^*| = (5 \cdot 2^q)^{n+1}$. Also, $|y|^{n+1} \geq (2^q)^{n+1}$. Therefore, for $(x,y) \in Q_-$ we have that

$$\begin{aligned}
&\left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{g}(x,y)\right|\right|^p \\
&\quad \leq \left(\frac{C_n}{(2^q)^{n+1}} \iint_{Q*} \left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{f}(u,v)\right|\right| \mathrm{d}v\, \mathrm{d}u_1 \cdots \mathrm{d}u_n\right)^p \\
&\quad = \left(\frac{5^{n+1}C_n}{|Q^*|} \iint_{Q^*} \left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{f}(u,v)\right|\right| \mathrm{d}u\mathrm{d}v\right)^p \\
&\quad = C_{p,n}\left(\frac{1}{|Q^*|} \iint_{Q^*} \left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)W^{\frac{1}{p}}(u,v)\overrightarrow{f}(u,v)\right|\right| \mathrm{d}u\mathrm{d}v\right)^p \\
&\quad \leq C_{p,n}\left(\frac{1}{|Q^*|} \iint_{Q^*} \left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right| \left|\left|W^{\frac{1}{p}}(u,v)\overrightarrow{f}(u,v)\right|\right| \mathrm{d}u\mathrm{d}v\right)^p \\
&\quad \leq C_{p,n}\left(\frac{1}{|Q^*|} \iint_{Q^*} \left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right|^{p'} \mathrm{d}u\mathrm{d}v\right)^{p/p'} \cdot \frac{1}{|Q^*|} \left|\left|\overrightarrow{f}\right|\right|^p_{L^p(Q^*,W)}
\end{aligned}$$

where the last inequality is a result of Hölder's inequality. We then calculate the L_p norm by summing over all such cubes. First, though, we remark that by doubling Q^*

just once we can contain Q_-. That is we can make the estimate

$$\begin{aligned}\iint_{Q_-} &\left(\frac{1}{|Q^*|}\iint_{Q^*}\left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right|^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\mathrm{d}y\mathrm{d}x\\ &\leq \iint_{Q_-}\left(\frac{2^{n+1}}{|2Q^*|}\iint_{2Q^*}\left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right|^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\mathrm{d}y\mathrm{d}x\\ &\leq C_{p,n}A_p(W)|2Q^*|\end{aligned}$$

by using the A_p condition. Lastly, note that Q^* is contained in at most $5\cdot 10^n$ cubes $Q_+ \in \mathcal{W}_+$. We then at long last have that

$$\begin{aligned}||\overrightarrow{g}||^p_{L^p\left(\mathbb{R}^{n+1}_-,W\right)}&\\ &\leq C_{p,n}\sum_{Q_-\in\mathcal{W}_-}\iint_{Q_-}\left[\begin{array}{c}\left(\frac{1}{|Q^*|}\iint_{Q^*}\left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right|^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\\ \cdot\frac{1}{|Q^*|}\left|\left|\overrightarrow{f}\right|\right|^p_{L^p(Q^*,W)}\end{array}\right]\mathrm{d}y\mathrm{d}x\\ &\leq C_{p,n}\sum_{Q_-\in\mathcal{W}_-}C_{p,n}A_p(W)|2Q^*|\frac{1}{|Q^*|}\iint_{Q^*}\left|\left|W^{\frac{1}{p}}(u,v)\overrightarrow{f}(u,v)\right|\right|^p\mathrm{d}u\mathrm{d}v\\ &= C_{p,n,W}\sum_{Q_-\in\mathcal{W}_-}2^{n+1}\iint_{Q^*}\left|\left|W^{\frac{1}{p}}(u,v)\overrightarrow{f}(u,v)\right|\right|^p\mathrm{d}u\mathrm{d}v\\ &\leq C_{p,n,W}(5\cdot 10^n)\sum_{Q_+\in\mathcal{W}_+}\iint_{Q_+}\left|\left|W^{\frac{1}{p}}(u,v)\overrightarrow{f}(u,v)\right|\right|^p\mathrm{d}u\mathrm{d}v\\ &= C_{p,n,W}||\overrightarrow{f}||^p_{L_p(\mathbb{R}^{n+1}_+,W)}\end{aligned}$$

The variable switching property given above also works when $p = 1$: By an application of Fubini's Theorem

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \left|\left|W(x)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\mathrm{d}x \\
&\qquad = \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \left|\left|W(x)W^{-1}(t)\right|\right| \left|\left|W(t)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\mathrm{d}x \\
&\qquad = \frac{1}{|Q|}\int_Q \left|\left|W(t)\overrightarrow{f}(t)\right|\right| \frac{1}{|Q|}\int_Q \left|\left|W(x)W^{-1}(t)\right|\right| \mathrm{d}x\mathrm{d}t \\
&\qquad \leq \left(\operatorname*{ess\,sup}_{t\in Q} \frac{1}{|Q|}\int_Q \left|\left|W(x)W^{-1}(t)\right|\right| \mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \left|\left|W(t)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\right)
\end{aligned}$$

where Q is a cube in $\mathbb{R}^{n+1}$. Thus, using (8.3),

$$\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \left|\left|W(x)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\,\mathrm{d}x \leq \frac{A_1(W)}{|Q|}\left|\left|\overrightarrow{f}\right|\right|_{L^1(Q,W)}.$$

So we can bound $\overrightarrow{g}$ in the L^p norm for all $1 \leq p < \infty$.

