12-1950

Electrical Analogue Computer for the Heat Pump

James Edward Goff
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by James Edward Goff entitled "Electrical Analogue Computer for the Heat Pump." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Electrical Engineering.

Paul C. Cromwell, Major Professor

We have read this thesis and recommend its acceptance:

Edgar D. Ecawls, C. H. Weaver, A. V. Schultz, C. T. Smith

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

TO THE COMMITTEE ON GRADUATE STUDY:

I am submitting to you a thesis written by James Edward Goff entitled "Electrical Analogue Computer for the Heat Pump". I recommment that it be accepted for nine quarter hours credit in partial fulfillment of the requirements for the degree of Master of Science, with a major in Electrical Engineering.

Paul C. Cromwell
Major Professor

We have read this thesis and recommend its acceptance.

Edgar D. Eaves

C. L. Carter

F. V. Schultz

C. H. Weaver

Accepted for the Committee

E. A. Waters
Dean of the Graduate School

ELECTRICAL ANALOGUE COMPUTER

FOR THE HEAT PUMP

A THESIS

Submitted to
The Committee on Graduate Study
of
The University of Tennessee
in
Partial Fulfillment of the Requirements
for the degree of
Master of Science

by

James Edward Goff

December, 1950

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SUMMARY

Because of the complex behavior of a cylindrical heat source-sink under intermittent operating conditions, analysis by analytical means is very difficult. Therefore an electrical analogy computer consisting of a distributed resistance capacitance model was designed and built. It was shown analytically that the behavior of this computer model was the same as that of the heat system. That the model was a true analogy is verified by the fact that data obtained from this model could be used to check certain heat flow problems previously solved by analytical means. Data were obtained for a cylindrical heat source in an infinite medium. These data were used to check the analytical solution presented by A. Gemant.

A generator was designed and built which could be used to solve two dimensional heat flow problems involving intermittent operation. Its theory of operation was set forth and the circuit diagram presented. With such equipment one may obtain numerical answers to many two dimensional problems.

INTRODUCTION

Some ninety-eight years ago Lord Kelvin first called attention to the fact that refrigeration systems are reversible, and that they possess unusual and interesting characteristics when operated as a heating system. Interest in the heat pump remained dormant until some twenty-six years ago, when the development of the domestic electric refrigerator and low electric rates began to make engineers think of electric heating and summer cooling as something more than a dream.

The use of the heat pump is, however, greatly restricted because of the available heat sources. Most city water supplies are at a high enough temperature in winter to be usable as a heat source, but the cost of water will usually exceed the power saving. The use of the heat of fusion of water by freezing ice presents the problem of disposing of tons of ice per day per home in very cold weather. Well water, where available, has been used successfully, but availability in the United States is very limited. Therefore this furnishes only a local solution to the problem. Thus it is generally recognized that success of the heat pump depends upon the discovery of a satisfactory heat source.

The earth has a large storage capacity, which remains at a relatively constant temperature, and is conveniently located. It has long been regarded as a possible source of heat, but lack of data on the characteristics of the earth has been a serious problem. Attempts at mathematical determination of such data lead to second order partial differential equations whose solutions are often complicated, and due to long duration of the transients, complete solutions of the equations are necessary.

Equations for the prediction of transient heat flow and temperature distribution in an infinite medium have been solved for the following cases, among others:

(a) An infinite line source with a constant rate of heat emission.[1]

(b) An infinite cylindrical source at constant temperature.[2]

(c) An infinite flat plate at constant temperature.[3]

These are cases where special boundary conditions permit solutions and the general two or three dimensional heat flow problem has not yet been solved.

Because of the mathematical difficulty involved, methods other than analytical were sought for obtaining the desired data. One method is to set up the heat system and measure these quantities directly. However the method is costly and much time is required to obtain the information.

Another method consists of obtaining the desired information by analogy. Electrical measurements can be made rapidly, accurately, and

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[1] L. R. Ingersoll and H. J. Plass, "Theory of Ground Pipe Heat Source for the Heat Pump", Heating, Piping, and Air Conditioning, 20:119-122, July, 1948.

[2] A. Gemant, "Transient Temperatures Around Heating Pipes Maintained at Constant Temperatures", Journal of Applied Physics, 17:1076-81, December, 1946.

[3] B. F. Raber, C. F. Boester, and F. V. Hutchinson, "The Heat Pump...How to Analyze Earth as a Heat Source", Heating, Piping, and Air Conditioning, 20:82-86, March, 1948.

at relatively low cost. Therefore, when possible, the use of an electrical model is warranted. In recent publications a number of investigations have presented analyses of the electrical analogy of heat flow. However in these cases, lumped parameter circuits were used.[4] Consequently, exact analogies were not realized, nor do lumped circuits lend themselves to determination of design data, such as the most economical pipe size, pipe spacing, and pipe arrangement.

This paper presents the mathematical proof that a distributed resistance capacitance system is described by the same differential equation as a two dimensional heat system, and describes a generator for use in obtaining data on heat pump operation. While the theory associated with the condenser was to be developed in this paper, actual construction of a suitable condenser was to be done by Sisson[5] in a companion thesis. Data were to be obtained using the condenser constructed by Sisson as an experimental verification of the theory developed in this paper. Such a system is capable of yielding numerical results with sufficient accuracy for normal engineering applications, and will in fact yield results as accurate as the associated equipment permits measurements to be made. The idea and basic model were suggested to the author by Professor P. C. Cromwell, of the Electrical Engineering Department of the University of Tennessee.

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[4] Victor Paschkis and H. C. Baker, "A Method for Determining Unsteady-State Heat Transfer by Means of an Electrical Analogy", *Transactions American Society of Mechanical Engineers*, 64:105, 1942.

[5] Austin Ray Sisson, "Calculations for Intermittent Operation of a Heat Pump by Means of an Electrical Analogy Computer", Unpublished Master's Thesis, University of Tennessee, 1950.

PART I

PROOF RESISTANCE CAPACITANCE SYSTEM EQUIVALENT TO HEAT SYSTEM

Consider the Maxwell Electromagnetic equation:

$$\nabla \times H = \frac{4\pi}{c}\left(\sigma E + \frac{1}{4\pi}\frac{\partial D}{\partial t}\right) \qquad (1\text{-}1)$$

By taking the divergence of both sides, we get:

$$\nabla \cdot \nabla \times H = \nabla \cdot \frac{4\pi}{c}\left(\sigma E + \frac{1}{4\pi}\frac{\partial D}{\partial t}\right)$$

But the divergence of the curl is identically zero

$$\nabla \cdot \nabla \times H \equiv 0$$

so that:

$$0 = \nabla \cdot \frac{4\pi}{c}\left(\sigma E + \frac{1}{4\pi}\frac{\partial D}{\partial t}\right) \qquad (1\text{-}2)$$

Partial derivatives of a continuous function may be taken in any order. Therefore we may write Eq(1-2) as:

$$0 = \nabla \cdot \sigma E + \frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla \cdot D) \qquad (1\text{-}2a)$$

Poisson's equation states that:

$$\nabla \cdot D = 4\pi\rho \qquad (1\text{-}3)$$

Substituting Eq(1-3) in Eq(1-2) we get:

$$\sigma \nabla \cdot E + \frac{\partial \rho}{\partial t} = 0$$

or:

$$\sigma \nabla \cdot E = -\frac{\partial \rho}{\partial t} \tag{1-4}$$

This is known in the literature as the equation of current continuity. Let us now apply this equation to the proposed physical system.

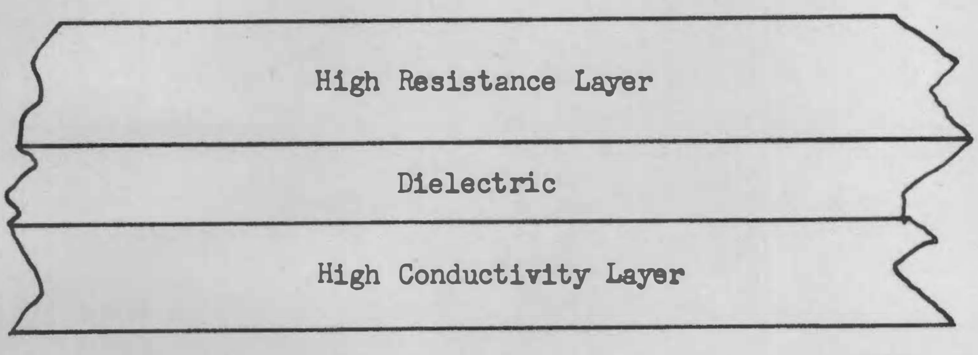

Fig. 1. Section of proposed model.

