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The Congruence-Based Zero-Divisor Graph

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To the Graduate Council:

I am submitting herewith a dissertation written by Elizabeth Fowler Lewis entitled "The Congruence-Based Zero-Divisor Graph." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

David F. Anderson, Major Professor

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The Congruence-Based Zero-Divisor Graph

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To my daughter Amy,

The time I spent writing this dissertation was time I would have otherwise been spending with you. My hope is that you will always be curious and possess a love of learning, and that you will understand the importance of hard work, patience, and persistence.

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Abstract

Let R be a commutative ring with nonzero identity and $\sim$ a multiplicative congruence relation on R. Then, $R/\sim$ is a semigroup with multiplication $[x][y] = [xy]$, where $[x]$ is the congruence class of an element x of R. We define the congruence-based zero-divisor graph of R to be the simple graph with vertices the nonzero zero-divisors of $R/\sim$ and with an edge between distinct vertices $[x]$ and $[y]$ if and only if $[x][y] = [0]$. Examples include the usual zero-divisor graph of R, compressed zero-divisor graph of R, and ideal-based zero-divisor graph of R. We study relationships among congruence-based zero-divisor graphs for various congruence relations on R. In particular, we study connections between ring-theoretic properties of R and graph-theoretic properties of congruence-based zero-divisor graphs for various congruence relations on R.

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Chapter 1

Introduction

1.1 Prelude

Ring theory is one of the fundamental areas of study in abstract algebra. More specifically, we will remain in the realm of commutative ring theory. For a general ring, multiplication does not have to be commutative and a multiplicative identity element does not have to exist. However, throughout this dissertation, we will let R be a commutative ring with nonzero identity. A general reference for abstract algebra is T. W. Hungerford's book [22], and a general reference for commutative ring theory is I. Kaplansky's book [23].

An important subset of R is the set of all **zero-divisors** of R, and we denote this set by $Z(R) = \{x \in R \setminus \{0\} \mid xy = 0 \text{ for some } y \in R \setminus \{0\}\} \cup \{0\}$. If R does not have any nonzero zero-divisors, we say that R is an **integral domain**. We may associate with each R the **zero-divisor graph** $\Gamma(R)$ of R. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and an edge exists between distinct vertices x and y if and only if $xy = 0$. This definition of a zero-divisor graph of a commutative ring with nonzero identity was introduced in P. S. Livingston's 1997 Master's thesis in [24] and appeared in the 1999 paper of D. F. Anderson and P. S. Livingston that followed in [8].

A **semigroup** is a set with an associative binary operation. We say that a multiplicative semigroup S has a **zero element**, 0, if for all $s \in S$, we have that $0s = s0 = 0$, and we say that S has an **identity element**, 1, if for all $s \in S$, we have that $1s = s1 = s$. Any semigroup that has an identity element is called a **monoid**. Semigroups are of particular interest to us in ring theory since every ring under the operation of multiplication is a semigroup, or more specifically, a semigroup with zero. Thus, the commutative rings with nonzero identity that we discuss throughout this dissertation may also be viewed as commutative monoids with zero under the operation of multiplication. Additional semigroup theory terminology and results can be found in the two-volume work by A. H. Clifford and G. B. Preston in [14] and [15] and in R. Gilmer's book in [21].

Let S be a commutative semigroup with zero. The set of all **zero-divisors** of S may be defined analogously to the set of all zero-divisors of R. More explicitly, $Z(S) = \{x \in S \setminus \{0\} \mid xy = 0 \text{ for some } y \in S \setminus \{0\}\} \cup \{0\}$. Furthermore, we may also associate with each S the **zero-divisor graph** $\Gamma(S)$ of S, defined analogously to the zero-divisor graph of R. The set of vertices of $\Gamma(S)$ is $Z(S)^* = Z(S) \setminus \{0\}$, and an edge exists between distinct vertices x and y if and only if $xy = 0$. This definition of a zero-divisor graph of a commutative semigroup with zero was introduced in 2002 by F. R. DeMeyer, T. McKenzie, and K. Schneider in [17]. Since every commutative ring R with nonzero identity may be viewed as a commutative semigroup with zero under the operation of multiplication, the usual zero-divisor graph $\Gamma(R)$ is a special case of the semigroup zero-divisor graph $\Gamma(S)$.

The goal of this dissertation is to provide a generalization of the zero-divisor graph of a commutative ring R with nonzero identity. We do this by imposing a multiplicative congruence relation $\sim$ on R and then studying the zero-divisor graph of the associated semigroup $R/\sim$. We introduce the concept we call the **congruence-based zero-divisor graph** of a commutative ring R with nonzero identity, and we denote this graph by $\Gamma_\sim(R)$. Our congruence-based zero-divisor graph unifies several of the graphs associated to a commutative ring, including the compressed zero-divisor graph and the ideal-based zero-divisor graph that are discussed later in this chapter. We study connections between ring-theoretic properties of R and graph-theoretic properties of $\Gamma_{\sim}(R)$ for various congruence relations $\sim$ on R. More generally, we study relationships among congruence-based zero-divisor graphs for various congruence relations on R.

1.2 Overview of Graph Theory Concepts

We need a few fundamental concepts from graph theory throughout our work. We introduce those concepts and the corresponding notation here. Additional graph theory terminology and results can be found in the book by B. Bollobás in [13].

For a graph G, we let $V(G)$ denote the set of vertices of G and $E(G)$ denote the collection of edges of G. If $V(G) = \emptyset$, then G is the **empty graph**, and we write $G = \emptyset$. In an **undirected graph**, there is no distinction between the two vertices associated with each edge so that an edge between the vertices x and y can be equivalently denoted by $x - y$ and $y - x$. Any edge of the form $x - x$ is called a **loop**, and any edge $x - y$ that occurs more than once in $E(G)$ is called a **multiple edge**. A graph G is a **weighted graph** if each edge in $E(G)$ is assigned a value and is an **unweighted graph** if no edge in $E(G)$ is assigned a value. With these definitions in mind, we can now define a **simple graph** to be an undirected, unweighted graph that has no loops or multiple edges.

In a graph G, distinct vertices x and y in $V(G)$ are **adjacent** if there exists an edge $x - y$ in $E(G)$ between x and y. If x and y are adjacent vertices, they are said to be **incident** with the edge between them. Furthermore, G is **complete** if for all distinct $x, y \in V(G)$, x and y are adjacent, and we write $G = K^n$, where $n = |V(G)|$.

The graph G' is a **subgraph** of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Additionally, if for every pair of distinct vertices in $V(G')$ that are adjacent in G we also have that they are adjacent in G', then G' is an **induced subgraph** of G. A special type of a subgraph of G is a complete subgraph of G, and we call such a subgraph a **clique**. The **clique number** of G, denoted by $cl(G)$, is defined to be $cl(G) = sup\{r \in \mathbb{Z}_+ \mid K^r$ is a subgraph of $G\}$, where $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ denotes the set of positive integers.

A **graph homomorphism** $f : G \longrightarrow G'$ is a mapping $f : V(G) \longrightarrow V(G')$ such that $x - y \in E(G)$ implies that $f(x) - f(y) \in E(G')$. Two graphs G and G' are said to be **isomorphic graphs**, and we write $G \cong G'$, if there exists a bijection $f : V(G) \longrightarrow V(G')$ such that both f and f^{-1} are graph homomorphisms. In other words, G and G' are isomorphic graphs if distinct vertices x and y of G are adjacent in G if and only if $f(x)$ and $f(y)$ are adjacent in G'.

We say that G is **connected** if G contains a path of edges between any two distinct vertices. The **distance** between distinct vertices x and y of G, denoted by $d(x, y)$, is the length of a shortest path between x and y. If no such path exists, we say that $d(x, y) = \infty$. We also say that $d(x, x) = 0$ for completeness. With this definition established, we say that the **diameter** of G, denoted by $diam(G)$, is defined to be $diam(G) = sup\{d(x, y) \mid x$ and y are (not necessarily distinct) vertices of $G\}$. Thus, $diam(G) = 0$ if and only if $|V(G)| = 1$, and if $|V(G)| \geq 2$, then G is complete if and only if $diam(G) = 1$. A **cycle** is a closed path $x_1 - x_2 - \cdots - x_n - x_1$ in G of $n \geq 3$ distinct vertices in G. If G contains a cycle, the **girth** of G, denoted by $gr(G)$, is the length of a shortest cycle in G. However, if G does not contain a cycle, we say that $gr(G) = \infty$. Thus, for any graph G, we have that $gr(G) \in \{3, 4, 5, \ldots\} \cup \{\infty\}$.

1.3 Foundations

The zero-divisor graph of a commutative ring R with nonzero identity, denoted by $\Gamma(R)$, and the zero-divisor graph of a commutative semigroup S with zero, denoted by $\Gamma(S)$, are simple graphs. We begin with some fundamental results for $\Gamma(R)$ and $\Gamma(S)$. Note that Theorem 1.1 is actually a special case of Theorem 1.2 since any ring may be viewed as a semigroup under the operation of multiplication.

Theorem 1.1. *Let R be a commutative ring with nonzero identity.*

(a) ([8, Theorem 2.3]) $\Gamma(R)$ *is connected and diam*$(\Gamma(R)) \leq 3$.

(b) ([19, Theorem 1.6], [25, (1.4)]) If $\Gamma(R)$ *contains a cycle, then gr*$(\Gamma(R)) \leq 4$.

Theorem 1.2. *Let S be a commutative semigroup with zero.*

(a) ([17, Theorem 1.2]) $\Gamma(S)$ *is connected and diam*$(\Gamma(S)) \leq 3$.

(b) ([17, Theorem 1.5]) If $\Gamma(S)$ *contains a cycle, then gr*$(\Gamma(S)) \leq 4$.

In order to discuss two generalizations of zero-divisor graphs of R that are of particular interest to us, we must also establish a few algebra concepts for completeness. An **equivalence relation** $\sim$ on a set X is a binary relation on X that satisfies the following three properties:

1. **reflexive property**: $x \sim x$ for all $x \in X$

2. **symmetric property**: $x \sim y$ implies that $y \sim x$

3. **transitive property**: $x \sim y$ and $y \sim z$ implies that $x \sim z$

For all $x \in X$, the **equivalence class** of x with respect to the equivalence relation $\sim$ on X is given by $[x]_\sim = \{y \in X \mid x \sim y\}$. Furthermore, the set of all equivalence classes determined by an equivalence relation on X forms a partition of X. A basic example of an equivalence relation on X is the familiar relation of equality that we will often denote as $=_X$ for clarity. The equivalence classes for this equivalence relation are $[x]_{=_X} = \{x\}$ for all $x \in X$. We may view an equivalence relation $\sim$ on a set X as a subset of $X \times X$ by saying that $(x, y) \in \sim$ if and only if $x \sim y$. Note that the equivalence relation $=_X$ on X is the diagonal $\{(x, x) \mid x \in X\}$ of $X \times X$. Also, $X \times X$ is the equivalence relation defined by $x \sim y$ for all $x, y \in X$.

Let $\sim_1$ and $\sim_2$ be equivalence relations on X. We have that $\sim_1 \subseteq \sim_2$ if and only if $x \sim_1 y$ implies that $x \sim_2 y$. So, we may define a partial order $\leq$ on the set of all equivalence relations on X by saying that for equivalence relations $\sim_1$ and $\sim_2$ on X, $\sim_1 \leq \sim_2$ if and only if $\sim_1 \subseteq \sim_2$. Note that a consequence of this definition is that $\sim_1 \leq \sim_2$ if and only if $[x]_{\sim_1} \subseteq [x]_{\sim_2}$ for all $x \in X$. Furthermore, since each equivalence relation $\sim$ on X must satisfy the reflexive property, we have that $=_X \leq \sim \leq X \times X$.

Let R be a commutative ring with nonzero identity. For each $x \in R$, the **annihilator** of x is defined to be $ann_R(x) = \{r \in R \mid rx = 0\}$, and it is a ring ideal of R. Now, define the relation $\sim$ on R by $x \sim y$ if and only if $ann_R(x) = ann_R(y)$, and note that $\sim$ is an equivalence relation on R. With this definition of $\sim$, we have that $[0]_\sim = \{0\}$ and $[1]_\sim = R \setminus Z(R)$. Thus, for each $x \in R \setminus ([0]_\sim \cup [1]_\sim)$, we have that $[x]_\sim \subseteq Z(R)^*$.

For each commutative ring R with nonzero identity, the **compressed zero-divisor graph** $\Gamma_E(R)$ of R is defined by utilizing the equivalence relation $\sim$ on R given by $x \sim y$ if and only if $ann_R(x) = ann_R(y)$. The set of vertices of $\Gamma_E(R)$ is $V(\Gamma_E(R)) = \{[x]_\sim \mid x \in R \setminus ([0]_\sim \cup [1]_\sim)\}$, and two distinct vertices $[x]_\sim$ and $[y]_\sim$ are connected by an edge in $\Gamma_E(R)$ if and only if $xy = 0$. The concept of the compressed zero-divisor graph was introduced in 2002 by S. B. Mulay in [25].

The equivalence relation $\sim$ on R given by $x \sim y$ if and only if $ann_R(x) = ann_R(y)$ is more specifically a congruence relation on R. This will be further discussed in Chapter 2. It turns out that $\Gamma_E(R) = \Gamma(S)$ for the semigroup $S = R/\sim$, where the multiplication on $S = R/\sim$ is well-defined due to $\sim$ being a congruence relation. Thus, by Theorem 1.2, we may automatically conclude that $\Gamma_E(R)$ is connected and $diam(\Gamma_E(R)) \leq 3$ ([28, Proposition 1.4]). Furthermore, we also have that if $\Gamma_E(R)$ contains a cycle, then $gr(\Gamma_E(R)) \leq 4$. However, D. F. Anderson and J. D. LaGrange recently improved upon this result in [4, Theorem 3.1] where they showed that $gr(\Gamma_E(R)) \in \{3, \infty\}$.

Let R be a commutative ring with nonzero identity, and let I be a ring ideal of R. The **ideal-based zero-divisor graph** $\Gamma_I(R)$ of R has as its set of vertices $V(\Gamma_I(R)) = \{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, with distinct vertices x and

y connected by an edge if and only if $xy \in I$. This definition and initial results surrounding ideal-based zero-divisor graphs were introduced by S. P. Redmond in his 2001 doctoral dissertation in [26] and in his paper that followed in 2003 in [27].

As in the case of the compressed zero-divisor graph, the ideal-based zero-divisor graph may also be viewed as a zero-divisor graph of a semigroup. Thus, $\Gamma(S)$ is a unifying concept for $\Gamma(R)$, $\Gamma_E(R)$, and $\Gamma_I(R)$. In order to describe $\Gamma_I(R)$ as a zero-divisor graph of a semigroup, we must first provide some additional background information from semigroup theory.

Let S be a commutative semigroup under the operation of multiplication, and let J be a nonempty subset of S. We say that J is a **semigroup ideal** of S if for all $x \in S$ and $y \in J$, we have that $xy \in J$. We have mentioned before that any ring R may be viewed as a semigroup under the operation of multiplication. So, it is worth noting that any ring ideal I of a ring R is also a semigroup ideal of R when we view R as a multiplicative semigroup. However, there exist semigroup ideals that are not ring ideals. In fact, for a commutative ring R with nonzero identity, $Z(R)$ is always a semigroup ideal of R, but $Z(R)$ need not be a ring ideal of R, in general. For example, in the ring $R = \mathbb{Z}/6\mathbb{Z}$, we have that $J = Z(R) = \{\bar{0}, \bar{2}, \bar{3}, \bar{4}\}$ is a semigroup ideal of R that is not a ring ideal of R since $\bar{2}, \bar{3} \in J$, but $\bar{2} + \bar{3} = \bar{5} \notin J$. Additional examples of semigroup ideals of a ring that are not ring ideals can be found in Example 2.12.

Now, for every commutative semigroup S and every semigroup ideal J of S, we may construct the **Rees semigroup**, commonly denoted by S/J. The construction of S/J is based on the congruence relation $\sim$ on S given by $x \sim y$ if and only if $x = y$ or $x, y \in J$. So, the Rees semigroup consists of the elements $x \in S \setminus J$ and a new element J that acts as the zero element of S/J. Conceptually, all of the elements in the semigroup ideal J of S collapse to the single zero element J of the Rees semigroup. Multiplication in S/J is defined as follows. For any $x \in S/J$, we have that $xJ = Jx = J$, thus demonstrating that the element J of the Rees semigroup S/J is its zero element. Now, for any $x, y \in S \setminus J$, the product $xy = J$ if and only if $xy \in J$. Otherwise, the product xy is simply the element $xy \in S \setminus J$.

With these semigroup theory concepts established, we may now make the claim that $\Gamma_I(R) = \Gamma(S)$, where S is the Rees semigroup R/I. It is important to note that the Rees semigroup R/I should not be confused with the quotient ring that is also denoted by R/I. Since $\Gamma_I(R)$ may be viewed as a zero-divisor graph of a commutative semigroup with zero, we may again automatically deduce some initial results. We write these results in the following theorem for future reference.

Theorem 1.3. *Let R be a commutative ring with nonzero identity, and let I be an ideal of R.*

(a) ([27, Theorem 2.4]) $\Gamma_I(R)$ is connected and diam$(\Gamma_I(R)) \leq 3$.

(b) If $\Gamma_I(R)$ contains a cycle, then gr$(\Gamma_I(R)) \leq 4$.

1.4 History

Zero-divisor graphs were first introduced by I. Beck in 1988 in [11]. In his definition of a zero-divisor graph, every element of R is a vertex, and distinct vertices x and y are adjacent if and only if $xy = 0$. With this definition, the zero element of R is adjacent to every other vertex. Beck's focus was primarily on colorings of zero-divisor graphs, and D. D. Anderson and M. Naseer furthered his work in 1993 in [1].

The zero-devisor graph was redefined in the 1997 Master's thesis of P. S. Livingston in [24] and in the 1999 paper of D. F. Anderson and P. S. Livingston that followed in [8]. They restricted the vertices to the set of nonzero zero-divisors of R, making their zero-divisor graph $\Gamma(R)$ an induced subgraph of Beck's zero-divisor graph. Furthermore, their focus was the investigation of the relationship between ring-theoretic properties of the commutative ring R and graph-theoretic properties of the zero-divisor graph $\Gamma(R)$. D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston furthered the study of the interplay between R and $\Gamma(R)$ in 2001 in [3]. Then, the work of S. B. Mulay in [25] and F. R. DeMeyer and K. Schneider in [19] provided additional depth to the study of $\Gamma(R)$ in 2002. In 2003, D. F. Anderson, R. Levy, and J. Shapiro restricted their focus to a particular type of ring R and investigated the corresponding zero-divisor graph $\Gamma(R)$ in [7].

Two distinct generalizations of zero-divisor graphs of rings originated in the early 2000s. S. B. Mulay's 2002 paper provided the framework and initial results for the compressed zero-divisor graph $\Gamma_E(R)$ in [25]; however, he used an alternative notation. In 2011, S. Spiroff and C. Wickham introduced the $\Gamma_E(R)$ notation for what they called the graph of equivalence classes of zero-divisors of a ring R in [28]. Their work included many fundamental results and a comparison of $\Gamma_E(R)$ to the standard zero-divisor graph $\Gamma(R)$. Recent work by D. F. Anderson and J. D. LaGrange in [5] and [4] furthered the study of $\Gamma_E(R)$ and introduced the terminology we use, the compressed zero-divisor graph. A second generalization, the ideal-based zero-divisor graph $\Gamma_I(R)$, was introduced by S. P. Redmond in his 2001 doctoral dissertation in [26] and in his paper that followed in 2003 in [27]. Additional results for $\Gamma_I(R)$ were provided in 2014 by D. F. Anderson and S. Shirinkam in [9].

In 2002, F. R. DeMeyer, T. McKenzie, and K. Schneider introduced what turns out to be a unifying concept, the zero-divisor graph of a semigroup, denoted by $\Gamma(S)$, in [17]. The study of $\Gamma(S)$ was continued by F. R. DeMeyer and L. DeMeyer in 2005 in [18], and by L. DeMeyer, L. Greve, A. Sabbaghi, and J. Wang in 2010 in [20].

There have also been many other ways of associating graphs to rings. For example, A. Badawi introduced the **annihilator graph** $AG(R)$ of a commutative ring R with nonzero identity in 2014 in [10]. The set of vertices of $AG(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and an edge exists between distinct vertices x and y if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. Note that $V(\Gamma(R)) = V(AG(R))$; however, it turns out that $E(\Gamma(R)) \subseteq E(AG(R))$, where equality does not hold, in general. To demonstrate this, consider the ring $R = \mathbb{Z}/8\mathbb{Z}$. We have that $2 \cdot 6 = 4 \neq 0$, implying that $2 - 6 \notin E(\Gamma(R))$. But, $ann_R(2) \cup ann_R(6) = \{0, 4\} \cup \{0, 4\} = \{0, 4\} \neq \{0, 2, 4, 6\} = ann_R(4) = ann_R(2 \cdot 6)$, implying that $2 - 6 \in E(AG(R))$. Thus, $AG(\mathbb{Z}/8\mathbb{Z}) \neq \Gamma(\mathbb{Z}/8\mathbb{Z})$ ([10, Example 2.7]). In fact, $AG(\mathbb{Z}/8\mathbb{Z}) \neq \Gamma_\sim(\mathbb{Z}/8\mathbb{Z})$ for any multiplicative congruence relation $\sim$ on R.

Over 200 papers associating graphs to rings have been published. Additional results surrounding zero-divisor graphs as well as an extensive bibliography can be found in the 2011 survey article by D. F. Anderson, M. C. Axtell, and J. A. Stickles in [2] and in the 2012 survey article by J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, and S. Spiroff in [16].

1.5 Dissertation Organization

In Chapter 2, we define a multiplicative congruence relation $\sim$ on a commutative ring R with nonzero identity and justify that $R/{\sim}$ is a commutative semigroup with zero. Then, we use this semigroup to define the congruence-based zero-divisor graph $\Gamma_\sim(R) = \Gamma(R/{\sim})$ and state a couple of fundamental properties that can be deduced immediately from this definition. Furthermore, we show that the usual zero-divisor graph of R, the compressed zero-divisor graph of R, and the ideal-based zero-divisor graph of R are examples of congruence-based zero-divisor graphs of R. We conclude by investigating the congruence class of $[0]_\sim$ and particular types of semigroup ideals of R. Many of the examples that we develop throughout Chapter 2 will be utilized in later chapters, as well.

In Chapter 3, we establish the notation $\mathcal{C}(R)$ for the set of all multiplicative congruence relations on R, and we define a partial order on the set. We begin by focusing on the elements $\sim \in \mathcal{C}(R)$ such that $[0]_\sim$ is a fixed semigroup ideal of R, and then we investigate the case when $[0]_\sim$ is, more specifically, a fixed ring ideal of R. We also develop the relationship between certain pairs of congruence relations that were introduced in Chapter 2, and we use these relationships to provide bounds on the number of multiplicative congruence relations on R as well as provide the necessary foundation for our work in Chapter 4 and Chapter 5.

In Chapter 4, we consider $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $[0]_{\sim_1} = [0]_{\sim_2}$ and $\sim_1 \leq \sim_2$. We show how natural maps between the semigroups $R/{\sim_1}$ and $R/{\sim_2}$ induce maps between the congruence-based zero-divisor graphs $\Gamma_{\sim_1}(R)$ and $\Gamma_{\sim_2}(R)$. We also show that $\Gamma_{\sim_2}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim_1}(R)$. Furthermore, with the relationships between pairs of familiar congruence relations that were established in Chapter 3, we deduce corresponding relationships between pairs of familiar congruence-based zero-divisor graphs.

In Chapter 5, we extend our work in Chapter 4 by considering commutative rings R and T such that $R \subseteq T$ and R and T have the same nonzero identity. We work with compatible congruence relations $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$, and we show when natural maps between the semigroups $R/{\sim_R}$ and $T/{\sim_T}$ induce maps between the congruence-based zero-divisor graphs $\Gamma_{\sim_R}(R)$ and $\Gamma_{\sim_T}(T)$. Furthermore, we show that if $\sim_R = \sim_T \cap (R \times R)$, then $\Gamma_{\sim_R}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim_T}(T)$. While not all of our familiar congruence relations behave nicely, we are able to make some conclusions regarding familiar congruence-based zero-divisor graphs here, as well.

In Chapter 6, we restrict our focus to a particular congruence-based zero-divisor graph, the ideal-based zero-divisor graph. We start by considering rings formed as the direct product of two rings. Then, we conclude by working with rings formed as an idealization of a module.

In Chapter 7, we conclude by providing a summary of our work as well as open questions related to our work.

Chapter 2

Congruence Relations and Congruence-Based Zero-Divisor Graphs

In this chapter, we define a multiplicative congruence relation $\sim$ on a commutative ring R with nonzero identity and develop the necessary concepts to define our congruence-based zero-divisor graph. We provide several examples of multiplicative congruence relations and describe the congruence-based zero-divisor graphs for the congruence relations that produce familiar zero-divisor graphs. We also discuss the zero element $[0]_\sim$ of the multiplicative semigroup $R/\sim$ for each of these examples of congruence relations. Furthermore, we investigate what can be said about $[0]_\sim$ in general, and we utilize special types of semigroup ideals in this discussion.

2.1 Multiplicative Congruence Relations $\sim$ on a Ring R and the Associated Semigroups $R/\sim$

Let R be a commutative ring with nonzero identity. A **multiplicative congruence relation** $\sim$ on R is an equivalence relation on the multiplicative monoid R with the additional property that if $x, y, z, w \in R$ with $x \sim y$ and $z \sim w$, then $xz \sim yw$.

Remark 2.1. *Let R be a commutative ring with nonzero identity, and let $\sim$ be an equivalence relation on R. Then, the following two properties are equivalent:*

(1) If $x, y, z, w \in R$ with $x \sim y$ and $z \sim w$, then $xz \sim yw$.