(D) To bound $D^\alpha g$, we again consider (9.3):

$$D^\alpha \overrightarrow{g}(x,y) = \sum \int_{[-1,1]^n}\int_{1/2}^2 \mu_1^{l_1}\cdots\mu_n^{l_n}(-\lambda)^{l_{n+1}} D^\beta \overrightarrow{f}(x+\mu y, -\lambda y)\mathrm{d}\lambda\mathrm{d}\mu$$

where $l_i \geq 0$ for each $1 \leq i \leq n+1$ and $|\beta| = |\alpha|$. For each term of this sum, a similar estimate to (9.4) then follows:

$$\begin{aligned}
&\left|\left|W^{1/p}(x,y)D^\alpha\overrightarrow{g}(x,y)\right|\right| \\
&\quad \leq C_{\alpha,n}\int_{[-1,1]^n}\int_{1/2}^2 \left|\left|W^{1/p}(x,y)D^\beta\overrightarrow{f}(x+\mu y, -\lambda y)\right|\right| \mathrm{d}\lambda\mathrm{d}\mu \\
&\quad \leq \frac{C_{\alpha,n}}{|y|^{n+1}}\int_{x_n+y}^{x_n-y}\cdots\int_{x_1+y}^{x_1-y}\int_{-(1/2)y}^{-2y} \left|\left|W^{1/p}(x,y)D^\beta\overrightarrow{f}(u_1,\ldots,u_n,v)\right|\right| \mathrm{d}v\,\mathrm{d}u_1\cdots\mathrm{d}u_n
\end{aligned}$$

which is similar to (9.4). Thus we can use Whitney cube estimates to finish bounding each term of the sum and hence the entire derivative. Thus $\overrightarrow{g}$ and all of its derivatives are bounded in the L^p norm. That is, given the smoothness assumptions on $\overrightarrow{f}$, (9.1) is satisfied.

(E) It then further follows that we can extend any $\overrightarrow{f} \in L^p_k(\mathbb{R}^{n+1}_+, W)$ given Lemma 8.4 and the argument provided in part (E) of the proof of Theorem 4.1.

(F) We now proceed as we did in part (F) of the proofs of Theorems 4.1 and 6.2. Adjusting notation only moderately, we again consider the extension operator:

$$E(\overrightarrow{f}) = \Lambda_+ \left\{ \frac{\sum_{i=1}^{\infty} \lambda_i E_i(\lambda_i \overrightarrow{f})}{\sum_{i=1}^{\infty} \lambda_i^2} \right\} + \Lambda_- \overrightarrow{f} \tag{9.5}$$

where, for $i = 1, 2, \ldots$, U_i are the open sets associated with the smooth domain Ω and Ω_i are the associated special Lipschitz domains. Also $\lambda_i, \lambda_0, \lambda_+, \lambda_- : \mathbb{R}^{n+1} \to \mathbb{R}$ are all smooth with bounded derivatives independent of i such that

- supp $\lambda_i \subset U_i$ and $\lambda_i = 1$ on $U_i^{\epsilon/2}$
- supp $\lambda_o \subset \epsilon/2$ neighborhood of Ω and $\lambda_0 = 1$ on $\overline{\Omega}$
- supp $\lambda_+ \subset \epsilon$ neighborhood of $\partial\Omega$ and $\lambda_+ = 1$ if $\text{dist}(x, \partial\Omega) \leq \epsilon/2$
- supp $\lambda_- \subset \Omega$ and $\lambda_- = 1$ if $\text{dist}(x, \partial\Omega) \geq \epsilon/2$

and, setting $\Lambda_+ = \lambda_0 \left(\frac{\lambda_+}{\lambda_+ + \lambda_-} \right)$ and $\Lambda_- = \lambda_0 \left(\frac{\lambda_-}{\lambda_+ + \lambda_-} \right)$,

- Λ_+ and Λ_- are smooth with bounded derivatives
- $\Lambda_+ + \Lambda_- = 1$ on $\overline{\Omega}$
- $\Lambda_+ + \Lambda_- = 0$ outside the $\epsilon/2$ neighborhood of Ω.

Based on the quotient rule we can consider

$$E\left(\overrightarrow{f}\right) = \sum_{i=1}^{\infty} H_i E_i \left(\lambda_i \overrightarrow{f}\right) + \Lambda_- \overrightarrow{f}$$

as a replacement for (9.5). Then

$$\left\|E\left(\overrightarrow{f}\right)\right\|_{L_k^p(\mathbb{R}^{n+1},W)} \leq \sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\left\|W^{\frac{1}{p}}(x)D^\alpha\left(\sum_{i=1}^\infty H_i(x)E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right)\right\|^p \mathrm{d}x\right)^{1/p}$$
$$+\sum_{|\alpha|\leq k}\left\|D^\alpha\left(\Lambda_-\overrightarrow{f}\right)\right\|_{L^p(\mathbb{R}^{n+1},W)}.$$

