5-2016

# Non-Compact Solutions to Inverse Mean Curvature Flow in Hyperbolic Space

Brian Daniel Allen
*University of Tennessee - Knoxville*, ballen25@vols.utk.edu

To the Graduate Council:

I am submitting herewith a dissertation written by Brian Daniel Allen entitled "Non-Compact Solutions to Inverse Mean Curvature Flow in Hyperbolic Space." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Alexandre S. Freire, Major Professor

We have read this dissertation and recommend its acceptance:

Jochen Denzler, Mike Frazier, Seddik Djouadi

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

# Non-Compact Solutions to Inverse Mean Curvature Flow in Hyperbolic Space

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Brian Daniel Allen

May 2016

*To my father and mother, who worked tirelessly and sacrificed their own desires to give me the opportunities that they never had themselves.*

# Acknowledgements

I would like to express my gratitude to my advisor, Dr. Alex Freire, for his support and guidance throughout my time in graduate school. For much of my time in graduate school my resources in geometric analysis were limited and Alex more than made up for this with fruitful conversations and useful seminars. I would also like to thank my father and mother, Daniel and Cindy Allen, my brother and sister, John Allen and Jenna Allen, as well as my extended family for encouraging and supporting me through this completely unfamiliar journey for all of us. A special thanks goes out to all of my fellow graduate students at the University of Tennessee and in particular I would like to thank Kyle Austin, Greg Clark, Nate Dejong, Amanda Diegel, Peter Jantsch, Ernest Jum and Eddie Tu as well as my fellow geometric analysis students Caleb Castleberry, Josh Mike and Kevin Sonnanburg. Lastly, no University of Tennessee graduate student can graduate without being deeply grateful to Pam Armentrout for the guidance and support that she provided throughout the whole graduate school experience as well as the rest of the wonderful faculty and staff at the University of Tennessee.

*" And this I believe: that the free, exploring mind of the individual human is the most valuable thing in the world. And this I would fight for: the freedom of the mind to take any direction it wishes, undirected. And this I must fight against: any idea, religion, or government which limits or destroys the individual. This is what I am and what I am about."*

*John Steinbeck, East of Eden*

# Abstract

We investigate Inverse Mean Curvature Flow (IMCF) of non-compact hypersurfaces in hyperbolic space. Specifically, we look at bounded graphs over horospheres in $\mathbb{H}^{n+1}$ and show long time existence of the flow as well as asymptotic convergence to horospheres. Along the way many important interior estimates as well as global estimates are obtained. In addition, we develop a useful family of cutoff functions for IMCF as well as a non-compact ODE maximum principle at infinity which are integral tools used throughout the document.

# Table of Contents

# Chapter 1

# Introduction

In this dissertation we will be studying solutions to a geometric evolution equation called Inverse Mean Curvature Flow (IMCF). Geometric evolution equations have been an exciting research area in Mathematics for some time now with many important applications to other areas of mathematics including Topology and Mathematical Physics. We start out this introduction by giving intuition for what a geometric evolution equation is and why it is useful and then we move on to define IMCF specifically as well as talk about some of the work that has already been done on IMCF.

A geometric evolution equations is a equation, similar to a differential equation or a partial differential equation, which starts with some hypersurface, $\Sigma^n$, and prescribes a rule for how it should evolve in some ambient space, say $\mathbb{R}^{n+1}$ for definiteness. These equations are designed to improve the hypersurface, in some way that will be made more concrete below, in order to smoothly deform $\Sigma$ into a hypersurface we know more information about.

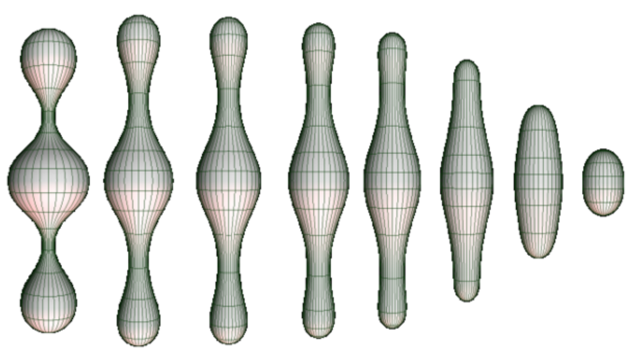

**Figure 1.1:** Example of a manifold evolving under a geometric evolution equation.

These geometric evolution equations are analogous to the heat equation which takes some initial heat distribution (in a room or thin rod, for instance) combined with a boundary condition (no heat can escape the room through the walls, or rod through the ends) and prescribes a rule for how the heat distribution should evolve in time that is consistent with our intuition of heat transfer (the heat should become uniform, throughout the room or rod, over time).

In the case of geometric evolution equations we want to think of the geometry (sectional curvature, principal curvature, mean curvature) of the hypersurface as being analogous to heat for the heat equation. In this way we expect the geometric evolution equation to cause the geometry/curvature of the hypersurface to become uniform (often meaning more and more like a sphere) or at least regularize the geometry of the hypersurface in some way.

Since geometric evolution equations are analogous to nonlinear heat equations (nonlinear parabolic PDEs) we expect the analogy described above to fail at times and cause singularities. These singularities can cause lots of problems and inspire many interesting questions that can make this area rich and exciting. For this reason, the first theorem one would often like to prove is that for a given set of initial hypersurfaces, satisfying some geometric conditions, the analogy between heat and curvature holds, without singularities.

We will investigate the geometric evolution of hypersurfaces $\Sigma^n$ through a one parameter family of embeddings $\varphi : \Sigma \times [0, T) \to N^{n+1}$, $\varphi$ satisfying inverse mean curvature flow

$$\begin{cases} \frac{\partial \varphi}{\partial t}(p,t) = \frac{\nu(p,t)}{H(p,t)} & \text{for } (p,t) \in \Sigma \times [0,T) \\ F(p,0) = \Sigma_0 & \text{for } p \in \Sigma \end{cases} \tag{1.1}$$

where $H$ is the mean curvature of $\Sigma_t := \varphi_t(\Sigma)$ and $\nu$ is a consistently chosen normal vector (we will be more specific later). In this paper we will specifically investigate $N = \mathbb{H}^{n+1}$ and we will require $\Sigma_0$ to be represented as a graph over a plane.

Claus Gerhardt (1990) and John Urbas (1990) were independently able to show that for mean convex ($H > 0$) and star-shaped hypersurfaces (See figure 3), the analogy between the heat equation holds for IMCF, which is more precisely stated in the following theorem.

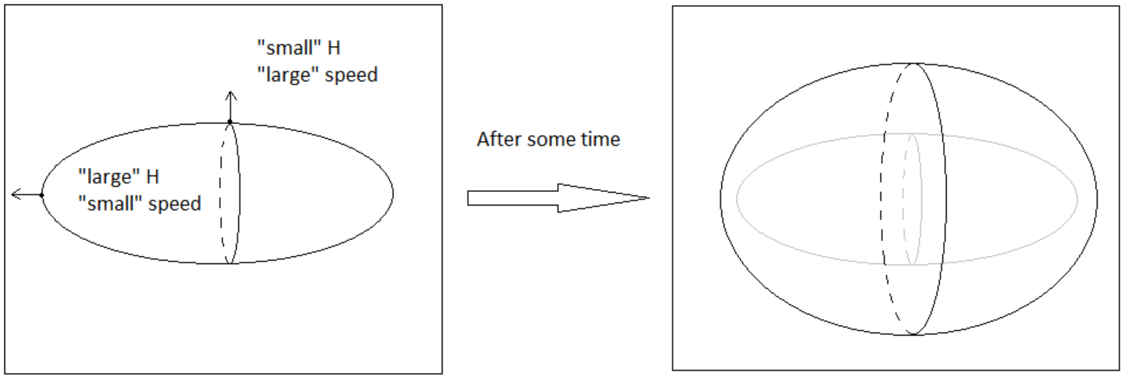


**Figure 1.2:** Illustration of a "pill" shaped hypersurface evolving under IMCF where its mean curvature is becoming more uniform over time.

**Theorem 1.** *(Gerhardt (1990); Urbas (1990)) Assume that $\Sigma^n \subset \mathbb{R}^{n+1}$ is a compact, embedded, mean convex and star-shaped hypersurface and let $\Sigma_t$ be the correspoding solution to IMCF. Then $\Sigma_t$ is well defined for all time ($T = \infty$) and the rescaled hypersurfaces $\tilde{\Sigma}_t$ ($\tilde{\varphi}(t) = e^{-t/n}\varphi(t)$) smoothly converge to spheres with radius $r = \frac{Area(\Sigma_0)}{Area(S^n)}$.*

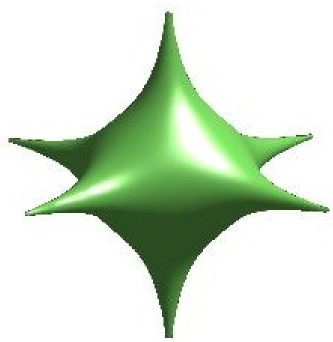

**Figure 1.3:** Example of a star-shaped and mean convex hypersurface.

Since then there have been extensions of this theorem to Lorentzian manifolds Gerhardt (2006), hyperbolic space Ding (2010); Gerhardt (2011) as well as to rotationally symmetric spaces with non-positive radial curvature Scheuer (2013). These theorems also demonstrate the analogy between the heat equation but also demonstrate the limitations of this analogy for different ambient spaces.

If one considers general mean convex hypersurfaces in $\mathbb{R}^{n+1}$ then one expects singularities to form which we would like to understand better. Since these singularities can be hard to understand directly for IMCF there has been a great deal of work on weak solutions of IMCF including viscosity solutions Chow and Gulliver (2001), weak level-set solutions through approximation by the p-Laplacian Moser (2007) as well as the most famous formulation of weak variational solutions to IMCF by Huisken and Ilmanen (2001) which were used to prove the Riemannian Penrose Inequality.

The Riemannian Penrose Inequality in General Relativity was the initial motivation for the introduction of IMCF by Geroch (1973). Roughly, the Riemannian Penrose inequality is a consistency statement for black holes which states that the area of the event horizon, $\text{Area}(\Sigma) := |\Sigma|$, is a lower bound for the mass of the black hole, $m$. More precisely the Riemannian Penrose inequality is stated for asymptotically flat manifolds 3-manifolds $M^3$.

**Definition 1.** *We say that $M^3$ is asymptotically flat if $\exists K \subset M$, compact, s.t. $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \bar{B}(0,1)$ and the metric tensor satisfies the following decay conditions*

$$|g_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, \quad |g_{ij,k}| \leq \frac{C}{|x|^2}$$

Then the Riemannian Penrose inequality, which gives a lower bound for the mass of the asymptotically flat manifold $M$ in terms of the area of its minimal surface boundary (the event horizon in this case), is given by

$$m \geq \sqrt{\frac{|\Sigma|}{16\pi}}$$

The proof idea was to consider the event horizon $\Sigma^2$ of a black hole $M^3$ as an initial hypersuraces for IMCF in the asymptotically flat 3-manifold $M^3$. Then you consider the Hawking mass defined as

$$m_H(\Sigma_t) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_\Sigma H^2 d\sigma\right)$$

So you notice that $m_H(\Sigma_0) = \sqrt{\frac{|\Sigma|}{16\pi}}$ (Since $H \equiv 0$ for event horizons in this case) and then you need to prove that $m_H(\Sigma_t)$ is montone in $t$ and that $\lim_{t\to\infty} m_H(\Sigma_t) = m$ in order to complete the proof idea.

Geroch was able to carry out his proof using IMCF in the rotationally symmetric case but the general case was left open. The problem was understanding the singularities that can occur under IMCF and his proof idea laid dormant for sometime until Huisken and Ilmanen were able to develop a variational definition of weak solutions of IMCF that was suitable for proving the Riemannian Penrose Inequality Huisken and Ilmanen (2001).

The development of IMCF for non-compact hypersurfaces inside Riemannian manifolds has not seen much consideration compared to the compact case but there are important and well understood results for non-compact Mean Curvature Flow (MCF) that should be mentioned. Ecker and Huisken (1989) were able to show convergence to a translating soliton for graphs over planes, satisfying certain initial growth conditions, in $\mathbb{R}^{n+1}$ under MCF by using a maximum principle which follows from Huisken's monotonicity formula. Later, they developed further interior estimates for non-compact MCF Ecker and Huisken (1991) as well as a non-compact maximum principle that works for a fairly general class of evolution equations with time dependent metrics including Ricci Flow.

The results of Ecker and Huisken were extended by Rasul in his dissertation thesis Rasul (2010), advised by Ecker, where he was able to relax the initial growth conditions at infinity and show "slow" convergence to a self-similar solution. Non-compact MCF was further extended to radial graphs in $\mathbb{H}^{n+1}$ by Unterberger (2003), who was able to show long time existence for locally Lipschitz entire radial graphs (graphs over $S^n_+ \subset \mathbb{H}^{n+1}$ parameterized using the upper half space model) and also convergence to a hyperplane $S^n_+$ in the upper half space model of $\mathbb{H}^{n+1}$ for entire radial graphs with bounded hyperbolic height.

The non-compact case of IMCF has seen almost no attention besides the specific examples given by Huisken and Ilmanen (2001) and the recent paper on solitons of IMCF by Drugan et al. (2015).In both cases the authors give examples of non-compact soliton solutions, which are solutions which only evolve in time be scaling some initial hypersurface. Besides these examples of special solutions there has been no work on showing convergence to a prototypical hypersurface for a class of initial data as has been done for compact IMCF for the sphere. The present work changes this by studying non-compact IMCF in Hyperbolic space.

Specifically, we look at initial hypersurfaces $\Sigma_0$ which can be represented as a graph of a bounded function, $y : \mathbb{R}^n \to \mathbb{R}$ with bounded gradient where $\mathbb{R}^n$ is the "plane at infinity" in the upper-half space model of $\mathbb{H}^{n+1}$. This hypersurface is then identified with the graph of $y(x)$, over $\mathbb{R}^n \times \{0\}$, and uniformly bounded away from $\mathbb{R}^n \times \{0\}$, in the upper half space model of hyperbolic space (i.e. $0 < y_0 < y(x) < y_1 < \infty$ for some constants $y_0, y_1 > 0$). We further assume that $\Sigma_0$ satisfies the following geometric bounds $0 < H_0 \leq H(x, 0) \leq H_1 < \infty$ and $|A|(x, 0) \leq A_0 < \infty$. For such initial hypersurfaces $\Sigma_0$ we are able to show the following

**Theorem 2.** *Let $\Sigma_t$ be a smooth solution of IMCF with initial hypersurface $\Sigma_0$ satisfying the following bounds $0 < H_0 \leq H(x, 0) \leq H_1 < \infty$ and $|A|(x, 0) \leq A_0 < \infty$. We further assume that $\Sigma_0$ can be represented as a graph of a bounded function with bounded gradient, over and uniformly bounded away from $\mathbb{R}^n \times \{0\}$ in the upper half space model of hyperbolic space. Then the IMCF starting at $\Sigma_0$ exists for all time $t \in [0, \infty)$ and asymptoticly converge to horospheres.*

We also obtain interior estimates in parabolic balls $U_R$ for solutions satisfying Theorem 2 which gives us detailed local control of the solution in terms of the initial data.

In chapter 2 we compute many important evolution equations for geometric quantities of interest. A lot of these evolution equations have also been calculated elsewhere but we put particular emphasis of the evolution equation for the support function that is defined to be well suited for studying the evolution of horospheres in hyperbolic space.

In chapter 3 we state and prove an ODE maximum principle at infinity which is the non-compact maximum principle that we will leverage throughout this document in order to obtain estimates of important geometric quantities. This theorem can be seen as a non-compact extension of the ODE maximum principle of Hamilton Hamilton (1986) which is also described in Mantegazza (2011). As far as we know, this theorem is new and we think that it is an interesting application of Omori-Yau's maximum principle at infinity for parabolic equations which should have applications elsewhere.

In chapter 4 we aim to prove short time existence for solutions of IMCF for a precise set of initial hypersurfaces, following similar arguments as in Gerhardt (2006). In this chapter we explicitly demonstrate that IMCF is a fully nonlinear, parabolic PDE by writing down a PDE, defined on $\mathbb{R}^n$, which is equivalent to IMCF up to tangential diffeomorphisms. Short time existence is proved (3) for this new PDE and the ODE relating the PDE and IMCF is explicitly stated and discussed.

In chapter 5 we extend the short time existence result of chapter 4 to a long time existence result. In this section we take advantage of the ODE maximum principle of chapter 3 to obtain upper and lower bounds on the graph function, mean curvature and the second fundamental form which culminate in a long time existence theorem (11)

In chapter 6 we extend our long time existence result of chapter 5 by studying the asymptotic behavior of solutions to IMCF. In this section we are able to show that initial hypersurfaces which satisfy the long time existence theorem will asymptotically converge to horospheres at a specified rate, demonstrating the analogy between the heat equation and non-compact solutions of IMCF in hyperbolic space.

In chapter 7 we develop cutoff function which are well suited for the study of IMCF. In doing this, we prove an important proposition (2) which allows us to find evolution equations on $\Sigma_t$ for extrinsically defined functions on $\mathbb{H}^{n+1}$ which ultimately leads to the definition of cutoff functions for IMCF. Then we use the cutoff functions to derive many local estimates, in parabolic balls relative to the cutoff functions, for important geometric quanties. This further demonstrates the relationship between IMCF and the heat equation by explicitly demonstrating the local control we have over the evolution of hypersurfaces evolving under IMCF.

# Chapter 2

# Evolution Equations under IMCF

In this section we will derive important geometric evolution equations that will be used throughout the document. We begin by specifying the notation we will use for various geometric quantities where we note that we will use bars to denote geometric quantities w.r.t $\mathbb{H}^{n+1}$, superscript 0 to denote quantities w.r.t. $\delta$, the flat metric of $\mathbb{R}^{n+1}$, and no bar or subscript to denote quantities w.r.t. $\Sigma_t$, throughout this document.

Our convention for defining the curvature tensor $\bar{R}$ of the manifold $(\mathbb{H}^{n+1}, \bar{g})$ is given by

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z$$

where $X, Y, Z \in T_p\mathbb{H}^{n+1}$ for $p \in \mathbb{H}^{n+1}$. Then for $W \in T_p\mathbb{H}^{n+1}$ we have that

$$R(X,Y,Z,W) = \bar{g}(R(X,Y)Z, W)$$

and the Ricci Tensor is defined as

$$Rc(X,Y) = \sum_{i=1}^{n+1} R(X, \bar{e}_i, \bar{e}_i, Y)$$

where $\{\bar{e}_1, ..., \bar{e}_{n+1}\}$ is an orthonormal frame for $T_p\mathbb{H}^{n+1}$. When it comes to extrinsic curvature quantities of $\Sigma_t$ we define the second fundamental form, $A$, of $\Sigma_t$ to be

$$A(X, Y) = \bar{g}(\bar{\nabla}_X \bar{\nu}, Y)$$

where $X, Y \in T_x\Sigma_t$ for $x \in \Sigma_t$ and $\bar{\nu}$ is the consistently chosen unit normal vector to $\Sigma_t$ in $\mathbb{H}^{n+1}$. From this definition we can define the mean curvature

$$H = \sum_{i=1}^{n} A(e_i, e_i)$$

where $\{e_1, ..., e_n\}$ is an orthonormal frame for $T_x\Sigma_t$.

Since we are concerned with studying IMCF inside of $\mathbb{H}^{n+1}$ it is convenient to have a particular model of Hyperbolic space in mind. For this paper it is useful to use the upper half space model of $\mathbb{H}^{n+1}$ which is defined on the space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ with coordinates $(x_1, ..., x_n, y)$ and the following metric

$$\bar{g} = \frac{1}{y^2}\left(dx_1^2 + ... + dx_n^2 + dy^2\right)$$

We denote the coordinate basis vectors as $\partial_{x_1}, ..., \partial_{x_n}, \partial_y = \partial_{x_{n+1}}$ which leads to the following notation $\bar{R}_{ijkl} = \bar{R}(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}, \partial_{x_l})$ , $\bar{R}_{ij} = Rc(\partial_{x_i}, \partial_{x_j})$ and $A_{ij} = A(\partial_{x_i}, \partial_{x_j})$. It then follows that we can find the following expressions for the curvature tensor, Ricci tensor and Scalar curvature in terms of the flat metric and the $y$ coordinate

$$\begin{aligned}\bar{R}_{ijkl} &= \frac{1}{y^4}\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)\\ \bar{R}_{ij} &= -n\frac{1}{y^2}\delta_{ij}\\ \bar{R} &= -n(n+1)\end{aligned}$$

One should note that this model is particularly convenient for us since Horospheres have such a simple parameterization as hyperplanes with constant $y$ coordinate. Since we are concerned with studying graphs over horospheres this means that we will be able to express the evolution of the hypersurfaces of interest as a fully nonlinear PDE defined on $\{y = y_0\} \cong \mathbb{R}^n$, the set which the hypersurface is a graph over.

Note that horospheres in the upper half space model of Hyperbolic space can also be parameterized as spheres that are tangent to the boundary plane $\{y = 0\}$. This is not the parameterization that we want to consider and fortunately there is an orientation reversing isometry, given by inversion through a half circle centered on the boundary plane $\{y = 0\}$, which sends horospheres parameterized by circles tangent to the boundary plane to horospheres parameterized by planes. So, without loss of generality, we may assume that we are considering a horosphere parameterized as a plane with constant $y$ coordinate.

By using well known formulas for conformal metrics we can find the following expression

$$\bar{\nabla}_X Y = \nabla^0_X Y - \frac{1}{y}\left(\langle X, \partial_y\rangle_0 Y + \langle \partial_y, Y\rangle_0 X - \langle X, Y\rangle_0 \partial_y\right) \tag{2.1}$$

which expresses the covariant derivative w.r.t. $\mathbb{H}^{n+1}$ in terms of the geometric of $\mathbb{R}^{n+1}_+$.

Using this, and the convention that we will put a bar over a vector field $\bar{Z} = yZ$ so that $\bar{Z}$ is a unit vector w.r.t. $\bar{g}$, we can obtain the following

$$\begin{aligned} d\bar{i}vX &= \sum_{i=1}^{n+1} \bar{g}(\bar{\nabla}_{\bar{e}_i} X, \bar{e}_i) \\ &= \sum_{i=1}^{n+1} \langle \nabla^0_{e_i} X - \frac{1}{y}(\langle e_i, \partial_y\rangle_0 X + \langle \partial_y, X\rangle_0 e_i - \langle e_i, X\rangle_0 \partial_y, e_i\rangle_0 \\ &= div^0 X - (n+1)\frac{1}{y}\langle X, \partial_y\rangle_0 \end{aligned}$$

where $\{e_1, ..., e_{n+1}\}$ is a orthonormal basis for $\mathbb{R}^{n+1}$ w.r.t the flat metric. Then by applying the equation above to the vector $\bar{\nu}$, the "downward" pointing unit normal vector which is well defined since $\Sigma_t$ will be a graph over the boundary plane in the upper half space model of $\mathbb{H}^{n+1}$, we obtain the following formula for the mean curvature $H$ where we are using the fact that $H = d\bar{i}v(\bar{\nu})$

$$H = -yH_0 - n\langle \nu_0, \partial_y \rangle_0 = -yH_0 + n\langle \nu_0, \eta \rangle_0 \tag{2.2}$$

where $\eta = -\partial_y$ and $\partial_y$ is the coordinate unit vector w.r.t $\delta$.