Let σ = volume conductivity of resistance layer.

$C(x,y,z)$ = volume capacitance of a section of high resistance layer to high conductivity layer.

$\rho(x,y,z)$ = charge density in high resistance layer.

The total charge in a differential volume of the resistance layer is given by the following equation:

$$q_D = \rho \, dx \, dy \, dz \tag{1-5}$$

The capacitance of the same volume is:

$$C_D = c \, dx \, dy \, dz \tag{1-6}$$

and in any condenser:

$$q_D = C_D V \tag{1-7}$$

Substituting (1-5) and (1-6) in (1-7) we get:

$$\rho \, dx \, dy \, dz = c \, dx \, dy \, dz \; V \tag{1-8}$$

Where: V = Voltage from differential volume of high resistance layer to high conductivity layer.

But dx dy dz represents the same continuous volume in both cases, so that Eq(1-8) becomes:

$$\rho = c V$$

or

$$\rho_{(x,y,z,t)} = c_{(x,y,z)} V_{(x,y,z,t)} \qquad (1\text{-}9)$$

Take the time derivative of both sides of Eq(1-9). Noting that only ρ and V are functions of time, we get:

$$\frac{\partial \rho}{\partial t} = c \frac{\partial V}{\partial t} \qquad (1\text{-}10)$$

but Eq(1-4) states:

$$\sigma \nabla \cdot E = -\frac{\partial \rho}{\partial t} \qquad (1\text{-}4)$$

Substitution of (1-10) in (1-4) yields:

$$\sigma \nabla \cdot E = -c \frac{\partial V}{\partial t} \qquad (1\text{-}11)$$

Let us consider the electric field intensity E. It can have a value only when there is a voltage gradient at the point considered.

$$E = -\nabla V^{1} \qquad (1\text{-}12)$$

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[1] $E = -\nabla V$, where V is a point function. It is obvious that V is a point function under the conditions outlined in this work. This may be easily demonstrated by the fact that the voltage between a point in the highly resistive layer and the high conductivity layer is the same regardless of the location of the volt meter leads.

In the model, it is proposed to connect a potential of several volts between the resistive layer and metal plate. A potential difference exists between the resistive layer and the high conductivity layer. Current will flow outward through the resistive layer from the probe contact, causing a definite voltage gradient. This gradient will be large compared to any gradients that may otherwise exist and we may substitute (1-12) for E in (1-11).

$$\sigma \nabla \cdot (-\nabla V) = - c \frac{\partial V}{\partial t}$$

or

$$\sigma \nabla \cdot \nabla V = c \frac{\partial V}{\partial t}$$

But $\nabla \cdot \nabla V$ is commonly written as $\nabla^2 V$,

therefore

$$\sigma \nabla^2 V = c \frac{\partial V}{\partial t}$$

or

$$\frac{\partial V}{\partial t} = \frac{\sigma}{c} \nabla^2 V$$

But note that when $\frac{\sigma}{c} \equiv \bar{\alpha}$ we have

$$\frac{\partial V}{\partial t} = \bar{\alpha} \nabla^2 V \tag{1-13}$$

which is identical to the heat flow equation in form.

$$\frac{\partial \phi}{\partial T} = \alpha \nabla^2 \phi \tag{1-14}$$

Where ϕ = temperature

T = time

$\alpha = \frac{k}{cd}$ = thermal diffusivity

k = thermal conductivity of material

d = density of material

c = specific heat of material

PART II

DERIVATION OF SCALE FACTORS

Given

$$\frac{\partial \phi}{\partial T} = \alpha \nabla^2 \phi \tag{2-1}$$

ϕ = Temperature in degrees Fahrenheit

T = Time in hours in heat system

α = Thermal diffusivity constant determined by earth under investigation

$$\frac{\partial V}{\partial t} = \bar{\alpha} \nabla^2 V \tag{2-2}$$

V = Electromotive force in volts

t = Time in seconds in electrical system

$\bar{\alpha}$ = Constant determined by model construction

Eq(2-1) and Eq(2-2) have the same form, so that a solution of one must be a solution to the second. Eq(2-1) is dimensionally consistent in degrees fahrenheit, hours, and feet, while Eq(2-2) is consistent in volts, seconds, and centimeters. Therefore one degree corresponds to one volt, one foot to one centimeter, and one hour to one second.

Since we wish to construct a scale model, let us introduce scale factors, p, a, and τ .

$$|\phi| \text{ degrees} = p|V| \text{ volts} \tag{2-3}$$

$$|X| \text{ feet} = a|\bar{X}| \text{ centimeters} \tag{2-4}$$

$$|T| \text{ hours} = \tau|t| \text{ seconds} \tag{2-5}$$

and

$$\alpha \frac{\text{square feet}}{\text{hours}} = f(\rho\ a\ r)\ \alpha\ \frac{\text{square centimeters}}{\text{seconds}} \qquad (2\text{-}6)$$

Consider $\frac{\partial \phi}{\partial T}$ and introduce the change of variable

$$\frac{\partial \phi}{\partial T} = \frac{\partial \phi}{\partial t}\frac{dt}{dT} \qquad (2\text{-}7)$$

But by Eq(2-5)

$$T = rt$$

so

$$\frac{\partial T}{\partial t} = r \qquad (2\text{-}8)$$

and

$$\frac{\partial \phi}{\partial T} = \frac{1}{r}\frac{\partial \phi}{\partial t} \qquad (2\text{-}9)$$

Similarly from Eq(2-3)

$$\phi = \rho V \qquad (2\text{-}10)$$

$$d\phi = \rho\, dV$$

Substitute Eq(2-10) in Eq(2-9) and Eq(2-9) becomes:

$$\frac{\partial \phi}{\partial T} = \frac{1}{r}\left(p\,\frac{\partial V}{\partial t}\right) \tag{2-11}$$

$$\frac{\partial \phi}{\partial T} = \frac{p}{r}\,\frac{\partial V}{\partial t} \tag{2-12}$$

Let us now consider $\frac{\partial^2 \phi}{\partial x^2}$ and introduce the change of variable

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial \bar{x}^2}\left(\frac{d\bar{x}}{dx}\right)^2 + \frac{\partial \phi}{\partial \bar{x}}\;\frac{d^2\bar{x}}{dx^2} \tag{2-13}$$

but $x = a\bar{x}$

$$\frac{\partial \bar{x}}{\partial x} = \frac{1}{a}$$

$$\frac{\partial^2 \bar{x}}{\partial x^2} = 0$$

Eq(2-13) becomes

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial \bar{x}^2}\left(\frac{1}{a}\right)^2 + 0$$

but

$$\phi = \rho V$$

Therefore Eq(2-13) becomes

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{p}{a^2}\frac{\partial^2 V}{\partial \bar{x}^2} \tag{2-14}$$

Similarly

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{p}{a^2}\frac{\partial^2 V}{\partial \bar{y}^2} \tag{2-15}$$

and

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{p}{a^2}\frac{\partial^2 V}{\partial \bar{z}^2} \tag{2-16}$$

Substituting Eq(2-12), Eq(2-14), Eq(2-15), and Eq(2-16) in Eq(2-1), we have the following:

$$\frac{p}{\tau}\frac{\partial^2 V}{\partial t} = \frac{p\alpha}{a^2}\left(\frac{\partial^2 V}{\partial \bar{x}^2} + \frac{\partial^2 V}{\partial \bar{y}^2} + \frac{\partial^2 V}{\partial \bar{z}^2}\right) \tag{2-17}$$

$$\frac{\partial V}{\partial t} = \frac{\tau\alpha}{a^2}\nabla^2 V \tag{2-18}$$

By similarity of Eq(2-18) and Eq(2-2) it is seen that

$$\frac{\tau\alpha}{a^2} = \bar{\alpha}$$

or

$$\tau = a^2 \frac{\bar{\alpha}}{\alpha} \tag{2-19}$$

Note: $\bar{\alpha}$ is fixed by construction of the model.

α is fixed by the nature of the heat conductor under investigation.

Thus one has three means of adjusting the equivalent size and equivalent time of operation of the model.

1. The scale factor (a) may be changed.
2. The time factor (τ) may be changed.
3. Both (a) and (τ) may be changed.

PART III

DERIVATION OF FACTOR RELATING CURRENT TO B.T.U.

