12-1960

Concentric Tori in the Three-Spere

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University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Charles Henry Edwards Jr. entitled "Concentric Tori in the Three-Spere." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Dr. O.G. Harrold, Jr., Major Professor

We have read this dissertation and recommend its acceptance:

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

December 2, 1960

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O. G. Harrold, Jr.

Major Professor

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CONCENTRIC TORI IN THE THREE-SPHERE

A Dissertation

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Charles Henry Edwards, Jr.

December 1960

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor O. G. Harrold, Jr., for his direction and assistance in the preparation of this dissertation. This research was supported in part through the National Science Foundation Grant 2793.

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CHAPTER I

INTRODUCTION

A torus is the topological product of two circles, while a solid torus is the topological product of a circle and a disk. Two solid tori B_1 and B_2 in the three-sphere S^3, with B_2 interior to B_1, are said to be concentric if and only if the closure of $B_1 - B_2$ (the set of points in B_1 but not in B_2) is homeomorphic to the topological product of a torus and a closed interval. Two tori in S^3 are concentric if and only if they are respectively the boundaries of two concentric solid tori.

If the polyhedral solid torus B_2 is interior to the polyhedral solid torus B_1, then the order of B_1 with respect to B_2 is defined to be the minimal number of points of intersection of a center line of B_2 with a meridianal disk of B_1 (see Definitions 2.5 and 2.7). The two solid tori B_1 and B_2 in S^3 are said to be equivalently knotted if and only if there exists an orientation-preserving semilinear homeomorphism of S^3 onto itself carrying a center line of B_1 onto a center line of B_2. The principal objective of Chapter II is to characterize pairs of concentric polyhedral solid tori in S^3 by proving that two polyhedral solid tori B_1 and B_2, with $B_2 \subset \operatorname{Int} B_1$ (the interior of a set S is denoted by $\operatorname{Int} S$), are concentric if and only if they are equivalently knotted and the order of B_1 with respect to B_2 is unity.

This characterization of concentricity is used in Chapter III to determine conditions under which a sequence $\{B_n\}_0^\infty$ of polyhedral solid tori has the property that there exists an integer N such that B_i and B_j are concentric whenever $i > j \geqq N$. It is proved, for instance,

that the sequence has this property if $B_0 = \bigcap_{n=1}^{\infty} B_n$ and $B_{n+1} \subset \operatorname{Int} B_n$ for $n = 1, 2, \ldots$.

In Chapter IV is proved the principal theorem of this thesis, to the effect that the three-sphere does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric. Results of Whyburn [13]* on non-separated cuttings of connected spaces are applied to show that, if $\mathcal{S}$ is an uncountable collection of mutually disjoint tori in S^3 , then there is a sequence $\{T_n\}_0^{\infty}$ of elements of $\mathcal{S}$ and a sequence $\{B_n\}_0^{\infty}$ of solid tori such that T_n is the boundary of B_n and such that $B_0 = \bigcap_{n=1}^{\infty} B_n$ with $B_{n+1} \subset \operatorname{Int} B_n$ for $n = 1, 2, \ldots$. A theorem of Bing [3] is then employed to find a homeomorphism f of S^3 onto itself which carries each of the solid tori B_n onto a polyhedral set. An application of the theorem quoted in the previous paragraph then yields the desired conclusion.

In Chapter V a generalization of the relation of concentricity is investigated. Two polyhedral solid tori B_1 and B_2 are said to be equivalently situated if and only if there is a polyhedral solid torus B which is interior to both B_1 and B_2 and is concentric with each. If B_1 is interior to B_2 , then the tori B_1 and B_2 are equivalently situated if and only if they are concentric. It is proved that, if $\{B_n\}_1^{\infty}$ is a sequence of polyhedral solid tori converging homeomorphically to the polyhedral solid torus B , there is an integer N such that B_n and B are equivalently situated for $n \geqq N$. It is also proved that, if B' and B'' are two

*Numbers in square brackets refer to numbers in the bibliography at the end of this paper.

concentric polyhedral solid tori, and if B_1 and B_2 are two polyhedral solid tori, each containing B' in its interior and each interior to B'', then there exists a semilinear homeomorphism f of S^3 onto itself carrying B_1 onto B_2 and such that f is the identity on $B' \bigcup (S^3 - B'')$.

In Chapter VI some concepts involving concentric tori are used to obtain a new characterization of tame simple closed curves in S^3 (a simple closed curve in S^3 is tame if it can be carried onto a polygonal knot by a homeomorphism of S^3). The simple closed curve J is said to have the concentral enclosure property if and only if there is a decreasing sequence of concentric polyhedral solid tori intersecting in J. It is shown that a simple closed curve is tame if and only if it has the concentral enclosure property and pierces a disk at each point (see Definition 6.6).

CHAPTER II

PAIRS OF CONCENTRIC TORI

Definition 2.1. A *torus* is the topological product of two circles.

Definition 2.2. A *solid torus* is the topological product of a circle and a disk.

Definition 2.3. Two solid tori B_1 and B_2 with $B_2 \subset \text{Int } B_1$ are *concentric* if and only if $\text{Cl}(B_1 - B_2)$ (the closure of a set S is denoted by $\text{Cl } S$) is the topological product of a torus and a closed interval.

Definition 2.4. Two tori T_1 and T_2 in S^3 are *concentric* if and only if there exist two concentric solid tori B_1 and B_2 in S^3 such that $T_1 = \text{Bd } B_1$ and $T_2 = \text{Bd } B_2$ (the boundary of a set S is denoted by $\text{Bd } S$).

Use will be made of some elementary results from the homology theory of polyhedral complexes, as discussed in Chapter 3 of the book *Lehrbuch der Topologie*, by Seifert and Threlfall [12]. In this thesis $r \sim s$ or $r \not\sim s$ will be written according as the cycles r and s are or are not homologous.

The first step towards the determination of a set of conditions, under which two polyhedral solid tori will be concentric, is provided by the following modification of the concentric toral theorem of Harrold, Griffith, and Posey [10].

Theorem 2.1. *Suppose that B_1 and B_2 are polyhedral solid tori in S^3 with boundaries T_1 and T_2 respectively such that*

(a) $B_2 \subset \text{Int } B_1$,

(b) *There is a polyhedral annular ring G in $\text{Cl}(B_1 - B_2)$ such*

that $G \cap T_i$ *is a simple closed curve* s_i *for* $i = 1, 2$,

(c) *There is a pair of disjoint polyhedral disks* D^a *and* D^b *such that* $D_i^a = D^a \cap B_i$ *and* $D_i^b = D^b \cap B_i$ *are disks with boundaries* r_i^a *and* r_i^b *respectively. Furthermore the* 1-*cycles* r_i^a, r_i^b, s_i *satisfy* $r_i^a \not\sim s_i$, $r_i^b \not\sim s_i$ *on* T_i, *and none is nullhomologous on* T_i, $i = 1, 2$. *Finally* s_i *meets* r_i^a *and* r_i^b *each in a single point,* $i = 1, 2$,

(d) $D_i^a \cap G$ *and* $D_i^b \cap G$, $i = 1, 2$, *are arcs spanning* Bd G (*each has exactly one endpoint on each of the two components of* Bd G).

Then B_1 *and* B_2 *are concentric.*

Proof. The statement of Theorem 2.1 differs from the statement of the concentric toral theorem only in that Harrold, Griffith, and Posey assumed G to be a disk rather than an annular ring. Since, however, this disk was used merely to obtain an annular ring with the properties described above, the proof of the concentric toral theorem applies here without essential alterations.

Definition 2.5. Suppose that σ is a semilinear mapping of a right prism P onto the solid torus B such that, if corresponding points of the two bases of P are identified, the mapping then induced by σ is a homeomorphism. Let e be the boundary of the lower base of P. Then $\sigma(e)$ is a simple closed curve on Bd B which is nullhomologous in B but not in Bd B. Those simple closed curves on Bd B which are homologous to $\sigma(e)$ are called *meridians* of B. It is easily seen that the meridians of B are the only paths on Bd B which bound in B but not in Bd B [11, p. 147]. A polyhedral disk D, such that Int $D \subset$ Int B and such that Bd D is a meridian of B, is called a

meridianal disk of B .

Definition 2.6. Suppose that K is a polyhedral 3-cell in S^3 . By a chord of K is meant an oriented polygonal arc u whose endpoints lie on Bd K but which is otherwise contained in the interior of K . Let the endpoints of u be joined by an arc w on Bd K . The chord u is said to be an unknotted chord of K if and only if $u \cup w$ is an unknotted simple closed curve (bounds a disk in S^3). It may be seen that the knot type of $u \cup w$ is independent of the choice of $w \subset \mathrm{Bd}\, K$ [11, p. 155].

Definition 2.7. Suppose that B is a polyhedral solid torus in S^3 . Let B be separated by two disjoint meridianal disks D_1 and D_2 into two 3-cells K_1 and K_2 . Choose two points $x \in \mathrm{Int}\, D_1$ and $y \in \mathrm{Int}\, D_2$. Let a_1 and a_2 be unknotted chords of K_1 and K_2 respectively joining the points x and y , a_1 directed from x to y and a_2 from y to x . The oriented simple closed curve $a_1 \cup a_2$ is called an oriented center line of the solid torus B . If the orientation of $a = a_1 \cup a_2$ is disregarded, then a is called simply a center line of B . It is easily seen that any two center lines a and a' of B are equivalent in the sense that there exists a semilinear mapping of S^3 onto itself, the identity on $S^3 - B$, carrying a onto a' [11, p. 158].

Definition 2.8. A solid torus B in S^3 is unknotted if and only if each of its center lines is unknotted.

Theorem 2.2. *Suppose that B_1 and B_2 are two polyhedral solid tori in S^3, with B_2 knotted and interior to B_1. Suppose further that there exist two polyhedral annular rings R_1 and R_2 in* $\mathrm{Cl}\,(B_1 - B_2)$

<u>with R_2 unknotted (its boundary components being unknotted curves) such that</u>

(a) $R_1 \cap \mathrm{Bd}\ B_i = s_i$ <u>and</u> $R_2 \cap \mathrm{Bd}\ B_i = r_i$ <u>are simple closed curves not nullhomologous on</u> $\mathrm{Bd}\ B_i$, <u>with</u> $r_i \not\sim s_i$ <u>and with</u> $r_i \cap s_i$ <u>being a single point for</u> $i = 1, 2$,

(b) $R_1 \cap R_2$ <u>is an arc spanning</u> $\mathrm{Bd}\ R_1$.

<u>Then</u> B_1 <u>and</u> B_2 <u>are concentric</u>.

<u>Proof</u>: An unknotted simple closed curve on the boundary $\mathrm{Bd}\ B$ of a knotted polyhedral solid torus B is either nullhomologous on $\mathrm{Bd}\ B$ or is a meridian of B [11, p. 164]. It therefore follows from the hypotheses that r_2 is a meridian of B_2, and hence bounds a polyhedral disk D_2^a whose interior is interior to B_2. Define $D^a = R_2 \cup D_2^a$, and rename $r_i = r_i^a$, $i = 1, 2$. Then D^a is a meridianal disk of B_1 whose intersection with B_2 is a meridianal disk of B_2.

If D^b is a meridianal disk of B_1 sufficiently near to D^a but disjoint with D^a, then $D^b \cap \mathrm{Bd}\ B_i = r_i^b$, $i = 1, 2$, will be simple closed curves not nullhomologous on $\mathrm{Bd}\ B_i$ and not homologous to s_i, and intersecting s_i in a single point. The hypotheses of Theorem 2.1 are then satisfied, so that it follows that B_1 and B_2 are concentric.

<u>Definition</u> 2.9. By a <u>longitude curve</u> of the polyhedral solid torus B is meant a curve on $\mathrm{Bd}\ B$ which is not nullhomologous in B and which crosses some meridian of B at exactly one point. A meridian curve and a longitude curve of B together generate the 1-dimensional homology group of $\mathrm{Bd}\ B$.

<u>Definition</u> 2.10. Suppose that B is a polyhedral solid torus in S^3. By a <u>latitude curve</u> of B is meant a simple closed curve on $\mathrm{Bd}\ B$ which is

not nullhomologous in B but which is nullhomologous in $Cl(S^3 - B)$. Any two latitude curves of B are homologous on Bd B [11, p. 161].

Theorem 2.3. Suppose that B_1 and B_2 are two polyhedral solid tori in S^3 with boundaries T_1 and T_2 respectively and with $B_2 \subset \text{Int } B_1$. If there exist two polyhedral annular rings R_1 and R_2 in $Cl(B_1 - B_2)$, with R_1 bounded by two longitude curves on T_1 and T_2 respectively, with R_2 bounded by two meridian curves on T_1 and T_2 respectively, and with $R_1 \cap R_2$ being an arc spanning Bd R_1, then B_1 and B_2 are concentric.

Proof: Define $s_i = R_1 \cap T_i$ and $r_i^a = R_2 \cap T_i$ for $i = 1, 2$. By hypothesis r_2^a is a meridian of B_2 and hence bounds a meridianal disk D_2^a of B_2. Then $D^a = R_2 \cup D_2^a$ is a meridianal disk of B_1 whose intersection with B_2 is a meridianal disk of B_2. It follows from the hypotheses that $s_i \not\sim 0$, $r_i^a \not\sim 0$, $s_i \not\sim r_i^a$ on T_i for $i = 1, 2$.

If D^b is a meridianal disk of B_1 sufficiently close to D^a but disjoint with D^a, then $D_2^b = D^b \cap B_2$ is a meridianal disk of B_2, and $D^b \cap T_i = r_i^b$ is a simple closed curve not nullhomologous on T_i and not homologous to s_i on T_i, and intersecting s_i in a single point for $i = 1, 2$. Then $D^b \cap R_1$ will be an arc spanning Bd R_1, just as does $D^a \cap R_1$.

Theorem 2.1 now applies directly to imply that B_1 and B_2 are concentric.

Definition 2.11. Suppose that B is a polyhedral solid torus with k a polygonal simple closed curve interior to B. The order of B with respect to k is defined to be the minimal number of points of intersection of a meridianal disk of B with the curve k (for all meridianal disks of B), and is denoted by $\mathcal{O}(B, k)$.

Definition 2.12. If B_1 and B_2 are polyhedral solid tori with $B_2 \subset \operatorname{Int} B_1$, the order of B_1 with respect to B_2 is defined to be the order of B_1 with respect to a center line of B_2, and is denoted by $\mathcal{O}(B_1, B_2)$. This definition of $\mathcal{O}(B_1, B_2)$ is independent of the choice of center line of B_2 [11, p. 172].

Definition 2.13. Two oriented polygonal simple closed curves in S^3 are said to be equivalent if and only if there is an orientation-preserving semilinear homeomorphism of S^3 onto S^3 carrying one onto the other. The equivalence classes of simple closed polygons which are defined by this relation are called knots.

Definition 2.14. Two polyhedral solid tori B_1 and B_2 in S^3 are said to be equivalently knotted if and only if any two center lines C_1 of B_1 and C_2 of B_2 can be so oriented as to be equivalent.

Definition 2.15. Let k_1 and k_2 be two knots in S^3. Let S be a polyhedral 2-sphere in S^3, and denote by C_1 and C_2 the closures of the two components of $S^3 - S$. Choose a polygonal arc w on S with endpoints x and y. Then choose chords u_1 (from x to y) and u_2 (from y to x) of C_1 and C_2 respectively, each with endpoints x and y, such that $u_1 \cup w$ (oriented as u_1) is a representative of the knot k_1 and $u_2 \cup w$ (oriented as u_2) is a representative of k_2. The knot represented by the oriented polygon $u_1 \cup u_2$ is then defined to be the product k_1k_2 of the knots k_1 and k_2.

This product of knots is associative and commutative. The knot represented by a plane circle plays the role of an identity: if one factor of a product of two knots is unknotted, then the product is equal to the other factor, and conversely. However, no knot other than this unique

identity has an inverse with respect to this multiplication--the identity knot cannot be expressed as a knot product containing non-identity factors [11, p. 156].

Theorem 2.4. *Let* B *and* B^* *be two polyhedral solid tori in* S^3 *with* $B \subset \operatorname{Int} B^*$. *Then* B *and* B^* *are concentric if and only if they are equivalently knotted with* $\bigcirc(B^*, B) = 1$.

Proof of Sufficiency: The construction here employed is quite similar to that used by Schubert in proving the weaker result that, if B and B^* are equivalently knotted with $\bigcirc(B^*, B) = 1$, then there is a semi-linear homeomorphism of S^3 onto itself carrying B^* onto B and leaving fixed points outside an arbitrary neighborhood of $\operatorname{Cl}(B^* - B)$ [11, p. 178].