(2) If $x, y \in R$ with $x \sim y$, then $zx \sim zy$ for all $z \in R$.

Proof. To show that (1) implies (2), let $x, y \in R$ with $x \sim y$. Since $\sim$ satisfies the reflexive property, we have that $z \sim z$ for all $z \in R$. Thus, $zx \sim zy$ for all $z \in R$. Conversely, to show that (2) implies (1), let $x, y, z, w \in R$ with $x \sim y$ and $z \sim w$. Now, $z \sim w$ implies that $xz \sim xw$ and $x \sim y$ implies that $wx \sim wy$. But, since R is commutative, we have that $xw \sim yw$. Then, $xz \sim yw$ by the transitive property of $\sim$. □

Before building upon this concept, we provide several examples of multiplicative congruence relations on a commutative ring R with nonzero identity. These examples will recur throughout this chapter, and many of them will be utilized in later chapters, as well.

Example 2.2. *Let R be a commutative ring with nonzero identity.*

1. *$=_R$ is clearly a multiplicative congruence relation on R.*

2. *$R \times R$ is clearly a multiplicative congruence relation on R.*

3. *Let I be a ring ideal of R, and define $\sim$ by $x \sim y$ if and only if $x - y \in I$. To show that $\sim$ is a multiplicative congruence relation on R, suppose that $a, b, c, d \in R$.*

Since $a - a = 0 \in I$, $a \sim a$. Suppose that $a \sim b$. Then, $a - b \in I$, implying that $b - a = -(a - b) \in I$ so that $b \sim a$, as well. Now, suppose that $a \sim b$ and $b \sim c$. Then, $a - b, b - c \in I$, implying that $a - c = (a - b) + (b - c) \in I$ so that $a \sim c$. Finally, suppose that $a \sim b$ and $c \sim d$. Then, $a - b, c - d \in I$. So, $ac - bd = ac - bc + bc - bd = (a - b)c + b(c - d) \in I$, implying that $ac \sim bd$, as desired.

4. *Let J be a semigroup ideal of R, and define $\sim$ by $x \sim y$ if and only if either $x = y$ or $x, y \in J$. Then, $\sim$ is clearly a multiplicative congruence relation on R. In fact, this is the congruence relation mentioned in the construction of the Rees semigroup in Chapter 1.*

5. *Define $\sim$ by $x \sim y$ if and only if $ann_R(x) = ann_R(y)$. Clearly, $\sim$ is an equivalence relation on R. To show that $\sim$ is also a multiplicative congruence relation on R, suppose that $a, b, c, d \in R$ with $a \sim b$ and $c \sim d$. Then, we have that $ann_R(a) = ann_R(b)$ and $ann_R(c) = ann_R(d)$, and we must show that $ann_R(ac) = ann_R(bd)$. Suppose that $x \in ann_R(ac)$. Then, $0 = x(ac) = (xa)c$, implying that $xa \in ann_R(c) = ann_R(d)$. Thus, $0 = (xa)d = (xd)a$, and we have that $xd \in ann_R(a) = ann_R(b)$. Now, $0 = (xd)b = x(bd)$, and $x \in ann_R(bd)$. So, $ann_R(ac) \subseteq ann_R(bd)$. The reverse inclusion can be shown in a similar manner, and we may conclude that $ac \sim bd$, as desired.*

6. *Let J be a semigroup ideal of R, and let $(J :_R x) = \{r \in R \mid rx \in J\}$ be the* ***conductor*** *semigroup ideal of R with respect to J. Define $\sim$ by $x \sim y$ if and only if $(J :_R x) = (J :_R y)$. Clearly, $\sim$ is an equivalence relation on R. To show that $\sim$ is also a multiplicative congruence relation on R, suppose that $a, b, c, d \in R$ with $a \sim b$ and $c \sim d$. Then, we have that $(J :_R a) = (J :_R b)$ and $(J :_R c) = (J :_R d)$, and we must show that $(J :_R ac) = (J :_R bd)$. Suppose that $x \in (J :_R ac)$. Then, $(xa)c = x(ac) \in J$, implying that $xa \in (J :_R c) = (J :_R d)$. Thus, $(xd)a = (xa)d \in J$, and we have that $xd \in (J :_R a) = (J :_R b)$. Now, $x(bd) = (xd)b \in J$, and $x \in (J :_R bd)$. So, $(J :_R ac) \subseteq (J :_R bd)$. The reverse*

inclusion can be shown in a similar manner, and we may conclude that that $ac \sim bd$, as desired.

Now, note that $(\{0\} :_R x) = \{r \in R \mid rx \in \{0\}\} = \{r \in R \mid rx = 0\} = ann_R(x)$, and $(R :_R x) = \{r \in R \mid rx \in R\} = R$. So, if $J = \{0\}$, we are reduced to the congruence relation found in Example 2.2.5, and if $J = R$, we are reduced to the congruence relation found in Example 2.2.2.

7. *Let $U(R) = \{u \in R \mid uv = 1$ for some $v \in R\}$ be the set of* ***units*** *of R, and note that $U(R)$ is a multiplicative group. Define $\sim$ by $x \sim y$ if and only if $x = uy$ for some $u \in U(R)$. To show that $\sim$ is a multiplicative congruence relation on R, suppose that $a, b, c, d \in R$. Since $a = 1a$ with $1 \in U(R)$, we have that $a \sim a$. Suppose that $a \sim b$. Then, $a = ub$ for some $u \in U(R)$. Since $U(R)$ is a multiplicative group, $u^{-1} \in U(R)$. Thus, since $b = u^{-1}a$, we have that $b \sim a$, as well. Now, suppose that $a \sim b$ and $b \sim c$. Then, $a = ub$ and $b = vc$ for some $u, v \in U(R)$. Since $uv \in U(R)$ and $a = ub = u(vc) = (uv)c$, we have that $a \sim c$. Finally, suppose that $a \sim b$ and $c \sim d$. Then, $a = ub$ and $c = vd$ for some $u, v \in U(R)$. Since $uv \in U(R)$ and $ac = (ub)(vd) = (uv)(bd)$, we have that $ac \sim bd$, as desired.*

   More generally, let G be a multiplicative subgroup of $U(R)$, and define $\sim_G$ by $x \sim_G y$ if and only if $x = uy$ for some $u \in G$. Following the same line of reasoning as above, $\sim_G$ is a multiplicative congruence relation on R. Furthermore, for $G = U(R)$, we have that $\sim_G = \sim$, where $\sim$ is defined above, and for $G = \{1\}$, we have that $\sim_G$ is the relation $=_R$ on R.

8. *Let $(x) = \{rx \mid r \in R\}$ be the* ***principal ring ideal*** *of R generated by the element $x \in R$. Define $\sim$ by $x \sim y$ if and only if $(x) = (y)$. Clearly, $\sim$ is an equivalence relation on R. To show that $\sim$ is also a multiplicative congruence relation on R, suppose that $a, b, c, d \in R$ with $a \sim b$ and $c \sim d$. Then, $(a) = (b)$ and $(c) = (d)$, and we must show that $(ac) = (bd)$. Suppose that $x \in (ac)$. Then, there exists a $y \in R$ such that $x = yac$. Furthermore, $ac \in (ac) = (bd)$, so there*

exists a $z \in R$ such that $ac = zbd$. Thus, $x = yac = yzbd \in (bd)$ since $yz \in R$. So, $(ac) \subseteq (bd)$. The reverse inclusion can be shown in a similar manner, and we may conclude that $ac \sim bd$, as desired.

Now, note that if R is an integral domain, $(x) = (y)$ if and only if $x = uy$ for some $u \in U(R)$. So, we are reduced to the congruence relation found in Example 2.2.7 in this case.

9. *Define $\sim$ by $x \sim y$ if and only if $rx = sy$ for some $r, s \in R \setminus Z(R)$. To show that $\sim$ is a multiplicative congruence relation on R, suppose that $a, b, c, d \in R$. Since $1a = 1a$ with $1 \in R \setminus Z(R)$, we have that $a \sim a$. Suppose that $a \sim b$. Then, $ra = sb$ for some $r, s \in R \setminus Z(R)$, implying that $sb = ra$ with $s, r \in R \setminus Z(R)$ so that $b \sim a$. Now, suppose that $a \sim b$ and $b \sim c$. Then, $ra = sb$ and $r'b = s'c$ for some $r, s, r', s' \in R \setminus Z(R)$. We have that $(rr')a = r'(ra) = r'(sb) = s(r'b) = s(s'c) = (ss')c$, but it remains to show that $rr', ss' \in R \setminus Z(R)$. Suppose that $rr' \in Z(R)$. First note that $rr' \neq 0$ since $r, r' \in R \setminus Z(R)$. Thus, there exists a $t \in Z(R)$ with $t \neq 0$ such that $(rr')t = 0$. But then, $0 = (rr')t = r(r't)$, implying that $r't = 0$ since $r \in R \setminus Z(R)$. However, that would imply that $t = 0$ since $r' \in R \setminus Z(R)$, and that is a contradiction. Thus, $rr' \in R \setminus Z(R)$, and a similar argument would show that $ss' \in R \setminus Z(R)$, as well, so that $a \sim c$. Finally, suppose that $a \sim b$ and $c \sim d$. Then, $ra = sb$ and $r'c = s'd$ for some $r, s, r', s' \in R \setminus Z(R)$. We have that $(rr')(ac) = (ra)(r'c) = (sb)(s'd) = (ss')(bd)$, and $rr', ss' \in R \setminus Z(R)$ by the argument above. Thus, $ac \sim bd$, as desired.*

   Now, note that if R is an integral domain, $Z(R) = \{0\}$. So, in this case, we would have that $x \sim y$ if and only if $rx = sy$ for some $r, s \in R \setminus \{0\}$. But, this happens if and only if $x = y = 0$ or $x, y \in R \setminus \{0\}$.

For any $x \in R$ and any multiplicative congruence relation $\sim$ on R, the set of elements that are congruent to x is called the **congruence class** of x and is denoted by $[x]_\sim = \{y \in R \mid x \sim y\}$. Consider the set $R/\sim = \{[x]_\sim \mid x \in R\}$ of all congruence classes with multiplication given by $[x]_\sim [y]_\sim = [xy]_\sim$ for all $x, y \in R$.

This multiplication is well-defined, for if $x \sim x'$ and $y \sim y'$, then $xy \sim x'y'$ since $\sim$ is a multiplicative congruence relation on R. Furthermore, $([x]_\sim[y]_\sim)[z]_\sim = [xy]_\sim[z]_\sim = [(xy)z]_\sim = [x(yz)]_\sim = [x]_\sim[yz]_\sim = [x]_\sim([y]_\sim[z]_\sim)$, so the multiplication is associative. Thus, $R/{\sim} = \{[x]_\sim \mid x \in R\}$ is a semigroup under the operation given by $[x]_\sim[y]_\sim = [xy]_\sim$ for all $x, y \in R$. Additionally, $[x]_\sim[y]_\sim = [xy]_\sim = [yx]_\sim = [y]_\sim[x]_\sim$ for all $x, y \in R$, $[1]_\sim[x]_\sim = [1x]_\sim = [x]_\sim$ for all $x \in R$, and $[0]_\sim[x]_\sim = [0x]_\sim = [0]_\sim$ for all $x \in R$. Thus, $R/{\sim}$ is a commutative monoid with identity $[1]_\sim$ and zero $[0]_\sim$.

Now, $R/{\sim} = \{[0]_\sim\}$ if and only if $[0]_\sim = [1]_\sim$. To see this, first note that if $R/{\sim} = \{[0]_\sim\}$, we clearly have that $[0]_\sim = [1]_\sim$. Conversely, if $[0]_\sim = [1]_\sim$, then $[x]_\sim = [1x]_\sim = [1]_\sim[x]_\sim = [0]_\sim[x]_\sim = [0x]_\sim = [0]_\sim$ for all $x \in R$. However, this occurs if and only if $\sim = R \times R$. Thus, for any other congruence relation $\sim$, we have $[0]_\sim \neq [1]_\sim$. So, $[1]_\sim$ would be the nonzero identity element of $R/{\sim}$.

2.2 Congruence-Based Zero-Divisor Graph $\Gamma_\sim(R)$

Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. Since $R/{\sim}$ is a commutative monoid with zero, we may define the **congruence-based zero-divisor graph** $\Gamma_\sim(R)$ of R as $\Gamma_\sim(R) = \Gamma(R/{\sim})$. More explicitly, the set of vertices of $\Gamma_\sim(R)$ is $Z(R/{\sim})^* = Z(R/{\sim}) \setminus \{[0]_\sim\}$, and the set of edges is defined such that for $[x]_\sim, [y]_\sim \in Z(R/{\sim})^*$ with $[x]_\sim \neq [y]_\sim$, there is an edge between $[x]_\sim$ and $[y]_\sim$ if and only if $[x]_\sim[y]_\sim = [0]_\sim$. Thus, there is an edge between distinct vertices $[x]_\sim$ and $[y]_\sim$ if and only if $xy \sim 0$. The definition of $\Gamma_\sim(R)$ as a zero-divisor graph of a commutative semigroup with zero allows us to deduce some preliminary results.

Theorem 2.3. *Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R.*

(a) $\Gamma_\sim(R)$ is connected and diam$(\Gamma_\sim(R)) \leq 3$.

(b) If $\Gamma_\sim(R)$ contains a cycle, then gr$(\Gamma_\sim(R)) \leq 4$.

Proof. Since $\Gamma_\sim(R) = \Gamma(R/\sim)$, this result follows from Theorem 1.2 and from [17, Theorems 1.2 and 1.5] with $S = R/\sim$. □

Several of the multiplicative congruence relations on a commutative ring R with nonzero identity provided in Example 2.2 lead to the familiar congruence-based zero-divisor graphs shown here.

Example 2.4. *Let R be a commutative ring with nonzero identity.*

1. $\Gamma_{=_R}(R) = \Gamma(R)$. *Actually, we technically have that $\Gamma_{=_R}(R) \cong \Gamma(R)$ since we are associating the congruence class $[x]_{=_R} = \{x\}$ with the element $x \in R$; however, we will write $=$ whenever this is the case.*

2. $\Gamma_{R\times R}(R) = \emptyset$.

3. *Let I be a ring ideal of R, and define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if $x - y \in I$. In the case where I is a proper ring ideal of R, $\Gamma_\sim(R) = \Gamma(R/I)$, where R/I denotes the quotient ring. To justify this equality, note that $[x]_\sim = \{y \in R \mid x - y \in I\} = \{y \in R \mid x + I = y + I\} = x + I$.*

4. *Let J be a semigroup ideal of R, and define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if either $x = y$ or $x, y \in J$. Then, $\Gamma_\sim(R) = \Gamma_J(R)$, where $\Gamma_J(R)$ is the natural extension of the historical ideal-based zero-divisor graph $\Gamma_I(R)$ with I more specifically a ring ideal of R. Again, technically we have that $\Gamma_\sim(R) \cong \Gamma_J(R)$ since each vertex of $\Gamma_\sim(R)$ is of the form $[x]_\sim = \{x\}$ with $x \notin J$, and we associate this congruence class with the element $x \in R \setminus J$.*

5. *For the multiplicative congruence relation $\sim$ on R defined by $x \sim y$ if and only if $ann_R(x) = ann_R(y)$, we have that $\Gamma_\sim(R) = \Gamma_E(R)$.*

6. *Let J be a semigroup ideal of R, and define the multiplicative congruence relation $\sim$ by $x \sim y$ if and only if $(J :_R x) = (J :_R y)$. By Example 2.2.6, we may conclude that if $J = \{0\}$, we have $\Gamma_\sim(R) = \Gamma_E(R)$, and if $J = R$, we have*

$\Gamma_\sim(R) = \emptyset$. Now, in the case where $J = I$ is a proper ring ideal of R, we have that $\Gamma_\sim(R) \cong \Gamma_E(R/I)$, where R/I denotes the quotient ring. This case is discussed in detail in Chapter 4.

For a commutative ring R with nonzero identity, Example 2.4 demonstrates that the congruence-based zero-divisor graph $\Gamma_\sim(R)$ generalizes the zero-divisor graph $\Gamma(R)$, the ideal-based zero-divisor graph $\Gamma_I(R)$, and the condensed zero-divisor graph $\Gamma_E(R)$. However, given a commutative ring R with nonzero identity and a commutative semigroup S_R with zero, $\Gamma(S_R)$ need not have the form $\Gamma_\sim(R)$ for any multiplicative congruence relation $\sim$ on R. The following example justifies that claim.

Example 2.5. *M. Behboodi and Z. Rakeei introduced the annihilating-ideal graph of a commutative ring R, denoted by $\mathbb{AG}(R)$, in [12]. $\mathbb{AG}(R)$ has as its set of vertices the nonzero ideals I of R such that $IJ = (0)$ for some nonzero ideal J of R, and $I - J$ is an edge in $\mathbb{AG}(R)$ if and only if $IJ = (0)$. Now, $\mathbb{AG}(R) = \Gamma(S_R)$ for the semigroup S_R that consists of the ring ideals of R under the operation of multiplication. Note that S_R is not a ring, in general.*

Let $R = \mathbb{Z}/2\mathbb{Z}[X,Y]/(X^2, XY, Y^2) = \mathbb{Z}/2\mathbb{Z}[x,y]$. Then, we have $V(\mathbb{AG}(R)) = \{(x),(y),(x+y),(x,y)\}$, and $\Gamma(S_R) = \mathbb{AG}(R) \cong K^4$. But, since $|\,V(\Gamma_\sim(R))\,| = |\,Z(R/{\sim})^*\,| \leq 3$ for each multiplicative congruence relation $\sim$ on R, we may conclude that $\Gamma(S_R) \ncong \Gamma_\sim(R)$ for any multiplicative congruence relation $\sim$ on R.

2.3 The Congruence Class $[0]_\sim$ and Semigroup Ideals of a Ring

Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. It turns out that the zero element $[0]_\sim$ of $R/{\sim}$ is a particularly important subset of the multiplicative monoid R. Theorem 2.6 tells us that the semigroup ideals of R are precisely the congruence classes $[0]_\sim$ for the multiplicative congruence relations $\sim$ on R.

Theorem 2.6. *Let R be a commutative ring with nonzero identity, and let J be a nonempty subset of R. Then, J is a semigroup ideal of R if and only if $J = [0]_\sim$ for some multiplicative congruence relation $\sim$ on R.*

Proof. Suppose that J is a semigroup ideal of R. Define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if either $x = y$ or $x, y \in J$. Then, $[0]_\sim = J$. Conversely, suppose that $J = [0]_\sim$ for some multiplicative congruence relation $\sim$ on R. To show that J is semigroup ideal of R, let $x \in R$ and $y \in [0]_\sim$. Since $y \sim 0$ and $\sim$ is a multiplicative congruence relation on R, we have that $xy \sim x0 = 0$. Thus, $xy \in [0]_\sim$, as desired. □

Now, the following example provides the description of $[0]_\sim$ for each $\sim$ introduced in Example 2.2.

Example 2.7. *Let R be a commutative ring with nonzero identity.*

1. $[0]_{=_R} = \{0\}$.

2. $[0]_{R \times R} = R$.

3. *Let I be a ring ideal of R, and define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if $x - y \in I$. Then, $[0]_\sim = \{r \in R \mid 0 - r \in I\} = \{r \in R \mid r \in I\} = I$.*

4. *Let J be a semigroup ideal of R, and define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if either $x = y$ or $x, y \in J$. Then, we have that $[0]_\sim = \{r \in R \mid 0 = r \text{ or } 0, r \in J\} = \{r \in R \mid r \in J\} = J$.*

5. *Define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if $ann_R(x) = ann_R(y)$. Then, we have that $[0]_\sim = \{r \in R \mid ann(0) = ann(r)\} = \{r \in R \mid R = ann(r)\} = \{0\}$.*

6. *Let J be a semigroup ideal of R, and define the multiplicative congruence relation $\sim$ on R by $x \sim y$ if and only if $(J :_R x) = (J :_R y)$. Then, we have that $[0]_\sim = \{r \in R \mid (J :_R 0) = (J :_R r)\} = \{r \in R \mid R = (J :_R r)\} = J$.*

7. *Define the multiplicative congruence relation* $\sim$ *on* R *by* $x \sim y$ *if and only if* $x = uy$ *for some* $u \in U(R)$. *Since* $Z(R) \cap U(R) = \emptyset$, *we can clearly see that* $[0]_\sim = \{r \in R \mid 0 = ur \text{ for some } u \in U(R)\} = \{0\}$. *More generally, let* G *be a multiplicative subgroup of* $U(R)$, *and define the multiplicative congruence relation* $\sim_G$ *on* R *by* $x \sim_G y$ *if and only if* $x = uy$ *for some* $u \in G$. *Then,* $[0]_{\sim_G} = \{0\}$, *as well.*

8. *Define the multiplicative congruence relation* $\sim$ *on* R *by* $x \sim y$ *if and only if* $(x) = (y)$. *Then,* $[0]_\sim = \{r \in R \mid (0) = (r)\} = \{r \in R \mid \{0\} = (r)\} = \{0\}$.

9. *Define the multiplicative congruence relation* $\sim$ *on* R *by* $x \sim y$ *if and only if* $rx = sy$ *for some* $r, s \in R \setminus Z(R)$. *By following the reasoning first mentioned in Example 2.7.7, we get* $[0]_\sim = \{y \in R \mid r0 = sy \text{ for some } r, s \in R \setminus Z(R)\} = \{r \in R \mid 0 = sy \text{ for some } s \in R \setminus Z(R)\} = \{0\}$.

It turns out that the **annihilator congruence relation** $\sim_{a_R}$ on R, defined by

$$x \sim_{a_R} y \qquad \text{if and only if} \qquad ann_R(x) = ann_R(y),$$

is of particular interest to us. We introdued $\sim_{a_R}$ in Example 2.2.5, pointed out that $\Gamma_{\sim_{a_R}}(R) = \Gamma_E(R)$ in Example 2.4.5, and justified that $[0]_{\sim_{a_R}} = \{0\}$ in Example 2.7.5. The following result provides some additional insight into the importance of this particular congruence relation.

Theorem 2.8. *Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. Then, the following statements are equivalent:*

(1) For all $x, y \in R$, $xy \sim 0$ implies that $xy = 0$.

(2) For all $x, y \in R$, $x \sim y$ implies that $ann_R(x) = ann_R(y)$.

(3) $\sim \leq \sim_{a_R}$.

(4) $[0]_\sim = \{0\}$.

Proof. First note that (2) and (3) are equivalent by definition of $\leq$. We show that (1), (2), and (4) are equivalent to complete the proof.

To show that (1) implies (2), let $x, y \in R$ with $x \sim y$. Suppose $a \in ann_R(x)$. Then, $ax = 0$. Since $x \sim y$, we have that $ax \sim ay$. Thus, $0 = ax \sim ay$. So, since $ay \sim 0$, we have that $ay = 0$, by assumption. Thus, $a \in ann_R(y)$ so that $ann_R(x) \subseteq ann_R(y)$. By similar argument, $ann_R(x) \supseteq ann_R(y)$, and we may conclude that $ann_R(x) = ann_R(y)$.

Next, we show that (2) implies (4). Clearly, $\{0\} \subseteq [0]_\sim$. To show the reverse inclusion, suppose that $a \in [0]_\sim \setminus \{0\}$. Then, $a \sim 0$ so that $ann_R(a) = ann_R(0)$, by assumption. But, $ann_R(0) = R$, implying that $ann_R(a) = R$. Thus, $ra = 0$ for all $r \in R$, and we may conclude that $a = 0$. This is a contradiction. Hence, $[0]_\sim = \{0\}$.

Finally, we show that (4) implies (1) to establish the equivalence of all four statements. Let $x, y \in R$ with $xy \sim 0$. Then, $xy \in [0]_\sim = \{0\}$ so that $xy = 0$. □

Since our focus is on zero-divisor graphs with vertices the nonzero zero-divisors, we naturally wondered if R could be restricted to $Z(R)$ in (1) and (2) of the Theorem 2.8 for two additional equivalent statements. More specifically, we wondered if we could add the following two statements to the list of equivalent statements of Theorem 2.8.

(1)* For all $x, y \in Z(R)$, $xy \sim 0$ implies that $xy = 0$.

(2)* For all $x, y \in Z(R)$, $x \sim y$ implies that $ann_R(x) = ann_R(y)$.

However, it turns out that we cannot. Example 2.9 demonstrates that $(1)^*$ and $(2)^*$ above are not even equivalent to each other. But, note that $(2)^*$ does imply $(1)^*$. To see this, assume $(2)^*$ holds, and let $x, y \in Z(R)$ with $xy \sim 0$. Then, $ann_R(xy) = ann_R(0) = R$, implying that $xy = 0$, as desired.

Example 2.9. *Let $R = \mathbb{Z}/4\mathbb{Z}$, and let $\sim\, = R \times R$. Note that $Z(R) = \{\bar{0}, \bar{2}\}$. Since $\bar{0} \cdot \bar{0} = \bar{0} \cdot \bar{2} = \bar{2} \cdot \bar{0} = \bar{2} \cdot \bar{2} = \bar{0}$, $(1)^*$ is trivially true. However, $\bar{0} \sim \bar{2}$ and $ann_R(\bar{0}) = R \neq \{\bar{0}, \bar{2}\} = ann_R(\bar{2})$, implying that $(2)^*$ is false.*

As in ring theory, semigroup ideals may be further described by additional properties that hold for them. Let S be a commutative semigroup under the operation of multiplication, and let J be a proper semigroup ideal of S. We say that J is a **prime semigroup ideal** of S if $xy \in J$ implies that $x \in J$ or $y \in J$. For example, the set of zero-divisors $J = Z(S)$ is a prime semigroup ideal of S whenever it is a proper subset of S. Also, we say that J is a **radical semigroup ideal** of S if for any $x \in S$, if $x^n \in J$ for some $n \in \mathbb{Z}_+$, then $x \in J$. Equivalently, J is a radical semigroup ideal of S if and only if for any $x \in S$, if $x^2 \in J$, then $x \in J$. Now, if a semigroup ideal is prime, it is automatically radical. However, a semigroup ideal may be radical but not prime. The following two results provide additional insight into semigroup ideals.