**Note:** The "downward" pointing normal is a good choice since Horospheres have mean curvature $H = n$ with this choice, instead of negative.

Throughout this document we will be using the fact that the hypersurfaces we are interested in can be expressed as graphs of a function $y : \mathbb{R}^n \to \mathbb{R}$, i.e. parameterized by $F : \mathbb{R}^n \to \mathbb{R}^{n+1}_+$ so that $F(x) = (x, y(x))$. From (2.2) we can find an important expression for $H$ in terms of the graph function $y(x)$ where $\bar{\nu} = \frac{y(\nabla^0 y, -1)}{\sqrt{1+|\nabla^0 y|^2}}$, $v = \sqrt{1 + |\nabla^0 y|^2}$, $\tilde{\delta}^{ij} = \delta^{ij} - \frac{y^i y^j}{v^2}$ and we denote $\frac{\partial y}{\partial x_i} := y_i$ and $\frac{\partial^2 y}{\partial x_i \partial x_j} := y_{ij}$

$$H = \frac{n + y\tilde{\delta}^{ij} y_{ij}}{v}$$

We note that $g^{ij} = y^2 \tilde{\delta}^{ij}$ gives an expression for the inverse induced metric ($g_{ij} = \frac{1}{y^2}(\delta_{ij} + y_i y_j)$) of the hypersurface $\Sigma_t$ in terms of the graph parameterization.

Now we can use the formula (2.1) to notice that the vector field $\partial_y$ satisfies the following equation in $\mathbb{H}^{n+1}$

$$\bar{\nabla}_X \partial_y = -\frac{1}{y} X \quad \forall X \tag{2.3}$$

and hence if we define $\eta := -\partial_y$ then we have $\bar{\nabla}_X \eta = \frac{1}{y} X$. The equation (2.3) is important because it will lead to a tame evolution equation for the support function which we will use to gain control on almost every other geometric quantity of interest. So if we define the support function $w(x) = \bar{g}(\eta, \bar{\nu}) = \frac{1}{yv}$ then we can find its evolution equation below where we will require the following lemma.

**Lemma 1.** *(Evolution equations under smooth IMCF)*

$$\frac{\partial g_{ij}}{\partial t} = 2\frac{A_{ij}}{H}$$
$$\frac{\partial g^{ij}}{\partial t} = -2\frac{A^{ij}}{H}$$
$$\frac{\partial \nu}{\partial t} = \frac{\nabla H}{H^2}$$

*Proof.* Let $\{e_1, ..., e_n\}$ be a normal frame at a point $p \in \Sigma_t$, then we compute

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \bar{\nabla}_{\frac{\partial \varphi}{\partial t}} \langle e_i, e_j \rangle = \langle \bar{\nabla}_{\frac{\partial \varphi}{\partial t}} e_i, e_j \rangle + \langle e_i, \bar{\nabla}_{\frac{\partial \varphi}{\partial t}} e_j \rangle \\
&= \langle \bar{\nabla}_{e_i} \frac{\partial \varphi}{\partial t}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \frac{\partial \varphi}{\partial t} \rangle \\
&= \langle \bar{\nabla}_{e_i} \left( \frac{\nu}{H} \right), e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \left( \frac{\nu}{H} \right) \rangle \\
&= \frac{1}{H} \langle \bar{\nabla}_{e_i} \nu, e_j \rangle + \frac{1}{H} \langle e_i, \bar{\nabla}_{e_j} \nu \rangle \\
&= 2\frac{A_{ij}}{H}
\end{aligned}$$

Then use $\frac{\partial g^{ij}}{\partial t} = -g^{ik} \frac{\partial g_{kl}}{\partial t} g^{lj}$ in order to obtain the second expression in the statement of the Lemma.

$$\begin{aligned}
\frac{\partial \nu}{\partial t} &= g^{ij} e_i \langle \frac{\partial \nu}{\partial t}, e_j \rangle = -g^{ij} e_i \langle \nu, \bar{\nabla}_{e_j} \frac{\partial \varphi}{\partial t} \rangle = -g^{ij} e_i \langle \nu, \bar{\nabla}_{e_j} \left( \frac{\nu}{H} \right) \rangle \\
&= -\nabla \left( \frac{1}{H} \right) = \frac{\nabla H}{H^2}
\end{aligned}$$

□

Now we can compute the evolution of the support function. One will notice that the evolution equation for $w$ involves the operator $\frac{1}{H^2}\Delta$ instead of $\Delta$ which has to do with the cancelation that occurs between the $\frac{\partial}{\partial t}$ terms and the $\frac{1}{H^2}\Delta$ terms in the computation below.

It turns out that $(\partial_t - \frac{1}{H^2}\Delta)$, where we denote $\partial_t = \frac{\partial}{\partial t}$, is the most natural heat operator for IMCF and hence will show up in the evolution equation of every geometric quantity of interest in the rest of this document.

**Lemma 2.**

$$(i)\ \ (\partial_t - \frac{1}{H^2}\Delta)w = \frac{|A|^2}{H^2}w$$
$$(ii)\ \ (\partial_t - \frac{1}{H^2}\Delta)w^{-1} = -\frac{|A|^2}{H^2}w^{-1} - \frac{2}{w^{-1}H^2}|\nabla w^{-1}|^2$$

**Note:** The evolution of the support function above is the same as the evolution of the support function $w(x) = \langle X, \nu_0 \rangle_0$ in Euclidean space where $X$ is the position vector of the hypersurface $\Sigma_t$. This is because the vector field $X$ satisfies a similar equation to (2.3) in Euclidean space.

*Proof.* In order to compute the laplacian of $w$ we will strongly take advantage of the fact that $\eta$ satisfies equation (2.3). Let $\{e_1, ..., e_n\}$ be a normal frame for $\Sigma_0$ and compute

$$\begin{aligned}\nabla_i w &= \bar{g}(\bar{\nabla}_i \eta, \bar{\nu}) + \bar{g}(\eta, \bar{\nabla}_i \bar{\nu})\\ &= \bar{g}(\frac{e_i}{y}, \bar{\nu}) + A(\eta^T, e_i) = A(\eta^T, e_i)\end{aligned}$$

where $\eta^T$ is the projection of $\eta$ onto the tangent space of $\Sigma_t$.

Then we can compute the Laplacian of $w$

$$\begin{aligned}\Delta w &= g^{ij}\nabla_i\nabla_j w\\ &= g^{ij}\nabla_i(A(\eta^T, e_j))\\ &= g^{ij}\left[(\nabla_i A)(\eta^T, e_i) + A(\nabla_i \eta^T, e_i)\right]\end{aligned}$$

Now in order to better understand the term $A(\nabla_i \eta^T, e_i)$ we find

$$
\begin{aligned}
\frac{1}{y}e_i = \bar{\nabla}_i\eta &= \bar{\nabla}_i(\eta^T + \bar{g}(\eta,\bar{\nu})\bar{\nu}) \\
&= \nabla_i\eta^T - A(e_i,\eta^T)\bar{\nu} + \bar{g}(\bar{\nabla}_i\eta,\bar{\nu})\bar{\nu} + \bar{g}(\eta,\bar{\nabla}_i\bar{\nu})\bar{\nu} + \bar{g}(\eta,\bar{\nu})\bar{\nabla}_i\bar{\nu} \\
&= \nabla_i\eta^T + \bar{g}(\eta,\bar{\nu})\bar{\nabla}_i\bar{\nu}
\end{aligned}
$$

If we rearrange this we find that

$$
\nabla_i\eta^T = \frac{1}{y}e_i - w\bar{\nabla}_i\bar{\nu}
$$

and so we can find

$$
\sum_{i=1}^{n} A(\nabla_i\eta^T, e_i) = \frac{1}{y}H - w|A|^2
$$

By using the Codazzi equation which says that $\nabla_X A(Y,Z) = \nabla_Y A(X,Z) - R(X,Y,Z,\bar{\nu})$ for $X, Y, Z \in T_x\Sigma_t$ and so in coordinates we find

$$
\begin{aligned}
\sum_{i=1}^{n}(\nabla_i A)(\eta^T, e_i) &= \sum_{i=1}^{n}(\nabla_{\eta^T}A)(e_i,e_i) - \bar{R}(e_i,\eta^T,e_i,\bar{\nu}) \\
&= \bar{g}(\nabla H,\eta^T) + \bar{R}c(\bar{\nu},\eta^T) = \bar{g}(\nabla H,\eta^T)
\end{aligned}
$$

where we are using the fact that $\bar{R}c(\bar{\nu},\eta^T) = \bar{g}(\bar{\nu},\eta^T) = 0$ for $\mathbb{H}^{n+1}$.

So now if we put all of this together we find

$$
\Delta w = \frac{1}{y}H - w|A|^2 + \bar{g}(\nabla H,\eta^T)
$$

Now if we compute the time derivative of $w$ under IMCF we find

$$
\begin{aligned}
\frac{\partial w}{\partial t} &= \bar{g}(\frac{\partial \eta}{\partial t}, \bar{\nu}) + \bar{g}(\eta, \frac{\partial \bar{\nu}}{\partial t}) \\
&= \frac{1}{H}\bar{g}(\bar{\nabla}_{\bar{\nu}}\eta, \bar{\nu}) + \frac{1}{H^2}\bar{g}(\eta^T, \nabla H) \\
&= \frac{1}{yH} + \frac{1}{H^2}\bar{g}(\eta^T, \nabla H)
\end{aligned}
$$

From these two equations the evolution equation for $w$ follows. From the evolution equation for $w$ we can also find the evolution equation for $w^{-1}$ which will help us get a lower bound on the support function $w$.

$$
(\partial_t - \frac{1}{H^2}\Delta)w^{-1} = -w^{-2}(\partial_t - \frac{1}{H^2}\Delta)w - \frac{2}{w^3H^2}|\nabla w|^2 = -\frac{|A|^2}{H^2}w^{-1} - \frac{2}{w^{-1}H^2}|\nabla w^{-1}|^2
$$

□

Now we will compute the rest of the evolution equations that we will use throughout the whole document. One particular quantity of interest will be $u = \frac{1}{wH}$ which is used to leverage the good evolution equation of $w$ in order to kill the bad terms in the evolution equation of $H$. This is important because it allows us to obtain a lower bound on $H$ which is the most important estimate for IMCF.

**Lemma 3.** *(Evolution equations under smooth IMCF in $\mathbb{H}^{n+1}$)*

$$
\begin{aligned}
(i)\quad & (\partial_t - \frac{1}{H^2}\Delta)A_{ij} = -\frac{2}{H^3}\nabla_i H \nabla_j H + \left(\frac{|A|^2}{H^2} + \frac{n}{H^2}\right)A_{ij} \\
(ii)\quad & (\partial_t - \frac{1}{H^2}\Delta)A_i^j = -\frac{2}{H^3}\nabla_i H \nabla^j H + \left(\frac{|A|^2}{H^2} + \frac{n}{H^2}\right)A_i^j - \frac{2}{H}A_{il}A^{jl} \\
(iii)\quad & (\partial_t - \frac{1}{H^2}\Delta)H = -2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} + \frac{n}{H} \\
(iv)\quad & (\partial_t - \frac{1}{H^2}\Delta)u = 2\frac{g(\nabla w, \nabla u)}{H^2 w^2} - \frac{nu}{H^2}
\end{aligned}
$$

**Note:** The terms $\frac{n}{H}$ or $\frac{n}{H^2}$ correspond to the curvature of $\mathbb{H}^{n+1}$.

*Proof.* In order to compute the evolution equation for $A_{ij}$ we start by computing its time derivative. Let $\{e_1, ..., e_n\}$ be a normal frame at a point $p \in \Sigma_t$ and denote $\nu$ as an input to a tensor with a subscript 0, then we compute

$$\begin{aligned}
\frac{\partial A_{ij}}{\partial t} &= -\bar{\nabla}_{\frac{\partial\varphi}{\partial t}} \langle \bar{\nabla}_{e_i} e_j, \nu \rangle \\
&= -\langle \bar{\nabla}_{\frac{\partial\varphi}{\partial t}} \bar{\nabla}_{e_i} e_j, \nu \rangle - \langle \bar{\nabla}_{e_i} e_j, \bar{\nabla}_{\frac{\partial\varphi}{\partial t}} \nu \rangle \\
&= -\langle \bar{\nabla}_{e_i} \bar{\nabla}_{\frac{\partial\varphi}{\partial t}} e_j, \nu \rangle - \frac{1}{H} \bar{R}_{0ij0} - \langle \bar{\nabla}_{e_i} e_j, \bar{\nabla}_{\frac{\partial\varphi}{\partial t}} \nu \rangle \\
&= -\langle \bar{\nabla}_{e_j} \bar{\nabla}_{e_i} \frac{\partial\varphi}{\partial t}, \nu \rangle - \frac{1}{H} \bar{R}_{0ij0} - \frac{1}{H^2} \cancel{\langle \bar{\nabla}_{e_i} e_j, \nabla H \rangle} \\
&= -\langle \bar{\nabla}_{e_j} \left( \frac{1}{H} \bar{\nabla}_{e_i} \nu + \nu \bar{\nabla}_{e_i} \left( \frac{1}{H} \right) \right), \nu \rangle - \frac{1}{H} \bar{R}_{0ij0} \\
&= \frac{1}{H} \langle \bar{\nabla}_{e_i} \nu, \bar{\nabla}_{e_j} \nu \rangle + \bar{\nabla}_{e_i} \left( \frac{1}{H} \right) \cancel{\langle \nu, \bar{\nabla}_{e_j} \nu \rangle} - \bar{\nabla}_{e_j} ( \frac{1}{H} \cancel{\langle \bar{\nabla}_{e_i} \nu, \nu \rangle}) - \nabla_{e_j} \nabla_{e_j} \left( \frac{1}{H} \right) - \frac{1}{H} \bar{R}_{0ij0} \\
&= \frac{A_{ik} A^k_j}{H} - \frac{\bar{R}_{0ij0}}{H} + \frac{\nabla_{e_j} \nabla_{e_j} H}{H^2} - 2 \frac{1}{H^3} \nabla_i H \nabla_j H
\end{aligned}$$

where we commute derivatives using the extrinsic curvature tensor in line 1, use the fact that $[\frac{\partial\varphi}{\partial t}, e_i] = 0$ since they are coordinate vectors in line 2, the definition of $\frac{\partial\varphi}{\partial t}$ in line 3 and 4, and identify and clean up terms in lines 4 and 5.

Then we combine with the well know formula for the laplacian of $A_{ij}$ known as Simons' equation

$$\Delta A_{ij} = \nabla_i \nabla_j H + H A_{ik} A^k_j - A_{ij} |A|^2 + H \bar{R}_{0i0j} - n A_{ij}$$

which can be derived by using the curvature tensor to commute derivatives, the Codazzi equation, and the relationship between the ambient derivative and the intrinsic derivative as follows:

$$
\begin{aligned}
\Delta A_{ij} &= g^{pq}\nabla_p\nabla_q A_{ij} \\
&= g^{pq}\nabla_p\nabla_i A_{qj} + g^{pq}\nabla_p \bar{R}_{qij0} \\
&= g^{pq}\left[\nabla_i\nabla_p A_{qj} + R^l_{piq}A_{jl} + R^l_{pij}A_{ql} + \nabla_p\bar{R}_{qij0}\right] \\
&= \nabla_i\nabla_j H + g^{pq}[R^l_{piq}A_{jl} + R^l_{pij}A_{ql} + \nabla_p\bar{R}_{qij0} + \nabla_i\bar{R}_{pjq0}] \\
&= \nabla_i\nabla_j H + g^{pq}[(A_{pq}A^l_i - A^l_p A_{iq})A_{jl} + (A_{pj}A^l_i - A^l_p A_{ij})A_{ql} - A_{pq}\bar{R}_{0ij0} + A_{ij}\bar{R}_{0pq0}] \\
&= \nabla_i\nabla_j H + HA_{ik}A^k_j - A_{ij}|A|^2 + H\bar{R}_{0i0j} - nA_{ij}
\end{aligned}
$$

where we use the Codazzi equation in line 2, the intrinsic curvature tensor in line 3 to permute second derivative of $A$, the Codazzi equation again in line 4 and the Gauss equations in line 4 together with the fact that $\bar{\nabla}\bar{R} = 0$ for $\mathbb{H}^{n+1}$. See Huisken (1986) for a version of Simons' identity for general Riemannian manifolds.

By combining the equations above we find the following evolution equation for $A_{ij}$

$$
(\partial_t - \frac{1}{H^2}\Delta)A_{ij} = -\frac{2}{H^3}\nabla_i H\nabla^j H + \frac{|A|^2}{H^2}A_{ij} + \frac{n}{H^2}A_{ij}
$$

The evolution equation for $H$ and $A^j_i$ is found by considering that $H = g^{ij}A_{ij}$, $A^j_i = g^{ik}A_{jk}$ and using the evolution equation for $A_{ij}$, $g_{ij}$ and $g^{ij}$.

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)H &= (\partial_t - \frac{1}{H^2}\Delta)(g^{ij}A_{ij}) \\
&= g^{ij}(\partial_t - \frac{1}{H^2}\Delta)A_{ij} + A_{ij}\frac{\partial g^{ij}}{\partial t} \\
&= -\frac{2}{H^3}|\nabla H|^2 + \frac{|A|^2}{H} + \frac{n}{H} - 2\frac{|A|^2}{H} \\
&= -\frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H} + \frac{n}{H}
\end{aligned}
$$

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)A_i^j &= (\partial_t - \frac{1}{H^2}\Delta)(g_{ik}A^{jk}) \\
&= g^{jk}(\partial_t - \frac{1}{H^2}\Delta)A_{ik} + A_{ik}\frac{\partial g^{jk}}{\partial t} \\
&= -\frac{2}{H^3}\nabla_i H\nabla^j H + \frac{|A|^2}{H^2}A_i^k + \frac{n}{H^2}A_i^j - 2\frac{A_{ik}A^{jk}}{H} \\
&= -\frac{2}{H^3}\nabla_i H\nabla^j H + \left(\frac{|A|^2}{H^2} + \frac{n}{H^2}\right)A_i^j - \frac{2}{H}A_{ik}A^{jk}
\end{aligned}
$$

Now we find the evolution equation for $u$

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)u &= \frac{-1}{Hw^2}(\partial_t - \frac{1}{H^2}\Delta)w - \frac{1}{H^2 w}(\partial_t - \frac{1}{H^2}\Delta)H - 2\frac{|\nabla H|}{H^5 w} - 2\frac{g(\nabla w, \nabla H)}{H^4 w^2} - 2\frac{|\nabla w|^2}{H^3 w^3} \\
&= 2\frac{g(\nabla w, \nabla u)}{H^2 w} - \frac{n}{H^3 w}
\end{aligned}
$$

where we have used the fact that $\nabla u = -\frac{\nabla H}{wH^2} - \frac{\nabla w}{Hw^2}$.

$\square$

# Chapter 3

# Short Time Existence

The goal of this chapter is to prove short time existence to IMCF in Theorem (3) for the case of non-compact graphs over the plane $\{y = 0\}$, satisfying certain conditions which we will be specific about later. We mostly follow the proof of short time existence given in Gerhardt (2006) where he shows short time existence in the compact case. We will prove this short time existence for functions in the following function space which we define through the following sequence of definitions.

**Definition 2.** *In $\mathbb{R}^n \times [0, T)$ we defined the parabolic distance between $p_1 = (\boldsymbol{x}_1, t_1)$ and $p_2 = (\boldsymbol{x}_2, t_2)$ as*

$$\rho(p_1, p_2) = |\boldsymbol{x}_1 - \boldsymbol{x}_2| + |t_1 - t_2|^{1/2}$$

**Definition 3.** *For $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$, $\alpha \in (0, 1)$ we define*

$$[u]_{\alpha/2,\alpha} = \sup_{p_1 \neq p_2} \frac{|u(p_1) - u(p_2)|}{\rho(p_1, p_2)^\alpha}$$

$$|u|_0 = \sup_{\mathbb{R}^n \times [0,T)} |u|$$

$$|u|_{\alpha/2,\alpha} = |u|_0 + [u]_{\alpha/2,\alpha}$$

**Definition 4.** *We define* $C^{\alpha/2,\alpha}(\mathbb{R}^n \times [0,T))$ *as the set of all functions* $u$ *so that* $|u|_{\alpha/2,\alpha} < \infty$

*Also, we define* $C^{1+\alpha/2,2+\alpha}$ *as the set of all funcions* $u$ *so that*

$$[u]_{1+\alpha/2,2+\alpha} := [u_t]_{\alpha/2,\alpha} + \sum_{i,j=1}^{n} [u_{x_i x_j}]_{\alpha/2,\alpha} < \infty$$

*and*

$$|u|_{1+\alpha/2,\alpha} := |u|_0 + |u_x|_0 + |u_t|_0 + \sum_{i,j=1}^{n} |u_{x_i x_j}|_0 + [u]_{1+\alpha/2,2+\alpha} < \infty$$

First we notice that for $\psi : \Sigma \times [0,T) \to \mathbb{H}^{n+1}$ the following flow

$$\left(\frac{\partial \psi}{\partial t}\right)^{\perp} = \frac{\nu}{H} \tag{3.1}$$

is, up to tangential diffeomorphisms, equivalent to IMCF (See Lemma (5) below ). So the point of this chapter is to prove short time existence to (3.1) which in turn gives us short times existence to (1.1) for bounded graphs in Hyperbolic space satisfying bounds mentioned below.

Now if we write $M_t$ as a graph over $\{y = 0\}$ using a function $y : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ then we have the expressions $\psi(x,t) = (x, y(x,t))$ and $\bar{\nu} = \frac{y(\nabla^0 y, -1)}{\sqrt{1+|\nabla^0 y|^2}}$ . So we notice that

$$\bar{g}\left(\frac{\partial \psi}{\partial t}, \nu\right) = \frac{-1}{y\sqrt{1+|\nabla^0 y|^2}} \frac{\partial y}{\partial t} = \frac{1}{H} \quad \Rightarrow \quad \frac{\partial y}{\partial t} = \frac{-y\sqrt{1+|\nabla^0 y|^2}}{H} = \frac{-1}{wH} = \frac{-vy}{H}$$

where we have used the notation $v := \sqrt{1+|\nabla^0 y|^2}$ and the fact that $w = \bar{g}(\partial_y, \bar{\nu}) = \frac{1}{vy}$.