Let $T = \operatorname{Bd} B$ and $T^* = \operatorname{Bd} B^*$. Since B^* is of order 1 with respect to B, there is a meridianal disk D^* of B^* which intersects B in a meridianal disk D [11, p. 174]. Let E^* be a second meridianal disk of B^* disjoint with D^* but sufficiently near D^* that E^* intersects B in a meridianal disk E. Then D^* and E^* will separate B^* into two 3-cells C_1^* and C_2^*, with C_i^* bounded by the union of the two disks D^* and E^* with an annular ring $G_i^* \subset T^*$ for $i = 1, 2$. Similarly D and E will separate B into two 3-cells $C_1 \subset C_1^*$ and $C_2 \subset C_2^*$, with C_i bounded by the union of the two disks D and E with an annular ring $G_i \subset T$ for $i = 1, 2$.

Now let a be a center line of the solid torus B, with $a = a_1 \cup a_2$, where a_1 and a_2 are unknotted chords of the 3-cells C_1 and C_2 respectively. Since $\bigcirc(B^*, B) = 1$ by hypothesis, it follows from a theorem of Schubert [11, p. 171] that either a is a center line of B^* or a is the product of a knot, different from the identity, with the knot

represented by a center line of B^*. Because a is a center line of B, it belongs to the same knot as any center line of B^*, since B and B^* are by hypothesis equivalently knotted. It then follows from the remarks following Definition 2.15 that a must be a center line of B^*. Therefore a_1 and a_2 are unknotted chords of C_1^* and C_2^* respectively [11, p. 161].

It follows finally that $U_1 = Cl(C_1^* - C_1)$ and $U_2 = Cl(C_2^* - C_2)$ are unknotted solid tori on each of which $d = Bd\ D$ and $e = Bd\ E$ are latitude curves [11, p. 161]. Then $d^* = Bd\ D^*$ and $e^* = Bd\ E^*$ are also latitude curves on U_1 and U_2 because d and d^* bound the annular ring $R = Cl(D^* - D)$ on $Bd\ U_1 \cap Bd\ U_2$, while e and e^* bound the annular ring $S = Cl(E^* - E)$ on $Bd\ U_1 \cap Bd\ U_2$.

Now let m_1 be a meridian curve of U_1 which intersects each of the curves d, d^*, e, e^* in exactly one point, and let M_1 be a meridianal disk of U_1 bounded by m_1. Let m_2 be a meridian curve of U_2 which intersects each of the latitude curves d, d^*, e, e^* in exactly one point and such that $m_1 \cap R = m_2 \cap R$ and $m_1 \cap S = m_2 \cap S$. That such a meridian of U_2 can be found follows from the elementary observation that, if the boundary T of a solid torus U is separated into two annular rings A_1 and A_2 by two latitude curves n_1 and n_2, and if a is an arc in A_1 with one endpoint on n_1 and the other on n_2, then there is an arc b in A_2 joining the endpoints of a in such a way that $a \cup b$ is a meridian curve of U (this is evident from an examination of the fundamental polygon of a torus). Then let M_2 be a meridianal disk of U_2 bounded by m_2.

Finally define $R_1 = Cl(D^* - D)$ and $R_2 = M_1 \cup M_2$. It is clear that R_1 is an annular ring with boundary components d and d^* which are

meridian curves of B and B^* respectively. If $\ell = (m_1 \cap T) \cup (m_2 \cap T)$ and $\ell^* = (m_1 \cap T^*) \cup (m_2 \cap T^*)$, then clearly ℓ and ℓ^* are longitude curves on T and T^* respectively (since $m_1 \cap d = m_2 \cap d$ and $m_1 \cap d^* = m_2 \cap d^*$ are single points) which together form the boundary of R_2. Since also R_2 is the union of the two disks M_1 and M_2 intersecting in a pair of arcs common to their boundaries, it follows that R_2 is an annular ring.

Since the annular rings R_1 and R_2 are both contained in $U_1 \cup U_2 = Cl(B^* - B)$, and since $R_1 \cap R_2$ is the arc $M_1 \cap R = M_2 \cap R$ spanning the boundaries of R_1 and R_2, it follows from Theorem 2.3 that B^* and B are concentric.

<u>Proof of Necessity</u>: Suppose now that B and B^* are concentric. Then $Cl(B^* - B)$ can be regarded as the topological product of T and the closed unit interval $[0, 1]$ with $T \times \{0\} = T$ and $T \times \{1\} = T^*$. Since B and B^* are polyhedral, it may be assumed that if $K \subset T$ is polyhedral, then $K \times [0, 1]$ is polyhedral.

Let m_1 and m_2 be two meridian curves on T bounding disjoint meridianal disks D_1 and D_2 respectively of B. Define R_i to be the annular ring $m_i \times [0, 1]$ for $i = 1, 2$. Then $D_1^* = D_1 \cup R_1$ and $D_2^* = D_2 \cup R_2$ are disks whose interiors are contained in Int B^* and whose boundaries are the simple closed curves $m_1^* = m_1 \times \{1\}$ and $m_2^* = m_2 \times \{1\}$ respectively on T^*. Since the projection mapping $p \times \{0\} \longrightarrow p \times \{1\}$, being a homeomorphism carrying the torus T onto the torus T^*, carries non-nullhomologous curves on T onto non-nullhomologous curves on T^*, it follows that m_1^* and m_2^* are not nullhomologous on T^*. Consequently D_1^* and D_2^* are disjoint meridianal disks of B^* intersecting B in the meridianal disks D_1 and D_2 respectively.

Hence D_1 and D_2 separate B into two closed 3-cells K_1 and K_2 respectively, while D_1^* and D_2^* separate B^* into two closed 3-cells K_1^* and K_2^* respectively. Let the subscripts be chosen in such a manner that $K_1 \subset K_1^*$ and $K_2 \subset K_2^*$.

Now let u be a center line of B which intersects the 3-cells K_1 and K_2 in the unknotted chords u_1 and u_2 respectively. Then u_1 and u_2 are chords of K_1^* and K_2^* respectively, and to show that u is also a centerline of B^*, it suffices to show that u_1 and u_2 are unknotted in K_1^* and K_2^* respectively (see Definition 2.7).

But since $Cl(K_i^* - K_i)$ is the solid torus $G_i \times [0, 1]$, where G_1 and G_2 are the two annular rings into which m_1 and m_2 separate T, it follows [11, p. 167] that u_i generates the same knot in K_i and K_i^* for $i = 1, 2$. Thus u_1 and u_2 are unknotted chords of K_1^* and K_2^* respectively, so that u is a center line of B^*. But the fact that B and B^* share the common center line u implies immediately that B and B^* are equivalently knotted with $\mathcal{O}(B^*, B) = 1$. This completes the proof of Theorem 2.4.

Lemma 2.1. Suppose that B *and* B^* *are polyhedral solid tori in* S^3 *with* $B \subset \operatorname{Int} B^*$. *Then* $\mathcal{O}(B^*, B) = 1$ *if and only if there is a polyhedral annular ring* R *in* $Cl(B^* - B)$ *bounded by two meridian curves* m *and* m^* *on* B *and* B^* *respectively.*

Proof: Suppose that $\mathcal{O}(B^*, B) = 1$. Then there is a meridianal disk D^* of B^* such that $D = D^* \cap B$ is a meridianal disk of B. Thus $R = Cl(D^* - D)$ is an annular ring in $Cl(B^* - B)$ bounded by two meridian curves on B and B^* respectively.

To prove the converse, assume the existence of the annular ring R. Let k be a center line of B, and let D be a meridianal disk of B which intersects k in a single point and whose boundary is m. Then, since the meridianal disk $D^* = D \cup R$ of B^* intersects the center line k of B in a single point, it follows from Definition 2.12 that $\bigcirc(B^*, B)$ is either 0 or 1. If $\bigcirc(B^*, B) = 0$, then the algebraic intersection number of D^* with k (to obtain the algebraic intersection number of a polyhedral disk with a polygonal line in general position, associate with each of the points of intersection the number $+1$ or -1 according to the sense of piercing, and compute the sum of the numbers thus defined) must be 0 [11, p. 170], so that D^* must intersect k in an even number of points. This contradiction proves that $\bigcirc(B^*, B) = 1$.

<u>Lemma</u> 2.2. <u>Suppose that</u> B <u>and</u> B^* <u>are polyhedral solid tori in</u> S^3 <u>with</u> $B \subset \operatorname{Int} B^*$. <u>If there is a polyhedral annular ring</u> R <u>in</u> $\operatorname{Cl}(B^* - B)$ <u>bounded by two longitude curves on</u> B <u>and</u> B^* <u>respectively, then</u> B <u>and</u> B^* <u>are equivalently knotted</u>.

<u>Proof</u>: This follows immediately from the fact that the two boundary curves of an annular ring in S^3 are equivalent and the fact that a center line and a longitude curve of a polyhedral solid torus are equivalent.

<u>Theorem</u> 2.5. <u>Let</u> B <u>and</u> B^* <u>be two polyhedral solid tori in</u> S^3 <u>with</u> $B \subset \operatorname{Int} B^*$. <u>Consider the following conditions</u>:

(a) $\bigcirc(B^*, B) = 1$.

(b) <u>There is a polyhedral annular ring</u> R <u>in</u> $\operatorname{Cl}(B^* - B)$ <u>bounded by two meridian curves on</u> B <u>and</u> B^* <u>respectively</u>.

(c) B <u>and</u> B^* <u>are equivalently knotted</u>.

(d) <u>There is a polyhedral annular ring</u> R <u>in</u> $Cl(B^* - B)$ <u>bounded by two longitude curves on</u> B <u>and</u> B^* <u>respectively.</u>

<u>If either of the first two conditions holds simultaneously with either of the last two conditions, then</u> B <u>and</u> B^* <u>are concentric.</u>

<u>Proof:</u> By Lemma 2.1, (b) implies (a). By Lemma 2.2, (d) implies (c). But, by Theorem 2.4, (a) and (c) suffice to imply that B and B^* are concentric.

<u>Corollary</u> 2.1. <u>Suppose that</u> B <u>and</u> B^* <u>are unknotted polyhedral solid tori in</u> S^3 <u>with</u> $B \subset \text{Int } B^*$. <u>If there is a polyhedral annular ring</u> R <u>in</u> $Cl(B^* - B)$ <u>bounded by two meridian curves on</u> B <u>and</u> B^* <u>respectively, then</u> B <u>and</u> B^* <u>are concentric.</u>

<u>Proof:</u> Lemma 2.1 implies that $O(B^*, B) = 1$. But B and B^* are equivalently knotted, both being unknotted. Theorem 2.4 then shows that B and B^* are concentric.

<u>Corollary</u> 2.2. <u>Suppose that</u> B <u>and</u> B^* <u>are knotted polyhedral solid tori in</u> S^3, B <u>interior to</u> B^*, <u>which are equivalently knotted. If there is a polyhedral annular ring</u> R <u>in</u> $Cl(B^* - B)$ <u>whose boundary components</u> m <u>and</u> m^* <u>are unknotted non-nullhomologous simple closed curves on</u> $\text{Bd } B$ <u>and</u> $\text{Bd } B^*$ <u>respectively, then</u> B <u>and</u> B^* <u>are concentric.</u>

<u>Proof:</u> Corollary 2.2 is an improvement of Theorem 2.2, and the proof is similar. Since an unknotted simple closed curve on the boundary of a knotted solid torus is either nullhomologous or is a meridian curve [11, p. 164], it follows that m is a meridian curve of B, and m^* is a meridian curve of B^*. Lemma 2.1 then applies to show that $O(B^*, B) = 1$. Since B and B^* are equivalently knotted by hypothesis, Theorem 2.4 implies that B and B^* are concentric.

Corollary 2.3. *Suppose that* B *and* B^* *are two unknotted solid polyhedral tori with* $B \subset \text{Int } B^*$. *If* $C = Cl(S^3 - B)$ *and* $C^* = Cl(S^3 - B^*)$, *then* $\bigcirc(B^*, B) = 1$ *if and only if* $\bigcirc(C, C^*) = 1$.

Proof: It is clear that C and C^* are indeed solid tori [11, p. 149] with $C^* \subset \text{Int } C$. If $\bigcirc(B^*, B) = 1$ then B^* and B are concentric by Theorem 2.4. Since $Cl(B^* - B) = Cl(C - C^*)$, C and C^* are also concentric. Hence $\bigcirc(C, C^*) = 1$ by Theorem 2.4. Similarly $\bigcirc(C, C^*) = 1$ implies $\bigcirc(B^*, B) = 1$.

CHAPTER III

SEQUENCES OF CONCENTRIC TORI

Definition 3.1. A simple closed curve k interior to the polyhedral solid torus B is contained *non-trivially* by B if and only if $\bigcirc(B, k) > 0$. It can be seen that B contains k non-trivially if and only if no 3-cell in B contains k in its interior [11, p. 171]. If B and B^* are two polyhedral solid tori with $B \subset \operatorname{Int} B^*$, then B^* contains B *non-trivially* if and only if B^* contains each center line of B non-trivially.

Lemma 3.1. *Suppose that B and B^* are polyhedral solid tori with $B \subset \operatorname{Int} B^*$, and let k be a polygonal simple closed curve interior to B. Then B^* contains k non-trivially if and only if B^* contains B non-trivially and B contains k non-trivially.*

Proof: This lemma follows immediately from the fact that $\bigcirc(B^*, k)$ is the product of $\bigcirc(B^*, B)$ and $\bigcirc(B, k)$ [11, p. 175].

Lemma 3.2. *Suppose that B and B^* are polyhedral solid tori, with $B \subset \operatorname{Int} B^*$, each containing the polygonal simple closed curve k non-trivially. Then $\bigcirc(B^*, k) \geqq \bigcirc(B, k)$.*

Proof: Since B^* contains k non-trivially, Lemma 3.1 gives $\bigcirc(B^*, B) \geqq 1$. But $\bigcirc(B^*, k)$ is the product of $\bigcirc(B^*, B)$ and $\bigcirc(B, k)$. Thus $\bigcirc(B^*, k) \geqq \bigcirc(B, k)$.

Lemma 3.3. *Suppose that $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori such that $B_{n+1} \subset \operatorname{Int} B_n$ for $n \geqq 1$, with each B_n containing the polygonal curve k in its interior. Then either every B_n contains k trivially or there exists an integer M such that $m > n \geqq M$ implies that $\bigcirc(B_n, B_m) = 1$.*

Proof: Supposing that not every B_n contains k trivially, choose an integer N such that $O(B_N, k) > 0$. Since $B_n \subset \operatorname{Int} B_N$ for $n \geqq N$, $n \geqq N$ implies that $O(B_n, k) > 0$ by Lemma 3.1. It then follows from Lemma 3.2 that

$$O(B_n, k) \geqq O(B_m, k) > 0$$

for $m > n \geqq N$. Thus $\{O(B_n, k)\}_{n=N}^{\infty}$ is a non-increasing sequence of positive integers. It follows that there exists an integer M such that

$$O(B_m, k) = O(B_n, k) > 0$$

for $m > n \geqq M$. But $O(B_n, k) = O(B_n, B_m) \cdot O(B_m, k)$ since $B_m \subset \operatorname{Int} B_n$ if $m > n$. Division gives

$$O(B_n, B_m) = \frac{O(B_n, k)}{O(B_m, k)} = 1$$

for $m > n \geqq M$.

Theorem 3.1. *Suppose that $\{B_n\}_1^{\infty}$ is a sequence of equivalently knotted polyhedral solid tori with $B_{n+1} \subset \operatorname{Int} B_n$ for $n \geqq 1$, and with every B_n containing the polygonal closed curve k in its interior. Then either every B_n contains k trivially or there exists an integer N such that $m, n \geqq N$ implies that B_m and B_n are concentric.*

Proof: Assuming that not every B_n contains k trivially, Lemma 3.3 gives the existence of an integer N such that $O(B_n, B_m) = 1$ for $m > n \geqq N$. Since $B_m \subset \operatorname{Int} B_n$ for $m > n$, and since by hypothesis

the solid tori are equivalently knotted, Theorem 2.4 applies to show that B_m and B_n are concentric for $m > n \geqq N$.

Lemma 3.4. *Suppose that $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori such that $B_{n+1} \subset B_n$ for $n \geqq 1$. Let k be a polygonal closed curve such that $\bigcap_{n=1}^\infty B_n = k$. Then all but a finite number of the solid tori B_n contain k non-trivially.*

Proof: Let B be a polyhedral solid torus with k as a center line (the existence of such a torus is given in [11, p. 177]). Thus in particular B is a polyhedral solid torus containing k non-trivially in its interior.

