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Some Aspects of Function Theory for Dirichlet-type Spaces

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To the Graduate Council:

I am submitting herewith a dissertation written by Shuaibing Luo entitled "Some Aspects of Function Theory for Dirichlet-type Spaces." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Some Aspects of Function Theory for Dirichlet-type Spaces

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To my family.

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Abstract

Let μ[mu] be a nonnegative Borel measure on the boundary $\mathbb{T}$[unit circle] of the unit disc and define φ_μ[phi mu] to be the harmonic function

$$\varphi_\mu(z) = \int \text{[integral]}_{\mathbb{T}} \frac{1-|z|^2\text{[square]}}{|\zeta\text{[zeta]}-z|^2} d\mu(\zeta).$$

The harmonically weighted Dirichlet space $D(\mu)$ is defined as the space of all analytic functions on the unit disc $\mathbb{D}$[unit disc] such that

$$\int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$$

is finite. When μ is the Lebesgue measure on $\mathbb{T}$, then $D(\mu)$ is the Dirichlet space D.

The harmonically weighted Dirichlet spaces were introduced by Richter in [50] as he was studying analytic two-isometries. These spaces have been studied extensively throughout the years, see e.g. [3], [21], [22], [23], [24], [52], [53], [62], [63], [64], [66] and [67].

The weak product of D denoted by $D \odot D$[DdotD] is the following set:

$$D \odot D = \{h \in \text{[in]}\mathrm{Hol}(\mathbb{D}) : h = \sum \text{[sum]} f_i g_i,$$
$$\sum \|f_i\|\|g_i\| < \infty\text{[infinity]}, f_i, g_i \in D\}.$$

The dual of $D \odot D$ has been characterized in 2010 by Arcozzi, Rochberg, Sawyer and Wick [9] as the space $\mathcal{X}$[script X](D) of analytic functions b on $\mathbb{D}$ such that $|b'|^2 dA$ is a Carleson measure for the Dirichlet space.

In this dissertation we show that for functions f in proper weak*-closed M_z[M sub z]*-invariant subspaces of $\mathcal{X}(D)$, the functions $(zf)'$ are in the Nevanlinna class of $\mathbb{D}$ and have meromorphic pseudocontinuations in the Nevanlinna class of the exterior disc. We then use this result to show that every nonzero M_z-invariant subspace $\mathcal{N}$[script N] of $D \odot D$ has index 1, i.e. satisfies $\dim \mathcal{N}/z\mathcal{N} = 1$.

In the second part of this dissertation, we study the corona theorem for the $D(\mu)$ spaces when μ is a finitely atomic measure. If μ is a finitely atomic measure, we use the observation from Richter and Sundberg [52] that $M(D(\mu)) = D(\mu) \cap$ [intersection]$H^\infty(\mathbb{D})$ to show that the set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D}$ is dense in the maximal ideal space of $M(D(\mu))$. Furthermore, we obtain the corona theorem for infinitely many functions in $M(D(\mu))$.

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Chapter 1

Introduction

In this introduction we will establish the definitions and background results that are necessary for our main theorems. In Section 1.1 we will talk about weak product spaces, how they arose and have been used in the literature and why we are interested in them at this time. Our new main theorems about weak product spaces will be presented in Chapter 2. In Section 1.3 we will introduce another main topic of this dissertation, namely maximal ideal spaces and the corona theorem. In Chapter 3 we will prove our generalization of the corona theorem to some of the harmonically weighted Dirichlet spaces.

1.1 Weak products of Hilbert spaces

Let $d \geq 1, \Omega \subseteq \mathbb{C}^d$ be a nonempty set, and let $\mathcal{H}$ be a Hilbert space on Ω. Suppose that for each point $z \in \Omega$, evaluation at z is continuous. Therefore there is an element $k_z \in \mathcal{H}$ with the property

$$\langle f, k_z \rangle_{\mathcal{H}} = f(z), \quad \forall f \in \mathcal{H}.$$

The vector k_z is called the reproducing kernel at z, and $\mathcal{H}$ is called a reproducing kernel Hilbert space (see [11] for a detailed discussion of reproducing kernel Hilbert space). Suppose $\{f_i\}_{i\in\mathcal{I}}$ is an orthonormal basis for $\mathcal{H}$, then by Parseval's identity, we have

$$k_\lambda(z) = \sum_{i\in\mathcal{I}} \overline{f_i(\lambda)} f_i(z),$$

and $k_\lambda(z) = \langle k_\lambda, k_z\rangle_{\mathcal{H}} = \overline{k_z(\lambda)}$, i.e. k is symmetric.

Considered as a function on $\Omega \times \Omega$, the kernel function k_z is positive definite, i.e. for any finite set $\{z_1, \cdots, z_n\} \subseteq \Omega$ and any complex numbers $\{a_1, \cdots, a_n\}$, we have

$$\begin{aligned}\sum_{i,j=1}^{n} a_i \overline{a_j} k_{z_i}(z_j) &= \langle \sum_{i}^{n} a_i k_{z_i}, \sum_{j}^{n} a_j k_{z_j} \rangle_{\mathcal{H}} \\ &= \| \sum_{i}^{n} a_i k_{z_i} \|_{\mathcal{H}}^2 \geq 0.\end{aligned}$$

Conversely we can start with a kernel and construct a Hilbert space that has the given kernel as its reproducing kernel.

Theorem 1.1.1 (Moore [46])**.** *Suppose k is a symmetric, positive definite kernel on a set Ω. Then there is a unique Hilbert space of functions on Ω for which k is a reproducing kernel.*

Let $d > 1, \Omega \subseteq \mathbb{C}^d$ be an open, connected, and nonempty set, and let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space with kernel k_z, the weak product of $\mathcal{H}$ denoted by $\mathcal{H} \odot \mathcal{H}$ is defined as

$$\mathcal{H} \odot \mathcal{H} = \{h \in \mathrm{Hol}(\Omega) : h = \sum_{i=1}^{\infty} f_i g_i, \sum_{i=1}^{\infty} \|f_i\|_{\mathcal{H}} \|g_i\|_{\mathcal{H}} < \infty, f_i, g_i \in \mathcal{H}\}.$$

If $h \in \mathcal{H} \odot \mathcal{H}$, the norm of h in $\mathcal{H} \odot \mathcal{H}$ is defined to be

$$\|h\|_* = \inf\{\sum_{i=1}^{\infty} \|f_i\|_{\mathcal{H}}\|g_i\|_{\mathcal{H}} : h = \sum_{i=1}^{\infty} f_i g_i, f_i, g_i \in \mathcal{H}\}. \tag{1.1.1}$$

Weak products first appeared in the work of Coifman, Rochberg and Weiss [25]. They arise in connection with the Hankel operators and they are used to establish the duality between $L^1_a(\mathbb{D})$ and the Bloch space and the duality of $H^1(\partial\mathbb{B}_d)$ and BMO. Later Ferguson and Lacey [31], Lacey and Terwilleger [37] studied the weak product of $H^2(\mathbb{D}^d)$, they showed that a Hankel form with symbol b is bounded on $H^2(\mathbb{D}^d)$ if and only if b is in $\mathrm{BMO}(\mathbb{D}^d)$.

Let D be the Dirichlet space of holomorphic functions f for which

$$\int_{\mathbb{D}} |f'|^2 \frac{dA}{\pi} < \infty,$$

and let D_h be the harmonic Dirichlet space, which consists of functions of the form $f + \overline{g}$ for $f, g \in D$.

In 2010, Arcozzi, Rochberg, Sawyer and Wick [9] characterized the dual space of the space of weak products of Dirichlet functions $D \odot D$. their result also involves the boundedness of the Hankel operators and it implies the characterization of the dual of $D_h \odot D_h$. The characterization of $D_h \odot D_h$ also follows from results of Maz'ya and Verbitsky [44]. This was observed by Richter and Sundberg [55], and they also studied the space of weak products systematically:

note that from (1.1.1) we have

$$|h(z)| \le \|k_z\|^2_{\mathcal{H}} \|h\|_*,$$

and it was shown in [55] that $\mathcal{H}\odot\mathcal{H}$ is a Banach space of analytic functions, thus $\mathcal{H}\odot\mathcal{H}$ is a Banach space of analytic functions such that point evaluations are bounded.

If $\mathrm{Hol}(\Omega^-)$ is densely contained in $\mathcal{H}$, define

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \exists C > 0, |\langle \varphi\psi, b\rangle| \leq C\|\varphi\|\|\psi\|, \forall \varphi, \psi \in \mathrm{Hol}(\Omega^-)\}.$$

If $b \in \mathcal{X}(\mathcal{H})$ write $\|b\|_{\mathcal{X}}$ for the infimum of all $C > 0$ such that $|\langle \varphi\psi, b\rangle| \leq C\|\varphi\|\|\psi\|$ for all $\varphi, \psi \in \mathrm{Hol}(\Omega^-)$. It is clear that $\mathcal{X}(\mathcal{H})$ is a Banach space (see [55]).

For a vector subspace $\mathcal{L} \subseteq \mathcal{H}$ let

$$\mathcal{L}\widehat{\odot}\mathcal{L} = \{\sum_{i=1}^{n} f_i g_i : f_1, \cdots, f_n, g_1, \cdots, g_n \in \mathcal{L}, n \in \mathbb{N}\}$$

be the set of finite sums of products of elements in $\mathcal{L}$ and define a norm on $\mathcal{L}\widehat{\odot}\mathcal{L}$ by

$$\|h\|_{\bullet\mathcal{L}} = \inf\{\sum_{i=1}^{n} \|f_i\|_{\mathcal{H}}\|g_i\|_{\mathcal{H}} : h = \sum_{i=1}^{n} f_i g_i, f_i, g_i \in \mathcal{L}\}.$$

Under mild assumptions on $\mathcal{H}$, the space $\mathcal{H} \odot \mathcal{H}$ always has $\mathcal{X}(\mathcal{H})$ as a dual space, and indeed we have the following theorem (see [55]):

Theorem 1.1.2 (Richter, Sundberg)**.** *Let $Hol(\Omega^-)$ be dense in $\mathcal{H}$, and suppose there is a linear subspace $\mathcal{L} \subseteq Hol(\Omega^-)$ which is dense in $\mathcal{H}$ and which satisfies $\|\varphi\|_* = \|\varphi\|_{\bullet\mathcal{L}}$ for all $\varphi \in \mathcal{L}\widehat{\odot}\mathcal{L}$. Then $(\mathcal{H} \odot \mathcal{H})^* = \mathcal{X}(\mathcal{H})$. This means if for $b \in \mathcal{X}(\mathcal{H})$, we define L_b on $\mathcal{H}$ by*

$$L_b(h) = \langle h, b\rangle_{\mathcal{H}}$$

then L_b extends to be bounded on $\mathcal{H} \odot \mathcal{H}$, and the map $b \to L_b$ is a conjugate linear isometric isomorphism of $\mathcal{X}(\mathcal{H})$ onto $(\mathcal{H} \odot \mathcal{H})^*$.

We can also embed $\mathcal{H} \odot \mathcal{H}$ into some Hilbert space.

Theorem 1.1.3 (Richter, Sundberg)**.** *If $\mathcal{H} = \mathcal{H}(k) \subseteq Hol(\Omega)$ has reproducing kernel k, then*

$$\mathcal{H} \odot \mathcal{H} \subseteq \mathcal{H}(k^2),$$

with $\|h\|_{\mathcal{H}(k^2)} \leq \|h\|_*$ for all $h \in \mathcal{H} \odot \mathcal{H}$, where $\mathcal{H}(k^2)$ is the Hilbert space with the reproducing kernel k^2.

In Chapter 2, we consider weak products of different spaces, and we also have the following theorem:

Theorem 1.1.4. *If $\mathcal{H} = \mathcal{H}(k^{\mathcal{H}}) \subseteq Hol(\Omega)$, $\mathcal{L} = \mathcal{L}(k^{\mathcal{L}}) \subseteq Hol(\Omega)$ have reproducing kernel $k^{\mathcal{H}}$ and $k^{\mathcal{L}}$ respectively, then*

$$\mathcal{H} \odot \mathcal{L} \subseteq \mathcal{H}(k^{\mathcal{H}} \cdot k^{\mathcal{L}}),$$

with $\|h\|_{\mathcal{H}(k^{\mathcal{H}} \cdot k^{\mathcal{L}})} \leq \|h\|_*$ for all $h \in \mathcal{H} \odot \mathcal{L}$, where $\mathcal{H}(k^{\mathcal{H}} \cdot k^{\mathcal{L}})$ is the Hilbert space with the reproducing kernel $k^{\mathcal{H}} \cdot k^{\mathcal{L}}$.

Let $\mathcal{H}$ satisfy the conditions in Theorem 1.1.2. Then for each $b \in \mathcal{X}(\mathcal{H})$, the rule $H_b(\varphi, \psi) = \langle \varphi\psi, b\rangle_H$, $\varphi, \psi \in \text{Hol}(\Omega^-)$ extends to be a bounded bilinear form on $\mathcal{H} \oplus \mathcal{H}$. This bilinear form defines a bounded operator H_b, the Hankel operator with symbol b. More precisely, if we let $\overline{\mathcal{H}} = \{\overline{f} : f \in \mathcal{H}\}$, $\|\overline{f}\|_{\overline{\mathcal{H}}} = \|f\|_{\mathcal{H}}$, then for $b \in \mathcal{X}(\mathcal{H})$, we have $H_b : \mathcal{H} \to \overline{\mathcal{H}}$, $\langle H_b\varphi, \overline{\psi}\rangle_{\overline{\mathcal{H}}} = H_b(\varphi, \psi) = \langle \varphi\psi, b\rangle_H$ for all $\varphi, \psi \in \text{Hol}(\Omega^-)$.

If $f \in \mathcal{H}$, we let $[f]$ be the smallest invariant subspace of the operator of multiplication by z. A function $f \in \mathcal{H}$ is called a cyclic vector if $[f] = \mathcal{B}$.

For the Dirichlet space D, we show in Chapter 2 that every nonzero M_z-invariant subspace $\mathcal{M}$ of D is the kernel of a Hankel operator, and then we use this fact to prove that if $f \in D$, then f is cyclic in D if and only if f is cyclic in $D \odot D$. We will give a few motivations in the next section.

1.2 M_z-Invariant Subspaces

Let $\Omega \subseteq \mathbb{C}$ be nonempty, open and connected. Let $\mathcal{B}$ ba a Banach space of analytic functions on Ω such that point evaluations are continuous, and $\mathcal{B}$ is M_z-invariant, i.e. for every $f \in \mathcal{B}$, we have $M_z f = zf \in \mathcal{B}$. If for every $\lambda \in \Omega$, $M_z - \lambda$ is bounded below on $\mathcal{B}$, then for every M_z- invariant subspace $\mathcal{M}$ of $\mathcal{B}$ we define the index of $\mathcal{M}$ to be the dimension of $\mathcal{M}/z\mathcal{M}$, i.e.

$$\text{ind}\mathcal{M} = \dim \mathcal{M}/z\mathcal{M}.$$

If $f \in \mathcal{B}$, we let $[f]$ be the smallest invariant subspace of the operator of multiplication by z. Thus $[f]$ is the closure of the polynomial multiples of f. A function $f \in \mathcal{B}$ is called a cyclic vector if $[f] = \mathcal{B}$.