Now if we use the fact that $H = \frac{n + y\tilde{\delta}^{ij} y_{ij}}{v}$, where we denote $\frac{\partial y}{\partial x_i} := y_i$, $\frac{\partial^2 y}{\partial x_i \partial x_j} := y_{ij}$ and recall that $\tilde{\delta}^{ij} = \delta^{ij} - \frac{y^i y^j}{v^2}$, then we find

$$\frac{\partial y}{\partial t} = \frac{-yv^2}{n + y\tilde{\delta}^{ij}y_{ij}} = F(x, y, \nabla^0 y, \nabla^0\nabla^0 y) \tag{3.2}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$, denoted $F(x, u, p_i, a_{ij})$, is a fully nonlinear operator and hence (3.2) is a fully nonlinear parabolic PDE.

$$\frac{\partial F}{\partial a_{kl}} = \frac{yv^2}{(n + y\tilde{\delta}^{ij}y_{ij})^2} y\tilde{\delta}^{kl} = \frac{y^2}{H^2}\tilde{\delta}^{kl}$$

So if our initial condition $y_0(x) \in \Lambda$ where

$$\Lambda := \{y \in C^2(\mathbb{R}^n) : 0 < H_1 < H(x) < H_2 < \infty, 0 < y_0 < y(x) \leq y_1 < \infty \text{ and } v(x) < v_0 < \infty\}$$

and $H(x)$ is the mean curvature of the graph of $y(x)$ then we have that $\frac{\partial F}{\partial a_{kl}} \geq \frac{y_0^2}{H_0^2}\delta_{kl}$ as symmetric matrices and so the linearized operator is uniformly parabolic for functions belonging to $\Lambda$.

Now we state and prove short time existence for (3.2) where we will use the notation that $U_T = \mathbb{R}^n \times [0, T)$ throughout.

**Theorem 3.** *Let $F$ be the operator defined above and let $y_0 \in \Lambda \cap C^{2+\alpha}(\mathbb{R}^n)$ where $\alpha \in (0, 1)$. Then, for any $0 < \beta < \alpha$, the initial value problem*

$$\begin{cases} y_t - F(x, y, \nabla^0 y, \nabla^0\nabla^0 y) & = 0 \\ y(x, 0) & = y_0(x) \end{cases} \tag{3.3}$$

*has a unique solution $y \in C^{\frac{2+\beta}{2}, 2+\beta}(U_\epsilon)$, where $\epsilon$ depends only on $\beta$ and $y_0$.*

*Proof.* This proof will be given in four steps.

**Step 1:** Let $\hat{y}$ be a solution to the linear parabolic problem

$$\begin{cases} \hat{y}_t - \Delta \hat{y} & = F(x, y_0, \nabla^0 y_0, \nabla^0 \nabla^0 y_0) - \Delta y_0 \\ \hat{y}(x, 0) & = y_0(x) \end{cases}$$

by standard linear PDE theory Krylov (1987a) we know that this PDE has a solution $\hat{y} \in C^{2+\alpha, \frac{2+\alpha}{2}}(U_T)$ (for any $T > 0$) with the following bound (independent of $T$)

$$\|\hat{y}\|_{\frac{2+\alpha}{2}, 2+\alpha} \leq N(n, \alpha) \left( \|y_0\|_{2+\alpha} + \|F(y_0)\|_\alpha + \|\Delta y_0\|_\alpha \right)$$

where we note that $\|y_0\|_{2+\alpha} \leq C_1, \|\Delta y_0\|_\alpha \leq C_2$ is implied by our assumptions on $y_0$.

The bound on $\|F(y_0)\|_\alpha$ follows from the fact that $\hat{y}, \nabla^0 \hat{y}, \nabla^0 \nabla^0 \hat{y} \in C^\alpha(\mathbb{R}^n)$ combined with the fact that if $u, v \in C^\alpha(\mathbb{R}^n)$ then $uv \in C^\alpha(\mathbb{R}^n)$ and $\frac{u}{v} \in C^\alpha(\mathbb{R}^n)$ as long as $v > v_0 > 0$, is bounded away from zero (Also $F(y_0) = -\frac{v(y_0) y_0}{H(y_0)}$).

Now we can choose $T_0 \leq T$ small enough so that for all $t \in [0, T_0]$

$$\hat{y}(\cdot, t) \in \Lambda \tag{3.4}$$

where this follows from the fact that $\|\hat{y}\|_{\frac{2+\alpha}{2}, 2+\alpha} \leq C$ and hence cannot immediately escape $\Lambda$ by continuity in $t$ of the $C^{2+\alpha}$ norm.

The idea is that we are going to linearize the nonlinear operator (3.2) at the solution $\hat{y}(\cdot, t)$ and so (3.4) implies that $F$ is parabolic at $\hat{y}$.

Now it will also be useful to define $\hat{f}(x, t) \in C^{\alpha, \frac{\alpha}{2}}(U_{T_0})$ to be

$$\hat{f} = \hat{y}_t - F(x, \hat{y}, \nabla^0\hat{y}, \nabla^0\nabla^0\hat{y})$$

from which we see that $\hat{f}(x, 0) = 0$.

**Step 2:** In this step we would like to employ the Inverse Function Theorem to the map $\Phi : V := \Lambda \cap C^{2+\beta, \frac{2+\beta}{2}}(U_{T_0}) \to W \subset C^{\beta, \frac{\beta}{2}}(U_{T_0}) \times C^{2+\beta}(\mathbb{R}^n)$ defined by

$$\Phi(y) = \big(y_t - F(x, y, \nabla^0 y, \nabla^0\nabla^0 y), y(x, 0)\big)$$

where $V$ is a neighborhood of $\hat{y}$ and $W$ is a neighborhood of $\Phi(\hat{y}) = (\hat{f}, y_0)$.

We notice that $\Phi$ is continuously differentiable on $V$ and its derivative, $D\Phi$ evaluated at $\hat{y} \in V$, is equal to the following operator

$$\begin{aligned} D\Phi(\hat{y}) &: C^{2+\beta, \frac{2+\beta}{2}} \to C^{\beta, \frac{\beta}{2}} \times C^{2+\beta} \\ D\Phi(\hat{y})[\eta] &= \left(\eta_t - \frac{\partial F}{\partial a_{ij}}\eta_{ij} - \frac{\partial F}{\partial p_i}\eta_i - \frac{\partial F}{\partial u}\eta, \eta(0)\right) \end{aligned}$$

defined for $\eta \in C^{2+\beta, \frac{2+\beta}{2}}(U_{T_0})$. We have already explicity computed $\frac{\partial F}{\partial a_{ij}}$ above and noticed that it was an elliptic operator but we can also calculate $\frac{\partial F}{\partial p_i}$ and $\frac{\partial F}{\partial u}$, as follows.

$$\begin{aligned} \frac{\partial F}{\partial p_i}|_{\hat{y}} &= \frac{-2yy_i}{n + y\tilde{\delta}^{ij}y_{ij}} + \frac{y^2v^2}{(n + y\tilde{\delta}^{ij}y_{ij})^2}\left(\frac{2y_iy_ky_j}{v^4}y_{kj} - \frac{y_j}{v^2}y_{ij}\right) \quad \Rightarrow \quad \left|\frac{\partial F}{\partial p_i}\right| \leq \frac{2y}{H} + \frac{y^2}{H^2}|\nabla^0\nabla^0 y| \\ \frac{\partial F}{\partial u}|_{\hat{y}} &= \frac{-v^2}{n + y\tilde{\delta}^{ij}y_{ij}} + \frac{yv^2}{(n + y\tilde{\delta}^{ij}y_{ij})^2}\tilde{\delta}^{ij}y_{ij} = \frac{-nv^2}{(n + y\tilde{\delta}^{ij}y_{ij})^2} = \frac{-n}{H^2} \quad \Rightarrow \quad \left|\frac{\partial F}{\partial u}\right| \leq \frac{n}{H^2} \end{aligned}$$

So we see that these coefficients do not present a problem as long as $\hat{y}(\cdot, t) \in \Lambda$, which we confirmed in Step 1, and so the first component of $D\Phi(\hat{y})[\eta]$ is a linear parabolic operator to which standard existence and uniqueness results for linear PDE applies Krylov (1987a) .

So by standard linear parabolic theory Krylov (1987a) we have that $D\Phi[\hat{y}]$ is one-to-one and onto. Then the inverse function theorem says that there is some $\rho > 0$ so that $\Phi$ is a $C^1$-diffeomorphism from $B_\rho(\hat{y}) \subset V$ onto a neighborhood $Z \subset W$ of $(\hat{f}, y_0)$.

**Step 3:** For this step, our goal is to show that the procedure in Step 2 gives us a solution to (3.2) for a short time. For this we let $\epsilon > 0$ and choose $\eta_\epsilon \in C^\infty([0,1])$ s.t. $0 \leq \eta_\epsilon \leq 1$, $0 \leq \frac{\partial \eta_\epsilon}{\partial t} \leq 2\epsilon^{-1}$,

$$\eta_\epsilon(t) = \begin{cases} 0, & 0 \leq t \leq \epsilon \\ 1, & 2\epsilon \leq t \leq 1 \end{cases}$$

and define $f_\epsilon = \hat{f}\eta_\epsilon$. Then, as we show in Lemma (4) below, $f_\epsilon \in C^{\alpha, \frac{\alpha}{2}}(U_{T_0})$ with uniformly bounded norm (in $\epsilon > 0$) and hence, by Ascoli's theorem we have that $f_\epsilon \to \hat{f}$ as $\epsilon \to 0$ in $C^{\beta, \frac{\beta}{2}}(U_{T_0})$ for all $0 < \beta < \alpha$.

So for small enough $\epsilon$ we have that the pair $(f_\epsilon, y_0) \in Z$ and hence by Step 2 there exists a unique solution $y^\epsilon \in B_\rho(\hat{y})$ of the equation

$$\Phi(y^\epsilon) = (f_\epsilon, y_0)$$

which is equivalent to saying that $y$ solves the initial value problem

$$y^\epsilon_t - F(x, y^\epsilon, \nabla^0 y^\epsilon, \nabla^0 \nabla^0 y^\epsilon) = f_\epsilon$$
$$y^\epsilon(x, 0) = y_0(x, 0)$$

and from the definition $f_\epsilon = \hat{f}\eta_\epsilon$ for $0 \leq t \leq \epsilon$ we have that $y^\epsilon$ solves the original nonlinear initial value problem (3.3) in $U_\epsilon = \mathbb{R}^n \times [0, \epsilon)$.

Then we also know that $y(\cdot,t) \in \Lambda$ for $t \in [0,\epsilon')$ for $0 < \epsilon' \le \epsilon$ since $y \in B_\rho(\hat{y})$ and hence cannot immediately escape $\Lambda$. This concludes the proof of existence in Theorem (3).

**Step 4:** It just remains to show that $y$ is the unique solution in the function class $\Lambda \cap C^{2+\beta,\frac{2+\beta}{2}}(U_\epsilon)$.

Let $y_1, y_2 \in C^{2+\beta,\frac{2+\beta}{2}}(U_\epsilon)$ be two solutions to (3) with the same initial condition. We will show that $y_1, y_2$ agree on a small time interval $0 \le t \le \delta$ for $\delta \le \epsilon$ where the argument relies on the fact that $y_0 \in C^{2+\alpha}(\mathbb{R}^n)$ and so the full uniqueness statement follows by iterating the argument since $y_1(\cdot,t), y_2(\cdot,t) \in C^{2+\alpha}(\mathbb{R}^n)$.

If $0 < \delta$ is small enough then the convex combination $y_\tau = \tau y_1 + (1-\tau)y_2 \in \Lambda$ for all $(t,\tau) \in [0,\delta] \times [0,1]$ since $\Lambda$ is an open, convex set.

Hence we find

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}(y_1 - y_2) + F(x, y_1, \nabla^0 y_1, \nabla^0\nabla^0 y_1) - F(x, y_2, \nabla^0 y_2, \nabla^0\nabla^0 y_2) \\
&= \frac{\partial}{\partial t}(y_1 - y_2) + \int_0^1 \frac{d}{d\tau} F(x, y_\tau, \nabla^0 y_\tau \nabla^0\nabla^0 y_\tau) d\tau \\
&= \frac{\partial}{\partial t}(y_1 - y_2) - a^{ij}(y_1 - y_2)_{ij} + b^i(y_1 - y_2)_i + c(y_1 - y_2),
\end{aligned}$$

where $a^{ij}$ is uniformly elliptic and $b, c$ bounded and hence $y_1 = y_2$ in $[0,\delta]$.

□

We finish this section with the proofs of two technical lemmas used above.

**Lemma 4.** *Let $\hat{f} \in C^{\alpha,\frac{\alpha}{2}}(U_{T_0})$ satisfying $\hat{f}(\cdot,0) = 0$ and let $\eta_\epsilon = \eta_\epsilon(t)$ be defined as above, then*

$$f_\epsilon = \hat{f}\eta_\epsilon \in C^{\alpha,\frac{\alpha}{2}}(U_{T_0}) \tag{3.5}$$

*with norm bounded independently of* $0 < \epsilon < 1$.

*Proof.* Since $\hat{f} \in C^{\alpha,\frac{\alpha}{2}}(U_{T_0})$ and $\eta_\epsilon$ is just a function of time we only need to worry about the holder semi-norm of $f_\epsilon$ w.r.t time.

So if we let $\gamma = \frac{\alpha}{2}$ and consider $t_1, t_2 \in [0, T_0]$ then we need to show that

$$|f_\epsilon(t_1) - f_\epsilon(t_2)| \leq K|t_1 - t_2|^\gamma$$

for some constant $K$ where we are ignoring the dependence of $f$ on $x \in \mathbb{R}^n$.

Without loss of generality we may assume that $0 < t_1 < t_2 < T_0$ and then notice

$$f_\epsilon(t_1) - f_\epsilon(t_2) = \left(\hat{f}(t_1) - \hat{f}(t_2)\right)\eta_\epsilon(t_2) + \hat{f}(t_1)\left(\eta_\epsilon(t_1) - \eta_\epsilon(t_2)\right) \tag{3.6}$$

where we notice that we already have the desired result for the first term in this sum so we just need to investigate the second term. Also, we may assume that $t_1 < 2\epsilon$ because otherwise the second term in (3.6) vanishes. Now we split the argument into three cases

**Case 1:** $t_2 \leq 3\epsilon$ and $|t_1 - t_2| \geq t_1$

$$\left|\hat{f}(t_1)\left(\eta_\epsilon(t_1) - \eta_\epsilon(t_2)\right)\right| \leq ct_1^\gamma \frac{|t_1 - t_2|}{\epsilon} \leq 3ct_1^\gamma \leq 3c|t_1 - t_2|^\gamma$$

where $c$ is a holder norm for $\hat{f}$ and the third inequality holds since this case implies $\frac{|t_1-t_2|}{\epsilon} \leq 3$.

**Case 2:** $t_2 \leq 3\epsilon$ and $|t_1 - t_2| < t_1$

$$\left|\hat{f}(t_1)\left(\eta_\epsilon(t_1)-\eta_\epsilon(t_2)\right)\right| \leq c\frac{t_1^\gamma}{\epsilon}|t_1-t_2| < 2ct_1^{\gamma-1}|t_1-t_2| \leq 2c|t_1-t_2|^\gamma$$

where $c$ is a holder norm for $\hat{f}$ and the last inequality holds since $\gamma - 1 < 0$.

**Case 3:** $t_2 > 3\epsilon \Rightarrow t_2 - t_1 \geq \epsilon$

$$\left|\hat{f}(t_1)\left(\eta_\epsilon(t_1)-\eta_\epsilon(t_2)\right)\right| \leq ct_1^\gamma \leq c(\epsilon 2)^\gamma \leq c2^\gamma|t_1-t_2|^\gamma$$

where $c$ is a holder constant for $\hat{f}$. ☐

Now our goal is to relate (1.1) to (3.1) by stating the system of ODEs that needs to be solved in order to relate the two equations.

**Lemma 5.** (3.1) *is, up to tangential diffeomorphisms, equivalent to* (1.1)

*Proof.* Given a solution $y(\bar{x},t)$ of (3.1) we let $\varphi(x,t) = (\bar{x}(x,t), y(\bar{x}(x,t),t))$ where $\bar{x} : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n$ and then we can find

$$\frac{\partial\varphi}{\partial t} = \left(\frac{\partial\bar{x}}{\partial t}, \frac{\partial y}{\partial t} + \left\langle \nabla^0 y, \frac{\partial\bar{x}}{\partial t}\right\rangle\right) = \frac{\nu}{H} = \frac{y}{vH}(\nabla^0 y, -1) \tag{3.7}$$

This implies that the ODE for $\bar{x}$ is given by

$$\begin{cases} \frac{\partial\bar{x}}{\partial t}(x,t) &= \frac{y}{vH}\nabla^0 y \\ \bar{x}(x,0) &= x \end{cases} \tag{3.8}$$

where we note that this is an ODE since we have already solved (3.1) and hence $\frac{y}{vH}\nabla^0 y$ is a predefined, well controlled function. We can confirm this by substituting the second equation given by (3.7) which shows us the following

$$\begin{aligned}\frac{\partial y}{\partial t} + \left\langle \nabla^0 y, \frac{\partial \bar{x}}{\partial t} \right\rangle = -\frac{y}{vH} \quad &\Rightarrow \quad \frac{\partial y}{\partial t} + \frac{y}{vH}|\nabla^0 y|^2 = -\frac{y}{vH} \\ &\Rightarrow \quad \frac{\partial y}{\partial t} = \frac{-y}{vH}\left(1 + |\nabla^0 y|^2\right) = \frac{-yv}{H}\end{aligned}$$

So if we define $\mathbf{G}(t, \bar{x}) = \frac{y}{vH}\nabla^0 y$ then Theorem (3) implies that this function is continuous for a short time and hence we can find short time existence to (3.8) by standard ODE Theorems. Combining Theorem (3) with the standard short time existence result for (3.8) we then obtain short time existence to (1.1), as desired. $\square$

# Chapter 4

# ODE Maximum Principle at Infinity

In this chapter we state and prove an ODE maximum principle that works for functions defined on complete non-compact manifolds and will be used a few times in this document for important estimates. We also state and prove a non-compact tensor maximum principle which will be used to obtain estimates for the second fundamental form in Chapter 5 and 6. Both theorems are extensions of the work of Hamilton (1986) which is described in detail in Mantegazza (2011).

The idea is to use the Omori-Yau maximum principle at infinity in order to get the necessary sign properties at a maximum of a bounded function which are crucial to proving a maximum principle. We state a version of the Omori-Yau maximum principle Cheng and Yau (1975); Omori (1967); Yau (1975) below but we also note that there are even more general versions Pigola (2003) which will also be applicable to (5) but we will not need those versions for our results. In fact, we will just need this theorem under the assumption that $Rc \geq -Cg$ on the manifold $M$ Cheng and Yau (1975); Yau (1975) which is a variation of the following theorem which is useful when you only need the comparison for the Laplacian $\Delta$ instead of for the full Hessian $\nabla_i \nabla_j$.

Since we will state a general version of the ODE maximum principle at infinity (5) which requires a maximum principle for the Hessian we will state a version of the Maximum principle at infinity which makes a requirement on sectional curvature and yields a result for the sign of the Hessian.

**Theorem 4.** *Pigola (2003) Let $(M, \langle , \rangle)$ be a complete, non-compact, Riemannian manifold. If $p \in M$ then define $r(x) : M \to \mathbb{R}$ to be the distance from $x$ to $p$ and assume that the sectional curvature of 2-planes containing $\nabla r$, $K_r$, satisfy the following bound*

$$K_r \geq -C(r^2 + 1)$$

*for some $C > 0$. Then for every bounded above function $u \in C^2(M)$ there is a sequence of points $\{x_n\} \subset M$ so that*

$$(i)u(x_n) > \sup_M u - \frac{1}{n} \quad (ii)|\nabla u|(x_n) < \frac{1}{n} \quad (iii)\nabla\nabla u(x_n) < \frac{1}{n}\langle \cdot, \cdot \rangle$$

Now we use this elliptic maximum principle to obtain a parabolic maximum principle which will be conducive to finding estimates for non-compact manifolds evolving under IMCF.

**Theorem 5.** *Assume for $t \in [0, T)$ that $g(t)$ is a family of Riemannian metrics defined on the manifold $M^n$ so that the dependence on $t$ is smooth. We also assume that $g_t$ is a metric to which the Omori-Yau maximum principle at infinity applies for each $t \in [0, T)$.*

*Let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function which is bounded for each time $t \in (0, T)$, i.e. $|u(x, t)| \leq C(t)$, satisfying*

$$\left(\partial_t - H^{ij}\nabla_i^{g_t}\nabla_j^{g_t}\right) u = \langle X(x, u, \nabla^{g_t} u, t), \nabla^{g_t} u\rangle_{g_t} + F(u)$$

*where $|X| \leq C_1(t)$, $F$ is a locally Lipschitz function on $\mathbb{R}$ and $H_{ij}$ is a symmetric, positive definite matrix so that $|H| \leq C_0$.*

*Setting $u_{sup}(t) = \sup_{x \in M} u(x, t)$ we have that the function, $u_{sup(t)}$ is locally Lipschitz and hence differentiable at almost every time $t \in [0, T)$. At every differentiable time we have that*

$$\frac{du_{sup}(t)}{dt} = \lim_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t} \qquad \textit{where } \{x_k\} \subset \mathbb{R}^n \textit{ is any sequence s.t.} \lim_{k\to\infty} u(x_k,t) = \sup_{x\in\mathbb{R}^n} u(x,t)$$

*If $\varphi : [0,T') \to \mathbb{R}$ is a maximal solution of the ODE*

$$\begin{cases} \varphi'(t) & = F(\varphi(t)) \\ \varphi(0) & = u_{sup}(0) \end{cases}$$

*then we have that $u(x,t) \leq \varphi(t)$ for $(x,t) \in M \times [0, \min\{T,T'\})$.*

Before we can prove this theorem we will need the following lemma.