Now $\bigcup_{n=1}^\infty (S^3 - B_n) = S^3 - \bigcap_{n=1}^\infty B_n = S^3 - k \supset S^3 - \text{Int } B$ so that $\{S^3 - B_n\}_1^\infty$ is an increasing sequence of open sets covering the compact set $S^3 - \text{Int } B$. Hence there is an integer N such that $S^3 - \text{Int } B \subset S^3 - B_n$ for $n \geqq N$. Then $B_n \subset \text{Int } B$ for $n \geqq N$. It then follows from Lemma 3.1 that B_n contains k non-trivially if $n \geqq N$. This can also be seen by noticing that if some B_m, $m > N$, contained k trivially, then k would be contained in the interior of some 3-cell C with $C \subset B_m \subset \text{Int } B$, which contradicts the fact that B contains k non-trivially (see Definition 3.1). Thus all but a finite number of the B_n contain k non-trivially.

Theorem 3.2. *Suppose that $\{B_n\}_1^\infty$ is a sequence of equivalently knotted polyhedral solid tori with $B_{n+1} \subset \text{Int } B_n$ for $n \geqq 1$, and with $\bigcap_{n=1}^\infty B_n$ being a polygonal closed curve k. Then there exists an integer N such that B_m and B_n are concentric for $m > n \geqq N$.*

Proof: According to Lemma 3.4, all but a finite number of the B_n contain k non-trivially. The existence of the integer N required by the

theorem is then implied by Theorem 3.1.

Lemma 3.5. *Suppose that* $\{B_n\}_1^\infty$ *is a sequence of polyhedral solid tori with* $B_{n+1} \subset \operatorname{Int} B_n$ *for* $n \geq 1$, *and with* $\bigcap_{n=1}^{\infty} B_n$ *being a polygonal closed curve* k *which is interior to each torus* B_n. *Then there exists an integer* N *such that* $\mathcal{O}(B_n, k) = 1$ *for* $n \geq N$.

Proof: As in the proof of Lemma 3.4 choose a polyhedral solid torus B having k as center line. Then $\mathcal{O}(B, k) = 1$. Choose an integer N sufficiently large that $B_n \subset \operatorname{Int} B$ for $n \geq N$. Then

$$\mathcal{O}(B, B_n)\,\mathcal{O}(B_n, k) = \mathcal{O}(B, k) = 1$$

for $n \geq N$. Since $\mathcal{O}(B, B_n)$ and $\mathcal{O}(B_n, k)$ are by definition non-negative integers, this implies that $\mathcal{O}(B, B_n) = \mathcal{O}(B_n, k) = 1$ for $n \geq N$.

Theorem 3.3. *Suppose that* $\{B_n\}_1^\infty$ *is a sequence of polyhedral solid tori, with* $B_{n+1} \subset \operatorname{Int} B_n$ *for* $n \geq 1$, *and with* $\bigcap_{n=1}^{\infty} B_n$ *being a polygonal closed curve* k *which is interior to each* B_n. *Then there exists an integer* N *such that* B_m *and* B_n *are concentric if* $m, n \geq N$.

Proof: Since the hypotheses of Lemma 3.5 are satisfied, $\mathcal{O}(B_n, k) = 1$ for n sufficiently large. There is therefore no loss in generality in assuming that $\mathcal{O}(B_n, k) = 1$ for all n. It then follows by a theorem of Schubert [11, p. 171] that, for each n, the knot $\overline{b_n}$ represented by a center line of the solid torus B_n is a factor of the knot $\overline{k}$ represented by the polygonal closed curve k, if k and b_n are appropriately oriented. In particular, an arbitrary fixed orientation of k having been assigned, each b_n is oriented positively with respect to k, that is, so that k is homologous in B_n to a positive multiple of b_n,

the generator of the 1-dimensional homology group of B_n. Since $\bigcirc(B_n, k) = 1$, it then follows [11, p. 171] that $k \sim b_n$ in B_n. If $m > n$, it follows that b_n is positively oriented with respect to b_m, since $k \sim b_n$ in B_n and $k \sim b_m$ in $B_m \subset B_n$, so that $b_m \sim b_n$ in B_n. Thus for each n the knot $\bar{k}$ can be represented as a knot product of the form

$$\bar{k} = \overline{b_n}\ \overline{x_n}$$

where $\overline{b_n}$ is represented by a center line of B_n.

Now, according to Lemma 3.3, there is no loss in assuming also that $\bigcirc(B_n, B_m) = 1$ for all values of m and n such that $m > n$. Consequently the same theorem of Schubert implies that, if $m > n$, $\overline{b_n}$ is a knot factor of $\overline{b_m}$. Hence to each pair of integers m and n with $m > n$ there corresponds a knot $\overline{y_{mn}}$ such that

$$\overline{b_m} = \overline{b_n}\ \overline{y_{mn}}$$

It follows that $\{b_n\}_1^\infty$ is a sequence of knots, each of which is a knot factor both of its successor in the sequence and of the knot $\bar{k}$.

If k is unknotted, then the relation $\bar{k} = \overline{b_n}\ \overline{x_n}$ implies that each of the solid tori B_n is unknotted, since the identity knot cannot be expressed as a non-trivial knot product [11, p. 156]. If k is knotted, then $\bar{k}$ has a unique factorization into finitely many prime knots [11, p. 232]. By a prime knot is meant a knot which cannot be represented as the product of two knots, neither of which is the identity. It follows that there does not exist an infinite sequence of factors of the knot $\bar{k}$, each of which is a proper factor of its successor in the sequence. There must therefore exist an integer N such that $m, n \geqq N$ implies $\overline{b_n} = \overline{b_m}$ so that the

solid tori B_m and B_n are equivalently knotted for $m, n \geq N$.

Theorem 3.1 now applies to show that B_m and B_n are concentric for m and n sufficiently large.

Theorem 3.4. *If $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori with $B_{n+1} \subset B_n$ for $n \geqq 1$, such that $\bigcap_{n=1}^\infty B_n$ is a polygonal closed curve k interior to each B_n, then, for n sufficiently large, the center line b_n of B_n is equivalent to k, to within orientation.*

Proof: Let B be a solid torus having k as a center line, so that $\bigcirc(B, k) = 1$. Since Int B is then a neighborhood of k , it follows as in the proof of Lemma 3.4 that there exists an integer N such that $B_n \subset \text{Int } B$ for $n \geqq N$. Then, since $\bigcirc(B, k) = \bigcirc(B, B_n) \cdot \bigcirc(B_n, k) = 1$, and since $\bigcirc(B, B_n)$ and $\bigcirc(B_n, k)$ are non-negative integers, it follows that $\bigcirc(B, B_n) = \bigcirc(B_n, k) = 1$.

Since k is thus a polygonal closed curve interior to B_n with $\bigcirc(B_n, k) = 1$ for $n \geqq N$, it follows that the knot $\overline{b_n}$ represented by b_n is a knot factor of the knot $\overline{k}$ represented by k [11, p. 171], if k and b_n are so oriented that $k \sim b_n$ in B_n . Hence there exists, for each $n \geqq N$, a knot $\overline{x_n}$ such that

$$\overline{k} = \overline{b_n}\,\overline{x_n}$$

On the other hand, since $\bigcirc(B, b_n) = 1$, and since k is the center line of B , it follows similarly that there is a knot $\overline{y_n}$ such that

$$\overline{b_n} = \overline{k}\,\overline{y_n}$$

Substitution gives

$$\overline{k} = \overline{k}\,\overline{x_n}\,\overline{y_n}$$

It follows that $\overline{x_n}\,\overline{y_n}$ is the identity knot [11, p. 156]. Since the identity knot cannot be expressed as a non-trivial product, it follows that $\overline{x_n}$ and $\overline{y_n}$ are each the identity knot. Hence $\overline{b_n} = \overline{k}$ for $n \geqq N$, so that all but finitely many of the center lines b_n are equivalent to k, to within orientation.

Corollary 3.1. *A properly decreasing sequence of unknotted (knotted) polyhedral solid tori, each contained in the interior of its predecessor, cannot intersect in a knotted (unknotted) polygonal closed curve.*

Proof: This is an immediate consequence of Theorem 3.4.

Lemma 3.6. *If B is a polyhedral solid torus in S^3, then there exist two polyhedral solid tori B_1 and B_2 such that*

(a) $B_1 \subset \operatorname{Int} B$,

(b) $B \subset \operatorname{Int} B_2$,

(c) $\mathcal{O}(B_2, B) = \mathcal{O}(B, B_1) = 1$,

(d) B, B_1, B_2 *are equivalently knotted.*

Proof: Let b be a center line of the solid torus B, and let k be a polygonal closed curve in S^3 equivalent to b. Since, given any curve k, there is in each neighborhood of k a polyhedral solid torus having k as a center line [11, p. 177], choose a polyhedral solid torus C_2 with k as center line. Then let C be a polyhedral solid torus contained in $\operatorname{Int} C_2$ with k as a center line, and let C_1 be a polyhedral solid torus contained in $\operatorname{Int} C$ with k as center line.

Since B and C are polyhedral solid tori having equivalent center lines b and k respectively, there is a semilinear homeomorphism f of S^3 onto itself which maps C onto B and k onto b [11, p. 180]. If $B_i = f(C_i)$ for $i = 1, 2$, then clearly $B_1 \subset \operatorname{Int} B$ and $B \subset \operatorname{Int} B_2$.

Since B, B_1, and B_2 share the common center line b (center lines are preserved under semilinear maps of solid tori) it follows immediately that they are equivalently knotted with $\mathcal{O}(B_2, B) = \mathcal{O}(B, B_1) = 1$.

Lemma 3.7. *Suppose that $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori, with $B_{n+1} \subset \operatorname{Int} B_n$ for $n \geqq 1$, such that each B_n contains the polyhedral solid torus B. Then either every B_n contains B trivially or there exists an integer M such that $\mathcal{O}(B_n, B_m) = 1$ if $m > n \geqq N$.*

Proof: If not every B_n contains B trivially, then there is an integer N such that B_n contains the center line k of B non-trivially. The existence of the required integer M is then implied by Lemma 3.3.

Lemma 3.8. *Suppose that $\{B_n\}_1^\infty$ is a sequence of equivalently knotted polyhedral solid tori, with $B_{n+1} \subset \operatorname{Int} B_n$ for $n \geqq 1$, such that each B_n contains the polyhedral solid torus B. If at least one of the tori $\{B_n\}_1^\infty$ contains B non-trivially, then there exists an integer N such that B_m and B_n are concentric for $m, n \geqq N$.*

Proof: If not every B_n contains B trivially, then not every B_n contains a center line k of B trivially. The existence of the required integer N therefore follows from Theorem 3.1.

Lemma 3.9. *Suppose that $\{B_n\}_1^\infty$ is a decreasing sequence of polyhedral solid tori intersecting in the polyhedral solid torus B interior to each B_n. Then there is an integer N such that $\mathcal{O}(B_n, B) = 1$ if $n \geqq N$.*

Proof: By Lemma 3.6, choose a polyhedral solid torus B^* containing B in its interior and such that $\mathcal{O}(B^*, B) = 1$. Since $\{S^3 - B_n\}_1^\infty$ is an increasing sequence of open sets covering the compact set $S^3 - \operatorname{Int} B^*$,

there is an integer N such that $S^3 - \text{Int } B^* \subset S^3 - B_n$ or $B_n \subset \text{Int } B^*$ for $n \geqq N$. Since $B \subset \text{Int } B_n$ for every n, it follows that

$$\mathcal{O}(B^*, B) = \mathcal{O}(B^*, B_n) \cdot \mathcal{O}(B_n, B)$$

for $n \geq N$. Hence, since $\mathcal{O}(B^*, B) = 1$, and since $\mathcal{O}(B^*, B_n)$ and $\mathcal{O}(B_n, B)$ are non-negative integers, it follows that $\mathcal{O}(B_n, B) = 1$ for $n \geqq N$.

Lemma 3.10. <u>Suppose that $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori, with $B_{n+1} \subset \text{Int } B_n$ for $n \geqq 1$, such that $\bigcap_{n=1}^{\infty} B_n$ is a polyhedral solid torus B. Then there exists an integer N such that B_n and B are equivalently knotted for $n \geqq N$.</u>

Proof: By Lemma 3.6 there is a polyhedral solid torus B^* containing B in its interior such that B and B^* are equivalently knotted with $\mathcal{O}(B^*, B) = 1$. Then by Lemma 3.9 there is an integer N such that $B_n \subset \text{Int } B^*$ with $\mathcal{O}(B_n, B) = 1$ for $n \geqq N$. The relation

$$\mathcal{O}(B^*, B) = \mathcal{O}(B^*, B_n) \cdot \mathcal{O}(B_n, B)$$

implies that $\mathcal{O}(B^*, B_n) = 1$ also for $n \geq N$.

Let k be a center line of B. Since B and B^* are equivalently knotted with $\mathcal{O}(B^*, B) = 1$, the proof of Theorem 2.4 shows that k is also a center line of B^*. Let $\bar{k}$ be the knot represented by k, and for each $n \geqq N$ denote by $\overline{b_n}$ the knot represented by a center line b_n of the solid torus B_n, with k and b_n so oriented that $k \sim b_n$ in B_n. Since k is interior to B_n and since $\mathcal{O}(B_n, k) = \mathcal{O}(B_n, B) = 1$ for $n \geq N$, it follows from [11, p. 171] that to each $n \geqq N$ there corresponds a knot $\overline{x_n}$ such that

$$\overline{k} = \overline{b_n}\ \overline{x_n} .$$

On the other hand, since b_n is interior to the solid torus B^* with $O(B^*, b_n) = O(B^*, B_n) = 1$ for $n \geqq N$, it follows similarly that to each $n \geqq N$ there corresponds a knot $\overline{y_n}$ such that

$$\overline{b_n} = \overline{k}\ \overline{y_n} .$$

But these two relations imply as in the proof of Theorem 3.4 that the knots $\overline{k}$ and $\overline{b_n}$ are the same. Consequently the solid tori B_n and B are equivalently knotted whenever $n \geqq N$.

Theorem 3.5. _Suppose that $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori, with $B_{n+1} \subset \operatorname{Int} B_n$ for each $n \geqq 1$, such that $\bigcap_{n=1}^{\infty} B_n$ is a polyhedral solid torus B. Then there exists an integer N such that for $m, n \geqq N$ the solid tori B_m and B_n are concentric with each other and with B._

Proof: By Lemma 3.9 there is an integer N_1 such that $O(B_n, B) = 1$ if $n \geqq N_1$. By Lemma 3.10 there is an integer N_2 such that the solid tori B_n and B are equivalently knotted if $n \geqq N_2$. If $N = \max(N_1, N_2)$ it then follows from Theorem 2.4 that B_n and B are concentric if $n \geqq N$.

Now suppose that $m > n \geqq N$. Since both B_n and B_m are equivalently knotted with B, it is clear that B_m and B_n are equivalently knotted. Since $O(B_n, B) = O(B_m, B) = 1$, it follows from the product relation

$$O(B_n, B) = O(B_n, B_m) \cdot O(B_m, B)$$

that $O(B_n, B_m) = 1$. Hence another application of Theorem 2.4 suffices to show that B_n and B_m are concentric if $m > n \geqq N$.

Lemma 3.11. Suppose that $\{B_n\}_1^\infty$ *is an increasing sequence of polyhedral solid tori, with* $B_n \subset \operatorname{Int} B_{n+1}$ *for* $n \geqq 1$, *such that* $\bigcup_{n=1}^{\infty} B_n = \operatorname{Int} B$, *with* B *being a polyhedral solid torus. Then there exists an integer* N *such that* $\bigcirc(B, B_n) = 1$ *for* $n \geqq N$.

Proof: There exists by Lemma 3.6 a polyhedral solid torus B^* such that $B^* \subset \operatorname{Int} B$ with $\bigcirc(B, B^*) = 1$. Since $\{\operatorname{Int} B_n\}_1^\infty$ is an increasing sequence of open sets covering the compact set B^*, it follows that there is an integer N such that $B^* \subset \operatorname{Int} B_n$ for $n \geqq N$. Then, since $B_n \subset \operatorname{Int} B$,

$$\bigcirc(B, B^*) = \bigcirc(B, B_n) \cdot \bigcirc(B_n, B^*)$$

for $n \geqq N$. Since $\bigcirc(B, B^*) = 1$, and since $\bigcirc(B, B_n)$ and $\bigcirc(B_n, B^*)$ are non-negative integers, it follows that $\bigcirc(B, B_n) = 1$ for $n \geqq N$.