For $p > 0$, let $H^p(\mathbb{D})$ be the Hardy space of analtytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^p}^p = \sup_{r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \frac{|d\zeta|}{2\pi} < \infty,$$

let $L_a^2(\mathbb{D})$ be the Bergman space of holomorphic functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ such that

$$\int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} = \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} < \infty.,$$

and let D be the Dirichlet space of holomorphic functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ for which

$$\int_{\mathbb{D}} |f'|^2 \frac{dA}{\pi} = \sum_{n=1}^{\infty} n|\hat{f}(n)|^2 < \infty.$$

It can be shown that for every $\lambda \in \mathbb{D}$, $M_z - \lambda$ is bounded below on $H^p(\mathbb{D})$, $L_a^2(\mathbb{D})$ and D.

By the factorization theorem, every function f in $H^p(\mathbb{D})$ can be written as $f = BSF$, where $B(z) = \prod_{i=1}^{\infty} \frac{\overline{\alpha_i}}{|\alpha_i|} \frac{\alpha_i - z}{1-\overline{\alpha_i}z}$ is the Blaschke product, α_i's are the zeros of f, $S(z) = \exp(-\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} d\sigma(t))$ is the singular inner function for some measure σ which is singular with respect to the Lebesgue measure, and $F(z) = \exp(\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})| \frac{dt}{2\pi})$ is an outer function.

The Beurling's famous theorem says that every nonzero M_z-invariant subspace $\mathcal{M}$ in the Hardy space $H^2(\mathbb{D})$ is singly generated, and it has the form $\mathcal{M} = \varphi H^2(\mathbb{D}) = [\varphi]$, where $\varphi = BS$ is an inner function with B a Blaschke product and S a singular inner function (see [59, Page 348]). Beurling's theorem fully characterizes the M_z-invariant subspaces in $H^2(\mathbb{D})$, it also tells us that $\mathrm{ind}\mathcal{M} = 1$ for every M_z-invariant subspace $\mathcal{M}$ in $H^2(\mathbb{D})$ and $f \in H^2(\mathbb{D})$ is cyclic if and only if f is an outer function.

Since then there have been a lot of articles in the literature studying the M_z- invariant subspaces in the Banach space of analytic functions, see e. g. [8], [16], [18], [29], [30], [34], [35], [36], [48], [49] and [53]. One of the surprising facts is that in the Bergman space $L_a^2(\mathbb{D})$, if $n \in \mathbb{N} \cup \{\infty\}$, then there is an M_z-invariant subspace $\mathcal{M}$ in $L_a^2(\mathbb{D})$ such that $\mathrm{ind}\mathcal{M} = n$ (see [8], the authors in [8] proved this fact using abstract theory). Later, Hedenmalm ([35]) constructed an invariant subspace in $L_a^2(\mathbb{D})$ with index 2 using the span of zero set based invariant subspaces.

It turns out that for the Dirichlet space D, we have similar result as in the Hardy space $H^2(\mathbb{D})$: every nonzero M_z-invariant subspace $\mathcal{M}$ in D is singly generated, and it has the form $\mathcal{M} = \varphi D(m_\varphi) = [\varphi]$, where $dm_\varphi(z) = |\varphi(z)|^2 \frac{|dz|}{2\pi}$ and φ is a multiplier of D (see [49], [53]). But there is still an open question about the cyclic vectors in D:

Conjecture 1.2.1 (Brown-Shields [16])**.** *Let $f \in D$, f is cyclic in D if and only if f is an outer function and its boundary zero set is of capacity zero.*

Some authors have studied this conjecture and they obtained some partial results, see e.g. [29], [30], [34] and [53].

Recall that $D \odot D$ is the space of weak products of functions of D, i.e.

$$D \odot D = \{h \in \mathrm{Hol}(\Omega) : h = \sum_{i=1}^{\infty} f_i g_i, \sum_{i=1}^{\infty} \|f_i\|_D \|g_i\|_D < \infty, f_i, g_i \in D\}.$$

Note that by the Riesz factorization theorem, every function in $H^1(\mathbb{D})$ is a product of two $H^2(\mathbb{D})$ functions, we have $H^1(\mathbb{D}) = H^2(\mathbb{D}) \cdot H^2(\mathbb{D}) = H^2(\mathbb{D}) \odot H^2(\mathbb{D})$. Thus we can think of $D \odot D$ as an analogue of $H^1(\mathbb{D})$ and it may be needed for a complete theory of the Dirihclet space.

The dual of $D \odot D$ has been characterized in 2010 by Arcozzi, Rochberg, Sawyer, and Wick [9] as the space $\mathcal{X}(D)$ of analytic functions b on $\mathbb{D}$ such that $|b'|^2 dA$ is a Carleson measure for the Dirichlet space.

In Chapter 2, we use the tools Cauchy dual, pseudocontinuation to study the index of M_z-invariant subspaces in $D \odot D$ (see section 2.3 for the definitions, see also [27] for the background on pseudocontinuations). We show that for functions f in proper weak*-closed M_z^*-invariant subspaces of $\mathcal{X}(D)$, the functions $(zf)'$ are in the Nevanlinna class of $\mathbb{D}$ and have meromorphic pseudocontinuations in the Nevanlinna class of the exterior disc. We then use this result to show that every nonzero M_z-invariant subspace $\mathcal{N}$ of $D \odot D$ has index 1, i.e. satisfies $\dim \mathcal{N}/z\mathcal{N} = 1$.

Note that for functions $f \in D \odot D$, we don't necessarily have $(zf)'$ are in the Nevanlinna class of $\mathbb{D}$. Also when a function f is in the Nevanlinna class of $\mathbb{D}$, it doesn't necessarily have a pseudocontinuation in the Nevanlinna class of the exterior disc. We attack those difficulties in section 2.3.

1.3 Corona Theorem

By the Gelfand theory every abelian Banach algebra $\mathcal{A}$ is isomorphic to a subspace of $C(\Delta)$, where Δ is a compact Hausdorff space. It is called the maximal ideal space of $\mathcal{A}$. In the following we denote the maximal ideal space of $\mathcal{A}$ by $\mathcal{M}_{\mathcal{A}}$. If $\mathcal{A} = H^\infty(\mathbb{D})$, then the open unit disc $\mathbb{D}$ is homeomorphic to a subset of $\mathcal{M}_{H^\infty}$ and for $f \in H^\infty(\mathbb{D})$ the identification of $f|_{\mathbb{D}}$ with its image in $C(\mathcal{M}_{H^\infty})$ is just the identity. Thus we will just write $\mathbb{D} \subseteq \mathcal{M}_{H^\infty}$ and $H^\infty(\mathbb{D}) \subseteq C(\mathcal{M}_{H^\infty})$. Carleson's famous corona theorem (see [19]) says that $\mathbb{D}$ is dense in $\mathcal{M}_{H^\infty}$. It is well-known that this theorem has an analytical reformulation which is as follows: If $\{\varphi_1, ..., \varphi_n\}$ is a finite set of functions in $H^\infty(D)$ satisfying

$$\sum_{j=1}^{n} |\varphi_j(z)|^2 \geq \eta > 0, \quad z \in \mathbb{D}, \quad \text{(Corona condition)}. \tag{1.3.1}$$

then there are functions $\{f_1, ..., f_n\} \subseteq H^\infty(D)$ with

$$\sum_{j=1}^{n} f_j(z)\varphi_j(z) = 1, \quad z \in \mathbb{D}, \quad \text{(Bezout equation)}. \tag{1.3.2}$$

Carleson's proof of the corona theorem was very complicated. He introduced what are now know as Carleson measures. They are an important tool in complex and harmonic analysis.

In 1979 Thomas Wolff gave a simplified (but unpublished) proof of the corona theorem using the $\overline{\partial}$-technique (see [32]).

If $\mathcal{H}(k)$ is a reproducing kernel Hilbert space of analytic functions on some region $\Omega \subseteq \mathbb{C}^d$, $d \geq 1$, then the multiplier algebra $M(\mathcal{H}(k))$ is defined by

$$M(\mathcal{H}(k)) = \{\varphi \in \mathcal{H}(k) : \varphi f \in \mathcal{H}(k), \forall f \in \mathcal{H}(k)\}.$$

For $\varphi \in M(\mathcal{H}(k))$, we denote it by M_φ, the multiplication operator by φ, and let M_φ^* be the adjoint of the operator M_φ. Then for any $f \in \mathcal{H}(k)$, $z \in \Omega$ we have

$$\langle M_\varphi^* k_z, f\rangle_{\mathcal{H}(k)} = \langle k_z, M_\varphi f\rangle_{\mathcal{H}(k)} = \overline{\varphi(z)} f(z) = \langle \overline{\varphi(z)} k_z, f\rangle_{\mathcal{H}(k)},$$

thus $M_\varphi^* k_z = \overline{\varphi(z)} k_z$, which implies $|\overline{\varphi(z)}| \leq \|M_\varphi^*\| = \|M_\varphi\|$. Taking the supremum over all $z \in \Omega$, we have $\|\varphi\|_{H^\infty} \leq \|M_\varphi\|$, therefore $M(\mathcal{H}(k))$ is always contained in $H^\infty(\Omega)$.

Notice $M(\mathcal{H}(k))$ is always an abelian Banach algebra and that $H^\infty(\mathbb{D})$ is the multiplier algebra of $H^2(\mathbb{D})$ and $L_a^2(\mathbb{D})$. On the other hand if $\mathcal{H}(k) = D$ or $\mathcal{H}(k) = H_d^2$, then $M(\mathcal{H}(k)) \neq H^\infty(\mathbb{D})$. Here H_d^2 is the reproducing kernel Hilbert space with kernel $k_\lambda(z) = \frac{1}{1-\sum_{i=1}^d z_i \overline{\lambda_i}}$ on the unit ball of $\mathbb{C}^d$. It is called the Drury-Arveson space. Thus for our understanding of the function theory on the spaces where $M(\mathcal{H}(k))$ does not equal $H^\infty(\mathbb{D})$ it becomes an interesting question whether or not a corona theorem holds for such a multiplier algebra.

It was shown that the corona theorem also holds for many other function algebras, such as $M(D)$ (see Tolokonnikov [69], Xiao [76]), $M(H_d^2)$ (see Costea, Sawyer and Wick [26]) and so on. In all of these cases the algebra is the multiplier algebra of a space with a complete Nevanlinna-Pick kernel (see Agler and McCarthy [2], also the definition below).

Definition 1.3.1. *Let B_d be the unit ball in $\mathbb{C}^d$. A reproducing kernel k on B_d is called a complete Nevanlinna-Pick kernel if $k_0(z) = 1$ for all $z \in B_d$ and if there exists a sequence of analytic functions $\{b_n\}_{n=1}^\infty$ on B_d such that*

$$1 - \frac{1}{k_\lambda(z)} = \sum_{n=1}^\infty b_n(z)\overline{b_n(\lambda)} \quad \textit{for all } \lambda, z \in B_d.$$

Thus since Shimorin [67] showed that for any Borel measure μ on $\mathbb{T}$, $D(\mu)$ has a complete Nevanlinna-Pick kernel, one wonders whether the corona theorem holds for $M(D(\mu))$.

Also the corona theorem has been generalized to infinitely many functions in $H^\infty(\mathbb{D})$ and $M(D)$ (see Rosenblum [58], Tolokonnikov [69] and Trent [73]). The infinite version, given by Rosenblum [58] and Tolokonnikov [69], can be formulated as follows (see Trent [72]):

Theorem 1.3.2. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq H^\infty(\mathbb{D})$. Suppose that*

$$0 < \epsilon^2 \le \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \le 1, \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $\{e_j\}_{j=1}^\infty \subseteq H^\infty(\mathbb{D})$ such that $\sum_{j=1}^\infty \varphi_j e_j = 1$ and $\sup_{z\in\mathbb{D}} \sum_{j=1}^\infty |e_j(z)|^2 \le \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2}$, where C_0 is a constant.

In this dissertation, we study the corona theorem for the $D(\mu)$ spaces when μ is a finitely atomic measure. Fix $\mu = \mu_k$, a finitely atomic measure, we observe that by [52, Lemma 5.3] we have $M(D(\mu_k)) = D(\mu_k) \cap H^\infty(\mathbb{D})$. We will see that a corona theorem holds for $M(D(\mu_k))$ if and only if every $\varphi \in \mathcal{M}_{M(D(\mu_k))}$ extends to some $\psi \in \mathcal{M}_{H^\infty}$ (see Theorem 3.1.12 which is due to Sundberg), we only need to show that every $\varphi \in \mathcal{M}_{M(D(\mu_k))}$ has an extension to some $\psi \in \mathcal{M}_{H^\infty}$.

As to infinitely many functions in $M(D(\mu_k))$, we consider $D_{l^2}(\mu_k)$, or $\oplus_1^\infty D(\mu_k)$, which can be considered as l^2-valued $D(\mu_k)$ space. If $F = (f_1, f_2, \ldots) \in \oplus_1^\infty D(\mu_k)$, then the norm is defined by

$$\begin{aligned}
&\|F\|^2_{\oplus_1^\infty D(\mu_k)} \\
&= \int_0^{2\pi} \|F(e^{it})\|^2_{l^2} \frac{dt}{2\pi} + \int_{\mathbb{T}} \int_0^{2\pi} \frac{\|F(e^{it}) - F(\zeta)\|^2_{l^2}}{|e^{it} - \zeta|^2} \frac{dt}{2\pi} d\mu_k(\zeta) \\
&= \sum_{j=1}^{\infty} \|f_j\|^2_{D(\mu_k)}.
\end{aligned}$$

Given $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$, we let $\Phi(z) = (\varphi_1(z), \varphi_2(z), \ldots)$. We use M_Φ to denote the (column) operator from $D(\mu_k)$ to $\oplus_1^\infty D(\mu_k)$ defined by

$$M_\Phi(f) = \{\varphi_j f\}_{j=1}^{\infty} \quad \text{for } f \in D(\mu_k).$$

Note that the pointwise hypothesis $\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ in Theorem 1.3.2 implies that the operator T_Φ defined on $H^2(\mathbb{D})$ in analogy to that of M_Φ is bounded and $\|T_\Phi\| = \sup_{z\in\mathbb{D}}(\sum_{j=1}^{\infty} |\varphi_j(z)|^2)^{\frac{1}{2}}$. Since $M(D(\mu_k)) = D(\mu_k) \cap H^\infty(\mathbb{D})$, the pointwise upper bound hypothesis will not be sufficient to conclude that M_Φ is bounded from $D(\mu_k)$ to $\oplus_1^\infty D(\mu_k)$. Thus, we will replace the assumption $\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ for $z \in \mathbb{D}$ by the condition $\|M_\Phi\| \leq 1$, and we have the following theorem:

Theorem 1.3.3. *Let $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$. Suppose that*

$$\|M_\Phi\| \leq 1 \quad \textit{and} \quad 0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \quad \textit{for all } z \in \mathbb{D}.$$

Then there exists $\{b_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$ such that

(i) $\Phi(z)B(z)^\top = 1$ *for all $z \in \mathbb{D}$, and*

(ii) $\|M_B\| \leq \frac{1}{\varepsilon}\left(2 + 16\|M_{B_{k-1}}\|^2\right)^{1/2}$, *where B_{k-1} is the solution for the corona theorem in $M(D(\mu_{k-1}))$.*

Chapter 2

Weak Products and Index

2.1 Weak Products

We will follow the general theory in [55, section 2].

Let $d \geq 1, \Omega \subseteq \mathbb{C}^d$ be a non-empty open connected set. Suppose $\mathcal{B}, \mathcal{E} \subseteq Hol(\Omega)$ are Banach spaces such that point evaluations in Ω are bounded, let $\mathcal{B}^*, \mathcal{E}^*$ be the dual spaces of $\mathcal{B}, \mathcal{E}$ respectively. For $z \in \Omega$, let $k_z^{\mathcal{B}^*} \in \mathcal{B}^*$ be the point evaluation map from $\mathcal{B}$ to $\mathbb{C}$, i.e $k_z^{\mathcal{B}^*}(f) = f(z), \forall f \in \mathcal{B}$. Similarly we let $k_z^{\mathcal{E}^*}$ be the point evaluation map from $\mathcal{E}$ to $\mathbb{C}$.

