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Geometry of Scales

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To the Graduate Council:

I am submitting herewith a dissertation written by Kyle Stephen Austin entitled "Geometry of Scales." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Geometry of Scales

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I would like to dedicate this document to my mother and father for giving me so many opportunities that they did not have.

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Life is hard. -Jerzy Dydak

Abstract

The geometry of coverings has widely been used throughout mathematics and it has recently been a promising tool for resolving longstanding problems in topological rigidity such as the Novikov conjecture and Gromov's positive scalar curvature conjecture. We discuss rigidity conjectures and how large scale geometry is being applied in order to resolve them for important cases.

Not only is small scale and large scale geometry very applicable to understanding global geometry of objects, but it is an interesting topic in its own right. The first chapter of this paper is devoted to building a framework for small scale geometry alongside large scale geometry so that the language between the two disciplines is the same. This way, it becomes easier to dualize concepts from one to the other and makes it easier for building bridges between the two.

The last chapter is devoted to my work on large scale n-to-1 functions. These functions have been shown to be canonical in large scale geometry in the sense that there are large scale analogues of the Hurewicz dimension raising theorems as well as an analogue of the theorem which states that an n-dimensional compact space admits a surjective n-to-1 map from the cantor set. My results show generalize some known results by showing that properties such as large scale finitism and metrizability are preserved by such functions.

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Chapter 1

Introduction

The title prompts two questions: What is the geometry of scales and what is it good for? The briefest possible answer to both questions is that scalar geometry is a global geometry and is useful for detecting global results. In this section, I will briefly introduce the ideas behind scales and will then proceed to offer the applications for which I find the most interesting. In particular, I will give some idea as to how large scale geometry is being applied to rigidity conjectures (Novikov conjecture, Borel conjecture, etc.), the nature of volume (amenability, property A, hyperbolicity, etc.), and to the Gromov-Lawson conjecture. I will not mention the applications of large scale geometry to big data analysis and networking problems in computer science.

1.0.1 What is the geometry of scales?

Before introducing the ideas behind the geometry of scales, we must introduce the necessary language involving coverings.

Given a set X, a collection $\mathcal{U}$ of subsets of X, and a subset V of X, the **star** of V with respect to $\mathcal{U}$, written $st(V,\mathcal{U})$, is defined as the union of all elements of $\mathcal{U}$ which intersect V. Given two collections of subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, the star of $\mathcal{V}_1$ with respect to $\mathcal{V}_2$ is the collection $st(\mathcal{V}_1,\mathcal{V}_2) = \{st(V,\mathcal{V}_2) : V \in \mathcal{V}_1\}$.

Given two collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, we say that $\mathcal{U}$ **refines** $\mathcal{V}$, denoted $\mathcal{U} \prec \mathcal{V}$, if for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ such that $U \subseteq V$. In this case we also say that $\mathcal{V}$ **coarsens** $\mathcal{U}$. We say that $\mathcal{U}$ **star refines** $\mathcal{V}$ if $st(\mathcal{U}, \mathcal{U}) \prec \mathcal{V}$.

Let X be a set. A **scale** of X is a cover of X. The motivating, and most essential examples of scales are covers of metric spaces by r-balls for some $r \geq 0$. Denote the cover of a metric space by $r-$balls by $\mathcal{B}_r$. Notice that $st(\mathcal{U}, \mathcal{B}_r) = \{B(U, 2r) : U \in \mathcal{U}\}$, where $B(U, 2r)$ stands for the $2r$ neighborhood of U, for any cover $\mathcal{U}$ in a metric space. In other words, staring a cover $\mathcal{U}$ against another yields a new cover which can be thought of like neighborhoods of $\mathcal{U}$. This is the whole meaning behind the geometry of scales. **If $\mathcal{U}$ star refines $\mathcal{V}$ then $\mathcal{U}$ is like a cover by points and $\mathcal{V}$ is a cover by neighborhoods of the points $\mathcal{U}$.**

- In small scale geometry, one consider collections of scales on X so that each scale can be interpreted as neighborhoods of a smaller scale.
- In large scale geometry, one considers collections of scales on X so that each scale can be intepreted as the points of a bigger scale.

In small scale geometry, one considers the geometry of smaller and smaller covers while, in large scale geometry, one considers the geometry of larger and larger covers.

1.0.2 How small scale geometry is related to topology

Small scale geometry, or uniform topology, has been an object of study since the beginning of the twentieth century. Roughly speaking, uniform structures are global topologies in the sense that they are defined as collections of coverings as opposed to just collections of neighborhoods about particular points. Instead of having neighborhood basis of points, one has neighborhood basis at all points simultaneously. Many concepts are easier to define from a topological point of view (like quotient space) and are very difficult to define in uniform topology. This is one of the reasons why topology became more mainstream and why uniform topology seems less useful.

It is less known that many concepts topologists have long been using, such as compactness, barycentric subdivision, and paracompactness, have origins in uniform category. The following proposition will make this more precise.

Proposition 1.0.1. *A topological Hausdorff space X is paracompact if and only if the collection of open covers of X forms a base for a uniform structure on X, and that uniform structure generates the original topology on X.*

Proof. It is well known, see (Austin and Dydak, 2014) theorem 3.4, that a Hausdorff space is paracompact if and only if every open cover has an open star refinement. ($\Leftarrow$) is then clear.

($\Rightarrow$) The set of all open covers of X satisfies condition 2 in the definition of ss-structure. We can take the ss-structure consisting of all the open covers and their coarsenings. One can readily see that this ss-structure generates the topology. $\square$

Proposition 1.0.1 reaffirms the idea that small scale structures are an ideal setting for global topology because much of the theory for paracompact spaces (which are arguably the best globally behaved spaces) coincides with the theory of the small scale structure in proposition 1.0.1 that generates its topology.

1.0.3 Applications of Large Scale Geometry

Gromov and Gromov-Lawson Conjectures

To me, Gromov's conjecture is one of the most interesting applications of large scale geometry. Even though it is similar to the rigidity questions in the next subsection, the main idea behind the conjecture is quite simple and does not require any knowledge of L-theory or analytical k-theories and so makes a nice starting point for getting a grasp on the appplications of coarse geometry.

Recall that the scalar curvature of a manifold M at a point $x \in M$, denoted $sc_M(x)$ is the sum of the section curvatures running over the two planes (in a chosen basis) in the tangent space T_xM at x. A manifold has positive scalar curvature if $sc_M(x) > 0$ for all $x \in M$; having positive scalar curvature is a global property. One similarly defines a manifold of negative (non-positive or non-negative) scalar curvature. It follows easily from the definition of scalar curvature that $sc_{M\times N}(x,y) = sc_M(x) + sc_N(y)$. Here is the motivating example.

Let M be a compact manifold and S^2 be the $2-$sphere. The curvature of S^2 can be made arbitrarily large by assuming it has arbitrarily small diameter. In particular, one can make the curvature of S^2 greater than $\max\{sc_M(x) : x \in X\}$. Once this is done, it follows from the additivity of curvature under taking products that $M \times S^2$ can be made to have positive scalar curvature. Gromov conjectured that this was essentially how one can obtain a metric of positive scalar curvature.

In order to state Gromov's conjecture, we need to introduce the notion of macroscopic dimension. We say that a map $f : X \to K$ where X is metric and K is a simplicial complex is **cobounded** if there exists some $R > 0$ such that $diam(f^{-1}(y)) < R$ for all $y \in K$. A metric space X has **macroscopic dimension** at most n, denoted $mc(X) \leq n$, if X admits a cobounded map to a simplicial complex of dimension n. It is not too difficult to show that macrosocpic dimension is bounded above by both covering and asymptotic dimension. Roughly put, the macroscopic dimension of a metric space is the number of dimensions that can be seen as one travels uniformly outward from the given space. In particular, the macroscopic dimension of any bounded space is 0 because the constant map to the 0 simplex is a cobounded map.

Notice that the universal covering of $M \times S^2$ is $\tilde{M} \times S^2$ where $\tilde{M}$ is the universal covering of M. It is easy to show that $mc(\tilde{M} \times S^2) = mc(\tilde{M})$ which is at least two less that the dimension of $\tilde{M} \times S^2$. This calculation is at the heart of Gromov's conjecture.

Conjecture 1.0.2 (Gromov's Positive Scalar Curvature Conjecture)**.** *If M be an n dimensional manifold of positive scalar curvature then $mc(\tilde{M}) \leq n-2$ where $\tilde{M}$ is the universal covering of M.*

So....How do we use the above conjecture, if true, to conclude anything about manifolds? It turns out that an affirmative to the above conjecture would yield an affirmative to the following conjecture of Gromov and Lawson.

Conjecture 1.0.3 (Gromov-Lawson Conjecture)**.** *A closed aspherical manifold M cannot carry a metric of positive scalar curvature.*

Here is how one obtains conjecture 1.0.3 from conjecture 1.0.2. A metric space X is said to be **uniformly contractible** if for every $r > 0$ there exists some $s > r$ such that $B(x, r)$ is contractible inside of $B(x, s)$ for every $x \in X$. One can show that uniformly contractible manifolds of dimension n have macroscopic dimenison equal to n. It is not too difficult to show that the universal covering of a closed aspherical n manifold M is uniformly contractible and hence must have macroscopic dimension equal to n. By conjecture 1.0.2, this cannot happen if M has positive scalar curvature.

Rigidity

One of the most fundamental problems in manifold theory (I am taking the use of the word manifold very broadly) is how much information about a given manifold M be determined from $\pi_1(M)$? More precisely,

Question 1.0.4. *Under what conditions on a manifold M and N does a homotopy equivalence between M and N imply that there exists a homeomorphism (diffeomorphism or otherwise) between M and N?*

A question of the kind given in 1.0.4 is known as a rigidity question and there are many remarkable results and famous unknown problems in rigidity. One of the first well known results in rigidity theory is the so called **Mostow Rigidity Theorem** which asserts that an isomorphism $\pi_1(M) \cong \pi_1(N)$ between two complete hyperbolic n manifolds M and N with finite volume is induced by a unique isometry (it is worth noting that hyperbolicity is a quasi-isometry invariant and, therefore, a large scale concept). This is especially powerful since this theorem translates to the statement that any smooth structure on such manifolds is totally determined by the homotopy type of the underlying space.

At nearly the same time in history, Borel posed his famous problem which is known as the Borel conjecture. The Borel conjecture is stated as follows: A homotopy equivalence f between closed aspherical topological manifolds X and Y is homotopic to a homeomorphism. In other words, the homotopy type of closed aspherical topological manifolds completely determines the topology. This conjecture has been open for over 60 years and has seen some partial results. For example, the stable Borel conjecture states that if f is a homotopy equivalence between closed aspherical topological n-manifolds X and Y then the induced map $\tilde{f} : X \times \mathbb{R}^n \to Y \times \mathbb{R}^n$ is homotopic to a homeomorphism. E. Guentner, R. Tessera, and G. Yu proved in (E. Guentner and Yu, 2012) that if X and Y have fundamental groups which have a large scale property called finite decomposition complexity then the stable Borel conjecture is true for X and Y.

Perhaps one of the most famous conjectures on the subject of rigidity is the so called Novikov conjecture which states that certain polynomials in the Pontryagin classes of the tangent bundle are homotopy invariants. The Novikov conjecture is actually intimately related to the Borel conjecture via the Baum-Connes conjecture, see (Roe, 1996). Two notable cases in which the conjecture has been proven for are the manifolds whose fundamental group has finite asymptotic dimension in (Yu, 2000) and, more generally, when the fundamental group of the manifold embeds coarsely into a Hilbert space, see (Yu, 2000).

The Baum-Connes conjecture is stronger than the Novikov or the Borel conjecture in the sense that they would be resolved if the Baum-Connes conjecture is answered in the affirmative. Unfortunately however, counterexamples do exist for the Baum-Connes conjecture. The conjecture states that for a locally compact Hausdorff and second countable group Γ the assembly map $\mu : KH_*^\Gamma(\underline{EG}) \to K_*(C_r^*(G))$ is an isomorphism where the left hand side is the K-homology of the classifying space for proper actions $\underline{EG}$ of G and the right hand side is the C^*-algebra K-theory for the reduced group C^*-algebra of Γ. The map μ is a generalization of the concept of analytical index for Fredholm operators and is known as an assembly map.

A promising method of resolving the conjecture is by perturbing it and resolving the coarse Baum-Connes conjecture. The idea comes from coarsening homology theories using anti-Čech approximations (or Vietoris-Rips complex approximations). Let H be a generalized homology theory on the category of metric spaces (Our constructions can be generalized, see below for large scale structures). For $r \geq 0$, we define the **Rips complex at scale** r of a metric space X to be the simplicial complex $P_r(X)$ whose 0-skeleton is X and a finite subset $\{x_1, x_2, \ldots, x_n\}$ span a simplex if $diam(\{x_1, x_2, \ldots, x_n\}) \leq r$. It is clear that if $s > r$, then we have an inclusion $P_r(X) \hookrightarrow P_s(X)$. We define the **coarse homology** of X, denoted by $RH(X)$, as $\varinjlim_{s \to \infty} H(P_s(X))$. For uniformly contractible spaces, like universal coverings of manifolds, we have an isomorphism $RH(X) \cong H(X)$ and therefore the process of taking coarse approximations is canonical for a large class of spaces.

Coarsening homology theories have been applied successfully to prove the Baum-Connes conjecture for uniformly contractible spaces with finite asymptotic dimension in (Yu, 1998) and for those groups which embed coarsely into the Hilbert space in (Yu, 2000), both of which are celebrated papers.

It is known that there exists an $s > 0$ such that $N_{r/s}(X) \subset P_r(X) \subset N_{rs}(X)$ where $N_t(X)$ is the nerve of the covering of X by t-balls. This means that we are really looking at simplicial approximations with respect to larger and larger coverings of a space and it therefore makes sense to understand the geometry of large coverings of a space.

Rigidity problems are a good example of the kinds of things one can apply uniform topology and coarse geometry. These geometries detect global invariants and, with some minor exceptions in uniform topology, are designed to ignore much local behavior. What are these so called global invariants? As mentioned in the introduction to the paper (Kyle Austin and Holloway, 2015), one fundamental such invariant is the nature of volume in an object.

The Nature of Volume

An essential concept throughout many different branches of mathematics is amenability, a quasi-isometry invariant. One of the first examples in which amenability lurks in the background is the Banach-Tarski paradox that says that a solid ball can be broken up into finitely many pieces and those pieces can be rearranged to form two balls of the same original size. The big idea behind the proof of the paradox is that the free group on two generators naturally doubles and this doubling induces doubling of the sphere in which it acts as rotations. It is now well known, (Runde, 2002), that a group G is amenable if it does not double; more precisely, there does not exist disjoint subsets $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \subset G$ and elements $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in G$ such that $\bigcup_{i=1}^{n} a_i \cdot A_i = \bigcup_{j=1}^{m} b_j \cdot B_j = G$.