Lemma 3.12. Suppose that $\{B_n\}_1^\infty$ *is an increasing sequence of polyhedral solid tori, with* $B_n \subset \operatorname{Int} B_{n+1}$ *for* $n \geqq 1$, *such that* $\bigcup_{n=1}^{\infty} B_n = \operatorname{Int} B$, *with* B *being a polyhedral solid torus. Then there exists an integer* N *such that* B *and* B_n *are equivalently knotted if* $n \geqq N$.

Proof: There is by Lemma 3.6 a polyhedral solid torus B^* interior to B such that B and B^* are equivalently knotted with $\bigcirc(B, B^*) = 1$. Then by the proof of Lemma 3.11 there is an integer N such that $B^* \subset \operatorname{Int} B_n$ with $\bigcirc(B_n, B^*) = 1$ for $n \geqq N$. The relation

$$\bigcirc(B, B^*) = \bigcirc(B, B_n) \cdot \bigcirc(B_n, B^*)$$

then implies that $\bigcirc(B^*, B_n) = 1$ for $n \geqq N$.

Let k be a center line of B. Since B and B^* are equivalently knotted with $\bigcirc(B, B^*) = 1$, the proof of Theorem 2.4 shows that k is also a center line of B^*. Let $\bar{k}$ be the knot represented by k, and for each

$n \geqq N$ denote by $\overline{b_n}$ the knot represented by a center line b_n of the solid torus B_n, with k and b_n so oriented that $k \sim b_n$ in B_n. Since k is interior to B_n and since $\bigcirc(B_n, k) = \bigcirc(B_n, B^*) = 1$ for each $n \geqq N$, it follows from [11, p. 171] that to each $n \geqq N$ there corresponds a knot $\overline{x_n}$ such that

$$\overline{k} = \overline{b_n}\, \overline{x_n} .$$

On the other hand, since b_n is interior to the solid torus B with $\bigcirc(B, b_n) = \bigcirc(B, B_n) = 1$ for $n \geqq N$, it follows similarly that to each $n \geqq N$ there corresponds a knot $\overline{y_n}$ such that

$$\overline{b_n} = \overline{k}\, \overline{y_n} .$$

But these two relations imply as in the proof of Theorem 3.4 that the knots $\overline{k}$ and $\overline{b_n}$ are the same. Consequently the solid tori B_n and B are equivalently knotted for $n \geqq N$.

Theorem 3.6. *If $\{B_n\}_1^\infty$ is an increasing sequence of polyhedral solid tori, with $B_n \subset \operatorname{Int} B_{n+1}$ for $n \geqq 1$, such that $\bigcup_{n=1}^{\infty} B_n = \operatorname{Int} B$, with B being a polyhedral solid torus, then there exists an integer N such that if $m, n \geqq N$ then the solid tori B_m and B_n are concentric with each other and with B.*

Proof: There is by Lemma 3.11 an integer N_1 such that $\bigcirc(B, B_n) = 1$ if $n \geqq N_1$. By Lemma 3.12 there is an integer N_2 such that B_n and B are equivalently knotted if $n \geqq N_2$. Thus, if $N = \max(N_1, N_2)$, it follows from Theorem 2.4 that B and B_n are concentric for $n \geqq N$. If $m > n \geqq N$, B_m is by the same token concentric with B and it then follows as in the proof of Theorem 3.5 that B_m

and B_n are concentric.

Corollary 3.2. *Let B be a polyhedral solid torus and consider two sequences $\{B_n\}_1^\infty$ and $\{C_m\}_1^\infty$ of polyhedral solid tori with*

(a) $B_n \subset \operatorname{Int} B_{n+1}$ *for* $n \geqq 1$ *and* $\bigcup_{n=1}^{\infty} B_n = \operatorname{Int} B$,

(b) $C_{m+1} \subset \operatorname{Int} C_m$ *for* $m \geqq 1$ *and* $\bigcap_{m=1}^{\infty} C_m = B$.

Then there exists an integer N such that if $m, n \geqq N$ then B_n and C_m are concentric with each other and with B.

Proof: By Theorem 3.6 there is an integer N_1 such that B and B_n are concentric if $n \geqq N_1$. By Theorem 3.5 there is an integer N_2 such that B and C_m are concentric if $m \geqq N_2$. Thus B is concentric with both B_n and C_m if $m, n \geqq N = \max(N_1, N_2)$, and it may be shown as in the proof of Theorem 3.5 that B_n and C_m are concentric.

CHAPTER IV

UNCOUNTABLE COLLECTIONS OF CONCENTRIC TORI

The object of this chapter is to show that the 3-sphere S^3 does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric. It will be shown in particular that, if $\mathcal{G}$ is an uncountable collection of pairwise exclusive tori in S^3, $\mathcal{G}$ contains an uncountable subcollection $\mathcal{G}'$, each pair of whose members are concentric. This will be accomplished by showing that uncountably many elements of $\mathcal{G}$ are limits from both sides (in an appropriate sense) of elements of $\mathcal{G}$. The results of Chapter III will then imply the existence of concentric tori in $\mathcal{G}$. These results will be obtained first under the assumption that the tori in $\mathcal{G}$ are polyhedral. This polyhedrality assumption will then be removed by application of Bing's theorem [3] on the deformation of locally tame sets onto polyhedral ones.

Theorem 4.1. *There does not exist an uncountable collection of mutually disjoint polyhedral tori in S^3, no two of which are concentric.*

Proof: Let $\mathcal{G}$ be an uncountable collection of mutually disjoint polyhedral tori in S^3. It will be shown that $\mathcal{G}$ contains a pair of concentric tori.

Suppose that $\mathcal{G}$ is indexed by a set Λ, so that $\mathcal{G} = \{T_\alpha : \alpha \in \Lambda\}$. Since the closure of at least one of the two domains in S^3 complementary to a polyhedral torus is a polyhedral solid torus [1], there corresponds to each $\alpha \in \Lambda$ a polyhedral solid torus B_α such that $T_\alpha = \text{Bd } B_\alpha$.

Now there exists a pair of points a and b in S^3 such that the collection $\mathcal{G}'$ of those tori in $\mathcal{G}$ which separate a and b in S^3 is uncountable [13]. It follows that at least one of the sets $\{\alpha \in \Lambda : a \in B_\alpha \text{ and } T_\alpha \in \mathcal{G}'\}$ and $\{\alpha \in \Lambda : b \in B_\alpha \text{ and } T_\alpha \in \mathcal{G}'\}$ is uncountable. Assume it is the former, and denote by $\mathcal{G}''$ the collection $\{T_\alpha \in \mathcal{G}' : a \in B_\alpha\}$. Thus $\mathcal{G}''$ is an uncountable subcollection of $\mathcal{G}$, each element of which separates a and b in S^3, and such that $T_\alpha \in \mathcal{G}''$ implies $a \in \text{Int } B_\alpha$. Therefore $\mathcal{G}''$ can be given a linear order by defining $T_\alpha < T_\beta$ if and only if $B_\alpha \subset \text{Int } B_\beta$ [8, p. 154]. Hence $\mathcal{G}''$ is an uncountable subcollection of $\mathcal{G}$ such that, if T_α and T_β are any two tori in $\mathcal{G}$, either $B_\alpha \subset \text{Int } B_\beta$ or $B_\beta \subset \text{Int } B_\alpha$. It therefore may be assumed without loss that the original collection $\mathcal{G}$ has this property.

Now Whyburn [13] has shown that every uncountable non-separated collection $\mathcal{G}$ of cuttings of a connected separable metric space M contains an uncountable subcollection $\mathcal{G}^*$ such that to each element T of $\mathcal{G}^*$ and each point p of $M - T$ there corresponds an element T' of $\mathcal{G}^*$ which separates T and p in M.

Thus let T be an element of the subcollection $\mathcal{G}^*$ of $\mathcal{G}$ given by Whyburn's theorem and denote by B the corresponding solid torus bounded by T (according to the correspondence defined above). It will next be shown that there is a sequence $\{T_n\}_1^\infty$ of tori in G^* such that the corresponding solid tori $\{B_n\}_1^\infty$ intersect in B. A sequence $\{C_n\}_1^\infty$ of solid tori with boundaries in $\mathcal{G}^*$, and such that $\bigcup_{n=1}^\infty C_n = \text{Int } B$ could be defined similarly.

There is in every neighborhood of a polyhedral solid torus B in S^3 a polyhedral solid torus C such that $B \subset \text{Int } C$ [11, p. 181]. It follows

that a sequence $\{C_n\}_1^\infty$ of solid tori can be chosen with $C_{n+1} \subset \text{Int } C_n$ for $n \geq 1$ and such that $B = \bigcap_{n=1}^\infty C_n$.

Now let $p \in \text{Bd } C_1$ be an arbitrary point of the boundary of the solid torus C_1 . There is then a torus T_p of the collection $\mathcal{S}^*$ which separates p and T in S^3 . Denote by D_p the corresponding solid torus bounded by T_p . By the order assumption on $\mathcal{S}$, either $D_p \subset \text{Int } B$ or $B \subset \text{Int } D_p$. But if $D_p \subset \text{Int } B$, $T_p = \text{Bd } D_p$ could not separate p and T in S^3 , because $p \in S^3 - B$. It must therefore be concluded that $B \subset \text{Int } D_p$ so that $p \in S^3 - D_p$.

Thus to each point p of $\text{Bd } C_1$ there corresponds a solid torus D_p , whose boundary T_p is an element of $\mathcal{S}$, such that $B \subset \text{Int } D_p$ while $p \in S^3 - D_p$. The collection

$$\left\{S^3 - D_p\right\}_{p \in \text{Bd } C_1}$$

is consequently an open cover of the compact set $\text{Bd } C_1$ and hence contains a finite subcover $\left\{S^3 - D_i\right\}_{i=1}^k$. Since the boundary T_i of each solid torus D_i is an element of $\mathcal{S}$, and since $\mathcal{S}$ is ordered by interior inclusion, there is among the solid tori $D_1, \ldots, D_k$ one, say D_j , such that $D_j \subset \text{Int } D_i$ if $i \neq j$, $1 \leqq i \leqq k$. Since

$$\text{Bd } C_1 \subset \bigcup_{i=1}^k (S^3 - D_i)$$

it follows that $\text{Bd } C_1 \subset S^3 - D_j$. Therefore D_j is a solid torus, whose boundary is an element of $\mathcal{S}$, such that $B \subset \text{Int } D_j$ and $D_j \subset \text{Int } C_1$. Now define $B_1 = D_j$.

The definition of the sequence $\{B_n\}_1^\infty$ is continued by induction. Suppose that the solid tori $B_1, \ldots, B_n$, with boundaries in $\mathcal{S}$, have been

defined in such a way that

$$B \subset \operatorname{Int} B_i \quad \text{for } i = 1, \ldots, n ;$$

$$B_{i+1} \subset \operatorname{Int} B_i \quad \text{for } i = 1, \ldots, n-1 ;$$

$$B_i \subset \operatorname{Int} C_i \quad \text{for } i = 1, \ldots, n .$$

Now let p be an arbitrary point of $\operatorname{Bd} B_n \cup \operatorname{Bd} C_{n+1}$. There exists as above a solid torus D_p, whose boundary T_p is an element of $\mathcal{G}$, such that $B \subset \operatorname{Int} D_p$ while $p \in S^3 - D_p$. The collection

$$\left\{S^3 - D_p\right\}_{p \in \operatorname{Bd} B_n \cup \operatorname{Bd} C_{n+1}}$$

is consequently an open cover of the compact set $\operatorname{Bd} B_n \cup \operatorname{Bd} C_{n+1}$, so that, using the linear order in G as above, a solid torus D_k is found, whose boundary is an element of $\mathcal{G}$, such that B is interior to D_k while D_k is interior to both B_n and C_{n+1}. Setting $B_{n+1} = D_k$, the solid tori $B_1, B_2, \ldots, B_{n+1}$ have now been defined in such a way that

$$B \subset \operatorname{Int} B_i \quad \text{for } i = 1, \ldots, n+1 ;$$

$$B_{i+1} \subset \operatorname{Int} B_i \quad \text{for } i = 1, \ldots, n ;$$

$$B_i \subset \operatorname{Int} C_i \quad \text{for } i = 1, \ldots, n+1 ;$$

and with the tori $T_i = \operatorname{Bd} B_i$ for $i = 1, \ldots, n+1$ being elements of $\mathcal{G}$. The sequence $\left\{B_n\right\}_1^\infty$ of solid polyhedral tori with boundaries $\left\{T_n\right\}_1^\infty$ in is thus defined in such a way that $B \subset \operatorname{Int} B_n$, $B_{n+1} \subset \operatorname{Int} B_n$, and $B \subset \operatorname{Int} C_n$ for each $n \geq 1$. Therefore

$$\begin{aligned} B &\subset \bigcap_{n=1}^{\infty} \operatorname{Int} B_n \\ &\subset \bigcap_{n=1}^{\infty} B_n \\ &\subset \bigcap_{n=1}^{\infty} \operatorname{Int} C_n \\ B &\subset \bigcap_{n=1}^{\infty} C_n \end{aligned}$$

so that $B = \bigcap_{n=1}^{\infty} B_n$.

Theorem 3.5 therefore applies to give an integer N such that, for $m > n \geqq N$, the solid tori B_m and B_n are concentric. But the boundaries of the solid tori B_n and B_m are tori in $\mathcal{G}$. This completes the proof that S^3 does not contain an uncountable collection of mutually disjoint polyhedral tori, no two of which are concentric.

Looking ahead to the problem of removing the polyhedrality hypothesis in Theorem 4.1, it might be well to note the steps in the proof at which this hypothesis was used. First, it was used in asserting that, if $T \in \mathcal{G}$, at least one of the components of $S^3 - T$ has a closure which is a solid torus. Insofar as this is concerned, it would suffice to assume merely that the elements of G are tamely imbedded in S^3 . Second, the assumption that the tori in $\mathcal{G}$ were polyhedral was used in finding a decreasing sequence $\{C_n\}_1^{\infty}$ of solid tori intersecting in a solid torus B bounded by a specified torus in $\mathcal{G}$. Here also the assumption that the tori in $\mathcal{G}$ are tame would suffice. Third, this polyhedrality assumption was used in applying Theorem 3.5. Since Bing [5] has shown that S^3 does not contain uncountably many mutually disjoint wild closed surfaces, there is no loss in generality in assuming that each torus in $\mathcal{G}$ is tamely imbedded in S^3 . But there still remains the problem of obtaining an extension of Theorem 3.5 for which no polyhedrality assumption is necessary.

Definition 4.1. A compact subset K of S^3 is said to be *tame* in S^3 if and only if there exists a homeomorphism of S^3 onto itself which carries K onto a polyhedron.

Lemma 4.1. *Suppose that* K *is a compact tame subset of* S^3 . *Then, given* $\epsilon > 0$, *there exists a homeomorphism* f *of* S^3 *onto itself*

which

(a) carries K onto a polyhedron,

(b) moves each point a distance less than ϵ ,

(c) is the identity on $S^3 - S(K, \epsilon)$, where $S(K, \epsilon)$ denotes the set of all points whose distance from K is less than ϵ .

Proof: Lemma 4.1 is a special case of Theorem 9 of [3], which states that, if K is a locally tame closed subset of a triangulated 3-manifold M with boundary (possibly vacuous), and if C is a closed subset of M such that K is locally polyhedral at each point of $K \cap C$, with ϕ a positive continuous function on $M - C$, then there exists a homeomorphism f of M onto itself such that $x = f(x)$ if $x \in C$, $f(K)$ is a polyhedron and $\rho(x,f(x)) < \phi(x)$ if $x \in M - C$ (ρ denotes the metric in M). In applying this theorem take $M = S^3$, $C = S^3 - S(K,\epsilon)$, and $\phi(x) \equiv \epsilon$ on $S(K,\epsilon)$. The homeomorphism f then given by Bing's theorem is clearly a homeomorphism of S^3 onto itself satisfying the requirements of the lemma.

Theorem 4.2. Suppose that $\{B_n\}_1^\infty$ is a sequence of tame solid tori in S^3 with $B_{n+1} \subset \text{Int } B_n$ for $n \geq 1$, such that $\bigcap_{n=1}^\infty B_n = K$ is either a tame simple closed curve or a tame solid torus. Then there exists an integer N such that the solid tori B_m and B_n are concentric if $m > n \geq N$.

Proof: Let g be a homeomorphism of S^3 onto itself which carries K onto a polygonal simple closed curve or a polyhedral solid torus, according as K is a tame simple closed curve or solid torus respectively.