Theorem 2.10. *Let R be a commutative ring with nonzero identity, and let J be a nonzero subset of R. Then, the following statements are equivalent:*

(1) J is a semigroup ideal of R.

(2) J is a union of principal ring ideals of R.

(3) J is a union of ring ideals of R.

Proof. To show that (1) implies (2), suppose that J is a semigroup ideal of R. We will show that $J = \bigcup_{x \in J}(x)$. First, note that since $J \neq \emptyset$, this union is nonempty, as well. Now, since $x \in (x)$ for each $x \in J$, we have that $J \subseteq \bigcup_{x \in J}(x)$. To show the reverse inclusion, note that for each $x \in J$, we have that $rx \in J$ for all $r \in R$ since J is a semigroup ideal of R. Thus, $(x) \subseteq J$ for each $x \in J$, implying that $\bigcup_{x\in J}(x) \subseteq J$. Now, $J = \bigcup_{x\in J}(x)$, and since each (x) is a principal ring ideal of R, we may conclude that J is a union of principal ring ideals of R.

Finally, since (2) clearly implies (3), it remains to show that (3) implies (1). To do this, suppose that J is a union of ring ideals of R. Then, $J = \bigcup_\alpha I_\alpha$, for ring ideals I_α of R. First, note that $J \neq \emptyset$ since each $I_\alpha \subseteq \bigcup_\alpha I_\alpha = J$ is nonempty. Suppose that $x \in R$ and $y \in J = \bigcup_\alpha I_\alpha$. Then, $y \in I_{\alpha_0}$ for some α_0. But, since I_{α_0} is a ring ideal of R, we have that $xy \in I_{\alpha_0} \subseteq \bigcup_\alpha I_\alpha = J$. So, J is a semigroup ideal of R. □

Theorem 2.11. *Let R be a commutative ring with nonzero identity.*

(a) A union of prime ring ideals of R is a prime semigroup ideal of R.

(b) A union of radical ring ideals of R is a radical semigroup ideal of R.

Proof. (a) Let $J = \bigcup_\alpha I_\alpha$, where each I_α is a prime ring ideal of R. By Theorem 2.10, J is a semigroup ideal of R. So, suppose that $x, y \in R$ with $xy \in J = \bigcup_\alpha I_\alpha$. Then, $xy \in I_{\alpha_0}$ for some α_0. But, since I_{α_0} is a prime ring ideal of R, either $x \in I_{\alpha_0} \subseteq \bigcup_\alpha I_\alpha = J$ or $y \in I_{\alpha_0} \subseteq \bigcup_\alpha I_\alpha = J$. Thus, J is a prime semigroup ideal of R, as desired.

(b) Let $J = \bigcup_\alpha I_\alpha$, where each I_α is a radical ring ideal of R. By Theorem 2.10, J is a semigroup ideal of R. So, suppose that $x \in R$ with $x^n \in J = \bigcup_\alpha I_\alpha$ for some $n \in \mathbb{Z}_+$. Then, $x^n \in I_{\alpha_0}$ for some α_0. But, since I_{α_0} is a radical ring ideal of R, we have that $x \in I_{\alpha_0} \subseteq \bigcup_\alpha I_\alpha = J$. Thus, J is a radical semigroup ideal of R, as desired.

□

In particular, Theorem 2.11 implies that a prime ring ideal of R is a prime semigroup ideal of R and that a radical ring ideal of R is a radical semigroup ideal of R; however, that is also clear by definition. Next, we show by example that for a fixed ring, not every prime semigroup ideal is a prime ring ideal and that not every radical semigroup ideal is a radical ring ideal.

Example 2.12. *Consider the commutative ring $R = \mathbb{Z}$ with nonzero identity.*

1. *First note that $2\mathbb{Z}$ and $3\mathbb{Z}$ are prime ring ideals of $R = \mathbb{Z}$ since 2 and 3 are prime integers. Thus, $J = 2\mathbb{Z} \cup 3\mathbb{Z}$ is a prime semigroup ideal of R by Theorem 2.11(a). But, $J = 2\mathbb{Z} \cup 3\mathbb{Z}$ is not a prime ring ideal of R since $2, 3 \in J$ but $2+3 = 5 \notin J$, implying that J is not even a ring ideal of R.*

2. *First note that $6\mathbb{Z}$ and $10\mathbb{Z}$ are radical ring ideals of $R = \mathbb{Z}$ since $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$ are products of distinct prime integers. Thus, $J = 6\mathbb{Z} \cup 10\mathbb{Z}$ is a radical semigroup ideal of R by Theorem 2.11(b). But, $J = 6\mathbb{Z} \cup 10\mathbb{Z}$ is not a radical ring ideal of R since $6, 10 \in J$ but $6 + 10 = 16 \notin J$, implying that J is not even a ring ideal of R.*

Now, even more can be said about the semigroup ideal $[0]_\sim$ of R. The following result generalizes the well-known results that $\Gamma(R) = \emptyset$ if and only if R is an integral domain, that $\Gamma_E(R) = \emptyset$ if and only if R is an integral domain, and that $\Gamma_I(R) = \emptyset$ if and only if $I = R$ or I is a prime ideal of R.

Theorem 2.13. *Let R be a commutative ring with nonzero identity, and let $\sim$ be a multiplicative congruence relation on R. Then, $\Gamma_\sim(R) = \emptyset$ if and only if $[0]_\sim = R$ or $[0]_\sim$ is a prime semigroup ideal of R.*

Proof. First, note that $[0]_\sim = R$ if and only if $\sim = R \times R$. For this case, $\Gamma_{R\times R}(R) = \emptyset$. Thus, we must show that if $[0]_\sim$ is a proper semigroup ideal of R, we have that $\Gamma_\sim(R) = \emptyset$ if and only if $[0]_\sim$ is a prime semigroup ideal of R. So, suppose that $[0]_\sim$ is a proper semigroup ideal of R that it is not prime. Then, there exist $x, y \in R \setminus [0]_\sim$ such that $xy \in [0]_\sim$. Thus, $[x]_\sim, [y]_\sim \in R/{\sim} \setminus \{[0]_\sim\}$ with $[x]_\sim[y]_\sim = [xy]_\sim = [0]_\sim$. So, $[x]_\sim, [y]_\sim \in Z(R/{\sim})^* = V(\Gamma_\sim(R))$, and $\Gamma_\sim(R) \neq \emptyset$.

Conversely, let $[0]_\sim$ be a prime semigroup ideal of R, and suppose that $\Gamma_\sim(R) \neq \emptyset$. Then, $Z(R/{\sim})^* = V(\Gamma_\sim(R)) \neq \emptyset$. So, there exist $[x]_\sim, [y]_\sim \in R/{\sim} \setminus \{[0]_\sim\}$ such that

$[x]_\sim [y]_\sim = [0]_\sim$. But, $[0]_\sim = [x]_\sim [y]_\sim = [xy]_\sim$. Thus, we have $x, y \in R$ with $xy \in [0]_\sim$, implying that either $x \in [0]_\sim$ or $y \in [0]_\sim$ since $[0]_\sim$ is a prime semigroup ideal of R. So, either $[x]_\sim = [0]_\sim$ or $[y]_\sim = [0]_\sim$, a contradiction. □

The following lemma provides the remaining information needed for us to arrive at our concluding result. Note that it generalizes the well-known analog for commutative rings with nonzero identity.

Lemma 2.14. *Let R be a commutative ring with nonzero identity. Then, R is a field if and only if for every semigroup ideal J of R, we have that $J = R$ or J is a prime semigroup ideal of R.*

Proof. Suppose that R is a field. Then, the only ring ideals of R are R and $\{0\}$. But, since every semigroup ideal J of R is a union of ring ideals of R by Theorem 2.10, it is also true that the only semigroup ideals of R are $J = R$ and $J = \{0\}$. But, since $J = \{0\}$ is a maximal ring ideal of R, it is also a prime ring ideal of R. Thus, $J = \{0\}$ is a prime semigroup ideal of R. So, for every semigroup ideal J of R, we have that $J = R$ or J is a prime semigroup ideal of R.

Conversely, suppose that for every semigroup ideal J of R, we have that $J = R$ or J is a prime semigroup ideal of R. First, note that since $\{0\}$ is actually a prime ring ideal of R, we have that R is an integral domain. Now, let $r \in R \setminus \{0\}$ and consider the semigroup ideal $J = (r^2)$ of R. If $J = R$, then $1 \in J = (r^2)$, implying that $1 = sr^2$ for some $s \in R$. But then, $1 = sr^2 = (sr)r$ so that $r^{-1} = sr \in R$. Otherwise, if $J \neq R$, then J is a prime semigroup ideal of R. So, since $r^2 \in (r^2) = J$, we also have that $r \in J = (r^2)$. Thus, $r = sr^2$ for some $s \in R$. But, since R is an integral domain, the cancellation law holds. So, since $r \neq 0$, we have that $1 = sr$ so that $r^{-1} = s \in R$. Thus, R is a field. □

Our final result for this chapter now follows immediately from Theorem 2.13, Theorem 2.6, and Lemma 2.14.

Theorem 2.15. *Let R be a commutative ring with nonzero identity. Then, we have that $\Gamma_{\sim}(R) = \emptyset$ for all multiplicative congruence relations $\sim$ on R if and only if R is a field.*

Proof. By Theorem 2.13, $\Gamma_{\sim}(R) = \emptyset$ for all multiplicative congruence relations $\sim$ on R if and only if $[0]_{\sim} = R$ or $[0]_{\sim}$ is a prime semigroup ideal of R for all multiplicative congruence relations $\sim$ on R. But, Theorem 2.6 tells us that this is true if and only if for every semigroup ideal J of R, we have that $J = R$ or J is a prime semigroup ideal of R. Thus, Lemma 2.14 allows us to conclude that this occurs if and only if R is a field. □

Chapter 3

Relationships Among Familiar Congruence Relations in $\mathcal{C}(R)$ and Bounds on $|\,\mathcal{C}(R)\,|$

Let R be a commutative ring with nonzero identity. To aid in our discussion, we define the set

$$\mathcal{C}(R) = \{\sim \mid \sim \text{ is a multiplicative congruence relation on } R\}.$$

As in the more general case of an equivalence relation, we may view each element $\sim\, \in \mathcal{C}(R)$ as a subset of $R \times R$ by saying that $(x, y) \in\, \sim$ if and only if $x \sim y$. Then, we may easily define a partial order $\leq$ on $\mathcal{C}(R)$ by $\sim_1 \leq \sim_2$ if and only if $\sim_1 \subseteq \sim_2$. Thus, for $\sim_1, \sim_2 \in \mathcal{C}(R)$ we have that $\sim_1 \leq \sim_2$ if and only if for $x, y \in R$, we have that $x \sim_1 y$ implies that $x \sim_2 y$. Also, for $x \in R$, if $\sim_1 \leq \sim_2$, then $[x]_{\sim_1} \subseteq [x]_{\sim_2}$. Furthermore, we have that $=_R\, \leq\, \sim\, \leq R \times R$ for all $\sim\, \in \mathcal{C}(R)$.

Now, Theorem 2.6 gives us that every semigroup ideal J of R is of the form $J = [0]_\sim$ for some $\sim\, \in \mathcal{C}(R)$. So, for every semigroup ideal J of R, we define the set

$$\mathcal{C}_J(R) = \{\sim\, \in \mathcal{C}(R) \mid [0]_\sim = J\}.$$

Note that Theorem 2.8 implies that $\mathcal{C}_{\{0\}}(R) = \{\sim\, \in \mathcal{C}(R) \mid\, \sim\, \leq\, \sim_{a_R}\}$, where $x \sim_{a_R} y$ if and only if $ann_R(x) = ann_R(y)$. We also have that $\mathcal{C}_R(R) = \{R \times R\}$. Furthermore, $\mathcal{C}(R)$ is the disjoint union

$$\mathcal{C}(R) = \bigcup \{\mathcal{C}_J(R) \mid J \text{ a semigroup ideal of } R\}.$$

In this chapter, we investigate the relationship among some of the elements of $\mathcal{C}(R)$ that were introduced in Chapter 2. We start by considering the set $\mathcal{C}_J(R)$, where J is a semigroup ideal of R, and then we move on to the special case where $J = I$ is a ring ideal of R. These results lead immediately to some bounds on the cardinality of $\mathcal{C}(R)$.

3.1 The Set $\mathcal{C}_J(R)$ for a Semigroup Ideal J of a Ring R

First, we maintain a broad perspective by letting J be a semigroup ideal of a commutative ring R with nonzero identity. We begin with a simple remark that provides some structure to the set $\mathcal{C}_J(R)$.

Remark 3.1. *Let J be a semigroup ideal of a commutative ring R with nonzero identity. Suppose that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ and $\sim\, \in \mathcal{C}(R)$. Then, if $\sim_1\, \leq\, \sim\, \leq\, \sim_2$, we have that $J = [0]_{\sim_1} \subseteq [0]_{\sim} \subseteq [0]_{\sim_2} = J$. So, $[0]_{\sim} = J$, implying that $\sim\, \in \mathcal{C}_J(R)$, as well.*

With this relationship in mind, we now investigate what can be said about various familiar congruence relations in $\mathcal{C}_J(R)$.

Lemma 3.2. *Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. Define $\sim_{R(J)}, \sim_{J_R} \in \mathcal{C}(R)$ by*

$$x \sim_{R(J)} y \quad \text{if and only if} \quad x = y \text{ or } x, y \in J, \quad \text{and}$$
$$x \sim_{J_R} y \quad \text{if and only if} \quad (J :_R x) = (J :_R y).$$

Then, $\sim_{R(J)}, \sim_{J_R} \in \mathcal{C}_J(R)$ and $\sim_{R(J)} \leq \sim_{J_R}$.

Proof. We have that $[0]_{\sim_{R(J)}} = J$ by Example 2.7.4, and $[0]_{\sim_{J_R}} = J$ by Example 2.7.6. Thus, $\sim_{R(J)}, \sim_{J_R} \in \mathcal{C}_J(R)$. Furthermore, if $x, y \in R$ with $x \sim_{R(J)} y$, then either $x = y$ or $x, y \in J$. If $x = y$, certainly $(J :_R x) = (J :_R y)$, and if $x, y \in J$, then $(J :_R x) = R = (J :_R y)$ since J is a semigroup ideal. Thus, $x \sim_{J_R} y$ so that $\sim_{R(J)} \leq \sim_{J_R}$. □

Utilizing these two congruence relations, we can get a partial converse to the claim made in Remark 3.1.

Theorem 3.3. *Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. Define $\sim_{R(J)}, \sim_{J_R} \in \mathcal{C}(R)$ by*

$$x \sim_{R(J)} y \quad \text{if and only if} \quad x = y \text{ or } x, y \in J, \quad \text{and}$$
$$x \sim_{J_R} y \quad \text{if and only if} \quad (J :_R x) = (J :_R y).$$

Then, for $\sim \in \mathcal{C}(R)$, we have that $\sim_{R(J)} \leq \sim \leq \sim_{J_R}$ if and only if $\sim \in \mathcal{C}_J(R)$.

Proof. First note that $\sim_{R(J)}, \sim_{J_R} \in \mathcal{C}_J(R)$ with $\sim_{R(J)} \leq \sim_{J_R}$ by Lemma 3.2. So, the forward implication follows from Remark 3.1. Conversely, let $\sim \in \mathcal{C}_J(R)$. Then, $[0]_\sim = J$. First, we show that $\sim_{R(J)} \leq \sim$. To do this, suppose that $x, y \in R$ with $x \sim_{R(J)} y$. Then, $x = y$ or $x, y \in J$. If $x = y$, certainly $x \sim y$, so suppose that $x, y \in J$. But, since $x, y, \in J = [0]_\sim$, we have that $0 \sim x$ and $0 \sim y$, implying that $x \sim y$. Thus, $\sim_{R(J)} \leq \sim$, as desired. Next, we show that $\sim \leq \sim_{J_R}$. To do this, suppose that $x, y \in R$ with $x \sim y$, and let $r \in (J :_R x)$. Then, $rx \in J = [0]_\sim$ so that $rx \sim 0$. But, we also have that $rx \sim ry$ since $\sim$ is a multiplicative congruence relation on R. Thus, $ry \sim 0$ so that $ry \in [0]_\sim = J$. Now, $r \in (J :_R y)$, and we have that $(J :_R x) \subseteq (J :_R y)$. The reverse inclusion may be shown in a similar manner, allowing us to conclude that $(J :_R x) = (J :_R y)$. Thus, $x \sim_{J_R} y$, implying that $\sim \leq \sim_{J_R}$, as desired. □

Corollary 3.4. *Let R be a commutative ring with nonzero identity, and define $=_R, \sim_{a_R} \in \mathcal{C}(R)$ by*

$$x =_R y \quad \text{if and only if} \quad x = y, \quad \text{and}$$
$$x \sim_{a_R} y \quad \text{if and only if} \quad ann_R(x) = ann_R(y).$$

Then, for $\sim \in \mathcal{C}(R)$, we have that $=_R \leq \sim \leq \sim_{a_R}$ if and only if $\sim \in \mathcal{C}_{\{0\}}(R)$.

Proof. Since $=_R$ is $\sim_{R(\{0\})}$ and $\sim_{a_R}$ is $\sim_{\{0\}_R}$, this result follows from Theorem 3.3 with $J = \{0\}$. □

It is worth mentioning that Corollary 3.4 simply reiterates the established fact that $\mathcal{C}_{\{0\}}(R) = \{\sim \in \mathcal{C}(R) \mid \sim \leq \sim_{a_R}\}$. More generally, Theorem 3.3 tells us that $\mathcal{C}_J(R) = \{\sim \in \mathcal{C}(R) \mid \sim_{R(J)} \leq \sim \leq \sim_{J_R}\}$. In other words, we now know that $\sim_{R(J)}$ is the smallest congruence relation in $\mathcal{C}_J(R)$ and that $\sim_{J_R}$ is the largest congruence relation in $\mathcal{C}_J(R)$. Next, we will further investigate what can be said about these two congruence relations as we let the semigroup ideal J vary.

Remark 3.5. *Let J be a semigroup ideal of a commutative ring R with nonzero identity, and define $\sim_{R(J)} \in \mathcal{C}(R)$ by $x \sim_{R(J)} y$ if and only if $x = y$ or $x, y \in J$. Then, for semigroup ideals J and J' of R, we have that $\sim_{R(J)} = \sim_{R(J')}$ if and only if $J = J'$ and $\sim_{R(J)} \leq \sim_{R(J')}$ if and only if $J \subseteq J'$.*

While the congruence relation $\sim_{R(J)}$ behaves nicely with respect to inclusion, it turns out that the congruence relation $\sim_{J_R}$ does not. Example 3.6 demonstrates that the inclusion relationship between two semigroup ideals J and J' of R does not tell us anything about the corresponding congruence relations $\sim_{J_R}$ and $\sim_{J'_R}$.

Example 3.6. *Let J be a semigroup ideal of a commutative ring R with nonzero identity, and define $\sim_{J_R} \in \mathcal{C}(R)$ by $x \sim_{J_R} y$ if and only if $(J :_R x) = (J :_R y)$.*

1. *Let $R = \mathbb{Z}$ and consider the semigroup ideals $J = 2\mathbb{Z}$ and $J' = 2\mathbb{Z} \cup 3\mathbb{Z}$. First, note that $J \subseteq J'$. Now, $(J :_R 1) = 2\mathbb{Z}$ and $(J :_R 3) = 2\mathbb{Z}$ so that $1 \sim_{J_R} 3$. However, $(J' :_R 1) = 2\mathbb{Z} \cup 3\mathbb{Z}$ and $(J' :_R 3) = \mathbb{Z}$ so that $1 \not\sim_{J'_R} 3$. Thus, $\sim_{J_R} \not\leq \sim_{J'_R}$.*

2. *Let $R = \mathbb{Z}$ and consider the semigroup ideals $J = 6\mathbb{Z}$ and $J' = 2\mathbb{Z}$. First, note that $J \subseteq J'$. Now, $(J :_R 2) = 3\mathbb{Z}$ and $(J :_R 6) = \mathbb{Z}$ so that $2 \not\sim_{J_R} 6$. However, $(J' :_R 2) = \mathbb{Z}$ and $(J' :_R 6) = \mathbb{Z}$ so that $2 \sim_{J'_R} 6$. Thus, $\sim_{J'_R} \not\leq \sim_{J_R}$.*

3.2 The Set $\mathcal{C}_I(R)$ for a Ring Ideal I of a Ring R

Now, we restrict our focus to the case where I is a ring ideal of a commutative ring R with nonzero identity, and we investigate what can be said about various familiar congruence relations in $\mathcal{C}_I(R)$.

Lemma 3.7. *Let R be a commutative ring with nonzero identity, and let I be a ring ideal of R. Define $\sim_{R(I)}, \sim_{R/I}, \sim_{I_R} \in \mathcal{C}(R)$ by*

$$\begin{array}{lll} x \sim_{R(I)} y & \text{if and only if} & x = y \text{ or } x, y \in I, \\ x \sim_{R/I} y & \text{if and only if} & x - y \in I, \qquad \text{and} \\ x \sim_{I_R} y & \text{if and only if} & (I :_R x) = (I :_R y). \end{array}$$

Then, $\sim_{R(I)}, \sim_{R/I}, \sim_{I_R} \in \mathcal{C}_I(R)$ and $\sim_{R(I)} \leq \sim_{R/I} \leq \sim_{I_R}$.

Proof. First note that $\sim_{R(I)}, \sim_{I_R} \in \mathcal{C}_I(R)$ with $\sim_{R(I)} \leq \sim_{I_R}$ by Lemma 3.2. Furthermore, we have that $[0]_{\sim_{R/I}} = I$ by Example 2.7.3, so $\sim_{R/I} \in \mathcal{C}_I(R)$, as well. Thus, Theorem 3.3 gives us that $\sim_{R(I)} \leq \sim_{R/I} \leq \sim_{I_R}$. □

In order to gain additional insight into the set $\mathcal{C}_I(R)$, we determine conditions under which $\sim_{R/I}$ is equal to the largest element $\sim_{I_R}$ in $\mathcal{C}_I(R)$ and conditions under which $\sim_{R/I}$ is equal to the smallest element $\sim_{R(I)}$ in $\mathcal{C}_I(R)$. But first, we establish a couple of facts.

Lemma 3.8. *Let R be a commutative ring with nonzero identity, and let I be a proper ring ideal of R. Define $\sim_{R/I}, \sim_{I_R} \in \mathcal{C}(R)$ by*

$$x \sim_{R/I} y \quad \text{if and only if} \quad x - y \in I, \qquad \text{and}$$
$$x \sim_{I_R} y \quad \text{if and only if} \quad (I :_R x) = (I :_R y).$$

Suppose $\sim_{R/I} = \sim_{I_R}$. Then,

(a) $U(R/I) = \{1 + I\}$, and

(b) I is a radical ring ideal of R.

Proof. Suppose that $\sim_{R/I} = \sim_{I_R}$.

(a) Clearly we have that $\{1 + I\} \subseteq U(R/I)$. To show the reverse inclusion, suppose that $x \in R$ such that $x + I \in U(R/I)$. Then, there exists a $y \in R$ such that $y + I \in U(R/I)$ and $(x + I)(y + I) = 1 + I$. So, $1 + I = xy + I$, implying that $1 - xy \in I$. We show that $(I :_R x) = (I :_R 1)$. Clearly, $(I :_R 1) = I \subseteq (I :_R x)$, so suppose that $r \in (I :_R x)$. We have that $r - (rx)y = r(1 - xy) \in I$ since $1 - xy \in I$. Thus, $r + I = (rx)y + I = I$ since $rx \in I$. So, $r \in I = (I :_R 1)$, implying that $(I :_R x) \subseteq (I :_R 1)$, as well. Now $x \sim_{I_R} 1$, so that $x \sim_{R/I} 1$ since $\sim_{R/I} = \sim_{I_R}$. Thus, $x - 1 \in I$ implying that $x + I = 1 + I$, and $U(R/I) \subseteq \{1 + I\}$.

(b) Let $x \in R$ be such that $x^2 \in I$. Then, $(1 + x + I)(1 - x + I) = 1 - x^2 + I = 1 + I$ So, $1 + x + I \in U(R/I) = \{1 + I\}$ by Lemma 3.8(a). Thus, $1 + x + I = 1 + I$ so that $x = (1 + x) - 1 \in I$, and we may conclude that I is a radical ring ideal of R.

□

A ring R is called a **Boolean ring** if $x^2 = x$ for all $x \in R$. Since we are interested in commutative rings, it is worth pointing out that a consequence of this definition is that every Boolean ring is commutative. An example of a Boolean ring is $\mathbb{Z}/2\mathbb{Z}$. In fact, $\mathbb{Z}/2\mathbb{Z}$ is the only Boolean ring that is also an integral domain. Furthermore, any finite Boolean ring with nonzero identity is isomorphic to a finite product of copies of $\mathbb{Z}/2\mathbb{Z}$. So, we may deduce from that fact that any finite Boolean ring with nonzero identity has 2^n elements for some $n \in \mathbb{Z}_+$. With this information, we are now prepared to show that $\sim_{R/I} = \sim_{I_R}$ precisely when the quotient ring R/I is a Boolean ring and then deduce additional results later in the chapter.

Theorem 3.9. *Let R be a commutative ring with nonzero identity, and let I be a proper ring ideal of R. Define $\sim_{R/I}, \sim_{I_R} \in \mathcal{C}(R)$ by*

$$x \sim_{R/I} y \quad \text{if and only if} \quad x - y \in I, \qquad \text{and}$$
$$x \sim_{I_R} y \quad \text{if and only if} \quad (I :_R x) = (I :_R y).$$

Then, $\sim_{R/I} = \sim_{I_R}$ if and only if R/I is a Boolean ring.