Another way to study the volume of a group (or metric space) is to look at the volume growth. I will outline the idea for finitely generated groups with metric induced by the Cayley graph metric just to sketch the big picture and make notice of a few highlights. For a finitely generated group, let $v : [0, \infty) \to [0, \infty)$ be the function defined by $v(r) = |B(1_G, r)|$

where || denotes the cardinality and 1_G denotes the identity element G. One can show that if this function is subexponential, then the group in question must be amenable! Another famous application of volume functions is due to Gromov (Gromov, 1981). He shows that a group has at most polynomial growth (v is bounded above by a polynomial function) if and only if the group is vitually nilpotent (has a finite index nilpotent subgroup and so is large scale equivalent to a nilpotent group). Some sharper results are known, but Gromov's polynomial growth theorem is a landmark in the theory of growth functions.

I suggest the reader look at the introduction to (Kyle Austin and Holloway, 2015) for a detailed explanation of more large scale invariants related to volume.

Chapter 2

Large and Small Scale Geometry

2.1 Scale Structures

2.1.1 An Introduction to Scales

One goal of this paper is to emphasize the concept of scale. In a metric space, the distance function gives a natural way to measure the scale of a cover. Namely, one can say how "thick" a cover is by using the Lebesgue number. Recall that the Lebesgue number of a cover is the supremum of the set of real numbers λ such that the collection of $\lambda/2$ balls of the space refines that cover. Equivalently, the Lebesgue number of the cover $\mathcal{U}$ is defined by

$$Leb(\mathcal{U}) = \sup\{\lambda > 0 : \forall x \in X \exists U \in \mathcal{U} \text{ such that } B(x, \lambda/2) \subseteq U\}.$$

From this point of view, the Lebesgue number lemma says that all covers of a compact metric space have positive thickness. The Lebesgue number is a small scale concept. Dual to the Lebesgue number is the large scale notion of the mesh of a cover. The mesh of a cover is the infimum of the real numbers M such that the cover refines a cover by $M/2$ balls. Notice that this is exactly dual to the notion of Lebesgue number, as can be seen by defining mesh as

$$mesh(\mathcal{U}) = \inf\{\lambda > 0 : \forall U \in \mathcal{U} \exists x \in X \text{ such that } U \subseteq B(x, \lambda/2)\}.$$

The Lebesgue number and the mesh of a cover both quantify the scale of that cover, and both are determined by comparing covers using refinement. Small scale and large scale structures on a space give a way to extend this notion of using refinements to determine scale to a more general class of spaces.

Covers give a way of determining scales in a metric space, so we generalize this notion by using the term **scale** for any cover of a space. A **scale structure** on X is a collection of scales of X. Closely related to scales are **entourages**. For a space X, an entourage is a subset of $X \times X$ containing the diagonal, which is the set $\{(x, x) : x \in X\}$. A collection of entourages forms an **entourage structure**.

For a small scale or large scale structure, we need not only scales but also a way to compare scales. For example, in the small scale, one should be able to "zoom in" to the space by going to a smaller scale, and in the large scale one should be able to "zoom out" to a larger scale. To capture this idea, we use the relation of star refinement. We say that $\mathcal{V}$ is a larger scale than $\mathcal{U}$ if $\mathcal{U}$ star refines $\mathcal{V}$ and that $\mathcal{V}$ is a smaller scale if $\mathcal{V}$ star refines $\mathcal{U}$. Every scale structure on a space can be made into a partially ordered set by using the relation of star refinement. That is, $\mathcal{U} \leq \mathcal{V}$ if and only if $st(\mathcal{U},\mathcal{U}) \prec \mathcal{V}$.

If $(P, \leq)$ is a poset, then a **filter** of P is a subset F of P which satisfies two conditions. First, if $x, y \in F$, then there exists $z \in F$ such that $z \leq x$ and $z \leq y$. Second, if $a \in F$ and $b \in P$ such that $a \leq b$, then $b \in F$. A subset B satisfying only the first condition is called a **filter base** and can be made into a filter by adding all elements $y \in P$ such that $x \leq y$ for some $x \in B$.

A small scale structure will be a scale structure in which one can always zoom into the space and view it from smaller and smaller scales. Using the language of filters, an **small scale structure** is a filter of the set of all covers of a space, ordered by star refinement, and a base for a small scale structure is a filter base of this poset. Dually, a large scale structure on a space is a scale structure which allows one to zoom out and view the space from farther and farther away. That is, a **large scale structure** is a filter of the set of all covers of a space, ordered by reverse star refinement, and a base for a large scale structure is a filter base of this poset (see Subsection 2.3.2 for a concrete description of bases for large and small scale). Another way to define these is to say that a small scale structure is a scale structure in which one can always decrease scale and a large scale structure is one in which one can always increase scale. In the remainder of this section, we shall first introduce the most important examples of scale structures, which are metric structures and translation structures and then we will introduce the general definitions of small and large scale structures.

One can similarly define an order relation on entourages using composition of entourages, see section 2.1.5 or J. Dydak (2008) for instance. That leads to the coarse structure in the sense of Roe Roe (2003). We will be doing most of our work with coverings. It is already known that the entourage approach and covering approach lead to isomorphic categories, see J. Dydak (2008).

2.1.2 Metric Scale Structures

Let (X, d) be a metric space. The metric naturally induces a scale structure by taking as scales the collections $\mathcal{B}_r = \{B(x, r) : x \in X\}$ for $r > 0$. This structure is perhaps the most important scale structure; see section 2.4. A metric small scale structure includes all $\mathcal{B}_r$ along with any cover which coarsens a $\mathcal{B}_r$. Thus, this structure contains all covers with positive Lebesgue number. Notice that these covers can become arbitrarily large, so long as they have large overlap. From the small scale point of view, all that is important is that the cover has some thickness, not that it is bounded.

A metric large scale structure consists of all $\mathcal{B}_r$ along with all collections of subsets which refine a $\mathcal{B}_r$, whether or not this collection is a cover. The idea is that anything which is smaller than a uniformly bounded collection is uniformly bounded. Notice that these collections can become arbitrarily thin, because from the large scale point of view, the only thing important is that there is a bound on the mesh of the cover.

2.1.3 Translation Structures

Let G be a group. There are two well known ways of creating invariant scale structures on G, both of which are induced by translations. The first corresponds to the uniform (small scale) structure on G, and the other is a generalization of the coarse (large scale) structure induced by the Cayey graph of G.

If G is a topological group then there exists a neighborhood base of the identity element, $\{U_\alpha : \alpha \in A\}$, where A is directed by square inclusion; i.e., $\alpha \geq \beta$ if and only if $U_\alpha \cdot U_\alpha \subset U_\beta$. This allows us to create a small scale structure on G by declaring the scales of G to be $\mathcal{U}_\alpha = \{gU_\alpha : g \in G\}$.

If G is a locally compact topological group, then considering scales of G of the form $\mathcal{U}_\alpha = \{gU_\alpha : g \in G\}$, where U_α is a pre-compact (i.e. $cl(U_\alpha)$ is compact) neighborhood of 1_G yields a large scale structure on G. If G is any group, then we can put the discrete topology on it and the above large scale structure on G has scales given by the collections $\mathcal{U}_F = \{gF : g \in G\}$ where F ranges over the finite subsets of G. It is known, see J. Dydak (2008) and Nikolay Brodskiy and Mitra (2007), that this scale structure is large scale equivalent to the metric scale structure induced by the Cayley graph in the case that G is a finitely generated group. The advantage of this approach, however, is that one does not need to define a Cayley graph or restrict to countable groups in order to use proper metrics.

The advantage of the above generalization becomes clear when one takes into consideration the object in which it is generalizing, the Cayley graph metric. The Cayley graph metric

It is notable that the concept of translation enables one to extend the methods of geometric group theory to metric spaces. The translation algebra (or Uniform Roe Algebra) of a metric space generalizes the action of a group on its space of characters and allows one to do representation theory and Fourier analysis on metric spaces. We view translations as a fundamental tool for connecting concepts.

The above approach to scales on a group G can be generalized to devise a general scheme of switching between scales on a set X and entourages, i.e. subsets of $X \times X$ containing the diagonal. Namely, given an entourage E in X, thought of as a neighborhood of the identity function on X, one can create its translates $g \circ E$, $g : X \to X$, and that leads to a scale on X^X. In turn, that scale, when restricted to X^*, gives a scale on X. Conversely, given a scale $\mathcal{U}$ on X, one can consider the entourage $\bigcup\limits_{U \in \mathcal{U}} U \times U$.

2.1.4 Small and Large Scale Structures

The general theme is that in small scale geometry, one zooms inward on a space through star refining covers, while in large scale geometry, one works outward on a space through star coarsening covers.

Let's recall basic definitions from the theory of uniform spaces. In order to exhibit the duality between large and small scales we will adjust the terminology accordingly.

Definition 2.1.1. A **small scale structure** (ss-structure for short) on a set X is a filter $\mathcal{SS}$ of scales on X whose elements are traditionally called **uniform covers**. In other words:

1) if $\mathcal{U}_1 \in \mathcal{SS}$ and $\mathcal{U}_2$ is a cover of X which coarsens $\mathcal{U}_1$, then $\mathcal{U}_2 \in \mathcal{SS}$;

2) if $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{SS}$ then there exists $\mathcal{U}_3 \in \mathcal{SS}$ which star refines both $\mathcal{U}_1$ and $\mathcal{U}_2$.

We say that a small scale structure $\mathcal{SS}$ is **Hausdorff** if for each $x, y \in X$ there exists a uniform cover $\mathcal{U} \in \mathcal{SS}$ which has no set containing both x and y.

Each pseudometric space (X, p) induces a small scale structure (called the **metric small scale structure**) that consists of all covers of X which coarsen a cover of the form $\{B(x, r) : x \in X\}$ for some $r > 0$, where $B(x, r) = \{y \in X : p(x, y) < r\}$. Notice that the metric small scale structure is Hausdorff if and only if p is a metric.

A function $f : X \to Y$ where X and Y are spaces with ss-structures is **small scale continuous** (ss-continuous for short) if for any uniform cover $\mathcal{V}$ of Y there is a uniform cover $\mathcal{U}$ of X such that $\{f(U) : U \in \mathcal{U}\} \prec \mathcal{V}$. Equivalently, the inverse image of a uniform cover of Y is a uniform cover of X.

Every small scale structure $\mathcal{SS}$ induces a topology on its underlying set as follows: a subset $U \subseteq X$ is open provided for each $x \in U$ there is a uniform cover $\mathcal{V} \in \mathcal{SS}$ such that $st(\{x\}, \mathcal{V}) \subset U$. A topological space is said to be **uniformizable** if there exists a small scale structure on the space which induces the topology of the space. It is well known that Hausdorff small scale structures induce a topology on their underlying space which is completely regular. Conversely, if X is a completely regular topological space, then one can define a small scale structure $\mathcal{SS}$ on X which generates its topology by taking as uniform covers all collections $\{f^{-1}(V) : V \in \mathcal{V}\}$ where $\mathcal{V}$ ranges over all uniform covers $\mathcal{V}$ of $\mathbb{R}$ with the metric small scale structure and f ranges over all continuous funtions from X to $\mathbb{R}$. Note that $\mathcal{SS}$ is contained in every ss-structure in which all continuous functions $f : X \to \mathbb{R}$ are ss-continuous.

It is possible for a topology on a set to be induced by different small scale structures. For example, the metric topology on $\mathbb{R}$ is induced by both the metric small scale structure and by the uniform structure whose base consists of all open covers of $\mathbb{R}$ (see Proposition 1.0.1 to see that this is actually a small scale structure). These structures are not the same since every cover in the first has positive Lebesgue number, while covers in the second can become arbitrarily thin. However, for a compact space, there is a unique uniformity which generates its topology. For metric spaces, having a unique small scale structure is equivalent to being compact, but in general, having a unique structure only implies that the space is locally compact; see Doss (1949). The long line is a good example of a completely regular space with unique uniform structure inducing its topology since it has only one compactification.

Proposition 2.1.2. *Let $(X, \mathcal{T})$ be a topological space and let X_S be a uniform structure on the underlying set X. If the interiors of each uniform cover is an open cover of X, then the topology $\mathcal{S}$ induced by the uniformity is contained in $\mathcal{T}$.*

Proof. Given an open set U in $(X, \mathcal{S})$ and given $x \in U$, there is a scale $\mathcal{V}$ in X_S such that $st(x, \mathcal{V}) \subset U$. Therefore $st(x, int(\mathcal{V})) \subset U$ and U is open in $(X, \mathcal{T})$. That means $id : (X, \mathcal{T}) \to (X, \mathcal{S})$ is continuous. □

The following definition of a large scale structure is a dualization of the definition of a small scale structure on a set.

Definition 2.1.3. J. Dydak (2008) A **large scale structure** on a set X (ls-structure for short) is a nonempty filter of scales on X (in the order being the reverse star refinement) along with all refinements (not necessarily covers) of those scales.

Alternatively, it is a collection $\mathcal{LS}$ of families of subsets of X satisfying the following properties:

1) $\mathcal{B}_1 \in \mathcal{LS}$ implies $\mathcal{B}_2 \in \mathcal{LS}$ if each nonsingleton element of $\mathcal{B}_2$ is contained in some element of $\mathcal{B}_1$;

2) if $\mathcal{B}_1$, $\mathcal{B}_2 \in \mathcal{LS}$ then $st(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LS}$.

If $\mathcal{B}_\infty \in \mathcal{LS}$, then we say that $\mathcal{B}_\infty$ is **uniformly bounded**.

Given a uniformly bounded family $\mathcal{B} \in \mathcal{LS}$ we define the **trivial extension** of $\mathcal{B}$ to be $\mathcal{B} \cup \{\{x\} : x \in X\}$. By 1 above, the trivial extension of any uniformly bounded family is also uniformly bounded, so although a uniformly bounded family may not be a cover of the underlying space, any such family may be extended to be a cover, i.e. a scale.

If (X, d) is an ∞-metric space, then the metric large scale structure $\mathcal{M}$ on X is defined by $\mathcal{B} \in \mathcal{M}$ if and only if there is $M > 0$ such that all elements of $\mathcal{B}$ are of diameter less than M.

A map $f : X \to Y$ between spaces with large scale structures is called **large scale continuous** (or **bornologous**) if the image of any uniformly bounded collection in X is uniformly bounded in Y.

Remark 2.1.4. Given any family of scale structures we can consider their union or their intersection. That quickly leads to the concepts of smallest or largest scale structures satisfying certain conditions.

2.1.5 Entourages Approach To Scale Structures

One may also define structures on set X using subsets of the product space $X \times X$.