For each $n \geq 1$, denote by T_n the tame torus bounding the solid torus B_n , and define

$$\epsilon_n = \min\left\{\frac{1}{n}, \frac{1}{3}\rho[g(T_n), g(T_{n-1})], \frac{1}{3}\rho[g(T_n), g(T_{n+1})]\right\}$$

If $U_n = S(g(T_n), \epsilon_n)$ for each $n \geqq 1$, then clearly $U_m \cap U_n = \Box$ (the empty set is denoted by $\Box$) whenever $m \neq n$.

For each $n \geqq 1$, let f_n be a homeomorphism of S^3 onto itself which

(a) carries $g(T_n)$ onto a polyhedron,

(b) moves each point a distance less than ϵ_n,

(c) is the identity on $S^3 - U_n$.

The existence of such homeomorphisms is given by Lemma 4.1. Then define a mapping f on S^3 by

$$f(x) = \begin{cases} x & \text{if } x \in S^3 - \bigcup_{n=1}^{\infty} U_n \\ f_n(x) & \text{if } x \in U_n \end{cases}$$

Since the identity is $1-1$ on $S^3 - \bigcup_{n=1}^{\infty} U_n$, and since f_n is a homeomorphism of U_n onto itself, it is clear that f is a $1-1$ mapping of S^3 onto itself. It is also clear that f is continuous at each point of $S^3 - g(K)$, since f agrees with the identity in some neighborhood of each point of $S^3 - g(K) - \bigcup_{n=1}^{\infty} \mathrm{Cl}\ U_n$, and with f_n in some neighborhood of a point of U_n, while f_n matches with the identity in a continuous manner at each point of $\mathrm{Bd}\ U_n$.

To show that f is a homeomorphism of S^3 onto itself, it therefore suffices to show that f is continuous at each point of $g(K)$, because S^3 is compact and Hausdorff. Since f is the identity on the closed set $g(K)$ it hence suffices to show that if $\{x_k\}_1^{\infty}$ is a sequence of points of $S^3 - g(K)$ converging to a point x of $\mathrm{Bd}\ g(K)$, then the sequence

$\{f(x_k)\}_1^{\infty}$ converges to x. Since $f(x_k) = x_k$ if $x_k \in S^3 - \bigcup_{n=1}^{\infty} U_n$, it may be assumed without loss that each point x_k of the sequence lies in one of the sets U_n, say U_{n_k}, so that

$$\rho(x_k, f(x_k)) = \rho(x_k, f_{n_k}(x_k)) < \epsilon_{n_k}$$

for each $k \geqq 1$. It is clear that $\lim_{k \to \infty} n_k = \infty$. For otherwise, if there were an integer N such that $n_k < N$ for infinitely many values of k, then infinitely many points of the sequence $\{x_k\}_1^{\infty}$ would lie in $S^3 - g(B_N)$, which would contradict the fact that $\{x_k\}_1^{\infty}$ converges to $x \in g(K)$, since $g(B_N)$ contains $g(K)$ in its interior. It follows that

$$\lim_{k \to \infty} \rho(x_k, f(x_k)) \leqq \lim_{k \to \infty} \epsilon_{n_k}$$

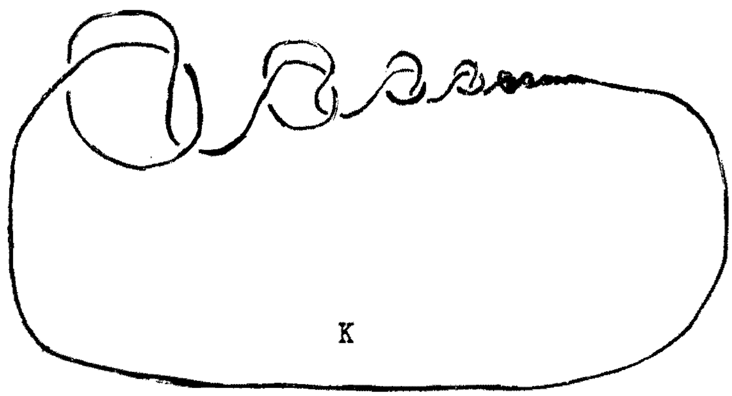
$$\leqq \lim_{k \to \infty} \frac{1}{n_k}$$

$$\lim_{k \to \infty} \rho(x_k, f(x_k)) = 0$$

Consequently $\lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} x_k = x = f(x)$ so that f is continuous at x. Therefore f is a homeomorphism of S^3 onto itself.

If $h = fg$, then h is a homeomorphism of S^3 onto itself carrying K and each of the solid tori B_n onto a polyhedron. Then $\{h(B_n)\}_1^{\infty}$ is a sequence of polyhedral solid tori such that $h(B_{n+1}) \subset \operatorname{Int} h(B_n)$ for $n \geqq 1$, and with $\bigcap_{n=1}^{\infty} h(B_n) = h(K) = g(K)$ being either a polygonal closed curve or a polyhedral solid torus. It then follows from Theorems 3.3 and 3.5 that there exists an integer N such that $h(B_m)$ and $h(B_n)$ are concentric for $m > n \geqq N$. But $Cl(B_n - B_m) = Cl\, h^{-1}[h(B_n) - h(B_m)] = h^{-1}\, Cl[h(B_n) - h(B_m)]$ so that B_m and B_n are concentric for $m > n \geqq N$.

Example 4.1. The conclusion of Theorem 4.2 does not necessarily hold if K is a wild simple closed curve. For instance suppose that K is a simple closed curve in which an infinite sequence of "trefoil knots" have been tied, in such a manner that they converge to a single point, as in the figure below. This curve is virtually equivalent to that obtained



by joining the endpoints of the wild arc described in Example 1.4 of [6]. It is apparent that K can be represented as the intersection of a decreasing sequence $\{B_n\}_1^\infty$ of polyhedral solid tori, with the center line of B_n being equivalent to the knot product of n trefoils. It follows from Theorem 2.4 that no two of these solid tori are concentric, since no two are equivalently knotted.

Lemma 4.2. *Suppose that* B_1, B_2, B_3 *are tame solid tori in* S^3 *with* $B_1 \subset \operatorname{Int} B_2$ *and* $B_2 \subset \operatorname{Int} B_3$. *Then* B_1 *and* B_3 *are concentric if and only if* B_2 *is concentric with both* B_1 *and* B_3.

Proof: It clearly involves no loss in generality to assume that B_1, B_2, B_3 are polyhedral, since they can be carried onto polyhedra by a homeomorphism of S^3 onto itself.

If B_2 is concentric with both B_1 and B_3, then B_1, B_2, B_3 are equivalently knotted with $\bigcirc(B_3, B_2) = \bigcirc(B_2, B_1) = 1$ by Theorem 2.4.

Then

$$\begin{aligned}\bigcirc(B_3, B_1) &= \bigcirc(B_3, k)\\ &= \bigcirc(B_3, B_2) \circ \bigcirc(B_2, k)\\ &= \bigcirc(B_3, B_2) \circ \bigcirc(B_2, B_1)\\ &= 1\end{aligned}$$

by [11, p. 175]. It then follows from Theorem 2.4 that B_1 and B_3 are concentric.

Conversely suppose that B_1 and B_3 are concentric. It follows from Theorem 2.4 that B_1 and B_3 are equivalently knotted with $\bigcirc(B_3, B_1) = 1$. Since $\bigcirc(B_3, B_1) = \bigcirc(B_3, B_2) \circ \bigcirc(B_2, B_1)$, it follows that $\bigcirc(B_3, B_2) = \bigcirc(B_2, B_1) = 1$. It can be shown as in the proof of Lemma 3.10 that B_1, B_2, B_3 are equivalently knotted, so that Theorem 2.4 applies to show that B_2 is concentric with both B_1 and B_3.

It is now possible to prove in complete generality the theorem on uncountable collections of tori in S^3.

Theorem 4.3. *Suppose that $\mathcal{G}$ is an uncountable collection of mutually disjoint tori in the 3-sphere S^3. Then $\mathcal{G}$ contains an uncountable subcollection $\mathcal{G}^*$ such that any two tori in $\mathcal{G}^*$ are concentric.*

Proof: Since S^3 does not contain uncountably many mutually disjoint wild closed 2-manifolds [5], it may be assumed that each torus in $\mathcal{G}$ is tame. Thus there corresponds to each torus $T_\alpha \in \mathcal{G}$ a solid torus B_α such that $T_\alpha = \mathrm{Bd}\, B_\alpha$ [1]. It may also be assumed as in the proof of Theorem 4.1 that, given T_α and T_β in $\mathcal{G}$, either $B_\alpha \subset \mathrm{Int}\, B_\beta$ or $B_\beta \subset \mathrm{Int}\, B_\alpha$. It is then shown by the method used in the proof of Theorem 4.1 that $\mathcal{G}$ contains a sequence $\{T_n\}_0^\infty$ of tori such that

$B_{n+1} \subset \text{Int } B_n$ for $n \geqq 1$, and such that $B_0 = \bigcap_{n=1}^{\infty} B_n$. Theorem 4.2 therefore applies to find an integer N such that T_m and T_n are concentric for $m > n \geqq N$. Consequently S^3 does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric.

Now define $T_\alpha \sim T_\beta$ if and only if either $T_\alpha = T_\beta$ or T_α and T_β are concentric. As a consequence of the assumption that $\mathcal{S}$ is linearly ordered by interior inclusion as above, Lemma 4.2 implies that this relation is an equivalence relation in $\mathcal{S}$. There is thereby induced a decomposition of $\mathcal{S}$ into mutually disjoint equivalence classes, such that two tori in $\mathcal{S}$ are elements of the same equivalence class if and only if they are concentric. An application of the Axiom of Choice gives a collection $\mathcal{H}$ containing exactly one element of each equivalence class. Since no two tori in $\mathcal{H}$ are concentric, it follows that $\mathcal{H}$ must be countable. Since the union of a countable number of sets is countable, one of the equivalence classes must be uncountable. Denote by $\mathcal{S}^*$ this uncountable class. Then $\mathcal{S}^*$ is an uncountable subcollection of $\mathcal{S}$ such that any two tori in $\mathcal{S}^*$ are concentric.

CHAPTER V

EQUIVALENTLY SITUATED TORI

In this chapter there is considered a relationship between polyhedral solid tori in S^3 which is a natural generalization of the relationship of concentricity.

Definition 5.1. Two polyhedral solid tori B_1 and B_2 in S^3 are said to be equivalently situated if and only if there exists a third polyhedral solid torus B which is interior to both B_1 and B_2 and is concentric with each.

Theorem 5.1. Two polyhedral solid tori B_1 and B_2 in S^3, one interior to the other are equivalently situated if and only if they are concentric.

Proof: To prove the necessity, suppose that B_1 and B_2 are equivalently situated with disjoint boundaries. Then, by Definition 5.1, there exists a polyhedral solid torus B which is interior to both B_1 and B_2 and which is concentric with both. Then any center line of B is also a center line of both B_1 and B_2 (see the proof of Theorem 2.4). Now, from the assumptions, it follows that either $B_1 \subset \text{Int } B_2$ or $B_2 \subset \text{Int } B_1$. Suppose the former inclusion holds. Theorem 2.4 then implies that B, B_1, and B_2 are equivalently knotted with $\bigcirc(B_1, B) = \bigcirc(B_2, B) = 1$. But then

$$\bigcirc(B_2, B_1) = \frac{\bigcirc(B_2, B)}{\bigcirc(B_1, B)} = 1$$

so Theorem 2.4 applies to show that B_1 and B_2 are concentric.

To prove the sufficiency, suppose that B_1 and B_2 are concentric

with $B_1 \subset \text{Int } B_2$. Lemma 3.6 then gives a polyhedral solid torus B which is interior to and concentric with B_1 . Lemma 4.2 then implies that B is also concentric with B_2 . Consequently B_1 and B_2 are equivalently situated.

Lemma 5.1. *If B and B^* are concentric polyhedral solid tori, and if B_1 and B_2 are two polyhedral solid tori, each containing B in its interior and each contained in the interior of B^*, then B_1 and B_2 are equivalently situated.*

Proof: From Lemma 4.2 it follows that B_1 and B_2 are both concentric with B . Therefore B_1 and B_2 are equivalently situated.

Definition 5.2. Suppose that $\{B_n\}_0^\infty$ is a sequence of homeomorphic compact sets in S^3 . It is said that the sequence $\{B_n\}_1^\infty$ *converges homeomorphically* to B_0 if and only if there exists a sequence $\{\epsilon_n\}_1^\infty$ of positive numbers, converging to zero, and a sequence $\{h_n\}_1^\infty$ of homeomorphisms, with h_n carrying B_n onto B_0 in such a way that $\rho(x, h_n(x)) \leqq \epsilon_n$ for every $x \in B_n$, where ρ is the standard metric for S^3 .

Theorem 5.2. *Suppose that $\{B_n\}_1^\infty$ is a sequence of polyhedral solid tori in S^3 converging homeomorphically to the polyhedral solid torus B . Then there exists an integer N such that B_n and B are equivalently situated for $n \geqq N$.*

Proof: By Lemma 3.6 choose two polyhedral solid tori B' and B'', each of which is concentric with B, with B' interior to B and B interior to B'' . It follows from the proof of Theorem 2.4 that each center line k of B' is also a common center line of B and B'' . Define

$$\alpha_1 = \min\left\{\rho(\mathrm{Bd}\ B,\ \mathrm{Bd}\ B'),\ \rho(\mathrm{Bd}\ B,\ \mathrm{Bd}\ B'')\right\}$$

Then choose a fixed interior point p' of B', a fixed point p'' of $S^3 - B''$, and define

$$\alpha = \min\left\{\alpha_1, \rho(p', \mathrm{Bd}B'), \rho(p'', \mathrm{Bd}\ B'')\right\} .$$

Now let $\{h_n\}_1^\infty$ be a sequence of homeomorphisms, h_n taking B_n onto B with

$$\max_{x \in B_n} \rho\,[x, h_n(x)] \leqq \epsilon_n$$

where $\{\epsilon_n\}_1^\infty$ is a sequence of positive numbers converging to zero. Choose an integer N such that $\epsilon_n < \alpha$ if $n \geqq N$.

If $n \geqq N$ and $y \in \mathrm{Bd}\ B_n$, it then follows that

$$\begin{aligned}\rho(y, \mathrm{Bd}\ B) &\leqq \rho(y, h_n(y))\\ &\leqq \epsilon_n < \alpha\\ \rho(y, \mathrm{Bd}\ B) &< \alpha_1 .\end{aligned}$$

Therefore $\mathrm{Bd}\ B_n$ is exterior to B' and interior to B''. Either $B' \subset \mathrm{Int}\ B_n$ or $B_n \subset S^3 - B'$. Suppose the latter inclusion holds. Since $p' \in B' \subset B$, there must exist a point $q \in B_n$ such that $h_n(q) = p'$. But this implies that

$$\begin{aligned}\rho(q, h_n(q)) &= \rho(q, p')\\ &\geqq \rho(p', \mathrm{Bd}\ B')\\ &\geqq \alpha\\ \rho(q, h_n(q)) &> \epsilon_n\end{aligned}$$

which is a contradiction. It follows that $B' \subset \text{Int } B_n$. It may be shown similarly that $B_n \subset \text{Int } B''$.

Thus B' is interior to both B and B_n and both B and B_n are interior to B'' , $n \geqq N$. Since B' and B'' are concentric, this implies by Lemma 5.1 that B and B_n are equivalently situated if $n \geqq N$.

Lemma 5.2. _The two polyhedral solid tori_ B_1 _and_ B_2 _in_ S^3 _are equivalently situated if and only if they have a common center line_ k .

Proof: To prove the necessity, suppose that B_1 and B_2 are equivalently situated. Then there is a polyhedral solid torus B interior to both B_1 and B_2 and concentric with both. If k is any center line of B , it may be seen as in the proof of Theorem 2.4 that k is a common center line of B_1 and B_2 .

Conversely, suppose that k is a common center line of B_1 and B_2. Then choose a solid torus B interior to both B_1 and B_2 and having k as a center line. It is then clear that B_1 and B_2 are both concentric with B , and are therefore equivalently situated.

Theorem 5.3. _The two polyhedral solid tori_ B_1 _and_ B_2 _in_ S^3 _are equivalently situated if and only if they are equivalently knotted and_ B_2 _has a center line_ b_2 _interior to_ B_1 _with_ $\mathcal{O}(B_1, b_2) = 1$.

Proof: Suppose that B_1 and B_2 are equivalently situated. Then by Lemma 5.2 they share a common center line so that the conclusion of the theorem is immediate.