**Lemma 6.** *Let $u : M^n \times (0,T) \to \mathbb{R}$ be a bounded $C^1$ function then $u_{sup} : (0,T) \to \mathbb{R}$, defined as $u_{sup}(t) = \sup_{x\in M} u(x,t)$, is a locally Lipschitz function in $(0,T)$. Also, at every differentiable time $t \in (0,T)$ we have that*

$$\frac{du_{sup}(t)}{dt} = \frac{\partial u(x,t)}{\partial t} \qquad \textit{where } x \in M \textit{ is a point where } u(\cdot,t) \textit{ attains its max}$$

*or*

$$\frac{du_{sup}(t)}{dt} = \lim_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t} \qquad \textit{where } \{x_k\} \subset M \textit{ is any sequence s.t.} \lim_{k\to\infty} u(x_k,t) = \sup_{x\in M} u(x,t)$$

*Proof.* **Proof of (6):**

Fix a $t \in (0,T)$ and then choose a $\delta > 0$ so that $[t-\delta, t+\delta] \subset (0,T)$. Then choose an $\epsilon$ so that $0 < \epsilon < \delta$ and note that since $u$ is bounded and $C^1$ on $M \times (0,T)$ we know that for every $x \in M$, there exists some Lipschitz constant $K > 0$, depending on $t$ and $\epsilon$, so that $u(x,t+\epsilon) - u(x,t) \leq K\epsilon$.

Now for each $\epsilon > 0$ we can find a sequence $\{x_k^\epsilon\}$ so that $u_{sup}(t+\epsilon) = \lim_{k\to\infty} u(x_k^\epsilon, t+\epsilon)$ and hence

$$\begin{aligned} u_{sup}(t+\epsilon) &= \lim_{k\to\infty} u(x_k^\epsilon, t+\epsilon) \leq \limsup_{k\to\infty} u(x_k^\epsilon, t) + K\epsilon \\ &\leq \lim_{k\to\infty} u(x_k^0, t) + K\epsilon = u_{sup}(t) + K\epsilon \end{aligned}$$

where the second inequality follows from the fact that $u_{sup}(t) = \lim_{k\to\infty} u(x_k^0, t)$. So we have found that $u_{sup}(t+\epsilon) - u_{sup}(t) \leq K\epsilon$. Repeating this argument for $-\delta < \epsilon < 0$ we conclude that $u_{sup}$ is a locally Lipschitz function on $(0, T)$ and hence differentiable at almost every time $t$.

**Note:** If $u$ attains its max at some point $x \in M$ then we can take the trivial sequence which is constantly equal to $x$.

Let $t \in (0, T)$ be a time where $u_{sup}$ is differentiable and let $\{x_k\}$ be a sequence so that $\lim_{k\to\infty} u(x_k, t) = \sup_{x\in M} u(x, t)$. Then by the Mean Value Theorem, for every $0 < \epsilon < \delta$ we can choose a $s_k \in (t, t+\epsilon)$ so that $u(x_k, t+\epsilon) = u(x_k, t) + \epsilon \frac{\partial u(x_k, s_k)}{\partial t}$ and so

$$\begin{aligned} u_{sup}(t+\epsilon) &\geq \limsup_{k\to\infty} u(x_k, t+\epsilon) = \limsup_{k\to\infty} \left[ u(x_k, t) + \epsilon \frac{\partial u(x_k, s_k)}{\partial t} \right] \\ &= u_{sup}(t) + \epsilon \limsup_{k\to\infty} \frac{\partial u(x_k, s_k)}{\partial t} \end{aligned}$$

so then by rearranging we find

$$\frac{u_{sup}(t+\epsilon) - u_{sup}(t)}{\epsilon} \geq \limsup_{k\to\infty} \frac{\partial u(x_k, s_k)}{\partial t}$$

and so by letting $\epsilon \to 0$ we find that $\frac{du_{sup}(t)}{dt} \geq \limsup_{k\to\infty} \frac{\partial u(x_k, t)}{\partial t}$.

Now if we repeat this argument for $-\delta < -\epsilon < 0$ we will get the following

$$\frac{du_{sup}(t)}{dt} \leq \liminf_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t}$$

Putting this all together we see that

$$\limsup_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t} \leq \frac{du_{sup}(t)}{dt} \leq \liminf_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t}$$

which tells us that $\lim_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t}$ must converge at a differentiable time of $u_{sup}(t)$ and equal its derivative.

□

*Proof.* **Proof of** (5)**:**

By the previous Lemma we know that $u_{sup}(t)$ is locally Lipschitz and hence differentiable almost everywhere in $[0,T)$. If we let $t \in [0,T)$ be a differentiable time and $\{x_k\}$ a sequence so that $\lim_{k\to\infty} u(x_k,t) = \sup_{x\in M} u(x,t)$, $|\nabla u(x_k,t)| < \frac{1}{k}$ and $\nabla_i\nabla_j u(x_k,t) < \frac{1}{k} g_{ij}$, which is guaranteed by the maximum principle at infinity, then we find

$$\begin{aligned}
\frac{du_{sup}}{dt}(t) &= \lim_{k\to\infty} \frac{\partial u}{\partial t}(x_k,t) \\
&\leq \limsup_{k\to\infty} \big( H^{ij}\nabla_i\nabla_j u(x_k,t) + \langle X(x_k,u,\nabla u,t), \nabla u(x_k,t)\rangle + F(u(x_k,t)) \big) \\
&\leq \limsup_{k\to\infty} \left( \frac{nC_0}{k} + \frac{|X|}{k} + F(u(x_k,t)) \right) \\
&\leq F\left( \limsup_{k\to\infty} u(x_k,t) \right) = F(u_{sup}(t))
\end{aligned}$$

and so we have that, at a differentiable time $t$

$$\frac{du_{sup}}{dt}(t) \leq F(u_{sup}(t))$$

At this point we follow the argument from Mantegazza (2011) where we will in more detail. Now let $\varphi : [0, T') \to \mathbb{R}$ be as in the statement of the Theorem and for $\epsilon > 0$ let $\varphi_\epsilon : [0, T_\epsilon) \to \mathbb{R}$ be the maximal solution of the family of ODEs

$$\begin{cases} \varphi_\epsilon'(t) & = F(\varphi_\epsilon(t)) \\ \varphi_\epsilon(0) & = u_{sup}(0) + \epsilon \end{cases}$$

Now we break the argument into three cases depending on the sign of $F(u_{sup}(0))$.

**Case 1:** $F(u_{sup}(0)) > 0$ For $\epsilon_1 \leq \epsilon_2$, small enough, we have that $\varphi_{\epsilon_1}(t) \leq \varphi_{\epsilon_2}(t)$ (by ODE comparison) and hence $T_{\epsilon_2} \leq T_{\epsilon_1}$. So $T_\epsilon$ is an increasing sequence of times as $\epsilon \to 0$ and so $T_\epsilon \nearrow T'$ as $\epsilon \to 0$ and by the upper semi-continuity of the existence time w.r.t. the initial condition we have that $T' \leq T_0$.

**Case 2:** $F(u_{sup}(0) < 0$

For $\epsilon_1 \leq \epsilon_2$, small enough, we have that $\varphi_{\epsilon_1}(t) \geq \varphi_{\epsilon_2}(t)$ (by ODE comparison) and hence $T_{\epsilon_2} \geq T_{\epsilon_1}$. So $T_\epsilon$ is an decreasing sequence of times as $\epsilon \to 0$ and so $T_\epsilon \searrow T'$ as $\epsilon \to 0$ and by the upper semi-continuity of the existence time w.r.t. the initial condition we have that $T' \leq T_0$.

**Case 3:** $F(u_{sup}(0)) = 0$

For $\epsilon$ small enough we have that $F(u_{sup}(0) + \epsilon) > 0$ or $F(u_{sup}(0) + \epsilon) < 0$ and then we are back in Case 1 or Case 2.

So we have shown that $T' \leq T_0$ and since $F$ is Lipschitz on compact sets we can restrict ourselves to $[0, T_\delta]$ for $T_\delta < T'$ where we know that $u$ and $\varphi_\epsilon$ are bounded, for small enough $\epsilon$, and hence solutions to the above ODE have continuous dependence on the initial conditions (over compact time intervals). Hence using the fact that the family of functions $\varphi_\epsilon$ is uniformly Lipschitz for small enough $\epsilon$ we find that $\varphi_\epsilon \to \varphi$ uniformly on $[0, T_\delta]$ for any $T_\delta < T'$ as $\epsilon \to 0$.

Now fix $\epsilon > 0$ and for sake of contradiction assume that there is some positive time so that $u_{sup}(t) > \varphi_\epsilon(t)$ and let $\bar{t} > 0$ be the infimum of all such times which we know is $\neq 0$

since $u_{sup}(0) = \varphi_\epsilon(0) - \epsilon$. So $u_{sup}(\bar{t}) = \varphi_\epsilon(\bar{t})$ and hence we can let $\Phi_\epsilon(t) = \varphi_\epsilon(t) - u_{sup}(t)$. Then at differentiable times for $u_{sup}(t)$ in the interval $[0, \bar{t})$ we know that $\Phi_\epsilon(t) > 0$ and

$$\Phi_\epsilon'(t) \geq F(\varphi_\epsilon(t)) - F(u_{sup}(t)) \geq -C_\epsilon(\varphi_\epsilon(t) - u_{sup}(t)) = -C_\epsilon \Phi_\epsilon(t)$$

where $C_\epsilon$ is a local Lipschitz constant for $F$ in the interval $\{\varphi_\epsilon(s) : 0 \leq s \leq \bar{t}\}$ and this differential inequality hold for a.e. $t \in [0, \bar{t}]$.

Then by integrating this equation we find that $\Phi_\epsilon(t) \geq \Phi_\epsilon(0)e^{-C_\epsilon t} = \epsilon e^{-C_\epsilon t}$ and so in particular $\Phi_\epsilon(\bar{t}) \geq \epsilon e^{-C_\epsilon \bar{t}} > 0$ but that contradicts the fact that $\Phi_\epsilon(\bar{t}) = 0 \longrightarrow\!\longleftarrow$.

So $u_{sup}(t) \leq \varphi_\epsilon(t)$ for every $t \in [0, T_\delta)$ and so if we let $\epsilon \to 0$ then we have that $u_{sup}(t) \leq \varphi(t)$ for every $t \in [0, T_\delta)$. Since $\delta > 0$ was arbitrary, we have proven the desired result for $[0, T')$. $\square$

Now we move on to state and prove a non-compact tensor maximum principle which will be used in Chapter 5 and 6 in order to obtain estimates for the eigenvalues of the second fundamental form $A_{ij}$.

**Theorem 6.** *Assume for $t \in [0, T]$ that $\Sigma_t \subset \mathbb{H}^{n+1}$ is a non-compact solution of IMCF given as a graph over $\{y = 0\}$ satisfying the bound on mean curvature $0 < H_0 \leq H(x,t) \leq H_1 < \infty$, an upper bound on the gradient $v(x,t) \leq v_0 < \infty$ and an upper bound on the graph function $y(x,t) \leq y_1 < \infty$, in the upper half space model of $\mathbb{H}^{n+1}$ for $t \in [0, T]$*

*Let $G$ be a smooth, bounded, symmetric (1,1) tensor s.t. $G$ satisfies the evolution inequality*

$$\left(\partial_t - \frac{1}{H^2}\Delta\right) G \geq \langle \nabla G, X \rangle + B(G)$$

*where $X$ is a bounded vector field, $|X| \leq X_0$, and $B(G)$ is a homogeneous polynomial in $G$ and hence satisfies the* ***null eigenvector condition****, i.e. for any vector s.t. $G_i^j V^i = 0$ we have that*

$$B_i^j V^i V_j \geq 0$$

*If $G(0) \geq 0$ then $G(t) \geq 0$ for all $t \in [0,T]$ such that the solution exists.*

*Proof.* Our method of proof is to consider a perturbed (1,1) tensor $G_\epsilon$, defined below, so that we know that $G_\epsilon$ can only attain zero eigenvectors at points of $\Sigma_t$, i.e. not on a sequence diverging to infinity, and then apply the usual tensor maximum principle to this new tensor in order to show it must remain positive semi-definite.

We begin by defining the (1,1) tensor $G_{\epsilon,R} = \varphi(\alpha_R)G + \epsilon(\eta_\epsilon + (t-t_0))\delta$ where $0 < \epsilon, \eta_\epsilon < 1$, $\delta$ is the identity (1,1) tensor, $\varphi(x) = e^{-\frac{1}{x}}$ and $\alpha_R = \left(R^2 - |x|^2 - \frac{2}{\mathcal{H}_0^R}(ny_1^2 + 4y_1 R)(t-t_0)\right)$ which is the cutoff function specifically designed for IMCF in $\mathbb{H}^{n+1}$ in Chapter 7. The evolution equation for $\varphi(\alpha_R)$ can be found by combining Lemma (8) as well as the proof of Lemma (12) in Chapter 7. Now we define $U_R = \{(x,t) \in \mathbb{R}^n \times [t_0, t_0+\tau] : \alpha_R(x,t) \geq 0\}$ and we notice that $0 \leq \varphi(\alpha_R) \leq 1$ on $U_R$.

Now if we assume $G_{\epsilon,R}(t_0) > 0$ (consider $t_0 = 0$ the first time around) then we will show that $G_{\epsilon,R}$ cannot attain a zero eigenvector at a point $(q,t) \in \overline{U}_R$, where $0 < \tau \leq 1$, by adapting the usual tensor maximum principle to the parabolic neighborhood $U_R$. More specifically, choose $(q,t') \in U_R$ to be a point, where a zero eigenvector of $G_{\epsilon,R}$ is attained, $t' \in [t_0, t_0+\tau]$, for the first time. First we will show that this cannot happen in $W_R$ and then we will show that it also cannot happen on $\{\varphi = 0\}$.

Let $V \in T_q\Sigma_{t'}$ be the zero eigenvector of $G_{\epsilon,R}$ and extend $V$ to a neighborhood $B_r(q)$, by parallel translation along geodesics emanating from $q$, where we choose $r$ small enough so that $B_r(q) \times \{t'\} \subset W_R$. Then we extend $V$ to be constant in time on the set $B_r(q) \times [t_0, t'] \subset W_R$. By this construction we see that

$$\nabla V|_{(q,t')} = 0 \qquad \frac{\partial V}{\partial t}|_{(q,t')} = 0 \qquad \frac{\partial}{\partial t}\left((G_{\epsilon,R})_i^j V_j V^i\right)_{(q,t')} \leq 0$$

and we can also calculate at the point $(q,t')$

$$
\begin{aligned}
\Delta\left((G_{\epsilon,R})_i^j V^i V_j\right)_{(q,t')} &= \Delta\left((G_{\epsilon,R})_i^j\right) V^i V_j|_{(q,t')} \\
&+ 2g^{lm}\left[(\nabla_l G_{\epsilon,R})_i^j V^i \nabla_m V_j + (G_{\epsilon,R})_i^j \nabla_l V^i \nabla_m V_j + (G_{\epsilon,R})_i^j V_j \nabla_l \nabla_m V^i\right]_{(q,t')} \\
&= \Delta\left((G_{\epsilon,R})_i^j\right) V^i V_j|_{(q,t')}
\end{aligned}
$$

where we use the fact that $V$ is a zero eigenvector for $G_{\epsilon,R}$ as well as the properties of the extension of $V$ noted above.

Putting these properties together we find, by the fact that $(G_{\epsilon,R})_i^j V^i V_j$ attains a minimum for the first time at $(q,t')$, the following inequality

$$
\left(\partial_t - \frac{1}{H^2}\Delta\right)\left((G_{\epsilon,R})_i^j V^i V_j\right)_{(q,t')} \leq 0
$$

Now let's consider the evolution of $G_{\epsilon,R}$ at the point $(q,t')$

$$
\begin{aligned}
&\left(\partial_t - \frac{1}{H^2}\Delta\right)\left((G_{\epsilon,R})_i^j V^i V_j\right)_{(q,t')} = \left(\partial_t - \frac{1}{H^2}\Delta\right)(G_{\epsilon,R})_i^j V^i V_j|_{(q,t')} \\
&= \left(\varphi\left(\partial_t - \frac{1}{H^2}\Delta\right)G_i^j + G_i^j\left(\partial_t - \frac{1}{H^2}\Delta\right)\varphi - \frac{2}{H^2}\langle\nabla G_i^j, \nabla\varphi\rangle\right) V^i V_j|_{(q,t')} \\
&+ \epsilon\delta_i^j V^i V_j|_{(q,t')} \\
&\geq -\frac{2}{H^2\varphi}\langle\nabla\left(\varphi G_i^j\right), \nabla\varphi\rangle V^i V_j|_{(q,t')} + \left[\varphi\langle\nabla G_i^j, X\rangle + \varphi B(G)_i^j + \epsilon\delta_i^j\right] V^i V_j|_{(q,t')} \\
&+ \left[\frac{G_i^j|\nabla\varphi|^2}{\varphi H^2} - \frac{2C_R G_i^j\varphi'}{\mathcal{H}_0^R}\right] V^i V_j|_{(q,t')} \\
&\geq \langle\nabla(\varphi G_i^j), X\rangle V^i V_j|_{(q,t')} + \left[\frac{|\nabla\varphi|^2}{\varphi H^2} - \langle\nabla\varphi, X\rangle - \frac{\varphi''|\nabla\beta|^2}{H^2} - \frac{2C_R\varphi'}{\mathcal{H}_0^R}\right] G_i^j V^i V_j|_{(q,t')} \\
&+ B(G_{\epsilon,R})_i^j V^i V_j|_{(q,t')} + \epsilon\delta_i^j V^i V_j|_{(q,t')} + (\varphi B(G) - B(G_{\epsilon,R}))_i^j V^i V_j|_{(q,t')}
\end{aligned}
$$

By assumption we know that $B(G_{\epsilon,R})_i^j V^i V_j|_{(q,t')} \geq 0$ so if we can deal with the gradient terms and choose our parameters so that the $\varphi B(G) - B(G_{\epsilon,R})$ term is strictly less, in absolute value, than $\epsilon\delta$ then we will be able to obtain a strict sign for this evolution inequality.

We start by dealing with the gradient terms where we notice that since $G_{\epsilon,R}|_{(q,t')} = 0$ we know that $G_i^j V^i V_j|_{(q,t')} \leq 0$ and hence

$$\left[\frac{|\nabla\varphi|^2}{\varphi H^2} - \langle\nabla\varphi, X\rangle - \frac{\varphi''|\nabla\alpha|^2}{H^2} - \frac{2C_R\varphi'}{\mathcal{H}_0^R}\right] G_i^j V^i V_j$$
$$\geq \left[\frac{|\nabla\varphi|^2}{\varphi\mathcal{H}_0^R} + |\nabla\varphi||X| - \frac{\varphi''|\nabla\alpha|^2}{H^2} - \frac{2C_R\eta'(\varphi)}{\mathcal{H}_0^R}\right] G_i^j V^i V_j$$

Now if we use the fact that $\Sigma_t$ is well defined hypersurface expressed as a graph over a plane ($\mathbb{R}^n$) with bounded gradient, $v$, in $W_R$ then we know that there exists some $D_R > 0$ so that $D_R^{-2}\delta \leq g \leq D_R^2\delta$, in $W_R$. Hence $|\nabla\alpha| \leq D_R|\nabla^0\alpha| \leq 2D_R\|x| \leq 2D_R R$. We note that $D_R$ depends on upper and lower bounds on $y$ and a upper bound on $v$ since $g_{ij} = \frac{1}{y^2}(\delta_{ij} + y_i y_j)$. We should note that Theorem (7) and Theorem (8) will give use the desired control in Chapter 5. Now we also use the fact that for compactly supported functions, Ecker (2004), we know that $\frac{|\nabla\varphi|^2}{\varphi} \leq \varphi'' \leq C'$ which means we can get rid of this term by exploiting the $-4D_R^2\varphi''|x|^2$ and so if we choose $C_R \geq 2\mathcal{H}_0^R D_R R X_0 + C' D_R R^2$ we can get rid of the gradient terms and rewrite the evolution inequality as

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\left((G_{\epsilon,R})_i^j V^i V_j\right)_{(q,t')} \geq \epsilon\delta_i^j V^i V_j|_{(q,t')} + (\eta(\varphi)B(G) - B(G_{\epsilon,R}))_i^j V^i V_j|_{(q,t')}$$

Now we consider the term $\varphi B(G) - B(G_{\epsilon,R})$ where we let $K$ be a locally Lipschitz constant for $B$, which will depend on a bound for $G$. Using the fact that $0 \leq \eta(\varphi) \leq 1$ and that $B$ is a homogeneous polynomial in $G$ we find

$$(\varphi B(G) - B(G_{\epsilon,R}))|_q \geq (B(\varphi G) - B(G_{\epsilon,R}))|_q \geq -K|\varphi G - G_{\epsilon,R}| \geq -\epsilon K\left[\eta_\epsilon + \tau\right]\delta$$

Since we required $\eta_\epsilon, \tau \leq 1$ we notice that $K$ only depends on a bound for $G$ since $\varphi \leq 1$ and

$$|\varphi G - G_{\epsilon,R}|_p = |\epsilon(\eta_\epsilon + (t-t_0))\delta| \leq \epsilon(1+\tau) \leq 2$$

for any point $(p,t) \in W_R$ where we also note that this choice does not depend on $R$.

Then if we choose $\eta_\epsilon \leq \frac{1}{4K}$ and let $\tau = \frac{1}{4K}$ we find that

$$(\eta(\varphi)B(G) - B(G_{\epsilon,R}))|_q \geq -\frac{\epsilon\delta}{2}$$

at the point $q \in M$ where the zero eigenvector occurs for $G_{\epsilon,R}$.

Putting all of this together we find the following inequality at $(q,t')$

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\left((G_{\epsilon,R})_i^j V^i V_j\right)_{(q,t')} > 0$$

which is a contradiction and hence $G_{\epsilon,R}$ cannot attain a zero eigenvector on $W_R$ for some $t \in [0,\tau]$. We also know that $G_{\epsilon,R} \neq 0$ on the set $\{\alpha = 0\}$ since $G_{\epsilon,R} = \epsilon(\eta_\epsilon + (t-t_0))\delta \geq 0$ on the set $\{\alpha = 0\}$ and hence $G_{\epsilon,R} \geq 0$ on $W_R$.

Then by letting $\epsilon \to 0$ we see that $\varphi \geq 0$ on $U_R$ for $t \in [0,\tau]$ which implies that $G \geq 0$ on $U_R$ for $t \in [0,\tau]$. Lastly, we let $R \to \infty$ in order to conclude that $G \geq 0$ on $\Sigma_t$ for $t \in [t_0, t_0+\tau]$. Then since $\tau = \frac{1}{4K}$ we can repeat this argument finitely many times on the intervals $[0,\tau]$, $[\tau, 2\tau]$, $[2\tau, 3\tau]$, etc. until we can conclude that $G \geq 0$ on $\Sigma_t$ for $t \in [0,T]$.