For a set X, the **diagonal** of X is defined to be $\Delta = \{(x, x) : x \in X\}$. For a subset $U \subseteq X \times X$, the **inverse** of U is defined to be $U^{-1} = \{(y, x) : (x, y) \in U\}$. For two sets $U, V \subseteq X \times X$, the **composition** (or the **product**) of U and V is defined as $U \circ V = \{(x, z) \mid (x, y) \in U \text{ and } (y, z) \in V \text{ for some } y \in X\}$. For a set $E \subseteq X \times X$ and $x \in X$, let $E[x] = \{y \in X : (y, x) \in E\}$.

Definition 2.1.5. A **uniform structure** on a set X is a collection $\mathcal{U}$ of subsets of $X \times X$ satisfying

1) $\Delta \subseteq U$ for all $U \in \mathcal{U}$;

2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

3) for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;

4) if $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$;

5) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

The elements of a uniform structure are called **entourages**.

For a pseudometric space (X, p), the metric uniform structure consists of all sets $E \subseteq X \times X$ which contain a set of the form $\{(a, b) : p(a, b) < r\}$ for some $r > 0$.

Definition 2.1.6. Roe (2003) A **coarse structure** on a set X is a collection $\mathcal{U}$ of subsets of $X \times X$ satisfying

1) $\Delta \in \mathcal{U}$;

2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

3) if $U, V \in \mathcal{U}$, then $U \circ V \in \mathcal{U}$;

4) if $U \in \mathcal{U}$ and $V \subseteq U$, then $V \in \mathcal{U}$;

5) if $U, V \in \mathcal{U}$, then $U \cup V \in \mathcal{U}$.

The elements of a coarse structure are called **controlled sets** or, more generally, **entourages** depending on the emphasis needed.

For an ∞-metric space (X, d), the metric coarse structure consists of all $E \subseteq X \times X$ such that $\sup\{d(x, y) : (x, y) \in E\} < M$ for some $M > 0$. The metric coarse structure is also called the **bounded coarse structure** associated to d.

It is known, see J. Dydak (2008), that coarse structures and ls-structures are equivalent concepts. We outline in the next section how one can visualize the equivalence through function spaces.

In terms of entourages, we define a small scale entourage base as a collection $\mathcal{B}$ of symmetric entourages such that if $E, F \in \mathcal{B}$, then there exists $G \in \mathcal{B}$ such that $G \circ G \subseteq E \cap F$. To make a small scale entourage base into a uniform structure, add all supersets of elements of $\mathcal{B}$. Similarly, a large scale entrourage base is a collection $\mathcal{B}$ of symmetric entourages such that if $E, F \in \mathcal{B}$, then there exists $G \in \mathcal{B}$ such that $E \circ F \subseteq G$. To make a large scale entourage base into a coarse structure, add all subsets of elements of $\mathcal{B}$.

2.2 Categorical Constructions

The small scale category has objects which are spaces with small scale structures and whose morphisms are small scale continuous maps. Dually, the large scale category is the category whose objects are spaces with large scale structures and whose morphisms are large scale continuous maps. In this section we show that these categories have products and coproducts for any collection of objects. We also consider several structures on function spaces, and construct structures which allow one to connect the product and coproduct structures.

2.2.1 Products and Coproducts

Motivated by the topological structures on the product of topological spaces, we define two small scale structures on $\prod_{\alpha \in A} X_\alpha$

Definition 2.2.1. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of ss-spaces. The **box small scale structure** is the ss-space which has a base of uniform covers as follows: choose uniform covers $\mathcal{U}_\alpha$ of X_α and consider all products $\prod_{\alpha \in A} U_\alpha$, where $U_\alpha \in \mathcal{U}_\alpha$.

The **product small scale structure** is the ss-space which has a base of uniform covers as follows: choose uniform covers $\mathcal{U}_\alpha$ of X_α such for all but finitely many $\alpha \in A$ one has $\mathcal{U}_\alpha = \{X_\alpha\}$, and consider all products $\prod_{\alpha \in A} U_\alpha$, where $U_\alpha \in \mathcal{U}_\alpha$.

Similar to the topological case, the product structure has the advantage over the box structure in that the product structure is a categorical product in the category of ss spaces. That is, given a collection of maps $f_\alpha : X \to X_\alpha$, there is a unique small scale continuous map $f : X \to \prod_{\alpha \in A} X_\alpha$ such that $f_\alpha = \pi_\alpha \circ f$ for each α. In some sense, the box small scale structure has too many covers to be a product. For example, if $X_n = \mathbb{R}$ for $n \geq 1$ and $f_n : \mathbb{R} \to X_n$ is the identity map for each n, then this gives a map $f : \mathbb{R} \to \prod_{n \geq 1} X_n$ defined by $f(x) = (x, x, x, \ldots)$. However, this map is not uniformly continuous when $\prod_{n \geq 1} X_n$ is considered with the box small scale structure since the inverse image of the cover whose element are the products of balls of radius $1/n$ in the nth component consists of just single points of $\mathbb{R}$.

The category of ss spaces also has coproducts; namely,

Definition 2.2.2. the disjoint union. Define the **disjoint union small scale structure** on $\coprod_{\alpha \in A} X_\alpha$ by letting a base be given by all covers of the form $\coprod_{\alpha \in A} \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is a uniform cover of X_α for each $\alpha \in A$. That is, the uniform covers consist of the above covers and all coarsenings of such covers.

Definition 2.2.3. For large scale structures $\{X_\alpha\}_{\alpha\in A}$, define the **product large scale structure** on $\prod_{\alpha\in A} X_\alpha$ by letting uniformly bounded covers be those covers $\mathcal{U}$ such that $\{\pi_\alpha(U) : U \in \mathcal{U}\}$ is uniformly bounded in X_α for each α.

This structure is indeed a categorical product in the category of ls spaces and bornologous maps.

We can define two different ls structures on the disjoint union of the X_α.

Definition 2.2.4. The **box disjoint union large scale structure** has covers of the form $\coprod_{\alpha\in A} \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is a uniform cover of X_α for each $\alpha \in A$.

However, this structure is not a coproduct in the category of ls spaces. If $X_n = \mathbb{R}$ for $n \geq 1$ and $f_n : X_n \to \mathbb{R}$ is the identity map for each n, then then induced map $f : \coprod_{n\geq 1} X_n \to \mathbb{R}$ is not bornologous since the image of $\coprod_{n\geq 1}\{B(x, n) \mid x \in X_n\}$ is not uniformly bounded in $\mathbb{R}$.

To get a coproduct, we define a new structure, analogous to the product small scale structure.

Definition 2.2.5. Define the **disjoint union large scale structure** on $\coprod_{\alpha\in A} X_\alpha$ by letting uniformly bounded covers be those covers of the form $\coprod_{\alpha\in A} \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is a uniformly bounded cover of X_α for each $\alpha \in A$ and for all but finitely many of $\alpha \in A$ the $\mathcal{U}_\alpha$ consist of only singletons.

Since starring together finitely many uniformly bounded covers gives a uniformly bounded cover of a large scale space, it follows that this is a coproduct in the category of ls spaces and bornologous maps.

Notice that in the large scale case we did not need to restrict the covers in the product structure in order to obtain a categorical product. On the other hand, to form a coproduct, we cannot mimic the construction for the small scale coproduct, but must put a restriction on the uniformly bounded families.

In the case of a collection of metric spaces $\{(X_\alpha, d_\alpha)\}_{\alpha\in A}$, we can also construct a **metric disjoint union** by defining an ∞-metric on $\coprod_{\alpha\in A} X_\alpha$ as follows. Let $d(x, y) = d_\alpha(x, y)$ if $x, y \in X_\alpha$ and $d(x, y) = \infty$ if $x \in X_\alpha$ and $y \in X_\beta$ where $\alpha \neq \beta$. To define the scale structure on the disjoint union, one simply takes the metric coarse or uniform structure induced by this ∞-metric. In the small scale, this structure is contained in the disjoint union structure. In the large scale, this structure sits between the box disjoint union and the disjoint union structures. Hence for an infinite collection of metric spaces, this is not a coproduct for either the large or small scale case.

2.2.2 Function Spaces

Given a scale structure on a set X and given any set Y, we define three different structures on X^Y.

Our motivation for this section come from how one puts a ss-structure on a topological group. Let G be a topological group and 1_G be the identity of G. One can equip an ss-structure on G by taking a neighborhood basis $\mathcal{U}$ of 1_G and transporting the neighborhoods of the identity to get covers of G. More precisely, one takes the covers of the form $\{gU : g \in G\}$ where $U \in \mathcal{U}$ as a base for the ss-structure. Our idea is that X^X is like a group and entourages (in both small and large scale) act like neighborhoods of the identity in X^X. One should aim to define covers on X^X in the same way that one obtains them for the case of topological groups.

The first structure we define starts with an entourage structure on X and gives a scale structure on X^Y which is an analogue of the uniform convergence topology. Consider two sets X and Y. Let E be a subset of $X \times X$ and let f be a function mapping Y to X. Define a subset $N(f, E)$ of X^Y by

$$N(f, E) = \{g : Y \to X \mid (f(y), g(y)) \in E \text{ for all } y \in Y\},$$

and let $\mathcal{U}_E$ be the collection

$$\mathcal{U}_E = \{N(f, E) \mid f \in X^Y\}.$$

Notice that if E contains the diagonal of $X \times X$, then $\mathcal{U}_E$ forms a cover of X^Y.

The following proposition follows immediately from the definitions.

Proposition 2.2.6. *Let E and F be subsets of $X \times X$, and Y be any set.*

1) If $E \subseteq F$, then $\mathcal{U}_E \prec \mathcal{U}_F$.

2) $\mathcal{U}_E \cup \mathcal{U}_F \prec \mathcal{U}_{E\cup F}$

3) $\mathcal{U}_E \cap \mathcal{U}_F \prec \mathcal{U}_{E\cap F}$

4) If $G \subseteq X \times X$ *such that* $G \circ G \circ G \circ G \subset E \cap F$*, then* $st(\mathcal{U}_G, \mathcal{U}_G) \prec \mathcal{U}_E \cap \mathcal{U}_F$

5) $st(\mathcal{U}_E, \mathcal{U}_F) \prec \mathcal{U}_{E\circ F\circ F}$*.*

Corollary 2.2.7. *Let* X *be a set with a uniform structure and let* Y *be any set. The collection of covers of* X^Y *of the form* $\mathcal{U}_E = \{N(f, E) : f \in X^Y\}$*, where* E *ranges over the entourages of* X*, forms a base for a small scale structure on* X^Y*. In particular, a uniform structure on* X *induces an ss-structure on* $X^{\{pt\}} = X$*.*

Corollary 2.2.8. *Let* X *be a set with a coarse structure and let* Y *be any set. The collection of families of subsets of* X^Y *of the form* $\mathcal{U}_E = \{N(f, E) : f \in X^Y\}$*, where* E *ranges over the entourages of* X*, forms a base for an ls-structure on* X^Y*. In particular, a coarse structure on* X *induces an ls-structure on* $X^{\{pt\}} = X$*.*

Recall that for any metric space (X, d) and Y any set, the uniform convergence ∞-metric on X^Y is given by $\hat{d}(f, g) = \sup\{d(f(y), g(y)) : y \in Y\}$. One can see that the ss-structure generated in Corollary 2.2.7 is the same as the metric ss-structure on X^Y generated by the uniform convergence metric. Thus, the above construction can be viewed as a generalization of the uniform convergence topology on the function space X^Y. It is also true that the ls-structure from Corollary 2.2.8 is the same as the ls-structure generated by the uniform convergence metric.

Proposition 2.2.9. *Let* X *be a set with a coarse structure. The evaluation function* $e : X^X \times X \to X$ *is bornologous where* $X^X \times X$ *is given the product ls-structure.*

Proof. Let $\mathcal{U}$ be a uniformly bounded cover of X^X, $\mathcal{V}$ be a uniformly bounded cover of X, and let E be a controlled set of X such that $\mathcal{U}_E = \{N(f, E) : f \in X^Y\}$ is a coarsening of $\mathcal{U}$. Notice that $\{e(V, U) : V \in \mathcal{V}$ and $U \in \mathcal{U}_E\} = \{\{g(U) : g \in N(f, E)\} : f \in X^X$ and $U \in \mathcal{V}\}$ Notice that $g(U)$ is the projection onto the second coordinate, denote the projection by π_2, of the set $U \times X \cap E$ which means that $\{\{g(U) : g \in N(f, E)\} : f \in X^X$ and $U \in \mathcal{V}\} = \{\pi_2(U \times X \cap E) : U \in \mathcal{V}\}$ which refines the cover induced by E and the cover $\mathcal{U}$ and must therefore be uniformly bounded. □

It is also possible to start with a scale structure on X and obtain an entourage structure on X^Y. Suppose that $\mathcal{U}$ is a scale of X. Define $E_{\mathcal{U}} = \{(f,g) \in X^Y \mid \forall y \in Y\ \exists U_y \in \mathcal{U}$ such that $f(y), g(y) \in U_y\}$. It is routine to verify that given a large scale (small scale) structure $\mathcal{S}$ on X, the collection $\mathcal{B} = \{E_{\mathcal{U}} : \mathcal{U} \in \mathcal{S}\}$ forms the base of a coarse (uniform) structure on X^Y. In particular, a large scale (small scale) structure on X gives rise to a coarse (uniform) structure on $X = X^{\{pt\}}$.

Theorem 2.2.10. *J. Dydak (2008) There is a one-to-one, order-preserving correspondence between the coarse (uniform) structures on a set X and the large scale (small scale) structures on X.*

Proof. As the discussion above shows, every scale structure on X induces an entourage structure on X and every entourage structure on X induces a scale structure on X. Using the notation from above, one can easily show that for a cover $\mathcal{U}$, it is the case $\mathcal{U} \prec \mathcal{U}_{E_{\mathcal{U}}} \prec st(\mathcal{U},\mathcal{U})$ and that for an entourage E, we have $E \subseteq E_{\mathcal{U}_E} \subseteq E \circ E$. These two facts show that starting with a scale structure, inducing the entourage structure, and then inducing the scale structure from the induced entrourage structure, one obtains the original scale structure. For example, let $\mathcal{LS}$ be a large scale structure on a set X, and let $\overline{\mathcal{LS}}$ be the large scale structure induced by the entrourage structure induced by $\mathcal{LS}$. If $\mathcal{U}$ is uniformly bounded in $\mathcal{LS}$, then $\mathcal{U} \prec \mathcal{U}_{E_{\mathcal{U}}}$, which implies that $\mathcal{LS} \subseteq \overline{\mathcal{LS}}$. On the other hand, $\mathcal{U}_{E_{\mathcal{U}}} \prec st(\mathcal{U},\mathcal{U})$, which implies that every basis element of $\overline{\mathcal{LS}}$ is contained in $\mathcal{LS}$. Hence, $\mathcal{LS} = \overline{\mathcal{LS}}$. Starting with an entourage structure, inducing a large scale structure, and then inducing an entourage structure will also return the original structure. This shows that entourage structures and scale structures are in one-to-one correspondence provided one can either zoom in or zoom out. □

Question 2.2.11. *Let X be a structure (large or small scale) and Y be a set. What is the relationship between properties of X and properties of X^Y?*

2.2.3 More Useful Structures on Function Spaces

We previously defined products and coproducts in the categories of ls-spaces and ss-spaces. Just as it is the case for topological spaces, the uniform convergence structure on function spaces defined earlier does not directly make the products and coproducts dual. We construct two alternative ls (ss) structures on function spaces so that one may realize the products and coproducts as dual via function spaces.