Conversely, suppose that B_1 and B_2 are equivalently knotted and that B_2 has a center line b_2 interior to B_2 with $\mathcal{O}(B_1, b_2) = 1$. Then choose a polyhedral solid torus B interior to both B_1 and B_2 and having b_2 as a center line. Since B and B_2 have a common center line

with $B \subset \text{Int } B_2$, it is clear that B and B_2 are equivalently knotted with $\bigcirc(B_2, B) = 1$. Therefore B and B_2 are concentric. Since B_1 and B_2 are equivalently knotted and B and B_2 have a common center line, it follows that B_1 and B are equivalently knotted. But $\bigcirc(B_1, B) = \bigcirc(B_1, b_2) = 1$. Hence B_1 and B are concentric. It follows that B_1 and B_2 are equivalently situated.

Throughout the remainder of this chapter any two polyhedral solid tori B_1 and B_2 considered together will be assumed to have boundaries T_1 and T_2 respectively which are in relative general position, so that $T_1 \cap T_2$ consists of a finite number of mutually disjoint simple closed polygons [9]. This assumption involves no loss of generality, since in any case such a situation can be obtained by a semilinear deformation on T_1 which is the identity outside an arbitrary preassigned neighborhood of $T_1 \cap T_2$ [9].

Lemma 5.3. *Suppose that B_1 and B_2 are equivalently situated polyhedral solid tori with boundaries T_1 and T_2 respectively. If m is a component of $T_1 \cap T_2$ which is a meridian on T_1, then m is not nullhomologous on T_2.*

Proof: Suppose to the contrary that m is nullhomologous on T_2, and let D be the disk on T_2 which is bounded by m. Then each component of $D \cap T_1$ is also nullhomologous on T_2, and each such component, being disjoint with the meridian m of T_1, is either a meridian on T_1 or is nullhomologous on T_1. Since there is only a finite number of these components of $D \cap T_1$, there must be at least one such component s (possibly m itself) which is a meridian on T_1, but which is such that each component of $D \cap T_1$, lying in the disk C on D bounded by s,

is nullhomologous on T_1. If $T_1 \cap \text{Int } C = \square$, then C is a meridian disk of B_1 and hence intersects every center line of B_1. Since C is a subdisk of D and D lies on T_2, it follows in this case that every center line of B_1 intersects T_2.

Suppose, on the other hand, that $T_1 \cap \text{Int } C \neq \square$, and let k be a center line of B_1. Then let a be a component of $T_1 \cap \text{Int } C$ which contains no other such component in its interior (on C). Denote by X the subdisk of C bounded by a and by Y the subdisk of T_1 bounded by a. Then define $C' = (C - X) \cup Y$, and deform C' semi-linearly away from T_1 in a sufficiently small neighborhood of Y that no new intersections with k are introduced. The disk C'' thus obtained is bounded by s and intersects T_1 in exactly those components, other than a, in which C intersected T_1. After a finite number of steps of this kind there is obtained a disk C^* which is bounded by s, whose interior does not intersect T_1, and which intersects k in only those points in which C intersected k. Since C^* is then a meridian disk of B_1, it must intersect k in at least one point. Hence C intersects k in at least one point. But C is a disk on T_2, so it follows that k and T_2 intersect in at least one point.

Thus it follows in either case that T_2 intersects every center line of B_1. But this contradicts the fact that, by Lemma 5.2, B_1 and B_2 have a common center line. Consequently m could not have been nullhomologous on T_2.

<u>Corollary 5.1.</u> <u>Suppose that B_1 and B_2 are knotted equivalently situated polyhedral solid tori with boundaries T_1 and T_2 respectively, and let m be a component of $T_1 \cap T_2$. Then m is a meridian curve of</u>

B_1 *if and only if it is a meridian curve of* B_2 .

Proof: Suppose that m is a meridian curve on B_1 . Since m is then an unknotted simple closed curve on the boundary T_2 of the knotted solid torus B_2 , it follows that m must be either nullhomologous on T_2 or a meridian curve of B_2 [11, p. 164]. Lemma 5.3 then implies that m is a meridian curve on B_2 . Similarly, if m is a meridian curve on B_2 , it must also be a meridian curve of B_1 .

Lemma 5.4. Let B_1 *and* B_2 *be two equivalently situated polyhedral solid tori which are both interior to and concentric with the polyhedral solid torus* B . *If* n *is a component of* $\mathrm{Bd}\, B_1 \cap \mathrm{Bd}\, B_2$ *which is a latitude curve on* $\mathrm{Bd}\, B_1$, *then* n *is not nullhomologous on* $\mathrm{Bd}\, B_2$.

Proof: Since, by Lemma 5.2, B_1 and B_2 have a common center line, either both are knotted or both are unknotted. If B_1 and B_2 are knotted solid tori, then n is a knotted simple closed curve, since a latitude curve of a polyhedral solid torus is equivalent to its center line [11, p. 160]. But a curve which is nullhomologous on a torus must necessarily be unknotted. Therefore n cannot be nullhomologous on $\mathrm{Bd}\, B_2$.

Suppose, on the other hand, that B_1 and B_2 are unknotted. Then of course B is also unknotted. Define $C = \mathrm{Cl}(S^3 - B)$ and $C_i = \mathrm{Cl}(S^3 - B_i)$ for $i = 1, 2$. Then C, C_1, and C_2 are unknotted polyhedral solid tori [11, p. 164] with C interior to both C_1 and C_2 and concentric with each. Thus C_1 and C_2 are equivalently situated. Since n is a latitude curve on B_1 , it is non-nullhomologous on $\mathrm{Bd}\, B_1$ and bounds a disk D whose interior lies in $S^3 - B_1 = \mathrm{Int}\, C_1$. It follows that n is a meridian curve on C_1 . Lemma 5.3 then implies that n cannot be nullhomologous on $\mathrm{Bd}\, B_2 = \mathrm{Bd}\, C_2$.

Lemma 5.5. *Suppose that* B_1 *and* B_2 *are two equivalently situated polyhedral solid tori which are interior to and concentric with the polyhedral solid torus* B. *If* s *is a component of* $\mathrm{Bd}\, B_1 \cap \mathrm{Bd}\, B_2$ *which is nullhomologous on* $\mathrm{Bd}\, B_1$, *then* s *is also nullhomologous on* $\mathrm{Bd}\, B_2$.

Proof: Let D be a disk on $\mathrm{Bd}\, B_1$ bounded by s, and suppose first that $\mathrm{Int}\, D \cap \mathrm{Bd}\, B_2 = \Box$. Then s is either a meridian curve on $\mathrm{Bd}\, B_2$, a latitude curve on $\mathrm{Bd}\, B_2$, or is nullhomologous in $\mathrm{Bd}\, B_2$. But if s were a meridian curve on $\mathrm{Bd}\, B_2$, Lemma 5.3 would imply that s is not nullhomologous on $\mathrm{Bd}\, B_1$, while if s were a latitude curve on $\mathrm{Bd}\, B_2$, Lemma 5.4 would imply that s is not nullhomologous in $\mathrm{Bd}\, B_1$. The curve s must therefore, in this case, be nullhomologous on $\mathrm{Bd}\, B_2$.

If, on the other hand, $\mathrm{Int}\, D \cap \mathrm{Bd}\, B_2 \neq \Box$, denote by $r_1, \ldots, r_k$ the components of $\mathrm{Int}\, D \cap \mathrm{Bd}\, B_2$. Each of these components must be nullhomologous on $\mathrm{Bd}\, B_1$, so denote by D_i the subdisk of D bounded by r_i for $1 \leqq i \leqq k$. Since the number of these components is finite, there must be at least one, say r_j, such that $\mathrm{Int}\, D_j \cap \mathrm{Bd}\, B_2 = \Box$. The argument of the preceding paragraph then shows that r_j is also nullhomologous on $\mathrm{Bd}\, B_2$. Hence denote by C_j the disk on $\mathrm{Bd}\, B_2$ which is bounded by r_j. Then define $D' = (D - D_j) \cup C_j$, and deform D' semilinearly away from $\mathrm{Bd}\, B_2$ in a neighborhood of C_j to obtain a disk D'' bounded by s and having only $k - 1$ components of intersection with $\mathrm{Bd}\, B_2$ (in its interior). After k steps of this kind a disk D^* is obtained which is bounded by s and whose interior does not intersect $\mathrm{Bd}\, B_2$. The argument of the preceding paragraph then applies to show that s is nullhomologous on $\mathrm{Bd}\, B_2$.

Lemma 5.6. *Suppose that* B_1 *and* B_2 *are two equivalently situated*

polyhedral solid tori with boundaries T_1 *and* T_2 *respectively. Let* B' *and* B'' *be two concentric polyhedral solid tori, with* B' *interior to both* B_1 *and* B_2 *, and with* B'' *containing both* B_1 *and* B_2 *in its interior. Then there exists a semilinear homeomorphism* h *of* S^3 *onto itself, which is the identity on* $B \cup (S^3 - B'')$ *, such that no component of* $h(T_1) \cap T_2$ *is nullhomologous on either* $h(T_1)$ *or* T_2 *.*

Proof: Let $r_1, \ldots, r_k$ be the components of $T_1 \cap T_2$, each of which is nullhomologous on either T_1 or T_2 . By Lemma 5.5 each of these components is nullhomologous on both T_1 and T_2 . Hence denote by C_i and D_i the disks on T_1 and T_2 respectively bounded by r_i , $i = 1, \ldots, k$.

Since the number of nullhomologous components of intersection of T_1 and T_2 is finite, there must exist a component r_j such that the disk D_j on T_2 contains no nullhomologous component of $T_1 \cap T_2$ in its interior. Then the two disks C_j and D_j intersect only in their common boundary, so that $C_j \cup D_j$ is a polyhedral 2-sphere. Since $C_j \cup D_j \subset \operatorname{Int} B''$, one of the components of $S^3 - (C_j \cup D_j)$ must be interior to B''; denote by A this open 3-cell bounded by $C_j \cup D_j$. Since $C_j \cup D_j \subset S^3 - B'$, clearly either $B' \subset A$ or $A \subset S^3 - B'$. But $B' \subset A \subset \operatorname{Int} B''$ implies that $\bigcirc(B'', B') = 0$ [11, p. 173], and by Theorem 2.4 this contradicts the fact that, by hypothesis, B' and B'' are concentric. It follows that $A \subset \operatorname{Int} (B'' - B')$.

Now define

$$M = B' \cup \operatorname{Cl}(S^3 - B'') \cup \operatorname{Cl}(T_1 - C_j) .$$

Since $\operatorname{Int} D_j \cap T_1 = \Box$, it is clear that M intersects the 2-sphere $C_j \cup D_j$ only in the simple closed polygon r_j , and that $M \cap A = \Box$.

A theorem of Graueb [7, Satz 12] then implies the existence of a semilinear homeomorphism f of S^3 onto itself which is the identity on M and which carries C_j onto D_j, so that $f(T_1) = (T_1 - C_j) \cup D_j$. Now let U be a neighborhood of D_j contained in $B'' - B'$ and let g be a semilinear homeomorphism of S^3 onto itself which deforms $f(T_1)$ away from T_2 in U and which is the identity outside U. If $h_1 = gf$, then h_1 is a semilinear homeomorphism of S^3 onto itself which is the identity on $B' \cup (S^3 - B'')$ and which is such that the nullhomologous components of $h_1(T_1) \cap T_2$ are exactly the nullhomologous components of $T_1 \cap T_2$, except for those which are contained in $C_j \cap T_2$. Consequently $h_1(T_1) \cap T_2$ has fewer nullhomologous components than does $T_1 \cap T_2$.

By a finite sequence of such semilinear homeomorphisms of S^3 onto itself, each the identity on $B' \cup (S^3 - B'')$, all of the nullhomologous components of $T_1 \cap T_2$ are finally eliminated.

Theorem 5.4. Suppose that B_1 and B_2 are two equivalently situated polyhedral solid tori in S^3. Let B' and B'' be two concentric polyhedral solid tori, with B' interior to both B_1 and B_2, and with B'' containing both B_1 and B_2 in its interior. Then there exists a semilinear homeomorphism h of S^3 onto itself carrying B_2 onto B_1 and such that h is the identity on $B' \cup (S^3 - B)$.

Proof: According to Lemma 5.6, it may be assumed that no component of $T_1 \cap T_2$, where $T_1 = \mathrm{Bd}\, B_1$ and $T_2 = \mathrm{Bd}\, B_2$, is nullhomologous on either T_1 or T_2. If $T_1 \cap T_2 = \square$, then Theorem 5.1 implies that B_1 and B_2 are concentric. It then follows from Theorem 2.4 that B_1 and B_2 are equivalently knotted and that $\mathcal{O}(B_2, B_1) = 1$. Consequently there exists [11, p. 178] a semilinear homeomorphism h of S^3 onto itself carrying B_2

onto B_1 and which is the identity outside a prescribed neighborhood of $Cl(B_2 - B_1)$.

Suppose on the other hand that $T_1 \cap T_2 \neq \square$, and let $s_1, \ldots, s_k$ be the components of $T_1 \cap T_2$. Since each of these components is non-nullhomologous on both T_1 and T_2 each pair of such components separates both T_1 and T_2 into a pair of annular rings. Enumerate the collection of all such annular rings on T_1 in a finite sequence $R_1, \ldots, R_m$. The number of these annular rings being finite, there must be at least one, say R_j , whose interior does not contain a component of $T_1 \cap T_2$. Then denote by S_j' and S_j'' the closures of the two annular rings into which T_2 is separated by $Bd\ R_j$. Since $S_j' \cap R_j = S_j'' \cap R_j = Bd\ R_j = Bd\ S_j' = Bd\ S_j''$, it follows [11, p. 165] that the tori $S_j' \cup R_j$ and $S_j'' \cup R_j$ bound solid tori A_j' and A_j'' respectively interior to B'' . It is immediate that $Int\ A_j' \cap Int\ A_j'' = \square$. Since clearly $B_2 \subset A_j' \cup A_j''$ and $B' \subset Int\ B_2$ by hypothesis, and since B' is a connected set intersecting neither $Bd\ A_j'$ nor $Bd\ A_j''$, it follows that either A_j' or A_j'' contains B' in its interior. If $B' \subset Int\ A_j'$ define $S_j = S_j''$, and if $B' \subset Int\ A_j''$ then define $S_j = S_j'$. Thus is defined an annular ring S_j on T_2 with $Bd\ S_j = Bd\ R_j$ and with $R_j \cup S_j$ bounding a polyhedral solid torus A_j which is interior to $B'' - B'$.

Now define

$$M = B' \cup Cl(S^3 - B'') \cup Cl(T_2 - S_j)$$

Since $T_2 \cap Int\ R_j = \square$, it is clear that M intersects the torus $R_j \cup S_j$ only in the pair of simple closed curves $Bd\ R_j = Bd\ S_j$, and

that M does not intersect $\operatorname{Int} A_j$. There is therefore [11, p. 165] a semilinear homeomorphism f of S^3 onto itself which is the identity on M and which carries S_j onto R_j, so that $f(T_2) = (T_2 - S_j) \cup R_j$. Now let U be a neighborhood of R_j contained in $B'' - B'$ and let g be a semilinear homeomorphism of S^3 onto itself which deforms $f(T_2)$ away from T_1 inside U and which is the identity outside U. If $h_1 = gf$, then h_1 is a semilinear homeomorphism of S^3 onto itself which is the identity on $B' \cup (S^3 - B'')$ and which is such that $h_1(T_2)$ intersects T_1 in precisely those components of $T_1 \cap T_2$ which are contained in $T_2 - S_j$. Thus $h_1(T_2)$ intersects T_1 in fewer components than did T_2. By a finite sequence of such semilinear homeomorphisms of S^3 onto itself, each the identity on $B' \cup (S^3 - B'')$, all of the components of intersection of T_1 and T_2 are consequently eliminated. Thus is obtained a semilinear homeomorphism ϕ of S^3 onto itself, the identity on $B' \cup (S^3 - B'')$, such that $\phi(T_2) \cap T_1 = \square$. It then follows from Theorem 5.1 that the polyhedral solid tori B_1 and $\phi(B_2)$ are concentric and hence that there exists a semilinear homeomorphism ψ of S^3 onto itself which is the identity on $B' \cup (S^3 - B'')$ and which carries $\phi(B_2)$ onto B_1.

If finally $h = \psi\phi$, then h is evidently the mapping required by the Theorem.

CHAPTER VI

TAME CURVES IN THE THREE-SPHERE

The purpose of this chapter is to obtain a new characterization of tame simple closed curves in S^3, using certain concepts involving concentric tori together with the characterization of Harrold, Griffith, and Posey [10].

Definition 6.1. The set X in S^3 is said to be locally polyhedral at $p \in S^3$ if and only if there is some closed neighborhood N of p in S^3 whose intersection with X is empty or is a finite polyhedron. The set X is called locally polyhedral modulo Y (mod Y) if and only if it is locally polyhedral at each point of $S^3 - Y$.