Consider the volume:

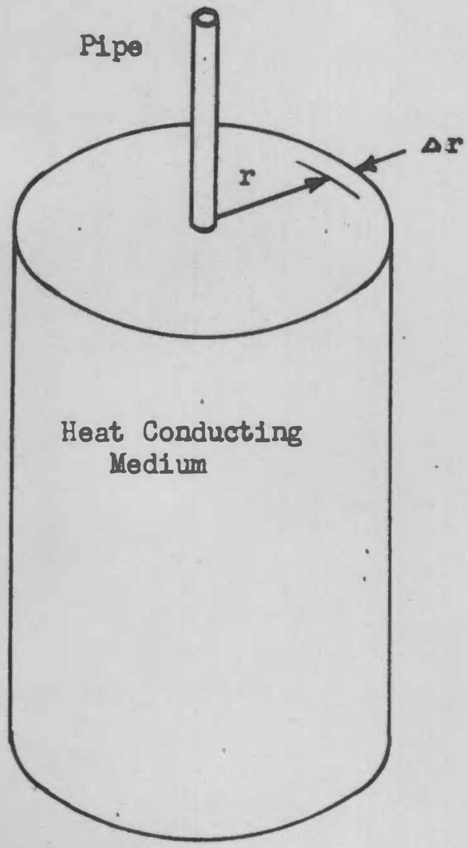

Fig. 2. Elemental Section Around Source

Definitions and Conventions:

k = Heat conductivity

σ = Electrical conductivity

$\Delta\phi$ = Temperature difference through Δr

ΔV = Voltage difference through $\Delta \bar{r}$

$r = a\bar{r}$ as before ($x = a\bar{x}$)

$\Delta\phi = p\,\Delta V$ as before

HEAT	ELECTRICAL
$\dfrac{Q}{T} = \dfrac{kA\Delta\phi}{\Delta r}$	$\dfrac{q}{t} = \dfrac{\sigma A \Delta V}{\Delta \bar{r}}$
$A = 2\pi r z$	$A = 2\pi \bar{r} \bar{z}$
Substitute for r and $\Delta\phi$	
$\dfrac{Q}{T} = \dfrac{k\, 2\pi\, a \bar{r} z\, \rho \Delta V}{a \Delta \bar{r}}$	$\dfrac{q}{t} = \dfrac{\sigma\, 2\pi \bar{r} \bar{z}\, \Delta V}{\Delta \bar{r}}$
$\dfrac{Q}{T z} = \dfrac{2\pi \bar{r} \Delta V\, k\rho}{\Delta \bar{r}}$	$\dfrac{q}{t \sigma \bar{z}} = \dfrac{2\pi \bar{r} \Delta V}{\Delta \bar{r}}$

Substituting for $\dfrac{2\pi \bar{r} \Delta V}{\Delta \bar{r}}$ we get

$$\frac{Q}{T z} = \frac{k\rho\, q}{t \sigma \bar{z}} = k\rho \left(\frac{1}{\sigma_0 \bar{z}}\right) \frac{q}{t}$$

but

$$\frac{Q}{T z} = \text{BTU/hr/ft. of pipe}$$

and

$$\frac{q}{t} = \text{Current in amperes}$$

while $\frac{1}{\sigma \frac{1}{2}} = \sigma_0$

σ_0 = Total electrical conductivity of the resistive layer of the model measured between a strip one centimeter wide and one centimeter long of any depth.

Therefore:

$$\text{BTU/hr/ft. pipe} = \frac{k\rho}{\sigma_0} I$$

I = Total current flowing into model pipe

Since the literature commonly presents this information in the form of the dimensionless group $\frac{Q}{k\Delta T}$ let us also express $\frac{Q}{k\Delta T}$ in current flow.

$$Q = \frac{k\rho}{\sigma_0} I$$

but

$$\frac{1}{\sigma_0} = R$$

Substitute for $\frac{1}{\sigma_0}$ and divide both sides by $k\Delta T$.

$$\frac{Q}{k\Delta T} = \frac{k\rho I R}{k\Delta T}$$

and

$$\frac{\rho}{\Delta T} = \frac{1}{\Delta V}$$

If the model is initially at zero voltage at $t = 0$

$$\Delta V = V$$

Therefore we get:

$$\frac{Q}{k\,\Delta T} = \frac{RI}{V} \qquad \text{Eq(3-1)}$$

PART IV

ELECTRICAL MODEL OF PIPE BURIED IN THE EARTH

Developed by A. R. Sisson[1]

The model condenser used in this work was constructed by using a 30" square polished brass plate as a base. A sheet of plastic approximately 0.002" thick was cemented to the brass plate as a dielectric. India ink proved to be the best resistive medium, and since the ink would not adhere to the plastic, a sheet of cellophane was cemented to the plastic to serve as a holder for the ink. The cellophane could not be used as the dielectric alone because its resistance was too low. The ink was spread by placing the plate on a whirling machine and pouring ink on the cellophane. The evaporator was simulated by a circular disc of aluminum cemented with India ink to the ink coating.

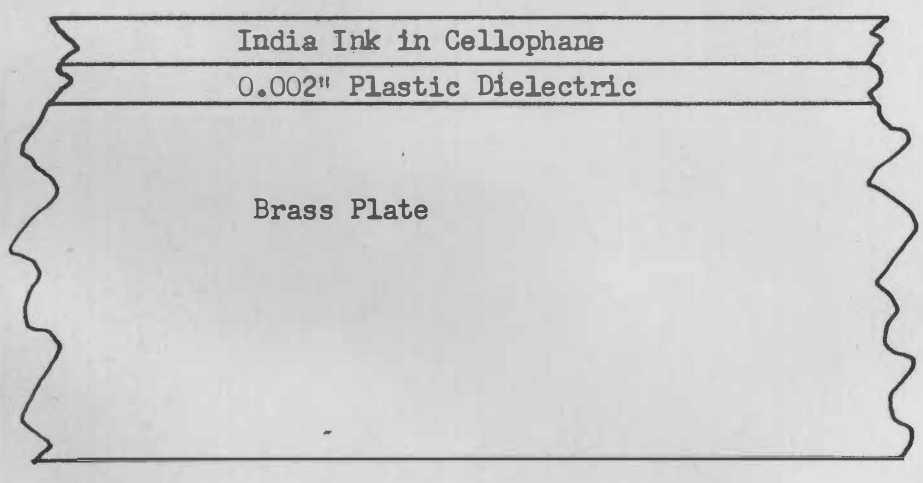

Fig. 3. Cross Section of Model

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[1]Sisson, op. cit.

It was desired to simulate a homogeneous earth so that several tests were conducted to determine that $\frac{1}{RC}$ was uniform.[2]

1. An alternating current voltage was applied to the aluminum disc in the center of the condenser. The voltage was measured at equal radius for points around the disc. These proved to be the same for equal radii.
2. It was feared that the condenser might possess circular symmetry due to process of applying ink, yet not be uniform. For this reason, a second test was made by applying a D. C. voltage between two edges of the model. The voltage drop for equal distance was measured and found to be equal.
3. In order to prove that inductance was negligible, the condenser was considered as a flat transmission line. An A. C. voltage was applied and phase shift between voltage and current was measured at X = 0 and found to be 45, which proved that inductance was negligible.
4. Voltage drop along the flat transmission line was measured. When plotted on semilog paper, this was a straight line and α was calculated from the slope of the plotted line and found to be 10,080 square centimeters per second.
5. R was determined from a relation between E & I at X = 0 and found to be 0.5×10^6 ohms.

While the various tests indicated that $\frac{1}{RC}$ was uniform throughout the high resistance layer, it has been found that the resistance was

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[2]Sisson, op. cit.

not uniform. It had the characteristics of a decrease in resistance with an increase in current as shown in Fig. 4. However, note that the change in resistance was small over the current range eight microamperes per inch to eighty microamperes per inch. In a typical case, the current flowing into the condenser after the initial peak was one hundred and sixty microamperes. With the five-eighths probe contact, the current was eighty microamperes per inch at the contact. The current decreased with an increase in distance from the probe contact, but one must move to a point at least six and one-half inches from the probe contact before the current drops to eight microamperes per inch. Phenomena this far from the probe contact had little or no effect on the current because of the short interval of operation. Therefore the nonlinear resistance introduced an experimental error of the type that was recognized and did not prohibit the use of the condenser. This effect was noted in the test data and the influence on test data is discussed later in this paper. The cause of this phenomenom was never determined.