□

# Chapter 5

# Long Time Existence

Now we look to build upon the short time existence theorem by obtaining some further estimates that will culminate with a long time existence result Theorem (11). For this purpose, we let $T$ be the maximal time of existence for the flow and consider $T' < T$ so that we know that the solution of IMCF on the interval $[0, T')$ is well controlled by short time existence (3). This control of the solution implies that the ODE maximum principle at infinity applies to $\Sigma_t$ for $t \in [0, T')$, since by Gauss's theorem $|Rc| \leq (n + H|A| + |A|^2) \leq C(T')$, which is vital to the estimates obtained in this section ($Rc \geq -C$ is good enough by Cheng and Yau (1975); Omori (1967); Yau (1975)). The estimates we obtain in this section will hold on $[0, T')$ and will be uniform in our choice of $T'$ which will ultimately lead to the desired result.

We start out with a concrete example of the evolution of horospheres in $\mathbb{H}^{n+1}$ and then we show that horospheres act as barriers for the flow.

**Example:** Consider the horosphere $y = y_0$ as a graph over $\mathbb{R}^n \times \{0\}$. Then $y$ is just a function of time and $H = n$ and so we find the ODE

$$\frac{dy}{dt} = \frac{-y}{n}$$

which has the solution $y(t) = y_0 e^{-t/n}$.

The following Theorem demonstrates that the above example acts as barriers for the flow.

**Theorem 7.** *If* $0 < \inf_{\mathbb{R}^n} y(x,0) = y_0$ *and* $\sup_{\mathbb{R}^n} y(x,0) \leq y_1$ *and we assume that* $\Sigma_0$ *is a hypersurface to which the hypotheses of Theorem* (3) *apply then we find that*

$$y_0 e^{-t/n} \leq y(x,t) \leq y_1 e^{-t/n}$$

*So horospheres act as barriers for the evolution of bounded graphs over* $\mathbb{R}^n$*.*

*Proof.* Notice that by construction and by Theorem (3) the function $y(x,t)$ is bounded above and below (for a short time) and hence we can use the result that $y_{inf}(t) = \inf_{\mathbb{R}^n} y(x,t)$ is a well defined, locally Lipschitz function. Then by the maximum principle at infinity Theorem (5) there exists a sequence $\{x_k\} \in \mathbb{R}^n$ so that $\lim_{k\to\infty} y(x_k,t) = \inf_{\mathbb{R}^n} y(x,t)$ and

$$|\nabla^0 y(x_k,t)| < \frac{1}{k} \qquad \nabla^0\nabla^0 y(x_k,t) > -\frac{1}{k}\delta$$

and so if we use the expressions for $H$ and $w$ in terms of graphs we find (bearing in mind that $\tilde{\delta}^{ij} = \delta^{ij} - \frac{y_i y_j}{1+|\nabla^0 y|^2}$)

$$H = \frac{n + y\tilde{\delta}^{ij}y_{ij}}{\sqrt{1+|\nabla^0 y|^2}} \quad \Rightarrow \quad H(x_k,t) \geq \frac{n - k^{-1}y\tilde{\delta}^{ij}\delta_{ij}}{\sqrt{1+\frac{1}{k^2}}} \quad \Rightarrow \quad \lim_{k\to\infty} H(x_k,t) \geq n$$

$$w = \frac{1}{y\sqrt{1+|\nabla^0 y|^2}} \quad \Rightarrow \quad w(x_k,t) = \frac{1}{y(x_k,t)\sqrt{1+|\nabla^0 y(x_k,t)|^2}} \quad \Rightarrow \quad \lim_{k\to\infty} w(x_k,t) = \frac{1}{y_{inf}(t)}$$

Now we have the following ODE for $y(x,t)$

$$\frac{\partial}{\partial t}\left(\frac{1}{y^2}\right) = \frac{\partial}{\partial t}\bar{g}(\partial_y,\partial_y) = \frac{2}{H}\bar{g}(\bar{\nabla}_{\bar{\nu}}\partial_y,\partial_y) = \frac{2}{H}\bar{g}(-\frac{\bar{\nu}}{y},\partial_y) = \frac{2}{yH}\bar{g}(\bar{\nu},\eta) = \frac{2w}{yH}\frac{\partial y}{\partial t} \quad = \frac{-y^2 w}{H}$$

Let $t$ be a point of differentiability of the locally Lipschitz function $y_{inf}(t)$. From (5) we have, for a sequence $\{x_k\}$ such that $y(x_k, t) \to y_{inf}(t)$ as above

$$\frac{dy_{inf}(t)}{dt} = \lim_{k\to\infty} \frac{\partial y}{\partial t}(x_k, t) = -\lim_{k\to\infty} \frac{y^2 w}{H} \geq -\frac{1}{n} y_{inf}(t)$$

,hence

$$\frac{\partial y_{inf}(t)}{\partial t} \geq \frac{-y_{inf}(t)}{n}$$

Then by integrating, since $y_{inf}(t)$ is absolutely continuous, we find

$$y_{inf} \geq y_0 e^{-t/n}$$

which yields the desired estimate.

Using a similar argument for $y_{sup}(t) = \sup_{\mathbb{R}^n} y(x, t)$ we find the upper bound. □

Our last goal is to extend the short time uniform bounds from Theorem (3) to give us long time existence. We start by obtaining $C^1$ bounds on $y$ through the support function $w$ since $w^{-1} = yv$ and we already have a $C^0$ bound from Theorem (7).

**Theorem 8.** *If we assume that $\Sigma_0$ is a hypersurface to which the hypotheses of (3) apply then we find that*

$$w^{-1}(x,t) \leq w_{sup}^{-1}(0) e^{-t/n} \text{ and hence } w(x,t) \geq w_{inf}(0) e^{t/n}$$
$$v(x,t) \leq \frac{y_{sup}(0)}{y_{inf}(0)} v_{sup}(0)$$

*Proof.* From the evolution equation for $w^{-1}$ given in Lemma (2)(ii)

$$(\partial_t - \frac{1}{H^2}\Delta)w^{-1} = -w^{-2}(\partial_t - \frac{1}{H^2}\Delta)w - \frac{2}{w^3H^2}|\nabla w|^2 = -\frac{|A|^2}{H^2}w^{-1} - \frac{2}{w^{-1}H^2}|\nabla w^{-1}|^2$$

from which, using $|A|^2 \geq H^2/n$, we can deduce the following differential inequality (at points of differentiability of $w_{sup}$, (5))

$$\frac{dw_{sup}^{-1}}{dt} \leq -\frac{1}{n}w_{sup}^{-1}$$

from which the first estimate follows. Then if we notice that $w^{-1} = vy$ we can find the second estimate by combining with the estimate for $y$ given in (7). $\square$

Now we get the required bounds on $H$ which will allow us to gain bounds on $|A|^2$ through (10) and which also tell us that the operator $F$ remains uniformly parabolic.

**Theorem 9.** *If we assume that $\Sigma_0$ is a hypersurface to which the hypotheses of Theorem (3) apply then we find*

$$c_0\sqrt{n^2 + C_0 e^{-2t/n}} \leq H(x,t) \leq \sqrt{C_0 e^{-2t/n} + n^2}$$

*where $C_0 = H_{sup}(0)^2 - n^2$ if $H_{sup}(0) > n$ and $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)H_{sup}(0)}$ or*

$$c_0 \leq H(x,t) \leq n$$

*where $H_{sup}(0) \leq n$ and $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$.*

*Proof.* We have the evolution equation for $H$ (3)

$$(\partial_t - \frac{1}{H^2}\Delta)H = -2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} + \frac{n}{H}$$

and by (3) we know that $H$ is bounded above for a short time $t$ and by using the ODE maximum principle at infinity (5) we obtain the differential inequality at points of differentiability of $H_{sup}(t)$

$$\frac{dH_{sup}}{dt} \leq \frac{1}{nH_{sup}}\left(n^2 - H_{sup}^2\right)$$

from which it follows by integration that $H_{sup}(t) \leq \sqrt{C_0 e^{-2t/n} + n^2}$ where $C_0 = H_{sup}(0)^2 - n^2$ if $H_{sup}(0) > n$ and $C_0 = 0$ if $H_{sup}(0) \leq n$.

Now to obtain the lower bound on $H$ we consider the evolution equation for $u$ given in Lemma (3) and by using the ODE maximum principle at infinity Theorem (5) we obtain the following differential inequality at points of differentiability of $u_{sup}$

$$\frac{du_{sup}}{dt} = -\frac{nu_{sup}}{H^2} \leq -\frac{n}{n^2 + C_0 e^{-2t/n}} u_{sup}$$

which implies, by integrating, that $u(x,t) \leq \frac{H_{sup}(0)u_{sup}(0)}{\sqrt{n^2 e^{2t/n} + C_0}}$ when $H_{sup}(0) > n$ and then by using the definition of $u = \frac{1}{Hw}$ and applying (7) we find

$$\begin{aligned} H &\geq \frac{w^{-1}\sqrt{n^2 e^{2t/n} + C_0}}{H_{sup}(0)u_{sup}(0)} = \frac{yv\sqrt{n^2 e^{2t/n} + C_0}}{H_{sup}(0)u_{sup}(0)} \\ &\geq \frac{y_{inf}(0)e^{-t/n}H_{inf}(0)w_{inf}(0)\sqrt{n^2 e^{2t/n} + C_0}}{H_{sup}(0)} = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)H_{sup}(0)}\sqrt{n^2 + C_0 e^{-2t/n}} \end{aligned}$$

which completes the lower estimate of $H$ when $H_{sup}(0) > n$ .

When $H_{sup}(0) \leq n$ we get the simpler differential inequality at points of differentiability of $u_{sup}$

$$\frac{du_{sup}}{dt} = -\frac{nu_{sup}}{H^2} \leq -\frac{u_{sup}}{n}$$

which implies, by integrating, that $u(x,t) \leq u_{sup}(0)e^{-t/n}$ and then by using the definition of $u = \frac{1}{Hw}$ and applying (7) we find

$$H \geq w^{-1}u_{sup}(0)e^{-t/n} = yvu_{sup}(0)^{-1}e^{t/n} \geq \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$$

which completes the lower estimate of $H$ when $H_{sup}(0) \leq n$.

□

So we will be able to take $H_1 = \max\{H_{sup}(0), n\}$ as an upper bound for H(x,t) and $H_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$ for a lower bound to apply the following theorem in order to obtain $C^2$ bounds through the second fundamental form.

**Theorem 10.** *Let $\Sigma_0$ be a hypersurface to which the hypotheses of Theorem (3) apply. Then consider the tensor $M_{ij} = HA_{ij}$ where $\{\kappa_1, ..., \kappa_n\}$ are the eigenvalues of $M_{ij}$ and $\{\lambda_1, ..., \lambda_n\}$ are the eigenvalues of $A_{ij}$. Then, assuming that $0 < H_0 \leq H(x,t) \leq H_1 < \infty$ on $\Sigma_t$ for $t \in [0, T')$, we have the following estimates for these eigenvalues*

$$\kappa_i \leq \frac{nH_1^2}{H_0^2} \cdot \frac{e^{\frac{2nt}{H_0^2}}}{C + e^{\frac{2nt}{H_0^2}}} \qquad \lambda_i \leq \frac{n}{H_0} \cdot \frac{H_1^2}{H_0^2} \cdot \frac{e^{\frac{2nt}{H_0^2}}}{C + e^{\frac{2nt}{H_0^2}}}$$

*for all $t \in [0, T)$ where $C = \frac{nH_1^2}{C_0 H_0^2}$ and $C_0 = \sup_{x \in \mathbb{R}^n} \kappa_{\max}(x, 0)$.*

*Proof.* We consider the evolution inequality for $M_i^j = HA_i^j$, Ding (2010), which can be deduced from the equations derived in chapter 2.

$$\left(\partial_t - \frac{1}{H^2}\Delta\right) M_i^j \leq -2\frac{\nabla_i H \nabla^j H}{H^2} - 2\frac{1}{H^3}\nabla^k M_i^j \nabla_k H - 2\frac{M^{jk}M_{ik}}{H_1^2} + \frac{2n}{H_0^2}M_i^j$$

Now if we let $\kappa(t) = \frac{nH_1^2}{H_0^2}\left(\frac{e^{\frac{2nt}{H_0^2}}}{C+e^{\frac{2nt}{H_0^2}}}\right)$ which is a solution of the ODE $\frac{\partial \kappa_{sup}}{\partial t} = \frac{2\kappa_{sup}}{H_1^2}\left(\frac{nH_1^2}{H_0^2} - \kappa_{sup}\right)$, the ODE corresponding to the zero order terms in the evolution of $M_i^j$. Then we define $G_i^j = \kappa(t)\delta_i^j - M_i^j$ and compute

$$\left(\partial_t - \frac{1}{H^2}\Delta\right) G_i^j \geq 2\frac{\nabla_i H \nabla^j H}{H^2} + 2\frac{1}{H^3}\nabla^k G_i^j \nabla_k H + 2\frac{G^{jk}G_{ik}}{H_1^2} - \frac{2n}{H_0^2}G_i^j$$

So we see that $B_i^j = 2\frac{G^{jk}G_{ik}}{H_1^2} - \frac{2n}{H_0^2}G_i^j$ satisfies the zero eigenvector condition and hence by the non-compact tensor maximum principle Theorem (6) we have that $G \geq 0$ and hence $M \leq \kappa(t)\delta$ which implies that $\kappa_i(t) \leq \frac{nH_1^2}{H_0^2}\left(\frac{e^{\frac{2nt}{H_0^2}}}{C+e^{\frac{2nt}{H_0^2}}}\right)$. The bound on $\lambda_i$ follows immediately since $\lambda_i = \frac{\kappa_i}{H}$ and $H \geq H_0$.

□

Now we prove long time existence in the following Theorem.

**Theorem 11.** *Let $\Sigma_t$ be a solution of IMCF with initial hypersurface $\Sigma_0$ satisfying the following bounds $0 < H_0 \leq H(x,0) \leq H_1 < \infty$ and $|A|(x,0) \leq A_0 < \infty$. We further assume that $\Sigma_0$ can be represented as a graph of a bounded function $y \in C^{2+\alpha}(\mathbb{R}^n)$, in the upper half space model of hyperbolic space, such that $y(x) > y_0 > 0$. Then the IMCF starting at $\Sigma_0$ exists for all time $t \in [0,\infty)$.*

*Proof.* The proof of Theorem (11) is now finished in the exact same way as Huisken and Ilmanen (2008) because we have lower and upper bounds on $H(x,t)$ for all time $t \in [0,T')$, from Theorem (10), we can use Theorem (9) to give us an upper bound on $|A|^2$ for all $t \in [0,T')$. More specifically from Theorem (9) and (10) we have the following bound

$$\lambda_i \leq \frac{n^2 H_1^2}{H_0^3} \cdot \frac{e^{\frac{2nt}{H_0^2}}}{C + e^{\frac{2nt}{H_0^2}}}$$

where $C = \frac{nH_1^2}{C_0 H_0^2}$, $H_1 = \max\{H_{sup}(0), n\}$ and $H_0 = \frac{ny_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)C_0}$.

This bound is uniform for all $T' < T$, the maximal time of existence. Now we can combine all of the estimates of this chapter to imply that $F$ is a uniformly parabolic operator which satisfies the requirements to apply the regularity results of Krylov (1987b) to find that $|u|_{1+\alpha/2,2+\alpha}$ is bounded. For higher order estimates see Gerhardt (1990), Gerhardt (2006), Krylov (1987b) and Urbas (1990).

Then we can extract a subsequence of times $t_i$ so that $\Sigma_{t_i} \to \Sigma_T$ as $i \to \infty$ where $\Sigma_T$ is a $C^{2+\alpha}$ or smooth, respectively, hypersurface with the same uniform bounds on $w$, $H$ and $A$. Then we can apply the short time existence results Theorem (3) with initial condition $\Sigma_T$ and hence continue the flow which contradicts the fact that $T$ was supposed to be maximal.

□

# Chapter 6

# Asymptotic Analysis

We have already shown the following bounds for $H$

$$c_0\sqrt{n^2 + C_0 e^{-2t/n}} \leq H(x,t) \leq \sqrt{C_0 e^{-2t/n} + n^2}$$

where $C_0 = H_{sup}(0)^2 - n^2$ if $H_{sup}(0) > n$ and $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)H_{sup}(0)}$ or

$$c_0 \leq H(x,t) \leq n$$

where $H_{sup}(0) \leq n$ and $c_0 = \frac{y_{inf}(0)H_{inf}(0)}{y_{sup}(0)v_{sup}(0)}$.

So now we would like to improve the lower bound so that we can show that $\lim_{t\to\infty} H(x,t) = n$.

More precisely we will prove the following theorem, using ideas from Brendle et al. (2016).

**Theorem 12.** *For hypersurfaces $\Sigma_0$ satisfying the long time existence theorem for IMCF in hyperbolic space we find that*

$$H = n + O(te^{-2t/n}) \qquad \textit{(Uniformly on } \Sigma_t\textit{)}$$

*Proof.* To this end we will consider the function $z = \frac{v}{H} = \frac{u}{y}$ and compute the following evolution equation

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)z &= y^{-1}(\partial_t - \frac{1}{H^2}\Delta)u - uy^{-2}(\partial_t - \frac{1}{H^2}\Delta)y - \frac{2}{H^2}g(\nabla u, \nabla y^{-1}) \\
&= \frac{2}{H^2 yw}g(\nabla w, \nabla u) - \frac{nu}{yH^2} + \frac{2}{y^2H^2}g(\nabla u, \nabla y) \\
&- \frac{u}{yH^2}\left((n-2) + 2\langle \partial y, \nu\rangle_0^2\right) + \frac{2u}{yH}\langle \nu, \partial y\rangle_0 \\
&= \frac{2}{H^2 yw}g(\nabla w, \nabla u) + \frac{2}{y^2H^2}g(\nabla u, \nabla y) + 2\frac{z^2}{v^2} - \frac{nz^3}{v^2} - \frac{(n-2)z}{H^2} - \frac{2z}{v^2H^2} \\
&= \frac{2}{H^2 yw}g(\nabla w, \nabla u) + \frac{2}{y^2H^2}g(\nabla u, \nabla y) + 2\frac{nz^2}{v^2}\left(\frac{1}{n} - z\right) + \frac{2z}{H^2} - \frac{2z}{v^2H^2}
\end{aligned}
$$

where we have taken advantage of the fact that $\langle \nu, \partial_y\rangle_0 = -\frac{1}{v}$. Now we would like to better understand the gradient terms.

$$
\frac{2}{H^2 yw}g(\nabla w, \nabla u) = 2\frac{g(\nabla w, \nabla(uy))}{H^2y^2w} - 2u\frac{g(\nabla w, \nabla y)}{H^2y^2w}
$$

and now we note that

$$
g(\nabla w, \nabla y) = \nabla_{\partial y^T} w = \frac{1}{y}\bar{g}(\partial_y^T, \nu) + A(\eta^T, \partial_y^T) = A(\eta^T, \partial_y^T)
$$

from which we find

$$
\frac{2}{H^2 yw}g(\nabla w, \nabla u) = 2\frac{g(\nabla w, \nabla(uy))}{y^2H^2w} - 2u\frac{A(\partial_y^T, \partial_y^T)}{y^2H^2w} \leq 2\frac{g(\nabla w, \nabla(uy))}{y^2H^2w} + 2v^2|\partial_y^T|^2\frac{|A|}{H^3}
$$

Now if we rearrange the other gradient term we find

$$
\frac{2}{y^2H^2}g(\nabla u, \nabla y) = \frac{2}{H^2y^3}g(\nabla(uy), \nabla y) - \frac{2u}{y^3H^2}g(\nabla y, \nabla y) = -\frac{2}{H^2y}g(\nabla(uy), \nabla y) - 2\frac{z}{H^2}|\partial y^T|^2
$$

where we used the fact that $\nabla y = y^2\partial_y^T$ and $g(\nabla y, \nabla y) = y^2|\partial_y^T|^2$.

When we put all of this together we find

$$(\partial_t - \frac{1}{H^2}\Delta)z = 2\frac{g(\nabla w, \nabla(uy^{-1}))}{H^2 w} - \frac{2}{H^2 y} g(\nabla(uy^{-1}), \nabla y) + 2\frac{nz^2}{v^2}\left(\frac{1}{n} - z\right)$$
$$+ \frac{2}{H^2}\left((v^2-1)\frac{|A|}{H} + z\frac{v^2-1}{v^2} - z\frac{(v^2-1)}{v^2}\right)$$
$$= 2\frac{g(\nabla w, \nabla(uy^{-1}))}{H^2 w} - \frac{2}{H^2 y} g(\nabla(uy^{-1}), \nabla y) + 2\frac{nz^2}{v^2}\left(\frac{1}{n} - z\right) + \frac{2|A|}{H^3}(v^2-1)$$

where we have used the fact that $|\partial_y^T|^2 = 1 - |\nu_0^y|^2 = 1 - \frac{1}{v^2} = \frac{v^2-1}{v^2}$.The maximum principle at infinity leads to the following differential inequality for $z_{sup}(t)$, at points of differentiability

$$\frac{dz_{sup}}{dt} \leq 2\frac{nz_{sup}^2}{v^2}\left(\frac{1}{n} - z_{sup}\right) + Ce^{-2t/n}$$

where we are using the fact that $\frac{2|A|}{H^3}$ is bounded by Theorem (9) and Theorem (10) and the decay of $v^2 - 1 = |\nabla^0 y|^2$ which follows from Lemma (7) below.

This implies by integration that $z \leq \frac{1}{n} + O(te^{-2t/n})$ and since $z = \frac{v}{H}$ and $v = 1 + O(te^{-2t/n})$, by Lemma 6, we can deduce the statement of the theorem

$$H = \frac{v}{z} \geq \frac{1 + O(te^{-2t/n})}{\frac{1}{n} + O(te^{-2t/n})} = n + O(te^{-2t/n})$$

where we note that we already had the necessary upper bound on $H$.

□

**Lemma 7.** *For hypersurfaces $\Sigma_0$ satisfying the long time existence theorem (11) for IMCF in hyperbolic space we find that*

$$v^2 - 1 = |\nabla^0 y|^2 \leq Ce^{-2t/n}$$

*Proof.* If we define $\psi = y_{x_1}^2 + ... + y_{x_n}^2 = |\nabla^0 y|^2$ and then differentiate the equation $\frac{\partial y}{\partial t} = \frac{-1}{F}$ w.r.t $y^k \nabla_k^0$, where $F = \frac{ny^{-1} + \tilde{\delta}^{ij} y_{ij}}{v^2} = \frac{H}{vy}$ we find

$$
\begin{aligned}
\frac{\partial \psi}{\partial t} &= y^k \left(\frac{\partial y}{\partial t}\right)_k = y^k \left(\frac{-1}{F}\right)_k = \frac{1}{F^2} y^k F_k \\
&= \frac{1}{v^2 F^2} y^k \left(-2F y^l y_{lk} - n y^{-2} y_k + \tilde{\delta}^{ij} y_{ijk} - 2\frac{y^i_k y^j y_{ij}}{v^2} + 2\frac{y^i y^j y_{ij} y^l y_{lk}}{v^4}\right) \\
&= \frac{1}{v^2 F^2}\left(\tilde{\delta}^{ij} y_{ijk} y^k + 2G^k \psi_k - \frac{2n\psi}{y^2}\right)
\end{aligned}
$$

where notice that $\psi_k = y^l y_{lk}$ and we have that $G^k = -F y_k - \frac{1}{v^2} y_{jk} y^j + \frac{1}{v^4} y^i y^j y_{ij} y_k$.