We first define an analogue of pointwise convergence for ss-space. Suppose that X has a small scale structure. For a cover $\mathcal{U}$ and a finite subset S of Y, let

$$N(f,\mathcal{U},S) = \{g : Y \to X \mid \forall y \in S,\ \exists U_y \in \mathcal{U} \text{ such that } f(y), g(y) \in U_y\}.$$

The collection $\mathcal{V}_{\mathcal{U},S} = \{N(f,\mathcal{U},S) : f \in X^Y\}$ is a cover of X^Y. Notice that if $\mathcal{W}_1$ star refines $\mathcal{W}_2$ and $\mathcal{W}_2$ star refines both $\mathcal{U}_1$ and $\mathcal{U}_2$, then $\mathcal{V}_{\mathcal{W}_1,S_1\cup S_2}$ star refines both $\mathcal{V}_{\mathcal{U}_1,S_1}$ and $\mathcal{V}_{\mathcal{U}_2,S_2}$. It then follows that the collection of all possible $\mathcal{V}_{\mathcal{U},S}$ forms a small scale base, call this ss-structure the **pointwise convergence ss-structure**.

Using the pointwise convergence ss-structure, we obtain a small scale structure on $\left(\coprod_{\alpha\in A} X_\alpha\right)^A$, and when restricted to $\{f : A \to \coprod_{\alpha\in A} X_\alpha \mid f(\alpha) \in X_\alpha \text{ for all } \alpha\}$ we get the product structure for the collection $\{X_\alpha\}$.

Finally, we define a new large scale structure on function spaces. For every collection of covers $\{\mathcal{U}_y\}_{y\in Y}$ and function $f : Y \to X$, define

$$N(f,\{\mathcal{U}_y\}) = \{g : Y \to X \mid \forall y \in Y,\ \exists U_y \in \mathcal{U}_y \text{ such that } f(y), g(y) \in U_y\}$$

and define a cover of X^Y by $\mathcal{V}_{\{\mathcal{U}_y\}} = \{N(f,\mathcal{U}_y) \mid f \in X^Y\}$. Note that $st(\mathcal{V}_{\{\mathcal{U}_y^{(1)}\}}, \mathcal{V}_{\{\mathcal{U}_y^{(2)}\}}) \prec \mathcal{V}_{\{st(\mathcal{U}_y^{(1)}, st(\mathcal{U}_y^{(2)}, \mathcal{U}_y^{(2)})\}}$. Hence, the collection of all $\mathcal{V}_{\{\mathcal{U}_y\}}$ forms a large scale base. In particular, we obtain a large scale structure on $\left(\coprod_{\alpha\in A} X_\alpha\right)^A$, and when restricted to $\{f : A \to \coprod_{\alpha\in A} X_\alpha \mid f(\alpha) \in X_\alpha \text{ for all } \alpha\}$ we get the product structure for the collection $\{X_\alpha\}$.

2.3 Connectivity at Scales

Connectivity in ls-scale geometry is a useful concept because, for example, a countable group with proper left-invariant metric is large scale 0-connected if and only if the group is finitely generated. The aim of this section is to explore different variants of connectivity for scale structures.

2.3.1 Connectivity

Recall that a topological space X is connected if and only if each map $f : X \to S^0$ is constant. The utility of this definition is that one does not need the notion of separation by open or closed sets. This definition easily allows for a definition of connectivity for ss-structures.

Definition 2.3.1. A ss-structure X is **ss-connected** if every ss-continuous $f : X \to S^0$ is constant.

We will show exactly how the above definition of connectivity using S^0 dualizes to large scale by first taking a traditional approach, see Roe (2003). If a set X is metric then it is clear that a finite union of bounded sets is again bounded. Some subtleties can arise by weakening the assumptions to X being an $\infty-$metric space. A prime example of this, which we will call **Large Scale** S^0 and denote by ls-S^0, is the only ∞-metric on the two point set $\{x_0, x_1\}$ for which $d(x_0, x_1) = \infty$. Notice that $\{x_i\}$ is bounded for $i = 1, 2$ but the union, $\{x_0, x_1\}$, is unbounded. It is necessary to make a distinction between ls-structures for which the finite union of bounded sets is bounded and those that do not enjoy that property.

Definition 2.3.2. A ls-structure on a space X is **ls-connected** if for all $x, y \in X$, the set $\{x, y\}$ is bounded. Equivalently, a coarse structure is coarsely connected if every finite subset of X is bounded. The **ls-connected component** of a point $x \in X$ is the set $\{y \in X : \{x, y\}$ is bounded$\}$.

It is clear that X is coarsely connected if and only if the coarse connected component of any point $x \in X$ is the whole set X.

Examples of ls-connected spaces are the bounded coarse structures associated to a metric and the discrete coarse structure $\mathcal{D}$ defined as $\mathcal{E} \in \mathcal{D}$ if and only if the cardinality of the union of the non-singleton sets in $\mathcal{E}$ is finite. An example of a non coarsely connected space is the bounded coarse structure associated to an ∞-metric which takes the value ∞ for at least one pair of points. For any space X, we may divide X into connected components, which are defined as equivalence classes of the relation $\sim$ where $x \sim y$ if and only if $\{x, y\}$ is bounded.

Proposition 2.3.3. *Let X be an ls-structure. The following are equivalent:*

1) X is ls-connected

2) Every finite union of bounded sets is bounded.

3) Every bornologous function $f : X \to$ ls-S^0 is constant.

Proof. (2) $\implies$ (1) is clear. (1) $\implies$ (2) Let $A_1, A_2, \dots, A_n$ be nonempty bounded subsets of X. There is a uniformly bounded cover $\mathcal{U}$ such that $\{A_1, A_2, \dots, A_n\} \subset \mathcal{U}$. Let $x_i \in A_i$ for $1 \le i \le n$. Notice that $\{x_1, x_2, \dots, x_n\}$ is a bounded set which implies that $st(\{x_1, x_2, \dots, x_n\}, \mathcal{U})$ is bounded. We are finished as the union $\bigcup_{i=1}^n A_i$ is contained in $st(\{x_1, x_2, \dots, x_n\}, \mathcal{U})$.

(1) $\iff$ (3). Let $\{C_i : i \in I\}$ be the coarse connected components of X. Choose $i_1 \in I$ and notice that $f : X \to ls - S^0 = \{0, 1\}$ given by $f(x) = 1$ for all $x \in C_{i_1}$ and $f(y) = 0$ for all $y \notin C_{i_1}$ is bornologous and onto if and only if I has more than one element. □

Remark 2.3.4. In analogy to topological spaces, i.e. sets with a topology, we can talk about small scale spaces (ss-spaces) or large scale spaces (ls-spaces).

2.3.2 0-Connectivity

Definition 2.3.5. A ls-structure X is **ls-0-connected** if there is a uniformly bounded cover $\mathcal{U}$ of X such that for every $x, y \in X$ there exists $\mathcal{U}$-chain connecting them. Dually, a ss-structure is **ss-0−connected** if for every $x, y \in X$ there exists a $\mathcal{U}$-chain connected them for every uniform cover $\mathcal{U}$.

The proof of the following proposition is routine and we shall leave the details to the reader.

Proposition 2.3.6. *Let X be a set with a cover $\mathcal{U}$. For each $x \in X$, set $P_{\mathcal{U},x} = \{y \in X :$ there exists $\mathcal{U}$-chain connecting x to $y\}$. We call $P_{\mathcal{U},x}$ the **$\mathcal{U}$-path component of** x. The sets $\{P_{\mathcal{U},x} : x \in X\}$ form a partition of X. Furthermore, if X is ls-space (ss-space), then X is ls-0-connected (ss-0-connected) if and only if $P_{\mathcal{U},x} = X$ for any $x \in X$ for some uniformly bounded (for every uniform) cover $\mathcal{U}$.*

Every ls(ss)-0-connected structure is ls(ss)-connected. We leave the verification of this fact to the reader.

An ls-structure which is ls-connected, but is not coarsely 0-connected would be the metric space $X = \{n^2 : n \in \mathbb{N}\}$. Observe that every uniformly bounded cover $\mathcal{U}$ of X consists of singletons outside of a sufficiently large ball about $1 \in X$. In fact, we can generalize this example to the following: Every large scale structure with asymptotic dimension 0 is not coarsely 0-connected. Recall that a metric space X has asymptotic dimension 0 if for every $R, S > 0$ there exists an $S-$disjoint uniformly bounded covering $\mathcal{U}$ of X which coarsens the covering by $R-$balls. Notice that the same proof that $\{n^2 : n \in \mathbb{N}\}$ is not coarsely 0-connected works any metric of asymptotic dimension 0. We can generalize further.

It is well known that a topological space X is 0-connected if and only if every continuous function to a zero dimensional space is constant. It is natural to extend this characterization to small and large scale structures.

Definition 2.3.7. Let $n \geq 0$. An ls-structure X is said to have **ls(or asymptotic)-dimension at most** n if for every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded coarsening $\mathcal{V}$ of $\mathcal{U}$ with multiplicity at most $n+1$. An ss-structure X is said to have **ss-dimension at most** n if for every uniform cover $\mathcal{U}$ of X there exists a uniform refinement $\mathcal{V}$ of $\mathcal{U}$ with multiplicity at most $n+1$.

We may simply say dimension instead of ss-dimension or ls-dimension when the context is clear which one we are talking about. By Proposition 1.0.1, a paracompact space has uniformity generated by all open covers. Notice that the above definition for ss-dimension does indeed generalize the well known covering dimension. We will need the following proposition from J. Dydak (2008) in order to prove the aforementioned analogue.

Proposition 2.3.8. *J. Dydak (2008) If $\mathcal{L}$ is a set of families in X such that $\mathcal{B}_\alpha, \mathcal{B}_\beta \in \mathcal{L}$ implies the existence of $\mathcal{B}_\gamma \in \mathcal{L}$ such that $\mathcal{B}_\alpha \cup \mathcal{B}_\beta \cup st(\mathcal{B}_\alpha, \mathcal{B}_\beta)$ refines $\mathcal{B}_\gamma$, then the family $\mathcal{LS}_X$ of all refinements of trivial extensions of elements of $\mathcal{L}$ forms a large scale structure on X.*

The families $\mathcal{L}$ in the previous proposition will be called a **base** for $\mathcal{LS}_X$. Similarly, a collection of covers $\mathcal{S}$ has the property that for every $\mathcal{U}, \mathcal{V} \in \mathcal{S}$ there exists some $\mathcal{W} \in \mathcal{S}$

such that $st(\mathcal{W}, \mathcal{W}) \prec \mathcal{U} \cap \mathcal{V}$ will be referred to as a base for $\mathcal{S}'$ where $\mathcal{S}'$ is the ss-structure consisting of all coarsenings of covers from $\mathcal{S}$.

Proposition 2.3.9. *Let X be a ss−structure. The following conditions are equivalent:*

1) X is ss-0-connected.

3) Every ss-continuous function $f : X \to Z$ is constant where Z a ss-space with ss-dimension zero.

Proof. (1) $\Rightarrow$ (2) Let $f : X \to Z$ be ss-continuous with Z a ss-space with dimension zero. Take a disjoint uniform cover $\mathcal{U}$ of Z and notice that the inverse image is a disjoint uniform cover of$X$. If the inverse image of the cover $\mathcal{U}$ consists of more than one set nonempty set then one can find two points which are not $f^{-1}(\mathcal{U})$-chainable. It follows that the image of X lies in one element of $\mathcal{U}$. Use the Hausdorff property of X and the fact that one can always refine covers to a disjoint one to show that the image of X is just a singleton.

(2) $\Rightarrow$ (1) The idea is to construct an appropriate asymptotically zero dimensional space out of X.

Claim 1: The collection $\mathcal{Z} = \{\{P_{\mathcal{U},x} : x \in X\} : \mathcal{U} \in \mathcal{LS}_X\}$ is a basis for zero dimensional ss-structure X_S on X. Furthermore, the identity map $id : X \to X_S$ is ss-continuous.

We need only show that for any two covers $P_{\mathcal{U}}, P_{\mathcal{V}} \in \mathcal{Z}$ there exists $P_{\mathcal{W}} \in \mathcal{Z}$ such that $st(\mathcal{W}) \leq \mathcal{U} \cap \mathcal{V}$. $\mathcal{U}$ and $\mathcal{V}$ are uniform covers of X, so there exists a uniform cover $\mathcal{W}$ which star refines $\mathcal{U} \cap \mathcal{V}$. Notice that $\{P_{\mathcal{W},x} : x \in X\}$ refines both $P_{\mathcal{U}}$ and $P_{\mathcal{V}}$ which is sufficient because $\{P_{\mathcal{B},x} : x \in X\}$ is a disjoint collection for any uniform cover $\mathcal{B}$ of X.

The structure X_S has ss-dimension 0 because it has a basis consisting of disjoint covers. The problem with X_S is that it may not satisfy the Hausdorff property. We can easily fix this.

Notice that we have an equivalence relation $\sim$ on X via $x \sim y$ if x and y are $\mathcal{U}$-chainable for all uniform covers $\mathcal{U}$ of X. Give $X_S/\sim$ the uniform structure induced by X_S; A cover $\mathcal{U}$ is uniform if and only if it is the image of a uniform cover from the quotient map $q : X \to X/\sim$. The identity map $id : X \to X_L$ is ss-continuous because the inverse image of $P_{\mathcal{U}}$ coarsens $\mathcal{U}$ for any uniform cover $\mathcal{U}$ of X. The quotient map $q : X_S \to X_S/\sim$ is ss-continuous and so $q \circ id : X \to X_S/\sim$ is ss-continuous.

By condition (3), the image of X is bounded and hence there must be some uniformly bounded cover $\mathcal{U}$ of X and $x \in X$ such that $X = P_{\mathcal{U},x}$. □

Proposition 2.3.10. *Let X be a LS−structure. The following conditions are equivalent:*

1) X is ls-0-connected.

3) The image of X is bounded for every bornologous map $f : X \to Z$ where Z a space with ls-dimension zero.

Proof. (1) ⇒ (2) Let $\mathcal{U}$ be a uniformly bounded cover of X such that $P_{\mathcal{U},x} = X$ for each $x \in X$. If $f : X \to Z$ is any bornologous function. We may coarsen $f(\mathcal{U})$ to a uniformly bounded cover $\mathcal{V}$ of Z with no double intersections. For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $f(U) \cap f(V) \neq \emptyset$. The same must be true for each element of $\mathcal{V}$ which contains an element of $f(\mathcal{U})$. As there are no double intersections in $\mathcal{V}$, it must be the case that $f(X)$ lies in a single element $V \in \mathcal{V}$ and hence $f(X)$ is bounded.