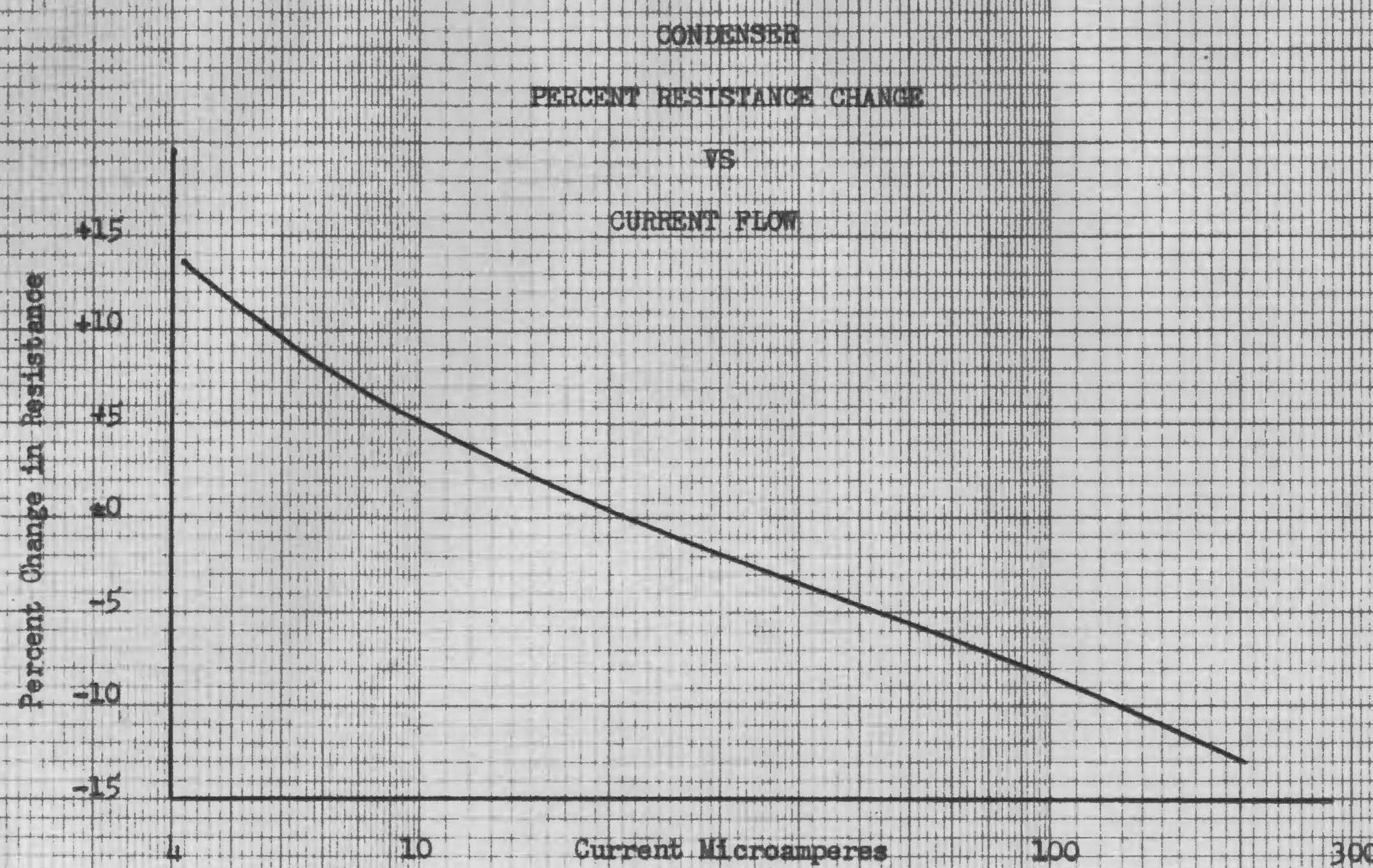

Fig. 4. Percent Resistance Change vs Current Flow In Condenser

PART V

GENERATOR

In the preceding work, it has been shown that a distributed resistance capacitance system is described by the same differential equations as a two dimensional heat system, and could therefore be used to solve certain heat flow problems, provided that use was made of corresponding electrical and heat flow quantities. It was shown in particular that voltage and temperature were corresponding quantities, so that any generator intended for use with an electrical model should be capable of generating a voltage with the same magnitude-time function as temperature in the heat system. Since the ultimate aim of this work was to present a method for solving heat pump problems by analogy, a generator was designed and constructed for that purpose. This paper however, will not present such data.[1] Let us therefore investigate the temperature variations of a pipe enclosed in a medium and used as a source of heat for the heat pump.

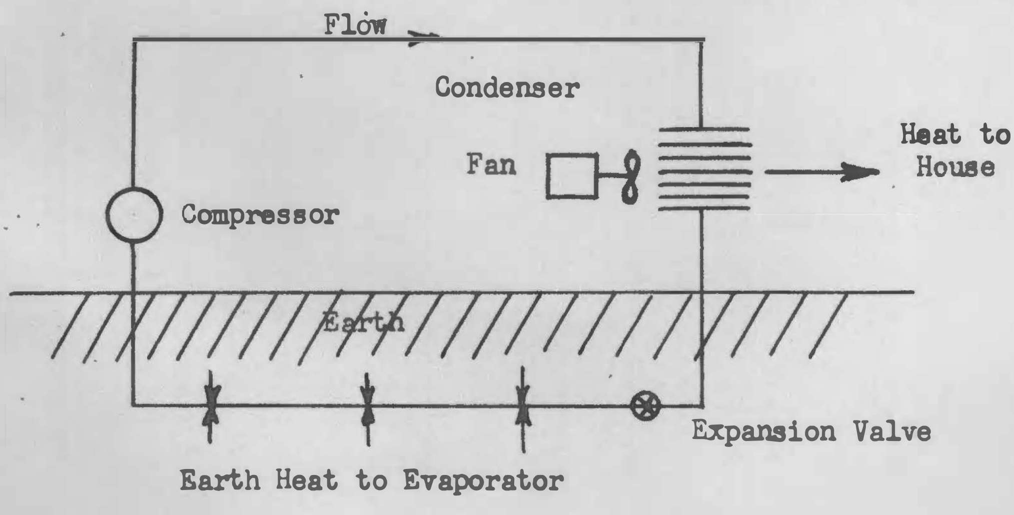

Fig. 5. Basic Heat Pump

[1] Sisson, op. cit.

Fig.5 shows a simplified schematic diagram for a direct expansion heat pump. When this system is operating as a source of heat, the compressor draws refrigerant gas from the evaporator and discharges it to the condensor at high enough pressure and temperature to condense and transfer its heat to the outgoing air stream. The liquid refrigerant having lost considerable heat to the air stream, is allowed to pass through the expansion valve into the evaporator, where the pressure and temperature are such as to allow the refrigerant to boil and extract from the heat source, its heat of vaporization. This of necessity requires that the refrigerant be at a lower temperature than the heat source. The gaseous refrigerant passes from evaporator into the compressor where the cycle is repeated.

When the heat pump is first set into operation, this circulation of refrigerant continues until enough heat has been extracted from the heat source to raise the temperature of the heated space to some preset level. The compressor is shut off; the refrigerant no longer circulates; no heat is extracted from the medium, and the evaporator temperature is allowed to approach that of the surrounding medium. As soon as more heat is required, the refrigerant is again circulated, sharply lowering evaporator temperature and extracting heat. This cycle continues for the whole of the heating season or operating interval, during which the evaporator is held at a very low temperature, or else approaching the medium temperature in a transient manner.

In order to facilitate experimental work and generator construction, it was decided to idealize the system to some extent. For

clearness, the operating conditions are enumerated as follows:

1. The refrigerant leaves the expansion valve at the same temperature T_2 for each and every <u>on</u> period.
2. The evaporator temperature drops sharply to temperature T_2 when refrigerant is circulated.
3. The <u>on</u> period is constant and the <u>off</u> period is constant over the full operating interval.

With the operating conditions as listed, one may readily sketch the temperature vs time of a particular foot of evaporator pipe.