Now if we also notice the following

$$
\tilde{\delta}^{ij}\psi_{ij} = \tilde{\delta}^{ij}(y_{kij} y^k + y^k_j y_{ki}) = \tilde{\delta}^{ij} y_{ijk} y^k + \tilde{\delta}^{ij} y^k_j y_{ki}
$$

where we notice that the difference between this case and the graph over a sphere case is that we don't get an extra term from commuting derivatives here.

We can also rewrite $\tilde{\delta}^{ij} y^k_j y_{ki}$ in the following way

$$
\tilde{\delta}^{ij} y^k_j y_{ki} = \delta^{lm}\tilde{\delta}^{ij} y_{li} y_{mj} = \delta^{lm}\delta^{ij} y_{li} y_{mj} - \frac{\delta^{lm}}{v^2} y^i y_{li} y^j y_{mj} = \delta^{ij}\delta^{lm} y_{il} y_{jm} - \frac{1}{v^2}\psi^k \psi_k
$$

So that we now obtain the desired evolution equation

$$
\frac{\partial \psi}{\partial t} = \frac{1}{v^2 F^2}\left(\tilde{\delta}^{ij}\psi_{ij} + 2G^k \psi_k + \frac{1}{v^2}\psi^k\psi_k - \frac{2n\psi}{y^2} - \delta^{ij}\delta^{lm} y_{il} y_{jm}\right)
$$

We can use this and the maximum principle at infinity to derive a differential inequality for $\psi_{sup}(t)$, at points of differentiability

$$
\frac{d\psi_{sup}}{dt} \leq \frac{-2n}{H^2}\psi_{sup} \leq \frac{-2n}{n^2 + C_0 e^{-2t/n}}\psi_{sup} \leq -2\left(\frac{1}{n} - \bar{C} e^{-2t/n}\right)\psi_{sup}(t)
$$

where we have used the bound $H^2 \leq n^2 + C_0 e^{-2t/n}$ and chosen a constant $\bar{C} > 0$. Now by integrating this differential inequality we find

$$\psi_{sup} \leq D e^{-2t/n - ne^{-2t/n}}$$

for some constant $D > 0$ which implies that $\psi = |\nabla^0 y|^2 = O(e^{-2t/n})$, as desired. □

Now we can show that, for suitable hypersurfaces, the IMCF converges asymptotically to horospheres.

**Proposition 1.** *For hypersurfaces $\Sigma_0$ satisfying the long time existence theorem for IMCF in hyperbolic space we find that*

$$|A_{ij} - g_{ij}| \leq Ct^2 e^{-2t/n}$$

*Proof.* We have previously calculated the evolution equation for $M_i^j = HA_i^j$

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\left(M_i^j\right) \leq -2\frac{\nabla_i H \nabla^j H}{H^2} - 2\frac{1}{H^3}\nabla^k M_i^j \nabla_k H - 2\frac{M^{jk}M_{ik}}{H_1^2} + \frac{2n}{H^2}M_i^j$$

and now using Theorem 1 we can adjust this equation as follows

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\left(M_i^j\right) \leq -2\frac{\nabla_i H \nabla^j H}{H^2} - 2\frac{1}{H^3}\nabla^k M_i^j \nabla_k H - 2\frac{M^{jk}M_{ik}}{n^2} + \frac{2}{n}M_i^j + Cte^{-2t/n}\delta_i^j$$

where we are using the fact that $|M|$ is bounded by the estimates obtained in Theorem (9) and Theorem (10).

Then by the non-compact tensor maximum principle Theorem (6) and an argument similar to that of the proof of Theorem (10) we just need to solve the following differential inequality for $\kappa_{sup}(t)$, at points of differentiability of $\kappa_{sup}(t)$

$$\frac{d\kappa_{sup}}{dt} \leq \frac{2\kappa_{sup}}{n^2}\left(n - \kappa_{sup}\right) + Cte^{-2t/n}$$

which implies, if you assume that $\kappa_{sup} \geq n$, by integrating that $\kappa_{sup}(t) \leq n + Ct^2e^{-2t/n}$ and so by Theorem 1 we have that

$$\lambda_{sup}(t) \leq \frac{\kappa_{sup}(t)}{H_{sup(t)}} \leq \frac{n + Ct^2e^{-2t/n}}{n + O(te^{-2t/n})} = 1 + O(t^2e^{-2t/n})$$

By combining this result with the fact that $H = n + O(te^{-2t/n})$ implies the statement of the proposition since it is not possible for any eigenvalue of $\lambda_i$, to be uniformly less than 1, $\lambda_i < \delta_0 < 1$ for $t \in [\tau, \infty)$ for some $\tau > 0, \delta_0 > 0$, $H = n + CO(te^{-2t/n})$ and $\lambda_{sup}(t) = 1 + CO(t^2e^{-2t/n})$ .

□

# Chapter 7

# Interior Estimates

In this chapter we are concerned with obtaining interior estimates for the flow which will demonstrate the analogous behavior of IMCF to the heat equation as well as potentially help us to obtain a stronger short time existence theorem where we allow the operator to become degenerate at infinity. This should be compared to the interior estimates obtained by Ecker and Huisken (1991) in the case of solutions to Mean Curvature Flow (MCF) of non-compact graphs in $\mathbb{R}^{n+1}$ as well as to Unterberger (2003) in the case of solutions to MCF of non-compact graphs over $S^n$ in $\mathbb{H}^{n+1}$. In both cases, the interior estimates were used to prove short time existence when the initial data is Lipschitz and the operator is allowed to become degenerate at infinity, i.e. for a sequence $x_n \in \Sigma_t$ diverging to infinity $H \to 0$.

In order to be able to prove any local estimates we need to develop well suited cutoff functions for IMCF and so we develop those tools in this chapter. We will start with important proposition which will allow us to compute useful evolution equations that will lead to a definition of cutoff functions which we will use throughout the second half of this chapter. In Unterberger (2003) extrinsically defined cutoff functions were also used in order to obtain interior estimates as well as Ecker (2004) but we will notice that the convenient cancellation that occurs in MCF for extrinsically defined functions does not occur when calculating evolution equations w.r.t. IMCF for extrinsically defined functions but we can still use this method to obtain useful cutoff functions.

The main result in this chapter are the interior estimates for $w$ (the support function), $H$ and $A$ in Theorem (13).

We consider the Riemannian manifold $N^{n+1}$ parameterized over $\mathbb{R}^b_a := \{(x_1, ..., x_n, y) \in \mathbb{R}^{n+1} : a < y < b\}$ where $a, b \in [-\infty, \infty]$ with the metric $\bar{g} = \lambda(y)^2\delta$ which is defined where $\lambda : (a, b) \to \mathbb{R}$ is defined. We will consider a $n$ dimensional, non-compact hypersurface $\Sigma_0 \subset \mathbb{R}^b_a$.

In line with our previous notation conventions, we will use bars to denote geometric quantities w.r.t $N^{n+1}$, superscript 0 to denote quantities w.r.t. $\delta$ and no bar or subscript to denote quantities w.r.t. $\Sigma_0$, endowed with the metric induced from $\bar{g}$.

By using well known formulas for conformal metrics, derived from Levi-Civita's formula for the connection, we can find the following expression

$$\bar{\nabla}_X Y = \nabla^0_X Y + \frac{\lambda'}{\lambda}\left(\langle X, \partial_y\rangle_0 Y + \langle \partial_y, Y\rangle_0 X - \langle X, Y\rangle_0 \partial_y\right) \tag{7.1}$$

Using this, and the convention that we will put a bar over a vector field $\bar{Z} = \lambda^{-1}(y)Z$ so that $\bar{Z}$ is a unit vector w.r.t. $\bar{g}$, we can obtain the following

$$\begin{aligned}
\bar{div} X &= \bar{g}(\bar{\nabla}_{\bar{e}_i} X, \bar{e}_i) \\
&= \langle \nabla^0_{e_i} X + \frac{\lambda'}{\lambda}(\langle e_i, \partial_y\rangle_0 X + \langle \partial_y, X\rangle_0 e_i - \langle e_i, X\rangle_0 \partial_y, e_i\rangle_0 \\
&= div^0 X + (n+1)\frac{\lambda'}{\lambda}\langle X, \partial_y\rangle_0
\end{aligned}$$

where $\{e_1, ..., e_{n+1}\}$ is a orthonormal basis for $\mathbb{R}^{n+1}$ w.r.t the flat metric. From which we obtain a useful expression for $H$ by using the fact that $H = \bar{div}(\bar{\nu})$

$$H = -\frac{H_0}{\lambda} + n\frac{\lambda'}{\lambda^2}\langle \nu_0, \partial_y\rangle_0 = -\frac{H_0}{\lambda} - n\frac{\lambda'}{\lambda^2}\langle \nu_0, \eta\rangle_0$$

where $\eta = -\partial_y$, which we will discuss in more detail later.

**Note:** We are choosing the "downward" pointing normal which is important to keep in mind.

Now we can obtain a useful Proposition which will allow us to find many important evolution equations w.r.t IMCF.

**Proposition 2.** *Let* $f : U \to \mathbb{R}$ *where* $U \subset \mathbb{R}^n \times \mathbb{R}_+$ *open, we consider the function* $g : \Sigma \times [0,T) \to \mathbb{R}$ *defined by* $g(p,t) = f(\varphi(p,t))$ *which has the following evolution equation under IMCF*

$$(\partial_t - \frac{1}{H^2}\Delta^{\Sigma_t})g = \frac{1}{\lambda^2 H^2}\left(\langle \nabla^0_\nu \nabla^0 f, \nu\rangle_0 - \Delta^0 f - (n-2)\frac{\lambda'}{\lambda}\langle \nabla^0 f, \partial_y\rangle_0 - 2\frac{\lambda'}{\lambda}\langle \nabla^0 f, \nu\rangle_0 \langle \nu, \partial_y\rangle_0\right) + \frac{2}{\lambda H}\nabla^0_\nu f$$

*Proof.* For any function $u$ and vector field $X$ we have that

$$\bar{div}(uX) = u\ \bar{div}(X) + X(u)$$
$$\bar{\nabla} u = \lambda^{-2}\nabla^0 u$$

Then we notice that

$$\partial_t g = \frac{1}{H}\bar{\nabla}_{\bar{\nu}} f$$
$$\Delta^{\Sigma_t} g = div(\nabla g) = div(\bar{\nabla} f - \bar{\nabla}_{\bar{\nu}} f \bar{\nu}) = div(\bar{\nabla} f) - H\bar{\nabla}_{\bar{\nu}} f$$

where $\Delta^{\Sigma_t} = g^{ij}\nabla^{\Sigma_t}\nabla^{\Sigma_t}$, the Laplacian w.r.t. the hypersurface $\Sigma_t$.

**Note:** Here is where we see a big difference between MCF and IMCF. When studying MCF there is a cancellation between the time derivative term and the first order term in the Laplacian which simplifies computations. In IMCF these two terms combine and hence give an extra term to deal with.

Now we can find the following expression where our goal is to first write all derivatives as extrinsic derivatives in $N^{n+1}$ and then convert all of those derivatives to derivatives on $\mathbb{R}^{n+1}$, using the formulas obtained above.

$$
\begin{aligned}
&(\partial_t - \frac{1}{H^2}\Delta^{\Sigma_t})g = \frac{2}{H}\bar{\nabla}_{\bar{\nu}} f - \frac{1}{H^2} div(\bar{\nabla} f) \\
&= \frac{1}{H^2}\left(-\bar{div}(\bar{\nabla} f) + \langle \bar{\nabla}_\nu \bar{\nabla} f, \nu\rangle_0\right) + \frac{2}{\lambda H}\nabla^0_\nu f \\
&= \frac{1}{H^2}\left(-\lambda^{-2}\bar{div}(\nabla^0 f) - \nabla^0 f(\lambda^{-2}) + \lambda^{-2}\langle \bar{\nabla}_\nu \nabla^0 f, \nu\rangle_0 + \nu(\lambda^{-2})\langle \nabla^0 f, \nu\rangle_0\right) + \frac{2}{\lambda H}\nabla^0_\nu f \\
&= \frac{1}{H^2}\left(-\lambda^{-2}\Delta^0 f - (n+1)\frac{\lambda'}{\lambda^3}\langle \nabla^0 f, \partial_y\rangle - \langle \nabla^0 f, \nabla^0(\lambda^{-2})\rangle_0 + \langle \nu, \nabla^0(\lambda^{-2})\rangle_0 \langle \nabla^0 f, \nu\rangle_0\right) \\
&+ \frac{1}{\lambda^2 H^2}\left(\langle \nabla^0_\nu \nabla^0 f + \frac{\lambda'}{\lambda}\left(\langle \nu, \partial_y\rangle_0 \nabla^0 f + \langle \partial_y, \nabla^0 f\rangle_0 \nu - \langle \nu, \nabla^0 f\rangle_0 \partial_y\right), \nu\rangle_0\right) + \frac{2}{\lambda H}\nabla^0_\nu f \\
&= \frac{1}{\lambda^2 H^2}\left(\langle \nabla^0_\nu \nabla^0 f, \nu\rangle_0 - \Delta^0 f - (n-2)\frac{\lambda'}{\lambda}\langle \nabla^0 f, \partial_y\rangle_0 - 2\frac{\lambda'}{\lambda}\langle \nabla^0 f, \nu\rangle_0 \langle \nu, \partial_y\rangle_0\right) + \frac{2}{\lambda H}\nabla^0_\nu f
\end{aligned}
$$

□

**Note:** $g$ depends on $t$ through the embedding function $\varphi_t$ but if it also independently depends on $t$ then there will be another term in the evolution equation for $g$ corresponding to the partial derivative w.r.t this aforementioned dependence on $t$.

**Note:** From now on we will be sloppy and just denote $g$, the function defined on $\Sigma_t$, and $f$, the extrinsically defined function on $N$, as the same function where the composition with the embedding function, $\varphi$, is implied.

Now we make the following definition which we will use throughout the rest of the document.

**Definition 5.** *Let $\Sigma_0$ be a hypersurface satisfying the conditions of Theorem (11) and let $\Sigma_t$ be the corresponding solution of IMCF which is guaranteed to exist for all time $t \in [0, \infty)$. Then for $T < \infty$ we let*

$$\Omega_{R,T} := B_R(0) \times [0, T)$$

*and then we also define*

$$\mathcal{H}_0^R = \inf_{\overline{\Omega}_{R,T}} \min(H, H^2) > 0$$

*If we consider a function* $\alpha(x_1, ..., x_n, y, t)$ *depending on* $R, \mathcal{H}_0^R$ *then we can also define*

$$U_R = \{(x,t) \in \Omega_{R,T} : \alpha(\varphi(x,t), t) > 0\}$$

**Note:** We will be interested in $\alpha$ such that $\overline{U}_R \subset \Omega_{R,T}$ so that we can use the fact that $\alpha \equiv 0$ on $\partial U_R$ to exclude the possibility that the maximum of a positive function $\alpha g$ can occur on $\partial U_R$ . We will impose this condition below through the constant $C_R$.

**Lemma 8.** *If we define* $\alpha = R^2 - |x|^2 - \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0 R + C_R)t$ *for* $N = \mathbb{H}^{n+1}$*, where* $y(x,0) \le y_0$*,* $C_R \ge 0$ *is arbitrary. Then* $\alpha$ *is a subsolution to the IMCF heat operator on* $\Sigma_t$*, i.e. for* $t \in [0,T)$*:*

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\alpha \le -\frac{2C_R}{\mathcal{H}_0^R} \le 0$$

*Proof.* If we let $|x|^2 = x_1^2 + ... + x_n^2$ for $N = \mathbb{H}^{n+1}$, in the upper half space model, then we find

$$(\partial_t - \frac{1}{H^2}\Delta)|x|^2 = \frac{1}{H^2}\left(y^2(2|\hat{\nu}|^2 - 2n) + 4y\langle x, \nu\rangle_0 \langle \nu, \partial y\rangle_0\right) + \frac{4y}{H}\langle x, \nu\rangle_0$$

where we have used the following relations as well as Proposition 2 (2)

$$\nabla^0|x|^2 = 2x \quad \nabla^0_\nu\nabla^0|x|^2 = 2\hat{\nu} \quad \Delta^0|x|^2 = 2n$$

where $\hat{\nu}$ is the projection of $\nu$ onto $\mathbb{R}^n \times \{0\}$.

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)\alpha &= \frac{-1}{H^2}\left(y^2(2|\hat{\nu}|^2 - 2n) + 4y\langle x,\nu\rangle_0\langle \nu, \partial y\rangle_0\right) - \frac{4y}{H}\langle x,\nu\rangle_0 - \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0R + C_R)\\ &\leq \frac{2ny^2}{H^2} + \frac{4yR}{H^2} + \frac{4yR}{H} - \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0R + C_R) \leq -\frac{2C_R}{\mathcal{H}_0^R}\end{aligned}$$

□

**Note:** We purposefully leave $C_R > 0$ undefined for now because we will choose it later depending on the estimate we are trying to achieve. We do note that in order to cause $\overline{U}_R \subset \Omega_{R,T}$ we may need to choose $C_R$ large enough to make this true. Since this choice can be made so that $C_R$ is $O(R^2)$ we will not worry about it in the future .

So now we have found good cutoff functions in the sense that they are subsolutions to the IMCF heat operator. Now we will use the cutoff functions we just derived in order to obtain detailed interior estimates for solutions of IMCF. This section ends with the full collection of local estimates for important geometric quantities in Theorem (13) as well as the corresponding global estimates as we let $R \to \infty$ in Corollary (1).

We start by obtaining evolution equations for $w^{-1}$ and $\alpha w^{-1}$ which will lead to our first local estimate for the lower bound of the support function which is equivalent to a bound of the derivative of the graph function $y$ since $w^{-1} = yv$.

**Lemma 9.** *If we let $\alpha$ be a subsolution to the IMCF heat operator defined above we find in $\Omega_{R,T}$ (See Definition (5) for the definition of $\mathcal{H}_0^R$)*

$$(\partial_t - \frac{1}{H^2}\Delta)(\alpha w^{-1}) \leq -\frac{2}{w^{-1}H^2}g(\nabla w^{-1}, \nabla(\alpha w^{-1})) - \frac{1}{n}(\alpha w^{-1}) - 2\frac{C_R}{\mathcal{H}_0^R}w^{-1}$$

*Proof.* We have the following evolution of $w^{-1}$ from Lemma (2) (ii) in $\Omega_{R,T}$

$$(\partial_t - \frac{1}{H^2}\Delta)w^{-1} = -w^{-2}(\partial_t - \frac{1}{H^2}\Delta)w - \frac{2}{w^3H^2}|\nabla w|^2 = -\frac{|A|^2}{H^2}w^{-1} - \frac{2}{w^{-1}H^2}|\nabla w^{-1}|^2$$

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\alpha w^{-1}) &= w^{-1}(\partial_t - \frac{1}{H^2}\Delta)\alpha + \alpha(\partial_t - \frac{1}{H^2}\Delta)w^{-1} - 2\frac{g(\nabla\alpha, \nabla w^{-1})}{H^2} \\ &\leq -\frac{|A|^2}{H^2}(\alpha w^{-1}) - \frac{2\alpha}{w^{-1}H^2}|\nabla w^{-1}|^2 - 2\frac{g(\nabla\alpha, \nabla w^{-1})}{H^2} \\ &\leq -\frac{2w}{H^2}g\left(\nabla\left(\alpha w^{-1}\right), \nabla w^{-1}\right) - \frac{|A|^2}{H^2}\alpha w^{-1} - 2\frac{C_R}{\mathcal{H}_0^R}w^{-1}\end{aligned}$$

where we are using the fact that

$$g\left(\nabla\left(\alpha w^{-1}\right), \nabla w^{-1}\right) = \alpha|\nabla w^{-1}|^2 + w^{-1}g(\nabla\alpha, \nabla w^{-1})$$

and then the last inequality follows from the fact that $|A|^2 \geq \frac{1}{n}H^2$.

□

Now we will take Lemma (9) and turn it into an estimate for $w^{-1}$ and hence a lower bound for $w$. First we make an important definition which will also be used throughout this chapter.

**Definition 6.** *Let $\Sigma_0$ be a hypersurface and let $\Sigma_t$ be the corresponding solution of IMCF which exists for all time $t \in [0, T)$, $T < \infty$. Then for $\theta \in (0, 1)$ given and $t \in [0, T]$ we let*

$$\begin{aligned}U_{R,\theta,t} &= \{(x,t) \in \Omega_{R,T} : t \text{ fixed and } |x|^2 + \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0R + C_R)t \leq \theta R^2\} \\ &= \{(x,t) \in \Omega_{R,T} : t \text{ fixed and } \alpha(\varphi(x,t),t) \geq (1-\theta)R^2\}\end{aligned}$$

**Lemma 10.** *If we assume that $\Sigma_0$ is a hypersurface to which the hypotheses of Theorem (11) hold on compact subsets, then we find that for all $t \in [0, T]$*

$$\max_{U_{R,\theta,t}} w^{-1} \leq (1-\theta)^{-1} \left( \max_{U_{R,1,0}} w^{-1} \right) e^{-t/n}$$

*Proof.* From Lemma (9) we have the following evolution equation

$$(\partial_t - \frac{1}{H^2}\Delta)(\alpha w^{-1}) = -\frac{2}{w^{-1}H^2} g(\nabla w^{-1}, \nabla(\alpha w^{-1})) - \frac{|A|^2}{H^2}(\alpha w^{-1})$$

So then by a Lemma of Hamilton applied in $U_R$ where we have $\alpha \equiv 0$ on the boundary and using that $|A|^2 \geq nH^2$ we find

$$\frac{d}{dt} \max_{U_{R,1,t}} (\alpha w^{-1}) \leq -\frac{1}{n} \max_{U_{R,1,t}} (\alpha w^{-1})$$

and so if we integrate this equation we find

$$\max_{U_{R,1,t}} (\alpha w^{-1}) \leq \max_{U_{R,1,0}} (\alpha w^{-1}) e^{-t/n}$$

and then using the fact that $\alpha \geq (1-\theta)R^2$ on $U_{R,\theta,t}$ we find

$$(1-\theta)R^2 \max_{U_{R,\theta,t}} w^{-1} \leq \max_{U_{R,1,0}} w^{-1} R^2 e^{-t/n}$$

which yields the desired result.