Definition 6.2. The simple closed curve J in S^3 is said to be locally peripherally unknotted if and only if, to each $\epsilon > 0$ and to each $x \in J$, there corresponds a topological 2-sphere K such that

(a) $K \subset S(x, \epsilon)$,

(b) K is locally polyhedral mod J,

(c) $x \in \operatorname{Int} K$,

(d) $K \cap J$ consists of exactly two points.

Definition 6.3. The simple closed curve J in S^3 is said to be locally unknotted if and only if, to each point $x \in J$, there corresponds a disk G such that

(a) G is locally polyhedral mod J,

(b) $G \cap J$ is an arc,

(c) $G \cap J$ is the closure of a neighborhood of x relative to J.

Definition 6.4. The simple closed curve J in S^3 is said to have the enclosure property if and only if, to each $\epsilon > 0$, there corresponds a polyhedral torus which is contained in $S(J, \epsilon)$ and which contains J in its interior.

Definition 6.5. The simple closed curve J in S^3 is said to have the concentral enclosure property if and only if there exists a decreasing sequence of mutually concentric polyhedral solid tori intersecting in J.

Some use will be made in this chapter of the following elementary portions of the theory of linking numbers as set forth in Chapter II of the book Topologie by Alexandroff and Hopf [2]. The linking number of two cycles z_1 and z_2 is denoted by $\mathcal{V}(z_1, z_2) = \mathcal{V}(z_2, z_1)$. If z_1 and z_2 are disjoint simple closed curves in S^3, and z_1 is the boundary of a disk D such that D and z_2 are in relative general position, then $|\mathcal{V}(z_1, z_2)|$ must be zero if $D \cap z_2 = \square$ and must be 1 if $D \cap z_2$ is a point. In the latter case it will be said that z_1 and z_2 link each other.

Definition 6.6. The simple closed curve J is said to pierce a disk at the point $x \in J$ if and only if there exists a disk D such that

(a) D is locally polyhedral mod J,

(b) $D \cap J = x$,

(c) Bd D links J.

As is well known, the examples of Fox and Artin [6] show that the properties of local peripheral unknottedness and local unknottedness are independent. The concentral enclosure property clearly implies the enclosure property, and Harrold [9] has shown that local peripheral unknottedness also implies the enclosure property. However, example 4.1 shows that neither of these implications can be reversed.

Theorem 6.1. *If J is a locally peripherally unknotted simple closed curve in S^3, then J pierces a disk at each point of a countable dense subset $\{x_n\}_1^\infty$ of J.*

Proof: The proof of this theorem is implicit in the proof of Lemma 5.2 of [10].

An arc A in S^3 is said to have the enclosure property if and only if, given $\epsilon > 0$, there exists a polyhedral 2-sphere K which contains A in its interior and which is contained in the ϵ-neighborhood of A. It is easily seen that $S^3 - A$ is homeomorphic to the complement of a standard segment, that is, is an open 3-cell, if and only if A has the enclosure property. The following theorem indicates that the concentral enclosure property plays somewhat the same role for simple closed curves as does the enclosure property for arcs.

Theorem 6.2. *If the simple closed curve J in S^3 has the concentral enclosure property, then there is a polygonal knot K in S^3 such that $S^3 - J$ and $S^3 - K$ are homeomorphic.*

Proof: Let $\{B_n\}_1^\infty$ be a sequence of concentric polyhedral solid tori intersecting in J. Define $U_1 = Cl(S^3 - B_1)$ and $U_n = Cl(B_{n-1} - B_n)$ for $n = 2, 3, \ldots$. Then let K be a polygonal knot of the same knot type as the tori B_n, and let $\{C_n\}_1^\infty$ be a sequence of concentric polyhedral solid tori intersecting in K, each having K as a center line [11, p. 177]. Define $V_1 = Cl(S^3 - C_1)$ and $V_n = Cl(C_{n-1} - C_n)$ for $n = 2, 3, \ldots$.

Since Schubert [11, p. 180] has shown that, given any two equivalently knotted polyhedral solid tori B and C in S^3, there is a semilinear homeomorphism of S^3 onto itself carrying B onto C, it follows

that there exists a semilinear homeomorphism g carrying U_1 onto V_1. Suppose now that the domain of g has been extended so as to carry $\bigcup_{n=1}^{i} U_n$ homeomorphically onto $\bigcup_{n=1}^{i} V_n$. Then, since B_i and B_{i+1} are concentric and C_i and C_{i+1} are concentric, U_{i+1} and V_{i+1} are both homeomorphic to the topological product of a torus and an interval. It follows that g can be further extended so as to carry $\bigcup_{n=1}^{i+1} U_n$ homeomorphically onto $\bigcup_{n=1}^{i+1} V_n$. Continuing by induction, g is extended homeomorphically so that finally

$$g\left(\bigcup_{n=1}^{\infty} U_n\right) = \bigcup_{n=1}^{\infty} V_n .$$

But $\bigcup_{n=1}^{\infty} U_n = S^3 - J$ and $\bigcup_{n=1}^{\infty} V_n = S^3 - K$, so that g is a homeomorphism of $S^3 - J$ onto $S^3 - K$.

<u>Lemma 6.1.</u> <u>Suppose that J is a simple closed curve in S^3 having the concentral enclosure property and let $\{B_n\}_1^{\infty}$ be a sequence of polyhedral solid tori such that</u>

(a) $B_{n+1} \subset \operatorname{Int} B_n$ <u>for</u> $n = 1, 2, \ldots,$

(b) $\bigcap_{n=1}^{\infty} B_n = J$.

<u>Then there is an integer N such that B_i and B_j are concentric if</u> $i > j \geqq N$.

<u>Proof:</u> Since J has the concentral enclosure property, there is a decreasing sequence $\{C_n\}_1^{\infty}$ of concentric polyhedral solid tori such that $\bigcap_{n=1}^{\infty} C_n = J$. Since $\{S^3 - B_n\}_1^{\infty}$ is an increasing sequence of open sets covering the compact set $S^3 - \operatorname{Int} C_1$, there is an integer N such that $S^3 - \operatorname{Int} C_1 \subset S^3 - B_N$. Then $B_n \subset \operatorname{Int} C_1$ for $n \geqq N$. Now let i and j be integers such that $i > j \geqq N$. Since $\{S^3 - C_n\}_1^{\infty}$ is an increasing sequence of open sets covering the compact set $S^3 - \operatorname{Int} B_i$, there is an integer M such that $C_M \subset \operatorname{Int} B_i$. Then B_i and B_j are

two polyhedral solid tori, each containing C_M in its interior and each contained in the interior of C_1. Since C_1 and C_M are concentric, it follows from Lemma 5.1 that B_i and B_j are equivalently situated. Since (a) implies that $\text{Bd } B_i \cap \text{Bd } B_j = \square$, Theorem 5.1 shows that B_i and B_j are concentric.

Lemma 6.2. Suppose that J is a simple closed curve in S^3 having the concentral enclosure property and which pierces a disk at each point of a countable dense subset. Then there is a sequence $\{D_i\}_1^\infty$ of disks and a sequence $\{B_i\}_0^\infty$ of concentric polyhedral solid tori such that for $i = 1, 2, \ldots,$

(a) $D_i \cap J = a_i$, a point,

(b) $\{a_i\}_1^\infty$ is a dense subset of J,

(c) D_i is locally polyhedral mod J,

(d) $\text{Bd } D_i$ links J,

(e) $D_i \subset S(a_i, 1/i)$,

(f) $D_i \cap D_j = \square$ if $i \neq j$,

(g) $J \subset \text{Int } B_i$,

(h) $B_i \subset S(J, 1/i) \cap \text{Int } B_{i-1}$

(i) $\text{Bd } D_i$ is a meridian curve on $\text{Bd } B_i$,

(j) $D_i \cap \text{Bd } B_j$ is a meridian curve on B_j for $j \geq i$, and $D_i \cap \text{Bd } B_j = \square$ if $j < i$.

Proof: The proof of this lemma follows closely the proofs of Lemmas 5.2 and 5.3 of [10]. The proof of the former applies without alteration to yield a sequence $\{D_i\}_1^\infty$ of disks satisfying conditions (a) through (f).

Now make the inductive assumption that the disks $D_1, \ldots, D_n$ have been properly altered and the solid tori $B_0, \ldots, B_{n-1}$ chosen (not neces-

sarily concentric) in such a way that conditions (g) through (j) are satisfied for these sets. On each of the disks D_i, $i = 1, \ldots, n$, choose a fixed subdisk D_i^n having a_i as an interior point and lying entirely in $S(J, 1/2n)$. For each i let γ_i denote $\rho(J, D_i - D_i^n)$ and then define

$$\epsilon_n = \min [\rho(J, \mathrm{Bd}\, B_{n-1}), \rho(J, \mathrm{Bd}\, D_n), \gamma_1, \ldots, \gamma_n, 1/2n].$$

Since J has the concentral enclosure property, there is a polyhedral solid torus B_n containing J in its interior and such that $B_n \subset S(J, \epsilon_n)$. It is then immediate that B_n satisfies conditions (g) and (h). Also

$$\mathrm{Bd}\, D_n \subset (S^3 - B_n) \cap \mathrm{Int}\, B_{n-1}$$

since it may be assumed that $D_n \subset \mathrm{Int}\, B_{n-1}$ (taking a subdisk if necessary). The inductive hypothesis together with $D_n \subset \mathrm{Int}\, B_{n-1}$ implies that (i) holds for $i < n$ and that (j) is satisfied for $j < n$ and all $i = 1, 2, \ldots, n$. Therefore only the sets $D_i \cap \mathrm{Bd}\, B_n$ for $1 \leqq i \leqq n$ need be considered in order to prove that the choice of B_n can be made as required.

It will be convenient to suppose that a disk D_0 has been chosen such that $D_0 \cap J$ is a single point a_0, $\mathrm{Bd}\, D_0$ links J, D_0 is locally polyhedral mod J, $D_0 \cap D_i = \square$ for $i = 1, 2, \ldots$. It is clear that no simple closed curve on any D_i, $i > 0$, can link $\mathrm{Bd}\, D_0$, and it may be assumed that $\mathrm{Bd}\, D_0 \subset S^3 - B_0$. It is clear that each meridian curve of a B_n links J. For if M were a meridianal disk of B_n not intersecting J then, since $M \cup \mathrm{Cl}(S^3 - B_n)$ is simply connected, it would follow that $\mathrm{Bd}\, D_0$ does not link J. On the other hand, each latitude curve of B_n links $\mathrm{Bd}\, D_0$, since $\mathrm{Bd}\, D_0$ links the curve $J \subset \mathrm{Int}\, B_n$. It follows that

every non-nullhomologous simple closed curve on Bd B_n must link either J or Bd D_o .

For any fixed i , $1 \leqq i \leqq n$, Bd D_i is a curve in $S^3 - B_n$ and $D_i \cap$ Bd B_n is the union of a finite collection $r'_{i1}, r'_{i2}, \ldots, r'_{in_i}$ of mutually disjoint simple closed curves. Since every simple closed curve on Bd B_n must link either J , Bd D_o , or neither, the curves r'_{ij} are divided into two types.

Type 1: r'_{ij} links neither J nor Bd D_o ,

Type 2: r'_{ij} links J but not Bd D_o .

It may now be shown by the argument on Page 20 of [10] that all of the components of Type 1 and all but a single component r_{in} of Type 2 can be eliminated by a deformation of B_n which is such that the new solid torus B_n obtained satisfies conditions (g) and (h). Since r_{in} links J it is clear that r_{in} cannot be nullhomologous on Bd B_n . Since r_{in} bounds a disk whose interior lies in Int B_n , it follows that r_{in} is a meridian curve on B_n .

This process is repeated for $i = 1, 2, 3, \ldots, n$ in turn, so that as a consequence (j) is satisfied for $j = n$ and $i \leqq n$. If D_n is replaced by $D_n \cap B_n$, then (i) will hold with $i = n$. Finally it may be assumed that $D_i \subset$ Int B_n for $i > n$, so that (j) is satisfied for $j = n$ and all i .

Thus a sequence $\{D_i\}_1^\infty$ of disks and a sequence $\{B_i\}_o^\infty$ of polyhedral solid tori are found satisfying conditions (a) through (j). It remains only to be seen that the solid tori $\{B_i\}_o^\infty$ may be taken concentric. It follows from (h) that $\bigcap_{i=o}^{\infty} B_i = J$. Since J has the concentral enclosure property it follows from Lemma 6.1 that there exists an integer N such that B_i and

B_j are concentric if $i, j \geqq N$. The desired sequences $\{D_i\}_1^\infty$ and $\{B_i\}_0^\infty$ are then obtained from those above by deleting the first N elements of each sequence.

Lemma 6.3. *Suppose that J is a simple closed curve in S^3 which has the concentral enclosure property and which pierces a disk at each point of a countable dense subset. Then J is locally peripherally unknotted.*

Proof: Given an arbitrary point $x \in J$, it suffices to find a sequence $\{K_n\}_1^\infty$ of 2-spheres such that for each $n \geqq 1$

(a) $x \in \operatorname{Int} K_n$,

(b) $J \cap K_n$ is a pair of points,

(c) K_n is locally polyhedral mod J,

(d) $K_{n+1} \subset \operatorname{Int} K_n$,

(e) $\bigcap_{n=1}^{\infty} \operatorname{Cl}(\operatorname{Int} K_n) = x$.

Let $\{D_n\}_1^\infty$ and $\{B_n\}_1^\infty$ be the sequences of disks and concentric solid tori respectively given by Lemma 6.2. Choose two subsequences $\{D_{n_i}\}_{i=1}^\infty$ and $\{D_{m_i}\}_{i=1}^\infty$ of $\{D_n\}_1^\infty$ such that (assuming that a subarc of J containing x as an interior point has been given a linear order) $\{a_{n_i}\}_1^\infty$ converges to x monotonely from the left and $\{a_{m_i}\}_1^\infty$ converges to x monotonely from the right. Denote by A_i that subarc of J which joins a_{n_i} and a_{m_i} and which has x as an interior point. Then clearly $\bigcap_{i=1}^{\infty} A_i = x$.

If $k_1 = \max(n_1, m_1)$ then, by condition (j) of Lemma 6.2, B_{k_1} intersects D_{n_1} and D_{m_1} in meridianal disks of B_{k_1}. Then denote by R_{k_1} the one of the two annular rings, into which $D_{n_1} \cup D_{m_1}$ separates $\operatorname{Bd} B_{k_1}$, such that the 2-sphere

$$K_1 = (D_{n_1} \cap B_{k_1}) \cup (D_{m_1} \cap B_{k_1}) \cup R_{k_1}$$

contains x in its interior. Clearly K_1 satisfies (a), (b), and (c) for $n = 1$.

If $K_1, \ldots, K_j$ and $R_{k_1}, \ldots, R_{k_j}$ have been similarly defined with the 2-spheres K_i satisfying conditions (a) through (d), let k_{j+1} be a fixed integer such that

$$k_{j+1} > \max(n_{j+1}, m_{j+1}, k_1, \ldots, k_j)$$

and denote by $R_{k_{j+1}}$ the one of the two annular rings, into which $D_{m_{j+1}} \cup D_{n_{j+1}}$ separates Bd $B_{k_{j+1}}$, such that the 2-sphere

$$K_{j+1} = (D_{n_{j+1}} \cap B_{k_{j+1}}) \cup (D_{m_{j+1}} \cap B_{k_{j+1}}) \cup R_{k_{j+1}}$$

contains x in its interior.

In this manner is defined by induction a sequence $\{K_j\}_1^\infty$ of 2-spheres satisfying conditions (a) through (d) with $J \cap \mathrm{Cl}(\mathrm{Int}\ K_j) = A_j$ and $\mathrm{Cl}(\mathrm{Int}\ K_j) \subset B_{k_j}$. Then

$$x \in \bigcap_{j=1}^{\infty} \mathrm{Cl}(\mathrm{Int}\ K_j) \subset \bigcap_{j=1}^{\infty} B_{k_j} = J$$

so that

$$\bigcap_{j=1}^{\infty} \mathrm{Cl}(\mathrm{Int}\ K_j) = \bigcap_{j=1}^{\infty} [J \cap \mathrm{Cl}(\mathrm{Int}\ K_j)]$$

$$= \bigcap_{j=1}^{\infty} A_j$$

$$\bigcap_{j=1}^{\infty} \mathrm{Cl}(\mathrm{Int}\ K_j) = x$$

Hence (e) also holds. Since x was an arbitrary point of J, it follows that J is locally peripherally unknotted.