The latter term is clearly bounded by $C\left\|\overrightarrow{f}\right\|_{L_k^p(\Omega,W)}$. Also, by Definition 2.4 (ii), $\sum_{i=1}^\infty \chi_{U_i}\leq N$ (also see (b) above). Thus the former term is

$$\sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\left\|W^{\frac{1}{p}}(x)\sum_{i=1}^\infty\sum_{\beta+\gamma=\alpha}D^\beta H_i(x)D^\gamma E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right\|^p\mathrm{d}x\right)^{1/p}$$
$$\leq \sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\left(\sum_{i=1}^\infty\chi_{U_i}\sum_{\beta+\gamma=\alpha}\left\|W^{\frac{1}{p}}(x)D^\beta H_i(x)D^\gamma E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right\|\right)^p\mathrm{d}x\right)^{1/p}$$
$$\leq \sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}N^{p/p'}\sum_{i=1}^\infty\left(\sum_{\beta+\gamma=\alpha}\left\|W^{\frac{1}{p}}(x)D^\beta H_i(x)D^\gamma E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right\|\right)^p\mathrm{d}x\right)^{1/p}$$
$$\leq C_{p,\Omega}\sum_{|\alpha|\leq k}\left(\int_{\mathbb{R}^{n+1}}\sum_{i=1}^\infty\sum_{\beta+\gamma=\alpha}\left\|W^{\frac{1}{p}}(x)D^\beta H_i(x)D^\gamma E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right\|^p\mathrm{d}x\right)^{1/p}$$
$$= C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^\infty\sum_{\beta+\gamma=\alpha}\int_{U_i}|D^\beta H_i(x)|^p\left\|W^{\frac{1}{p}}(x)D^\gamma E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right\|^p\mathrm{d}x\right)^{1/p}$$
$$\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^\infty\sum_{\beta+\gamma=\alpha}\int_{U_i}\left\|W^{\frac{1}{p}}(x)D^\gamma E_i\left(\lambda_i\overrightarrow{f}\right)(x)\right\|^p\mathrm{d}x\right)^{1/p}$$
$$\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^\infty\sum_{\beta+\gamma=\alpha}\sum_{|\kappa|\leq k}\int_{\Omega_i}\left\|W^{\frac{1}{p}}(x)D^\kappa\left(\lambda_i\overrightarrow{f}\right)(x)\right\|^p\mathrm{d}x\right)^{1/p}$$
$$\leq C_{p,k,\Omega}\sum_{|\alpha|\leq k}\left(\sum_{i=1}^\infty\sum_{\beta+\gamma=\alpha}\sum_{|\kappa|\leq k}\sum_{\epsilon+\delta=\kappa}\int_{U_i\cap\Omega}|D^\epsilon\lambda_i(x)|\left\|W^{\frac{1}{p}}(x)D^\delta\overrightarrow{f}(x)\right\|^p\mathrm{d}x\right)^{1/p}$$
$$\leq C_{p,k,\Omega,W}\left\|\overrightarrow{f}\right\|_{L_k^p(\Omega,W)}$$

via several applications of Hölder's inequality.

□

Chapter 10

The Matrix A_p Weighted Lipschitz Case

With all of the preliminary cases stated, we can now design a fairly straightforward proof of the penultimate result: Theorem 1.1.

Theorem. *Let D be a Lipschitz domain, $1 \leq p < \infty$, $k = 0, 1, 2, \ldots$, and $W \in A_p$. Given $\overrightarrow{f} \in L_k^p(D, W)$ there exists an extension $E\left(\overrightarrow{f}\right) \in L_k^p(\mathbb{R}^n, W)$, that is, there exists $E\left(\overrightarrow{f}\right) : \mathbb{R}^n \to \mathbb{R}$ such that $E\left(\overrightarrow{f}\right)|_D = \overrightarrow{f}$ and*

$$\left|\left|E\left(\overrightarrow{f}\right)\right|\right|_{L_k^p(\mathbb{R}^n, W)} \leq C_{p,k,D,W} \left|\left|\overrightarrow{f}\right|\right|_{L_k^p(D,W)}. \tag{10.1}$$

The proof of this follows by combining the scalar weighted Lipschitz domain case presented in Chapter 7 with the component wise arguments given in Chapter 9.

Proof. (A) We first assume that $\overrightarrow{f} \in C^\infty\left(\overline{D}\right)$ where $D \subset \mathbb{R}^{n+1}$ is a special Lipschitz domain with Lipschitz graph Γ and bound M.

Now, let us explicitly define our extension. Let $(x, y) \in \overline{D}^c$. Consider the vector function $\overrightarrow{g}(x, y) = (g_1(x, y), \ldots, g_m(x, y))$ where each component g_i of $\overrightarrow{g}$ has the form

$$g_i(x,y) = \int_{[-1,1]^n} \int_1^L f_i(x + c_1' \mu \Delta(x,y), y + c_2' \lambda \Delta(x,y))\, \phi(\lambda) \mathrm{d}\lambda\, \psi(\mu) \mathrm{d}\mu \tag{10.2}$$

for each corresponding component f_i of $\overrightarrow{f}$ with $x = (x_1, \ldots, x_n)$, $\mu = (\mu_1, \ldots, \mu_n)$, $\mathrm{d}\mu = \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_n$, and $\psi(\mu) = \psi_1(\mu_1) \cdots \psi_n(\mu_n)$. Here the $\phi, \psi_j \in C^\infty$ are compactly supported in $[1, L]$ and $[-1, 1]$, respectively, and satisfy the moment conditions

$$\begin{array}{cc} \int \phi(\lambda) \mathrm{d}\lambda = 1, & \int \lambda^l \phi(\lambda) \mathrm{d}\lambda = 0 \\ \int \psi_j(\mu_j) \mathrm{d}\mu_j = 1, & \int \mu^l \psi_j(\mu_j) \mathrm{d}\mu_j = 0 \end{array} \tag{10.3}$$

for all $1 \le j \le n$ and $1 \le l \le k$. Also, the constants c_1', c_2', L are chosen as in Lemma 7.2(a) making g well-defined. We then define

$$E\left(\overrightarrow{f}\right) = \begin{cases} \overrightarrow{f}(x,y) & (x,y) \in \overline{D} \\ \overrightarrow{g}(x,y) & (x,y) \in \overline{D}^c \end{cases} \tag{10.4}$$