Proof. Suppose that $\sim_{R/I} = \sim_{I_R}$. Let $x \in R$ so that $x + I \in R/I$. First, we show that $(I :_R x^2) = (I :_R x)$. Suppose that $r \in (I :_R x^2)$. Then, $rx^2 \in I$, implying that $(rx)^2 = r(rx^2) \in I$, as well. So, $rx \in I$ since I is a radical ring ideal of R by Lemma 3.8(b). Thus, $r \in (I :_R x)$, and we have that $(I :_R x^2) \subseteq (I :_R x)$. To show the reverse inclusion, suppose that $r \in (I :_R x)$. Then, $rx \in I$, implying that $rx^2 = (rx)x \in I$. Thus, $r \in (I :_R x^2)$, and we may conclude that $(I :_R x) \subseteq (I :_R x^2)$, as well. Now, $x^2 \sim_{I_R} x$, implying that $x^2 \sim_{R/I} x$ since $\sim_{R/I} = \sim_{I_R}$. Thus, $x^2 - x \in I$ so that $(x + I)^2 = x^2 + I = x + I$, and we may conclude that R/I is a Boolean ring.

Conversely, suppose that R/I is a Boolean ring. First note that $\sim_{R/I} \leq \sim_{I_R}$ by Lemma 3.7. To show the reverse inequality, suppose that $x, y \in R$ with $x \sim_{I_R} y$. Then, $(I :_R x) = (I :_R y)$. But, $x^2 + I = (x + I)^2 = x + I$ since R/I is a Boolean ring. So, $(x - 1)x = x^2 - x \in I$, implying that $x - 1 \in (I :_R x) = (I :_R y)$. Thus,

$xy - y = (x-1)y \in I$, and a similar argument gives us that $xy - x = yx - x \in I$. Now, $x - y = (xy - y) - (xy - x) \in I$, implying that $x \sim_{R/I} y$. Thus, $\sim_{I_R} \leq \sim_{R/I}$, and we may conclude that $\sim_{R/I} = \sim_{I_R}$. □

Corollary 3.10. *Let R be a commutative ring with nonzero identity, and define $=_R, \sim_{a_R} \in \mathcal{C}(R)$ by*

$$x =_R y \quad \text{if and only if} \quad x = y, \quad \text{and}$$
$$x \sim_{a_R} y \quad \text{if and only if} \quad ann_R(x) = ann_R(y).$$

Then, $=_R$ is $\sim_{a_R}$ if and only if R is a Boolean ring.

Proof. Since $=_R$ is $\sim_{R/\{0\}}$, $\sim_{a_R}$ is $\sim_{\{0\}_R}$, and $R/\{0\} \cong R$, this result follows from Theorem 3.9 with $I = \{0\}$. □

Note that Corollary 3.10 is essentially the same result as [5, Corollary 2.7], but in a different context. Next, we move on to show that $\sim_{R(I)} = \sim_{R/I}$ precisely when $I = \{0\}$.

Theorem 3.11. *Let R be a commutative ring with nonzero identity, and let I be a proper ring ideal of R. Define $\sim_{R(I)}, \sim_{R/I} \in \mathcal{C}(R)$ by*

$$x \sim_{R(I)} y \quad \text{if and only if} \quad x = y \text{ or } x, y \in I, \quad \text{and}$$
$$x \sim_{R/I} y \quad \text{if and only if} \quad x - y \in I.$$

Then, $\sim_{R(I)} = \sim_{R/I}$ if and only if $I = \{0\}$. Moreover, in this case, $\sim_{R(I)}$ and $\sim_{R/I}$ are each $=_R$.

Proof. First, suppose that $\sim_{R(I)} = \sim_{R/I}$, and let $x \in I$. Then, $x + 1 - 1 = x \in I$ so that $x + 1 \sim_{R/I} 1$. But then, $x + 1 \sim_{R(I)} 1$ since $\sim_{R/I} = \sim_{R(I)}$. Now, since I is a proper ring ideal of R, we must have that $1 \notin I$. So, $x + 1 = 1$, implying that $x = 0$. Thus, $I = \{0\}$. Conversely, suppose that $I = \{0\}$. Since $\sim_{R(\{0\})}$ and $\sim_{R/\{0\}}$ are each $=_R$, we have that $\sim_{R(I)} = \sim_{R/I}$, as desired. □

3.3 Bounds on $|\mathcal{C}(R)|$

With the results shown earlier in this chapter, we can immediately deduce a couple of facts about the number of elements in $\mathcal{C}_I(R)$ for a ring ideal I of a commutative ring R with nonzero identity. We have already established that $\sim_{R(I)} \in \mathcal{C}_I(R)$, so we automatically know that $|\mathcal{C}_I(R)| \geq 1$. Also, since $\mathcal{C}_I(R) \subseteq \mathcal{C}(R)$ for every ring ideal I of R, these bounds on $|\mathcal{C}_I(R)|$ implicitly provide bounds on $|\mathcal{C}(R)|$.

Corollary 3.12. *Let R be a commutative ring with nonzero identity, and let I be a ring ideal of R. Then, $|\mathcal{C}_I(R)| = 1$ if and only if either*

(a) $I = R$, or

(b) $I = \{0\}$ and R is a Boolean ring.

Proof. First, note that if $I = R$, we have that $|\mathcal{C}_R(R)| = |\{R \times R\}| = 1$. Thus, we must show that if I is a proper ring ideal of R, then $|\mathcal{C}_I(R)| = 1$ if and only if $I = \{0\}$ and R is a Boolean ring. So, suppose that I is a proper ring ideal of R. Theorem 3.3 implies that $\mathcal{C}_I(R) = \{\sim \in \mathcal{C}(R) \mid \sim_{R(I)} \leq \sim \leq \sim_{I_R}\}$. But, Theorem 3.11 tells us that $\sim_{R(I)} = \sim_{R/I}$ if and only if $I = \{0\}$, and Theorem 3.9 tells us that $\sim_{R/I} = \sim_{I_R}$ if and only if R/I is a Boolean ring. Combining these results, we have that $\sim_{R(I)} = \sim_{R/I} = \sim_{I_R}$ if and only if $I = \{0\}$ and $R \cong R/\{0\}$ is a Boolean ring. Thus, $|\mathcal{C}_I(R)| = 1$ if and only if $I = \{0\}$ and R is a Boolean ring, as desired. □

Corollary 3.13. *Let R be a commutative ring with nonzero identity, and let I be a proper, nonzero ring ideal of R. Then, $|\mathcal{C}_I(R)| \geq 2$.*

Proof. Since $\{0\} \subsetneq I \subsetneq R$, Corollary 3.12 tells us that $|\mathcal{C}_I(R)| \neq 1$. Thus, it must be true that $|\mathcal{C}_I(R)| \geq 2$. □

Now, every commutative ring R with nonzero identity has at least two elements, 0 and 1. Thus, $=_R < R \times R$, and we automatically know that $|\mathcal{C}(R)| \geq 2$. However, we can sharpen this bound if we require that R is not a field. But first, we establish a couple of facts.

Lemma 3.14. *Let R be a Boolean ring with nonzero identity. Then, for $x, y \in R$, we have that $(x) = (y)$ if and only if $x = y$. Moreover, R does not have a unique nonzero, proper semigroup ideal.*

Proof. Suppose that R is a Boolean ring with nonzero identity, and let $x, y \in R$. If $x = y$, then clearly $(x) = (y)$. Conversely, suppose that $(x) = (y)$. Then, $x \in (x) = (y)$, implying that $x = ry$ for some $r \in R$, and $y \in (y) = (x)$, implying that $y = sx$ for some $s \in R$. So, we have that $x = ry = ry^2 = (ry)y = xy = x(sx) = sx^2 = sx = y$.

Now, note that $\{0\} = (0)$ and $R = (1)$. So, for $x \in R \setminus \{0, 1\}$, we would have that (x) is a nonzero, proper ring ideal of R. Furthermore, if $x, y \in R \setminus \{0, 1\}$ with $x \neq y$, then the nonzero, proper ring ideals (x) and (y) would be distinct. Thus, if R did have a unique nonzero, proper ring ideal, then we must have that $R \setminus \{0, 1\}$ contains exactly one element. But then, the Boolean ring R must contain exactly 3 elements, and this is a contradiction since every finite Boolean ring has 2^n elements for some $n \in \mathbb{Z}_+$. Thus, we have demonstrated that R does not have a unique nonzero, proper ring ideal.

To conclude, we must further show that R does not have a unique nonzero, proper semigroup ideal. To do this, suppose that J is the unique nonzero, proper semigroup ideal of R that is necessarily not a ring ideal. Let $x \in J \setminus \{0\}$. Then, $(x) \subseteq J$. So, we have that $\{0\} = (0) \subsetneq (x) \subseteq J \subsetneq R$. But, the ring ideal (x) of R is also a semigroup ideal of R, so we must have that $(x) = J$ since J is assumed to be the unique nonzero, proper semigroup ideal of R. However, we would then have that $J = (x)$ is a ring ideal of R, a contradiction. □

Theorem 3.15. *Let R be a commutative ring with nonzero identity that is not a field. Then, $|\,\mathcal{C}(R)\,| \geq 5$.*

Proof. First, we will show that $|\,\mathcal{C}(R)\,| \geq 4$. Since R is not a field, there exists a nonzero, proper ring ideal I of R. Then, $|\,\mathcal{C}_I(R)\,| \geq 2$ by Corollary 3.13, and $|\,\mathcal{C}_R(R)\,| = 1$ by Corollary 3.12. Also, since $=_R \in \mathcal{C}_{\{0\}}(R)$, we have that $|\,\mathcal{C}_{\{0\}}(R)\,| \geq 1$. Now, $\mathcal{C}(R) = \bigcup \{\mathcal{C}_J(R) \mid J \text{ a semigroup ideal of } R\}$ is a disjoint union, and we have that $\{0\} \subsetneq I \subsetneq R$ are distinct semigroup ideals of R. So, combining these results gives us that $|\,\mathcal{C}(R)\,| \geq |\,\mathcal{C}_{\{0\}}(R)\,| + |\,\mathcal{C}_I(R)\,| + |\,\mathcal{C}_R(R)\,| \geq 1+2+1 = 4$.

Next, we will show that $|\,\mathcal{C}(R)\,| \neq 4$ so that we may conclude that $|\,\mathcal{C}(R)\,| \geq 5$. Since R is not a field, there exists a nonzero, proper ring ideal I of R. But, note that to get $|\,\mathcal{C}(R)\,| = 4$, it must be true that $|\,\mathcal{C}_{\{0\}}(R)\,| = 1$, and that the ring ideal I is the unique nonzero, proper semigroup ideal $J = I$ such that $|\,\mathcal{C}_J(R)\,| = 2$. Now, in order to get that $|\,\mathcal{C}_{\{0\}}(R)\,| = 1$, Corollary 3.12 tells us that R must be a Boolean ring. Then, Lemma 3.14 implies that R does not have a unique nonzero, proper semigroup ideal, a contradiction. Thus, $|\,\mathcal{C}(R)\,| \geq 5$, as desired. □

The following two examples demonstrate that the bound of 5 on $|\,\mathcal{C}(R)\,|$ can be achieved and that $\mathcal{C}(R)$ need not be finite or even countably infinite. So, the bound presented in Theorem 3.15 cannot be improved.

Example 3.16. *Let $R = \mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. First, note that $=_R$ gives us the four congruence classes $[\bar{0}]_{=_R} = \{\bar{0}\}$, $[\bar{1}]_{=_R} = \{\bar{1}\}$, $[\bar{2}]_{=_R} = \{\bar{2}\}$, and $[\bar{3}]_{=_R} = \{\bar{3}\}$, and that $R \times R$ gives us the single congruence class $[\bar{0}]_{R\times R} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. The annihilator congruence relation $\sim_{a_R}$ gives us the three congruence classes $[\bar{0}]_{\sim_{a_R}} = \{\bar{0}\}$, $[\bar{1}]_{\sim_{a_R}} = \{\bar{1}, \bar{3}\}$, and $[\bar{2}]_{\sim_{a_R}} = \{\bar{2}\}$. Since $\mathcal{C}_{\{\bar{0}\}}(R) = \{\sim \in \mathcal{C}(R) \mid =_R \leq \sim \leq \sim_{a_R}\}$, and since no other congruence relation on R fits strictly between $=_R$ and $\sim_{a_R}$, we have that $|\,\mathcal{C}_{\{\bar{0}\}}(R)\,| = 2$. Also, Corollary 3.12 implies that $|\,\mathcal{C}_R(R)\,| = |\,R \times R\,| = 1$.*

Now, the only nonzero, proper ring ideal of $\mathbb{Z}/4\mathbb{Z}$ is $I = 2\mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{2}\}$. In fact, I is the only nonzero, proper semigroup ideal of R. Define $\sim_{R(I)} \in \mathcal{C}(R)$ by $x \sim_{R(I)} y$ if and only if $x = y$ or $x, y \in I$, and define $\sim_{I_R} \in \mathcal{C}(R)$ by $x \sim_{I_R} y$ if and only if $(I :_R x) = (I :_R y)$. Furthermore, note that $\mathcal{C}_I(R) = \{\sim \in \mathcal{C}(R) \mid \sim_{R(I)} \leq \sim \leq \sim_{I_R}\}$ by Theorem 3.3. The congruence relation $\sim_{R(I)}$ gives us the three congruence classes $[\bar{0}]_{\sim_{R(I)}} = \{\bar{0}, \bar{2}\}$, $[\bar{1}]_{\sim_{R(I)}} = \{\bar{1}\}$, and $[\bar{3}]_{\sim_{R(I)}} = \{\bar{3}\}$, and the congruence relation $\sim_{I_R}$ gives us the two congruence classes $[\bar{0}]_{\sim_{I_R}} = \{\bar{0}, \bar{2}\}$ and $[\bar{1}]_{\sim_{I_R}} = \{\bar{1}, \bar{3}\}$. But, since no other congruence relation on R fits strictly between $\sim_{R(I)}$ and $\sim_{I_R}$, we have that

$|\,\mathcal{C}_I(R)\,| = 2$. *Finally, since* $\mathcal{C}(R) = \mathcal{C}_{\{0\}}(R) \cup \mathcal{C}_I(R) \cup \mathcal{C}_R(R)$ *is a disjoint union,* $|\,\mathcal{C}(R)\,| = |\,\mathcal{C}_{\{0\}}(R)\,| + |\,\mathcal{C}_I(R)\,| + |\,\mathcal{C}_R(R)\,| = 2 + 2 + 1 = 5$.

Example 3.17.

1. *Let R be a commutative ring with nonzero identity, and let G be a multiplicative subgroup of $U(R)$. Define the multiplicative congruence relation $\sim_G$ on R by $x \sim_G y$ if and only if $x = uy$ for some $u \in G$, as in Example 2.2.7. Then, $[0]_{\sim_G} = \{0\}$ by Example 2.7.7, implying that $\sim_G \in \mathcal{C}_{\{0\}}(R)$. Furthermore, note that $[1]_{\sim_G} = \{x \in R \mid 1 \sim_G x\} = \{x \in R \mid x = u \text{ for some } u \in G\} = G$. Now, for multiplicative subgroups G_1 and G_2 of $U(R)$, with $G_1 \subseteq G_2$, we automatically have that $\sim_{G_1} \leq \sim_{G_2}$. But, it turns out that the converse is also true. To see this, suppose that $\sim_{G_1} \leq \sim_{G_2}$, and let $x \in G_1$. Since $G_1 = [1]_{\sim_{G_1}}$, we have that $x \sim_{G_1} 1$, implying that $x \sim_{G_2} 1$, as well. So, $x \in [1]_{\sim_{G_2}} = G_2$, and $G_1 \subseteq G_2$, as desired. Thus, we have that $\sim_{G_1} = \sim_{G_2}$ if and only if $G_1 = G_2$ and $\sim_{G_1} \leq \sim_{G_2}$ if and only if $G_1 \subseteq G_2$. This implies that $\{\sim_G \mid G \text{ is a multiplicative subgroup of } U(R)\}$ is a family of distinct congruence relations in $\mathcal{C}_{\{0\}}(R) \subseteq \mathcal{C}(R)$.*

2. *Let R be a commutative ring that contains the field of rational numbers $\mathbb{Q}$, and let $\{p_x \mid x \in \mathbb{Z}_+\}$ be the set of all positive prime integers. For each $X \subseteq \mathbb{Z}_+$, define $G_X = \left\{\prod_{x \in X} p_x^{a_x} \mid a_x \in \mathbb{Z} \text{ and } a_x = 0 \text{ for all but finitely many } x \in X\right\}$. Now, G_X is a multiplicative subgroup of $\mathbb{Q} \setminus \{0\} \subseteq U(R)$ for each $X \subseteq \mathbb{Z}_+$. Furthermore, since $|\,\mathbb{Z}_+\,| = \aleph_0$, there are $2^{\aleph_0} = c$ possible subsets of $\mathbb{Z}_+$, implying that there are uncountably many distinct subgroups G_X of $U(R)$. So, Example 3.17.1 gives us that $\{\sim_{G_X} \mid X \subseteq \mathbb{Z}_+\}$ is an uncountable family of distinct congruence relations in $\mathcal{C}_{\{0\}}(R) \subseteq \mathcal{C}(R)$, implying that $|\,\mathcal{C}(R)\,| \geq c$. But, since we may also view $\mathcal{C}(R)$ as a subset of the power set of $R \times R$, we also have that $|\,\mathcal{C}(R)\,| \leq c$. Thus, $\mathcal{C}(R)$ is uncountable.*

Finally, we will show by example that it is possible to achieve values of 2, 3, and 4 for $|\,\mathcal{C}(R)\,|$. Note that R is necessarily a field in these examples by Theorem 3.15.

Now, the only ring ideals of a field R are $\{0\}$ and R, and a similar proof allows us to further conclude that the only semigroup ideals of a field R are $\{0\}$ and R. Thus, $|\,\mathcal{C}(R)\,| = |\,\mathcal{C}_{\{0\}}(R)\,| + |\,\mathcal{C}_R(R)\,|$ since $\mathcal{C}(R) = \mathcal{C}_{\{0\}}(R) \cup \mathcal{C}_R(R)$ is a disjoint union. Furthermore, since $|\,\mathcal{C}_R(R)\,| = 1$ by Corollary 3.12, our examples must have values of 1, 2, and 3 for $|\,\mathcal{C}_{\{0\}}(R)\,|$.

Example 3.18.

1. *Let $R = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. First, note that $=_R$ gives us the two congruence classes $[\bar{0}]_{=_R} = \{\bar{0}\}$ and $[\bar{1}]_{=_R} = \{\bar{1}\}$, and that $R \times R$ gives us the single congruence class $[\bar{0}]_{R\times R} = \{\bar{0}, \bar{1}\}$. So, $=_R < R \times R$, but no other congruence relation on R fits strictly between these two congruence relations. Thus, $|\,\mathcal{C}(R)\,| = 2$.*

2. *Let $R = \mathbb{Z}/3\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}\}$. First, note that $=_R$ gives us the three congruence classes $[\bar{0}]_{=_R} = \{\bar{0}\}$, $[\bar{1}]_{=_R} = \{\bar{1}\}$, and $[\bar{2}]_{=_R} = \{\bar{2}\}$, and that $R \times R$ gives us the single congruence class $[\bar{0}]_{R\times R} = \{\bar{0}, \bar{1}, \bar{2}\}$. The annihilator congruence relation $\sim_{a_R}$ gives us the two congruence classes $[\bar{0}]_{\sim_{a_R}} = \{\bar{0}\}$ and $[\bar{1}]_{\sim_{a_R}} = \{\bar{1}, \bar{2}\}$. But, since $\mathcal{C}_{\{\bar{0}\}}(R) = \{\sim\, \in \mathcal{C}(R) \mid\, =_R\, \leq\, \sim\, \leq\, \sim_{a_R}\}$, and since no other congruence relation on R fits strictly between $=_R$ and $\sim_{a_R}$, we have that $|\,\mathcal{C}_{\{\bar{0}\}}(R)\,| = 2$. Thus, $|\,\mathcal{C}(R)\,| = 3$.*

3. *Let $R = \mathbb{Z}/5\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. First, note that $=_R$ gives us the five congruence classes $[\bar{0}]_{=_R} = \{\bar{0}\}$, $[\bar{1}]_{=_R} = \{\bar{1}\}$, $[\bar{2}]_{=_R} = \{\bar{2}\}$, $[\bar{3}]_{=_R} = \{\bar{3}\}$, and $[\bar{4}]_{=_R} = \{\bar{4}\}$, and that $R \times R$ gives us the single congruence class $[\bar{0}]_{R\times R} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. The annihilator congruence relation $\sim_{a_R}$ gives us the two congruence classes $[\bar{0}]_{\sim_{a_R}} = \{\bar{0}\}$ and $[\bar{1}]_{\sim_{a_R}} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.*

   Consider the equivalence relation $\sim_0$ on R, given by the three equivalence classes $[\bar{0}]_{\sim_0} = \{\bar{0}\}$, $[\bar{1}]_{\sim_0} = \{\bar{1}, \bar{4}\}$, and $[\bar{2}]_{\sim_0} = \{\bar{2}, \bar{3}\}$. It is easy to check that $\sim_0$ is a congruence relation on R by Remark 2.1. Also, note that $=_R\, <\, \sim_0\, <\, \sim_{a_R}$. We show that $\sim_0$ is the only congruence relation on R that fits strictly between $=_R$ and $\sim_{a_R}$. To do this, suppose that $\sim\, \in \mathcal{C}(R)$ with $=_R\, <\, \sim\, <\, \sim_{a_R}$. Then, *we must have that* $[\bar{0}]_\sim = \{\bar{0}\}$. *Furthermore, for the inequality* $=_R < \sim$ *to hold, at least two of the elements in* $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$ *must be congruent to each other. If* $\bar{1} \sim \bar{2}$, *then we also have that* $\bar{2} = \bar{2}\cdot\bar{1} \sim \bar{2}\cdot\bar{2} = \bar{4}$ *and* $\bar{3} = \bar{3}\cdot\bar{1} \sim \bar{3}\cdot\bar{2} = \bar{1}$. *Combining these results gives us that* $\bar{3} \sim \bar{1} \sim \bar{2} \sim \bar{4}$, *implying that* $\sim\, = \,\sim_{a_R}$, *a contradiction. So,* $\bar{1} \not\sim \bar{2}$. *Now, if* $\bar{1} \sim \bar{3}$, *then we also have that* $\bar{2} = \bar{2}\cdot\bar{1} \sim \bar{2}\cdot\bar{3} = \bar{1}$, *if* $\bar{2} \sim \bar{4}$, *then we also have that* $\bar{1} = \bar{3}\cdot\bar{2} \sim \bar{3}\cdot\bar{4} = \bar{2}$, *and if* $\bar{3} \sim \bar{4}$, *then we also have that* $\bar{2} = \bar{4}\cdot\bar{3} \sim \bar{4}\cdot\bar{4} = \bar{1}$. *Each of these three cases leads to* $\bar{1} \sim \bar{2}$, *but we just determined that* $\bar{1} \not\sim \bar{2}$. *So,* $\bar{1} \not\sim \bar{3}$, $\bar{2} \not\sim \bar{4}$, *and* $\bar{3} \not\sim \bar{4}$. *However, if* $\bar{1} \sim \bar{4}$, *then we also have that* $\bar{2} = \bar{2}\cdot\bar{1} \sim \bar{2}\cdot\bar{4} = \bar{3}$, *and if* $\bar{2} \sim \bar{3}$, *then we also have that* $\bar{1} = \bar{3}\cdot\bar{2} \sim \bar{3}\cdot\bar{3} = \bar{4}$. *Thus,* $\bar{1} \sim \bar{4}$ *if and only if* $\bar{2} \sim \bar{3}$, *and this gives us that* $\sim\, = \,\sim_0$. *Then, since* $\mathcal{C}_{\{\bar{0}\}}(R) = \{\sim \in \mathcal{C}(R) \mid =_R \leq \sim \leq \sim_{a_R}\}$, *we have that* $|\,\mathcal{C}_{\{\bar{0}\}}(R)\,| = 3$, *implying that* $|\,\mathcal{C}(R)\,| = 4$.

We conclude this chapter by showing that there is essentially only one field R that gives us $|\,\mathcal{C}(R)\,| = 2$.

Corollary 3.19. *Let* R *be a commutative ring with nonzero identity. Then, we have that* $|\,\mathcal{C}(R)\,| = 2$ *if and only if* $R \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. If $R \cong \mathbb{Z}/2\mathbb{Z}$, then $|\,\mathcal{C}(R)\,| = 2$ by Example 3.18.1. Conversely, suppose that $|\,\mathcal{C}(R)\,| = 2$. Then, R must be a field by Theorem 3.15. So, $\{0\}$ is the only proper ideal of R. Furthermore, since $|\,\mathcal{C}_R(R)\,| = 1$ by Corollary 3.12, it must be true that $|\,\mathcal{C}_{\{0\}}(R)\,| = 1$, as well. But, Corollary 3.12 then gives us that R is a Boolean ring. Thus, $R \cong \mathbb{Z}/2\mathbb{Z}$ since this is the only Boolean ring that is also a field. ∎

Chapter 4

Induced Maps Between Congruence-Based Zero-Divisor Graphs of a Ring R

Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. For $\sim \in \mathcal{C}_J(R)$, we have that $[x]_\sim$ and $[y]_\sim$ are adjacent in $\Gamma_\sim(R)$ if and only if $xy \in J$. Thus, the nicest case is when $J = \{0\}$ so that $[x]_\sim$ and $[y]_\sim$ are adjacent in $\Gamma_\sim(R)$ if and only if $xy = 0$. In particular, this occurs for the graphs $\Gamma(R)$ and $\Gamma_E(R)$.