□

So now we have found our first local estimate which will give us a lower bound on the support function in a parabolic ball which says that $\Sigma_t$ remains a graph in the corresponding parabolic ball. In other words, this gives us an interior gradient bound for the solution $y(x,t)$ since $w^{-1} = yv$ In the next section we will obtain a series of local estimates, which require more work than the straight forward cutoff function techniques used above, culminating in Theorem (13) and Corollary (1).

Notice that we were able to choose $C_R = 0$ in the definition of $\alpha$ while obtaining the estimate for $w^{-1}$ which means that this estimate will extend as $R \to \infty$ easily since $U_R$ will be non-degenerate as $R \to \infty$.

Next our goal is to obtain a lower interior estimate on $H$ which we will obtain through the function $u = \frac{1}{wH}$ since we can leverage the good terms in the evolution equation of $w$ in order to kill the bad terms in the evolution equation of $H$ in order to obtain a useful evolution equation for $u$.

**Lemma 11.** *We have the following evolution inequality for* $\alpha u$

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha u) \leq &-\frac{2}{H^2} g(\nabla(w^{-1}), \nabla(\alpha u)) - \frac{2}{\alpha H^2} g(\nabla \alpha, \nabla(\alpha u)) \\
&+ \frac{2u}{H^2} g(\nabla(w^{-1}), \nabla \alpha) + \frac{2u}{\alpha H^2}|\nabla \alpha|^2 - \frac{n}{H^2}(\alpha u) - 2u\frac{C_R}{\mathcal{H}_0^R}
\end{aligned}
$$

*Proof.* From Lemma (8) and the evolution equation for $u$ obtained in Lemma (3) (iv)

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha u) &= u(\partial_t - \frac{1}{H^2}\Delta)\alpha + \alpha(\partial_t - \frac{1}{H^2}\Delta)u - 2\frac{g(\nabla \alpha, \nabla u)}{H^2} \\
&\leq -2u\frac{C_R}{\mathcal{H}_0^R} + 2\alpha\frac{g(\nabla w, \nabla u)}{H^2 w^2} - \frac{n}{H^2}(\alpha u) - 2\frac{g(\nabla \alpha, \nabla u)}{H^2}
\end{aligned}
$$

Then using the following two relations

$$
\begin{aligned}
-\frac{2}{H^2} g(\nabla(w^{-1}), \nabla(\alpha u)) &= -\frac{2\alpha}{H^2} g(\nabla(w^{-1}), \nabla u) - \frac{2u}{H^2} g(\nabla(w^{-1}), \nabla \alpha) \\
-\frac{2}{\alpha H^2} g(\nabla \alpha, \nabla(\alpha u)) &= -\frac{2}{H^2} g(\nabla \alpha, \nabla u) - \frac{2u}{\alpha H^2}|\nabla \alpha|^2
\end{aligned}
$$

we get the desired inequality.

□

Notice that we have an $\alpha$ in the denominator of the $|\nabla\alpha|^2$ term which we know causes problems so we will try to handle this by looking at $\eta(\alpha)$ for $\eta : \mathbb{R} \to \mathbb{R}$ increasing, where we will impose the condition that $\eta(0) = 0$, if we can.

**Lemma 12.** *We have the following evolution inequality for $\eta(\alpha)u$ (in $\Omega_{R,T}$), which we will often write as $\eta u$*

$$(\partial_t - \frac{1}{H^2}\Delta)(\eta u) \leq -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'}{\eta H^2}g(\nabla\alpha, \nabla(\eta u)) + \frac{2u\eta'}{H^2}g(\nabla(w^{-1}), \nabla\alpha) + \frac{u|\nabla\alpha|^2}{H^2}\left(\frac{2\eta'^2}{\eta} - \eta''\right) - \frac{n}{H^2}(\eta u) - 2u\eta'\frac{C_R}{\mathcal{H}_0^R}$$

*Proof.* This follows by noting the following formula

$$(\partial_t - \frac{1}{H^2}\Delta)(\eta u) = u(\partial_t - \frac{1}{H^2}\Delta)\eta + \eta(\partial_t - \frac{1}{H^2}\Delta)u - 2\eta'\frac{g(\nabla\alpha, \nabla u)}{H^2}$$

and from Lemma (8)

$$(\partial_t - \frac{1}{H^2}\Delta)\eta \leq -\frac{2C_R\eta'}{\mathcal{H}_0^R} - \frac{\eta''|\nabla\alpha|^2}{H^2}$$

and then following the same steps as in the proof of Lemma (11). □

As noted earlier, when obtaining the interior estimate on $w^{-1}$ we were able to set $C_R = 0$ in the definition of $\alpha$. For the next few estimates (except for the upper interior estimate on $H$) we will not be able to do this and so it will be vital to keep track of how our choice of $C_R$ depends on various geometric quantities as well as $R$. We choose to express $C_R$ as a combination of constants $C_R = C_0^R + C_1^R R + C_2^R R^2$ where $C_0^R, C_1^R, C_2^R$ don't depend on $R$ directly but rather depend on estimates of geometric quantities in $\Omega_{R,T}$.

As we obtain the following estimates we will lump various constants into the general constants $C_0^R, C_1^R, C_2^R$ and keep track of their dependence so that we can consider extending these estimates as $R \to \infty$ in Corollary (1) as well as consider proving a stronger short time existence theorem based on these estimates. In the statement of each estimate and at the end of each proof we will make sure to explicitly state the dependence of each of the constants $C_0^R, C_1^R, C_2^R$ on various geometric quantities.

**Lemma 13.** *Assume that $\Sigma_0$ is a hypersurface to which the hypothesis of (11) holds in $B_R \subset \mathbb{R}^n$ then we find that (for all $t \in [0,T]$), $\theta \in (0,1)$*

$$\max_{U_{R,\theta,t}} u \leq \left( \max_{U_{R,1,0}} u \right) e^{\frac{-nt}{H_1^2}} e^{\frac{\theta}{(1-\theta)R^2}} \leq \left( \max_{U_{R,1,0}} u \right) e^{\frac{\theta}{(1-\theta)R^2}}$$

*where $H \leq H_1$ on $U_R$. (Recall $u = \frac{1}{wH} = \frac{yv}{H}$)*

**Note:** $C_0^R$ depends on bounds for the cutoff function, $C_1^R$ depends on upper and lower bounds for $y$, an upper bound on $v$ and an upper bound on $|A|$ in $\Omega_{R,T}$ and $C_2^R = 0$.

*Proof.* If we consider $\eta(s) = e^{-frac1s}$ and use Lemma (12) we find that in $\Omega_{R,T}$

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\eta u) \leq & -\frac{2}{H^2} g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\eta H^2} g(\nabla\alpha, \nabla(\eta u)) \\ & + \frac{u}{H^2}\left(2\eta'|\nabla(w^{-1})||\nabla\alpha| + \left(\frac{2\eta'^2}{\eta} - \eta''\right)|\nabla\alpha|^2\right) - 2u\eta' \frac{C_R}{\mathcal{H}_0^R} - \frac{n}{H^2}(\eta u)\end{aligned}$$

Now if we use the fact that $\Sigma_t$ is well defined hypersurface expressed as a graph over a plane ($\mathbb{R}^n$) with bounded gradient, $v$, in $\Omega_{R,T}$ then we know that there exists some $D_R > 0$ so that $D_R^{-2}\delta \leq g \leq D_R^2\delta$, in $\Omega_{R,T}$. Hence $|\nabla\alpha| \leq D_R|\nabla^0\alpha| \leq 2D_R|x| \leq 2D_R R$. We note that $D_R$ depends on upper and lower bounds on $y$ and a upper bound on $v$ since $g_{ij} = \frac{1}{y^2}(\delta_{ij} + y_i y_j)$. We should note that Lemma (10) and Theorem (7) will give use the desired control.

We also know that $|\nabla w^{-1}| = \frac{|\nabla w|}{w^2} \leq D'_R$ in $\Omega_{R,T}$, which is equivalent to having a lower bound on $w$ Lemma (10) and a bound on $|A|^2$, which can be seen by choosing a vector $v$ tangent to $\Sigma_t$ and calculating

$$\nabla_v w = \nabla_v \bar{g}(\nu, \eta) = \bar{g}(\bar{\nabla}_v \nu, \eta^T) = A(v, \eta^T) \quad \Rightarrow \quad |\nabla w|^2 \leq |A|^2$$

where we have used similar calculation as to Lemma (2).

Lastly, we use the fact that $\frac{2\eta'^2}{\eta}|\nabla\alpha|^2 = \frac{|\nabla\eta|^2}{\eta} \leq \eta'' \leq C'$ for $C^2$, compactly supported functions Ecker (2004) and so we can rewrite the equation as

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\eta u) \leq &-\frac{2}{H^2} g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\eta H^2} g(\nabla\alpha, \nabla(\eta u)) \\ &+ \frac{u}{\mathcal{H}_0^R}\left(4D_R D'_R R\eta' + 2\eta'' - 4D_R^2|x|^2\eta'' - 2\eta' C_R\right) - \frac{n}{H^2}(\eta u)\end{aligned}$$

So now if we use the fact that $2\eta'' - 4D_R^2 R^2 \eta'' = (2 - 4D_R|x|^2)\eta'' \leq C''\eta'$ since $(2 - 4D_R|x|^2)$ can be chosen to be negative, for $|x|$ close to $R$, and then we can choose $C_R \geq 4D_R D'_R R$ to find that

$$(\partial_t - \frac{1}{H^2}\Delta)(\eta u) \leq -\frac{2}{H^2} g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\eta H^2} g(\nabla\alpha, \nabla(\eta u)) - \frac{n}{H^2}(\eta u)$$

where we note that $C_0^R = C''$, $C_1^R = 4D_R D'_R$ and $C_2^R = 0$.

Now if we know that $H \leq H_1$ on $U_R$ and then for $\delta \geq 0$ small define $\Phi_\delta(x,t) = c_0 e^{\frac{-n}{H_1^2}t} - \eta u + \delta t$ where $c_0 = \max_{U_{R,1,0}} \eta u > 0$ then we find on $U_R$

$$(\partial_t - \frac{1}{H^2}\Delta)\Phi_\delta \geq -\frac{2}{H^2} g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\eta H^2} g(\nabla\alpha, \nabla(\eta u)) - \frac{n}{H_1^2}\Phi_0 + \delta$$

Now let $(x_0, t_0) \in U_R$, for sake of contradiction, be such that $\Phi_\delta(x_0,t_0) = \min_{\bar{U}_R} \Phi_\delta(x,t) < 0$ and hence $\nabla\Phi_\delta(x_0,t_0) = 0$, $\Delta\Phi_\delta(x_0,t_0) \geq 0$ and $\frac{\partial\Phi_\delta}{\partial t}(x_0,t_0) = \frac{\partial\Phi_0}{\partial t}(x_0,t_0) + \delta \leq 0$

So by the evolution equation above we find that $(\partial_t - \frac{1}{H^2}\Delta)\Phi_\delta(x_0,t_0) > 0$ where we are using the fact that $\Phi_\delta(x_0,t_0) < 0 \Rightarrow \Phi_0(x_0,t_0) + \delta t_0 < 0 \Rightarrow -\Phi_0 > \delta t_0 > 0$ which shows us that $-\frac{n}{H_1^2}\Phi_0 + \delta > 0$, the non-gradient terms in the evolution of $\Phi_\delta$.

On the contrary by our deductions at $(x_0,t_0)$ we find that $(\partial_t - \frac{1}{H^2}\Delta)\Phi_\delta(x_0,t_0) \leq 0$ which is a contradiction and so $\Phi_\delta$ cannot attain a negative minimum on $U_R$. Hence by letting $\delta \to 0$ we find that $\Phi_0$ cannot attain a negative minimum on $U_R$ which implies that a negative minimum of $\Phi_0$ in $\overline{U}_R$ can only be attained on the set $\partial U_R = U_{R,1,0} \cup \{\alpha = 0\}$.

We know that the min cannot occur on $\{\alpha = 0\}$ since $\varphi = 0$ on this set and we see by the construction of $\Phi_0$ that it cannot attain a negative min on $U_{R,1,0}$ since $c_0 = \min_{U_{R,1,0}}(\eta u)$ and hence $\Phi_0(x,t) \geq 0$ which means that

$$\max_{U_{R,1,t}} \eta u \leq \left(\max_{U_{R,1,0}} \eta u\right) e^{\frac{-n}{H_1^2}t}$$

Since we know that $0 \leq \alpha \leq R^2$ on $U_R$ we know that $\eta(\alpha) \leq e^{-\frac{1}{R^2}}$ on $U_R$ and since $\alpha \geq (1-\theta)R^2$ on the set $U_{R,\theta,t}$ we know that $e^{-\frac{1}{(1-\theta)R^2}}$ and so we have

$$e^{-\frac{1}{(1-\theta)R^2}} \max_{U_{R,\theta,t}} u \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{-n}{H_1^2}t} e^{-\frac{1}{R^2}} \quad \Rightarrow \quad \max_{U_{R,\theta,t}} u \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{-n}{H_1^2}t} e^{\frac{\theta}{(1-\theta)R^2}}$$

and so we find that

$$\max_{U_{R,\theta,t}} u \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{-n}{H_1^2}t} \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{\theta}{(1-\theta)R^2}}$$

where the second inequality follows since $e^{\frac{-n}{H_1^2}t} \leq 1$ for all $t \in [0,T]$ which gives us the desired result. We also note that as $R \to \infty$ we see that $e^{\frac{\theta}{(1-\theta)R^2}} \to 1$.

□

Now we obtain a upper interior estimate on $H$ in $U_R$.

**Lemma 14.** *If we assume that* $\Sigma_0$ *is a hypersurface to which Theorem* (11) *applies on* $B_R$ *then we have the interior upper estimate for* $H$ *for* $t \in [0, T]$

$$\max_{U_{R,\theta,t}} H \leq (1-\theta)^{-1} \max\left(n, \max_{U_{R,1,0}} H\right)$$

**Note:** $C_R = 0$ in the definition of $\alpha$ for this estimate.

*Proof.* For this we will look at $\alpha H$ which has a nice evolution equation so we can choose $C_R = 0$ and find in $U_R$

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\alpha H) &= H(\partial_t - \frac{1}{H^2}\Delta)\alpha + \alpha(\partial_t - \frac{1}{H^2}\Delta)H - 2\frac{g(\nabla\alpha, \nabla H)}{H^2} \\ &\leq -2\alpha\frac{|\nabla H|^2}{H^3} - 2\frac{g(\nabla\alpha, \nabla H)}{H^2} + \left(\frac{n}{H^2} - \frac{|A|^2}{H^2}\right)(\alpha H) \\ &\leq -2\frac{g(\nabla(\alpha H), \nabla H)}{H^3} + n\left(\frac{1}{H^2} - \frac{1}{n^2}\right)(\alpha H) \\ &\leq -2\frac{g(\nabla(\alpha H), \nabla H)}{H^3} + n\alpha\left(\frac{1}{H} + \frac{1}{n}\right)\left(\frac{1}{\alpha H} - \frac{1}{\alpha n}\right)(\alpha H)\end{aligned}$$

Since we know that $H$ is bounded from below in $U_R$ we can proceed by the ODE maximum principle for $\alpha H$ in $U_R$, where we note that $\alpha \equiv 0$ on $\partial U_R \setminus U_{R,1,0}$, to obtain the following evolution inequality in $U_R$

$$\frac{d}{dt}\max_{U_{R,1,t}}(\alpha H) \leq n\alpha\left(\frac{1}{\mathcal{H}_R} + \frac{1}{n}\right)\left(\frac{1}{\max_{U_{R,1,0}}(\alpha H)} - \frac{1}{\alpha n}\right)\max_{U_{R,1,0}}(\alpha H)$$

Now we notice that if $\left(\frac{1}{\max_{U_{R,1,0}}(\alpha H)} - \frac{1}{\alpha n}\right) \geq 0 \iff \alpha n \geq \max_{U_{R,1,0}}(\alpha H) \iff H \leq n$ on $U_{R,1,0}$ and if $\left(\frac{1}{\max_{U_{R,1,0}}(\alpha H)} - \frac{1}{\alpha n}\right) < 0 \iff \alpha n < \max_{U_{R,1,0}}(\alpha H) \iff H > n$ on $U_{R,1,0}$. This tells us that $\frac{d}{dt}\max_{U_{R,1,t}}(\alpha H) \geq 0$ when $H \leq n$ and $\frac{d}{dt}\max_{U_{R,1,t}}(\alpha H) < 0$ when $H > n$ which leads to the following inequality

$$\max_{U_{R,1,t}}(\alpha H) \leq \max\left(\alpha n, \max_{U_{R,1,0}}(\alpha H)\right)$$

and then using the fact that $\alpha \geq (1-\theta)R^2$ on $U_{R,\theta,t}$ we find

$$(1-\theta)R^2 \max_{U_{R,\theta,t}} H \leq \max\left(n, \max_{U_{R,1,0}} H\right) R^2$$

which yields the desired result.

□

The last local estimate we will obtain is a second order estimate for the graph function $y$ which we will obtain through bounding $A_{ij}$. Again we will need to consider $P_i^j = w^{-1}A_i^j$ instead of $A_i^j$ directly because we need to leverage the good evolution equation for $w^{-1}$ in order to kill the bad terms in the evolution of $A_i^j$ and obtain a useful evolution equation for $P_i^j$. We start by obtaining important evolution equations and then obtain the estimate in Lemma (16).

**Lemma 15.** *If we define $P_i^j = w^{-1}A_i^j$ then we will find the following evolution equation*

$$(\partial_t - \frac{1}{H^2}\Delta)P_i^j = -\frac{2}{wH^3}\nabla_i H \nabla^j H - \frac{2w}{H^2}g(\nabla w^{-1}, \nabla P_i^j) + \frac{n}{H^2}P_i^j - \frac{2w}{H}(P^2)_i^j$$

*Now if we consider $\eta : \mathbb{R} \to \mathbb{R}$ and $\alpha$ the cutoff function from Lemma 1 or 2 then we find the following evolution equation for $\eta(\alpha)P_i^j$*

$$(\partial_t - \frac{1}{H^2}\Delta)(\eta P_i^j) = -\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\eta P_i^j)) - \frac{2}{\eta H^2}g(\nabla\eta, \nabla(\eta P_i^j))$$

$$- \frac{2\eta}{wH^3}\nabla_i H\nabla^j H + \frac{2w\eta'}{H^2}P_i^j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j}{H^2}\frac{\eta'^2}{\eta}|\nabla\alpha|^2 - \frac{\eta'' P_i^j}{H^2}|\nabla\alpha|^2$$

$$+ \frac{n}{H^2}(\eta P_i^j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j - \frac{2C_R\eta' P_i^j}{\mathcal{H}_0^R}$$

*Proof.*

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)P_i^j &= w^{-1}(\partial_t - \frac{1}{H^2}\Delta)A_i^j + A_i^j(\partial_t - \frac{1}{H^2}\Delta)w^{-1} - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j) \\ &= w^{-1}\left(-\frac{2}{H^3}\nabla_i H\nabla^j H + \frac{|A|^2}{H^2}A_i^j + \frac{n}{H^2}A_i^j - \frac{2}{H}(A^2)_i^j\right) \\ &- \frac{|A|^2}{H^2}w^{-1}A_i^j - \frac{2wA_i^j}{H^2}|\nabla w^{-1}|^2 - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j) \\ &= -\frac{2}{wH^3}\nabla_i H\nabla^j H - \frac{2wA_i^j}{H^2}|\nabla w^{-1}|^2 - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j) + \frac{n}{H^2}P_i^j - \frac{2w}{H}(P^2)_i^j\end{aligned}$$

Now we will use the fact that

$$-\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(w^{-1}A_i^j)) = -\frac{2wA_i^j}{H^2}|\nabla w^{-1}|^2 - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j)$$

to find the following

$$(\partial_t - \frac{1}{H^2}\Delta)P_i^j = -\frac{2}{wH^3}\nabla_i H\nabla^j H - \frac{2w}{H^2}g(\nabla w^{-1}, \nabla P_i^j) + \frac{n}{H^2}P_i^j - \frac{2w}{H}(P^2)_i^j$$

**Note:** We are not worried about the $\nabla_i H\nabla^j H$ term since at some point in this argument we are going to look at the maximum eigenvalue of $P_i^j$ in which case this term will be negative.

Now if we let $\alpha$ be the cutoff function from (8) so that $(\partial_t - \frac{1}{H^2}\Delta)\alpha \leq -\frac{2C_R}{\mathcal{H}_0^R}$ then we can compute the following evolution inequality for $\alpha P_i^j$

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha P_i^j) \leq \alpha(\partial_t - \frac{1}{H^2}\Delta)P_i^j + P_i^j(\partial_t - \frac{1}{H^2}\Delta)\alpha - \frac{2}{H^2}g(\nabla\alpha, \nabla P_i^j) \\
= -\frac{2\alpha}{wH^3}\nabla_i H \nabla^j H - \frac{2w\alpha}{H^2}g(\nabla w^{-1}, \nabla P_i^j) - \frac{2}{H^2}g(\nabla\alpha, \nabla P_i^j) \\
+ \frac{n\alpha}{H^2}P_i^j - \frac{2w\alpha}{H}(P^2)_i^j - \frac{2C_R P_i^j}{\mathcal{H}_0^R}
\end{aligned}
$$

Now we again compute some gradient terms

$$
\begin{aligned}
-\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\alpha P_i^j)) = -\frac{2\alpha w}{H^2}g(\nabla w^{-1}, \nabla P_i^j) - \frac{2w}{H^2}P_i^j g(\nabla w^{-1}\nabla\alpha) \\
-\frac{2}{\alpha H^2}g(\nabla\alpha, \nabla(\alpha P_i^j)) = -\frac{2}{H^2}g(\nabla\alpha, \nabla P_i^j) - \frac{2P_i^j}{\alpha H^2}|\nabla\alpha|^2
\end{aligned}
$$

from which we find

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha P_i^j) \leq -\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\alpha P_i^j)) - \frac{2}{\alpha H^2}g(\nabla\alpha, \nabla(\alpha P_i^j)) \\
- \frac{2\alpha}{wH^3}\nabla_i H \nabla^j H + \frac{2w}{H^2}P_i^j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j}{\alpha H^2}|\nabla\alpha|^2 \\
+ \frac{n}{H^2}(\alpha P_i^j) - \frac{2w}{\alpha H}(\alpha^2 P^2)_i^j - \frac{2C_R P_i^j}{\mathcal{H}_0^R}
\end{aligned}
$$

To deal with the $\alpha$ that shows up in the denominator of the $|\nabla\alpha|^2$ term we consider, as before, a function $\eta : \mathbb{R} \to \mathbb{R}$ non-decreasing and compute the following evolution for $\eta(\alpha)P_i^j$

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\eta P_i^j) = -\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\eta P_i^j)) - \frac{2}{\eta H^2}g(\nabla\eta, \nabla(\eta P_i^j)) \\
- \frac{2\eta}{wH^3}\nabla_i H \nabla^j H + \frac{2w\eta'}{H^2}P_i^j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j}{H^2}\frac{\eta'^2}{\eta}|\nabla\alpha|^2 - \frac{\eta'' P_i^j}{H^2}|\nabla\alpha|^2 \\
+ \frac{n}{H^2}(\eta P_i^j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j - \frac{2C_R \eta' P_i^j}{\mathcal{H}_0^R}
\end{aligned}
$$

□

Now we are ready to prove an estimate for $P_i^j$ which will imply an estimate for $A_i^j$.