(B) Considering (10.2) we fix i and omit the index by writing g and f, respectively, since all terms are estimated in the same manner. By using Lemma 6.2 we can find the derivatives of g by differentiating under the integral sign in (10.2) since we assumed f was smooth. In general, we have that

$$D^\alpha g = \sum_{\ell=1}^{|\alpha|} \sum_{\{\beta_1, \ldots, \beta_\ell\} \in \mathcal{B}_{\alpha,\ell}} \sum_{n_1=1}^{n+1} \cdots \sum_{n_\ell=1}^{n+1} \iint \prod_{r=1}^{\ell} \left(D^{\beta_r} x_{n_r} + c' \mu_{n_r} D^{\beta_r} \Delta\right) \left(D^{\sum_{s=1}^{\ell} e_{n_s}} f\right) \phi \psi \, \mathrm{d}\mu \tag{10.5}$$

where $\mu_{n+1} = \lambda$ and $c' = c_1'$ or c_2' as the case may be.

By assuming the smoothness of each component f, we know that each component g will also be smooth. To gain the weak differentiability of $E\left(\overrightarrow{f}\right)$ in then follows by a standard argument (see, for instance, the appendix of [15]) that we need only show that the following trace property is satisfied for all derivatives α up to order k:

$$D^\alpha g(x,y) \to D^\alpha f(x_0,y_0) \text{ as } (x,y) \to (x_0,y_0) \text{ for any } (x_0,y_0) = (x_0,\Gamma(x_0)) \quad (10.6)$$

Since $\delta \approx \Delta$ and using (10.3) it follows that (10.6) holds for $\alpha = 0$. Assume $1 \leq |\alpha| \leq k$. Then, in a similar way, each derivative $D^\alpha g$ will have a term of the form $\iint D^\alpha f \phi\psi$ which will satisfy (10.6). Explicitly, if $\sum_{s=1}^{\ell} e_{n_s} = \alpha$, then $\ell = |\alpha|$. Thus $\alpha = \sum_{r=1}^{\ell} \beta_r$ and thus for each r there exists a t such that $\beta_r = e_{n_t}$. Also, exactly one set of n_r will be such that $n_r = n_t$. When this occurs, $D^{\beta_r} x_{n_r} = 1$ for all r and thus the leading term in the product in (10.5) will yield a single term of 1.
So it remains to show that all other terms must go to zero. By fixing α, ℓ, $\beta_1, \ldots, \beta_\ell$, and $n_1, \ldots, n_\ell$ we have that the terms under the integral in (10.5) all have the form

$$D^\gamma f \prod_{r=1}^{\ell} D^{\beta_r}[x_{n_r} + c' \mu_{n_r} \Delta].$$

We need only examine the highest order derivatives of $\mu_{n_r}\Delta$ since bounding the lower order terms follows similarly. Thus, by writing $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) = (\alpha', \alpha_{n+1})$, and, similarly, $\beta_r = ((\beta_r)', (\beta_r)_{n+1})$ for each $1 \leq r \leq \ell$, and $\gamma = (\gamma', \gamma_{n+1})$ we have that

$$D^\gamma f \prod_{r=1}^{\ell} c' \mu_{n_r} D^{\beta_r} \Delta = C\, D^{(\gamma', \gamma_{n+1})} f\, \mu^{\gamma'} \lambda^{\gamma_{n+1}} D^{\beta_1}\Delta \cdots D^{\beta_\ell}\Delta \quad (10.7)$$

where $|\gamma| = \ell \geq 1$ and $\sum_{r=1}^{\ell} (\beta_r)_i = \alpha^i$ for each $1 \leq i \leq n+1$ because of the product and chain rules involved in calculating (10.5). (See Lemma 6.2) For simplicity, denote

$$\mathcal{R}_\alpha = D^{\beta_1}\Delta \cdots D^{\beta_l}\Delta$$

for this combination of regularized distance derivatives. It is important to note here that by Lemma 7.1 (b)

$$|\mathcal{R}_\alpha| \leq B_\alpha \delta^{1-|\beta_1|} \cdots \delta^{1-|\beta_l|} = B_\alpha \delta^{|\gamma|-|\alpha|} \tag{10.8}$$

which is singular for exponents $|\gamma| \leq |\alpha| - 1$ as we approach the boundary.