We define $\mathcal{B} \odot \mathcal{E}$ to be the weak products of functions in $\mathcal{B}$ and $\mathcal{E}$ as follows

$$\mathcal{B} \odot \mathcal{E} = \{h = \sum_{i=1}^{\infty} f_i g_i : f_i \in \mathcal{B}, g_i \in \mathcal{E}, \sum_{i=1}^{\infty} \|f_i\|_{\mathcal{B}} \|g_i\|_{\mathcal{E}} < \infty\}.$$

The norm in $\mathcal{B} \odot \mathcal{E}$ is defined by

$$\|h\|_* = \inf\{\sum_{i=1}^{\infty} \|f_i\|_{\mathcal{B}} \|g_i\|_{\mathcal{E}} : h = \sum_{i=1}^{\infty} f_i g_i\}, \; h \in \mathcal{B} \odot \mathcal{E}.$$

Lemma 2.1.1. *If $\mathcal{B}, \mathcal{E} \subseteq Hol(\Omega)$ are Banach spaces such that point evaluations in Ω are bounded, then $(\mathcal{B} \odot \mathcal{E}, \|\cdot\|_*)$ is a Banach space of analytic functions such that point evaluations in Ω are continuous.*

Proof. As in [55], we note that $|h(z)| \leq \|h\|_* \|k_z^{\mathcal{B}^*}\|_{\mathcal{B}^*} \|k_z^{\mathcal{E}^*}\|_{\mathcal{E}^*}$ for all $h \in \mathcal{B} \odot \mathcal{E}$, and $z \in \Omega$. This implies that if $\|h\|_* = 0$, then $h = 0$. The proof of completeness is the same as in [55].

We only show that the norm in $\mathcal{B} \odot \mathcal{E}$ satisfies the triangle inequality. Let $h, k \in \mathcal{B} \odot \mathcal{E}$, $\forall \varepsilon > 0$, we can find $f_i, a_i \in \mathcal{B}, g_i, b_i \in \mathcal{E}$ such that $h = \sum_{i=1}^\infty f_i g_i, k = \sum_{i=1}^\infty a_i b_i$ with $\|h\|_* \geq \sum_{i=1}^\infty \|f_i\|_{\mathcal{B}} \|g_i\|_{\mathcal{E}} - \varepsilon, \|k\|_* \geq \sum_{i=1}^\infty \|a_i\|_{\mathcal{B}} \|b_i\|_{\mathcal{E}} - \varepsilon$. then

$$h + k = \sum_{i,j} f_i g_i + a_j b_j,$$

thus $\|h + k\|_* \leq \|h\|_* + \|k\|_* + 2\varepsilon$, letting $\varepsilon \to 0$, we see that the norm satisfies the triangle inequality. □

Remark 2.1.2. (1) *Let $\mathcal{B} = \mathcal{E} = H^1(\mathbb{D})$, then $H^1(\mathbb{D}) \odot H^1(\mathbb{D})$ is a Banach space, and $H^{1/2}(\mathbb{D}) \subseteq H^1(\mathbb{D}) \odot H^1(\mathbb{D})$.*

(2) *Of course, one could define a weak product for some other Banach spaces. If $\mathcal{B} = l^p = \{\alpha = (\alpha_i) : \|\alpha\|_{l^p}^p = \sum_{i=1}^\infty |\alpha_i|^p < \infty\}, p \geq 1$, then $l^p \subseteq l^\infty$ with $\|\alpha\|_{l^\infty} \leq \|\alpha\|_{l^p}$. For $p, q \geq 1$, and $\beta = (\beta_i) \in l^p, \gamma = (\gamma_i) \in l^q$, let $\beta \cdot \gamma = (\beta_i \gamma_i)$, then we define*

$$l^p \odot l^q = \{\alpha = \sum_{j=1}^\infty \beta_j \cdot \gamma_j : \sum_{j=1}^\infty \|\beta_j\|_{l^p} \|\gamma_j\|_{l^q} < \infty, \beta_j \in l^p, \gamma_j \in l^q\}$$

the norm is defined by

$$\|\alpha\|_* = \inf\{\sum_{j=1}^\infty \|\beta_j\|_{l^p} \|\gamma_j\|_{l^q} : \alpha = \sum_{j=1}^\infty \beta_j \cdot \gamma_j\}.$$

Then for every i, $|\alpha_i| \leq \|\alpha\|_*$, thus $\|\alpha\|_* = 0$ if and only if $\alpha = 0$. Therefore $l^p \odot l^q$ is a Banach space, $p, q \geq 1$.

Similarly, we have $l^{1/2} \subseteq l^1 \odot l^1$.

Let $\mathcal{A} \subseteq \mathcal{B}, \mathcal{L} \subseteq \mathcal{E}$ be vector subspaces, we define

$$\mathcal{A}\hat{\odot}\mathcal{L} = \{h = \sum_{i=1}^{n} f_i g_i : f_i \in \mathcal{B}, g_i \in \mathcal{E}, n \in \mathbb{N}\},$$

and the norm is defined by

$$\|h\|_\bullet = \inf\{\sum_{i=1}^{n} \|f_i\|_\mathcal{B} \|g_i\|_\mathcal{E} : h = \sum_{i=1}^{n} f_i g_i\}, \ h \in \mathcal{A}\hat{\odot}\mathcal{L}.$$

Then $\mathcal{A}\hat{\odot}\mathcal{L} \subseteq \mathcal{B} \odot \mathcal{E}$ and $\|h\|_* \leq \|h\|_\bullet$ for every $h \in \mathcal{A}\hat{\odot}\mathcal{L}$. If we use $(\mathcal{A}\hat{\odot}\mathcal{L})_\bullet$ to denote the completion of $\mathcal{A}\hat{\odot}\mathcal{L}$ with respect to the norm $\|\cdot\|_\bullet$, then the inclusion of $\mathcal{A}\hat{\odot}\mathcal{L}$ into $\mathcal{B} \odot \mathcal{E}$ extends to be a contraction $V : (\mathcal{A}\hat{\odot}\mathcal{L})_\bullet \to \mathcal{B} \odot \mathcal{E}$.

The following three results are the Banach space analogues of Hilbert space results that were proved in [55]. The same proofs will establish the following Lemma and Theorems.

Lemma 2.1.3. *If $\mathcal{A} \subseteq \mathcal{B}, \mathcal{L} \subseteq \mathcal{E}$ are dense vector subspaces, then $\mathcal{A}\hat{\odot}\mathcal{L}$ is dense in $\mathcal{B} \odot \mathcal{E}$ in the $\|\cdot\|_*$ norm and*

$$\|h\|_* = \inf\{\sum_{i=1}^{\infty} \|f_i\|_\mathcal{B} \|g_i\|_\mathcal{E} : h = \sum_{i=1}^{\infty} f_i g_i, f_i \in \mathcal{A}, g_i \in \mathcal{L}\}, \ h \in \mathcal{B} \odot \mathcal{E}.$$

Theorem 2.1.4. *If $\mathcal{A} \subseteq \mathcal{B}, \mathcal{L} \subseteq \mathcal{E}$ are dense vector subspaces, then V is onto and the induced map $\tilde{V} : (\mathcal{A}\hat{\odot}\mathcal{L})_\bullet / \ker V \to \mathcal{B} \odot \mathcal{E}$ is isometric.*

For $d \geq 1$, let $\mathbb{B}_d$ denote the open unit ball in $\mathbb{C}^d$, for $0 \leq r < 1, z \in \mathbb{B}_d$, let $f_r(z) = f(rz)$. Write $Hol(\mathbb{B}_d^-)$ for the algebra of all functions f on $\mathbb{B}_d$ such that f extends to be analytic in a neighborhood of the closure of $\mathbb{B}_d$.

Theorem 2.1.5. *Let $\mathcal{B}, \mathcal{E} \subseteq Hol(\mathbb{B}_d)$ be Banach spaces of analytic functions which satisfy the following conditions*

(a) *Point evaluations in $\mathbb{B}_d$ are bounded and $\mathcal{B}$ is reflexive.*

(b) *$\mathcal{B}$ contains $Hol(\mathbb{B}_d^-)$.*

(c) *If $f_n, f \in Hol(\mathbb{B}_d^-)$ such that $f_n \to f$ uniformly in some open neighborhood of $\mathbb{B}_d$, then $f_n \to f$ in $\mathcal{B}$.*

(d) *There is a $C > 0$ such that if $0 < r < 1$ and if $f \in \mathcal{B}$, then $f_r \in \mathcal{B}$ and $\|f_r\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{B}}$.*

Then $Hol(\mathbb{B}_d^-)$ is dense in $\mathcal{B} \odot \mathcal{E}$ and

$$\|h\|_* = \|h\|_\bullet = \inf\{\sum_{i=1}^n \|f_i\|_{\mathcal{B}}\|g_i\|_{\mathcal{E}} : h = \sum_{i=1}^n f_i g_i, f_i, g_i \in Hol(\mathbb{B}_d^-)\}$$

for all $h \in Hol(\mathbb{B}_d^-)$.

Let $d = 1, \Omega \subseteq \mathbb{C}$ be open, connected and nonempty set. Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space, i.e. point evaluations in Ω are bounded. If $\mathcal{H}$ is M_z-invariant, let

$$\mathcal{K} = (zH)' = \{(zf)' : f \in \mathcal{H}\},$$

and the norm in $\mathcal{K}$ is defined by $\langle (zf)', (zg)' \rangle_{\mathcal{K}} = \langle f, g \rangle_{\mathcal{H}}$.

If $\mathrm{Hol}(\Omega^-)$ is densely contained in $\mathcal{H}$, we define

$$\mathcal{X}(\mathcal{H}, \mathcal{K}) = \{b \in \mathcal{H} : \exists C > 0, \left|\langle \varphi(z\psi)', (zb)' \rangle_{\mathcal{K}}\right| \leq C\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}, \quad \forall \varphi, \psi \in \mathrm{Hol}(\Omega^-)\},$$

for $b \in \mathcal{X}(\mathcal{H}, \mathcal{K})$, write $\|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}$ for the infimum of all $C > 0$ such that $\left|\langle \varphi(z\psi)', (zb)' \rangle_{\mathcal{K}}\right| \leq C\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}$ for all $\varphi, \psi \in \mathrm{Hol}(\Omega^-)$.

Lemma 2.1.6. *If $\mathcal{H} \subseteq Hol(\Omega)$ is a reproducing kernel Hilbert space, and if $Hol(\Omega^-)$ is densely contained in $\mathcal{H}$, then $\mathcal{X}(\mathcal{H},\mathcal{K})$ is a Banach space of analytic functions on Ω such that*

$$|b(z)| \leq \|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}\|k_z^{\mathcal{H}}\|_{\mathcal{H}}\|1\|_{\mathcal{H}}.$$

Proof. Note that $\mathrm{Hol}(\Omega^-)$ is dense in $\mathcal{H}$ and $b(w) = \langle b, k_w^{\mathcal{H}}\rangle_{\mathcal{H}} = \langle (zb)', 1 \cdot (zk_w^{\mathcal{H}})'\rangle_{\mathcal{K}}$, the inequality follows.

Also

$$\|b\|_{\mathcal{H}}^2 = \|(zb)'\|_{\mathcal{K}}^2 = \langle 1 \cdot (zb)', (zb)'\rangle_{\mathcal{K}} \leq \|1\|_{\mathcal{H}}\|b\|_{\mathcal{H}}\|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})},$$

thus $\|b\|_{\mathcal{H}} \leq \|1\|_{\mathcal{H}}\|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}$.

The completeness follows from the same argument as in Lemma 2.2 in [55]. It is clear that $\|\cdot\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}$ is a norm in $\mathcal{X}(\mathcal{H},\mathcal{K})$. □

Theorem 2.1.7. *Let $Hol(\Omega^-)$ be dense in $\mathcal{H}$, let $\mathcal{K} = (z\mathcal{H})'$, and suppose there is a linear subspace $\mathcal{A} \subseteq Hol(\Omega^-)$ which is dense in $\mathcal{H}$ and which satisfies $\|\varphi\|_* = \|\varphi\|_\bullet$ for all $\varphi \in \mathcal{A}\hat{\odot}(z\mathcal{A})'$. Then $(\mathcal{H} \odot \mathcal{K})^* = \mathcal{X}(\mathcal{H},\mathcal{K})$, this means if $b \in \mathcal{X}(\mathcal{H},\mathcal{K})$, define L_b on $\mathcal{K}$ by*

$$L_b(k) = \langle k, (zb)'\rangle_{\mathcal{K}},$$

then L_b extends to be bounded on $\mathcal{H} \odot \mathcal{K}$ and the map $b \mapsto L_b$ is a conjugate linear isometric isomorphism of $\mathcal{X}(\mathcal{H},\mathcal{K})$ onto $(\mathcal{H} \odot \mathcal{K})^*$.

Proof. We will follow the argument in [55, Theorem 1.3].

Let $b \in \mathcal{X}(\mathcal{H},\mathcal{K})$, $h \in \mathcal{A}\hat{\odot}(z\mathcal{A})' \subseteq Hol(\Omega^-) \subseteq \mathcal{K}$. Then $h = \sum_{i=1}^{n} f_i(zg_i)'$ for some $f_i, g_i \in \mathcal{A}$, $i = 1,\dots,n$, and

$$\begin{aligned}|L_b(h)| = |\langle \sum_{i=1}^{n} f_i(zg_i)', (zb)'\rangle_{\mathcal{K}}| &\le \sum_{i=1}^{n} |\langle f_i(zg_i)', (zb)'\rangle_{\mathcal{K}}| \\ &\le \sum_{i=1}^{n} \|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}\|f_i\|_{\mathcal{H}}\|g_i\|_{\mathcal{H}},\end{aligned}$$

thus $|L_b(h)| \le \|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}\|h\|_{\bullet} = \|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}\|h\|_{*}$. By Lemma 2.1.3 it follows that L_b extends to be bounded on $\mathcal{H}\odot\mathcal{K}$ with $\|L_b\| \le \|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})}$.

If $L \in (\mathcal{H}\odot\mathcal{K})^*$, then for $(zf)' \in \mathcal{K}$ we have

$$|L((zf)')| \le \|L\|\|(zf)'\|_* \le \|L\|\|f\|_{\mathcal{H}}\|1\|_{\mathcal{H}},$$

thus $\exists (zb)' \in \mathcal{K}$, such that $L((zf)') = \langle (zf)', (zb)'\rangle_{\mathcal{K}} := L_b((zf)')$.

If $\varphi, \psi \in \mathrm{Hol}(\Omega^-)$, then

$$\begin{aligned}|\langle \varphi(z\psi)', (zb)'\rangle_{\mathcal{K}}| = |L(\varphi(z\psi)')| &\le \|L\|\|\varphi(z\psi)'\|_* \\ &\le \|L\|\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}},\end{aligned}$$

thus $b \in \mathcal{X}(\mathcal{H},\mathcal{K})$, and $\|b\|_{\mathcal{X}(\mathcal{H},\mathcal{K})} \le \|L\| = \|L_b\|$. □

Remark 2.1.8. *If $\mathcal{H}$ is a reproducing kernel Hilbert space satisfying the four conditions $(a)-(d)$ in Theorem 2.1.5. If $\mathcal{H}$ is M_z-invariant, then $\mathcal{K} = (z\mathcal{H})'$ also satisfies the four conditions $(a)-(d)$ in Theorem 2.1.5, and so by Theorem 2.1.5, we have $\|\varphi\|_* = \|\varphi\|_\bullet$ for all $\varphi \in Hol(\Omega^-)$. Thus, in this case we have $(\mathcal{H}\odot\mathcal{K})^* = \mathcal{X}(\mathcal{H},\mathcal{K})$.*

Proof. It is clear that $\mathcal{K}$ satisfies conditions $(a)-(c)$ in Theorem 2.1.5.