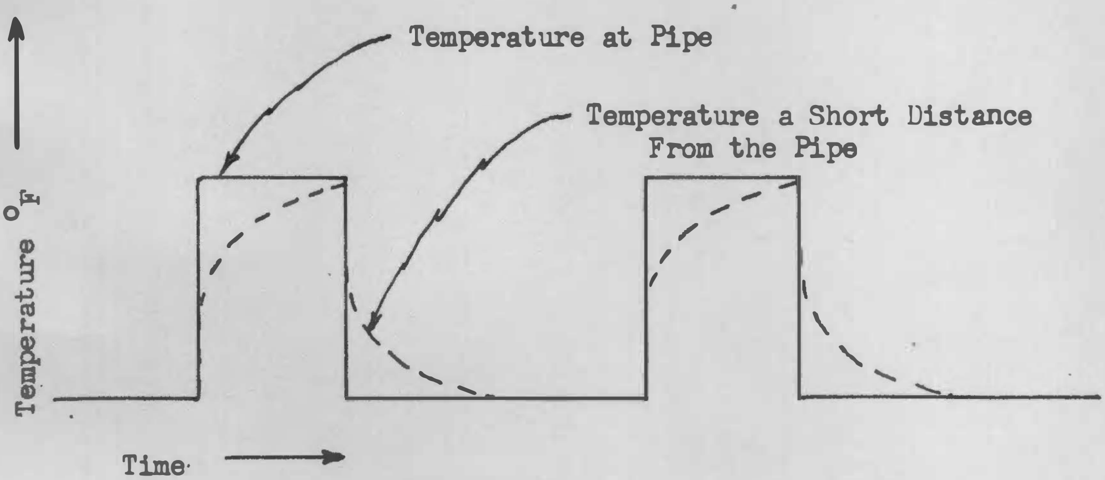

Fig. 6. Evaporator Temperature

This solid curve is the applied temperature variation while the dotted lines show the probable effect of transient change in the earth temperature on the evaporator temperature. The basic principle of the generator is therefore simple. It must generate a voltage whose time variation is the same as the solid curve of Fig.6 . There are however,

several complicating factors, as follows:

1. During the on period, it must act as a low impedance generator. This is necessary since the pipe is in direct contact with the earth and will act as a low impedance heat sink.
2. During the off cycle, it must act as an infinite impedance generator or else it will allow the model to discharge through the generator. This would not represent the off period in the heat pump during which the medium cannot regain the heat lost.
3. It was decided to view the various voltage and current patterns on a cathode ray oscilloscope. This necessitated the use of repeated cycles of the operating interval with the corresponding requirement that the model be discharged and restored to its original uncharged state after each operating interval.

This generator is realized through the use of a combination of multivibrators, whose outputs were either mixed or superimposed one upon the other to give the desired wave form, while a special output circuit was used to give the correct output impedance.

The operation of the generator begins with the tube V_1, which is a normally free running multivibrator[2], with a frequency of 10 to 15 cycles per second. Multivibrators, due to the nature of their operation, are inherently unstable, so that a line frequency synchronizing signal was introduced by connecting the cathodes of V_1 to 6.3 volts. This synchronization used in conjunction with the positive grid return, combines to form a very

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[2]F. E. Terman, "Radio Engineers Handbook", Page 512, McGraw-Hill Book Company, New York, London, 1943.

stable generator, eliminating jitter due to random cycling.[3] The square wave output from V_1 is differentiated by network A,[4] giving a synchronizing pulse with sharp rise time. This pulse is amplified by V_2b, and used to initiate the sweep of the oscilloscope, also to synchronize the external square wave generator so that it begins a positive pulse just as the sweep begins. The grid of V_2a is connected to the grid of V_1b, so that the same pulse is applied to both tubes. The output of the cathode follower V_2a is fed to tubes V_3 and V_5.

V_3 is a slave multivibrator that generates one square pulse of controllable length for each positive pulse applied to its grid. This square wave is also differentiated and amplified by V_4. However, the output section of V_4 is biased such that it eliminates the pulse formed by the leading edge of the square pulse. Since the length of the square wave generated by V_3 is controllable, one may control the time at which the pulse occurs. This gives a delayed synchronizing pulse that can be used to start the sweep at any time during the operating interval. Thus one may examine phenomena, after a period of operation, on a large scale.

V_5 is also a one shot multivibrator and is used to generate a gate pulse. This gating pulse is used to turn V_6 on and off at a rapid rate, by controlling the bias, much as a switch is turned on and off rapidly. The external square wave generator is cathode coupled to V_6. When the gating pulse is applied to V_6 grid, the bias is raised to the cut off

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[3]Terman, op. cit.

[4]F. E. Terman, "Radio Engineering", Page 599, McGraw-Hill Book Company, New York, London, 1947.

level so that the square waves are passed and amplified by V_6. Conversely, when the gating pulse is not present, tube V_6 bias is below cut off and the tube is inoperative.

In this manner, we achieve two important requirements. The frequency of the external square wave generator can be adjusted so that the length of each positive 1/2 cycle corresponds to the on-time in the heat pump system, while the length of the gating pulse is made equal to the operating interval. These on-pulses are amplified by the video amplifier V_7, and fed to the output system through the low impedance cathode follower V_8a. V_9 acts as a double clamp circuit and limiter so that positive pulses of the same amplitude only are fed to V_{10}a. The diode V_{10}a acts as a unilateral device. It offers a low impedance to the positive on-pulses yet offers a very high impedance to current flow in the opposite direction.[5] Note how effectively this simulates heat pump operation. During the on-period, the refrigerant is circulated and heat is extracted from the medium by lowering evaporator temperature the same as current flows into the model with each positive on-pulse. When the off-period arrives, the voltage at the plate of V_{10}a is at a positive value, due to charging of the model. This charge will flow outward from the electrical contact as current, similar to heat flow in the heat system. The voltage in model $V = \frac{Q}{C}$ will be raised slightly by each increment of charge left by each pulse, in the same way as temperature is displaced

[5]The output impedance of the model is approximately equal to the plate resistance of V_{10}a in series with the output impedance of V_8a. Tests show this to be below 1,000 ohms while the input impedance of the model is about 50,000 ohms to the initially applied pulse.

slightly each on-period as shown in Fig. 6. Due to this charge, the cathode of $V_{10}a$ will be positive, and the unilateral action of $V_{10}a$ prevents the model from discharging. Likewise, the heat medium cannot regain heat during off periods when no refrigerant is circulated.

V_8b is a discharge tube required as a direct result of condition 3, listed on Page 20. Many discharge circuits were considered, however the circuit shown was by far the most simple and satisfactory.

When the gating pulse is applied to V_6a, a second pulse of sufficient negative value to drive the grid below cut off is applied to the grid of V_8b. Since the tube is cut off it will be a very high impedance connected across the output and will not affect the operation. When the gating pulse stops, a positive pulse is applied to V_8b, raising the grid well above cut off. A low impedance[6] is now connected directly across the model allowing it to discharge and prepare for the next operating interval.

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[6] Plate resistance of 6SN7 with positive grid is 5000 ohms or lower.