(2) ⇒ (1) The idea is to construct an appropriate asymptotically zero dimensional space out of X.

<u>Claim 1:</u> The collection $\{\{P_{\mathcal{U},x} : x \in X\} : \mathcal{U} \in \mathcal{LS}_X\}$ is a basis for an asymptotically zero dimensional large scale structure $\mathcal{LS}_1$ on X. Furthermore, the identity map $id : (X, \mathcal{LS}) \to (X, \mathcal{LS}_1)$ is bornologous.

We need only show that this collection the subadditivity condition presented in 3.3.1. If $\mathcal{U}$ and $\mathcal{V}$ are uniformly bounded covers of X then there exists a uniformly bounded cover $\mathcal{W}$ which coarsens both $\mathcal{U}$ and $\mathcal{V}$. Notice that $\{P_{\mathcal{W},x} : x \in X\}$ coarsens both $\{P_{\mathcal{U},x} : x \in X\}$ and $\{P_{\mathcal{V},x} : x \in X\}$ which is sufficient because $\{P_{\mathcal{B},x} : x \in X\}$ is a disjoint collection for any uniformly bounded cover $\mathcal{B}$ of X.

The structure $(X, \mathcal{LS}_1)$ has asymptotic dimension 0 because it has a large scale basis consisting of disjoint covers.

To see that the identity map $id : (X, \mathcal{LS}) \to (X, \mathcal{LS}_1)$ is bornologous: Let $\mathcal{U}$ be a uniformly bounded cover of $(X, \mathcal{LS})$. Notice that $f(\mathcal{U})$ refines $\{P_{\mathcal{U},x} : x \in X\}$.

By condition (3), the image of X is bounded and hence there must be some uniformly bounded cover $\mathcal{U}$ of X and $x \in X$ such that $X = P_{\mathcal{U},x}$. □

2.4 Covariant And Contravariant Approaches

It is the nature of Svarč - Milnor Lemma to induce an ls-structure on a group from a metric space that the group is acting on. A result of N. Brodskiy, J. Dydak, and A. Mitra in Nikolay Brodskiy and Mitra (2008) (Proposition 3.1) can be interpreted as a converse to Svarč-Milnor Lemma. What they show is that there is a unique way to put a ls-structure on a set in which a group G (with the translation ls-structure introduced in subsection 2.1.3) acts if that action satisfies certain criterion. The following proposition is a generalization of their result and it is a motivating example for the philosophy of this section.

Proposition 2.4.1. *Let X be a space with a large scale structure, Y any set, and $f : X \to Y$ a surjective map such that the collection $\{f^{-1}(\{y\}) \mid y \in Y\}$ is uniformly bounded in X. Then there is a unique large scale structure on Y such that f is a coarse equivalence.*

Proof. Define the large scale structure $\mathcal{LS}$ on Y as follows. A collection $\mathcal{U}$ is uniformly bounded if and only if $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is uniformly bounded in X. This is a large scale structure since the surjectivity of f implies that $f^{-1}(st(\mathcal{B}_1, \mathcal{B}_2)) = st(f^{-1}(\mathcal{B}_1), f^{-1}(\mathcal{B}_2))$, and $\mathcal{A} \prec \mathcal{B}$ implies that $f^{-1}(\mathcal{A}) \prec f^{-1}(\mathcal{B})$.

First, we show that f is bornologous. Suppose that $\mathcal{U}$ is uniformly bounded in X. We need to show that $f^{-1}(f(\mathcal{U})$ is uniformly bounded. This is true since if $A = f^{-1}(f(U))$, then $A \subseteq st(U, \{f^{-1}(\{y\}) \mid y \in Y\}$. Thus, $f^{-1}f(\mathcal{U}) \prec st(\mathcal{U}, \{f^{-1}(\{y\}))$, which implies that it is uniformly bounded in X.

Let g be any right inverse of f. Then $g(\mathcal{U}) \subseteq f^{-1}(\mathcal{U})$, so is uniformly bounded in X for any uniformly bounded collection $\mathcal{U}$ of Y.

Note that $f \circ g = 1_X$ and that $\{\{x, g \circ f(x)\} \mid x \in X\} \prec \{f^{-1}(\{y\}) \mid y \in Y\}$ so is uniformly bounded. Thus, f is a coarse equivalence.

Now suppose that $\mathcal{S}$ is any other ls structure on Y such that $f : X \to Y$ is a coarse equivalence with respect to $\mathcal{S}$. Let $\mathcal{U} \in \mathcal{LS}$. Then $\mathcal{U} = ff^{-1}(\mathcal{U})$ is uniformly bounded with respect to $\mathcal{S}$. Hence $\mathcal{LS} \subseteq \mathcal{S}$. By symmetry we get that $\mathcal{LS} = \mathcal{S}$, proving the uniqueness. □

Remark 2.4.2. The hypothesis of the above proposition essentially says that the map f is "coarsely" one-to-one and onto to begin with, so in order to dualize the above result to small scale geometry, we would need to assume the function f was one-to-one and onto to begin with and so the dualization is not interesting.

It is common practice in topology to try classify a property by mapping well known spaces into a given space. For example, path and higher connectivity theories are built on mappings of spheres into a given space. Dually, it is also common to classify a property of spaces by mapping a given space to known spaces. For example, cohomology groups are determined by mapping into Eilenberg-MacLane spaces. A property of a space X is said to be **covariant** if it is determined by mapping known spaces into X and the property will be said to be **contravariant** if can be defined by mapping X to a known class of spaces. See Dydak (1999) for a general treatment of topological concepts determined in this way.

Compactifications, being totally determined by continuous and bounded real valued functions, are a contravariant property. We show in a concurrent paper, see Kyle Austin and Holloway (2015), that certain ls-structures are contravariantly defined by the small scale category and that certain ss-structures are defined covariantly by the category of ls-structures.

In this section we will show that large and small scale geometry can be defined in a covariant and contravariant way by looking at functions to and from metric spaces.

2.4.1 General Philosophy

The purpose of this subsection is to outline general ways of inducing ls (ss)-structures on sets with no structure via mapping to or from known structures.

Recall from topology that the indiscrete topology on a set X is the only topology on X so that any function $f : Y \to X$ to a topological space Y is continuous. Dually, the discrete topology on X is the unique topology so that any function $f : X \to Z$ is continuous where Z is any topological space. These two topologies are a staple in development of covariant and contravariant topologies as introduced by J. Dydak in Dydak (1999). We need analogues of these two topologies in the setting of ss (ls)-structures to define covariant and contravariant ss (ls)-structures.

Let X be a set as before. The **discrete ss (ls)-structure** on X is the unique ss (ls)-structure so that an function $f : X \to Y$ is uniformly continuous (bornologous) for any ss (ls)-structure Y. Dually the **indiscrete ss (ls)-structure** on X is the unique ss (ls)-structure so that any function $f : Y \to X$ is uniformly continuous (bornologous) for any ss-structure Y. The following proposition shows that each of these structures exists and gives a concrete description of each structure.

Proposition 2.4.3. *Let X be a set.*

1) The discrete ss-structure is the ss-structure whose only uniform cover is $\{X\}$.

2) The indiscrete ss-structure is the ss-structure in which every cover is a uniform cover.

3) The discrete ls-structure is the ls-structure whose only uniformly bounded cover is the cover by singletons.

4) The indiscrete ls-structure is the ls-structure in which every family of subsets is uniformly bounded.

Let X be a set and let $Co = \{(Y_\delta, f_\delta) : \delta \in \Delta\}$ be a family of pairs consisting of an ss (ls)-structures Y_δ and an ss (ls)-continuous function $f_\delta : Y_\delta \to X$. The **covariant ss (ls)-structure with respect to** Co is the smallest ss (ls)-structure so that f_δ is ss (ls)-continuous for all $\delta \in \Delta$. Dually, if $Contra = \{(Y_\delta, f_\delta) : \delta \in \Delta\}$ is a family of pairs consisting of an ss (ls)-structures Y_δ and an ss (ls)-continuous function $f_\delta : X \to Y_\delta$ then one defines the **contravariant ss (ls)-structure with respect to** $Contra$ is the smallest ss (ls)-structure so that f_δ is ss (ls)-continuous for all $\delta \in \Delta$.

Proposition 2.4.4. *The covariant and contrariant ss-structures on a set X exist and are unique.*

Proof. Recall that a sequence $\{\mathcal{U}_i : i \geq 0\}$ of covers of X is called a normal sequence if $\mathcal{U}_i$ star refines $\mathcal{U}_{i-1}$ for each $i \geq 1$. It is a common trick in uniform space theory (see Isbell (1964) page 12) to use normal covers to build ss-structures with desirable properties.

Consider the covariant case first; Let Co be as in the above paragraph. Let NS be the collection of all normal sequences $\{\mathcal{U}_i\}_{i\geq 0}$ such that the inverse images $f_\delta^{-1}(\mathcal{U}_i)$ is a uniform cover for each $i \geq 0$ and for each $\delta \in \Delta$. Notice that the intersection of two normal sequences in NS also has to lie in NS. It follows that the collection $\mathcal{SS}$ of all covers which are in some sequence in NS forms a small scale structure on X and it is clear that this is the largest structure making all maps f_δ ss-continuous.

For the contravariant case, first notice that if $f : A \to B$ is a map from any set to a small scale space B with structure $\mathcal{SS}$, then the collection $\{f^{-1}(\mathcal{U}) \mid \mathcal{U} \in \mathcal{SS}\}$ forms a small scale structure on A. So to build the contravariant structure with respect to a collection of spaces a maps, simply take the collection of inverse images of uniform covers of $\prod_{\delta\in\Delta} Y_\delta$. This makes all maps f_δ uniformly continuous since $f_\delta^{-1}(\mathcal{U}_\delta) = f^{-1}(\mathcal{U}_\delta \times \prod_{\alpha\neq\delta}\{X_\alpha\})$, there f is induced map to the product. It is straightforward to show that this is the smallest structure making each f_δ uniformly continuous. □

Proposition 2.4.5. *The covariant and contrariant ls-structures on a set X exist and are unique.*

Proof. Consider the covariant case first and let Co be as in the previous proof. One can construct an ls-structure on X by taking the collection $LS = \bigcup_{\delta\in\Delta}\{f_\delta(\mathcal{U}_\delta) : \mathcal{U}_\delta$ is a uniformly bounded cover of $Y_\delta\}$ and closing it under union, star, and refinements. It is straghtforward to show that this is the smallest ls-structure making each function $f_\delta : Y_\delta \to X$ ls-continuous.

For the contravariant case, let $Contra$ be as in the previous proof. Let $LS = \{\mathcal{U} : f_\delta(\mathcal{U})$ is uniformly bounded for each $\delta \in \Delta\}$. Using the fact that the image of the star of two covers refines the star of the image of those covers, one can show that LS is closed under starring covers. It follows that LS is a ls-structure and it is clearly the largest ls-structure making all of the functions f_δ ls-continuous. □

There is a third way to define an ss-structure which uses the covariant structure inherited from families of pseudometrics. Given a family of pseudometrics $Contra = \{((X, d), id_X) : d \in \Delta\}$ of pairs consisting of a pseudometric space (X, d) and the identity mapping on a set X, we can take the contravariant ss-structure with respect to $Contra$. In fact, any ss-structure is induced by some family of pseudometrics (Bourbaki (1998), IX.1.4). This construction is another source of duality between large and small scale geometry. We show that every ls-structure is the covariant ls-structure induced by $\infty-$metric spaces and identity mappings.

One can dualize the above approach to ls-structures. However, we do not know if every ls-structure can be described that way.

Definition 2.4.6. Let D be a family of ∞-metrics on a space X. For each $d \in D$, let $\mathcal{B}_d$ be the metric coarse structure induced by d. The coarse structure **generated by** D is defined as $\bigcap_{d \in D} \mathcal{B}_d$. We say that a coarse space is **submetrizable** if it is generated by a family of ∞-metrics.

A family $\mathcal{B}$ of subsets of X is uniformly bounded in the coarse structure generated by a collection of ∞-metrics if and only if it is uniformly bounded in all ∞-metrics in the collection.

John Roe gives an example (Roe (2003) Example 2.44) of a nonmetrizable large scale structure. Given an infinite set X, declare a family of subsets $\mathcal{U}$ to be uniformly bounded if there exists an integer $n_{\mathcal{U}}$ such that $|U| < n_{\mathcal{U}}$ for all $U \in \mathcal{U}$. One can show that this structure is not metrizable.

It is also true that a submetrizable large scale structure need not be metrizable. Jesus Moreno-Damas shows in Moreno-Damas (2014) that if one takes on $\mathbb{R}^{\mathbb{N}}$ the collection of metrics $D = \{d_n(x, y) = |x_n - y_n|\}$, then the coarse structure generated by D is not metrizable.

There are several dual results concerning uniform spaces induced by pseudometrics and coarse spaces induced by ∞-metrics. For example, an ss-structure is induced by a single pseudometric if and only if it has a countable fundamental system of entourages (Bourbaki (1998) IX.1.4). On the other hand, a coarse structure is the bounded coarse structure of an ∞-metric if and only if it is countably generated (Roe (2003) Thm. 2.55). In the subsections below, we will explore several other dual results between the small scale and the large scale.

We present two points of view for coarse structures.

2.4.2 Covariant Approach

Let X_S be a small scale structure on a set X. Let $\mathcal{S}$ be the collection of pseudometrics d on X for which the identity map $id : X_S \to (X, d)$ is uniformly continuous. We recall from the theory of uniform structures that the uniform structure X_S on X is the smallest uniform structure for which the the identity $X \to (X,d)$ is uniformly continuous for all $d \in \mathcal{S}$. See Isbell (1964), James (1990), or Joshi (1983).

We dualize the above result for large scale structures X_L on a set X: Let $\mathcal{L}$ be the collection of ∞-metrics d on X for which the identity function $id : (X,d) \to X_L$ is bornologous.

Theorem 2.4.7. *The large scale structure X_L is the smallest large scale structure for which satisfies*

$*$: the identity function $id : (X,d) \to X_L$ is bornologous for all $d \in \mathcal{L}$.

The following lemma will be useful.

Lemma 2.4.8. *Let X_L be a large scale structure on a set X and $\mathcal{B}$ a uniformly bounded family of X_L. There exists an ∞-metric d on X such that $\mathcal{B}$ is uniformly bounded in (X,d) and $id : (X,d) \to X_L$ is bornologous.*

Before proving the lemma, we recall some known results about large scale basis and metrizability.