In this chapter, we consider $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ with $\sim_1 \leq \sim_2$. We establish that natural surjective semigroup homomorphism from $R/\sim_1$ to $R/\sim_2$ induces a surjective map from $\Gamma_{\sim_1}(R)$ to $\Gamma_{\sim_2}(R)$ that is not necessarily a graph homomorphism. Furthermore, we define an injective map from $R/\sim_2$ to $R/\sim_1$ that is not necessarily a semigroup homomorphism, but we show that it induces an injective graph homomorphism from $\Gamma_{\sim_2}(R)$ to $\Gamma_{\sim_1}(R)$. This injective graph homomorphism turns out to be a graph isomorphism from $\Gamma_{\sim_2}(R)$ onto its image, allowing us to conclude that $\Gamma_{\sim_2}(R)$ may be viewed as an induced subgraph of $\Gamma_{\sim_1}(R)$. Finally, we conclude by studying graph homomorphisms between pairs of congruence-based zero-divisor graphs for congruence relations that were introduced in Chapter 2 and studied in detail in Chapter 3.

4.1 Semigroup Homomorphisms and the Induced Maps Between Graphs

Let R be a commutative ring with nonzero identity. We consider $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ with $\sim_1 \leq \sim_2$ throughout this chapter, so we begin by calling attention to these two assumptions. Note that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ tells us that $[0]_{\sim_1} = J = [0]_{\sim_2}$. However, the assumption that $\sim_1 \leq \sim_2$ automatically gives us $[0]_{\sim_1} \subseteq [0]_{\sim_2}$. Thus, if $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $\sim_1 \leq \sim_2$, we have that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ if and only if $[0]_{\sim_2} \subseteq [0]_{\sim_1}$. The following example demonstrates that it is possible to have two congruence relations $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ that are not comparable.

Example 4.1. *Let R be a commutative ring that contains the field of rational numbers $\mathbb{Q}$, and consider the multiplicative **cyclic subgroups** $G = \langle 2 \rangle = \{2^n \mid n \in \mathbb{Z}\}$ and $H = \langle 3 \rangle = \{3^n \mid n \in \mathbb{Z}\}$ of $\mathbb{Q} \setminus \{0\} \subseteq U(R)$. Define the multiplicative congruence relations $\sim_G$ and $\sim_H$ on R by $x \sim_G y$ if and only if $x = uy$ for some $u \in G$ and $x \sim_H y$ if and only if $x = uy$ for some $u \in H$, as in Example 2.2.7 and Example 3.17.1. Then, $[0]_{\sim_G} = \{0\} = [0]_{\sim_H}$ by Example 2.7.7, implying that $\sim_G, \sim_H \in \mathcal{C}_{\{0\}}(R)$. However, $\sim_G$ and $\sim_H$ are not comparable.*

We now establish a basic fact that will simplify the proof of our first theorem of this chapter.

Lemma 4.2. *Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. Suppose that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$. Then, $[x]_{\sim_1} \in Z(R/\sim_1)^*$ if and only if $[x]_{\sim_2} \in Z(R/\sim_2)^*$.*

Proof. First note that since $\sim_1, \sim_2 \in \mathcal{C}_J(R)$, we have that $[0]_{\sim_1} = J = [0]_{\sim_2}$. Suppose that $[x]_{\sim_1} \in Z(R/\sim_1)^*$. Then, there exists a $[y]_{\sim_1} \in Z(R/\sim_1)^*$ such that $[x]_{\sim_1}[y]_{\sim_1} = [xy]_{\sim_1} = [0]_{\sim_1}$. Thus, $xy \in [0]_{\sim_1} = [0]_{\sim_2}$ so that $[0]_{\sim_2} = [xy]_{\sim_2} = [x]_{\sim_2}[y]_{\sim_2}$. Furthermore, since $[x]_{\sim_1} \neq [0]_{\sim_1}$ and $[y]_{\sim_1} \neq [0]_{\sim_1}$, we have that $x \notin [0]_{\sim_1} = [0]_{\sim_2}$ and $y \notin [0]_{\sim_1} = [0]_{\sim_2}$. So, $[x]_{\sim_2} \neq [0]_{\sim_2}$ and $[y]_{\sim_2} \neq [0]_{\sim_2}$, implying that $[x]_{\sim_2} \in Z(R/{\sim_2})^*$. Thus, $[x]_{\sim_1} \in Z(R/{\sim_1})^*$ implies that $[x]_{\sim_2} \in Z(R/{\sim_2})^*$. The reverse implication may be shown by a similar argument, so we may conclude that $[x]_{\sim_1} \in Z(R/{\sim_1})^*$ if and only if $[x]_{\sim_2} \in Z(R/{\sim_2})^*$. □

The following example demonstrates that it is possible to have $x \in Z(R)^*$ with $[x]_\sim \notin Z(R/{\sim})^*$. But, note that $[0]_{=_R} = \{0\} \subsetneq [0]_\sim$ for this example.

Example 4.3. *Let $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then, $I = \{(0,0),(1,0)\}$ is a prime ring ideal of R. Define the congruence relation $\sim_{R(I)} \in \mathcal{C}(R)$ by $x \sim_{R(I)} y$ if and only if $x = y$ or $x, y \in I$. By Example 2.7.4, $[0]_{\sim_{R(I)}} = I \neq \{(0,0)\}$. Also, $R/{\sim_{R(I)}} = \{[(0,0)]_{\sim_{R(I)}}, [(0,1)]_{\sim_{R(I)}}, [(1,1)]_{\sim_{R(I)}}\}$. Now, $(0,1) \in Z(R)^*$ since $(0,1)(1,0) = (0,0)$. But, $[(0,1)]_{\sim_{R(I)}} \notin Z(R/{\sim})^*$ since $[(0,1)]_{\sim_{R(I)}}[(0,1)]_{\sim_{R(I)}} = [(0,1)]_{\sim_{R(I)}} \neq [(0,0)]_{\sim_{R(I)}}$ and $[(0,1)]_{\sim_{R(I)}}[(1,1)]_{\sim_{R(I)}} = [(0,1)]_{\sim_{R(I)}} \neq [(0,0)]_{\sim_{R(I)}}$. However, note that $\Gamma_{\sim_{R(I)}}(R) = \Gamma_I(R) = \emptyset$ for this example.*

Let R be a commutative ring with nonzero identity, and let $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $\sim_1 \leq \sim_2$. Consider the function $f : R/{\sim_1} \longrightarrow R/{\sim_2}$ defined by $f([x]_{\sim_1}) = [x]_{\sim_2}$. First note that f is well-defined if and only if $[x]_{\sim_1} = [y]_{\sim_1}$ implies that $f([x]_{\sim_1}) = f([y]_{\sim_1})$. But this occurs if and only if $[x]_{\sim_1} = [y]_{\sim_1}$ implies that $[x]_{\sim_2} = [y]_{\sim_2}$, or in other words, if and only if for $x, y \in R$, we have that $x \sim_1 y$ implies that $x \sim_2 y$. Thus, f is well-defined if and only if $\sim_1 \leq \sim_2$. Furthermore, f is clearly surjective, and f is a semigroup homomorphism since $f([x]_{\sim_1}[y]_{\sim_1}) = f([xy]_{\sim_1}) = [xy]_{\sim_2} = [x]_{\sim_2}[y]_{\sim_2} = f([x]_{\sim_1})f([y]_{\sim_1})$. We also have that $f([0]_{\sim_1}) = [0]_{\sim_2}$ and $f([1]_{\sim_1}) = [1]_{\sim_2}$.

Theorem 4.4. *Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. Suppose that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ with $\sim_1 \leq \sim_2$. Then, the surjective semigroup homomorphism $f : R/{\sim_1} \longrightarrow R/{\sim_2}$ defined by $f([x]_{\sim_1}) = [x]_{\sim_2}$ induces the surjective map $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ defined by $F = f|_{Z(R/{\sim_1})^*}$.*

Proof. First, note that since $\sim_1, \sim_2 \in \mathcal{C}_J(R)$, we have that $[0]_{\sim_1} = J = [0]_{\sim_2}$. Now, define the induced function $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ explicitly by $F([x]_{\sim_1}) = [x]_{\sim_2}$, and note that $F = f|_{Z(R/\sim_1)^*}$ is well-defined since f is well-defined. To show that F maps into $V(\Gamma_{\sim_2}(R)) = Z(R/\sim_2)^*$, let $[x]_{\sim_1} \in V(\Gamma_{\sim_1}(R)) = Z(R/\sim_1)^*$. Then, Lemma 4.2 gives us that $F([x]_{\sim_1}) = [x]_{\sim_2} \in Z(R/\sim_2)^* = V(\Gamma_{\sim_2}(R))$, as desired. Finally, to show that F is surjective, let $[x]_{\sim_2} \in V(\Gamma_{\sim_2}(R)) = Z(R/\sim_2)^* \subseteq R/\sim_2$. Then, $[x]_{\sim_1} \in R/\sim_1$ with $f([x]_{\sim_1}) = [x]_{\sim_2}$ since f is surjective. But, since $[x]_{\sim_2} \in Z(R/\sim_2)^*$, Lemma 4.2 gives us that $[x]_{\sim_1} \in Z(R/\sim_1)^* = V(\Gamma_{\sim_1}(R))$. So, $F([x]_{\sim_1}) = f([x]_{\sim_1}) = [x]_{\sim_2}$, as desired. □

Remark 4.5. *The induced function F in Theorem 4.4 is not necessarily a graph homomorphism. However, more can be said about F. Suppose we have that $[x]_{\sim_1}, [y]_{\sim_1} \in V(\Gamma_{\sim_1}(R)) = Z(R/\sim_1)^*$ with $[x]_{\sim_1} - [y]_{\sim_1} \in E(\Gamma_{\sim_1}(R))$. Then, $[x]_{\sim_1}[y]_{\sim_1} = [0]_{\sim_1}$. So, $F([x]_{\sim_1}), F([y]_{\sim_1}) \in V(\Gamma_{\sim_2}(R)) = Z(R/\sim_2)^*$ with $F([x]_{\sim_1})F([y]_{\sim_1}) = f([x]_{\sim_1})f([y]_{\sim_1}) = f([x]_{\sim_1}[y]_{\sim_1}) = f([0]_{\sim_1}) = [0]_{\sim_2}$. Thus, either $F([x]_{\sim_1}) = F([y]_{\sim_1})$ or $F([x]_{\sim_1}) - F([y]_{\sim_1}) \in E(\Gamma_{\sim_2}(R))$.*

The following result provides a sufficient condition to conclude that the induced map F is a graph homomorphism.

Theorem 4.6. *Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. Suppose that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ with $\sim_1 \leq \sim_2$ so that we have the map $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ defined by $F([x]_{\sim_1}) = [x]_{\sim_2}$. Then, F is a graph homomorphism if J is a radical semigroup ideal of R.*

Proof. First note that since $\sim_1, \sim_2 \in \mathcal{C}_J(R)$, we have that $[0]_{\sim_1} = J = [0]_{\sim_2}$. Let J be a radical semigroup ideal of R. Suppose that $[x]_{\sim_1}, [y]_{\sim_1} \in V(\Gamma_{\sim_1}(R))$ with $[x]_{\sim_1} - [y]_{\sim_1} \in E(\Gamma_{\sim_1}(R))$. In particular, note that $[x]_{\sim_1} \neq [0]_{\sim_1}$. Remark 4.5 asserts that either $F([x]_{\sim_1}) = F([y]_{\sim_1})$ or $F([x]_{\sim_1}) - F([y]_{\sim_1}) \in E(\Gamma_{\sim_2}(R))$. But, if $F([x]_{\sim_1}) = F([y]_{\sim_1})$, we have that $[x]_{\sim_2} = [y]_{\sim_2}$. Thus, $x \sim_2 y$, implying that $x^2 \sim_2 xy$ since $\sim_2 \in \mathcal{C}(R)$. Furthermore, since $[x]_{\sim_1} - [y]_{\sim_1} \in E(\Gamma_{\sim_1}(R))$, we have that

$[x]_{\sim_1}[y]_{\sim_1} = [xy]_{\sim_1} = [0]_{\sim_1}$. So, $xy \in [0]_{\sim_1} = [0]_{\sim_2}$, implying that $xy \sim_2 0$. Combining these facts gives us $x^2 \sim_2 0$. So, $x^2 \in [0]_{\sim_2} = J$, implying that $x \in J = [0]_{\sim_1}$ since J is a radical semigroup ideal of R. Thus, $[x]_{\sim_1} = [0]_{\sim_1}$, a contradiction. So, $F([x]_{\sim_1}) \neq F([y]_{\sim_1})$, implying that $F([x]_{\sim_1}) - F([y]_{\sim_1}) \in E(\Gamma_{\sim_2}(R))$. Thus, F is a graph homomorphism. □

We conclude this section by considering the special case of $J = \{0\}$.

Corollary 4.7. *Let R be a commutative ring with nonzero identity, and suppose that $\sim \in \mathcal{C}_{\{0\}}(R)$. Then, we have the surjective maps $\Gamma(R) \xrightarrow{F} \Gamma_\sim(R) \xrightarrow{F'} \Gamma_E(R)$ defined by $x \xmapsto{F} [x]_\sim \xmapsto{F'} [x]_{\sim_{a_R}}$. Moreover, if $\{0\}$ is a radical ring ideal of R, then F and F' are graph homomorphisms.*

Proof. Since $\sim \in \mathcal{C}_{\{0\}}(R)$, Corollary 3.4 gives us that $=_R \leq \sim \leq \sim_{a_R}$, where $\sim_{a_R}$ is the annihilator congruence relation on R. Then, Theorem 4.4 gives us the surjective maps $F : \Gamma_{=_R}(R) \longrightarrow \Gamma_\sim(R)$ defined by $F([x]_{=_R}) = [x]_\sim$ and $F' : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim_{a_R}}(R)$ defined by $F'([x]_\sim) = [x]_{\sim_{a_R}}$ that are graph homomorphisms if $\{0\}$ is a radical ideal of R by Theorem 4.6. But, Example 2.4.1 gives us that $\Gamma(R) \cong \Gamma_{=_R}(R)$, where the vertex x of $\Gamma(R)$ is identified with the vertex $[x]_{=_R} = \{x\}$ of $\Gamma_{=_R}(R)$. Also, Example 2.4.5 gives us that $\Gamma_E(R) = \Gamma_{\sim_{a_R}}(R)$. Thus, we may redefine the functions F and F' so that $F : \Gamma(R) \longrightarrow \Gamma_\sim(R)$ is defined by $F(x) = [x]_\sim$ and $F' : \Gamma_\sim(R) \longrightarrow \Gamma_E(R)$ is defined by $F'([x]_\sim) = [x]_{\sim_{a_R}}$. □

4.2 Induced Subgraphs of General Congruence-Based Zero-Divisor Graphs

Let R be a commutative ring with nonzero identity, and let $\sim_1, \sim_2 \in \mathcal{C}(R)$ with $\sim_1 \leq \sim_2$. For each congruence class $[x]_{\sim_2} \in R/\sim_2$, choose a single representative $x_0 \in R$ with $x_0 \in [x]_{\sim_2}$. Then, define $g : R/\sim_2 \longrightarrow R/\sim_1$ by $g([x]_{\sim_2}) = [x_0]_{\sim_1}$. First, note that g is well-defined since each congruence class has a single representative chosen for it. To show that g is injective, suppose that $[x]_{\sim_2}, [y]_{\sim_2} \in R/{\sim_2}$ with $[x]_{\sim_2} \neq [y]_{\sim_2}$. Let $x_0 \in [x]_{\sim_2}$ and $y_0 \in [y]_{\sim_2}$ be the chosen representatives. Thus, $[x_0]_{\sim_2} \neq [y_0]_{\sim_2}$, implying that $x_0 \not\sim_2 y_0$. Then, since $\sim_1 \leq \sim_2$, we also have that $x_0 \not\sim_1 y_0$. So, $g([x]_{\sim_2}) = [x_0]_{\sim_1} \neq [y_0]_{\sim_1} = g([y]_{\sim_2})$, as desired.

Now, g is not necessarily a semigroup homomorphism. But, it turns out that $f \circ g = 1_{R/\sim_2}$, where $f : R/{\sim_1} \longrightarrow R/{\sim_2}$ is the surjective semigroup homomorphism defined by $f([x]_{\sim_1}) = [x]_{\sim_2}$, as in the statement of Theorem 4.4. To see this, suppose that $[x]_{\sim_2} \in R/{\sim_2}$ has chosen representative x_0. Then, $f(g([x]_{\sim_2})) = f([x_0]_{\sim_1}) = [x_0]_{\sim_2} = [x]_{\sim_2}$, as desired.

In the following result, we will show that g induces the injective graph homomorphism $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ defined by $G = g|_{Z(R/\sim_2)^*}$. This map will turn out to be a graph isomorphism from $\Gamma_{\sim_2}(R)$ onto its image under G, allowing us to conclude that $\Gamma_{\sim_2}(R)$ may be viewed as an induced subgraph of $\Gamma_{\sim_1}(R)$.

Theorem 4.8. *Let R be a commutative ring with nonzero identity, and let J be a semigroup ideal of R. Suppose that $\sim_1, \sim_2 \in \mathcal{C}_J(R)$ with $\sim_1 \leq \sim_2$. Then, there exists a surjective map $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ and an injective graph homomorphism $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ such that $F \circ G = 1_{\Gamma_{\sim_2}(R)}$. Furthermore, $\Gamma_{\sim_2}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim_1}(R)$.*

Proof. First note that since $\sim_1, \sim_2 \in \mathcal{C}_J(R)$, we have that $[0]_{\sim_1} = J = [0]_{\sim_2}$. Furthermore, Theorem 4.4 gives us that the surjective semigroup homomorphism $f : R/{\sim_1} \longrightarrow R/{\sim_2}$ defined by $f([x]_{\sim_1}) = [x]_{\sim_2}$ induces the surjective map $F : \Gamma_{\sim_1}(R) \longrightarrow \Gamma_{\sim_2}(R)$ defined by $F = f|_{Z(R/\sim_1)^*}$.

For each congruence class $[x]_{\sim_2} \in R/{\sim_2}$, choose a single representative $x_0 \in R$ with $x_0 \in [x]_{\sim_2}$. Then, define the injective map $g : R/{\sim_2} \longrightarrow R/{\sim_1}$ by $g([x]_{\sim_2}) = [x_0]_{\sim_1}$. Now, define $G : \Gamma_{\sim_2}(R) \longrightarrow \Gamma_{\sim_1}(R)$ by $G = g|_{Z(R/\sim_2)^*}$. First, note that G is well-defined and injective since g is well-defined and injective. To show that G maps into $V(\Gamma_{\sim_1}(R)) = Z(R/{\sim_1})^*$, let $[x]_{\sim_2} \in V(\Gamma_{\sim_2}(R)) = Z(R/{\sim_2})^*$, and let $x_0 \in [x]_{\sim_2}$ be the chosen representative. Then, $[x_0]_{\sim_2} = [x]_{\sim_2} \in Z(R/{\sim_2})^*$. So, Lemma 4.2

gives us that $G([x]_{\sim_2}) = [x_0]_{\sim_1} \in Z(R/{\sim_1})^* = V(\Gamma_{\sim_1}(R))$, as desired. Furthermore, $F(G([x]_{\sim_2})) = F([x_0]_{\sim_1}) = [x_0]_{\sim_2} = [x]_{\sim_2}$.

Finally, we simultaneously show that G is a graph homomorphism and that $\Gamma_{\sim_2}(R)$ and its image under G are graph isomorphic so that $\Gamma_{\sim_2}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim_1}(R)$. Let $[x]_{\sim_2}, [y]_{\sim_2} \in V(\Gamma_{\sim_2}(R))$ have chosen representatives $x_0 \in [x]_{\sim_2}$ and $y_0 \in [y]_{\sim_2}$ so that $G([x]_{\sim_2}) = [x_0]_{\sim_1}$ and $G([y]_{\sim_2}) = [y_0]_{\sim_1}$. First note that $[x]_{\sim_2} \neq [y]_{\sim_2}$ if and only if $G([x]_{\sim_2}) \neq G([y]_{\sim_2})$ since G is well-defined and injective. Furthermore, $[x]_{\sim_2} - [y]_{\sim_2} \in E(\Gamma_{\sim_2}(R))$ if and only if $[x]_{\sim_2}[y]_{\sim_2} = [xy]_{\sim_2} = [0]_{\sim_2}$, which occurs if and only if $xy \sim_2 0$. But, since $x_0 \sim_2 x$ and $y_0 \sim_2 y$, we have that $x_0y_0 \sim_2 xy$ since $\sim_2 \in \mathcal{C}(R)$. So, $xy \sim_2 0$ if and only if $x_0y_0 \sim_2 0$. This happens if and only if $x_0y_0 \in [0]_{\sim_2} = [0]_{\sim_1}$, which occurs if and only if $[x_0y_0]_{\sim_1} = [0]_{\sim_1}$. Finally, we may conclude that $G([x]_{\sim_2})G([y]_{\sim_2}) = [x_0]_{\sim_1}[y_0]_{\sim_1} = [x_0y_0]_{\sim_1} = [0]_{\sim_1}$ if and only if $G([x]_{\sim_2}) - G([y]_{\sim_2})$ is an edge in the image of $\Gamma_{\sim_2}(R)$ under the map G. □

Again, we conclude this section by considering the special case of $J = \{0\}$. Note that the fact that $\Gamma_E(R)$ is graph isomorphic to an induced subgraph of $\Gamma(R)$ was originally given in [6, page 4].

Corollary 4.9. *Let R be a commutative ring with nonzero identity, and suppose that $\sim \in \mathcal{C}_{\{0\}}(R)$. Then, we have surjective maps $\Gamma(R) \xrightarrow{F} \Gamma_\sim(R) \xrightarrow{F'} \Gamma_E(R)$ and injective graph homomorphisms $\Gamma_E(R) \xrightarrow{G'} \Gamma_\sim(R) \xrightarrow{G} \Gamma(R)$ such that $F \circ G = 1_{\Gamma_\sim(R)}$, $F' \circ G' = 1_{\Gamma_E(R)}$, and $F' \circ F \circ G \circ G' = 1_{\Gamma_E(R)}$. Furthermore, $\Gamma_\sim(R)$ is graph isomorphic to an induced subgraph of $\Gamma(R)$, and $\Gamma_E(R)$ is graph isomorphic to an induced subgraph of $\Gamma_\sim(R)$ and of $\Gamma(R)$.*

Proof. Since $\sim \in \mathcal{C}_{\{0\}}(R)$, Corollary 3.4 gives us that $=_R \leq \sim \leq \sim_{a_R}$, where $\sim_{a_R}$ is the annihilator congruence relation on R. Thus, Theorem 4.8 gives us the surjective maps $F : \Gamma_{=_R}(R) \longrightarrow \Gamma_\sim(R)$ and $F' : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim_{a_R}}(R)$ and the injective graph homomorphisms $G : \Gamma_\sim(R) \longrightarrow \Gamma_{=_R}(R)$ and $G' : \Gamma_{\sim_{a_R}}(R) \longrightarrow \Gamma_\sim(R)$ such that $F \circ G = 1_{\Gamma_\sim(R)}$ and $F' \circ G' = 1_{\Gamma_{\sim_{a_R}}(R)}$. Theorem 4.8 also implies that $\Gamma_\sim(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{=_R}(R)$ and that $\Gamma_{\sim_{a_R}}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim}(R)$. However, Example 2.4.1 gives us that $\Gamma(R) \cong \Gamma_{=_R}(R)$, where the vertex x of $\Gamma(R)$ is identified with the vertex $[x]_{=_R} = \{x\}$ of $\Gamma_{=_R}(R)$. Furthermore, Example 2.4.5 gives us that $\Gamma_E(R) = \Gamma_{\sim_{a_R}}(R)$. Thus, we may redefine the functions F, F', G, and G' so that we have $F : \Gamma(R) \longrightarrow \Gamma_{\sim}(R)$, $F' : \Gamma_{\sim}(R) \longrightarrow \Gamma_E(R)$, $G : \Gamma_{\sim}(R) \longrightarrow \Gamma(R)$, and $G' : \Gamma_E(R) \longrightarrow \Gamma_{\sim}(R)$. Then, the compositions become $F \circ G = 1_{\Gamma_{\sim}(R)}$ and $F' \circ G' = 1_{\Gamma_E(R)}$, allowing us to conclude that $\Gamma_{\sim}(R)$ is graph isomorphic to an induced subgraph of $\Gamma(R)$ and that $\Gamma_E(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim}(R)$. Finally, note that we also have the surjective map $F' \circ F : \Gamma(R) \longrightarrow \Gamma_E(R)$ and the injective graph homomorphism $G \circ G' : \Gamma_E(R) \longrightarrow \Gamma(R)$ such that $F' \circ F \circ G \circ G' = F' \circ 1_{\Gamma_{\sim}(R)} \circ G' = F' \circ G' = 1_{\Gamma_E(R)}$. This gives us the fact that $\Gamma_E(R)$ is graph isomorphic to an induced subgraph of $\Gamma(R)$, as well. □

4.3 Induced Maps Between Familiar Congruence-Based Zero-Divisor Graphs and Induced Subgraphs

For a commutative ring R with nonzero identity and $\sim \in \mathcal{C}(R)$, it is always true that $[0]_{\sim}$ is a semigroup ideal of R. However, we are particularly interested in the special case when $[0]_{\sim} = I$ is also a ring ideal of R. Let I be a ring ideal of a commutative ring R with nonzero identity. Then, for $\sim \in \mathcal{C}_I(R)$, we have that $[x]_{\sim}$ and $[y]_{\sim}$ are adjacent in $\Gamma_{\sim}(R)$ if and only if $xy \in I$.