For any $(zf)' \in \mathcal{K}$, suppose $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, then

$$[(zf)']_r(z) = \sum_{n=0}^{\infty} (n+1)\hat{f}(n) r^n z^n = (zf_r)'(z),$$

thus $[(zf)']_r \in \mathcal{K}$ and

$$\begin{aligned} \|[(zf)']_r\|_{\mathcal{K}} &= \|(zf_r)'\|_{\mathcal{K}} = \|f_r\|_{\mathcal{H}} \leq C\|f\|_{\mathcal{H}} \\ &= C\|(zf)'\|_{\mathcal{K}} \end{aligned}$$

□

If $\mathcal{H} \subseteq Hol(\Omega)$ is a reproducing kernel Hilbert space such that $Hol(\Omega^-)$ is densely contained in $\mathcal{H}$, define

$$\begin{aligned} \mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \exists C > 0, \left|\langle \varphi\psi, b\rangle_{\mathcal{H}}\right| \leq C\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}, \\ \forall \varphi, \psi \in \mathrm{Hol}(\Omega^-)\}, \end{aligned}$$

and for $b \in \mathcal{X}(\mathcal{H})$, write $\|b\|_{\mathcal{X}(\mathcal{H})}$ for the infimum of all $C > 0$ such that $\left|\langle \varphi\psi, b\rangle_{\mathcal{H}}\right| \leq C\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}$ for all $\varphi, \psi \in \mathrm{Hol}(\Omega^-)$.

If $\mathcal{H} \subseteq Hol(\Omega)$ is a reproducing kernel Hilbert space, we say a positive Borel measure μ on Ω is a Carleson measure for $\mathcal{H}$, $\mu \in \mathrm{CM}(\mathcal{H})$, if there is a $C > 0$ such that for all $f \in \mathcal{H}$,

$$\int_{\Omega} |f|^2 d\mu \leq C^2 \|f\|_{\mathcal{H}}^2.$$

The smallest such C is the Carleson measure norm of μ.

Remark 2.1.9. (a) *It is shown in [55] that* $\mathcal{X}(\mathcal{H})$ *is a Banach space of analytic functions and it is the dual of* $\mathcal{H} \odot \mathcal{H}$*, that is* $(\mathcal{H} \odot \mathcal{H})^* = \mathcal{X}(\mathcal{H})$.

(b) Let $\alpha \in [0,1]$*,* $D_\alpha = \{f \in Hol(\mathbb{D}) : \|f\|^2_{D_\alpha} = \sum_{n=0}^\infty (n+1)^\alpha |\hat{f}(n)|^2 < \infty\}$*. Let*

$$\mathcal{C}(D_\alpha) = \{f \in D_\alpha : |f'|^2(1-|z|^2)^{1-\alpha} dA \text{ is a Carleson measure for } D_\alpha\}.$$

It is shown in [75] that $(D_\alpha \odot (zD_\alpha)')^* = \mathcal{C}(D_\alpha)$*.*

(c) If $\mathcal{H} = H^2$ *is the Hardy space, then* $H^2 \odot H^2 = H^1$*, and by the well known Fefferman's theorem (see [32]),* $(H^1)^* = BMOA$*, thus* $\mathcal{X}(H^2) = BMOA$*. By the result in [75], we have* $H^2 \odot (zH^2)' = \partial H^1$*, and* $\mathcal{X}(H^2, (zH^2)') = BMOA = \mathcal{X}(H^2)$*.*

(d) If $\mathcal{H} = D$ *is the Dirichlet space, then by the result in [9], we have* $(D \odot D)^* = \mathcal{C}(D)$*, thus* $\mathcal{X}(D) = \mathcal{C}(D)$*. By the result in [75], we have* $D \odot (zD)' = \partial(D \odot D)$*, and* $\mathcal{X}(D, (zD)') = \mathcal{C}(D) = \mathcal{X}(D)$*.*

(e) In (c) and (d), we see that for $\alpha = 0, 1$*,* $\mathcal{X}(D_\alpha, (zD_\alpha)') = \mathcal{X}(D_\alpha)$*. Then we have the following question:*

Question 2.1.10. *For* $\alpha \in (0,1)$*, is* $\mathcal{X}(D_\alpha, (zD_\alpha)') = \mathcal{X}(D_\alpha)$*?*

The following Theorem can be derived from Theorem 3.1 in [55].

Theorem 2.1.11. *If* $\mathcal{H} = \mathcal{H}(k^{\mathcal{H}}) \subseteq Hol(\Omega)$*,* $\mathcal{L} = \mathcal{L}(k^{\mathcal{L}}) \subseteq Hol(\Omega)$ *have reproducing kernel* $k^{\mathcal{H}}$ *and* $k^{\mathcal{L}}$ *respectively, then*

$$\mathcal{H} \odot \mathcal{L} \subseteq \mathcal{H}(k^{\mathcal{H}} \cdot k^{\mathcal{L}}),$$

with $\|h\|_{\mathcal{H}(k^{\mathcal{H}} \cdot k^{\mathcal{L}})} \leq \|h\|_*$ *for all* $h \in \mathcal{H} \odot \mathcal{L}$*, where* $\mathcal{H}(k^{\mathcal{H}} \cdot k^{\mathcal{L}})$ *is the Hilbert space with reproducing kernel* $k^{\mathcal{H}} \cdot k^{\mathcal{L}}$*.*

2.2 Hankel Operators and Cyclicity

In this section, we prove that every nonzero M_z-invariant subspace $\mathcal{M}$ of D is the kernel of a Hankel operator, and then conclude that: If $f \in D$, then f is cyclic in D if and only if f is cyclic in $D \odot D$. This is a joint work with Stefan Richter.

Recall that if $\mathcal{H} \subseteq Hol(\Omega)$ is a reproducing kernel Hilbert space such that $Hol(\Omega^-)$ is densely contained in $\mathcal{H}$,

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \exists C > 0, \left|\langle \varphi\psi, b\rangle_{\mathcal{H}}\right| \leq C\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}, \quad \forall \varphi, \psi \in \mathrm{Hol}(\Omega^-)\}.$$

The following Theorem is in [55] and it is stated as Theorem 1.1.2 in the introduction, we include it here for the convenience of the reader.

Theorem 2.2.1. *Let $Hol(\Omega^-)$ be dense in $\mathcal{H}$, and suppose there is a linear subspace $\mathcal{L} \subseteq Hol(\Omega^-)$ which is dense in $\mathcal{H}$ and which satisfies $\|\varphi\|_* = \|\varphi\|_{\bullet\mathcal{L}}$ for all $\varphi \in \mathcal{L}\widehat{\odot}\mathcal{L}$. Then $(\mathcal{H} \odot \mathcal{H})^* = \mathcal{X}(\mathcal{H})$. This means if for $b \in \mathcal{X}(\mathcal{H})$, we define L_b on $\mathcal{H}$ by*

$$L_b(h) = \langle h, b\rangle_{\mathcal{H}}$$

then L_b extends to be bounded on $\mathcal{H} \odot \mathcal{H}$, and the map $b \to L_b$ is a conjugate linear isometric isomorphism of $\mathcal{X}(\mathcal{H})$ onto $(\mathcal{H} \odot \mathcal{H})^*$.

Let $\mathbb{D}$ be the open unit disc with boundary $\mathbb{T}$ and let μ be a nonnegative Borel measure on the closed unit disc, define

$$D(\mu) = \{f \in H^2(\mathbb{D}) : \int_{\mathbb{D}^-} D_\lambda(f) d\mu(\lambda) < \infty\},$$

where $D_\lambda(f) = \int_{\mathbb{T}} |\frac{f(\zeta)-f(\lambda)}{\zeta-\lambda}|^2 \frac{|d\zeta|}{2\pi}$ is the local Dirichlet integral of f at $\lambda \in \mathbb{D}^-$. The norm of f in $D(\mu)$ is defined by

$$\|f\|^2_{D(\mu)} = \|f\|^2_{H^2} + \int_{\mathbb{D}^-} D_\lambda(f)d\mu(\lambda).$$

For μ supported in $\mathbb{T}$, $D(\mu)$ spaces were introduced by Richter in [51] as he was studying analytic two-isometries, and then they were generalized by Aleman [3] to the μ on the closed unit disc. When $\mu = \frac{dt}{2\pi}$, $D(\frac{dt}{2\pi})$ is the Dirichlet space D (see [50], [52]). We can verify that $D(\mu)$ spaces satisfy all the conditions in Theorem 2.2.1.

Let $\mathcal{H}$ satisfy the conditions in Theorem 2.2.1. Recall that $\overline{\mathcal{H}} = \{\overline{f} : f \in \mathcal{H}\}, \|\overline{f}\|_{\overline{\mathcal{H}}} = \|f\|_{\mathcal{H}}$, then for $b \in \mathcal{X}(\mathcal{H})$, we have $H_b : \mathcal{H} \to \overline{\mathcal{H}}, \langle H_b\varphi, \overline{\psi}\rangle_{\overline{\mathcal{H}}} = H_b(\varphi,\psi) = \langle \varphi\psi, b\rangle_{\mathcal{H}}$ for all $\varphi,\psi \in \mathrm{Hol}(\Omega^-)$.

Also recall that $M(\mathcal{H})$ is the multiplier algebra of $\mathcal{H}$, i.e.

$$M(\mathcal{H}) = \{\varphi \in \mathcal{H} : \varphi f \in \mathcal{H}, \forall f \in \mathcal{H}\}.$$

Lemma 2.2.2. *Suppose Hol*(Ω^-) *is densely contained in* H*, and Hol*$(\Omega^-) \subseteq M(\mathcal{H})$*. If* $b \in \mathcal{X}(\mathcal{H})$ *and* $f \in \mathcal{H}$*, then for every* $\varphi \in$ *Hol*(Ω^-) *we have*

$$\langle H_b f, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} = \langle \varphi f, b\rangle_{\mathcal{H}} = \langle f, M^*_\varphi b\rangle_{\mathcal{H}}. \tag{2.2.1}$$

Thus $\ker H_b = \{f \in \mathcal{H} : \langle f, M^*_\varphi b\rangle_{\mathcal{H}} = 0, \forall\varphi \in \mathit{Hol}(\Omega^-)\} = [b]^\perp_*$*, where* $[b]_*$ *denotes the smallest subspace that contains* b *and is invariant under* M^*_φ *for every* $\varphi \in$ *Hol*(Ω^-)*.*

Proof. Let $\psi_n \in \mathrm{Hol}(\Omega^-)$ and $\psi_n \to f$ in $\mathcal{H}$, as $n \to \infty$, then for $\varphi \in \mathrm{Hol}(\Omega^-)$ we have

$$\begin{aligned}\langle H_b f, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} &= \lim_{n\to\infty} \langle H_b \psi_n, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} \\ &= \lim_{n\to\infty} \langle \varphi\psi_n, b\rangle_{\mathcal{H}} = \langle \varphi f, b\rangle_{\mathcal{H}} \\ &= \langle f, M_\varphi^* b\rangle_{\mathcal{H}},\end{aligned}$$

thus $f \in \ker H_b$ if and only if $f \in [b]_*^\perp$ and the lemma follows. □

Remark 2.2.3. *If for $\lambda \in \Omega$ we have $k_\lambda \in Hol(\Omega^-)$, then*

$$H_b f(\lambda) = \langle H_b f, \overline{k_\lambda}\rangle_{\overline{\mathcal{H}}} = \langle f k_\lambda, b\rangle_{\mathcal{H}}.$$

Furthermore, if $f \in Hol(\Omega^-)$, then

$$\begin{aligned}H_{k_z} f(\lambda) &= \langle f k_\lambda, k_z\rangle_{\mathcal{H}} = \overline{M_f^* k_z(\lambda)} \\ &= f(z)\overline{k_z(\lambda)},\end{aligned}$$

thus $H_{k_z} f = \langle f, k_z\rangle_{\mathcal{H}} \overline{k_z}$ is a rank one operator.

If $\mathcal{B}$ is a Banach space, and $T \in B(\mathcal{B})$, we let $LatT$ be the lattice of T-invariant subspaces of $\mathcal{B}$. The following Theorem is in [50].

Theorem 2.2.4. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(H)$ satisfy*

$$\bigcap_{n>0} T^n H = (0), \tag{2.2.2}$$

$$\|x\|^2 + \|T^2 x\|^2 \leq 2\|Tx\|^2, \quad \forall x \in H. \tag{2.2.3}$$

If $\mathcal{M} \in LatT$, then $[\ker(T|_{\mathcal{M}})^*]_T = \mathcal{M}$, where $[\ker(T|_{\mathcal{M}})^*]_T$ is the smallest T-invariant subspace of $\mathcal{H}$ that contains $\ker(T|_{\mathcal{M}})^*$.

The following Theorem is known to experts, but we cannot locate a reference, we include a proof here for completeness.

Theorem 2.2.5. *Let $\mathcal{H}$ be the $D(\mu)$ space and let $T = M_z$, then T satisfies (2.2.2) and (2.2.3) in Theorem 2.2.4 (see [3]). $\forall \lambda_0 \in \mathbb{D}$, let $T_{\lambda_0} := (T - \lambda_0)(I - \overline{\lambda_0}T)^{-1}$, then T_{λ_0} also satisfies (2.2.2) and (2.2.3), and $\ker(T_{\lambda_0}|_{\mathcal{M}})^* = \ker((T-\lambda_0)|_{\mathcal{M}})^*$. Thus $\forall \mathcal{M} \in Lat(M_z, D(\mu)), \mathcal{M} = [\ker(T_{\lambda_0}|_{\mathcal{M}})^*]_{T_{\lambda_0}} = [\ker((T-\lambda_0)|_{\mathcal{M}})^*]_T$.*

Proof. $\forall \lambda_0 \in \mathbb{D}$, let $\Delta^2_{\lambda_0} = T^*_{\lambda_0} T_{\lambda_0} - I$, and $\Delta^2 = T^*T - I$.

Note that $\sigma(T) \subseteq \mathbb{D}^-$ (see [50]), where $\sigma(T)$ is the spectrum of T. $\forall x \in D(\mu)$, let $y = (1 - \overline{\lambda_0}T)^{-1}x$, then $x = (1 - \overline{\lambda_0}T)y$ and we have

$$\begin{aligned}
&\|T_{\lambda_0}x\|^2 - \|x\|^2 = \|(T-\lambda_0)y\|^2 - \|(1-\overline{\lambda_0}T)y\|^2 \\
&= (\|Ty\|^2 - 2Re\langle Ty, \lambda_0 y\rangle + |\lambda_0|^2\|y\|^2) \\
&- (\|y\|^2 - 2Re\langle y, \overline{\lambda_0}Ty\rangle + |\lambda_0|^2\|Ty\|^2) \\
&= (1-|\lambda_0|^2)(\|Ty\|^2 - \|y\|^2) \\
&= (1-|\lambda_0|^2)\Big(\|T(1-\overline{\lambda_0}T)^{-1}x\|^2 - \|(1-\overline{\lambda_0}T)^{-1}x\|^2\Big),
\end{aligned}$$

this implies $\Delta^2_{\lambda_0} = (1-|\lambda_0|^2)(I - \lambda_0 T^*)^{-1}\Delta^2(1-\overline{\lambda_0}T)^{-1}$.