Since the term $D^\gamma f$ occurs only in the lower order terms, we can perform a Taylor expansion about $(\mu, \lambda) = (0, 1)$:

$$\begin{aligned} & D^\gamma f(x + c_1'\mu\Delta, y + c_2'\lambda\Delta) \\ & \quad = \sum_{|\zeta| \leq |\alpha|-|\gamma|} \frac{D^{\zeta+\gamma} f(x, y + c_2'\Delta)(c_1'\Delta)^{|\zeta'|}(c_2'\Delta)^{\zeta_{n+1}}}{\zeta!} \mu^{\zeta'} (\lambda - 1)^{\zeta_{n+1}} \\ & \qquad + \sum_{|\zeta| = |\alpha|-|\gamma|+1} \frac{D^{\zeta+\gamma} f\left(\theta(x, y, \mu, \lambda)\right)(c_1'\Delta)^{|\zeta'|}(c_2'\Delta)^{\zeta_{n+1}}}{\zeta!} \mu^{\zeta'} (\lambda - 1)^{\zeta_{n+1}} . \end{aligned}$$

for some θ on the the line segment between $(x, y + c_2'\Delta)$ and $(x + c_1'\mu\Delta, y + c_2'\lambda\Delta)$. Thus integrating (10.7) yields

$$\begin{aligned} & \iint D^\gamma f \, \mu^{\gamma'} \lambda^{\gamma_{n+1}} \mathcal{R}_\alpha \psi(\mu)\phi(\lambda) \mathrm{d}\mu \mathrm{d}\lambda \\ & \quad = \sum_{|\zeta| \leq |\alpha|-|\gamma|} \left\{ \begin{array}{l} \frac{(c_1'\Delta)^{|\zeta'|}(c_2'\Delta)^{\zeta_{n+1}} \mathcal{R}_\alpha}{\zeta!} D^{\zeta+\gamma} f\,(x, y + c_2'\Delta) \\ \cdot \int \mu^{\gamma'+\zeta'} \psi(\mu) \mathrm{d}\mu \int \lambda^{\gamma_{n+1}} (\lambda - 1)^{\zeta_{n+1}} \phi(\lambda) \mathrm{d}\lambda \end{array} \right\} + O(\Delta^{|\alpha|-|\gamma|+1} \mathcal{R}_\alpha) \end{aligned}$$

since ψ and ϕ are smooth with compact support and f is bounded at the boundary. Due to the moment conditions in (10.3) all of the first sum terms are identically zero.The remainder also goes to zero as $(x, y) \to (x_0, y_0)$ using (10.8) since $|\Delta|^{|\alpha|-|\gamma|+1} |\mathcal{R}_\alpha| \leq B_\alpha \delta$.

(C) Now that we have the derivatives, we must bound each one in the L^p norm. First let $1 < p < \infty$ and consider bounding $\overrightarrow{g}$ pointwise. Fix $(x, y) \in \overline{D}^c$. Then, by applying the change of variables $u = x + c_1'\mu\Delta$, $v = y + c_2'\lambda\Delta$ we have that

$\mathrm{d}u = (c_1'\Delta)^n \mathrm{d}\mu$, $\mathrm{d}v = c_2'\Delta \mathrm{d}\lambda$ and thus

$$\left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{g}(x,y)\right|\right| = \left|\left|W^{\frac{1}{p}}(x,y)\iint \overrightarrow{f}(u,v)\phi\left(\frac{v-y}{c_2'\Delta}\right)\frac{\mathrm{d}v}{c_2'\Delta}\,\psi\left(\frac{u-x}{c_1'\Delta}\right)\frac{\mathrm{d}u}{(c_1'\Delta)^n}\right|\right| \tag{10.9}$$

$$\leq \frac{C_{n,M}}{|\Delta|^{n+1}}\iint_{R(x,y)}\left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{f}(u,v)\right|\right|\mathrm{d}v\mathrm{d}u$$

where $R(x,y)$ is the cube defined in Lemma 7.2(a). Let $Q = [a_1,b_1]\times\cdots\times[a_n,b_n]\times[c,d]$ be a Whitney cube for $\overline{D}^c$. Fix $(x,y)\in Q$. Letting $l = b_j - a_j = d-c$ for all $1\leq j\leq n$ it follows that $l\approx\delta^*(x,y)\approx\Delta(x,y)\approx\overline{\Delta}(Q)\approx\underline{\Delta}(Q)$. Thus, using (7.1), we have that

$$\left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{g}(x,y)\right|\right|^p \leq C_{p,n,M}\left(\frac{1}{|Q^*|}\iint_{Q^*}\left|\left|W^{\frac{1}{p}}(x,y)\overrightarrow{f}(u,v)\right|\right|\mathrm{d}v\mathrm{d}u\right)^p$$

where $\bigcup_{(x,y)\in Q} R(x,y)\subset Q^*\subset D$ is given by Lemma 7.2(b) and 7.2(c).
By Lemma 7.2(d), Q^* can be enlarged to a cube Q^{**} containing Q and whose size is equivalent to that of Q^*. Thus

$$\iint_Q\left(\frac{1}{|Q^*|}\iint_{Q^*}\left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right|^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\mathrm{d}y\mathrm{d}x$$

$$\leq \iint_{Q^{**}}\left(\frac{C_{n,M}}{|Q^{**}|}\iint_{Q^{**}}\left|\left|W^{\frac{1}{p}}(x,y)W^{-\frac{1}{p}}(u,v)\right|\right|^{p'}\mathrm{d}u\mathrm{d}v\right)^{p/p'}\mathrm{d}y\mathrm{d}x$$