The following elements of $\mathcal{C}(R)$ were introduced in Chapter 2, and various relationships between pairs of these congruence relations were established in Chapter 3. We list these particular elements of $\mathcal{C}(R)$ here for convenience to establish the notation that will be used throughout this section.

$$
\begin{array}{llll}
x =_R y & \text{if and only if} & x = y, & \\
x \sim_{a_R} y & \text{if and only if} & ann_R(x) = ann_R(y), & \\
x \sim_{R(I)} y & \text{if and only if} & x = y \text{ or } x, y \in I, & \\
x \sim_{R/I} y & \text{if and only if} & x - y \in I, & \text{and} \\
x \sim_{I_R} y & \text{if and only if} & (I :_R x) = (I :_R y). &
\end{array}
$$

In Example 2.4.1, we justified the identification of $\Gamma_{=_R}(R)$ with the zero-divisor graph $\Gamma(R)$ by associating each congruence class $[x]_{=_R} = \{x\} \in V(\Gamma_{=_R}(R))$ with the element $x \in V(\Gamma(R))$. Similarly, in Example 2.4.4, we justified the identification of $\Gamma_{\sim_{R(I)}}(R)$ with the ideal-based zero-divisor graph $\Gamma_I(R)$ by associating each congruence class $[x]_{\sim_{R(I)}} = \{x\} \in V(\Gamma_{\sim_{R(I)}}(R))$ with the element $x \in V(\Gamma_I(R))$. Finally, in Example 2.4.3, we pointed out that the equality of $\Gamma_{\sim_{R/I}}(R)$ and the zero-divisor graph $\Gamma(R/I)$ is due to the fact that $[x]_{\sim_{R/I}} = x + I$.

Now, Example 2.4.5 asserts that $\Gamma_{\sim_{a_R}}(R)$ is the compressed zero-divisor graph $\Gamma_E(R)$, by definition. Also, we mentioned in Example 2.4.6 that $\Gamma_{\sim_{I_R}}(R)$ may be identified with the compressed zero-divisor graph $\Gamma_E(R/I)$; however, we did not provide any justification. We will now build the foundation that is necessary to establish this fact.

Theorem 4.10. *Let R be a commutative ring with nonzero identity, and let I be a proper ring ideal of R. Define $\sim_{R/I} \in \mathcal{C}(R)$ by*

$$x \sim_{R/I} y \qquad \text{if and only if} \qquad x - y \in I.$$

Suppose that $\sim \in \mathcal{C}_I(R)$ with $\sim_{R/I} \leq \sim$, and define $\sim' \in \mathcal{C}(R/I)$ by

$$x + I \sim' y + I \qquad \text{if and only if} \qquad x \sim y.$$

Then, $\Gamma_\sim(R) \cong \Gamma_{\sim'}(R/I)$.

Proof. Let $\sim \in \mathcal{C}_I(R)$ with $\sim_{R/I} \leq \sim$. First note that if $x, x' \in R$ with $x + I = x' + I$, then $x - x' \in I$ so that $x \sim_{R/I} x'$. But, since $\sim_{R/I} \leq \sim$, we also have that $x \sim x'$. So, for $x + I = x' + I$ and $y + I = y' + I$, we have that $x \sim y$ if and only if $x' \sim y'$. Thus, $x + I \sim' y + I$ if and only if $x' + I \sim' y' + I$ so that $\sim' \in \mathcal{C}(R/I)$ is well-defined. Furthermore, $[0]_{\sim'} = \{x + I \in R/I \mid 0 + I \sim' x + I\} = \{x + I \in R/I \mid 0 \sim x\} = \{x + I \in R/I \mid x \in [0]_\sim = I\} = \{I\}$ contains only the zero element of R/I.

Define the map $H : \Gamma_\sim(R) \longrightarrow \Gamma_{\sim'}(R/I)$ by $H([x]_\sim) = [x + I]_{\sim'}$, and note that $H([0]_\sim) = [0]_{\sim'}$. To simultaneously show that H is well-defined and injective, let $x, y \in R$ be such that $[x]_\sim, [y]_\sim \in V(\Gamma_\sim(R)) = Z(R/{\sim})^*$. Then, $[x]_\sim = [y]_\sim$ if and only if $x \sim y$, which occurs if and only if $x + I \sim' y + I$, by the definition of $\sim'$. But, that happens if and only if $H([x]_\sim) = [x + I]_{\sim'} = [y + I]_{\sim'} = H([y]_\sim)$. Thus, H is well-defined and injective.

To show that H is surjective, let $x \in R$ be such that $[x + I]_{\sim'} \in V(\Gamma_{\sim'}(R/I)) = Z(R/I/{\sim'})^*$. Then, there exists a $y \in R$ such that $[y + I]_{\sim'} \in Z(R/I/{\sim'})^*$ and $[x + I]_{\sim'}[y + I]_{\sim'} = [(x + I)(y + I)]_{\sim'} = [xy + I]_{\sim'} = [0]_{\sim'}$, implying that $xy \in I$. But, since $\sim \in \mathcal{C}_I(R)$, we have that $[0]_\sim = I$. So, $xy \in I = [0]_\sim$, implying that $[x]_\sim[y]_\sim = [xy]_\sim = [0]_\sim$. Now, since $[x + I]_{\sim'} \neq [0]_{\sim'}$ and $[y + I]_{\sim'} \neq [0]_{\sim'}$, we have that $x, y \notin I = [0]_\sim$. So, $[x]_\sim \neq [0]_\sim$ and $[y]_\sim \neq [0]_\sim$, implying that $[x]_\sim \in Z(R/{\sim})^* = V(\Gamma_\sim(R))$. Then, since $H([x]_\sim) = [x + I]_{\sim'}$, we may conclude that H is surjective.

Finally, to show that H is a graph isomorphism, let $x, y \in R$. Note that $\sim \in \mathcal{C}_I(R)$ implies that $[x]_\sim[y]_\sim = [xy]_\sim = [0]_\sim = I$ if and only if $xy \in I$. However, $H([x]_\sim)H([y]_\sim) = [x+I]_{\sim'}[y+I]_{\sim'} = [(x+I)(y+I)]_{\sim'} = [xy+I]_{\sim'}$, so $H([x]_\sim)H([y]_\sim) = [0]_{\sim'}$ if and only if $xy \in I$, as well. Furthermore, since H is well-defined and injective, $[x]_\sim \neq [y]_\sim$ if and only if $H([x]_\sim) \neq H([y]_\sim)$. Thus, $[x]_\sim - [y]_\sim \in E(\Gamma_\sim(R))$ if and only if $H([x]_\sim) - H([y]_\sim) \in E(\Gamma_{\sim'}(R/I))$. Thus, H is a graph isomorphism, implying that $\Gamma_\sim(R) \cong \Gamma_{\sim'}(R/I)$, as desired. □

Corollary 4.11. *Let R be a commutative ring with nonzero identity, and let I be a proper ring ideal of R. Define $\sim_{I_R} \in \mathcal{C}(R)$ by*

$$x \sim_{I_R} y \qquad \text{if and only if} \qquad (I :_R x) = (I :_R y).$$

Then, for $\sim'_{I_R} \in \mathcal{C}(R/I)$, defined by

$$x + I \sim'_{I_R} y + I \qquad \text{if and only if} \qquad x \sim_{I_R} y,$$

we have that $\Gamma_{\sim_{I_R}}(R) \cong \Gamma_{\sim'_{I_R}}(R/I) \cong \Gamma_E(R/I)$.

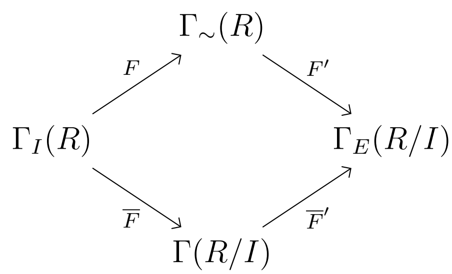
Proof. Lemma 3.7 tells us that $\sim_{I_R} \in \mathcal{C}_I(R)$ with $\sim_{R/I} \leq \sim_{I_R}$. So, the fact that $\Gamma_{\sim_{I_R}}(R) \cong \Gamma_{\sim'_{I_R}}(R/I)$ follows immediately from Theorem 4.10. Now, to show that $\Gamma_{\sim'_{I_R}}(R/I) \cong \Gamma_E(R/I)$, let $x, y \in R$ be such that $x+I, y+I \in R/I$. Then, by definition, $x + I \sim'_{I_R} y + I$ if and only if $(I :_R x) = (I :_R y)$. However, $(I :_R x) = \{r \in R \mid rx \in I\}$ and $ann_{R/I}(x+I) = \{r + I \in R/I \mid (r+I)(x+I) = 0 + I\} = \{r + I \in R/I \mid rx + I = 0 + I\}$. Then, since $s \in \{r \in R \mid rx \in I\}$ if and only if $s + I \in \{r + I \in R/I \mid rx + I = 0 + I\}$, we have that $(I :_R x) = (I :_R y)$ if and only if $ann_{R/I}(x+I) = ann_{R/I}(y+I)$. So, we may conclude that $x+I \sim'_{I_R} y+I$ if and only if $ann_{R/I}(x+I) = ann_{R/I}(y+I)$, implying that $\Gamma_{\sim'_{I_R}}(R/I) \cong \Gamma_{\sim_{a_{R/I}}}(R/I) = \Gamma_E(R/I)$, as well. □

Now, Example 2.4 and Corollary 4.11 provide us with graph isomorphisms between particular congruence-based zero-divisor graphs and familiar zero-divisor graphs from the literature. We explicitly list those graph isomorphisms here for convenience.

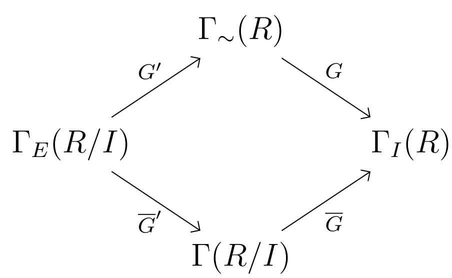
$$
\begin{array}{lll}
\Gamma_{=_R}(R) \longleftrightarrow \Gamma(R) & \text{given by} & [x]_{=_R} \leftrightarrow x, \\
\Gamma_{\sim_{a_R}}(R) \longleftrightarrow \Gamma_E(R) & \text{given by} & [x]_{\sim_{a_R}} \leftrightarrow [x]_{\sim_{a_R}}, \\
\Gamma_{\sim_{R(I)}}(R) \longleftrightarrow \Gamma_I(R) & \text{given by} & [x]_{\sim_{R(I)}} \leftrightarrow x, \\
\Gamma_{\sim_{R/I}}(R) \longleftrightarrow \Gamma(R/I) & \text{given by} & [x]_{\sim_{R/I}} \leftrightarrow x + I, \qquad \text{and} \\
\Gamma_{\sim_{I_R}}(R) \longleftrightarrow \Gamma_E(R/I) & \text{given by} & [x]_{\sim_{I_R}} \leftrightarrow [x + I]_{\sim_{a_{R/I}}}.
\end{array}
$$

We conclude this chapter with a result that follows immediately from Theorem 4.8 and the above graph isomorphisms. Note that the special case of $I = \{0\}$ was given in Corollary 4.9. While additional relationships between pairs of congruence-based zero-divisor graphs are given in Corollary 4.12, a couple of these relationships are well-known. The fact that $\Gamma(R/I)$ is graph isomorphic to an induced subgraph of $\Gamma_I(R)$ was originally shown in [27, Corollary 2.7], and the fact that $\Gamma_E(R/I)$ is graph isomorphic to an induced subgraph of $\Gamma(R/I)$ can be deduced from [6, page 4].

Corollary 4.12. *Let R be a commutative ring with nonzero identity, and let I be a proper ring ideal of R. Suppose that $\sim \in \mathcal{C}_I(R)$. Then, we have surjective maps*



and the injective graph homomorphisms



such that the following compositions hold:

$$F \circ G = 1_{\Gamma_\sim(R)},$$
$$\overline{F} \circ \overline{G} = 1_{\Gamma(R/I)},$$
$$F' \circ G' = 1_{\Gamma_E(R/I)},$$
$$\overline{F}' \circ \overline{G}' = 1_{\Gamma_E(R/I)}, \qquad \text{and}$$
$$F' \circ F \circ G \circ G' = \overline{F}' \circ \overline{F} \circ \overline{G} \circ \overline{G}' = 1_{\Gamma_E(R/I)}.$$

Furthermore, $\Gamma_\sim(R)$ and $\Gamma(R/I)$ are each graph isomorphic to an induced subgraph of $\Gamma_I(R)$, and $\Gamma_E(R/I)$ is graph isomorphic to an induced subgraph of $\Gamma_\sim(R)$, of $\Gamma(R/I)$, and of $\Gamma_I(R)$.

Proof. The fact that $\sim_{R(I)}, \sim_{I_R} \in \mathcal{C}_I(R)$ is given in Lemma 3.2, and since $\sim \in \mathcal{C}_I(R)$, Theorem 3.3 gives us that $\sim_{R(I)} \leq \sim \leq \sim_{I_R}$. Furthermore, Lemma 3.7 gives us that

$\sim_{R/I} \in \mathcal{C}_I(R)$ with $\sim_{R(I)} \leq \sim_{R/I} \leq \sim_{I_R}$. Thus, Theorem 4.8 gives us the surjective maps

$$\begin{aligned} F &: \Gamma_{\sim_{R(I)}}(R) \longrightarrow \Gamma_{\sim}(R), \\ \overline{F} &: \Gamma_{\sim_{R(I)}}(R) \longrightarrow \Gamma_{\sim_{R/I}}(R), \\ F' &: \Gamma_{\sim}(R) \longrightarrow \Gamma_{\sim_{I_R}}(R), \qquad \text{and} \\ \overline{F}' &: \Gamma_{\sim_{R/I}}(R) \longrightarrow \Gamma_{\sim_{I_R}}(R), \end{aligned}$$

and the injective graph homomorphisms

$$\begin{aligned} G &: \Gamma_{\sim}(R) \longrightarrow \Gamma_{\sim_{R(I)}}(R), \\ \overline{G} &: \Gamma_{\sim_{R/I}}(R) \longrightarrow \Gamma_{\sim_{R(I)}}(R), \\ G' &: \Gamma_{\sim_{I_R}}(R) \longrightarrow \Gamma_{\sim}(R), \qquad \text{and} \\ \overline{G}' &: \Gamma_{\sim_{I_R}}(R) \longrightarrow \Gamma_{\sim_{R/I}}(R), \end{aligned}$$

such that

$$\begin{aligned} F \circ G &= 1_{\Gamma_{\sim}(R)}, \\ \overline{F} \circ \overline{G} &= 1_{\Gamma_{\sim_{R/I}}(R)}, \\ F' \circ G' &= 1_{\Gamma_{\sim_{I_R}}(R)}, \qquad \text{and} \\ \overline{F}' \circ \overline{G}' &= 1_{\Gamma_{\sim_{I_R}}(R)}. \end{aligned}$$

Theorem 4.8 also implies that $\Gamma_{\sim}(R)$ and $\Gamma_{\sim_{R/I}}(R)$ are each graph isomorphic to an induced subgraph of $\Gamma_{\sim_{R(I)}}(R)$, and that $\Gamma_{\sim_{I_R}}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim}(R)$ and of $\Gamma_{\sim_{R/I}}(R)$. However, the isomorphisms listed above allow us to redefine the functions so that we have

$$\begin{aligned}
&F : \Gamma_I(R) \longrightarrow \Gamma_\sim(R),\\
&\overline{F} : \Gamma_I(R) \longrightarrow \Gamma(R/I),\\
&F' : \Gamma_\sim(R) \longrightarrow \Gamma_E(R/I),\\
&\overline{F}' : \Gamma(R/I) \longrightarrow \Gamma_E(R/I),\\
&G : \Gamma_\sim(R) \longrightarrow \Gamma_I(R),\\
&\overline{G} : \Gamma(R/I) \longrightarrow \Gamma_I(R),\\
&G' : \Gamma_E(R/I) \longrightarrow \Gamma_\sim(R), \qquad \text{and}\\
&\overline{G}' : \Gamma_E(R/I) \longrightarrow \Gamma(R/I).
\end{aligned}$$

Then, the compositions become

$$\begin{aligned}
&F \circ G = 1_{\Gamma_\sim(R)},\\
&\overline{F} \circ \overline{G} = 1_{\Gamma(R/I)},\\
&F' \circ G' = 1_{\Gamma_E(R/I)}, \qquad \text{and}\\
&\overline{F}' \circ \overline{G}' = 1_{\Gamma_E(R/I)},
\end{aligned}$$

allowing us to conclude that $\Gamma_\sim(R)$ and $\Gamma(R/I)$ are each graph isomorphic to an induced subgraph of $\Gamma_I(R)$, and that $\Gamma_E(R/I)$ is graph isomorphic to an induced subgraph of $\Gamma_\sim(R)$ and of $\Gamma(R/I)$. Finally, note that we also have the surjective maps

$$\begin{aligned}
&F' \circ F : \Gamma_I(R) \longrightarrow \Gamma_E(R/I) \qquad \text{and}\\
&\overline{F}' \circ \overline{F} : \Gamma_I(R) \longrightarrow \Gamma_E(R/I),
\end{aligned}$$

and the injective graph homomorphisms

$$G \circ G' : \Gamma_E(R/I) \longrightarrow \Gamma_I(R) \qquad \text{and}$$
$$\overline{G} \circ \overline{G}' : \Gamma_E(R/I) \longrightarrow \Gamma_I(R),$$

such that

$$F' \circ F \circ G \circ G' = F' \circ 1_{\Gamma_\sim(R)} \circ G' = F' \circ G' = 1_{\Gamma_E(R/I)} \qquad \text{and}$$
$$\overline{F}' \circ \overline{F} \circ \overline{G} \circ \overline{G}' = \overline{F}' \circ 1_{\Gamma(R/I)} \circ \overline{G}' = \overline{F}' \circ \overline{G}' = 1_{\Gamma_E(R/I)}.$$

Then, each of these compositions gives us the fact that $\Gamma_E(R/I)$ is graph isomorphic to an induced subgraph of $\Gamma_I(R)$, as well. □

Chapter 5

Induced Maps Between Congruence-Based Zero-Divisor Graphs of Rings R and T with $R \subseteq T$

Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$. We say that $\sim_R$ and $\sim_T$ are **compatible** if for $x, y \in R$, $x \sim_R y$ implies that $x \sim_T y$. Thus, $\sim_R$ and $\sim_T$ are compatible if and only if $\sim_R \subseteq \sim_T$, which holds if and only if $\sim_R \subseteq \sim_T \cap (R \times R)$ since $\sim_R \subseteq R \times R$.

The assumption that $\sim_R$ and $\sim_T$ are compatible automatically gives us that $[0]_{\sim_R} \subseteq [0]_{\sim_T} \cap R$. Thus, if $\sim_R$ and $\sim_T$ are compatible, we have that $[0]_{\sim_R} = [0]_{\sim_T} \cap R$ if and only if $[0]_{\sim_T} \cap R \subseteq [0]_{\sim_R}$. However, it is possible that compatible congruence relations $\sim_R$ and $\sim_T$ are such that $[0]_{\sim_R} \subsetneq [0]_{\sim_T} \cap R$. For example, $=_R \in \mathcal{C}(R)$ and $T \times T \in \mathcal{C}(T)$ are compatible. But, $[0]_{=_R} = \{0\}$ while $0, 1 \in [0]_{T \times T} \cap R$.

In this chapter, we consider rings $R \subseteq T$ with compatible congruence relations $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ as an extension of our work in Chapter 4. We establish when the natural semigroup homomorphism from $R/{\sim_R}$ to $T/{\sim_T}$ induces a map from $\Gamma_{\sim_R}(R)$ to $\Gamma_{\sim_T}(T)$ that is not necessarily a graph homomorphism. Then, we discuss congruence relations on R that are induced by congruence relations on T and establish that induced congruence relations lead to induced subgraphs of congruence-based zero-divisor graphs. Finally, we conclude by presenting a commutative diagram of various congruence-based zero-divisor graphs.

5.1 Compatible Congruence Relations, Semigroup Homomorphisms, and the Induced Maps Between Graphs

Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ be compatible congruence relations. Consider the function $f : R/{\sim_R} \longrightarrow T/{\sim_T}$ defined by $f([x]_{\sim_R}) = [x]_{\sim_T}$. First note that f is well-defined if and only if $[x]_{\sim_R} = [y]_{\sim_R}$ implies that $f([x]_{\sim_R}) = f([y]_{\sim_R})$. But this occurs if and only if $[x]_{\sim_R} = [y]_{\sim_R}$ implies that $[x]_{\sim_T} = [y]_{\sim_T}$, or in other words, if and only if for $x, y \in R$, we have that $x \sim_R y$ implies that $x \sim_T y$. Thus, f is well-defined if and only if $\sim_R$ and $\sim_T$ are compatible. Furthermore, f is a semigroup homomorphism since $f([x]_{\sim_R}[y]_{\sim_R}) = f([xy]_{\sim_R}) = [xy]_{\sim_T} = [x]_{\sim_T}[y]_{\sim_T} = f([x]_{\sim_R})f([y]_{\sim_R})$. We also have that $f([0]_{\sim_R}) = [0]_{\sim_T}$ and $f([1]_{\sim_R}) = [1]_{\sim_T}$.

Now, f is injective if and only if $f([x]_{\sim_R}) = f([y]_{\sim_R})$ implies that $[x]_{\sim_R} = [y]_{\sim_R}$. But, this occurs if and only if for $x, y \in R$, we have that $[x]_{\sim_T} = [y]_{\sim_T}$ implies that $[x]_{\sim_R} = [y]_{\sim_R}$, or in other words, if and only if for $x, y \in R$, we have that $x \sim_T y$ implies that $x \sim_R y$. Thus, f is injective if and only if $\sim_T \cap (R \times R) \subseteq \sim_R$. Note that in this case, we have that $[0]_{\sim_T} \cap R \subseteq [0]_{\sim_R}$.

The preceding observations allow us to conclude the following result.

Theorem 5.1. *Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$. Then, the function $f : R/{\sim_R} \longrightarrow T/{\sim_T}$ defined by $f([x]_{\sim_R}) = [x]_{\sim_T}$ is an injective semigroup homomorphism if and only if $\sim_R = \sim_T \cap (R \times R)$. Moreover, $[0]_{\sim_R} = [0]_{\sim_T} \cap R$ in this case.*

As in Chapter 4, we utilize the semigroup homomorphism above to induce a map between corresponding congruence-based zero-divisor graphs.

Theorem 5.2. *Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ be compatible, and let $f : R/{\sim_R} \longrightarrow T/{\sim_T}$ be the semigroup homomorphism defined by $f([x]_{\sim_R}) = [x]_{\sim_T}$. Then, f induces the function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ defined by $F = f|_{Z(R/\sim_R)^*}$ if $[0]_{\sim_R} = [0]_{\sim_T} \cap R$.*

Proof. First, suppose that $[0]_{\sim_R} = [0]_{\sim_T} \cap R$. Then, define the induced function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ explicitly by $F([x]_{\sim_R}) = [x]_{\sim_T}$, and note that $F = f|_{Z(R/\sim_R)^*}$ is well-defined since f is well-defined. To show that F maps into $V(\Gamma_{\sim_T}(T)) = Z(T/{\sim_T})^*$, let $[x]_{\sim_R} \in V(\Gamma_{\sim_R}(R)) = Z(R/{\sim_R})^*$. So, there exists a $[y]_{\sim_R} \in Z(R/{\sim_R})^*$ such that $[x]_{\sim_R}[y]_{\sim_R} = [xy]_{\sim_R} = [0]_{\sim_R}$. Thus, $xy \in [0]_{\sim_R} = [0]_{\sim_T} \cap R$, implying that $xy \in [0]_{\sim_T}$. So, $[x]_{\sim_T}[y]_{\sim_T} = [xy]_{\sim_T} = [0]_{\sim_T}$. Furthermore, since $[x]_{\sim_R} \neq [0]_{\sim_R}$ and $[y]_{\sim_R} \neq [0]_{\sim_R}$, we have that $x \notin [0]_{\sim_R} = [0]_{\sim_T} \cap R$ and $y \notin [0]_{\sim_R} = [0]_{\sim_T} \cap R$. However, since $x, y \in R$, we may conclude that $x \notin [0]_{\sim_T}$ and $y \notin [0]_{\sim_T}$. Thus, $[x]_{\sim_T} \neq [0]_{\sim_T}$ and $[y]_{\sim_T} \neq [0]_{\sim_T}$, implying that $F([x]_{\sim_R}) = [x]_{\sim_T} \in Z(T/{\sim_T})^* = V(\Gamma_{\sim_T}(T))$. □

Remark 5.3. *The induced function F in Theorem 5.2 is not necessarily a graph homomorphism. However, more can be said about F. Suppose we have that $[x]_{\sim_R}, [y]_{\sim_R} \in V(\Gamma_{\sim_R}(R)) = Z(R/{\sim_R})^*$ with $[x]_{\sim_R} - [y]_{\sim_R} \in E(\Gamma_{\sim_R}(R))$. Then, $[x]_{\sim_R}[y]_{\sim_R} = [0]_{\sim_R}$. So, $F([x]_{\sim_R}), F([y]_{\sim_R}) \in V(\Gamma_{\sim_T}(T)) = Z(T/{\sim_T})^*$ with $F([x]_{\sim_R})F([y]_{\sim_R}) = f([x]_{\sim_R})f([y]_{\sim_R}) = f([x]_{\sim_R}[y]_{\sim_R}) = f([0]_{\sim_R}) = [0]_{\sim_T}$. Thus, either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R}) - F([y]_{\sim_R}) \in E(\Gamma_{\sim_T}(T))$.*

The following result provides three different sufficient conditions to conclude that the induced map F is a graph homomorphism.