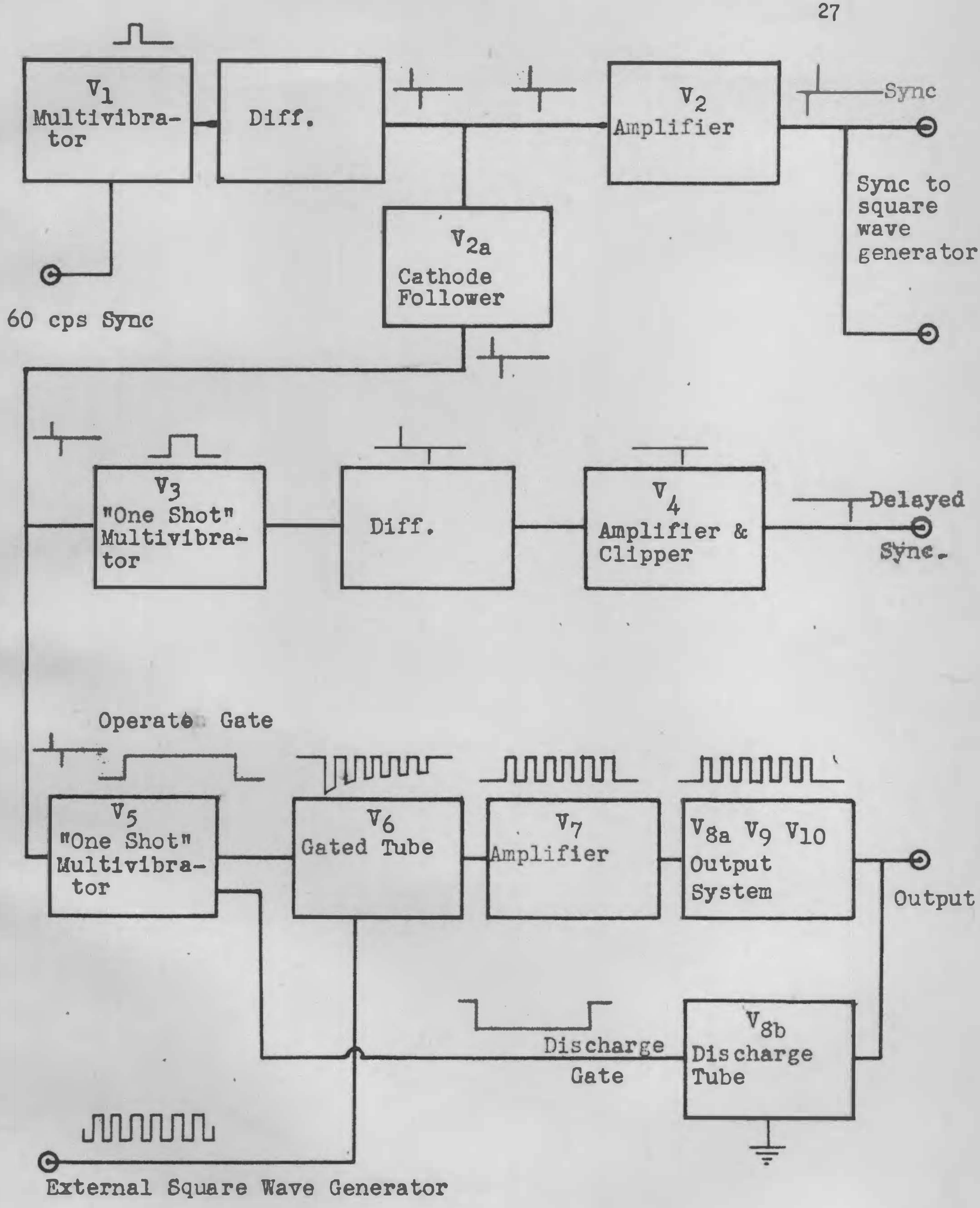

Fig. 7. Block Diagram of Generator

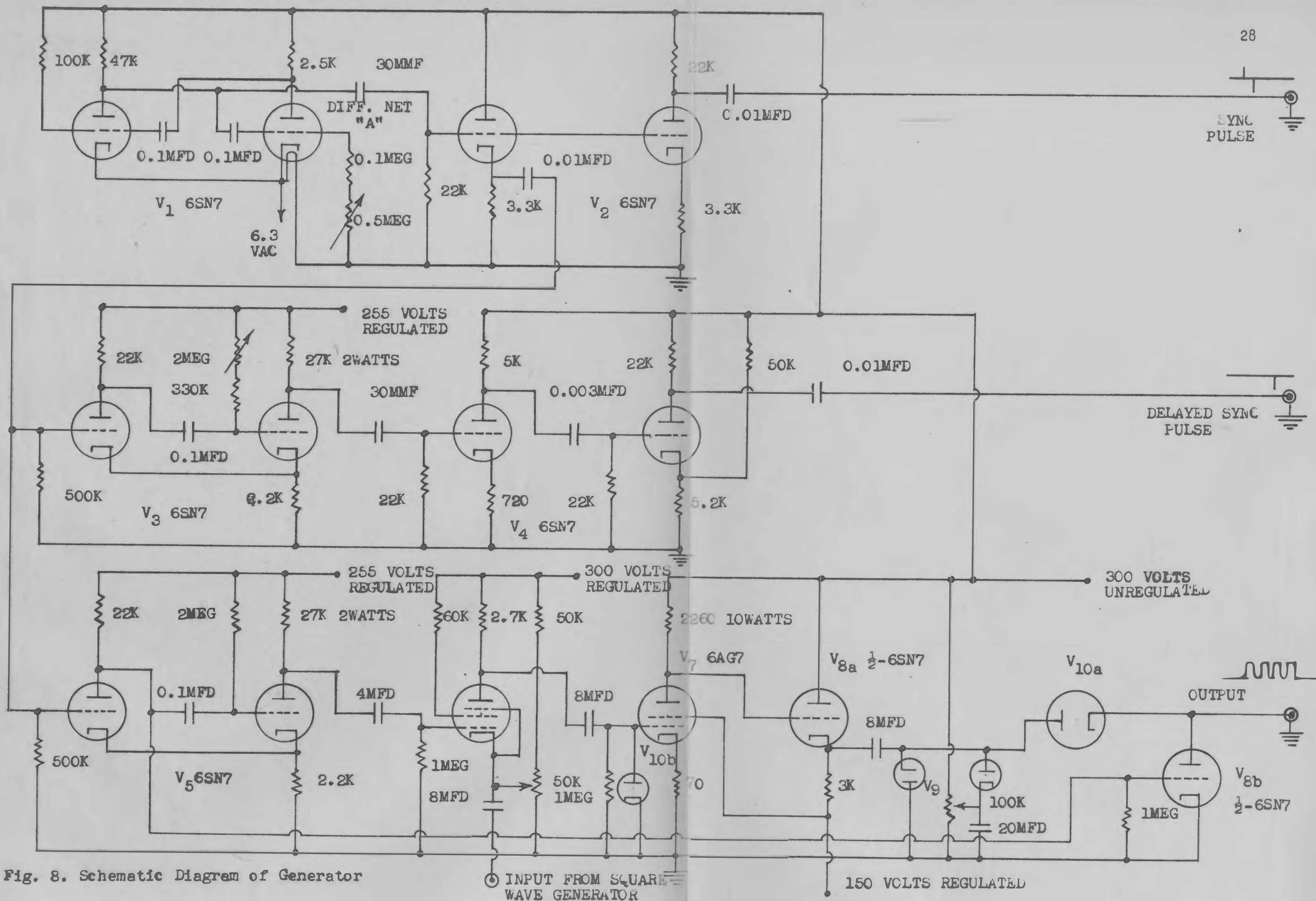

Fig. 8. Schematic Diagram of Generator

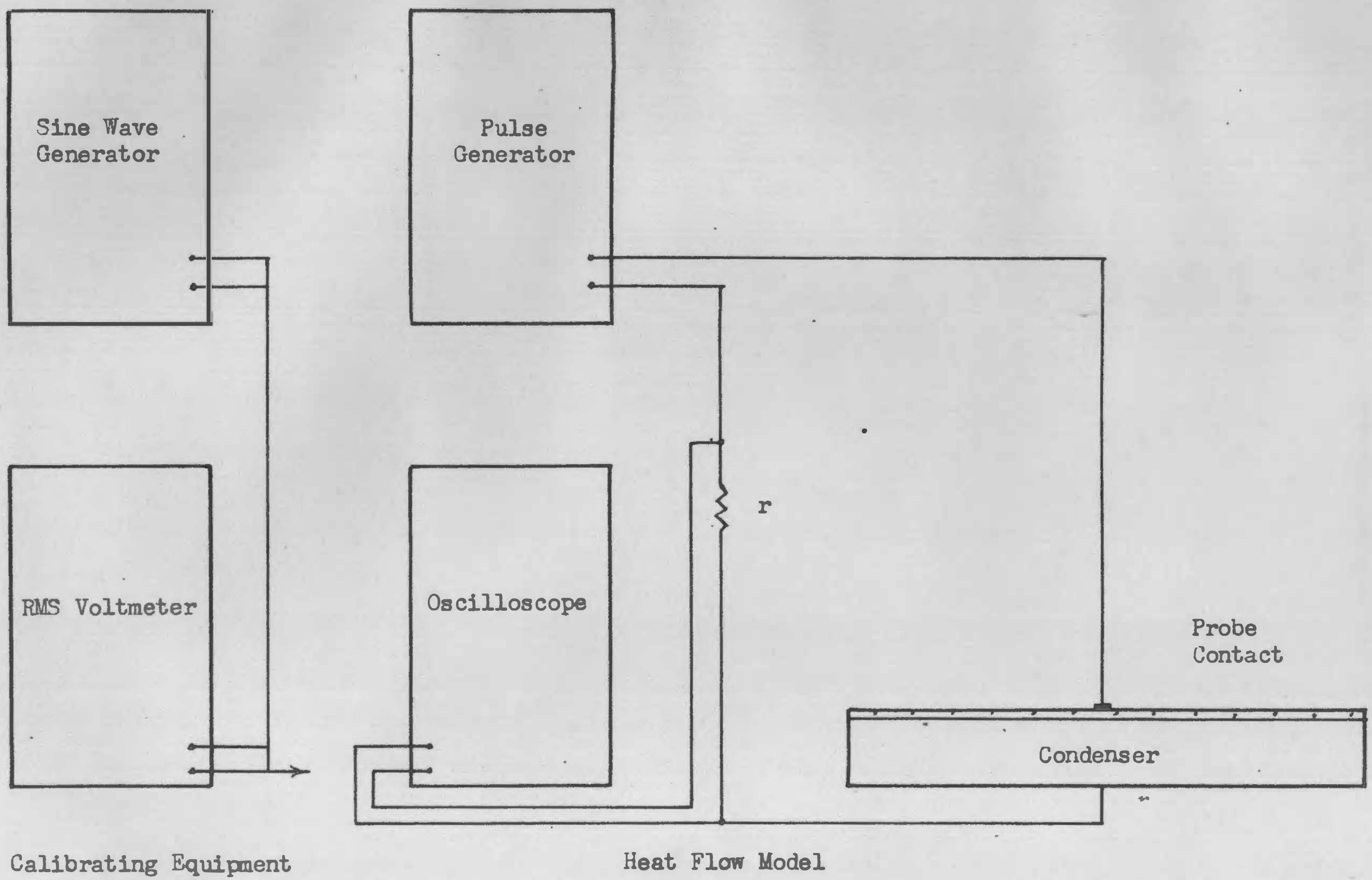

Fig. 9. System Wiring Diagram.