Proposition 2.4.9. *J. Dydak (2008) Let $(X, \mathcal{LS}_X)$ be a large scale structure. The following are equivalent:*

1) $(X, \mathcal{LS}_X)$ has a countable large scale basis $\mathcal{LS}'_X$

2) There exists an ∞-metric d such that the uniformly bounded covers with respect to the metric coincide with $\mathcal{LS}_X$.

Recall that we can iductively define higher stars via $st^0(\mathcal{U}) = \mathcal{U}$ and $st^n(\mathcal{U}) = st(st^{n-1}(\mathcal{U}), \mathcal{U})$ for $n \geq 1$.

Lemma 2.4.10. *Let $\mathcal{B}$ be a family of subsets of a space X. Then $st(st^n(\mathcal{B}), st^m(\mathcal{B}))$ refines $st^{n+2m+1}(\mathcal{B})$.*

Proof. We define $st^k(\mathcal{U}, \mathcal{V})$ to be the cover obtained by staring $\mathcal{U}$ against $\mathcal{V}$ k times . Let $U \in st^n(\mathcal{B})$ and $V \in st^m(\mathcal{B})$). There exists $U_1, V_1 \in \mathcal{B}$ such that $U = st^{n-1}(st(U_1, \mathcal{B}), \mathcal{B})$ and $V = st^{n-1}(st(V_1, \mathcal{B}), \mathcal{B})$. If $U \cap V \neq \emptyset$ then it follows that $st^{m+n}(U_1, \mathcal{B}) \cap V_1 \neq \emptyset$ which further implies that $st^{m+n+1}(U_1, \mathcal{B}) \supset V_1$. It is clear then that $st^{2m+n+1}(U_1, \mathcal{B}) \supset st^m(V_1, \mathcal{B}) = V$. Therefore $st^{2m+n+1}(U_1, \mathcal{B}) \supset st(U, st^m(\mathcal{B}))$. $\square$

Proof of Lemma 2.4.8. We will show that the collection of families $\{st^n(\mathcal{B}) : n \geq 1\}$ is a large scale basis for a metrizable large scale structure on X which will be denoted by $\langle \mathcal{B} \rangle$.

To see that $\{st^n(\mathcal{B}) : n \geq 1\}$ satisfied the sub-additivity condition in proposition 1.6 in J. Dydak (2008), just notice that $st(st^n(\mathcal{B}), st^m(\mathcal{B})) = st^{n+2m+1}(\mathcal{B})$ by Lemma 2.4.10 and that $st^n(\mathcal{B}) \cup st^m(\mathcal{B}) \cup st(st^n(\mathcal{B}), st^m(\mathcal{B}))$ refines $st^{n+2m+1}(\mathcal{B})$.

Notice that $(X, \langle \mathcal{B} \rangle)$ has a countable large scale basis and is therefore metrizable. Also, $st^n(\mathcal{B})$ is uniformly bounded in X_L for every n since $\mathcal{B}$ was uniformly bounded in X_L, so $id : (X, \langle \mathcal{B} \rangle) \to X_L$ is bornologous. □

Proof of Theorem 2.4.7. Let $X_{L'}$ be a large scale structure on X satisfyting $*$. Let $\mathcal{B} \in X_L$. Then by Lemma 2.4.8, there is a metric $d \in \mathcal{L}$ such that $\mathcal{B}$ is uniformly bounded in (X, d). Thus, $\mathcal{B} \in X_{L'}$, implying that $X_L \subseteq X_{L'}$. □

2.4.3 Contravariant Approach

Let X and Y be coarse spaces. The product coarse structure on $X \times Y$ is defined by Roe in Roe (2003) as follows: a set $E \subseteq (X \times Y) \times (X \times Y)$ is controlled if and only if its projections onto $X \times X$ and $Y \times Y$ are controlled. We now define the product large scale structure; a collection $\mathcal{B}$ is uniformly bounded if and only if $\{\pi_X(B) : B \in \mathcal{B}\}$ and $\{\pi_Y(B) : B \in \mathcal{B}\}$ are uniformly bounded where π_X and π_Y are the projections to X and Y. It is straightforward to verify that the product coarse structure induces the product large scale structure and so they are equivalent.

If $\{X_\alpha\}_{\alpha \in I}$ is a family of spaces indexed by some set I where each space has some large scale structure $\mathcal{L}_\alpha$, then we can define a large scale structure $\mathcal{L}$ on $\prod_{\alpha \in I} X_\alpha$ be letting $\mathcal{B} \in \mathcal{L}$ if and only if $\{\pi_\alpha(B) : B \in \mathcal{B}\}$ is uniformly bounded for each α, where π_α is the projection map to X_α. In the case that each $X_\alpha = X$, we may view X as a subspace of $\prod_{\alpha \in I} X_\alpha$ by the injection $i : X \to \prod_{\alpha \in I} X_\alpha$ where $i(x) = (x_\alpha)_{\alpha \in I}$, where each $x_\alpha = x$. The space X, viewed as a subset of the product then inherits a large scale structure, which is equal to $\bigcap_{\alpha \in I} \mathcal{L}_\alpha$.

For a pseudometric space (X, d) with the induced uniformity from d, the map $d : X \times X \to \mathbb{R}$ is uniformly continuous, where $X \times X$ has the product uniformity (Joshi (1983) Ex 14.1.2). Also from Joshi (1983) is that if D is the collection of all pseudometrics on a uniform space

X with uniform structure $\mathcal{U}$ such that $d : X \times X \to \mathbb{R}$ is uniformly continuous, then the uniformity generated by D is equal to $\mathcal{U}$. We now consider the situation for coarse structures.

Let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ with metric extending the normal metric on $\mathbb{R}$ and defining $d(\infty, p) = \infty$ for $p \in \mathbb{R}$.

Proposition 2.4.11. *Let (X, d) be an ∞-metric space and let $\mathcal{C}$ be the bounded coarse structure induced by d. Then $d : (X \times X)_{\mathcal{C}\times\mathcal{C}} \to \mathbb{R}^*$ is bornologous where $\mathbb{R}^*$ has the bounded coarse structure.*

Proof. Let $E \subseteq (X \times X) \times (X \times X)$ be controlled. Then there are $M, N > 0$ such that $\sup\{d(a, c) : ((a, b), (c, d)) \in E\} < M$ and $\sup\{d(b, d) : ((a, b), (c, d)) \in E\} < N$. First assume that $d(a, b) = \infty$. Then since $d(a, b) \leq d(a, c) + d(c, d) + d(d, b)$ and $d(a, c), d(d, b) < \infty$, it follows that $d(c, d) = \infty$. So in $\mathbb{R}^*$, the distance between $d(a, b)$ and $d(c, d)$ is 0. Now assume that $d(a, b)$ and $d(c, d)$ are finite. Note that by the triangle inequality we have

$$d(a, b) - d(c, d) \leq d(a, c) + d(c, d) + d(b, d) - d(c, d) < M + N$$

and similarly $d(c, d) - d(a, b) < M + N$. Hence $\sup\{|d(a, b) - d(c, d)| : ((a, b), (c, d)) \in E\} < M + N$. Thus, $(d \times d)(E)$ is controlled, implying that $d : X \times X \to \mathbb{R}^*$ is bornologous.

□

Lemma 2.4.12. *Let (X, d) be a space with an ∞-metric d and let $\mathcal{L}$ be a coarse structure on X. Then $d : (X \times X)_{\mathcal{L}\times\mathcal{L}} \to \mathbb{R}^*$ is bornologous if and only if $id : X_{\mathcal{L}} \to (X, d)$ is bornologous.*

Proof. First assume that $\mathcal{L}$ is finer than the bounded coarse structrure induced by d. Then the coarse structure on $X \times X$ induced by $\mathcal{L}$ is finer than the structure induced by the metric. Then since $d : X \times X \to \mathbb{R}^*$ is bornologous when $X \times X$ is given the structure induced by the metric is it also the case that it is bornologous when given the smaller structure induced by $\mathcal{L}$.

Now suppose that $\mathcal{L}$ is a coarse structure on X such that $d : (X \times X)_{\mathcal{L}} \to \mathbb{R}^*$ is bornologous. Let E be an entourage of L. Then $E \cup E^{-1} \cup \Delta$ is also contained in L and so $F = \{(a, b, c, d) : (a, c), (b, d) \in E \cup E^{-1} \cup \Delta\}$ is a controlled subset of $(X \times X) \times (X \times X)$. Thus there is an $M > 0$ such that $\sup\{|d(a, b) - d(c, d)| : (a, b, c, d) \in F\} < M$. Let $(x, y) \in E$.

Then $(x, y, y, y) \in F$, so $d(x, y) < M$. Thus, $\sup\{d(x, y) : (x, y) \in E\} < M$, implying that E is controlled in the coarse structure induced by d. □

Corollary 2.4.13. *The coarse structure generated by a family D of ∞-metrics is the coarsest such that for each $d \in D$, the map $d : X \times X \to \mathbb{R}^*$ is bornologous.*

Question 2.4.14. *Is every coarse space submetrizable?*

Conjecture 2.4.15. *Let $\mathcal{L}$ and $\mathcal{M}$ be two coarse structures on the same space X with $\mathcal{L} \subsetneq \mathcal{M}$. Then there exists an ∞-metric $d : X \times X \to \mathbb{R}$ such that $d : (X \times X)_{\mathcal{L}} \to \mathbb{R}$ is bornologous and $d : (X \times X)_{\mathcal{M}} \to \mathbb{R}$ is not bornologous.*

Note that every coarse structure on a space X is a subset of the coarse structure consisting of the power set of X. The power set of X is generated by the metric $d : X \times X \to \mathbb{R}$ defined by $d(x, x) = 0$ for all $x \in X$ and $d(x, y) = 1$ for $x \neq y$. Thus, for any coarse structure $\mathcal{L}$ on X there is at least one metric d such that $d : (X \times X)_{\mathcal{L} \times \mathcal{L}} \to \mathbb{R}$ is bornologous. Our next theorem shows that the collection of all such metrics generates the given coarse structure.

Theorem 2.4.16. *Let X be a space with coarse structure $\mathcal{L}$. If D is the family of all ∞-metrics on X which are bornologous as functions from $(X \times X)_{\mathcal{L}}$ to $\mathbb{R}$, then D generates $\mathcal{C}$.*

Proof. Let $\mathcal{L}$ be a coarse structure, let D be the collection of all ∞-metrics d such that $d : (X \times X)_{\mathcal{L}} \to \mathbb{R}$ is bornologous, and let $\mathcal{M}$ be the coarse structure generated by D. Then by Corollary 2.4.13, $\mathcal{L} \subseteq \mathcal{M}$. That $\mathcal{L} = \mathcal{M}$ follows from Conjecture 2.4.15 □

Chapter 3

Large Scale Surjections

3.1 Introduction

T. Miyata and Ž. Virk introduced coarse analogs of the Hurewicz dimension raising theorems in T. Miyata (2013). There they showed that finite asymptotic dimension was preserved by functions which have a property that is coarsely analogous to n-to-1 functions which they called $(B)_n$. In particular, they proved that for a metric space X, the property that $asdim(X) \leq n$ is equivalent to the existence of a $(B)_n$ function $f : Y \to X$ from a space Y with $asdim(Y) = 0$. Their results show that there are analogues to the classical n-to-1 maps from the Cantor set onto an n dimensional compact space. We aim to study these functions that satisfy the $(B)_n$ property which we will call coarsely n-to-1.

The large scale analogue of a surjection in coarse geometry is a function that becomes indistinguishable from a surjection when viewed from ever increasing distances. More precisely, a function $f : X \to Y$ of metric spaces X and Y is coarsely surjective if the image of X is an R-net in Y for some $R > 0$. In order to define what a coarsely n-to-1 map of metric spaces is, we need to generalize the notion of a point. The points of a metric space X are any collection of subsets of X which become indistinguishable from points when viewed from ever increasing distances. So points of X look like a uniformly bounded family of subsets of X. A function $f : X \to Y$ of metric spaces is **coarsely** n-to-1 if for every $R > 0$ there exists $S > 0$ such that the point inverse of any set of diameter at most R can be covered by at most n sets of diameter at most S.

The focus of this note is to further establish the canonical nature of large scale n-to-1 functions in coarse geometry. Properties of these large scale surjections will be developed in the most general setting of large scale structures. The main resluts of this paper are as follows:

Theorem 3.1.1. *Let X and Y be coarse structures and $f : X \to Y$ coarse and coarsely n-to-1 and satisfies the following additional property:*

($*$) For every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover V of X such that for each element $U \in \mathcal{U}$ there exists $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $st^n(f(U_i), f(\mathcal{V})) \supset V$ for each $1 \leq i \leq n$.

If X then Y is metrizable.

Theorem 3.1.2. *Let $f : X \to Y$ be coarse and coarsely n-to-1 between general large scale structures X and Y. X has finite asymptotic dimension if and only if Y has finite asymptotic dimension.*

Theorem 3.1.3. *Let $f : X \to Y$ be coarse and coarsely n-to-1. X is large scale finitistic if and only if Y is large scale finitistic.*

Theorem 3.1.4. *Let X and Y be spaces and $f : X \to Y$ coarse and coarsely finite-to-1. If X is large scale weakly paracompact then so is Y.*

Theorem 3.1.5. *Let X and Y be spaces and $f : X \to Y$ coarse and large scale n-to-1. If X is of bounded geometry then Y is of bounded geometry. Conversely, if Y is of bounded geometry then there exists a bounded geometry subspace $Z \subset X$ for which the inclusion iduces a coarse equivalence from Z to X.*

All of these concepts are introduced for the metric setting in the survey Matija Cencelj and Vavpetic (2014a)(see also Matija Cencelj and Vavpetic (2014b)). In the process of proving the above assertions, we show that large scale n-to-1 functions are optimal for pushing forward certain discrete collections of sets. It is therefore another goal of this paper to generalize certain discreteness properties to large scale structures.

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3.2 Surjections of Large Scale Structures

In J. Dydak (2008) the authors define **Large Scale Structures** for any set X. Their motivation was that the definition of coarse structures given by Roe **?** should be equivalent to specifying which collections of subsets of X are uniformly bounded.

Given some set X and $\mathcal{U} \subset 2^X$(the power set of X) the **star** of some subset U of X with respect to $\mathcal{U}$ is defined by by $st(U,\mathcal{U}) = \cup\{V \in \mathcal{U} : V \cap U \neq \emptyset\}$. Given two collections $\mathcal{V}_1, \mathcal{V}_2 \subset 2^X$ the star of $\mathcal{V}_1$ with respect to $\mathcal{V}_2$ is denoted by $st(\mathcal{V}_1, \mathcal{V}_2)$ and is the new collection $\{st(V, \mathcal{V}_2) : V \in \mathcal{V}_1\}$. We define $st(\mathcal{U}) = st(\mathcal{U},\mathcal{U})$ and inductively define higher stars by $st^n(\mathcal{U}) = st(st^{n-1}(\mathcal{U}),\mathcal{U})$. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of some set X, we say that $\mathcal{U}$ **coarsens** $\mathcal{V}$(equivalently $\mathcal{V}$ **refines** $\mathcal{U}$), denoted by $\mathcal{U} \geq \mathcal{V}$, if each element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$. Recall that the **Lebesgue number** of a cover $\mathcal{U}$ of a metric space X is $\sup\{R \in \mathbb{R}_{\geq 0} : \mathcal{U} \geq \{B(x, R) : x \in X\}\}$. Also recall that the **multiplicity** of a cover $\mathcal{U}$ of some set X is the maximum number of elements of $\mathcal{U}$ which contain a point in common and ∞ if no such maximum exists.