Theorem 5.4. *Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_R \in \mathcal{C}(R)$ and $\sim_T \in \mathcal{C}(T)$ be compatible, and suppose that $[0]_{\sim_R} = [0]_{\sim_T} \cap R$ so that we have the map $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ defined by $F([x]_{\sim_R}) = [x]_{\sim_T}$.*

(a) If $[0]_{\sim_R}$ is a radical semigroup ideal of R, then F is a graph homomorphism.

(b) If $[0]_{\sim_T}$ is a radical semigroup ideal of T, then F is a graph homomorphism.

(c) If $\sim_T \cap (R \times R) \subseteq \sim_R$, then F is an injective graph homomorphism.

Proof.

(a) Suppose $[0]_{\sim_R}$ is a radical semigroup ideal of R. Let $[x]_{\sim_R}, [y]_{\sim_R} \in V(\Gamma_{\sim_R}(R))$ with $[x]_{\sim_R} - [y]_{\sim_R} \in E(\Gamma_{\sim_R}(R))$. In particular, note that this implies that $[x]_{\sim_R} \neq [0]_{\sim_R}$. Remark 5.3 asserts that either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R}) - F([y]_{\sim_R}) \in E(\Gamma_{\sim_T}(T))$. But, if $F([x]_{\sim_R}) = F([y]_{\sim_R})$, we have that $[x]_{\sim_T} = [y]_{\sim_T}$. Thus, $x \sim_T y$, implying that $x^2 \sim_T xy$ since $\sim_T \in \mathcal{C}(T)$. Furthermore, since $[x]_{\sim_R} - [y]_{\sim_R} \in E(\Gamma_{\sim_R}(R))$, we have that $xy \sim_R 0$, implying that $xy \sim_T 0$ since $\sim_R$ and $\sim_T$ are compatible. Combining these facts gives us $x^2 \sim_T 0$ so that $x^2 \in [0]_{\sim_T}$. But, since $x \in R$, we also have that $x^2 \in R$, implying that $x^2 \in [0]_{\sim_T} \cap R = [0]_{\sim_R}$. Thus, $x \in [0]_{\sim_R}$ since $[0]_{\sim_R}$ is a radical semigroup ideal of R, and we have that $[x]_{\sim_R} = [0]_{\sim_R}$, a contradiction. So, we may conclude that F is a graph homomorphism.

(b) Suppose $[0]_{\sim_T}$ is a radical semigroup ideal of T. We will show that the semigroup ideal $[0]_{\sim_R}$ of R is a radical semigroup ideal of R so that we may conclude that F is a graph homomorphism by Theorem 5.4(a). Suppose that $x \in R$ with $x^n \in [0]_{\sim_R}$ for some $n \in \mathbb{Z}_+$. Then, $x^n \sim_R 0$ so that $x^n \sim_T 0$ since $\sim_R$ and $\sim_T$ are compatible. Thus, $x^n \in [0]_{\sim_T}$, implying that $x \in [0]_{\sim_T}$ since $[0]_{\sim_T}$ is a radical semigroup ideal of T. But, since $x \in R$, we have that $x \in [0]_{\sim_T} \cap R = [0]_{\sim_R}$. Thus, $[0]_{\sim_R}$ is a radical semigroup ideal of R, as desired.

(c) Suppose $\sim_T \cap (R \times R) \subseteq \sim_R$. Since $\sim_R$ and $\sim_T$ are compatible, we also have that $\sim_R \subseteq \sim_T \cap (R \times R)$. Thus, $\sim_R = \sim_T \cap (R \times R)$, implying that the semigroup homomorphism $f : R/\sim_R \longrightarrow T/\sim_T$ defined by $f([x]_{\sim_R}) = [x]_{\sim_T}$ is injective by Theorem 5.1. Thus, the induced function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ defined by $F = f|_{Z(R/\sim_R)^*}$ is also injective. Suppose that $[x]_{\sim_R}, [y]_{\sim_R} \in V(\Gamma_{\sim_R}(R))$ with $[x]_{\sim_R} - [y]_{\sim_R} \in E(\Gamma_{\sim_R}(R))$. Remark 5.3 asserts that either $F([x]_{\sim_R}) = F([y]_{\sim_R})$ or $F([x]_{\sim_R}) - F([y]_{\sim_R}) \in E(\Gamma_{\sim_T}(T))$. But, if $F([x]_{\sim_R}) = F([y]_{\sim_R})$, we have that $[x]_{\sim_R} = [y]_{\sim_R}$ since F is injective, and this is a contradiction. Thus, F is an injective graph homomorphism.

□

Now, if $T = R$, then $\sim_R$ and $\sim_T$ are compatible if and only if $\sim_R \leq \sim_T$. Thus, Theorem 4.4, Remark 4.5, and Theorem 4.6 are special cases of Theorem 5.2, Remark 5.3, and Theorem 5.4(a)(b), respectively.

5.2 Induced Congruence Relations and Induced Subgraphs

Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Each congruence relation $\sim_T \in \mathcal{C}(T)$ induces the congruence relation $\sim_R \in \mathcal{C}(R)$ defined by $\sim_R = \sim_T \cap (R \times R)$. Then, Theorem 5.1 gives us that the function $f : R/\sim_R \longrightarrow T/\sim_T$ defined by $f([x]_{\sim_R}) = [x]_{\sim_T}$ is an injective semigroup homomorphism and that $[0]_{\sim_R} = [0]_{\sim_T} \cap R$. Furthermore, Theorem 5.4(c) gives us that the induced function $F : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T)$ defined by $F = f|_{Z(R/\sim_R)^*}$ is an injective graph homomorphism.

Now, suppose we have $F([x]_{\sim_R}), F([y]_{\sim_R}) \in V(\Gamma_{\sim_T}(T)) = Z(T/\sim_T)^*$ with $F([x]_{\sim_R}) - F([y]_{\sim_R}) \in E(\Gamma_{\sim_T}(T))$. Then, $F([x]_{\sim_R})F([y]_{\sim_R}) = [0]_{\sim_T}$, implying that

$[x]_{\sim_T}[y]_{\sim_T} = [xy]_{\sim_T} = [0]_{\sim_T}$. So, $xy \in [0]_{\sim_T}$, but since $x, y \in R$, we also have that $xy \in R$. Thus, $xy \in [0]_{\sim_T} \cap R = [0]_{\sim_R}$, implying that $[x]_{\sim_R}[y]_{\sim_R} = [xy]_{\sim_R} = [0]_{\sim_R}$. So, $[x]_{\sim_R} - [y]_{\sim_R} \in E(\Gamma_{\sim_R}(R))$. With this final piece of information, we have that F is a graph isomorphism between $\Gamma_{\sim_R}(R)$ and its image in $\Gamma_{\sim_T}(T)$.

The preceding observations allow us to conclude the following result.

Theorem 5.5. *Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Suppose that $\sim_T \in \mathcal{C}(T)$, and let $\sim_R \in \mathcal{C}(R)$ be the induced congruence relation on R defined by $\sim_R = \sim_T \cap (R \times R)$. Then, $\Gamma_{\sim_R}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{\sim_T}(T)$.*

Next, we utilize Theorem 5.5 to provide examples of induced subgraphs of a few familiar graphs. These graphs were introduced in Chapter 2 as examples of congruence-based zero-divisor graphs for various congruence relations.

Example 5.6. *Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity.*

1. *The congruence relation $=_T$ on T induces the congruence relation $=_R$ on R. Thus, we have that $\Gamma(R)$ is graph isomorphic to an induced subgraph of $\Gamma(T)$.*

2. *Let I be a proper ring ideal of T. Then, $I \cap R$ is an induced proper ring ideal of R. So, the congruence relation $\sim_{T/I} \in \mathcal{C}(T)$ defined by $x \sim_{T/I} y$ for $x, y \in T$ if and only if $x - y \in I$ induces the congruence relation $\sim_{R/I \cap R} \in \mathcal{C}(R)$ defined by $x \sim_{R/I \cap R} y$ for $x, y \in R$ if and only if $x - y \in I \cap R$. Thus, we have that $\Gamma(R/I \cap R)$ is graph isomorphic to an induced subgraph of $\Gamma(T/I)$.*

3. *Let J be a semigroup ideal of T. Then, $J \cap R$ is an induced semigroup ideal of R. So, the congruence relation $\sim_{T(J)} \in \mathcal{C}(T)$ defined by $x \sim_{T(J)} y$ for $x, y \in T$ if and only if $x = y$ or $x, y \in J$ induces the congruence relation $\sim_{R(J \cap R)} \in \mathcal{C}(R)$ defined by $x \sim_{R(J \cap R)} y$ for $x, y \in R$ if and only if $x = y$ or $x, y \in J \cap R$. Thus, we have that $\Gamma_{J \cap R}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_J(T)$.*

Now, not all of the familiar congruence relations and corresponding congruence-based zero-divisor graphs we considered in Chapter 2 behave this nicely. The following example provides a case when the natural candidate for the induced congruence relation does not turn out to be the induced congruence relation.

Example 5.7. *Let J be a semigroup ideal of T. Then, $J \cap R$ is an induced semigroup ideal of R. So, the congruence relation $\sim_{J_T} \in \mathcal{C}(T)$ defined by $x \sim_{J_T} y$ for $x, y \in T$ if and only if $(J :_T x) = (J :_T y)$ corresponds to the congruence relation $\sim_{(J\cap R)_R} \in \mathcal{C}(R)$ defined by $x \sim_{(J\cap R)_R} y$ for $x, y \in R$ if and only if $(J \cap R :_R x) = (J \cap R :_R y)$. In this case, $x \sim_{J_T} y$ for $x, y \in R$ implies that $x \sim_{(J\cap R)_R} y$ so that $\sim_{J_T} \cap (R \times R) \subseteq \sim_{(J\cap R)_R}$. But this inclusion may be strict. Thus, $\sim_{J_T} \in \mathcal{C}(T)$ does not necessarily induce the natural candidate $\sim_{(J\cap R)_R} \in \mathcal{C}(R)$.*

To demonstrate this claim, suppose that $J = \{0\}$, implying that $J \cap R = \{0\}$, as well. Then, $\sim_{J_T} = \sim_{a_T}$ is defined by $x \sim_{a_T} y$ for $x, y \in T$ if and only if $ann_T(x) = ann_T(y)$, and $\sim_{(J\cap R)_R} = \sim_{a_R}$ is defined by $x \sim_{a_R} y$ for $x, y \in R$ if and only if $ann_R(x) = ann_R(y)$. Certainly, $ann_T(x) = ann_T(y)$ for $x, y \in R$ implies that $ann_R(x) = ann_R(y)$. But, we may have $ann_R(x) = ann_R(y)$ for $x, y \in R$ with $ann_T(x) \neq ann_T(y)$. For example, suppose that $R = F[X, Y, Z]/(XZ, YZ) = F[x, y, z]$ and $T = F[X, Y, Z, W]/(XZ, YZ, YW) = F[x, y, z, w]$. Then, we may view R as a subring of T in the natural way. However, $ann_R(x) = ann_R(y)$ while $ann_T(x) \neq ann_T(y)$ ([5, Section 3]). Thus, $\sim_{a_T} \cap (R \times R) \subsetneq \sim_{a_R}$, implying that $\sim_{a_T} \in \mathcal{C}(T)$ does not induce the natural candidate $\sim_{a_R} \in \mathcal{C}(R)$.

It turns out that we may resolve this issue in a special case, but we must first introduce a new ring. Let $S \subseteq R$ be multiplicatively closed with $S \cap Z(R) = \emptyset$ and $1 \in S$. The **localization** of R with respect to S, denoted by R_S, has as its set of elements the equivalence classes of fractions $\frac{r}{s}$ with $r \in R$ and $s \in S$. Two such fractions $\frac{r}{s}$ and $\frac{r'}{s'}$ are equivalent if $s_0(sr' - s'r) = 0$ for some $s_0 \in S$, and the operations of addition and multiplication are defined as they are for the rational numbers. Then, the map $f : R \longrightarrow R_S$ defined by $f(x) = \frac{x}{1}$ is injective, so we may view R as a subring of $T = R_S$. Thus, for $\sim_T \in \mathcal{C}(T)$, we have the induced congruence relation $\sim_R \in \mathcal{C}(R)$ defined by $\sim_R = \sim_T \cap (R \times R)$.

Example 5.8. *Let $S \subseteq R$ be multiplicatively closed with $S \cap Z(R) = \emptyset$ and $1 \in S$, and let $T = R_S$ be the localization of R with respect to S.*

1. *Let $\sim_{a_T} \in \mathcal{C}(T)$ be defined by $x \sim_{a_T} y$ for $x, y \in T$ if and only if $ann_T(x) = ann_T(y)$. In this case, $\sim_{a_T}$ induces the congruence relation $\sim_{a_R} \in \mathcal{C}(R)$ defined by $x \sim_{a_R} y$ for $x, y \in R$ if and only if $ann_R(x) = ann_R(y)$ since $(ann_R(x))_S = ann_{R_S}(\frac{x}{1})$. Thus, we have that $\Gamma_E(R)$ is graph isomorphic to an induced subgraph of $\Gamma_E(R_S)$. However, since $[\frac{x}{s}]_{\sim_{a_T}} = [\frac{x}{1}]_{\sim_{a_T}}$ for all $x \in R$ and $s \in S$, we actually have that $\Gamma_E(R) \cong \Gamma_E(R_S)$ ([5, Theorem 3.2]).*

2. *Let I be a ring ideal of $T = R_S$. Then, $I \cap R$ is an induced ring ideal of R. Note that we have $(I \cap R)_S = I$ in this case. So, the congruence relation $\sim_{T(I)} \in \mathcal{C}(T)$ defined by $x \sim_{T(I)} y$ for $x, y \in T$ if and only if $x = y$ or $x, y \in I$ induces the congruence relation $\sim_{R(I \cap R)} \in \mathcal{C}(R)$ defined by $x \sim_{R(I \cap R)} y$ for $x, y \in R$ if and only if $x = y$ or $x, y \in I \cap R$. Thus, we have that $\Gamma_{I \cap R}(R)$ is graph isomorphic to an induced subgraph of $\Gamma_{(I \cap R)_S}(R_S)$. This is a special case of Example 5.6.3.*

   Realize that in the above set-up, we started with a ring ideal I of R_S and then defined the ring ideal $I \cap R$ of R. However, if we start instead with a ring ideal I' of R and then define the ideal I'_S of R_S, then it need not be true that $I' = I'_S \cap R$. But, if $I' = \{x \in R \mid \frac{x}{1} \in I'_S\}$, then it is true that $I' = I'_S \cap R$.

5.3 Commutative Diagram of Congruence-Based Zero-Divisor Graphs

Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_T, \sim'_T \in \mathcal{C}(T)$ with $\sim_T \leq \sim'_T$. Then, $\sim_T$ induces $\sim_R \in \mathcal{C}(R)$ defined by $\sim_R = \sim_T \cap (R \times R)$, and $\sim'_T$ induces $\sim'_R \in \mathcal{C}(R)$ defined by

$\sim'_R = \sim'_T \cap (R \times R)$. Note that the relationship $\sim_R \leq \sim'_R$ automatically holds. Thus, we have the following two surjective semigroup homomorphisms:

$$f_R : R/{\sim_R} \longrightarrow R/{\sim'_R}, \qquad f_R([x]_{\sim_R}) = [x]_{\sim'_R}, \qquad \text{and}$$
$$f_T : T/{\sim_T} \longrightarrow T/{\sim'_T}, \qquad f_T([x]_{\sim_T}) = [x]_{\sim'_T}.$$

Furthermore, Theorem 5.1 gives us the following two injective semigroup homomorphisms:

$$g : R/{\sim_R} \longrightarrow T/{\sim_T}, \qquad g([x]_{\sim_R}) = [x]_{\sim_T}, \qquad \text{and}$$
$$g' : R/{\sim'_R} \longrightarrow T/{\sim'_T}, \qquad g'([x]_{\sim'_R}) = [x]_{\sim'_T}.$$

Now, $f_T \circ g : R/{\sim_R} \longrightarrow T/{\sim'_T}$ is such that $f_T\big(g([x]_{\sim_R})\big) = f_T([x]_{\sim_T}) = [x]_{\sim'_T}$, and $g' \circ f_R : R/{\sim_R} \longrightarrow T/{\sim'_T}$ is such that $g'\big(f_R([x]_{\sim_R})\big) = g'([x]_{\sim'_R}) = [x]_{\sim'_T}$. So, we may conclude that $f_T \circ g = g' \circ f_R$.

Suppose we also have that $[0]_{\sim_T} = [0]_{\sim_{T'}}$. Then, Theorem 5.1 allows us to conclude that $[0]_{\sim_R} = [0]_{\sim_T} \cap R = [0]_{\sim'_T} \cap R = [0]_{\sim'_R}$, as well. Thus, Theorem 4.4 gives us the following two surjective maps:

$$F_R : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim'_R}(R), \qquad F_R([x]_{\sim_R}) = [x]_{\sim'_R}, \qquad \text{and}$$
$$F_T : \Gamma_{\sim_T}(T) \longrightarrow \Gamma_{\sim'_T}(T), \qquad F_T([x]_{\sim_T}) = [x]_{\sim'_T}.$$

Since Theorem 5.1 gives us that $[0]_{\sim_R} = [0]_{\sim_T} \cap R$ and $[0]_{\sim'_R} = [0]_{\sim'_T} \cap R$, we may apply Theorem 5.4(c) to give us the following two injective graph homomorphisms:

$$G : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim_T}(T), \qquad G([x]_{\sim_R}) = [x]_{\sim_T}, \qquad \text{and}$$
$$G' : \Gamma_{\sim'_R}(R) \longrightarrow \Gamma_{\sim'_T}(T), \qquad G'([x]_{\sim'_R}) = [x]_{\sim'_T}.$$

Now, $F_T \circ G : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim'_T}(T)$ is such that $F_T\big(G([x]_{\sim_R})\big) = F_T([x]_{\sim_T}) = [x]_{\sim'_T}$, and $G' \circ F_R : \Gamma_{\sim_R}(R) \longrightarrow \Gamma_{\sim'_T}(T)$ is such that $G'\big(F_R([x]_{\sim_R})\big) = G'([x]_{\sim'_R}) = [x]_{\sim'_T}$. So, we may conclude that $F_T \circ G = G' \circ F_R$.

The preceding observations allow us to conclude the following result for the maps defined above.

Theorem 5.9. *Let R be a subring of a commutative ring T, and assume that R and T have the same nonzero identity. Let $\sim_T, \sim'_T \in \mathcal{C}(T)$ with $\sim_T \leq \sim'_T$, and suppose that $\sim_R, \sim'_R \in \mathcal{C}(R)$ are defined by $\sim_R = \sim_T \cap (R \times R)$ and $\sim'_R = \sim'_T \cap (R \times R)$. Then, the following diagram of semigroup homomorphisms commutes.*

$$\begin{array}{ccc} R/{\sim_R} & \overset{g}{\hookrightarrow} & T/{\sim_T} \\ \Big\downarrow{\scriptstyle f_R} & & \Big\downarrow{\scriptstyle f_T} \\ R/{\sim'_R} & \overset{g'}{\hookrightarrow} & T/{\sim'_T} \end{array}$$

Furthermore, if $[0]_{\sim_T} = [0]_{\sim'_T}$, then the following diagram of induced maps of congruence-based zero-divisor graphs commutes.

$$\begin{array}{ccc} \Gamma_{\sim_R}(R) & \overset{G}{\hookrightarrow} & \Gamma_{\sim_T}(T) \\ \Big\downarrow{\scriptstyle F_R} & & \Big\downarrow{\scriptstyle F_T} \\ \Gamma_{\sim'_R}(R) & \overset{G'}{\hookrightarrow} & \Gamma_{\sim'_T}(T) \end{array}$$

Moreover, G and G' are graph homomorphisms.

Chapter 6

Results for Ideal-Based Zero-Divisor Graphs

In this chapter, we restrict our focus to a particular type of congruence-based zero-divisor graph, the ideal-based zero-divisor graph. For a commutative ring R with nonzero identity and a ring ideal I of R, the ideal-based zero-divisor graph $\Gamma_I(R)$ is the congruence-based zero-divisor graph $\Gamma_\sim(R)$, where the multiplicative congruence relation $\sim$ on R is defined by $x \sim y$ if and only if $x = y$ or $x, y \in I$, as discussed in Example 2.4.4. We focus on two specific types of ideal-based zero-divisor graphs. In particular, we consider rings formed as a direct product of rings, and we also investigate rings formed as an idealization of a module.

6.1 Direct Products

From commutative rings R and S, each with nonzero identity, we may define another commutative ring $R \times S$ with nonzero identity called the **direct product** of R and S. The elements of $R \times S$ are ordered pairs, and for $(r_1, s_1), (r_2, s_2) \in R \times S$, addition is defined by $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and multiplication is defined by $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2)$. With the operations defined in this way, it is clear that the zero element of $R \times S$ is $(0,0)$ and that the nonzero identity element of $R \times S$ is $(1,1)$. Furthermore, every ring ideal of $R \times S$ is of the form $I \times J$, where I is a ring ideal of R and J is a ring ideal of S. So, we may consider how $\Gamma_{I \times J}(R \times S)$ is related to $\Gamma_I(R)$ and $\Gamma_J(S)$.

The following result discusses the vertices of $\Gamma_{I \times J}(R \times S)$ in terms of the vertices of $\Gamma_I(R)$ and $\Gamma_J(S)$.

Theorem 6.1. *Let R and S be commutative rings, each with nonzero identity, and let I and J be proper ring ideals of R and S, respectively. Then,*

(a) $V(\Gamma_{I \times S}(R \times S)) = V(\Gamma_I(R)) \times S$,

(b) $V(\Gamma_{R \times J}(R \times S)) = R \times V(\Gamma_J(S))$, *and*

(c) $V(\Gamma_{I \times J}(R \times S)) = ((R \setminus I) \times J) \cup (I \times (S \setminus J)) \cup (V(\Gamma_I(R)) \times S) \cup (R \times V(\Gamma_J(S)))$.

Proof.

(a) Let $(x,y) \in V(\Gamma_I(R)) \times S \subseteq (R \setminus I) \times S = (R \times S) \setminus (I \times S)$. Since $x \in V(\Gamma_I(R))$, there exists an $a \in R \setminus I$ such that $xa \in I$. Then, $(a,0) \in (R \setminus I) \times S = (R \times S) \setminus (I \times S)$, and $(x,y)(a,0) = (xa,0) \in I \times S$. Thus, $(x,y) \in V(\Gamma_{I \times S}(R \times S))$, and hence $V(\Gamma_{I \times S}(R \times S)) \supseteq V(\Gamma_I(R)) \times S$.

To show the reverse inclusion, suppose that $(x,y) \in V(\Gamma_{I \times S}(R \times S))$. Then, $(x,y) \in (R \times S) \setminus (I \times S) = (R \setminus I) \times S$. Furthermore, there exists an element $(a,b) \in (R \times S) \setminus (I \times S) = (R \setminus I) \times S$ such that $(xa, yb) = (x,y)(a,b) \in I \times S$. Thus, $xa \in I$, implying that $x \in V(\Gamma_I(R))$. Then, $(x,y) \in V(\Gamma_I(R)) \times S$ so that $V(\Gamma_{I \times S}(R \times S)) \subseteq V(\Gamma_I(R)) \times S$, as well.

(b) This proof is similar to the proof of Theorem 6.1(a).

(c) Let $(x,y) \in ((R \setminus I) \times J) \cup (I \times (S \setminus J)) \cup (V(\Gamma_I(R)) \times S) \cup (R \times V(\Gamma_J(S)))$. If $(x,y) \in (R \setminus I) \times J$, then $(x,y) \in (R \times S) \setminus (I \times J)$. Since J is a proper ideal of S, there exists an element $b \in S \setminus J$. Then, $(0,b) \in (R \times S) \setminus (I \times J)$, and

$(x, y)(0, b) = (0, yb) \in I \times J$ since $y \in J$. Thus, $(x, y) \in V(\Gamma_{I\times J}(R\times S))$. Similarly, if $(x, y) \in I \times (S \setminus J)$, then $(x, y) \in V(\Gamma_{I\times J}(R\times S))$. If $(x, y) \in V(\Gamma_I(R)) \times S$, then $x \in V(\Gamma_I(R)) \subseteq R \setminus I$ so that $(x, y) \in (R\times S) \setminus (I \times J)$. Also, since $x \in V(\Gamma_I(R))$, there exists an $a \in R \setminus I$ such that $xa \in I$. Then, $(a, 0) \in (R \times S) \setminus (I \times J)$, and $(x, y)(a, 0) = (xa, 0) \in I \times J$. Thus, $(x, y) \in V(\Gamma_{I\times J}(R \times S))$. Similarly, if $(x, y) \in R \times V(\Gamma_J(S))$, then $(x, y) \in V(\Gamma_{I\times J}(R \times S))$. So, we may conclude that $V(\Gamma_{I\times J}(R\times S)) \supseteq ((R\setminus I)\times J)\cup(I\times(S\setminus J))\cup(V(\Gamma_I(R))\times S)\cup(R\times V(\Gamma_J(S)))$.