Note that we have the following equivalent relationships:

$$\begin{aligned}
&T^*_{\lambda_0}\Delta^2_{\lambda_0}T_{\lambda_0} \leq \Delta^2_{\lambda_0} \\
&\Leftrightarrow (T^* - \overline{\lambda_0})\Delta^2_{\lambda_0}(T-\lambda_0) \leq (I - \lambda_0 T^*)\Delta^2_{\lambda_0}(I - \overline{\lambda_0}T) \\
&i.e.\ T^*\Delta^2_{\lambda_0}T - \lambda_0 T^*\Delta^2_{\lambda_0} - \overline{\lambda_0}\Delta^2_{\lambda_0}T + |\lambda_0|^2\Delta^2_{\lambda_0} \\
&\leq \Delta^2_{\lambda_0} - \lambda_0 T^*\Delta^2_{\lambda_0} - \overline{\lambda_0}\Delta^2_{\lambda_0}T + |\lambda_0|^2 T^*\Delta^2_{\lambda_0}T \\
&\Leftrightarrow (1-|\lambda_0|^2)T^*\Delta^2_{\lambda_0}T \leq (1-|\lambda_0|^2)\Delta^2_{\lambda_0} \\
&i.e.\ (1-|\lambda_0|^2)(I-\lambda_0 T^*)^{-1}T^*\Delta^2 T(1-\overline{\lambda_0}T)^{-1} \\
&\leq (1-|\lambda_0|^2)(I-\lambda_0 T^*)^{-1}\Delta^2(1-\overline{\lambda_0}T)^{-1},
\end{aligned}$$

by the assumption, we have $T^*\Delta^2 T \leq \Delta^2$, thus $T^*_{\lambda_0}\Delta^2_{\lambda_0}T_{\lambda_0} \leq \Delta^2_{\lambda_0}$. Also $\bigcap\limits_{n>0} T^n_{\lambda_0}H = (0)$.

Therefore, the conclusion follows from Theorem 2.2.4. ◻

Lemma 2.2.6. *$M(D(\mu)) \subseteq \mathcal{X}(D(\mu))$.*

Proof. Note that by Theorem 1.9 in [3, p74], we have

$$\int_{\mathbb{D}^-} D_\lambda(f)d\mu(\lambda) = \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)dA(z),$$

where $U_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right|\frac{d\mu(w)}{1-|w|^2} + \int_{\mathbb{T}} \frac{1-|z|^2}{|w-z|^2}d\mu(w)$.

If $\varphi \in M(D(\mu))$, then φ is bounded and $f\varphi' = (f\varphi)' - f'\varphi$. Hence the measure $|\varphi'|^2 U_\mu dA$ is a Carleson measure for $D(\mu)$. This property implies that $\varphi \in \mathcal{X}(D(\mu))$. ◻

Theorem 2.2.7. *Let $\mathcal{H}$ be a Hilbert space, suppose polynomials are multipliers and $\mathcal{H}$ has the following wandering subspace property: If $\mathcal{N}$ is a non-zero multiplier invariant subspace, then dim$\mathcal{N} \ominus z\mathcal{N} = 1$ and $[\mathcal{N} \ominus z\mathcal{N}] = \mathcal{N}$. Let k_0 denote the reproducing kernel for 0. If $\mathcal{M}$ is a multiplier invariant subspace such that 0 is not a common zero of all functions in $\mathcal{M}$, then*

$$\mathcal{M}^\perp = [P_{\mathcal{M}^\perp}k_0]_*.$$

Proof. Since $P_{\mathcal{M}^\perp}k_0 \in \mathcal{M}^\perp$ we have $[P_{\mathcal{M}^\perp}k_0]_* \subseteq \mathcal{M}^\perp$. For the reverse inclusion we let $\mathcal{N} = [P_{\mathcal{M}^\perp}k_0]_*^\perp$. Then $\mathcal{M} \subseteq \mathcal{N}$, and $\mathcal{N}$ is a nonzero multiplier invariant subspace. Thus by the hypothesis it suffices to show that $\mathcal{M} \ominus z\mathcal{M} = \mathcal{N} \ominus z\mathcal{N}$.

Since 0 is not common zero of all functions in $\mathcal{M}$ we have that $P_{\mathcal{M}}k_0$ is a basis for the 1-dimensional space $\mathcal{M} \ominus z\mathcal{M}$. We have to show that $P_{\mathcal{M}}k_0 \in \mathcal{N} \ominus z\mathcal{N}$. We clearly have $P_{\mathcal{M}}k_0 \in \mathcal{N}$, thus it suffices to show that $\langle P_{\mathcal{M}}k_0, zf\rangle = 0$ for all $f \in \mathcal{N}$. Since

$M_z^* k_0 = 0$ we have $M_z^* P_{\mathcal{M}} k_0 = -M_z^* P_{\mathcal{M}^\perp} k_0 \in \mathcal{N}^\perp$. Thus

$$\langle P_{\mathcal{M}} k_0, zf \rangle = \langle M_z^* P_{\mathcal{M}} k_0, f \rangle = \langle -M_z^* P_{\mathcal{M}^\perp} k_0, f \rangle = 0.$$

□

Proposition 2.2.8. *If $\mathcal{H}$ is a reproducing kernel Hilbert space on $\mathbb{D}$ with a complete Nevanlinna-Pick kernel k_λ, then $\forall \lambda_0 \in \mathbb{D}$, $P_{\mathcal{M}} k_{\lambda_0} \in M(\mathcal{H})$, where $\mathcal{M} \subseteq \mathcal{H}$ is a multiplier invariant subspace.*

Proof. If k_λ is a complete Nevanlinna-Pick kernel, then for any multiplier invariant subspace $\mathcal{M} \subseteq \mathcal{H}$, we have $l_\lambda(z) := \frac{P_{\mathcal{M}} k_\lambda(z)}{k_\lambda(z)}$ is positive definite. Then by a Theorem of McCullough and Trent ([45]), we have $\forall \lambda_0 \in \mathbb{D}$, $l_{\lambda_0} \in M(\mathcal{H})$. Also by [33, Lemma 2.2], we have $k_{\lambda_0} \in M(\mathcal{H})$, thus $P_{\mathcal{M}} k_{\lambda_0} = l_{\lambda_0} k_{\lambda_0} \in M(\mathcal{H})$, $\forall \lambda_0 \in \mathbb{D}$.

□

Proposition 2.2.9. *If $\varphi \in M(D)$, then so is $M_z^* \varphi$.*

Proof. Note that $M_z^* z^n = \frac{n+1}{n} z^{n-1} = z^{n-1} + \int_0^1 (tz)^{n-1} dt, n > 0$.

Let $Lf = \frac{f - f(0)}{z}$ be the backward shift, then $\forall f \in D, M_z^* f = Lf + \int_0^1 (Lf)_t dt$, where $f_t(z) := f(tz)$.

If $\varphi \in M(D)$, then $L\varphi \in M(D)$, also $\exists C$, such that $\forall f \in D$, $\|\varphi_t f\|_D \leq C \|\varphi f\|_D$, the result follows. □

Now we show that every nonzero M_z-invariant subspace $\mathcal{M}$ of $D(\mu)$ is the kernel of some Hankel operator with symbol $b \in \mathcal{X}(D(\mu))$.

Proposition 2.2.10. *Suppose $\mathcal{M} \in Lat(M_z, D(\mu)), \mathcal{M} \neq 0$, then there is a function $b \in M(D(\mu)) \cap \mathcal{M}^\perp$, such that $[b]_* = \mathcal{M}^\perp$ and $\ker H_b = \mathcal{M}$.*

Proof. Suppose $z_0 \notin Z(\mathcal{M})$, and $\varphi \in \mathcal{M} \ominus (z - z_0)\mathcal{M}$, $\|\varphi\| = 1$.

Note that $k_{z_0} = P_{\mathcal{M}}k_{z_0} + P_{\mathcal{M}^\perp}k_{z_0}$, $P_{\mathcal{M}}k_{z_0} = c\varphi$, where c is a constant, let $b = P_{\mathcal{M}^\perp}k_{z_0}$, we konw that by Shimorin's result ([67]), $D(\mu)$ has a complete Nevanlinna-Pick kernel, then by Proposition 2.2.8, we have $b \in M(D(\mu))$. We will show that $[b]_* = \mathcal{M}^\perp$.

Let $g = (M_z^* - \overline{z_0})\varphi$, then for any p polynomial, $\langle g, p\varphi\rangle = \langle \varphi, (z - z_0)p\varphi\rangle = 0$, which implies $g \in \mathcal{M}^\perp$, so $[g]_* \subseteq \mathcal{M}^\perp$.

Let $\mathcal{L} = [g]_*^\perp$, then $\mathcal{M} \subseteq \mathcal{L}$. Note that for any $f \in \mathcal{L}$, $\langle \varphi, (z - z_0)f\rangle = \langle g, f\rangle = 0$, by Theorem 2.2.5, $\mathcal{M} = [\varphi] = \mathcal{L}$, thus $[g]_* = \mathcal{M}^\perp$.

From $(M_z^* - \overline{z_0})b = (M_z^* - \overline{z_0})(k_{z_0} - c\varphi) = 0 - c(M_z^* - \overline{z_0})\varphi = -cg$, we get

$$\begin{aligned}\mathcal{M}^\perp &\supseteq Span\{b, (M_z^* - \overline{z_0})b, \ldots\} \supseteq Span\{g, (M_z^* - \overline{z_0})g, \ldots\} \\ &= [g]_* = \mathcal{M}^\perp,\end{aligned}$$

therefore $[b]_* = \mathcal{M}^\perp$, and $\ker H_b = \mathcal{M}$ by Lemma 2.2.2.

□

Remark 2.2.11. *(a) If $\mathcal{M} \in Lat(M_z, D(\mu)), \mathcal{M} \neq 0$ and $0 \notin Z(\mathcal{M})$, then $b := P_{\mathcal{M}^\perp}k_0 = 1 - \overline{\varphi(0)}\varphi$, where $\varphi \in \mathcal{M} \ominus z\mathcal{M}$ is the extremal function. Then the conclusion of Proposition 2.2.10 follows from Theorem 2.2.7.*

(b) If μ is the Lebesgue measure on $\mathbb{T}$, then $D(\mu) = D$. Note that by proposition 2.2.9, If $\varphi \in M(D)$, then $M_z^*\varphi \in M(D)$. In this case, we can let $b = (M_z^* - \overline{z_0})\varphi$ in Proposition 2.2.10.

Lemma 2.2.12. *Let $\mathcal{M}$ be a multiplier invariant subspace of $D(\mu)$, let $\overline{\mathcal{M}}^{\odot}$ be its closure in $D(\mu) \odot D(\mu)$. Let $\mathcal{N} = \overline{\mathcal{M}}^{\odot} \cap D(\mu)$, then $\mathcal{N}$ is closed in $D(\mu)$ and $\overline{\mathcal{N}}^{\odot} = \overline{\mathcal{M}}^{\odot}$.*

Proof. Let $f_n \in \mathcal{N}, f_n \to f$ in $D(\mu)$, then $f_n \in \overline{\mathcal{M}}^{\odot}$, and note that $\|f_n - f\|_* \leq \|f_n - f\|_{D(\mu)} \to 0$ as $n \to \infty$, thus $f \in \overline{\mathcal{M}}^{\odot}$, and so $f \in \mathcal{N}$.

Note that $\mathcal{M} \subseteq \overline{\mathcal{M}}^{\odot} \cap D(\mu) = \mathcal{N}$, we have $\overline{\mathcal{M}}^{\odot} \subseteq \overline{\mathcal{N}}^{\odot}$. On the other hand, let $f \in \overline{\mathcal{N}}^{\odot}$, there exist f_n's in $\mathcal{N}$, such that $f_n \to f$ in $D(\mu) \odot D(\mu)$ as $n \to \infty$, note that $f_n \in \overline{\mathcal{M}}^{\odot}$, thus $f \in \overline{\mathcal{M}}^{\odot}$. □

Corollary 2.2.13. *Let $\mathcal{M}, \mathcal{N}$ be multiplier invariant subspaces of $D(\mu)$, $\mathcal{M} \neq \mathcal{N}$, and let $\overline{\mathcal{M}}^{\odot}$ and $\overline{\mathcal{N}}^{\odot}$ be their closures in $D(\mu) \odot D(\mu)$. Then $\overline{\mathcal{M}}^{\odot} \neq \overline{\mathcal{N}}^{\odot}$.*

It follows that for any $\mathcal{M}$ we have $\mathcal{M} = \overline{\mathcal{M}}^{\odot} \cap D(\mu)$.

Proof. Without loss of generality we suppose that there is an $f \in \mathcal{N}$ such that $f \notin \mathcal{M}$. By Proposition 2.2.10 we pick $b \in \mathcal{X}(D(\mu))$ such that $\ker H_b = \mathcal{M}$. Then the functional that b defines in the dual of $D(\mu) \odot D(\mu)$ annihilates $\mathcal{M}$ and hence it annihilates $\overline{\mathcal{M}}^{\odot}$. However, since $f \in \mathcal{N} \setminus \ker H_b$ we have $H_b f \neq 0$. Then there is a $\varphi \in \mathrm{Hol}(\mathbb{D}^-)$ such that $\langle \varphi f, b \rangle_{D(\mu)} = \langle H_b f, \overline{\varphi} \rangle_{\overline{D(\mu)}} \neq 0$. Thus b does not annihilate all of $\overline{\mathcal{N}}^{\odot}$. □

Theorem 2.2.14. *Let $f \in D(\mu)$. Then f is cyclic in $D(\mu)$ if and only if f is cyclic in $D(\mu) \odot D(\mu)$.*

Proof. Since for any polynomial p, $\|pf - 1\|_* \leq \|pf - 1\|_{D(\mu)}$, it is clear that cyclic vectors in $D(\mu)$ are cyclic in $D(\mu) \odot D(\mu)$.

If f is not cyclic in $D(\mu)$, then we can take $\mathcal{M} = [f]$ and $N = D(\mu)$ and apply the previous Corollary to conclude that f is not cyclic in $D(\mu) \odot D(\mu)$. □

We see that in Proposition 2.2.10, for any $\mathcal{M} \in Lat(M_z, D(\mu)), \mathcal{M} \neq 0$, there is a $b \in M(D(\mu))$, such that $\ker H_b = \mathcal{M}$. But in general, for an Hilbert space $\mathcal{H}$, and $\mathcal{M} \in Lat(M_z, \mathcal{H})$, $\mathcal{M} \neq 0$, we may not find a function $b \in \mathcal{X}(\mathcal{H})$, such that $\ker H_b = \mathcal{M}$.

Example 2.2.15. Consider $H^2(\mathbb{D}^2)$, the Hardy space on the bidisc,

(i) If $\mathcal{M} = \{f \in H^2(\mathbb{D}^2) : f(0,0) = 0\} = [\{z,w\}] = zH^2(z) \oplus \sum_{n=1}^{\infty} w^n H^2(z)$, then $\mathcal{M}^\perp = \text{span}\{1\}$, where span$\{1\}$ is the space spanned by 1. Let $b = 1, H_b : H^2(\mathbb{D}^2) \to \overline{H^2(\mathbb{D}^2)}$, then $\ker H_1 = [1]_*^\perp = \{f \in H^2(\mathbb{D}^2) : \langle fg, 1\rangle = 0, \forall g \in M(H^\infty(\mathbb{D}^2))\} = \mathcal{M}$.