Definition 3.2.1 (J. Dydak (2008))**.** A **Large Scale Structure** on a set X is a nonempty set of families $\mathcal{B}$ of subsets of X satisfying

1) $\mathcal{B}_1 \in \mathcal{B}$ implies $\mathcal{B}_2 \in \mathcal{B}$ if each nonsingleton element of $\mathcal{B}_2$ is contained in some element of $\mathcal{B}_1$.

2) If $\mathcal{B}_1$, $\mathcal{B}_2 \in \mathcal{B}$ then $st(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{B}$.

Let $(X, \mathcal{LSS}_X)$ be a large scale structure. Given a uniformly bounded family $\mathcal{B} \in \mathcal{LSS}_X$ we define the trivial extension of $\mathcal{B}$ to be $\mathcal{B} \cup \{\{x\} : x \in X\}$. By 1 above, we have that the collection of singletons is in any large scale structure. By trivially extending all elements of $\mathcal{LSS}$, we simply refer to its elements as uniformly bounded covers instead of collections. This is the equivalent of the diagonal being a controlled set in the setting of coarse structures of Roe **?**. **From now on, we will write only X for the large scale structure $(X, \mathcal{LSS})$ and refer to elements of $\mathcal{LSS}$ as uniformly bounded covers.**

An important example of large scale structures that will be used in the last section is the metric large scale structure. If X is metric then the uniformly bounded collections are collections of subsets of X with bounded mesh; i.e the collections are precisely the refinements of the covers of X by R-balls where $R \geq 0$. We begin to motivate the definition of coarsely n-to-1 maps using this large scale structure as a base model. Recall that for a metric X and $R > 0$ a collection of subsets $\mathcal{B}$ of X is said to be $R-$**discrete** if $dist(U, V) \geq R$ for all $U, V \in \mathcal{B}$. The proof of the following proposition can be found in J. Dydak (2015).

Proposition 3.2.2. *J. Dydak (2015) A function $f : X \to Y$ of metric spaces is coarsely n-to-1 if and only if for every $R, S > 0$ there is a uniformly bounded cover $\mathcal{V}$ of X such that the preimage of an $S-$ball in Y can be covered by at most n elements of $\mathcal{V}$ that are $R-$disjoint.*

The following lemma sheds some light on how to define R-disjointness in the general large scale setting. We will require metric spaces to have midpoints in the following lemma. Recall that a metric space has the midpoint property if for every $x, y \in X$ there exists a point $z \in X$ such that $d(x, z) = d(y, z) = \frac{d(x,y)}{2}$. Every metric space with midpoint property can be completed to be a geodesic metric space, so, up to coarse equivalence, every metric with midpoint property is geodesic.

Lemma 3.2.3. *Let X be a metric space with midpoint property. A collection of closed subsets $\mathcal{V}$ is $R-$disjoint if and only if the collection $st(\mathcal{V}, \{B(x, \frac{R}{2}) : x \in X\})$ is a disjoint family.*

Proof. ($\Rightarrow$) If $\mathcal{V}$ is R-disjoint then the collection $st(\mathcal{V}, \{B(x, \frac{R}{2}) : x \in X\})$ is disjoint by the triangle inequality.

($\Leftarrow$) Assume $st(\mathcal{V}, \{B(x, \frac{R}{2}) : x \in X\})$ is a disjoint family and suppose there is $A, B \in \mathcal{V}$ with $d(A, B) < R$. It follows that there is $x \in A$ and $y \in B$ such that $d(x, y) < R$. Let z be a midpoint between x and y. Notice then that $z \in st(A, \{B(x, \frac{R}{2}) : x \in X\})$ and $z \in st(B, \{B(x, \frac{R}{2}) : x \in X\})$ contrary to our assumption. □

In view of the above proposition, it makes sense to define coarsely n-to-1 maps in two ways for general large scale structures. By the lemma, the notion of R-discreteness can be generalized to large scale structures as follows: Let $\mathcal{W}$ be a uniformly bounded collection of a large scale structure X. A collection $\mathcal{V}$ is said to be $\mathcal{W}-$**discrete** if $st(\mathcal{V}, \mathcal{W})$ is a disjoint family.

Definition 3.2.4. Let X and Y be large scale structures on sets X and Y. A function $f : X \to Y$ is **coarse**(or **bornologous**) if for every $\mathcal{B}_X \in \mathcal{LSS}_X$ there exists some $\mathcal{B}_Y \in \mathcal{LSS}_Y$ such that $f(\mathcal{B}_x)$ refines $\mathcal{B}_Y$.

A function $f : X \to Y$ is **coarsely** n**-to-**1(or **large scale** n**-to-**1) if for every uniformly bounded cover $\mathcal{U}_Y$ of Y there exists a uniformly bounded cover $\mathcal{U}_X$ of X such that for every $B \in \mathcal{U}_Y$ there exists $B_1, B_2, \ldots, B_n \in \mathcal{U}_X$ with $f^{-1}(B) \subset \bigcup_{i=1}^n B_i$.

A function $f : X \to Y$ is **discretely** n**-to-**1 if for every uniformly bounded cover $\mathcal{U}_Y$ of Y and uniformly bounded cover $\mathcal{W}_X$ of X there exists a uniformly bounded cover $\mathcal{U}_X$ of X such that for every $B \in \mathcal{U}_Y$ there exists a $\mathcal{W}_X$-discrete collection $\{B_1, B_2, \ldots, B_n\} \subset \mathcal{U}_{\mathcal{X}}$ with $f^{-1}(B) \subset \bigcup_{i=1}^n B_i$.

Remark 3.2.5. This definition of a coarse map does not assume that the preimage of a bounded set is bounded.

Remark 3.2.6. We will later show that discretely n-to-1 maps are the same as coarsely n-to-1 maps.

The following proposition will be useful later on and its proof is imediate.

Proposition 3.2.7. *Let $f : X \to Y$ be a function between large scale structures. f is coarsely n-to-1 if and only if for every uniformly bounded cover $\mathcal{U}_Y$ of Y there exists a uniformly bounded cover $\mathcal{U}_X$ of X such that for every $B \in \mathcal{U}_Y$ there exists $B_1, B_2, \ldots, B_n \in \mathcal{U}_X$ with $f^{-1}(B) = \bigcup_{i=1}^n B_i$.*

3.3 Metrizability

In this section, we show that metrizability is pushed forward by a certain class of coarsely n-to-1 maps. We recall some results about metrizability for large scale structures.

We also make use of the construction of large scale structures out of metric spaces introduced Chapter 2 section 2.4. We recall some items from that chapter both for the readers convenience and for the opportunity to state some things in different ways.

Proposition 3.3.1. *J. Dydak (2008) If $\mathcal{LSS}'_X$ is a set of families in X such that $\mathcal{B}_\alpha, \mathcal{B}_\beta \in \mathcal{LSS}'_X$ implies the existence of $\mathcal{B}_\gamma \in \mathcal{LSS}'_X$ such that $\mathcal{B}_\alpha \cup \mathcal{B}_\beta \cup st(\mathcal{B}_\alpha, \mathcal{B}_\beta)$ refines $\mathcal{B}_\gamma$, then* *the family $\mathcal{LSS}_X$ of all refinements of trivial extensions of elements of $\mathcal{LSS}'_X$ forms a large scale structure on X.*

A standard set of families that satisfy the above criterion if the collection of covers of a metric space by $n-$balls for $n \in \mathbb{N}$. The large scale structure of a metric space is precisely the the set of all refinements of these covers. It will be convenient to define large scale structures as those generated by a certain collection of uniformly bounded covers just as one defines a topology by specifying a basis. This leads to the following definition.

Definition 3.3.2. Let $(X, \mathcal{LSS}_X)$ be a large scale structure. If there exists a set of families $\mathcal{LSS}$ which satisfy the cirterion of propostion 3.3.1 above then we will say that $\mathcal{LSS}$ is a **large scale basis** for $(X, \mathcal{LSS}_X)$.

The following is a nice characterization of metrizability of a LS-structure in terms of large scale basis. The proof can be found in J. Dydak (2008) and in **?**.

Proposition 3.3.3. *J. Dydak (2008) Let $(X, \mathcal{LSS}_X)$ be a large scale structure. The following are equivalent:*

1) $(X, \mathcal{LSS}_X)$ has a countable large scale basis $\mathcal{LSS}'_X$

2) There exists an ∞ metric d such that the uniformly bounded covers with respect to the metric coincide with $\mathcal{LSS}_X$.

Definition 3.3.4. Let $(X, \mathcal{LSS}_X)$ be a large scale structure. Then X is called **metrizable** if X admits an ∞ metric such that $\mathcal{LSS}_X$ consists of all uniformly bounded collections of subsets of X

Theorem 3.3.5. *Let X and Y be coarse structures and $f : X \to Y$ coarse and coarsely n-to-1 and satisfies the following additional property:*

($*$) For every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover V of X such that for each element $U \in \mathcal{U}$ there exists $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $st^n(f(U_i), f(\mathcal{V})) \supset V$ for each $1 \leq i \leq n$.

If X then Y is metrizable.

Proof. By Theorem 3.3.3 we need only prove that $\mathcal{LSS}_X$ has a countable large scale basis if and only if $\mathcal{LSS}_Y$ has a countable large scale basis.

Let $\mathcal{LSS}'_X$ be a countable large scale basis for X. Consider the collection $\mathcal{LSS}'_Y$ consisting of all possible finite stars of elements of $f(\mathcal{LSS}'_X) := \{f(\mathcal{B}) : \mathcal{B} \in \mathcal{LSS}_X\}$ where $f(\mathcal{B}) := \{f(B) : B \in \mathcal{B}\}$. $\mathcal{LSS}'_Y$ is a countable collection for Y which satisfies the finite additivity condition of proposition 3.3.1. It needs to be shown that the large scale structure on Y generated by $\mathcal{LSS}'_Y$ is $\mathcal{LSS}_Y$. It suffices to show that every $\mathcal{B} \in \mathcal{LSS}_Y$ refines some collection in $\mathcal{LSS}'_Y$.

Let $\mathcal{B}_Y \in \mathcal{LSS}_Y$. We have $\mathcal{B}_X \in \mathcal{LSS}_X$ such that for each $B \in \mathcal{B}_Y$ there is $B_1, B_2, \ldots B_n \in \mathcal{B}_X$ with $B \subset \bigcup_{i=1}^n B_i$ such that $st^n(f(U_i), f(\mathcal{V})) \supset V$ for each $1 \leq i \leq n$. Let $\mathcal{B}'_X \in \mathcal{LSS}'_X$ be a coarsening of $\mathcal{B}_X$. Notice that $st^n(f(\mathcal{B}'_X))$ coarsens the cover $\mathcal{B}_Y$ and $st^n(f(\mathcal{B}'_X)) \in \mathcal{LSS}'_Y$ which completes the claim. □

Question 3.3.6. *Let X and Y be large scale structures and $f : X \to Y$ coarse. Is f is coarsely n-to-1 if and only if for every uniformly bounded cover $\mathcal{U}$ there is a uniformly bounded cover V of X such that for each element $U \in \mathcal{U}$ there exists $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $st^n(f(U_i), f(\mathcal{V})) \supset V$ for each $1 \leq i \leq n$?*

Proposition 3.3.7. *Let X and Y be large scale structures and $f : X \to Y$ a function. f is discretely n-to-1 if and only if f is coarsely n-to-1.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) Let $\mathcal{U}$ be a uniformly bounded cover of Y and $\mathcal{W}_X$ a uniformly bounded cover of X. Let $\mathcal{V}_0$ be a uniformly bounded cover of X such that for every $B \in \mathcal{U}$ there exists $B_1, B_2, \ldots, B_n \in \mathcal{V}$ with $f^{-1}(B) \subset \bigcup_{i=1}^n B_i$. Inductively define $\mathcal{V}_n$ as a uniformly bounded cover of X such that for each element $U \in st^n(\mathcal{U})$ there exists $U_1, U_2, \ldots, U_n \in \mathcal{V}_n$ with $f^{-1}(U) \subset \bigcup_{i=1}^n U_i$.

$\{st^n(\mathcal{U})\}$ is a large scale basis for a metrizable large scale structure (Y, d). The collection of all finite stars of elements of the collection $\{\mathcal{W}_X, \mathcal{V}_n : n \geq 0\}$ is a large scale basis for a metrizable large scale structure (X, d'). (X, d') and (Y, d) have been designed so that $f : (X, d') \to (Y, d)$ is coarsely n-to-1 which means, by proposition 3.2.2, that there exists a uniformly bounded cover $\mathcal{W}$ of (X, d) such that for every $B \in \mathcal{U}$ there exists a $\mathcal{W}_X$-discrete collection $\{B_1, B_2, \ldots, B_n\} \subset \mathcal{U}_{\mathcal{X}}$ with $f^{-1}(B) \subset \bigcup_{i=1}^n B_i$. □

3.4 Asymptotic Dimension and Finitism

The purpose of this section is to translate some ideas from T. Miyata (2013) into the language of large scale structures. We will motivate our definition of asymptotic dimension with the metric case. Let X be metric and $n \geq 0$ an integer.

1) X is said to have **asymptotic dimension at most** n provided that for each $R > 0$ there exists a uniformly bounded cover of X with Lebesgue number greater than R and having multiplicity at most $n + 1$.

Following T. Miyata (2013), we opt for different definition of asymptotic dimension.

2) X is said to have **asymptotic dimension at most** n provided that for each $R > 0$ there exists a uniformly bounded cover $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ where $\mathcal{V}_i$ is an R-disjoint family for each $i = 1, 2, \ldots, n + 1$.

Again, based on the equivalence of the previous two definitions in the metric case, it makes sense to define two notions of asymptotic dimension for large scale structures.

Definition 3.4.1. A large scale structure X is said to have **asymptotic dimension** at most n, denoted by $asdim(X) \leq n$ if For every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded cover $\mathcal{V}$ coarsening $\mathcal{U}$ with multiplicity at most $n + 1$.

A large scale structure X is said to be **large scale finitistic** if for every uniformly bounded cover $\mathcal{U}$ of X there exists $m \geq 1$ and a uniformly bounded cover $\mathcal{V}$ which coarsens $\mathcal{U}$ with multiplicity at most m.