PART VI

EXPERIMENTAL PROCEDURE

Once a theory had been formulated, immediate work was started to determine a method of verifying it with experimental evidence. Consideration of analytically solved problems indicated that few works could be used for this purpose. The line source of Ingersoll and Plass could not be duplicated physically, while any work with an infinite flat plate would rapidly disintegrate to a one dimensional problem. Thus Gemant's work for an infinite pipe in an infinite medium appeared to be the most promising.

Gemant[1] has shown that the heat flow from an infinitely long cylinder in an infinite medium held at a constant temperature, is given by the following equation:

$$\frac{Q}{k\Delta T} = 4\int_0^\infty \epsilon^{-\alpha\beta^2 t} \frac{J_1(\beta r)\, N_0(\beta r_p) - J_0(\beta r_p)\, N_1(\beta r)}{J_0^2(\beta r_p) + N_0^2(\beta r_p)}\, d\beta \qquad (6\text{-}1)$$

where

ΔT	=	Temperature
Q	=	Heat flow
k	=	Thermal conductivity
J_o, N_o	=	Bessel function of 1st and 2nd kind order 0
J_1, N_1	=	Bessel function of 1st and 2nd kind order 1
α	=	Thermal diffusivity of medium
r	=	Distance from center line of cylinder
r_p	=	Radius of cylinder

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[1] Gemant, op. cit.

t = Time

β = A variable of integration

Gemant's graph of $\frac{Q}{k\Delta T}$ vs. $\frac{\alpha T}{r_p^2}$ prepared by numerical integration of Eq(6-1) is represented by the solid line in Fig. 12. A model of the cylindrical source can be easily constructed and current flow can be measured and converted to heat flow.

Previous experimenters have found the earth to possess considerable heat capacity and resistance to heat flow. These combine to cause a relatively small section of medium to act as an infinite earth for thousands of hours. Thus a small model such as previously described can be used to simulate an infinite medium for many hours, while the method in which $\sigma \bar{z}$ was defined as σ_0 in all derivations produces a model that extends to $\pm \infty$ in the z plane. After careful consideration of these facts, it was decided to attempt to check Gemant's work as experimental verification of the theory set forth in this paper.

The model condenser, generator, square wave generator, and oscilloscope, were assembled and connected as shown by Fig. 9, and operated as follows:

The external square wave generator was disconnected and the bias of V_6 adjusted, so that the gating pulse was amplified and appeared as the output pulse. It had a duration of 0.066 second and simulated continuous operation, for that interval in the electrical system, which was converted to an equivalent value of $\frac{\alpha T}{r_p^2}$ in heat systems.

The gain of the oscilloscope was adjusted so that the voltage wave pattern could be viewed on the screen. A camera was placed on a stand in front of the screen and the film exposed once. The signal from

the model was disconnected and a sine wave generator was connected to the oscilloscope. Care was exercised not to change the gain setting on the oscilloscope. The film was again exposed. The voltage magnitude and frequency of the sine wave generator were then recorded. The sine wave generator was disconnected and a straight line appeared on the screen. The film was exposed for a third time to give a base line for the picture. By this procedure, the time-magnitude function of the current flow into the model was recorded along with a sine wave calibrating voltage and time scale.

Values of vertical deflection d_V and horizontal deflection d_H were taken from Fig. 10 and Fig. 11, and tabulated in Table I and Table II respectively. These were converted to values of $\frac{\alpha t}{r_p^2}$ and $\frac{Q}{k \Delta T}$ and plotted with the theorectical work of Gemant. This is shown in Fig. 12.

Data:

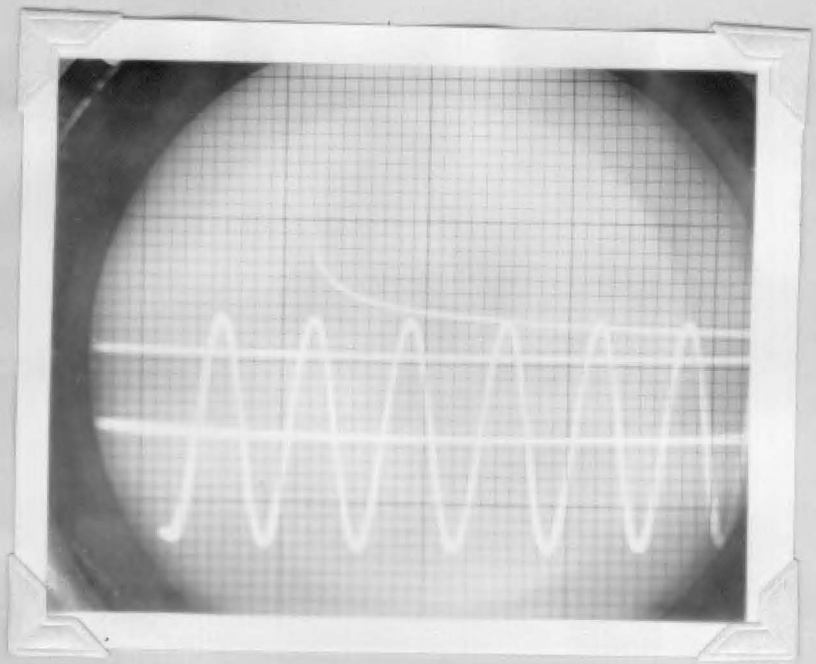

Fig. 10. $\frac{Q}{k\Delta T}$ vs $\frac{\alpha \dot{T}}{r_p^2}$ Taken from Oscilloscope with a High Speed Sweep.

Sine Wave Voltage = 0.035 RMS Volts

Sine Wave Frequency = 2370 Cycles per Second

TABLE I

VALUES OF $\frac{Q}{k\,\Delta T}$ AND $\frac{\alpha T}{r_p^2}$

TAKEN FROM FIG. 10

Vertical Deflection Inches	$\frac{Q}{k\Delta T}$	Horizontal Deflection Inches	$\frac{\alpha T}{r_p^2}$
1.175	3.29	0.2	8.36
0.95	2.77	0.5	20.9
0.90	2.52	1.0	41.8
0.80	2.24	2.0	83.6

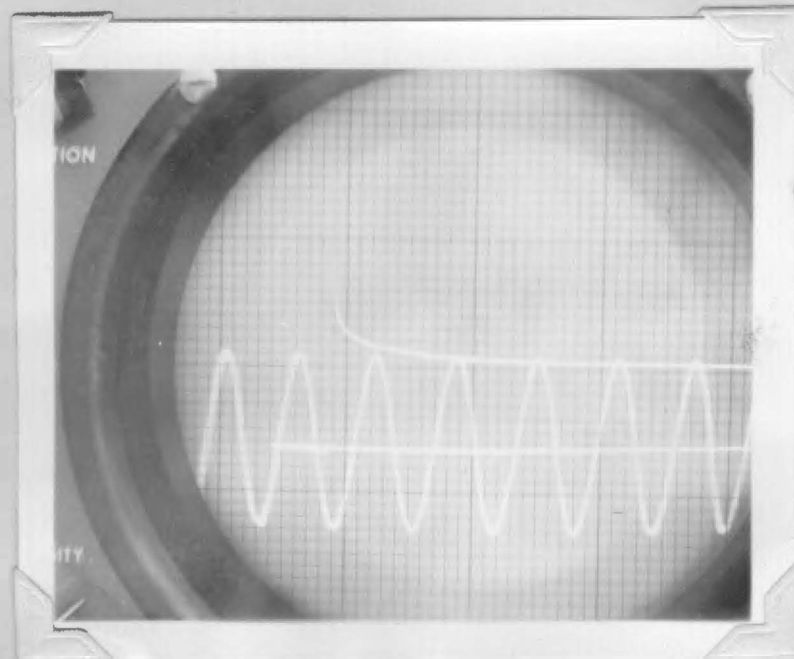

Fig. 11. $\frac{Q}{K\Delta T}$ vs $\frac{\alpha T}{r_p^2}$ Taken from Oscilloscope With a Low Speed Sweep.