$$\leq C_{p,n,M}A_p^p(W)|Q^{**}|$$

by applying the A_p condition (8.2). Thus, we have a "variable switching property:"

$$\begin{aligned}
&\iint_Q \left(\frac{1}{|Q^*|} \iint_{Q^*} \left|\left| W^{\frac{1}{p}}(x,y) \overrightarrow{f}(u,v) \right|\right| \mathrm{d}v\mathrm{d}u \right)^p \mathrm{d}y\mathrm{d}x \\
&\leq \iint_Q \left(\frac{1}{|Q^*|} \iint_{Q^*} \left|\left| W^{\frac{1}{p}}(x,y) W^{-\frac{1}{p}}(u,v) \right|\right| \left|\left| W^{\frac{1}{p}}(u,v) \overrightarrow{f}(u,v) \right|\right| \mathrm{d}v\mathrm{d}u \right)^p \mathrm{d}y\mathrm{d}x \\
&\leq \iint_Q \left(\frac{1}{|Q^*|} \iint_{Q^*} \left|\left| W^{\frac{1}{p}}(x,y) W^{-\frac{1}{p}}(u,v) \right|\right|^{p'} \mathrm{d}v\mathrm{d}u \right)^{p/p'} \frac{1}{|Q^*|} \left|\left| \overrightarrow{f} \right|\right|^p_{L^p(Q^*,W)} \mathrm{d}y\mathrm{d}x \\
&\leq C_{n,M} A_p^p(W) \left|\left| \overrightarrow{f} \right|\right|^p_{L^p(Q^*,W)}
\end{aligned}$$

By Lemma 4(b), each point of D is contained in at most $C_{n,M} < \infty$ regions Q^*. Letting $\{Q_i\}_{i=1}^\infty$ be the Whitney decomposition for $\overline{D}^c$, it then follows that

$$\begin{aligned}
||\overrightarrow{g}||^p_{L^p\left(\overline{D}^c,W\right)} &= \sum_{i=1}^\infty ||\overrightarrow{g}||^p_{L^p(Q_i,W)} \\
&\leq C_{p,n,M} \sum_{i=1}^\infty \iint_{Q_i} \left(\frac{1}{|Q_i^*|} \iint_{Q_i^*} \left|\left| W^{\frac{1}{p}}(x,y) \overrightarrow{f}(u,v) \right|\right| \mathrm{d}v\mathrm{d}u \right)^p \mathrm{d}x\mathrm{d}y \\
&\leq C_{p,n,M,W} \sum_{i=1}^\infty \iint_{Q_i^*} \left|\left| W^{\frac{1}{p}}(u,v) \overrightarrow{f}(u,v) \right|\right|^p \mathrm{d}u\mathrm{d}v \\
&= C_{p,n,M,W} \sum_{i=1}^\infty \iint_D \chi_{Q_i^*} \left|\left| W^{\frac{1}{p}}(u,v) \overrightarrow{f}(u,v) \right|\right|^p \mathrm{d}u\mathrm{d}v \\
&\leq C_{p,n,M,W} \iint_D \left|\left| W^{\frac{1}{p}}(u,v) \overrightarrow{f}(u,v) \right|\right|^p \mathrm{d}u\mathrm{d}v \\
&= C_{p,n,M,W} ||\overrightarrow{f}||^p_{L^p(D,W)}
\end{aligned}$$

The variable switching property given above also works when $p = 1$: By an application of Fubini's Theorem

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \left|\left|W(x)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\mathrm{d}x \\
&\leq \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \left|\left|W(x)W^{-1}(t)\right|\right| \left|\left|W(t)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\mathrm{d}x \\
&= \frac{1}{|Q|}\int_Q \left|\left|W(t)\overrightarrow{f}(t)\right|\right| \frac{1}{|Q|}\int_Q \left|\left|W(x)W^{-1}(t)\right|\right| \mathrm{d}x\mathrm{d}t \\
&\leq \left(\operatorname*{ess\,sup}_{t\in Q} \frac{1}{|Q|}\int_Q \left|\left|W(x)W^{-1}(t)\right|\right| \mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \left|\left|W(t)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\right)
\end{aligned}
$$

where Q is a cube in $\mathbb{R}^{n+1}$. Thus using (8.3)

$$\frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q \left|\left|W(x)\overrightarrow{f}(t)\right|\right| \mathrm{d}t\,\mathrm{d}x \leq \frac{A_1(W)}{|Q|}\left|\left|\overrightarrow{f}\right|\right|_{L^1(Q,W)}.$$

So we can bound $\overrightarrow{g}$ in the L^p norm for all $1 \leq p < \infty$.

(D) Next we aim to bound the derivatives in a like manner. In doing so we can again assume that our derivative has the form of (10.7), that is,

$$\mathcal{R}_\alpha(x,y)\int_{[-1,1]^n}\int_1^L D^{(\gamma',\gamma_{n+1})}f(x+c_1'\mu\Delta(x,y),y+c_2'\lambda\Delta(x,y))\mu^{\gamma'}\lambda^{\gamma_{n+1}}\phi(\lambda)\mathrm{d}\lambda\psi(\mu)\mathrm{d}\mu \tag{10.10}$$

where $1 \leq |\gamma| \leq |\alpha|$. This follows since the lower order derivative terms will be bounded by similar arguments and we have already bounded the $\gamma = 0$ term.