To show the reverse inclusion, let $(x, y) \in V(\Gamma_{I\times J}(R \times S))$. Then, we have that $(x, y) \in (R \times S) \setminus (I \times J)$, and there exists an $(a, b) \in (R \times S) \setminus (I \times J)$ such that $(xa, yb) = (x, y)(a, b) \in I \times J$. Since $(x, y) \in (R \times S) \setminus (I \times J)$, either $x \in R \setminus I$ or $y \in S \setminus J$. First, suppose that $x \in R \setminus I$. If $a \in R \setminus I$, then $x \in V(\Gamma_I(R))$ since $xa \in I$, and $(x, y) \in V(\Gamma_I(R)) \times S$. However, if $a \in I$, then $b \in S \setminus J$ since $(a, b) \in (R \times S) \setminus (I \times J)$. Then, if $y \in S \setminus J$, we have that $y \in V(\Gamma_J(S))$ since $yb \in J$, and $(x, y) \in R \times V(\Gamma_J(S))$. But, if $y \in J$, then $(x, y) \in (R \setminus I) \times J$. Therefore, if $x \in R \setminus I$, we may conclude that $(x, y) \in ((R \setminus I) \times J) \cup (V(\Gamma_I(R)) \times S) \cup (R \times V(\Gamma_J(S)))$. Similarly, if $y \in S \setminus J$, we may conclude that $(x, y) \in (I \times (S \setminus J)) \cup (V(\Gamma_I(R)) \times S) \cup (R \times V(\Gamma_J(S)))$. Thus, $(x, y) \in ((R\setminus I)\times J)\cup(I\times(S\setminus J))\cup(V(\Gamma_I(R))\times S)\cup(R\times V(\Gamma_J(S)))$, and $V(\Gamma_{I\times J}(R\times S)) \subseteq ((R\setminus I)\times J)\cup(I\times(S\setminus J))\cup(V(\Gamma_I(R))\times S)\cup(R\times V(\Gamma_J(S)))$, as well.

□

With the vertices of $\Gamma_{I\times J}(R \times S)$ determined above, it remains to show how these vertices are connected to one another. The following result gives us this information by providing the conditions required for two vertices of $\Gamma_{I\times J}(R \times S)$ to be connected by an edge.

Theorem 6.2. *Let R and S be commutative rings, each with nonzero identity, and let I and J be proper ring ideals of R and S, respectively.*

(a) Suppose that $(r_1, s_1), (r_2, s_2) \in V(\Gamma_{I\times S}(R \times S))$ with $(r_1, s_1) \neq (r_2, s_2)$. Then, $(r_1, s_1) - (r_2, s_2) \in E(\Gamma_{I\times S}(R \times S))$ if and only if $r_1r_2 \in I$.

(b) Suppose that $(r_1, s_1), (r_2, s_2) \in V(\Gamma_{R\times J}(R \times S))$ with $(r_1, s_1) \neq (r_2, s_2)$. Then, $(r_1, s_1) - (r_2, s_2) \in E(\Gamma_{R\times J}(R \times S))$ if and only if $s_1s_2 \in J$.

(c) Suppose that $(r_1, s_1), (r_2, s_2) \in V(\Gamma_{I\times J}(R \times S))$ with $(r_1, s_1) \neq (r_2, s_2)$. Then, $(r_1, s_1) - (r_2, s_2) \in E(\Gamma_{I\times J}(R \times S))$ if and only if $r_1r_2 \in I$ and $s_1s_2 \in J$.

Proof.

(a) Let $(r_1, s_1), (r_2, s_2) \in V(\Gamma_{I\times S}(R \times S))$ with $(r_1, s_1) \neq (r_2, s_2)$. First note that $(r_1, s_1), (r_2, s_2) \in V(\Gamma_I(R)) \times S$ by Theorem 6.1(a) so that $s_1s_2 \in S$. Then, we have that $(r_1, s_1) - (r_2, s_2) \in E(\Gamma_{I\times S}(R \times S))$ if and only if $(r_1r_2, s_1s_2) = (r_1, s_1)(r_2, s_2) \in I \times S$, which occurs if and only if $r_1r_2 \in I$.

(b) This proof is similar to the proof of Theorem 6.2(a).

(c) Let $(r_1, s_1), (r_2, s_2) \in V(\Gamma_{I\times J}(R \times S))$ with $(r_1, s_1) \neq (r_2, s_2)$. Then, we have that $(r_1, s_1) - (r_2, s_2) \in E(\Gamma_{I\times J}(R \times S))$ if and only if we have $(r_1r_2, s_1s_2) = (r_1, s_1)(r_2, s_2) \in I \times J$, which occurs if and only if $r_1r_2 \in I$ and $s_1s_2 \in J$.

□

Fortunately, we can say a bit more about $\Gamma_{I\times J}(R \times S)$. Suppose we have that $(r_1, s_1), (r_2, s_2) \in V(\Gamma_{I\times J}(R \times S))$ with $(r_1, s_1) - (r_2, s_2) \in E(\Gamma_{I\times J}(R \times S))$. Then, $(r_1, s_1), (r_2, s_2) \in ((R \setminus I) \times J) \cup (I \times (S \setminus J)) \cup (V(\Gamma_I(R)) \times S) \cup (R \times V(\Gamma_J(S)))$ by Theorem 6.1(c). Figure 6.1 provides a visualization of this region broken into the more specific regions indicated by $A, B, C, D, E, F,$ and G. Now, Theorem 6.2(c) tells us that we have $r_1r_2 \in I$ and $s_1s_2 \in J$. We can use this information to further investigate how the vertices are connected by edges in $\Gamma_{I\times J}(R \times S)$.

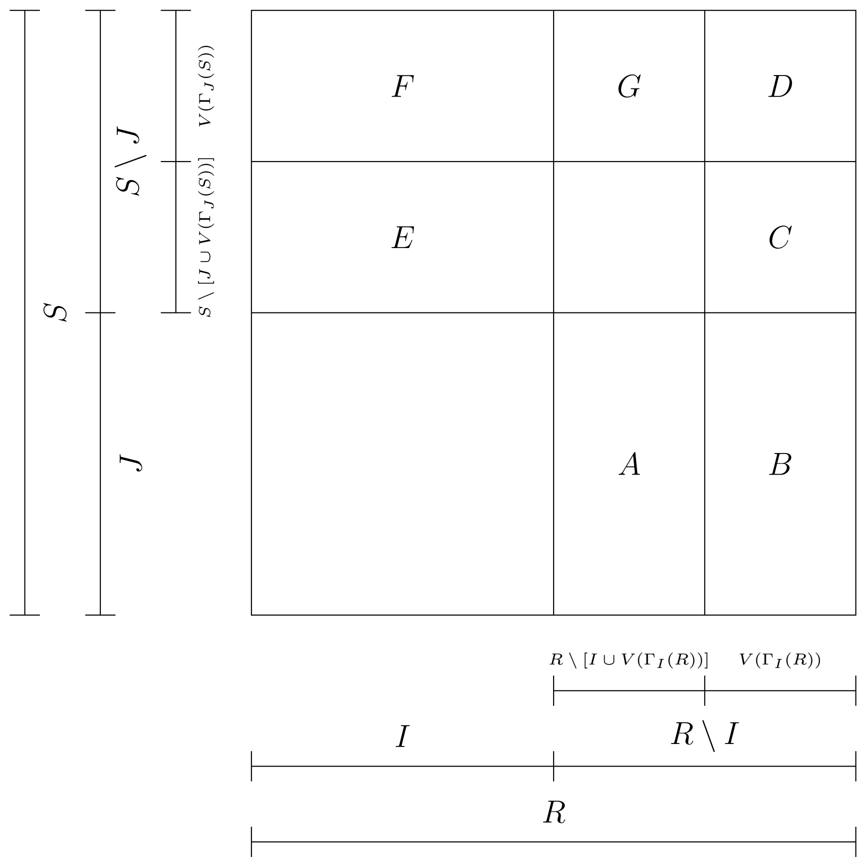


Figure 6.1: Visualization of Subregions of $R \times S$

First, suppose that $(r_1, s_1) \in A$. Then, since $r_1 \notin I \cup V(\Gamma_I(R))$, the only way to have $r_1 r_2 \in I$ is to have $r_2 \in I$. But then, since $(r_2, s_2) \notin I \times J$, we must have that $s_2 \in S \setminus J$ but cannot be more specific as to whether or not $s_2 \in V(\Gamma_J(S))$. Thus, $(r_2, s_2) \in E \cup F$. Similarly, if $(r_1, s_1) \in E$, we may conclude that $(r_2, s_2) \in A \cup B$.

Next, suppose that $(r_1, s_1) \in B$. Since $s_1 \in J$, we will always have $s_1 s_2 \in J$ for any $s_2 \in S$. Now, it is possible for $r_2 \in I$, and if this is the case, then all we can say is that $s_2 \notin J$ since we have that $(r_2, s_2) \notin I \times J$. Thus, our conclusion would be that $(r_2, s_2) \in E \cup F$. However, it is also possible for $r_2 \in V(\Gamma_I(R))$ since $r_1 \in V(\Gamma_I(R))$, and if this is the case, then we would not have any restriction on s_2 since $r_2 \notin I$ would imply that $(r_2, s_2) \notin I \times J$. Thus, our conclusion would be that $(r_2, s_2) \in B \cup C \cup D$. Combining these cases gives us that $(r_2, s_2) \in B \cup C \cup D \cup E \cup F$. Similarly, if $(r_1, s_1) \in F$, we may conclude that $(r_2, s_2) \in A \cup B \cup D \cup F \cup G$.

Now, suppose that $(r_1, s_1) \in C$. Then, since $s_1 \notin J \cup V(\Gamma_J(S))$, the only way to have $s_1 s_2 \in J$ is to have $s_2 \in J$. But then, since $(r_2, s_2) \notin I \times J$, we must have that $r_2 \in R \setminus I$. Furthermore, since $r_1 \in V(\Gamma_I(R))$, it must be true that $r_2 \in V(\Gamma_I(R))$, as well. Thus, $(r_2, s_2) \in B$. Similarly, if $(r_1, s_1) \in G$, we may conclude that $(r_2, s_2) \in F$.

Finally, suppose that $(r_1, s_1) \in D$. It is possible for $r_2 \in I$, and if this is the case, we must have that $s_2 \notin J$ since $(r_2, s_2) \notin I \times J$. Furthermore, since $s_1 \in V(\Gamma_J(S))$, it must be true that $s_2 \in V(\Gamma_J(S))$, as well. Thus, our conclusion would be that $(r_2, s_2) \in F$. However, it is also possible for $r_2 \in V(\Gamma_I(R))$ since $r_1 \in V(\Gamma_I(R))$. If this is the case, it is possible for $s_2 \in J$ since $r_2 \notin I$. But, if $s_2 \notin J$, then we would again have to have that $s_2 \in V(\Gamma_J(S))$ since $s_1 \in V(\Gamma_J(S))$. Thus, our conclusion would be that $(r_2, s_2) \in B \cup D$. Combining these cases gives us that $(r_2, s_2) \in B \cup D \cup F$.

6.2 Idealization

Given a commutative ring R with nonzero identity and an R-module M, we may define another commutative ring $R(+)M$ with nonzero identity called the **idealization** of M. The elements of $R(+)M$ are ordered pairs in $R \times M$. Furthermore, for $(r_1, m_1), (r_2, m_2) \in R(+)M$, the operations of addition and multiplication are defined by $(r_1, m_1)+(r_2, m_2) = (r_1+r_2, m_1+m_2)$ and by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2+r_2m_1)$, respectively. With the operations defined in this way, it is clear that the zero element of $R(+)M$ is $(0, 0)$ and that the nonzero identity element of $R(+)M$ is $(1, 0)$. Also, every ring ideal I of R gives rise to the ring ideal $I(+)M$ of $R(+)M$. So, we may consider the relationship between $\Gamma_I(R)$ and $\Gamma_{I(+)M}(R(+)M)$.

The following result discusses the vertices of $\Gamma_{I(+)M}(R(+)M)$ in terms of the vertices of $\Gamma_I(R)$.

Theorem 6.3. *Let R be a commutative ring with nonzero identity, I a ring ideal of R, and M an R-module. Then, $V(\Gamma_{I(+)M}(R(+)M)) = V(\Gamma_I(R)) \times M$.*

Proof. Let $(r, m) \in V(\Gamma_{I(+)M}(R(+)M))$. Then, $(r, m) \in (R(+)M)) \setminus (I(+)M) = (R \setminus I) \times M$ and there exists an $(s, n) \in (R(+)M)) \setminus (I(+)M) = (R \setminus I) \times M$ such that $(r, m)(s, n) \in I(+)M$. But, $(r, m)(s, n) = (rs, rn + sm)$, implying that $rs \in I$. Now, since $r, s \in R \setminus I$ and $rs \in I$, we have that $r \in V(\Gamma_I(R))$. Thus, $(r, m) \in V(\Gamma_I(R)) \times M$, giving us $V(\Gamma_{I(+)M}(R(+)M)) \subseteq V(\Gamma_I(R)) \times M$.

To show the reverse inclusion, let $(r, m) \in V(\Gamma_I(R)) \times M$. Since $r \in V(\Gamma_I(R))$, we have that $r \in R \setminus I$ and there exists an $s \in R \setminus I$ such that $rs \in I$. Let $n \in M$. Then, $(r, m), (s, n) \in (R \setminus I) \times M = (R(+)M)) \setminus (I(+)M)$. Furthermore, $(r, m)(s, n) = (rs, rn + sm) \in I(+)M$. Thus, $(r, m) \in V(\Gamma_{I(+)M}(R(+)M))$, giving us $V(\Gamma_{I(+)M}(R(+)M)) \supseteq V(\Gamma_I(R)) \times M$, as well. □

Corollary 6.4. *Let R be a commutative ring with nonzero identity, and let M be an R-module. Then, $V(\Gamma_{\{0\}(+)M}(R(+)M)) = V(\Gamma(R)) \times M$.*

Proof. Let $I = \{0\}$. Then, Theorem 6.3 gives us that $V(\Gamma_{\{0\}(+)M}(R(+)M)) = V(\Gamma_{\{0\}}(R)) \times M = V(\Gamma(R)) \times M$. □

Now that we have investigated the vertices of $\Gamma_{I(+)M}(R(+)M)$, we must determine how these vertices are connected to one another. The following result provides us with this information by describing the edges of $\Gamma_{I(+)M}(R(+)M)$.

Theorem 6.5. *Let R be a commutative ring with nonzero identity, I a ring ideal of R, and M an R-module. Suppose that $(r_1, m_1), (r_2, m_2) \in V(\Gamma_{I(+)M}(R(+)M))$ with $(r_1, m_1) \neq (r_2, m_2)$. Then, $(r_1, m_1) - (r_2, m_2) \in E(\Gamma_{I(+)M}(R(+)M))$ if and only if $r_1 r_2 \in I$.*

Proof. Let $(r_1, m_1), (r_2, m_2) \in V(\Gamma_{I(+)M}(R(+)M))$ with $(r_1, m_1) \neq (r_2, m_2)$. By the definition of multiplication, we have that $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. Now, we always have $r_1 m_2 + r_2 m_1 \in M$. So, $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1) \in I(+)M$ if and only if $r_1 r_2 \in I$. Thus, $(r_1, m_1) - (r_2, m_2) \in E(\Gamma_{I(+)M}(R(+)M))$ if and only if $(r_1, m_1)(r_2, m_2) \in I(+)M$, if and only if $r_1 r_2 \in I$. □

Corollary 6.6. *Let R be a commutative ring with nonzero identity, and let M be an R-module. Suppose that $(r_1, m_1), (r_2, m_2) \in V(\Gamma_{\{0\}(+)M}(R(+)M))$ with $(r_1, m_1) \neq (r_2, m_2)$. Then, $(r_1, m_1) - (r_2, m_2) \in E(\Gamma_{\{0\}(+)M}(R(+)M))$ if and only if $r_1 r_2 = 0$.*

Proof. Let $I = \{0\}$. Then, $(r_1, m_1) - (r_2, m_2) \in E(\Gamma_{\{0\}(+)M}(R(+)M))$ if and only if $r_1 r_2 \in \{0\}$, by Theorem 6.5. However, this occurs if and only if $r_1 r_2 = 0$. □

While Theorem 6.3 and Theorem 6.5 do not provide much insight into the nature of the ideal-based zero-divisor graph $\Gamma_{I(+)M}(R(+)M)$, they do convey the message that much of the structure of $\Gamma_{I(+)M}(R(+)M)$ is dictated by the structure of $\Gamma_I(R)$. In fact, let us fix $m_0 \in M$. Then, Theorem 6.3 gives us that for every $r \in V(\Gamma_I(R))$, we have that $(r, m_0) \in V(\Gamma_{I(+)M}(R(+)M))$. Let $r_1, r_2 \in V(\Gamma_I(R))$ with $r_1 \neq r_2$. Then, $(r_1, m_0), (r_2, m_0) \in V(\Gamma_{I(+)M}(R(+)M))$ with $(r_1, m_0) \neq (r_2, m_0)$. Furthermore, $r_1 - r_2 \in E(\Gamma_I(R))$ if and only if $r_1 r_2 \in I$. However, Theorem 6.5 gives us that $(r_1, m_0) - (r_2, m_0) \in E(\Gamma_{I(+)M}(R(+)M))$ if and only if $r_1 r_2 \in I$, as well. So, the graph $\Gamma_{I(+)M}(R(+)M)$ contains an isomorphic copy of $\Gamma_I(R)$ as an induced subgraph with vertex set $\{(r, m_0) \mid r \in \Gamma_I(R)\}$. We may think of the this induced subgraph as the m_0 "layer" of $\Gamma_{I(+)M}(R(+)M)$. Furthermore, for $m_1, m_2 \in M$ with $m_1 \neq m_2$, we have that the set of vertices for the m_1 "layer" is disjoint from the set of vertices for the m_2 "layer". Thus, we have $|M|$ disjoint induced subgraphs of $\Gamma_{I(+)M}(R(+)M)$ that are isomorphic to $\Gamma_I(R)$, and we may think of each such subgraph as a "layer" of $\Gamma_{I(+)M}(R(+)M)$.

Now, Theorem 1.3(a) gives us that $\Gamma_{I(+)M}(R(+)M)$ is connected. Thus, these "layers" must be connected to one another, and it turns out that the structure of $\Gamma_I(R)$ gives us most of the information we need to determine how the "layers" are connected to each other. Let $r_1, r_2 \in V(\Gamma_I(R))$ with $r_1 \neq r_2$, and let $m_1, m_2 \in M$ with $m_1 \neq m_2$. Then, $(r_1, m_1), (r_2, m_2) \in V(\Gamma_{I(+)M}(R(+)M))$ by Theorem 6.3. Furthermore, note that (r_1, m_1) is in the m_1 "layer" and (r_2, m_2) is in the m_2 "layer". We have that $(r_1, m_1) - (r_2, m_2) \in E(\Gamma_{I(+)M}(R(+)M))$ if and only if $r_1 r_2 \in I$ by Theorem 6.5, but this is true if and only if $r_1 - r_2 \in E(\Gamma_I(R))$. Finally, the last case we need to consider is when $r_1 = r_2 = r$. It follows from Theorem 6.5 that $(r, m_1) - (r, m_2) \in E(\Gamma_{I(+)M}(R(+)M))$ if and only if $r^2 \in I$, but realize that this information cannot be drawn directly from the structure of $\Gamma_I(R)$.

The following result can be deduced from what we know about the "layered" structure of $\Gamma_{I(+)M}(R(+)M)$.

Theorem 6.7. *Let R be a commutative ring with nonzero identity, I a ring ideal of R, and M a nonzero R-module. Then, $\Gamma_{I(+)M}(R(+)M)$ is complete if and only if $\Gamma_I(R)$ is complete and $r^2 \in I$ for all $r \in V(\Gamma_I(R))$. Moreover, if $\Gamma_I(R) \cong K^n$ and $r^2 \in I$ for all $r \in V(\Gamma_I(R))$, then $\Gamma_{I(+)M}(R(+)M) \cong K^{n|M|}$.*

Proof. Suppose that $\Gamma_{I(+)M}(R(+)M)$ is a complete graph. First note that if $|V(\Gamma_I(R))| = 1$, then $\Gamma_I(R)$ is complete and for the single vertex $r \in V(\Gamma_I(R))$, it must be true that $r^2 \in I$. So, suppose that $r_1, r_2 \in V(\Gamma_I(R))$ with $r_1 \neq r_2$. Fix $m_0 \in M$ so that we have $(r_1, m_0), (r_2, m_0) \in V(\Gamma_{I(+)M}(R(+)M))$ by Theorem 6.3. Furthermore, note that $(r_1, m_0) \neq (r_2, m_0)$. Since $\Gamma_{I(+)M}(R(+)M)$ is complete by assumption, we have that $(r_1, m_0) - (r_2, m_0) \in E(\Gamma_{I(+)M}(R(+)M))$. But then, Theorem 6.5 gives us that $r_1r_2 \in I$. So, $r_1 - r_2 \in E(\Gamma_I(R))$, and we may conclude that $\Gamma_I(R)$ is complete. Now, let $r \in V(\Gamma_I(R))$ and let $m_1, m_2 \in M$ with $m_1 \neq m_2$ so that $(r, m_1) \neq (r, m_2)$. By Theorem 6.3, $(r, m_1), (r, m_2) \in V(\Gamma_{I(+)M}(R(+)M))$. Thus, $(r, m_1) - (r, m_2) \in E(\Gamma_{I(+)M}(R(+)M))$ since $\Gamma_{I(+)M}(R(+)M)$ is complete by assumption. Then, $r^2 = rr \in I$ by Theorem 6.5.

Conversely, suppose that $\Gamma_I(R)$ is complete and $r^2 \in I$ for all $r \in V(\Gamma_I(R))$. Let $(r_1, m_1), (r_2, m_2) \in V(\Gamma_{I(+)M}(R(+)M))$ with $(r_1, m_1) \neq (r_2, m_2)$. Then, $(r_1, m_1), (r_2, m_2) \in V(\Gamma_I(R)) \times M$ by Theorem 6.3. So, $r_1, r_2 \in V(\Gamma_I(R))$. If $r_1 \neq r_2$, then $r_1r_2 \in I$ since $\Gamma_I(R)$ is complete by assumption. Also, if $r_1 = r_2 = r$, then $r_1r_2 = r^2 \in I$ by assumption. So, $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1) \in I(+)M$, and we have that $(r_1, m_1) - (r_2, m_2) \in E(\Gamma_{I(+)M}(R(+)M))$. Thus, $\Gamma_{I(+)M}(R(+)M)$ is complete. □

Chapter 7

Conclusion

7.1 Dissertation Summary

In this dissertation, we defined the congruence-based zero-divisor graph of a commutative ring with nonzero identity as the zero-divisor graph of a particular commutative semigroup with zero. Thus, we were able to immediately deduce various graph-theoretic properties of congruence-based zero-divisor graphs from the corresponding results for semigroup zero-divisor graphs. We also justified that several familiar zero-divisor graphs are examples of our more general congruence-based zero-divisor graph. In particular, for a commutative ring with nonzero identity, the usual zero-divisor graph, the compressed zero-divisor graph, and the ideal-based zero-divisor graph are all examples of congruence-based zero-divisor graphs. Thus, the congruence-based zero-divisor graph generalizes each of these distinct types of zero-divisor graphs. In the final content chapter, we more carefully considered a couple of particular types of ideal-based zero-divisor graphs as a warm-up for research in the area of zero-divisor graphs.

We also investigated various maps between pairs of congruence-based zero-divisor graphs. Some of those maps led to conclusions surrounding isomorphic copies of congruence-based zero-divisor graphs being induced subgraphs of other congruence-based zero-divisor graphs. We developed the relationship between various pairs of familiar multiplicative congruence relations, and those relationships led to maps between familiar examples of congruence-based zero-divisor graphs. Furthermore, some of those maps allowed us to deduce well-known subgraphs of various zero-divisor graphs using our framework.

7.2 Open Questions

For a commutative ring R with nonzero identity and a multiplicative congruence relation $\sim$ on R, the congruence-based zero-divisor graph $\Gamma_\sim(R)$ is the zero-divisor graph of the semigroup $S = R/\sim$. So, we know that every congruence-based zero-divisor graph is a zero-divisor graph of a semigroup with zero. However, we have not actually demonstrated that the concept of a congruence-based zero-divisor graph is distinct from the concept of a semigroup zero-divisor graph.

Question 7.1. *Given a monoid S, when do we have that $\Gamma(S) \cong \Gamma_\sim(R)$ for some ring R and some congruence relation $\sim$ on R?*

Note that $S \cong R/\sim$ implies that $\Gamma(S) \cong \Gamma_\sim(R)$. However, it is certainly possible to have $\Gamma(S) \cong \Gamma_\sim(R)$ while $S \ncong R/\sim$. Thus, a related, but more difficult, question follows.

Question 7.2. *Given a monoid S, when do we have that $S \cong R/\sim$ for some ring R and some congruence relation $\sim$ on R?*

The defined equality $\Gamma_\sim(R) = \Gamma(R/\sim)$, where $R/\sim$ is a commutative semigroup with zero, allowed us to deduce some graph-theoretic properties of $\Gamma_\sim(R)$ without additional work. However, we have not investigated other connections between ring-theoretic properties of R and graph-theoretic properties of $\Gamma_\sim(R)$. The following question points out some graph-theoretic properties that would be worth considering.

Note that it is known when the usual zero-divisor graph and the ideal-based zero-divisor graph are each complete, complete bipartite, a star graph, or planar and when the compressed zero-divisor graph is complete. However, general results have not been established.

Question 7.3. *Let R be a commutative ring with nonzero identity. Under what conditions on R do we have that $\Gamma_{\sim}(R)$ is complete, complete bipartite, a star graph, or planar?*

Finally, for a commutative ring R with nonzero identity, $\mathcal{C}(R)$ is a complete lattice with $inf_{\alpha}\{\sim_{\alpha}\} = \cap_{\alpha}\sim_{\alpha}$. The following question is related to the structure of the set of all congruence-based zero-divisor graphs of R.

Question 7.4. *Let R be a commutative ring with nonzero identity, and let $\{\sim_{\alpha}\}$ be a family of multiplicative congruence relations on R. Suppose that $\sim\, = inf_{\alpha}\{\sim_{\alpha}\}$. How is $\Gamma_{\sim}(R)$ related to each $\Gamma_{\sim_{\alpha}}(R)$? In particular, how is $\Gamma_{\sim_1\cap\sim_2}(R)$ related to $\Gamma_{\sim_1}(R)$ and to $\Gamma_{\sim_2}(R)$?*

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