Sine Wave Voltage = 0.11 RMS Volts

Sine Wave Frequency = 550 Cycles Per Second

TABLE II

VALUES OF $\frac{Q}{k\Delta T}$ AND $\frac{\alpha T}{r_p^2}$

TAKEN FROM FIG. 11

Vertical Deflection Inches	$\frac{Q}{k\Delta T}$	Horizontal Deflection Inches	$\frac{\alpha T}{r_p^2}$
1.075	2.55	0.1	29.6
0.95	2.25	0.2	59.2
0.84	1.99	0.5	148
0.77	1.82	1.0	296
0.71	1.68	2.0	592
0.68	1.61	2.5	740

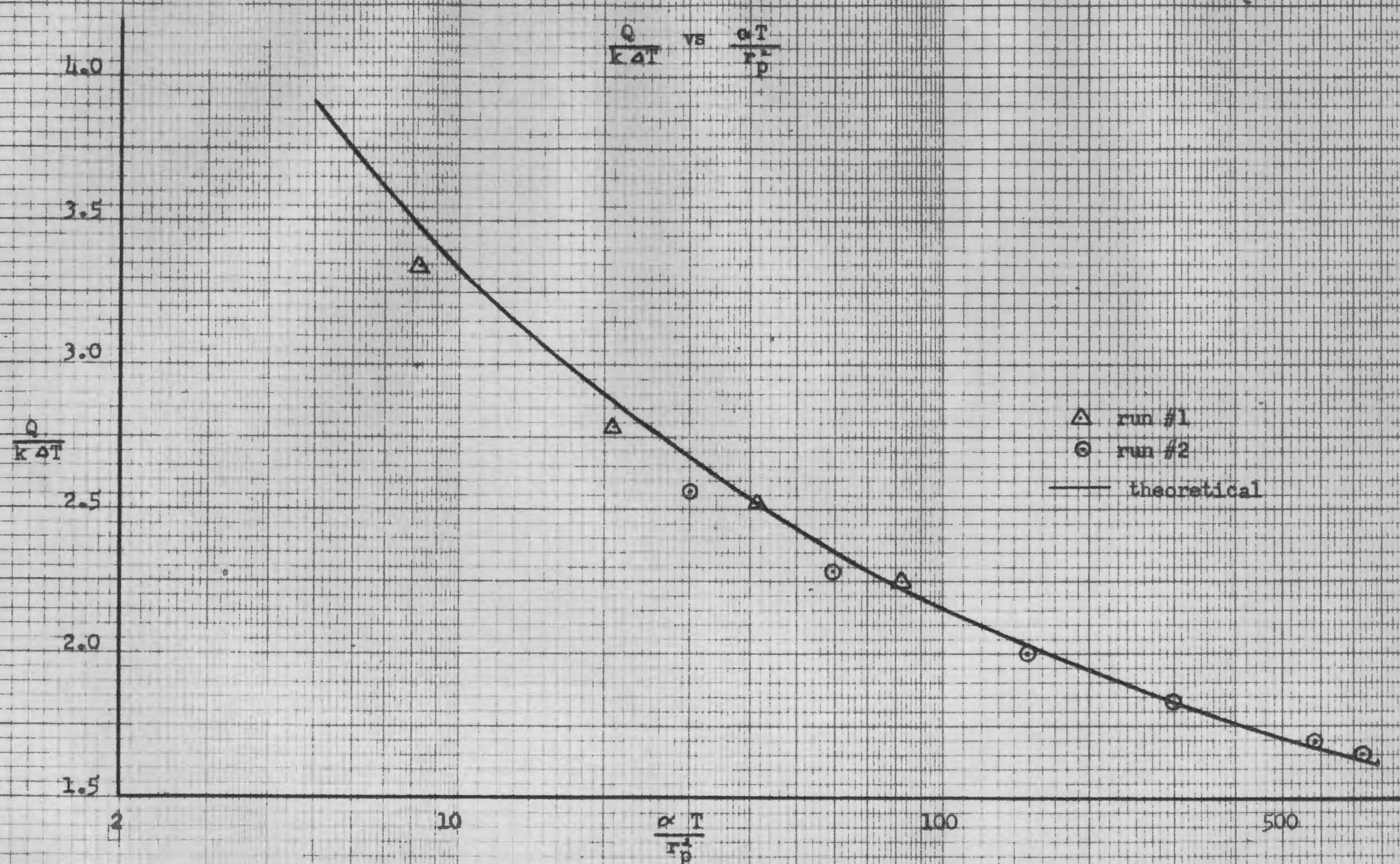

Fig. 12. $\frac{Q}{k \Delta T}$ vs $\frac{\alpha T}{r_p^2}$ For Theoretical And Experimental Data

Gemant's graph of $\frac{Q}{k\Delta T}$ vs. $\frac{\alpha T}{r_p^2}$ prepared by numerical integration of Eq(6-1) is represented by the solid line in Fig. 12, and gives the heat flow from an infinitely long cylinder in an infinite medium as a function of time.

In the tests presented in Fig. 10 and Fig. 11, it was attempted to simulate these conditions in the model. Fig. 10 and Fig. 11 were used to obtain the data in Table I and Table II and these data were plotted on Fig. 12 and are indicated by the test points on Fig. 12. The test data agrees with Gemant's data within an experimental error of 4.5% maximum and generally the error is less than 3%. There is, however, one important point that should be noted.

The condenser non-linear resistance is involved in the factor converting current to heat flow as follows:

$$\frac{Q}{k\Delta T} = \frac{RI}{V} \qquad \text{Eq(3-1)}$$

This causes the experimental points to be less than theoretical values for low values of $\frac{\alpha T}{r_p^2}$ and to be greater as $\frac{\alpha T}{r_p^2}$ becomes larger and current decreases causing the resistance to increase. This effect is quite noticeable and is consistent with known facts.

It will also be noted that data from Fig. 10 and Fig. 11 overlap. These data should exactly match at the points where they overlap, but they do not match, and indicate an error of 2.4% between the two tests.

Since two calibrating instruments and four readings are involved

in every point located on the curve, it is expected that this error was caused by inability to read the instruments more closely.

PART VII

CONCLUSION

A method for solving two dimensional heat flow problems has been presented, and the theory involved was derived by analatical means. Gemant's work with an infinitely long cylinder in an infinite medium was checked experimentally, using a model constructed by Sisson. The data are presented in Fig. 12, and show very good agreement between experimental data and Gemant's data.

Using the distributed resistance capacitance model heat pump data such as pipe size, pipe spacing, and shape factor can be determined accurately enough for normal engineering practice. These data are of extreme importance in the design of heat pumps, and the lack of data has been a large factor in retarding growth of heat pump installations. Much work is required along this line, and it is believed that this model presents a practical method for obtaining these data.

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APPENDIX

APPENDIX

SAMPLE CALCULATIONS

Given:[1] $R = 500{,}000$ ohms

$\bar{\alpha} = 10{,}080$

A Solve for $\frac{Q}{k\Delta T}$ for slow speed sweep

Data: Sine wave voltage = 0.11

Sine wave frequency = 550 CPS

$$\frac{Q}{k\Delta T} = \frac{RI}{V} \qquad \text{Eq(3-1)}$$

but

$$I = \frac{\text{Deflection voltage (E)}}{r}$$

and

$$E = \frac{\text{Inches Deflection (Peak to Peak Sine Wave Voltage)}}{\text{Peak to Peak Deflection of Sine Wave}}$$

Therefore substituting

$$\frac{Q}{k\Delta T} = \frac{500{,}000}{44.5}\left(\frac{dv}{1.38}\right)\frac{(0.11(2)(\ \ 2)}{1070}$$

$$\frac{Q}{k\Delta T} = 2.37\ dv$$

B Solve for $\frac{\alpha T}{r_p^2}$ for slow speed sweep

Data: $\bar{\alpha} = 10{,}080$

$f = 550$ CPS

$r_p = 5/16"$

$$\frac{\alpha T}{r_p^2} = \frac{\bar{\alpha}\, t}{\bar{r}_p^2}$$

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[1]Sisson, *op. cit.*

$$\frac{\alpha T}{r_p^2} = \frac{10,080}{(5/16)^2}\, t$$

but

$$\frac{\alpha t}{r_p^2} = \frac{\text{Deflection along Axis}}{\text{Length of Cycle}} \left(\frac{1}{f} \right)$$

therefore

$$\frac{\alpha t}{r_p^2} = \frac{10,080}{(5/16)^2} \quad \frac{(d_H \quad)}{0.633} \quad \frac{1}{550}$$

$$\frac{\alpha t}{r_p^2} = 296\, d_H$$