It is important to note that $|\mathcal{R}_\alpha| \leq B_\alpha\delta^{1-|\beta_1|}\cdots\delta^{1-|\beta_l|} = B_\alpha\delta^{l-|\alpha|}$. Thus, if $|\gamma| = |\alpha|$, then $|\mathcal{R}_\alpha| = B_\alpha$. Hence, since the derivatives of $\overrightarrow{f}$ are given by the derivatives of its components f, we recover a pointwise bound similar to (10.9):

$$\begin{aligned}
&\left\|\left|W^{1/p}(x,y)\mathcal{R}_\alpha \iint D^\gamma \overrightarrow{f}(\cdot,\cdot)\mu^{\gamma'}\lambda^{\gamma_{n+1}}\phi\psi \mathrm{d}\lambda\mathrm{d}\mu\right|\right\| \\
&\leq |\mathcal{R}_\alpha| \iint \left\|\left|W^{1/p}(x,y)D^\gamma \overrightarrow{f}(x+c_1'u\Delta, y+c_2'v\Delta)\right|\right\| |\mu|^{\gamma'} |\lambda|^{\gamma_{n+1}} |\psi(\mu)| |\phi(\lambda)| \mathrm{d}\lambda\mathrm{d}\mu \\
&\leq C_{\alpha,n,M}\frac{1}{|\Delta|^{n+1}} \iint_{R(x,y)} \left\|\left|W^{1/p}(x,y)D^\gamma \overrightarrow{f}(u,v)\right|\right\| \mathrm{d}v\mathrm{d}u
\end{aligned}$$

since $\psi, \phi \in C_c^\infty$. Thus we have an estimate similar to (10.9) with $\overrightarrow{f}$ replaced by $D^\gamma \overrightarrow{f}$ and thus we can apply the Whitney cube estimates to finish bounding this particular derivative.

If $|\gamma| \leq |\alpha|$, we need $|\alpha| - |\gamma|$ powers of Δ to cancel the powers in the estimate (10.8) for $\mathcal{R}_\alpha$. Since $|\gamma| \geq 1$, there exists $j \in \{1, \ldots, n+1\}$ such that $\gamma_j \geq 1$. For such j, we then do a Taylor expansion while employing the integral form of the remainder which produces the powers of Δ necessary for bounding $\mathcal{R}_\alpha$ as follows:

If $j \in \{1, \ldots, n\}$, then for each component f of $\overrightarrow{f}$ by expanding about $\mu_j = -1$ we have that

$$\begin{aligned}
&D^\gamma f(\cdot, x_j + c_2'\mu_j\Delta, \cdot) \\
&\qquad = \sum_{\ell=0}^{|\alpha|-|\gamma|-1} \frac{(c_1'\Delta)^\ell(\mu_j+1)^\ell}{\ell!} D^{\gamma+\ell e_j} f(\cdot, x_j - c_1'\Delta, \cdot) \\
&\qquad + \frac{(c_1'\Delta)^{|\alpha|-|\gamma|}}{(|\alpha|-|\gamma|-1)!} \int_{-1}^{\mu_j} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1'u\Delta, \cdot)(\mu_j - u)^{|\alpha|-|\gamma|-1}\mathrm{d}u
\end{aligned}$$

Thus the integration with respect to μ_j in (10.10) becomes

$$\begin{aligned}
&\int_{-1}^{1} D^\gamma f(\cdot, x_j + c_1'\mu_j\Delta, \cdot)\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j \\
&= \sum_{\ell=0}^{|\alpha|-|\gamma|-1} \frac{(c_1'\Delta)^\ell}{\ell!} D^{\gamma+\ell e_j} f(\cdot, x_j - c_1'\Delta, \cdot) \int_{-1}^{1} (\mu_j+1)^\ell \mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j \\
&+ \frac{(c_1'\Delta)^{|\alpha|-|\gamma|}}{(|\alpha|-|\gamma|-1)!} \int_{-1}^{1}\int_{-1}^{\mu_j} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1'u\Delta, \cdot)(\mu_j - u)^{|\alpha|-|\gamma|-1}\mathrm{d}u\mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j
\end{aligned}$$

Here the sum term is identically zero using the moment conditions in (10.3) since $\gamma_j \geq 1$. Moreover, we can now apply Fubini's Theorem to get that

$$\int_{-1}^{1}\int_{-1}^{\mu_j} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1' u\Delta, \cdot)(\mu_j - u)^{|\alpha|-|\gamma|-1} \mathrm{d}u \mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j$$
$$= \int_{-1}^{1} D^{\gamma+(|\alpha|-|\gamma|)e_j} f(\cdot, x_j + c_1' u\Delta, \cdot) \int_{u}^{1} (\mu_j - u)^{|\alpha|-|\gamma|-1} \mu_j^{\gamma_j}\psi_j(\mu_j)\mathrm{d}\mu_j \mathrm{d}u$$

where the inside integral will be bounded by a constant depending only on α since $\psi, \phi \in C_c^{\infty}$. Therefore, by denoting $\mu' = (\mu_1, \cdots, \mu_{j-1}, \mu_{j+1}, \cdots, \mu_n)$, we have that