**Lemma 16.** *If we define $P_i^j = w^{-1}A_i^j$ and we assume that $\Sigma_0$ is a hypersurface to which Theorem (11) applies on $B_R$ then we find for $t \in [0,T]$, $\theta \in (0,1)$*

$$\max_{U_{R,\theta,t}} P_i^j \leq \max\left(\max_{U_{R,1,0}} P_i^j, \frac{nc_0}{2}\right) e^{\frac{\theta}{(1-\theta)R^2}}$$

*where $\max_U P_i^j$ refers to the maximum eigenvalue of $P$ over the set $U$ and $c_0$ is a upper bound on $w^{-1}H$ in $U_R$, guaranteed by previous estimates. Specifically $w^{-1}H \leq 4\left(\max_{U_{R,1,0}} w^{-1}\right)\max\left(n, \max_{U_{R,1,0}} H\right)e^{-t/n}$.*

**Note:** $C_0^R$ depends on a bound for the cutoff function, $C_1^R$ depends on upper and lower bounds for $y$, an upper bound on $v$ and an upper bound on $|A|$ in $\Omega_{R,T}$ and $C_2^R = 0$.

*Proof.* Now we would like to better understand some terms in the equation given in Lemma (15) and estimate the bad terms

$$\frac{2w\eta'}{H^2}g(\nabla w^{-1}, \nabla\alpha) = \frac{-2w\eta'}{H^2}g(\nabla w^{-1}, \nabla\alpha) \leq \frac{2w\eta'}{H^2}|\nabla w^{-1}||\nabla\alpha|$$

Now we notice that as before in Lemma (13) we can use the fact that $\Sigma_t$ is well defined hypersurface expressed as a graph over a plane ($\mathbb{R}^n$) with bounded gradient, $v$, in $\Omega_{R,T}$ so we know that there exists some $D_R > 0$ so that $D_R^{-2}\delta \leq g \leq D_R^2\delta$, in $\Omega_{R,T}$. Hence $|\nabla\alpha| \leq D_R|\nabla^0\alpha| \leq 2D_R|x| \leq 2D_R R$. We note that $D_R$ depends on upper and lower bounds on $y$ and a upper bound on $v$ since $g_{ij} = \frac{1}{y^2}(\delta_{ij} + y_i y_j)$. We should note that Lemma (10) and Theorem (7) will give use the desired control.

Also as in Lemma (13), we know that $|\nabla w^{-1}| = \frac{|\nabla w|}{w^2} \leq D_R'$ in $\Omega_{R,T}$, which is equivalent to having a lower bound on $w$ Lemma (10) and a bound on $|A|^2$, which can be seen by choosing a vector $v$ tangent to $\Sigma_t$ and calculating

$$\nabla_v w = \nabla_v \bar{g}(\nu, \eta) = \bar{g}(\bar{\nabla}_v \nu, \eta^T) = A(v, \eta^T) \quad \Rightarrow \quad |\nabla w|^2 \leq |A|^2$$

where we have used similar calculation as to Lemma (2). Now if we choose $\eta(s) = e^{-\frac{1}{s}}$ then we can use the same argument as in Lemma (13) to bound the term $\frac{2\eta'^2}{\eta} - |x|^2\eta'' = C''\eta'$ and so we can find

$$\begin{aligned}\frac{2\eta' w}{H^2}|\nabla w^{-1}||\nabla \alpha| + \frac{|\nabla \alpha|^2}{H^2}\left(\frac{2\eta'^2}{\eta} - \eta''\right) - \frac{2C_R\eta'}{\mathcal{H}_0^R} &\leq \frac{1}{\mathcal{H}_0^R}\left(2\eta' w D_R D_R' R + 4\eta'' D_R^2 R^2 - 2C_R\eta'\right) \\ &\leq \frac{2\eta'}{\mathcal{H}_0^R}\left(D_R D_R' D_R'' R + C'' - C_R\right)\end{aligned}$$

where $D_R''$ is a upper bound on $w$ (implied by Theorem (7)). Now we can choose $C_R \geq D_R D_R' D_R'' R + C''$ in order to get rid of the bad gradient terms that come from the cutoff function and hence $C_0^R = C''$, $C_1^R = D_R D_R' D_R''$ and $C_2^R = 0$ .

Now we look to understanding the zero order terms $\frac{n}{H^2}(\eta P_i^j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j$. Now if we let $\lambda$ be the largest eigenvalue of $P_i^j$ at a point $(x,t) \in U_R$, then we find the following

$$-\frac{2w}{\eta H}(\eta\lambda)^2 + \frac{n}{H^2}(\eta\lambda) = -\frac{2w}{H^2}\lambda\left(\eta\lambda - \frac{1}{2}n\eta H w^{-1}\right) \leq -\frac{2w}{H^2}\lambda\left(\eta\lambda - \frac{1}{2}nc_0\eta\right)$$

at the point $(x,t)$ where $c_0$ is an upper bound on $Hw^{-1}$, as in the statement of the theorem. So we notice that this term is negative when $\eta\lambda > \frac{1}{2}nc_0\eta$ and hence decreasing. We will use this intuition about the zero order terms later but we just take note of it now and move on to make this argument rigorous.

Now we are ready to give the proof of the lemma. Let $\Phi_i^j = C\delta_i^j - \eta P_i^j + \delta(t-\tau)$ where $C = \max\left(C_0, \frac{nc_0}{2}\eta\right)$ and $C_0$ is the maximum eigenvalue of $\eta P_i^j$ in the set $U_{R,1,0}$ and $\tau \geq 0$ will be chosen later. The goal is to show that the minimum eigenvalue of $C\delta_i^j - \eta P_i^j$ is positive.

For sake of contradiction assume that the minimum eigenvalue over $\overline{U}_R$ of $\Phi_i^j$ is negative. Then we first consider the case where there is a point $(x_0, t_0) \in U_R$ where $\Phi_i^j$ has a zero eigenvector, call it $\beta$, for the first time with eigenvector $v \in T_{x_0}\Sigma_{t_0}$. Then we use parallel translation to extend $v$ along radial geodesics emanating from $x_0 \in \Sigma_t$ in a neighborhood of $x_0$ and then extend it to be constant in time for a short amount of time. From this construction we find the following inequalities

$$\frac{\partial v}{\partial t}|_{(x_0,t_0)} = 0 \quad \nabla v|_{(x_0,t_0)} = 0 \quad \frac{\partial \Phi(v,v)}{\partial t}|_{(x_0,t_0)} \leq 0 \quad \nabla\Phi(v,v)|_{(x_0,t_0)} = 0 \quad \Delta\Phi(v,v)|_{(x_0,t_0)} \geq 0$$

We can also compute that

$$\begin{aligned}\Delta(\Phi(v,v)) &= g^{ij}\nabla_i\left((\nabla_j\Phi)(v,v) + 2\Phi(\nabla_j v, v)\right)\\ &= g^{ij}\left((\nabla_i\nabla_j\Phi)(v,v) + 4(\nabla_j\Phi)(\nabla_i v, v) + 2\Phi(\nabla_i\nabla_j v, v) + 2\Phi(\nabla_i v, \nabla_j c)\right)\\ &= (\Delta\Phi)(v,v) + 4(\nabla\Phi)(\nabla v, v) + 2\Phi(\Delta v, v) + 2\Phi(\nabla v, \nabla v)\end{aligned}$$

$$\Delta(\Phi(v,v))|_{(x_0,t_0)} \geq 2\Phi(\Delta v, v)|_{(x_0,t_0)} = 0$$

where we used the fact that $v$ is a zero eigenvector for $\Phi$ at the point $(x_0, t_0)$ in the last equality.

Then we find the following evolution inequality at the point $(x_0, t_0)$

$$(\partial_t - \frac{1}{H^2}\Delta)(\eta\Phi_i^j v^i v_j) \geq \frac{2w}{\eta H}(\eta^2 P^2)_i^j v^i v_j - \frac{n}{H^2}(\eta P_i^j v^i v_j) + \delta \geq 0$$

where we notice the inequality since at the point $(x_0, t_0)$, if we let $\lambda = P_i^j v^i v_j$, we find

$$\frac{2w}{\eta H}(\eta^2 P^2)_i^j v^i v_j - \frac{n}{H^2}(\eta P_i^j v^i v_j) + \delta = \frac{2w}{H^2}\lambda\left(\eta\lambda - \frac{1}{2}nc_0\eta\right) + \delta > 0$$

where the strict inequality follows since $\beta = C - \lambda\eta + \delta(t_0 - \tau) = 0$ so $\lambda\eta = C + \delta(t_0 - \tau)$ and now $\tau$ can be chosen so that $t_0 - \tau < 1$ and $C$ was chosen to be larger than $\frac{1}{2}nc_0\eta$.

By our assumptions though we know that $\frac{\partial\Phi(v,v)}{\partial t}|_{(x_0,t_0)} \leq 0$ and $\Delta\Phi(v,v)|_{(x_0,t_0)} \geq 0$ and hence we find

$$(\partial_t - \frac{1}{H^2}\Delta)\Phi(v,v) \leq 0$$

which is a contradiction so if we let $\delta \to 0$ we see that $C\delta_i^j - \eta P_i^j$ cannot attain a zero eigenvector in $U_R$ which implies that $C\delta_i^j - \eta P_i^j$ cannot attain a strictly negative eigenvalue on $U_R$.

Now we know that $C\delta_i^j - \eta P_i^j$ cannot obtain a strictly negative minimum eigenvalue on $\{\alpha = 0\}$ and we see by construction that $C\delta_i^j - \eta P_i^j$ does not obtain a negative eigenvalue at time $t = 0$ since $C$ was chosen to be less than $C_0$, the minimum eigenvalue of $\eta P_i^j$ in the set $U_{R,1,0}$. So it doesn't obtain one anywhere on $U_R$ and hence $\eta P_i^j$ is bounded from above, as desired.

More specifically we have that

$$\max_{U_{R,1,t}} \eta P_i^j \leq C = \max\left(\max_{U_{R,1,0}} \eta P_i^j, \frac{nc_0}{2}\eta\right)$$

Since we know that $0 \leq \alpha \leq R^2$ on $U_R$ we know that $\eta(\alpha) \leq e^{-\frac{1}{R^2}}$ on $U_R$ and since $\alpha \geq (1-\theta)R^2$ on the set $U_{R,\theta,t}$ we know that $e^{-\frac{1}{(1-\theta)R^2}}$ and so we have

$$\max_{U_{R,\theta,t}} P_i^j \leq C = \max\left(\max_{U_{R,1,0}} P_i^j, \frac{nc_0}{2}\right) e^{\frac{\theta}{(1-\theta)R^2}}$$

which yields the desired result.

□

Now we interpret all of the previous lemmas of this chapter in order to see what they say about important geometric quantities.

**Theorem 13.** *If we assume that $\Sigma_0$ is a hypersurface so that the hypotheses of Theorem (11) hold on $B_R$ and $0 < y_0 \leq y(x,t) \leq y_1 < \infty$ then we find the following estimates for $\theta \in (0,1)$*

$$
\begin{aligned}
v(x,t) &\leq (1-\theta)^{-1}\frac{y_1}{y_0}\left(\max_{U_{R,1,0}} v\right)\\
H(x,t) &\geq \left(e^{-\frac{\theta}{(1-\theta)R^2}}\right)\frac{y_0}{y_1}\left(\min_{U_{R,1,0}} H\right)\left(\max_{U_{R,1,0}} v\right)^{-1}\\
H(x,t) &\leq (1-\theta)^{-1}\max\left(n, \max_{U_{R,1,0}} H\right)\\
A_i^j(x,t) &\leq \left(e^{\frac{\theta}{(1-\theta)R^2}}\right)\frac{y_1}{y_0}\max\left(\max_{U_{R,1,0}} A_i^j e^{t/n}, \frac{n}{2}\max\left(n, \max_{U_{\bar{R},1/2,0}} H\right)\right)\left(\max_{U_{R,1,0}} v\right)
\end{aligned}
$$

*where all of the estimates are valid on $U_{R,\theta,t}$ and the last inequality means that the largest eigenvalue of $A_i^j$ is less than the quantity on the right side.*

*Proof.* This Theorem is a culmination of the Lemmas of this chapter and hence follows from combining and unpacking Lemmas (10) (13) (16) (14) as follows.

In this case $w^{-1} = yv$ and hence (10) tells us that

$$v \leq (1-\theta)^{-1}y^{-1}\left(\max_{U_{R,1,0}} w^{-1}\right)e^{-t/n} \leq (1-\theta)^{-1}\frac{y_1}{y_0}\left(\max_{U_{R,1,0}} v\right)e^{-t/n}$$

and hence we find the first estimate.

Lemma (13) tells us that

$$\frac{w^{-1}}{\max_{U_{R,1,0}} u}e^{t/n} \leq H$$

and so if we notice that $w \leq \frac{1}{y}$ we find

$$H(x,t) \geq \frac{\min_{U_{R,\theta,t}} y}{\max_{U_{R,1,0}} u} e^{t/n} = \frac{\min_{U_{R,\theta,t}} y}{\max_{U_{R,1,0}} \frac{1}{Hw}} e^{t/n} \geq y_0 \left( \min_{U_{R,1,0}} H \right) \left( \min_{U_{R,1,0}} w \right) \geq \frac{y_0}{y_1} \left( \min_{U_{R,1,0}} H \right) \left( \max_{U_{R,1,0}} v \right)^{-1}$$

The third estimate follows directly from (14) and the last estimate follows from combining (7) with (16). $\square$

In the next Corollary to (13) we show that all the local estimates obtained in this chapter which are stated in Theorem (13) are uniformly controlled as $R \to \infty$ and hence extend to estimates on all of $\Sigma_t$.

**Corollary 1.** *If we assume that $\Sigma_0$ is a hypersurface to which the hypotheses of Theorem (11) apply then all the estimates of (13) remain true over the set $\overline{U} = \lim_{R\to\infty} U_R$ which is non-degenerate, i.e. $\mathbb{R}^n \times [0,\epsilon) \subset \overline{U}$ for some $\epsilon > 0$.*

$$\begin{aligned}
v(x,t) &\leq \frac{y_1}{y_0} \left( \max_{\Sigma_0} v \right) \\
H(x,t) &\geq \frac{y_0}{y_1} \left( \min_{\Sigma_0} H \right) \left( \max_{\Sigma_0} v \right)^{-1} \\
H(x,t) &\leq \max \left( n, \max_{\Sigma_0} H \right) \\
A_i^j &\leq \frac{y_1}{y_0} \max \left( \max_{\Sigma_0} A_i^j e^{t/n}, \frac{n}{2} \max \left( n, \max_{\Sigma_0} H \right) \right) \left( \max_{\Sigma_0} v \right)
\end{aligned}$$

*where all of the estimates are valid on $U$ where $\mathbb{R}^n \times [0,\infty) = U$. The last inequality means that the largest eigenvalue of $A_i^j$ is less than the quantity on the right side.*

*Proof.* As we attempt to extend the estimates we noted in Theorem (13) we first notice that we do not need to worry $C_0^R, C_1^R, C_2^R$ becoming unbounded as $R \to \infty$ since by Theorem (13) the quantities that $C_0^R, C_1^R, C_2^R$ depend on are uniformly controlled by the initial data on $U_R$ for each $R$ and we are assuming uniformly controlled initial data. The one issue we need to resolve is that if $C_R$ grows too quickly then it is possible for $U$ to be degenerate, i.e. $\Sigma_0 \times [0,\epsilon) \not\subset U$ for any $\epsilon > 0$, as $R \to \infty$.

We chose $C_R = C_0^R + C_1^R R + C_2^R R^2$ precisely for this reason since, as we will show below, this will guarantee that $U$ cannot be degenerate. In particular we can find the following characterization of the largest time $t$ that can occur in $U_R$

$$\begin{aligned} \alpha > 0 \quad &\Rightarrow \quad R^2 - |x|^2 - \frac{2}{\mathcal{H}_0^R}\left(ny_0^2 + 4y_0 R + C_R\right) t > 0 \\ &\Rightarrow \quad t < \frac{\mathcal{H}_0^R (R^2 - |x|^2)}{2(ny_0^2 + 4y_0 R + C_0^R + C_1^R R + C_2^R R^2)} \end{aligned}$$

which for fixed $x$ has a limit as $R \to \infty$ and tells us that $t < \infty$ since $C_2^R = 0$ and hence $U$ is non-degenerate. □

# Bibliography

Brendle, S., Hung, P., and Wang, M. (2016). A minkowski inequality for hypersurfaces in the anti-desitter-schwarschild manifold. *Comm. Pure Appl. Math.*, 69:124–144. 48

Cheng, S. and Yau, S. (1975). Differential equation on riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28:333–354. 29, 40

Chow, B. and Gulliver, R. (2001). Aleksandrov reflection and geometric evolution of hypersurfaces. *J. Math. Phys.*, 17:141–145. 4

Ding, Q. (2010). The inverse mean curvature flow in rotationally symmetric spaces. *Chinese Annals of Mathematics - Series B*, pages 1–18. 3, 45

Drugan, G., Lee, H., and Wheeler, G. (2015). Solitons for the inverse mean curvature flow. *Arxiv:1505.00183v2 [math.DG]*. 5

Ecker, K. (2004). *Regularity Theory For Mean Curvature Flow*. Birkhäuser. 38, 54, 65

Ecker, K. and Huisken, G. (1989). Mean curvature evolution of entire graphs. *Ann. Math.*, 130:453–471. 5

Ecker, K. and Huisken, G. (1991). Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.*, 105:547–569. 5, 54

Gerhardt, C. (1990). Flow of nonconvex hypersurfaces into spheres. *J. Diff. Geom.*, 32:299–314. 2, 3, 47

Gerhardt, C. (2006). *Curvature Problems*, volume 39 of *Ser. in Geom. and Topol.* International Press, Somerville, MA. 3, 7, 19, 47

Gerhardt, C. (2011). Inverse curvature flows in hyperbolic space. *J. Diff. Geom.*, 89:487–527. 3

Geroch, R. (1973). Energy extraction. *Ann. New York Acad. Sci.*, 224:108–117. 4

Hamilton, R. (1986). Four-manifolds with positive curvature operator. *J. Diff. Geom.*, 24:153–179. 6, 29

Huisken, G. (1986). Contracting convex hypersurfaces in riemannian manifolds by their mean curvature. *Invent. math.*, 84:463–480. 17

Huisken, G. and Ilmanen, T. (2001). The inverse mean curvature flow and the riemannian penrose inequality. *J. Differential Geom.*, 59(3):353–437. 4, 5

Huisken, G. and Ilmanen, T. (2008). Higher regularity of the inverse mean curvature flow. *J. Differential Geom.*, 80(3):433–451. 46

Krylov, N. (1987a). *Lectures on Elliptic and Parabolic Equations in Holder Spaces*, volume 12 of *Graduate Studies in Math.* American Mathematical Society. 22, 23, 24

Krylov, N. (1987b). *Nonlinear elliptic and parabolic equations of the second order*. Dordrecht: Reidel. 47

Mantegazza, C. (2011). *Lecture Notes on Mean Curvature Flow.* Progress in Math. Birkhäuser. 6, 29, 34

Moser, R. (2007). The invese mean curvature flow and p-harmonic functions. *J. Eur. Math. Soc.*, 9:77–83. 4

Omori, H. (1967). Isometric immersions of riemannian manifolds. *J. Math. Soc. Japan*, 19:205–214. 29, 40

Pigola, S. (2003). Maximum and comparison principles at infinity on riemannian manifolds. *PhD Thesis, Univesitá degli Studi Milano.* 29

Rasul, K. (2010). Slow convergence of graphs under mean curvature flow. *Comm. in Anal. and Geom.*, 18:987–1008. 5

Scheuer, J. (2013). The inverse mean curvature flow in warped cylinders of non-positive radial curvature. *arxiv:1312.5662.* 3

Unterberger, P. (2003). Evolution of radial graphs in hyperbolic space by their mean curvature. *Comm. in Ana. and Geom.*, 11:675–695. 5, 54

Urbas, J. (1990). On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. *Math. Z.*, 205:355–372. 2, 3, 47

Yau, S. (1975). Harmonic functions on complete riemannian manifolds. *Comm. Pure Appl. Math.*, 28:201–228. 29, 40

# Vita

Brian Allen was born in Stony Point, NY to the parents of Daniel and Cindy Allen. He is the first of three siblings: John and Jenna. He attended Stony Point Elementary School, continued on to Farley Middle School and then onto North Rockland High School where he became interested in mathematics, physics and philosophy. After graduation he headed to Pennsylvania to attend Messiah College in order to major in mathematics and minor in computer science and philosophy. During college Brian was able to take part in independent studies with passionate professors, Dr. Christopher Staecker and Dr. Gene Chase, as well as attend an REU program at RIT (Rochester Institute of Technology) , under the direction of Dr. William Basener, which all prepared him as well as encouraged him to attend graduate school. In addition, he spent his last semester of college at Temple University where he was able to expand his course offerings before attending grad school as well as attend a differential geometry class taught by Dr. David Futer, which greatly influenced his research interests in the future. After graduating from college, Brian continued his journey south by moving to Knoxville, TN in order to start graduate school, as a graduate teaching assistant, at the University of Tennessee. After passing preliminary exams in topology and PDEs and attending classes on Riemannain geometry, Ricci flow and mean curvature flow, taught by Dr. Alex Freire, Brian chose to work with Alex in the area of geometric analysis which ultimately led to the research contained in this dissertation. After graduation, Brian will be moving back to New York in order to start a three year teaching position at the United States Military Academy in West Point, NY.