(ii) If $\mathcal{M} = z^2H^2(z) \oplus \sum_{n=1}^{\infty} w^n H^2(z)$, then $\mathcal{M}^\perp = \text{span}\{1, z\}$. Let $b = z$, then $\ker H_z = [z]_*^\perp = \mathcal{M}$.

Let $\mathcal{H}$ be a Hilbert space, $T_1, T_2 \in B(\mathcal{H})$. For a subset $\mathcal{E}$ of $\mathcal{H}$, we denote by $[\mathcal{E}]_{T_1,T_2}$ the smallest invariant subspace of $\mathcal{H}$ for both T_1 and T_2 containing $\mathcal{E}$. If $[\mathcal{E}]_{T_1,T_2} = \mathcal{H}$, then $\mathcal{E}$ is called a generating set for operators T_1 and T_2. The minimum number of elements in generating sets is called the rank of $\mathcal{H}$ for T_1, T_2, and we denote it by rank $_{\{T_1,T_2\}}\mathcal{H}$.

(iii) Let $\{\varphi_n(z)\}$ be a sequence of inner functions in $H^\infty(z)$ such that $\varphi_n(z)/\varphi_{n+1}(z) \in H^\infty(z)(n \geq 0)$, and functions in $\{\varphi_n(z)\}$ have no nonconstant common inner divisors. Let $\mathcal{M} = \sum_{n=0}^{\infty} \oplus w^n \varphi_n(z) H^2(z)$, then $\mathcal{M}^\perp = \sum_{n=0}^{\infty} \oplus w^n (H^2(z) \ominus \varphi_n(z)H^2(z))$.

When $\varphi_0(z)$ is a Blaschke product, it is shown in [38] that rank $_{\{M_z^*, M_w^*\}}$ $\mathcal{M}^\perp$ varies from 1 to ∞:

$$\text{rank }_{\{M_z^*, M_w^*\}}\mathcal{M}^\perp = \sup_{\alpha \in \mathbb{D}} \#\{n : \zeta_n(\alpha) = 0, n \geq 0\},$$

where $\zeta_n(z) = \frac{\varphi_n(z)}{\varphi_{n+1}(z)}$.

If $\zeta_n(z)$ is mutually prime (i.e., there are nonconstant common inner divisors of $\zeta_n(z)$ and $\zeta_i(z)$ with $n \neq i$), then rank $_{\{M_z^*, M_w^*\}}\mathcal{M}^\perp = 1$ (see [39]). It is also shown in [39] if $\{a_n\}$ is a positive sequence and $\{a_n\} \in l^1$, then $b = \sum_{n=0}^{\infty} \oplus a_n w^n M_z^* \varphi_n(z) \in M^\perp$, and $[b]_* = \mathcal{M}^\perp$. In this case, it is easy to see that $b \in H^\infty(\mathbb{D}^2) \subset BMOA(H^2(\mathbb{D}^2)) = \mathcal{X}(H^2(\mathbb{D}^2))$ (where $\mathcal{X}(H^2(\mathbb{D}^2))$ is the dual space of $H^2(\mathbb{D}^2) \odot H^2(\mathbb{D}^2)$), and $\ker H_b = [b]_*^\perp = \mathcal{M}$.

But if rank ${}_{\{M_z^*, M_w^*\}}\mathcal{M}^\perp \geq 2$, since $\ker H_b = [b]_*^\perp$, there is no $b \in \mathcal{X}(H^2(\mathbb{D}^2))$ such that $\ker H_b = \mathcal{M}$.

For the Bergman space $L_a^2(\mathbb{D})$, we have a similar result.

If $\mathcal{M} \in Lat(M_z, L_a^2(\mathbb{D}))$, we write

$$M_z = \begin{pmatrix} M_z|_{\mathcal{M}} & B \\ 0 & P_{\mathcal{M}^\perp} M_z|_{\mathcal{M}^\perp} \end{pmatrix}$$

with respect to the decomposition $L_a^2(\mathbb{D}) = \mathcal{M} \oplus \mathcal{M}^\perp$.

The following Lemma is in [5].

Lemma 2.2.16. *Let* $\mathcal{M} \in Lat(M_z, L_a^2(\mathbb{D})), \mathcal{M} \neq L_a^2(\mathbb{D}), \lambda \in \mathbb{D}$. *Then* $M_z^*|_{\mathcal{M}^\perp} - \lambda$ *is a semi-Fredholm operator, and* $\text{ind}\mathcal{M} = 1 - \text{ind}(M_z^*|_{\mathcal{M}^\perp} - \lambda)$.

Theorem 2.2.17. *There exists an* $\mathcal{M} \in Lat(M_z, L_a^2(\mathbb{D}))$*, such that* $\ker H_b \neq \mathcal{M}$ *for all* $b \in \mathcal{X}(L_a^2(\mathbb{D}))$.

Proof. Let $\mathcal{M} \in Lat(M_z, L_a^2(\mathbb{D}))$ with $\text{ind}\mathcal{M} \geq 3$ (see [8], [36]). Suppose that there exists a function $b \in \mathcal{X}(L_a^2(\mathbb{D}))$ such that $\ker H_b = \mathcal{M}$, then from Lemma 2.2.2, we have $\mathcal{M} = [b]_*^\perp$. Thus $\mathcal{M}^\perp = [b]_*$, and so $\text{ind}M_z^*|_{\mathcal{M}^\perp} \subseteq \{-1, 0, 1\}$, which contradicts Lemma 2.2.16. Therefore the conclusion follows. ☐

2.3 Index of Invariant Subspaces in $D \odot D$

Recall that the Dirichlet space D is the space of holomorphic functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ for which

$$\int_{\mathbb{D}} |f'|^2 \frac{dA}{\pi} = \sum_{n=1}^{\infty} n |\hat{f}(n)|^2 < \infty.$$

The norm on D is given by $\|f\|_D^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'|^2 \frac{dA}{\pi} = \sum_{n=0}^{\infty} (n+1)|\hat{f}(n)|^2$. When $f, g \in D$,

$$\begin{aligned} \langle f, g \rangle_D &= \langle f, g \rangle_{H^2} + \int_{\mathbb{D}} f' \overline{g'} \frac{dA}{\pi} \\ &= \sum_{n=0}^{\infty} (n+1) \hat{f}(n) \overline{\hat{g}(n)}. \end{aligned}$$

A positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for D, $\mu \in \mathrm{CM}(D)$, if there is a $C > 0$ such that for all $f \in D$,

$$\int_{\mathbb{D}} |f|^2 d\mu \leq C^2 \|f\|_D^2.$$

The smallest such C is the Carleson measure norm of μ.

In this section we show that every nonezero M_z-invariant subspace in $D \odot D$ has index one. A part of this is a joint work with Stefan Richter.

Definition 2.3.1. (*a*) *Let* $\mathbb{D}$ *be the open unit disc with boundary* $\mathbb{T}$ *and* $\mathbb{D}_e = \mathbb{C}_\infty \backslash \mathbb{D}^-$ *be the exterior disc, where* $\mathbb{D}^-$ *is the closure of* $\mathbb{D}$ *and* $\mathbb{C}_\infty$ *is the Riemann sphere.*

(*b*) *Let* $N(\mathbb{D})$ *and* $N(\mathbb{D}_e)$ *be the set of Nevanlinna functions* (*i.e., the quotients of two bounded analytic functions*) *on* $\mathbb{D}$ *and* $\mathbb{D}_e$ *respectively.*

(*c*) *For $p > 0$, let $H^p(\mathbb{D})$ be the Hardy space of analtytic functions f on $\mathbb{D}$ such that*

$$\|f\|_{H^p}^p = \sup_{r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \frac{|d\zeta|}{2\pi} < \infty.$$

(*d*) *A function $F \in N(\mathbb{D}_e)$ is called a pseudocontinuation of $f \in N(\mathbb{D})$ across $\mathbb{T}$, if $f(e^{it}) = F(e^{it})$ a.e..*

By Privalov's uniqueness Theorem ([41, p. 82-84]), whenever a pseudocontinuation exists, it is unique.

By Lemma 2.1 in [55], $D \odot D$ is a Banach space of analytic functions such that point evaluations at points of $\mathbb{D}$ are continuous, and it is clear that $D \odot D$ is M_z invariant. Moreover, we have

Lemma 2.3.2. *If $h \in D \odot D$ and $h(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, then there is a function $g \in D \odot D$ such that $(z-\lambda)g = h$.*

Proof. Suppose $h \in D \odot D$ and $h(\lambda) = 0$. Note that the norm of h in $D \odot D$ is defined by (**??**), and

$$fg = (\frac{f+g}{2})^2 - (\frac{f-g}{2})^2.$$

Thus we have $\forall \varepsilon > 0$, $\exists f_i's \in D$, such that $h = \sum_i f_i^2$, with $\|h\|_* \geq \sum_i \|f_i\|_D^2 - \varepsilon$. Then $h(\lambda) = \sum_i f_i^2(\lambda) = 0$ and

$$\frac{h}{z-\lambda} = \sum_i \frac{f_i^2 - f_i^2(\lambda)}{z-\lambda} = \sum_i \frac{f_i - f_i(\lambda)}{z-\lambda}(f_i + f_i(\lambda)).$$

Note that $\|\frac{f_i - f_i(\lambda)}{z-\lambda}\|_D \leq C_1(\lambda)\|f_i\|_D$, $\|(f_i + f_i(\lambda))\|_D \leq \|f_i\|_D + |f_i(\lambda)| \leq C_2(\lambda)\|f_i\|_D$ for some constants $C_1(\lambda)$ and $C_2(\lambda)$.

Thus $\|\frac{h}{z-\lambda}\|_* \leq C_1(\lambda)C_2(\lambda)(\|h\|_* + \varepsilon)$, which shows $\frac{h}{z-\lambda} \in D \odot D$. Let $g = \frac{h}{z-\lambda}$, then $(z-\lambda)g = h$. ∎

From Lemma 2.3.2, we see that for $\lambda \in \mathbb{D}$, there is a constant $C(\lambda)$ such that $\|(M_z - \lambda)h\|_* \geq C(\lambda)\|h\|_*$ for any $h \in D \odot D$. Thus for any M_z-invariant subspace $\mathcal{N}$ of $D \odot D$, we have $(M_z - \lambda)|_{\mathcal{N}}$ is a semi-Fredholm operator, and by the Fredholm theory, $\dim(\mathcal{N}/(z-\lambda)\mathcal{N})$ does not depend on $\lambda \in \mathbb{D}$ (see [48]). We then define

$$\operatorname{ind}(\mathcal{N}) = \dim(\mathcal{N}/z\mathcal{N}).$$

2.3.1 Duality

It is shown in [9] that $(D \odot D)^* = \mathcal{X}(D)$ (see also [20]), the dual pairing is with respect to the Dirichlet pairing, and

$$\mathcal{X}(D) = \{b \in Hol(\mathbb{D}) : |b'|^2 dA \text{ is a Carleson measure for } D\},$$

and the norm in $\mathcal{X}(D)$ is $\|b\|^2_{\mathcal{X}(D)} = |b(0)|^2 + \||b'|^2 dA\|_{CM(D)}$.

Let $\mathcal{Y} = \{(zb)' : b \in \mathcal{X}(D)\}$, the norm in $\mathcal{Y}$ is defined by

$$\|(zb)'\|_{\mathcal{Y}} = \|b\|_{\mathcal{X}(D)}.$$

Let $\mathcal{S} = \{(zh)' : h \in D \odot D\}$, the norm in $\mathcal{S}$ is defined by

$$\|(zh)'\|_{\mathcal{S}} = \|h\|_{D \odot D}.$$

By Corollary 1.4 in [9], $\mathcal{S} = (\partial D) \odot D = \partial(D \odot D)$.

Let $V : \mathcal{X}(D) \to \mathcal{Y}$, $Vb = (zb)'$, then V is an isometry. Similarly, $U : D \odot D \to \mathcal{S}$, $Uh = (zh)'$ is an isometry.

Lemma 2.3.3. *$\mathcal{Y}$ is the dual space of $D \odot D$ realized by the H^2 pairing, i.e., for $h \in D \odot D, (zb)' \in \mathcal{Y}$,*

$$\langle h, (zb)' \rangle_{(D \odot D, \mathcal{Y})} := \lim_{r \to 1^-} \int_0^{2\pi} h(re^{it}) \overline{(zb)'(re^{it})} \frac{dt}{2\pi}.$$

Proof. Note that $\langle p, q \rangle_D = \langle p, (zq)' \rangle_{H^2}$ for all polynomials p and q.

Let $(zb)' \in \mathcal{Y}$, then $b \in \mathcal{X}(D)$, and $\langle h, (zb)' \rangle_{(D \odot D, \mathcal{Y})} = \langle h, b \rangle_D$ for $h \in D \odot D$. Then

$$\begin{aligned} \left| \langle h, (zb)' \rangle_{(D \odot D, \mathcal{Y})} \right| &= \left| \langle h, b \rangle_D \right| \\ &\leq \|h\|_* \|b\|_{\mathcal{X}(D)} \\ &= \|h\|_* \|(zb)'\|_{\mathcal{Y}}, \end{aligned}$$

thus $(zb)' \in (D \odot D)^*$.

Let $T \in (D \odot D)^*$, then there is a $b \in \mathcal{X}(D)$, such that $T(h) = \langle h, b \rangle_D, h \in D \odot D$.

Then $(zb)' \in \mathcal{Y}$, and $T(h) = \langle h, (zb)' \rangle_{(D \odot D, \mathcal{Y})}, h \in D \odot D$. It is clear that the map $T \mapsto (zb)'$ from $(D \odot D)^*$ to $\mathcal{Y}$ is bounded and one-to-one, and so we can identify T with $(zb)' \in \mathcal{Y}$. □

Because of Lemma 2.3.3, we call $\mathcal{Y}$ the Cauchy dual of $D \odot D$, see [4, section 5] for a detailed discussion of Cauchy dual.

Let $L_a^2(\mathbb{D}) = \{f \in Hol(\mathbb{D}) : \|f\|^2_{L_a^2(\mathbb{D})} = \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty\}$ be the Bergman space, then we have

Lemma 2.3.4. *$\mathcal{Y}$ is the dual space of $\mathcal{S}$ realized by the L_a^2 pairing, i.e., for $(zh)' \in \mathcal{S}, (zb)' \in \mathcal{Y}$,*

$$\langle (zh)', (zb)' \rangle_{L_a^2(\mathbb{D})} := \int_{\mathbb{D}} (zh)'(z) \overline{(zb)'(z)} \frac{dA(z)}{\pi}.$$

Proof. Note that $\langle p, q \rangle_D = \langle (zp)', (zq)' \rangle_{L_a^2(\mathbb{D})}$ for all polynomials p and q.

Let $(zb)' \in \mathcal{Y}$, by the same arguments as in Lemma 2.3.3, we have $(zb)' \in \mathcal{S}^*$.

Let $T \in \mathcal{S}^*$, then by [75, Theorem 1], there is a $b \in \mathcal{X}(D)$ with $b(0) = 0$, such that $T((zh)') = \int_{\mathbb{D}} (zh)' \overline{b'(z)} \frac{dA(z)}{\pi}$, where $(zh)' \in \mathcal{S}$.

Let $c = \frac{b}{z}$, then $T((zh)') = \int_{\mathbb{D}} (zh)' \overline{b'(z)} \frac{dA(z)}{\pi} = \int_{\mathbb{D}} (zh)' \overline{(zc)'(z)} \frac{dA(z)}{\pi}$.