$$\begin{aligned}
&\left|\left|W^{1/p}(x,y)\mathcal{R}_\alpha \iint D^\gamma \overrightarrow{f}(\cdot,\cdot)\mu^{\gamma'}\lambda^{\gamma_{n+1}}\phi\psi \mathrm{d}\lambda\mathrm{d}\mu\right|\right| \\
&\quad\leq C_{\alpha,n,M}|\Delta|^{|\alpha|-|\gamma|}|\mathcal{R}_\alpha| \int_{[-1,1]^{n-1}} \int_1^L \int_{-1}^1 \left|\left|W^{1/p}(x,y)D^\beta \overrightarrow{f}(\cdot, x_j + c_1' u\Delta, \cdot)\right|\right| \\
&\qquad\qquad \cdot \int_{-1}^1 |\mu_j - u|^{|\alpha|-|\gamma|-1}|\mu_j|^{\gamma_j}|\psi_j(\mu_j)|\mathrm{d}\mu_j \mathrm{d}u\, \mathrm{d}\lambda\, \mathrm{d}\mu' \\
&\quad\leq C_{\alpha,n,M}\mathcal{B}_\alpha \int_{[-1,1]^{n-1}} \int_{-1}^1 \int_1^L \left|\left|W^{1/p}(x,y)D^\beta \overrightarrow{f}(\cdot, x_j + c_1' u\Delta, \cdot)\right|\right| \mathrm{d}\lambda\, \mathrm{d}u\, \mathrm{d}\mu' \\
&\quad\leq \frac{C_{\alpha,n,M}}{|\Delta|^{n+1}} \iint_{R(x,y)} \left|\left|W^{1/p}(x,y)D^\beta \overrightarrow{f}(u,v)\right|\right| \mathrm{d}v\mathrm{d}u
\end{aligned}$$

where $|\beta| = |\alpha|$ and the next to last inequality follows using (10.8). Thus we again have an estimate similar to (10.9) with $\overrightarrow{f}$ replaced by $D^\beta\overrightarrow{f}$. As a final case, if $j = n+1$, then we similarly expand about $\lambda = 1$ to again gain the above result.
Thus

$$||D^\alpha \overrightarrow{g}||_{L^p(\overline{D}^c, W)} \leq C_{p,k,D,W} \left|\left|\overrightarrow{f}\right|\right|_{L_k^p(D,W)}.$$

In a similar manner to the above, all lesser order derivatives will be bounded. Therefore, under the assumption that $\overrightarrow{f} \in C^\infty\left(\overline{D}\right)$, the theorem holds.

(E) It then further follows that we can extend any $\overrightarrow{f} \in L_k^p(D, W)$ given Lemma 8.4 and the argument provided in part (E) of the proof of Theorem 4.1.

(F) We can also finally relax the special Lipschitz domain assumption by repeating the argument given in part (F) of the proof of Theorem 7.1. (Here, one need only replace each symbol Ω with a D) $\square$

Chapter 11

Open Questions

In conclusion, let us consider some open questions which could provide further avenues for research in the near future. There are three angles of attack worth considering. The first is to slowly generalize our domain, which would seem standard given our results for both smooth and Lipschitz domains. The second possible way to generalize is in terms of the type of weight considered. There is also a third place for generalization: a more general space than Sobolev, such as Besov space.

11.1 A Domain Question

Perhaps a more obvious option is to try to extend to the case of (ϵ, δ) - domains. Some alteration would be required, though, because the original extension operator was defined as a sum of "fitted polynomials" which approximate the function on disjoint cubes. Still, because it is defined using cubes, there does seem to be hope that this method will work in the matrix A_p case.

11.2 A Weight Question

One natural question is whether the A_p restriction placed on the weight W in the weighted extension theorem can be relaxed. For instance, since any scalar A_p weight must also be a doubling weight, it is natural to want to relax to doubling. Also, there is a doubling condition for matrix weights (see [16]). Chua has done significant research trying to relax to the doubling condition in the scalar weighted case for extension theorems (see [4]), but the matrix weighted case has been left untouched. So this would be one avenue to pursue.

11.3 Generalizing the Space

The third and most attractive option for generalization is to work on the extension problem in the context of different function spaces. A lot of recent work has been done with defining matrix weighted Besov spaces (see [16] and [9]). Also, unweighted extension theorems have been proved for both fractional Besov spaces, also called Slobodeckij spaces (see [2]) and even the function space called BMO (see [10]). All in all, the extension problem is a rich one with much still unexplored and a host of applications.

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Vita

Ryan Loga was born Christopher Ryan Loga in Fort Oglethorpe, GA, to his parents Michael and Janene Loga. He is the middle child with an older brother, Scott, and a younger sister, Melissa. He lived the majority of his life in Apison, TN where he went to school at A. W. Spalding Elementary and continued to Collegedale Academy. For his undergraduate studies, he attended Southern Adventist University where he double majored in Mathematics and Engineering with the prospect of eventually working as an engineer. He graduated in May 2010 with an Associate of Science degree in Engineering Studies and a Bachelor of Science degree in Mathematics. Having originally planned to do graduate work in Nuclear Engineering, he applied to The University of Tennessee. After realizing his preference for math, he was able to update his application and was still able to obtain a graduate teaching assistantship. While at The University of Tennessee, he graduated with a Masters of Science degree in Mathematics in May 2014. He finally obtained his Doctor of Philosophy degree in May 2016. Post graduation he continues to teach and do research at the university level.