The forward direction of the following proof is an adaptation of the proof for metric spaces found in Bell and Dranishnikov (2011). There G. Bell and A. Dranishnikov prove that these two definitions coincide for the case of metric spaces. The idea for the converse was suggested to the author by Jerzy Dydak. We obtain that the two criterion aforementioned for metric spaces are also equivalent in the class of general large scale structures.

Proposition 3.4.2. *Let $(X, \mathcal{LSS}_X)$ be a large scale structure and $\mathcal{U}$ a uniformly bounded cover of X. The following are equivalent:*

1) There exists a uniformly bounded cover $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ where $st(\mathcal{V}_i, \mathcal{U})$ is a disjoint collection for each $i = 1, 2, \ldots, n$.

2) There exists a uniformly bounded cover $\mathcal{V}$ which coarsens $\mathcal{U}$ with multiplicity at most $n+1$.

Proof. ($\Rightarrow$) Let $\mathcal{U}$ be a uniformly bounded cover of X. Choose a uniformly bounded cover $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ where $st(\mathcal{V}_i, st(\mathcal{U},\mathcal{U}))$ is a disjoint collection for each $i = 1, 2, \ldots, n$. Consider the cover $\mathcal{W} = st(\mathcal{V},\mathcal{U})$. This is a coarsening of $\mathcal{U}$ and we claim that the multiplicity is at most $n+1$. Notice that $st(\mathcal{V}_i,\mathcal{U})$ is disjoint for each $i = 1, 2, \ldots n$ which means that each element of X lies in at most one element of $st(\mathcal{V}_i,\mathcal{U}) \subset \mathcal{W}$. It follows that each element of X belongs to at most $n+1$ elements of $\mathcal{W}$.

($\Leftarrow$) Let $\mathcal{U}$ be a uniformly bounded cover of X. We need to construct a metric model that captures the dimension for X in terms of $\mathcal{U}$. Let $\mathcal{U}_0 = \mathcal{U}$. Let $\mathcal{U}_1$ be a uniformly bounded cover of X which coarsens $\mathcal{U}_0$ and has multiplicity at most $n+1$. Define $\mathcal{U}_2 = st(\mathcal{U}_1,\mathcal{U}_0)$. Notice that this cover coarsens both $\mathcal{U}_0$ and $\mathcal{U}_1$. Let $\mathcal{U}_3$ be a uniformly bounded cover of X which coarsens $\mathcal{U}_2$ and has multiplicity at most $n+1$. Continue as follows: for $k \geq 3$ define U_k as a uniformly bounded cover of X which coarsens $st(\mathcal{U}_{k-1},\mathcal{U}_{k-2})$ and has multiplicity at most $n+1$. Observe that $\mathcal{U}_{i+1}$ coarsens $\mathcal{U}_i$ for $i \geq 0$.

<u>Claim :</u> The collection $\{\mathcal{U}_i : i \geq 0\}$ is a large scale basis for a metric large scale structure on X with asymptotic dimension at most n. Furthermore, the uniformly bounded covers of this metric large scale structure generated by $\mathcal{U}$ are uniformly bounded in $(X, \mathcal{LSS}_X)$

proof of claim: Indeed, let $k > l \geq 0$ be integers and notice that $\mathcal{U}_k$ and $\mathcal{U}_l$ has the property that $\mathcal{U}_k \cup \mathcal{U}_l \cup st(\mathcal{U}_l,\mathcal{U}_k)$ refines $\mathcal{U}_{k+1}$. To see this, observe that $\mathcal{U}_l$ is coarsened by $\mathcal{U}_{k-1}$ and we have $\mathcal{U}_{k+1}$ coarsens $st(\mathcal{U}_k,\mathcal{U}_{k-1})$ which in turn coarsens $st(\mathcal{U}_k,\mathcal{U}_l)$. In this case, $st(\mathcal{U}_k,\mathcal{U}_l)$ coarsens $st(\mathcal{U}_k,\mathcal{U}_l) \cup \mathcal{U}_k \cup \mathcal{U}_l$ because the collection $\{\mathcal{U}_i : i \geq 0\}$ is nested.

$\{\mathcal{U}_i : i \geq 0\}$ is countable and so generates a metric large scale structure by proposition 3.3.3. Use the fact that every element of the large scale basis generating this metric structure has multiplicity at most $n+1$ to see that it has dimension at most n. □

X with the metric structure generated by $\mathcal{U}$ is metric of dimension at most n. There exists a uniformly bounded cover $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ of X with the metric stucture where $st(\mathcal{V}_i,\mathcal{U})$ is a disjoint collection for each $i = 1, 2, \ldots, n$. $\mathcal{V}$ is uniformly bounded in $(X, \mathcal{LSS}_X)$ which completes the proof.

□

Theorem 3.4.3. *Let $f : X \to Y$ be coarse and coarsely n-to-1. X has finite asymptotic dimension if and only if Y has finite asymptotic dimension.*

Proof. ($\Rightarrow$) Say, $asdim_d(X) \leq m-1$. Let $\mathcal{U}$ be a uniformly bounded cover of Y. Let $\mathcal{W}$ be a uniformly bounded cover of X such that the preimage of element of $\mathcal{U}$ can be covered by at most n elements of $\mathcal{W}$. Let $\mathcal{V} = \bigcup_{i=0}^{m} \mathcal{V}_i$ be a cover of X such that $\mathcal{V}_i$ is $\mathcal{W}$-disjoint for each $i = 0, 1, 2, \ldots, m$. Consider the cover $\mathcal{W}_0 := st(f(\mathcal{V}), \mathcal{U})$ of Y. We claim that this cover has multiplicity bounded by $n \cdot m$. Consider the multiplicity of $st(f(\mathcal{V}_i), \mathcal{U})$. Let $y \in Y$ and consider how many elements of $st(f(\mathcal{V}_i), \mathcal{U})$ could contain y. $f^{-1}(y)$ can be covered by at most n elements of $\mathcal{W}$ and $\mathcal{V}_i$ is $\mathcal{W}-$disjoint which means that y could only belong to at most n elements of $st(f(\mathcal{V}_i), \mathcal{U})$. Notice that this implies y belongs to at most $n \cdot m$ elements of $\mathcal{W}_0$.

($\Leftarrow$) Say $asdim_d(Y) \leq m-1$. Let $\mathcal{U}$ be a uniformly bounded cover of X. Then $f(\mathcal{U})$ is a uniformly bounded cover of Y and so we can find a cover $\mathcal{V} = \bigcup_{i=1}^{m} \mathcal{V}_i$ where $\mathcal{V}_i$ is $f(\mathcal{U})$-disjoint. Let $\mathcal{W}_0$ be a uniformly bounded cover of X such that for every $B \in \mathcal{V}$ there exists $B_1, B_2, \ldots, B_n \in \mathcal{W}_0$ with $f^{-1}(B) = \bigcup_{i=1}^{n} B_i$. Let $\mathcal{W}_i = \{B_1, B_2, \ldots, B_n : B \in \mathcal{V}_i\}$ and let $\mathcal{W} = \bigcup_{i=1}^{n} \mathcal{W}_i$

Consider $\mathcal{B} = st(\mathcal{W}, \mathcal{U}) = \bigcup_{i=1}^{n} st(\mathcal{W}_i, \mathcal{U})$; it is a coarsening of $\mathcal{U}$. We need only show it has bounded multiplicity. Fix $i \in \{1, 2, \ldots n\}$ and consider the multiplicity of $st(\mathcal{W}_i, \mathcal{U})$. Notice that for distinct $A, B \in \mathcal{V}_i$ their preimages A_i and B_j are $\mathcal{U}$ disjoint for $i, j = 1, 2, \ldots n$. It follows that the multiplicity of $st(\mathcal{W}_i, \mathcal{U})$ is at most n. Thus the multiplicity of $\mathcal{B}$ is at most mn and so $asdim(X) \leq m \cdot n$. □

The proof of 3.4.3 could be used verbatum to prove the following

Theorem 3.4.4. *Let $f : X \to Y$ be coarse and coarsely n-to-1. X is large scale finitistic if and only if Y is large scale finitistic.*

3.5 Invariance of Large Scale Weak Paracompactness

In Matija Cencelj and Vavpetic (2014a) the authors show that one can define a metric space X to be large scale weakly paracompact if for every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded cover $\mathcal{V}$ so that each element of $\mathcal{U}$ intersects at most finitely many elements of $\mathcal{V}$. We approach large scale weak paracompactness for large scale structures using this train of thought.

Definition 3.5.1. A space X is **large scale weakly paracompact** for every uniformly bounded cover $\mathcal{U}$ of X there exists a uniformly bounded cover $\mathcal{V}$ so that each element of $\mathcal{U}$ intersects at most finitely many elements of $\mathcal{V}$.

Theorem 3.5.2. *Let X and Y be spaces and $f : X \to Y$ coarse and coarsely finite-to-*1*. X is large scale weakly paracompact then so is Y.*

Proof. Let $\mathcal{U}$ be a uniformly bounded cover of Y. Let $\mathcal{V}$ be a uniformly bounded cover of X such that for every $U \in \mathcal{U}$ there exists $U_1, U_2, \ldots, U_n \in \mathcal{V}$ with $f^{-1}(U) = \bigcup_{i=1}^{n} U_i$. Let $\mathcal{W}$ be a uniformly bounded covering of X for which every element of $\mathcal{U}$ interesects only finitely many elements of $\mathcal{W}$. Consider the uniformly bounded cover $f(\mathcal{W})$ of Y. We claim that each element of $\mathcal{V}$ intersects only finitely many elements of $f(\mathcal{W})$. To see this, let $V \in \mathcal{V}$. We have that $f^{-1}(V) \subset \bigcup_{i=1}^{n} V_i$ for $V_1, V_2, \ldots, V_n \in \mathcal{U}$. V_i intersects at most finitely many, say at most m for some $m \geq 1$, elements of $\mathcal{W}$ for each $i = 1, 2, \ldots, n$. It follows that $\mathcal{U}$ intersects at most $n \cdot m$ elements of $f(\mathcal{U})$. □

3.6 Spaces of Bounded Geometry

From now on, all work will be conducted in the metric setting. The purpose of this section is show that large scale n to 1 maps preserve the property of being of bounded geometry. To cut down on wordiness we will use the following termonology: Let $R > 0$. An R-point in a metric space X is a subset $A \subset X$ with $diam(A) \leq R$.

Definition 3.6.1. A space X is said to have **bounded geometry** if for every $R > 0$ there is $m(R) > 0$ so that each R-ball contains no more than $m(R)$ elements.

Typical examples of bouded geometry spaces of finitely generated groups with the Cayley graph metric. In view of the Svarč Milnor Lemma, this gives a wealth of examples of metric spaces X which admit proper and cocompact actions by finitely generated groups.

Recall that a map $f : X \to Y$ of metric spaces is called a **coarse embedding** if there exists nondecreasing functions $p_-^+ : [0,\infty) \to [0,\infty)$ such that $p_-(d(x,y)) \leq d(f(x), f(y)) \leq p^+(d(x,y))$; i.e. f preserves the coarsening of coverings. If is a **coarse equivalence** if the image of X is an R net in Y for some $R > 0$. Note in particular that an inclusion $i : X \hookrightarrow Y$ is a coarse equivalence if and only if X is an R-net in Y for some $R > 0$.

A reason to consider spaces of bounded geometry is that the the Rips Complex simplicial approxiations are locally finite and hence metrizable. Typically, a metric space X is approximated by Rips complexes of a coarsely equivalent bounded geometry subspace. The main result of this section says that if there exists a large scale n-to-1 function $f : X \to Y$ then X can be approximated by metrizable Rips complexes of coarsely equivalent subspaces if and only if the same is true for Y.

Theorem 3.6.2. *Let X and Y be spaces and $f : X \to Y$ coarse and large scale n-to-1. If X is of bounded geometry then Y is of bounded geometry. Conversely, if Y is of bounded geometry then there exists a bounded geometry subspace $Z \subset X$ for which the inclusion iduces a coarse equivalence from Z to X.*

Proof. ($\Rightarrow$) Let $R > 0$. Define $l(R) > 0$ so that point inverse of R-points in Y can be covered by at most n $l(R)$-points in X. Notice that the number of elements in the original R-point cannot exceed the number of elements in the cover of the point inverse by n $l(R)$-points and each $l(R)$-point in X has at most $m((l(R))$ elements. It follows that each R-point has at most $n \cdot m(l(R))$ points and so Y has bounded geometry.

($\Leftarrow$) Given $R > 0$ there exists $m > 0$ such that the inverse image of R-points in Y can be covered by n m-points in X. For each $y \in Y$ we define the subset $R_y \subset X$ to be a set containing one point from each of the m-points covering $f^{-1}(B(y,R))$.

Consider $Z = \bigcup_{y \in Y} R_y$. To see that Z is of bounded geometry, let $n > 0$ and $\mathcal{U}_n$ be the uniformly bounded cover of Z by n-balls and consider $|U|$, the cardinality of U, for some $U \in \mathcal{U}_n$. Notice that f takes $\mathcal{U}_n$ to a uniformly bounded cover of Y which can be coarsened by the uniformly bounded cover of Y by S-balls for some $S > R$. Y is of bounded geometry(and hence is coarsely doubling, see Matija Cencelj and Vavpetic (2014a)) which means that there is $l(S) > 0$ for which each ball of radius S can be covered by at most $l(S)$ R-points. Let $p(R)$ be the maximum number of elements contained in a R-point in Y. Thus the image of U lies in a S-ball which has has at most $l(S) \cdot p(R)$ points. If follows that U contains at most $n \cdot l(S) \cdot p(R)$ points and so Z is of bounded geometry.

To see that Z is coarsely equivalent to X, just observe that Z forms a net in X. Indeed, every element of the cover by $2m$ balls of X contains a point of Z. □

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Vita

Kyle Austin was born in New Albany Indiana which is just across the Ohio River from Louisville, Kentucky. His father, Mike , is a mason by trade and he and my two uncles are the last in the long line of masons coming from the Austin side of my family. His mother, Cookie, is an outstanding member of the local community. She has personally touched the lives of many underprivileged children as well as me ad my three siblings; Erica, Leslie, and Alan.

Kyle attended Indiana University Southeast in fall 2007 and received his B.S. in mathematics in the spring of 2011. He had orinally planned on studying philosophy, but he changed my mind after witnissing the enthusiasm of my mathematcis lecturer. He had never experienced someone enjoying mathematics and, for the first time in his life, found that he had a deep love for mathematics. Kyle decided to attend the University of Tennessee because one of my undergraduate advisors, John Lagrange, had received his doctorate there. Kyle took my first official topology course from Jerzy Dydak in the fall 2011 and was very attracted to his style of thinking. Dydak's style could best be described as that of an extension theoretic topologist which is the style of thinking and arguments Kyle uses this dissertation.

Kyle plans on persuing the study of topology and extension theory as a postdoctoral researcher at Ben Gurion University of the Negev in Israel with a colleague of Dydak's, Michael Levin.