Lemma 6.4. *Suppose that J is a simple closed curve in S^3 which has the concentral enclosure property and which pierces a disk at each point of a countable dense subset. Then J is locally unknotted.*

Proof: It will be shown that J is locally unknotted by defining an annular ring G which has J as one component of its boundary and which is locally polyhedral modulo J. Let $\{a_n\}_1^\infty$ be the given countable dense subset of J and let $\{D_n\}_1^\infty$ and $\{B_n\}_1^\infty$ be the sequences of disks and concentric solid tori respectively given by Lemma 6.2.

A dyadic system of notation will be convenient. Let $a(0) = a_1$ and $a(1) = a_2$. Let H and K be the two subarcs of J from a_1 to a_2. Let a_{n_3} be the first element of $\{a_3, a_4, \ldots\}$ on H and let a_{n_4} be the first on K. Set $a(0, 0) = a(0) = a_{n_1}$, $a(1, 0) = a(1) = a_{n_2}$; $a(0, 1) = a_{n_3}$, $a(1, 1) = a_{n_4}$. The set $a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}$ divides J into four subarcs. Continuing in the obvious manner, at the nth stage 2^n points $a(\alpha_1, \ldots, \alpha_n)$, $\alpha_i = 0$ or 1, have been selected from $\{a_1, a_2, \ldots\}$. Each a_i occurs first at a unique stage. As before set $a(\alpha_1, \ldots, \alpha_{n-1}, 0) = a(\alpha_1, \ldots, \alpha_{n-1})$. The disk D_n which meets J at $a_n = a(\alpha_1, \ldots, \alpha_p)$ is denoted by $D(\alpha_1, \ldots, \alpha_p)$.

Let m_1 be the least positive integer such that the solid torus B_{m_1} meets both $D(0)$ and $D(1)$. Let m_2 be the least integer greater than m_1 such that B_{m_2} meets each $D(\alpha_1, \alpha_2)$, $\alpha_i = 0, 1$. In general let m_p be the least integer greater than m_{p-1} such that B_{m_p} meets each of the disks $D(\alpha_1, \ldots, \alpha_p)$, $\alpha_i = 0$ or 1. A new sequence

$\{C_p\}_{p=1}^{\infty}$ of concentric polyhedral solid tori intersecting in J is obtained by defining $C_p = B_{mp}$. The torus $\mathrm{Bd}\ C_n$ then meets each of the disks $D(\alpha_1, \ldots, \alpha_n)$ in a meridian curve. The disk $C_n \cap D(\alpha_1, \ldots, \alpha_n)$ will be denoted by $D_n(\alpha_1, \ldots, \alpha_n)$.

Define $U_n = \mathrm{Cl}(C_n - C_{n+1})$. Since C_n and C_{n+1} are concentric, U_n is the topological product of a torus and an interval. Clearly U_n is separated into 2^n components by the union of the disks $D(\alpha_1, \ldots, \alpha_n)$, $\alpha_i = 0, 1$. Denote by $Q(\alpha_1, \ldots, \alpha_n)$ the component determined by $D(\alpha_1, \ldots, \alpha_n)$ and $D(\beta_1, \ldots, \beta_n)$, where $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are consecutive n-tuples, with n-tuples ordered by $(\alpha_1, \ldots, \alpha_n) < (\beta_1, \ldots, \beta_n)$ if and only if $\alpha_j = \beta_j$, $j < i$, and $\alpha_i < \beta_i$ for $i \leq n$. The boundary of $Q(\alpha_1, \ldots, \alpha_n)$ then consists of the four annular rings

$$T(\alpha_1, \ldots, \alpha_n) = U_n \cap D(\alpha_1, \ldots, \alpha_n)$$

$$T(\beta_1, \ldots, \beta_n) = U_n \cap D(\beta_1, \ldots, \beta_n)$$

$$R_n(\alpha_1, \ldots, \alpha_n) \subset \mathrm{Bd}\ C_n$$

$$R_{n+1}(\alpha_1, \ldots, \alpha_n) \subset \mathrm{Bd}\ C_{n+1}$$

Since C_n and C_{n+1} are concentric, they have a common center line k which intersects each of the meridianal disks $D(\alpha_1, \ldots, \alpha_n)$ and $D(\beta_1, \ldots, \beta_n)$ in a single point. If k' is the subarc of k which is interior to each of the two 2-spheres $D(\alpha) \cup D(\beta) \cup R_n(\alpha)$ and $[D(\alpha) - T(\alpha)] \cup [D(\beta) - T(\beta)] \cup R_{n+1}(\alpha)$, then clearly k' is an unknotted chord of each of these spheres, so that $Q(\alpha_1, \ldots, \alpha_n)$ is an unknotted solid torus on which the simple closed polygons

$$r_n(\alpha_1, \ldots, \alpha_n) = D(\alpha_1, \ldots, \alpha_n) \cap \text{Bd } C_n$$

$$r_n(\beta_1, \ldots, \beta_n) = D(\beta_1, \ldots, \beta_n) \cap \text{Bd } C_n$$

$$r_{n+1}(\alpha_1, \ldots, \alpha_n) = D(\alpha_1, \ldots, \alpha_n) \cap \text{Bd } C_{n+1}$$

$$r_{n+1}(\beta_1, \ldots, \beta_n) = D(\beta_1, \ldots, \beta_n) \cap \text{Bd } C_{n+1}$$

are latitude curves.

The annular ring G will be defined as

$$G = J \cup \bigcup_{n=1}^{\infty} G_n$$

where G_n is an annular ring in U_n intersecting each $T(\alpha_1, \ldots, \alpha_n)$ in an arc, with Bd G_n consisting of two simple closed curves, one on Bd C_n and the other on Bd C_{n+1}, such that $G_n \cap \text{Bd } C_{n+1} = G_{n+1} \cap \text{Bd } C_{n+1}$.

The set U_1 is separated by $D(0)$ and $D(1)$ into the two solid polyhedral tori $Q(0)$ and $Q(1)$. Let $s_1(0)$ and $s_1(1)$ be polygonal arcs spanning $R_1(0)$ and $R_1(1)$ respectively, with common endpoints on $r_1(0)$ and $r_1(1)$, so that $s_1(0) \cup s_1(1)$ is a longitude curve on Bd C_1. Let $t(\alpha_1)$, $\alpha_1 = 0, 1$, be a polygonal arc spanning $T(\alpha_1)$ and having an endpoint in common with $s_1(\alpha_1)$. Then join the endpoints of $t(0)$ and $t(1)$ by polygonal arcs $s_2(0)$ and $s_2(1)$ on $R_2(0)$ and $R_2(1)$ respectively, each intersecting each of the meridian curves $r_2(\alpha_1, \alpha_2)$ in a single point, and such that $s_1(\alpha_1) \cup s_2(\alpha_2) \cup t(0) \cup t(1)$ is a meridian curve on $Q(\alpha_1)$, $\alpha_1 = 0, 1$. Let $A(\alpha_1)$, $\alpha_1 = 0, 1$, be a polyhedral meridianal disk bounded by $s_1(\alpha_1) \cup s_2(\alpha_2) \cup t(0) \cup t(1)$. Then $G_1 = A(0) \cup A(1)$ is a polyhedral annular ring in U_1 whose boundary consists of the curves $s_1(0) \cup s_1(1)$ on Bd C_1 and $s_2(0) \cup s_2(1)$ on Bd C_2.

Suppose that the annular rings $G_1, \ldots, G_{n-1}$ have now been defined in such a way that $\bigcup_{i=1}^{n-1} G_i$ is an annular ring in $\bigcup_{i=1}^{n-1} U_i$ with boundary curves $s_1(0) \cup s_1(1)$ on Bd C_1 and $\bigcup s_n(\alpha_1, \ldots, \alpha_{n-1})$ on Bd C_n, such that the longitude curve $\bigcup s_n(\alpha_1, \ldots, \alpha_{n-1})$ intersects each annular ring $R_n(\alpha_1, \ldots, \alpha_n)$ in an arc $s_n(\alpha_1, \ldots, \alpha_n)$ spanning its boundary. Define 2^n polygonal arcs $t(\alpha_1, \ldots, \alpha_n)$, $\alpha_i = 0, 1$, with $t(\alpha_1, \ldots, \alpha_n)$ spanning the annular ring $T(\alpha_1, \ldots, \alpha_n)$ and having one endpoint on $\bigcup s_n(\alpha_1, \ldots, \alpha_{n-1})$.

To define a meridianal disk $A(\alpha_1, \ldots, \alpha_n)$ of the solid torus $Q(\alpha_1, \ldots, \alpha_n)$, join the endpoints of $t(\alpha_1, \ldots, \alpha_n) \cup s_n(\alpha_1, \ldots, \alpha_n) \cup t(\beta_1, \ldots, \beta_n)$ by a polygonal arc $s_{n+1}(\alpha_1, \ldots, \alpha_n)$, such that $t(\alpha) \cup t(\beta) \cup s_n(\alpha) \cup s_{n+1}(\alpha)$ is a meridian curve on $Q(\alpha_1, \ldots, \alpha_n)$ and $s_{n+1}(\alpha_1, \ldots, \alpha_n)$ intersects each of the meridian curves $r_{n+1}(\alpha_1, \ldots, \alpha_{n+1})$ in a single point. Then let $A(\alpha_1, \ldots, \alpha_n)$ be a polyhedral meridianal disk of $Q(\alpha_1, \ldots, \alpha_n)$ with $t(\alpha) \cup t(\beta) \cup s_n(\alpha) \cup s_{n+1}(\alpha)$ as its boundary. The annular ring G_n is now defined by

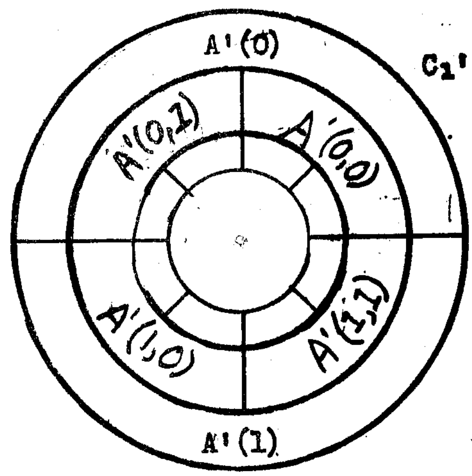
$$G_n = \bigcup_{\alpha_i = 0,1} A(\alpha_1, \ldots, \alpha_n)$$

It will now be proved that the set

$$G = J \cup \bigcup_{n=1}^{\infty} G_n$$

is indeed an annular ring having J as one component of its boundary and which is locally polyhedral modulo J. Denote by G' the annular ring in the xy-plane bounded by the circles $r = 1$ and $r = 2$. Denote by J'

the circle $r = 1$ and by C_n' the circle $r = 1 + 1/n$, $n = 1, 2, \ldots$. Denote by G_n' , the annular ring bounded by C_n' and C_{n+1}' . Let g be a homeomorphism of J onto J' and denote by $a'(\alpha_1, \ldots, \alpha_n)$ the image under g of $a(\alpha_1, \ldots, \alpha_n)$, and denote by $L'(\alpha_1, \ldots, \alpha_n)$ the radial line in the xy-plane determined by the



point $a'(\alpha_1, \ldots, \alpha_n)$. Finally denote by $A'(\alpha_1, \ldots, \alpha_n)$ the sector of G_n' determined by $L'(\alpha_1, \ldots, \alpha_n)$ and $L'(\beta_1, \ldots, \beta_n)$, where $(\beta_1, \ldots, \beta_n)$ is the n-tuple immediately succeeding $(\alpha_1, \ldots, \alpha_n)$. See the figure above.

Let g_1 be a homeomorphism of G_1 onto G_1' carrying $A(\alpha_1)$ onto $A'(\alpha_1)$, $\alpha_1 = 0, 1$. In general, let g_n be a homeomorphism of G_n onto G_n' which carries $A(\alpha_1, \ldots, \alpha_n)$ onto $A'(\alpha_1, \ldots, \alpha_n)$ and agrees with g_{n-1} on $G_n \cap G_{n-1}$. This gives a homeomorphism g of $G - J$ onto $G' - J'$ defined by $g(x) = g_n(x)$ if $x \in G_n$. Then g is clearly a $1 - 1$ map of G onto g' and is continuous at each point of $G - J$. It remains to be seen that g is also continuous at each point of J .

For each point $p \in J$ and each integer n let $D_n(p)$ denote

$\bigcup D(\alpha_1, \ldots, \alpha_n)$ if p lies on no disk $D(\alpha_1, \ldots, \alpha_n)$, but let $D_n(p)$ denote $[\bigcup D(\alpha_1, \ldots, \alpha_n)] \setminus D(\beta_1, \ldots, \beta_n)$ if $p \in D(\beta_1, \ldots, \beta_n)$. Then define $V(p)$ to be the component of $(\text{Int } C_n) - D_n(p)$ which contains p. Similarly for each point $p' \in J'$ and each integer n let $L_n'(p')$ denote $\bigcup L'(\alpha_1, \ldots, \alpha_n)$ if p lies on no line $L'(\alpha_1, \ldots, \alpha_n)$, but let $L_n'(p')$ denote $[\bigcup L'(\alpha_1, \ldots, \alpha_n)] - L'(\beta_1, \ldots, \beta_n)$ if $p' \in L'(\beta_1, \ldots, \beta_n)$. If B_n' is the set of points inside C_n' but not inside J', denote by $V_n'(p')$ that component of $B_n' - L_n'(p')$ which contains p'. It is readily seen that $p = \bigcap_{n=1}^{\infty} V_n(p)$ and $p' = \bigcap_{n=1}^{\infty} V_n'(p')$.

Now let $\{p_n\}_1^{\infty}$ be a sequence of points of $G - J$ converging to $p \in J$. Then to any neighborhood N of $p' = g(p)$ in G' there corresponds an index n such that $V_n'(p') \subset N$. Since $p_n \to p$, it follows that all but a finite number of the points $\{p_n\}$ lie in $[V_n(p)] \cap (G - J)$. Since g is a homeomorphism on $G - J$, it follows that all but a finite number of the points $g(p_n)$ lie in $V_n'(p') \subset N$. Consequently g is continuous at the point $p \in J$.

Thus g is a $1 - 1$ continuous map of the set G onto the annular ring G'. But G is clearly closed and hence compact. For obviously any any limit point of $G - J$ not in $G - J$ must lie in $\bigcap_{n=1}^{\infty} C_n = J$, so that $Cl(G - J) \subset G$. Then $G = Cl(G - J) \cup J$, so that G is the union of two closed sets and is hence itself closed. It follows that g is a homeomorphism of G onto G'.

Thus J is a component of the boundary of the annular ring G, which is locally polyhedral mod J. It follows immediately that J is locally unknotted.

<u>Theorem 6.3.</u> <u>The simple closed curve</u> J <u>in</u> S^3 <u>is tame if and only</u>

if J has the concentral enclosure property and pierces a disk at every point.

Proof of Sufficiency: If J has the concentral enclosure property and pierces a disk at every point, then, by Lemmas 6.3 and 6.4, J is locally peripherally unknotted and locally unknotted. It follows from Theorem VII of [10] that J is tame.

Proof of Necessity: If J is tame then there is a homeomorphism f of S^3 onto itself such that $K = f(J)$ is a polygonal closed curve. Given $x \in J$, K clearly pierces a disk D at $f(x)$. Then J pierces the disk $f^{-1}(D)$ at x. By the polyhedral surface approximation Theorem [4], it may be assumed that $f^{-1}(D)$ is locally polyhedral modulo J.

There is a sequence $\{C_n\}_1^\infty$ of concentric polyhedral solid tori such that $\bigcap_{n=1}^{\infty} C_n = K$ [11, p. 177]. If $B_n = f^{-1}(C_n)$ for $n = 1, 2, \ldots$, then $\{B_n\}_1^\infty$ is a sequence of concentric tame solid tori such that $\bigcap_{n=1}^{\infty} B_n = J$. By the proof of Theorem 4.2, there is a homeomorphism g of S^3 onto itself which leaves J fixed and carries each B_n onto a polyhedral solid torus. Consequently J has the concentral enclosure property.

Corollary 6.1. The simple closed curve J in S^3 is tame if and only if J is locally peripherally unknotted and has the concentral enclosure property.

Proof: If J is tame, then J is locally peripherally unknotted by Theorem VII of [10] and has the concentral enclosure property by Theorem 6.3.

Conversely, if J is locally peripherally unknotted, then J pierces a disk at each point of a countable dense subset by Theorem 6.1. By

Lemma 6.4, J is then locally unknotted. It follows that J is tame [10].

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