It is easy to see that $c \in \mathcal{X}(D)$, and the map $T \mapsto (zc)'$ from $\mathcal{S}^*$ to $\mathcal{Y}$ is bounded and one-to-one, and so we can identify T with $(zc)' \in \mathcal{Y}$. □

Let $h \in D \odot D, b \in \mathcal{X}(D)$, and let $H = (zh)', B = (zb)'$, then

$$\begin{aligned} \langle M_z h, b \rangle_D &= \langle h, M_z^* b \rangle_D = \langle h, (z M_z^* b)' \rangle_{(D \odot D, \mathcal{Y})} \\ &= \langle h, V M_z^* b \rangle_{(D \odot D, \mathcal{Y})} \\ &= \langle H, V M_z^* b \rangle_{L_a^2(\mathbb{D})}, \end{aligned}$$

and

$$\begin{aligned} \langle zh, b \rangle_D &= \langle zh, B \rangle_{(D \odot D, \mathcal{Y})} \\ &= \lim_{r \to 1^-} \int_0^{2\pi} (zh)(e^{it}) \overline{B(re^{it})} \frac{dt}{2\pi} \\ &= \langle h, LB \rangle_{(D \odot D, \mathcal{Y})} \\ &= \langle H, LB \rangle_{L_a^2(\mathbb{D})} \\ &= \langle H, LVb \rangle_{L_a^2(\mathbb{D})}, \end{aligned}$$

where $LB(z) = \frac{B(z) - B(0)}{z}$ is the backward shift, therefore $V M_z^* = LV$, which says $M_z^*|_{\mathcal{X}(D)}$ is isometrically isomorphic to $L|_{\mathcal{Y}}$.

Note that $\mathcal{X}(D)$ is continuously embedded in D, which implies $\mathcal{Y}$ is continuously embedded in $L_a^2(\mathbb{D})$. It is clear that $\mathcal{Y}$ is M_z invariant, and $1 \in \mathcal{Y}$.

Lemma 2.3.5. *$\mathcal{Y}$ satisfies the following three conditions:*

(i) $\forall \lambda \in \mathbb{D}, L_\lambda \mathcal{Y} \subseteq \mathcal{Y}$, *where* $L_\lambda f = \frac{f-f(\lambda)}{z-\lambda}$,

(ii) $\mathcal{Y} = \{g \in Hol(\mathbb{D}) : \exists C > 0,$ *for all* $f \in D, \int_{\mathbb{D}} |fg|^2 dA \leq C\|f\|_D^2\}$,

(iii) $\sigma(M_z|_{\mathcal{Y}}) = \mathbb{D}^-$.

Proof. (i) Since $D \odot D$ is M_z invariant and $VM_z^* = LV$, we have $L\mathcal{Y} \subseteq \mathcal{Y}$, where $Lf = \frac{f-f(0)}{z}$.

Note that for any $\lambda \in \mathbb{D}$, $M_z - \lambda$ is not onto on $D \odot D$, which implies $\sigma(M_z|_{D\odot D}) \supseteq \mathbb{D}^-$. $\forall h \in D \odot D, \lambda \in \mathbb{D}$, we have $\frac{1}{1-\bar{\lambda} z} h \in D \odot D$, thus $\sigma(M_z|_{D\odot D}) = \mathbb{D}^-$.

Therefore $\sigma(M_z^*|_{\mathcal{X}(D)}) = \mathbb{D}^-$, which implies $\forall \lambda \in \mathbb{D}, (I - \lambda L)^{-1}$ exists. An elementary calculation shows that $(I - \lambda L)^{-1} L f = \frac{f(z)-f(\lambda)}{z-\lambda} = L_\lambda f$. Thus $\forall f \in \mathcal{Y}, L_\lambda f \in \mathcal{Y}$.

(ii) It is clear that (ii) is true, we include a proof here for completeness.

Note that if $b \in \mathcal{X}(D)$, then $b \in D$ and there is a C such that for all $f \in D, \int_{\mathbb{D}} |fb|^2 dA \leq C\|f\|_D^2$. Thus if $g = (zb)' = b + zb' \in \mathcal{Y}$, applying the triangle inequality, there is a C such that for all $f \in D, \int_{\mathbb{D}} |fg|^2 dA \leq C\|f\|_D^2$.

On the other hand, suppose $g \in Hol(\mathbb{D})$ and there is a C such that for all $f \in D, \int_{\mathbb{D}} |fg|^2 dA \leq C\|f\|_D^2$. Then $g \in L_a^2(\mathbb{D})$.

Let $b \in Hol(\mathbb{D})$ be $(zb)' = g$, then $b \in D$ and $b' = \frac{g-b}{z}$. Let $f \in D$ with $f(0) = 0$, then

$$\begin{aligned}\int_{\mathbb{D}} |f\frac{g-b}{z}|^2 dA &\leq C\Big(\int_{\mathbb{D}} |\frac{f}{z} g|^2 dA + \int_{\mathbb{D}} |\frac{f}{z} b|^2 dA\Big) \\ &\leq C\|\frac{f}{z}\|_D^2 \\ &\leq C\|f\|_D^2,\end{aligned}$$

thus for $f \in D$,

$$\int_{\mathbb{D}} |fb'|^2 dA = \int_{\mathbb{D}} |f\frac{g-b}{z}|^2 dA \leq C\|f\|_D^2,$$

and so $b \in \mathcal{X}(D)$ which implies $g \in \mathcal{Y}$.

(iii) If $\lambda \in \mathbb{D}, g \in \mathcal{Y}$, then $|\frac{1}{1-\lambda z}| \leq \frac{1}{1-|\lambda|}$ for all $z \in \mathbb{D}$. Thus for $f \in D$,

$$\int_{\mathbb{D}} |f \frac{1}{1-\lambda z} g|^2 dA \leq \frac{C}{(1-|\lambda|)^2} \|f\|_D^2,$$

this implies $\frac{1}{1-\lambda z} b \in \mathcal{Y}$, therefore $\sigma(M_z|_{\mathcal{Y}}) \supseteq \mathbb{D}^-$.

Note that, $1 \in \mathcal{Y}$ implies $\forall \lambda \in \mathbb{D}, M_z - \lambda$ is not onto, thus $\sigma(M_z|_{\mathcal{Y}}) = \mathbb{D}^-$. □

From Lemma 2.3.5, we see that $\sigma(L|_{\mathcal{Y}}) = \mathbb{D}^-$, and so

$$\forall \mathcal{M} \in Lat(L, \mathcal{Y}), \sigma(L|_{\mathcal{M}}) \subseteq \mathbb{D}^-,$$

where $Lat(L, \mathcal{Y})$ is the lattice of L-invariant subspace of $\mathcal{Y}$.

2.3.2 Pseudocontinuation method 1

In this subsection we show that if $\mathcal{M}$ is a weak*-closed L-invariant subspace of $\mathcal{Y}$ with $\mathcal{M} \neq \mathcal{Y}$, then every $f \in \mathcal{M}$ is contained in the Nevanlinna class $N(\mathbb{D})$ and has a pseudocontinuation in $N(\mathbb{D}_e)$.

First, we introduce the Cauchy transform.

For $\lambda \in \mathbb{C}$, $\mu \in M(\mathbb{D}^-)$, the complex regular Borel measures with support in $\mathbb{D}^-$, define the Newtonian potential $U_\mu(\lambda) = \int_{\mathbb{D}^-} \frac{d|\mu|(z)}{|z-\lambda|}$. Then $U_\mu \in L^1_{loc}(dA)$. Let $E_\mu = \{r \in (0, \infty) : U_\mu(r\zeta) \in L^1(\mathbb{T}, |d\zeta|)\}$, then by Fubini's Theorem, E_μ has full measure in $\mathbb{R}^+$ (see [4, p. 239]).

Since $U_\mu \in L^1_{loc}(dA)$, we can (at least $[dA]-a.e.$) define the Cauchy transform of μ by $C_\mu(\lambda) = \int_{\mathbb{D}^-} \frac{d\mu(z)}{z-\lambda}$. It is clear that $C_\mu(\lambda)$ is analytic off the support of μ. In fact, it is well-known that the Cauchy transform of any measure with support in $\mathbb{T}$ is in $H^p(\mathbb{D})$ and $H^p(\mathbb{D}_e)$ for all $p < 1$ (see [28, p. 39]). Let $\mu \in M(\mathbb{D}^-)$, by considering the sweep of μ into $\mathbb{T}$, we see that $C_\mu \in H^p(\mathbb{D}_e)$ for any $0 < p < 1$. This also follows from the following Lemma. Let $C_\mu^+(\zeta)$ be the nontangential limit values of this function which exists a.e. $\zeta \in \mathbb{T}$.

The following Lemma is the Lemma 3.2 in [4].

Lemma 2.3.6. *Let $\mu \in M(\mathbb{D}^-)$. Then for each $0 < p < 1$, there is a constant $c_p > 0$ with*

$$\Big(\int_{|\lambda|=r} |C_\mu(\lambda)|^p \frac{|d\lambda|}{2\pi r}\Big)^{1/p} \le \frac{c_p}{r}\|\mu\|, \quad \forall r \in E_\mu.$$

Lemma 2.3.7. *The polynomials are weak* dense in $\mathcal{X}(D)$.*

Proof. Let $\mathcal{M}$ be the weak* closure of the polynomials in $\mathcal{X}(D)$. Let $f \in D \odot D$, $f \in {}^\perp\mathcal{M} = \{g \in D \odot D, \langle g, b\rangle_D = 0, \forall b \in \mathcal{M}\}$, and suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $0 = \langle f, z^i\rangle_D = (i+1)a_i, i = 0, 1, \cdots$, thus $f = 0$ and so $\mathcal{M} = \mathcal{X}(D)$. $\square$

Recall $\mathcal{S} = \{(zh)' : h \in D \odot D\}$, and the norm in $\mathcal{S}$ is defined by $\|(zh)'\|_{\mathcal{S}} = \|h\|_{D\odot D}$.

Lemma 2.3.8. *Let $f \in \mathcal{Y}, \phi \in \mathcal{S}$, then $\phi, f\phi \in L^1(\mathbb{D})$, where $L^1(\mathbb{D}) = \{f : \int_{\mathbb{D}} |f| dA < \infty\}$.*

Proof. Let $\phi \in \mathcal{S}$, then there exists an $h \in D \odot D$, such that $\phi = (zh)'$. Suppose $h = \sum_{i=1}^{\infty} f_i^2, f_i \in D$ with $\sum_{i=1}^{\infty} \|f_i\|_D^2 < \infty$. Since $f_i, f_i' \in L_a^2(\mathbb{D})$, we have $\phi = (zh)' \in L^1(\mathbb{D})$.

By Lemma 2.3.5, if $f \in \mathcal{Y}$, we have

$$\begin{aligned}\int_{\mathbb{D}} |f\phi| dA &= \int_{\mathbb{D}} |f(zh)'| dA = \int_{\mathbb{D}} |f \sum_{i=1}^{\infty} (f_i^2 + 2z f_i f_i')| dA \\ &\leq C \sum_{i=1}^{\infty} \Big(\int_{\mathbb{D}} |f f_i^2| dA + \int_{\mathbb{D}} |f f_i f_i'| dA \Big) \\ &\leq C \sum_{i=1}^{\infty} \Big[(\int_{\mathbb{D}} |f f_i|^2 dA)^{1/2} (\int_{\mathbb{D}} |f_i|^2 dA)^{1/2} \\ &\qquad\qquad + (\int_{\mathbb{D}} |f f_i|^2 dA)^{1/2} (\int_{\mathbb{D}} |f_i'|^2 dA)^{1/2} \Big] \\ &\leq C \sum_{i=1}^{\infty} \|f_i\|_D^2 < \infty.\end{aligned}$$

□

Proposition 2.3.9. *Let $f \in \mathcal{Y}, \phi \in \mathcal{S}$. Let $d\mu_1 = f\overline{\phi} dA, d\mu_2 = \overline{\phi} dA$, where dA is the area measure on $\mathbb{D}$. Then $C_{\mu_i}(\lambda) = \int_{\mathbb{D}} \frac{d\mu_i(z)}{z-\lambda}$ is continuous on $\mathbb{D}$, $i = 1, 2$.*

Proof. Note that if $g \in L^\infty(dA)$ has compact support, then $\int_{\mathbb{D}} \frac{g(z)}{z-\lambda} dA(z)$ is continuous on $\mathbb{D}$.

Let $\lambda \in \mathbb{D}$, suppose $|\lambda| < r < 1$, then

$$\begin{aligned}\int_{\mathbb{D}} \frac{d\mu_1(z)}{z-\lambda} &= \int_{\mathbb{D}} \frac{f(z)\overline{\phi(z)}}{z-\lambda} dA(z) \\ &= \int_{|z|<r} \frac{f(z)\overline{\phi(z)}}{z-\lambda} dA(z) + \int_{r<|z|<1} \frac{f(z)\overline{\phi(z)}}{z-\lambda} dA(z) \\ &:= I_1 + I_2,\end{aligned}$$

I_1 is continuous at λ since $f\overline{\phi}\chi_{|z|<r} \in L^\infty(dA)$. By Lemma 2.3.8, $f\phi \in L^1(\mathbb{D})$, we have I_2 is also continuous at λ. Thus $C_{\mu_1}(\lambda) = \int_{\mathbb{D}} \frac{f(z)\overline{\phi(z)}}{z-\lambda} dA(z)$ is continuous on $\mathbb{D}$. Similarly, $C_{\mu_2}(\lambda) = \int_{\mathbb{D}} \frac{\overline{\phi(z)}}{z-\lambda} dA(z)$ is continuous on $\mathbb{D}$.

□

We now prove the main Theorem about the pseudocontinuations of the functions in the proper weak*-closed L-invariant subspace of $\mathcal{Y}$. The proof is similar to Theorem 4.8 in [4], the main difference being that we need to work with weak* continuous linear functionals instead of continuous functionals. Also, there are certain simplifications in our case, because the authors in [4] consider more general measures.

Theorem 2.3.10. *Let $\mathcal{M}$ be a weak*-closed L-invariant subspace of $\mathcal{Y}$ with $\mathcal{M} \neq \mathcal{Y}$. Then every $f \in \mathcal{M}$ is contained in the Nevanlinna class $N(\mathbb{D})$ and has a pseudocontinuation in $N(\mathbb{D}_e)$.*

Proof. Note that ${}^{\perp}\mathcal{M} = \{h \in D \odot D : \langle h, f\rangle_{(D\odot D,\mathcal{Y})} = 0, \forall f \in \mathcal{M}\}$. Let

$$\mathcal{A} = U({}^{\perp}\mathcal{M}) = \{\phi \in \mathcal{S} : \phi = (zh)', h \in {}^{\perp}\mathcal{M}\}, \tag{2.3.1}$$

where $U : D \odot D \to \mathcal{S}$, $Uh = (zh)'$ is isometric. Then ${}^{\perp}\mathcal{M} = U^{-1}\mathcal{A}$, and for $\phi \in \mathcal{A}$, there is an $h \in {}^{\perp}\mathcal{M}$, such that $\phi = (zh)'$,

$$\langle \phi, f\rangle_{L^2_a(\mathbb{D})} = \langle h, f\rangle_{(D\odot D,\mathcal{Y})} = 0, \quad f \in \mathcal{M}. \tag{2.3.2}$$

Let $f \in \mathcal{M}$, note that when $|\lambda| < 1$, $\frac{f-f(\lambda)}{z-\lambda} = (I - \lambda L|_{\mathcal{M}})^{-1}L|_{\mathcal{M}}f \in \mathcal{M}$. Thus for nonzero $\phi \in \mathcal{A}$, by (2.3.2), we have $\langle \frac{f-f(\lambda)}{z-\lambda}, \phi\rangle_{L^2_a(\mathbb{D})} = 0$ for all $|\lambda| < 1$.

By Proposition 2.3.9, we have for each $r \in (0, 1)$ and $|\zeta| = 1$

$$f(r\zeta)\int_{\mathbb{D}} \frac{\overline{\phi}}{z - r\zeta} dA(z) = \int_{\mathbb{D}} \frac{f\overline{\phi}}{z - r\zeta} dA(z). \tag{2.3.3}$$

For $r \in (0, 1)$, $|\lambda| < 1$ define $\Phi_r(\lambda) = \int_{|z|<r} \frac{\phi(z)}{\overline{z}-r/\lambda} dA(z)$.

By Lemma 2.3.8, $f \in \mathcal{M}, \phi \in \mathcal{S}$ implies $f\phi, \phi \in L^1(\mathbb{D})$, and by Lemma 2.3.6, $\Phi_r \in H^p(\mathbb{D})$ for all $0 < p < 1$ and

$$\|\Phi_r\|_{H^p} \leq c_p \int_{\mathbb{D}} |\phi| dA, \quad \forall r \in (0,1). \tag{2.3.4}$$

Moreover $\Phi_r(\lambda) \to \Phi(\lambda) := \int_{\mathbb{D}} \frac{\phi(z)}{\overline{z} - 1/\lambda} dA(z)$ uniformly on compact sets as $r \to 1^-$. By Fatou's Lemma and (2.3.4) $\forall s < 1$

$$\int_0^{2\pi} |\Phi(se^{it})|^p \frac{dt}{2\pi} \leq \liminf_{r \to 1} \int_0^{2\pi} |\Phi_r(se^{it})|^p \frac{dt}{2\pi} \leq \Big(c_p \int_{\mathbb{D}} |\phi| dA\Big)^p, \tag{2.3.5}$$

thus $\Phi \in H^p(\mathbb{D})$.

Using power series we have $\Phi(\lambda) = -\lambda \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{D}} \overline{z}^n \phi(z) dA(z)$. Since the polynomials are weak* dense in $\mathcal{X}(D)$ and $\langle h, \frac{z^n}{n+1} \rangle_D = \langle \phi, z^n \rangle_{L^2_a(\mathbb{D})}$, we have $\int_{\mathbb{D}} \overline{z}^n \phi(z) dA(z) = 0, \forall n$ implies $\phi = 0$. But $\mathcal{M} \neq \mathcal{Y}$, we conclude that $\Phi \not\equiv 0$ whenever $\phi \neq 0$.

Note that if $s < 1, |z| < r < 1$, then $\left| \frac{\overline{z} - r/\zeta}{\overline{z} - r/(s\zeta)} \right| \leq 2$. An application of the dominated convergence theorem shows that the non-tangential boundary values of Φ_r are given $[|d\zeta|]$-a.e. by $\Phi_r(\zeta) = \int_{|z|<r} \frac{\phi(z)}{\overline{z} - r\overline{\zeta}} dA(z)$.

Now we show that for any $0 < p < 1$

$$\int_{|\zeta|=1} |f(r\zeta)\Phi_r(\zeta)|^p |d\zeta| \tag{2.3.6}$$

is uniformly bounded for $r \in (0,1)$.

Note that since ϕ is analytic, we have $\int_{|z|>r} \frac{\overline{\phi(z)}}{z - r\zeta} dA(z) = 0$, then

$$|\Phi_r(\zeta)| = |\overline{\Phi_r(\zeta)}| = \Big| \int_{|z|<r} \frac{\overline{\phi(z)}}{z - r\zeta} dA(z) \Big| = \Big| \int_{\mathbb{D}} \frac{\overline{\phi(z)}}{z - r\zeta} dA(z) \Big|.$$

By (2.3.3), we get for any $0 < p < 1$

$$|f(r\zeta)\Phi_r(\zeta)|^p = \left|f(r\zeta)\int_{\mathbb{D}}\frac{\overline{\phi(z)}}{z-r\zeta}dA(z)\right|^p$$
$$= \left|\int_{\mathbb{D}}\frac{f(z)\overline{\phi(z)}}{z-r\zeta}dA(z)\right|^p,$$

thus for any $0 < p < 1$

$$\int_{|\zeta|=1}|f(r\zeta)\Phi_r(\zeta)|^p|d\zeta| = \int_{|\zeta|=1}\left|\int_{\mathbb{D}}\frac{f(z)\overline{\phi(z)}}{z-r\zeta}dA(z)\right|^p|d\zeta|$$
$$\leq c_p\int_{\mathbb{D}}|f\phi|dA \quad \text{by Lemma 2.3.6},$$

therefore (2.3.6) is uniformly bounded for $0 < p < 1$ and $r \in (0,1)$.

Now we can show that $f \in N(\mathbb{D})$.

The uniform boundedness of (2.3.6) implies that $f(r\lambda)\Phi_r(\lambda)$ is uniformly bounded on compact subsets of $\mathbb{D}$ (see e.g [28, p. 36]). Hence by a normal family argument $f(r_n\lambda)\Phi_{r_n}(\lambda) \to H(\lambda)$ uniformly on compact sets for some sequence $r_n \to 1$ with $r_n \in (0,1)$. By Fatou's Lemma, $H \in H^p(\mathbb{D})$. Note that $\Phi_r(\lambda) \to \Phi(\lambda)$ uniformly on compact sets as $r \to 1^-$ and $\Phi \in H^p(\mathbb{D}), \Phi \not\equiv 0$, we have $f\Phi = H$ with $\Phi \not\equiv 0$ and so $f = H/\Phi$ is a Nevanlinna function.

Now, we show that f has a pseudocontinuation across $\mathbb{T}$.

Let $d\mu_1(z) = f(z)\overline{\phi}(z)dA(z), d\mu_2(z) = \overline{\phi}(z)dA(z)$, where $\phi \in \mathcal{A}$, then by Corollary 3.4 in [4], $C_{\mu_i}(r\zeta) \to C^+_{\mu_i}(\zeta)$ in measure $[|d\zeta|]$ as $r \to 1^-$, $i = 1, 2$, where $C^+_{\mu_i}(\zeta)$ are the nontangential limit values of $C_{\mu_i}|_{\mathbb{D}_e}, i = 1, 2$.

Note that by (2.3.3), for $|\lambda| < 1$, $f(\lambda)\int_{\mathbb{D}}\frac{\overline{\phi}}{z-\lambda}dA = \int_{\mathbb{D}}\frac{f\overline{\phi}}{z-\lambda}dA$, therefore $C^+_{\mu_2}(\zeta)f(r\zeta) \to C^+_{\mu_1}(\zeta), a.e.[|d\zeta|]$, as $r \to 1^-$.

When $|\lambda| > 1$, $C_{\mu_2}(\lambda) = \int_{\mathbb{D}} \frac{\overline{\phi}}{z-\lambda} dA = \frac{-1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \int_{\mathbb{D}} z^n \overline{\phi(z)} dA(z)$, then $C_{\mu_2}(\lambda) \not\equiv 0$ in $\mathbb{D}_e$. Note that by Lemma 2.3.6, $C_{\mu_i} \in H^p(\mathbb{D}_e), i = 1, 2$ for any $0 < p < 1$, thus $\frac{C_{\mu_1}(\lambda)}{C_{\mu_2}(\lambda)}|_{\mathbb{D}_e}$ is a pseudocontinuation of f. □

2.3.3 Pseudocontinuation method 2

In this subsection we use the idea in [54] to show that if $\mathcal{M}$ is a weak*-closed L-invariant subspace of $\mathcal{Y}$ with $\mathcal{M} \neq \mathcal{Y}$, then every $f \in \mathcal{M}$ is contained in the Nevanlinna class $N(\mathbb{D})$ and has a pseudocontinuation in $N(\mathbb{D}_e)$.

Lemma 2.3.11. *Let $g \in Hol(\mathbb{D})$, then $|g'|^2 dA$ is a Carleson measure for D if and only if*

$$\int_{\mathbb{T}} |f(z)|^2 D_z(g) \frac{|dz|}{2\pi} \leq C\|f\|^2, \forall f \in D,$$

where $D_z(g) = \int_{\mathbb{T}} \frac{|g(z)-g(w)|^2}{|z-w|^2} \frac{|dw|}{2\pi}$ is the local Dirichlet integral of g at z.

Proof. Note that $D_z(g) = \int_{\mathbb{D}} |g'(w)|^2 \frac{1-|w|^2}{|1-\overline{z}w|^2} \frac{dA(w)}{\pi}$ (see [52]), we have

$$\begin{aligned}
&\int_{\mathbb{T}} |f(z)|^2 D_z(g) \frac{|dz|}{2\pi} - \int_{\mathbb{D}} |f(w)|^2 |g'(w)|^2 \frac{dA(w)}{\pi} \\
&= \int_{\mathbb{D}} |g'(w)|^2 (\int_{\mathbb{T}} |f(z)|^2 \frac{1-|w|^2}{|1-\overline{z}w|^2} \frac{|dz|}{2\pi} - |f(w)|^2) \frac{dA(w)}{\pi} \\
&= \int_{\mathbb{D}} |g'(w)|^2 (P_w(|f|^2) - |f(w)|^2) \frac{dA(w)}{\pi} \\
&\leq \|f\|_D^2 \|g\|_D^2,
\end{aligned}$$

where in the second to the last inequality we used $f \in D$ implies $f \in BMOA$, the conclusion follows. □

Lemma 2.3.12. *Let $h \in D \odot D, b \in \mathcal{X}(D)$, then $D_\zeta(h, b) \in L^1(\mathbb{T})$ and*

$$\int_{\mathbb{T}} D_\zeta(h, b) \frac{|d\zeta|}{2\pi} = \int_{\mathbb{D}} h'(z) \overline{b'(z)} \frac{dA(z)}{\pi},$$

where $D_\zeta(h,b) = \int_{\mathbb{T}} \frac{(h(w)-h(\zeta))\overline{(b(w)-b(\zeta))}}{|\zeta-w|^2}\frac{|dw|}{2\pi}$.

Therefore $\langle h,b\rangle_D = \int_{\mathbb{T}} h(z)\overline{b(z)}\frac{|dz|}{2\pi} + \int_{\mathbb{T}} D_\zeta(h,b)\frac{|d\zeta|}{2\pi}$.

Proof. Let $\mathcal{P} = \{p : p \text{ is a polynomial}\}$, then $\mathcal{P}$ is dense in D and $\mathcal{P} \subseteq Hol(\mathbb{D}^-)$. By Theorem 1.3 in [55], we have $\mathcal{P}\widehat{\odot}\mathcal{P}$ is dense in $D \odot D$.

If $h \in \mathcal{P}\widehat{\odot}\mathcal{P}, h = \sum_{i=1}^{n} f_i g_i$, then

$$\begin{aligned}
D_\zeta(h,b) &= \int_{\mathbb{T}} \frac{(h(w)-h(\zeta))\overline{(b(w)-b(\zeta))}}{|\zeta-w|^2}\frac{|dw|}{2\pi}\\
&= \int_{\mathbb{T}} \sum_{i=1}^{n} \frac{(f_i(w)g_i(w)-f_i(\zeta)g_i(\zeta))\overline{(b(w)-b(\zeta))}}{|\zeta-w|^2}\frac{|dw|}{2\pi}\\
&= \sum_{i=1}^{n}\Big(\int_{\mathbb{T}} f_i(w)\frac{(g_i(w)-g_i(\zeta))\overline{(b(w)-b(\zeta))}}{|\zeta-w|^2}\frac{|dw|}{2\pi}\\
&\qquad + \int_{\mathbb{T}} g_i(\zeta)\frac{(f_i(w)-f_i(\zeta))\overline{(b(w)-b(\zeta))}}{|\zeta-w|^2}\frac{|dw|}{2\pi}\Big)
\end{aligned}$$

If $b \in \mathcal{X}(D)$, then apply Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{T}} |D_\zeta(h,b)|\frac{|d\zeta|}{2\pi}\\
&\le \sum_{i=1}^{n}\Big(\int_{\mathbb{T}} |f_i(w)|(D_w(b))^{\frac12}(D_w(g_i))^{\frac12}\frac{|dw|}{2\pi}\\
&\qquad + \int_{\mathbb{T}} |g_i(\zeta)|(D_\zeta(b))^{\frac12}(D_\zeta(f_i))^{\frac12}\frac{|dw|}{2\pi}\Big)\\
&\le \sum_{i=1}^{n}\Big[\Big(\int_{\mathbb{T}} |f_i(w)|^2 D_w(b)\frac{|dw|}{2\pi}\Big)^{\frac12}\Big(\int_{\mathbb{T}} D_w(g_i)\frac{|dw|}{2\pi}\Big)^{\frac12}\\
&+ \Big(\int_{\mathbb{T}} |g_i(\zeta)|^2 D_\zeta(b)\frac{|d\zeta|}{2\pi}\Big)^{\frac12}\Big(\int_{\mathbb{T}} D_\zeta(f_i)\frac{|d\zeta|}{2\pi}\Big)^{\frac12}\Big]\\
&\le \sum_{i=1}^{n} C\|f_i\|_D\|g_i\|_D,
\end{aligned}$$

where in the last inequality we used Lemma 2.3.11.

For any $h \in D \odot D$, using Fatou's Lemma, we have $\int_{\mathbb{T}} |D_\zeta(h,b)| \frac{|d\zeta|}{2\pi} < \infty$.

Note that

$$\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathbb{T}} f_i(w) \frac{(g_i(w)-g_i(\zeta))\overline{(b(w)-b(\zeta))}}{|\zeta - w|^2} \frac{|dw|}{2\pi}\frac{|d\zeta|}{2\pi} \\
&= \int_{\mathbb{T}} f_i(w) \int_{\mathbb{D}} g_i'(z)\overline{b'(z)} \frac{1-|z|^2}{|1-\overline{w}z|^2} \frac{dA(z)}{\pi}\frac{|dw|}{2\pi} \\
&= \int_{\mathbb{D}} g_i'(z)\overline{b'(z)} \int_{\mathbb{T}} f_i(w) \frac{1-|z|^2}{|1-\overline{w}z|^2} \frac{|dw|}{2\pi}\frac{dA(z)}{\pi} \\
&= \int_{\mathbb{D}} f_i(z) g_i'(z)\overline{b'(z)} \frac{dA(z)}{\pi},
\end{aligned}$$

we have for any $h \in \mathcal{P}\widehat{\odot}\mathcal{P}$, $\int_{\mathbb{T}} D_\zeta(h,b)\frac{|d\zeta|}{2\pi} = \int_{\mathbb{D}} h'(z)\overline{b'(z)}\frac{dA(z)}{\pi}$. Thus, $\int_{\mathbb{T}} D_\zeta(h,b)\frac{|d\zeta|}{2\pi} = \int_{\mathbb{D}} h'(z)\overline{b'(z)}\frac{dA(z)}{\pi}$ holds for any $h \in D \odot D$ and $b \in \mathcal{X}(D)$. □

Lemma 2.3.13. *Let $h \in D \odot D, b \in \mathcal{X}(D)$, then*

$$\sup_{0<r<1} \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \frac{\overline{re^{it}}z}{(1-\overline{re^{it}}z)^2} (h(z) - h(re^{it}))\overline{(b(z)-b(re^{it}))} \frac{|dz|}{2\pi} \right| \frac{dt}{2\pi} < \infty.$$

Proof. For $r \in (0,1)$, let

$$I_r = \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \frac{\overline{re^{it}}z}{(1-\overline{re^{it}}z)^2} (h(z) - h(re^{it}))\overline{(b(z)-b(re^{it}))} \frac{|dz|}{2\pi} \right| \frac{dt}{2\pi}.$$

Suppose $h = \sum\limits_{j=1}^{\infty} f_j g_j$, where $f_j, g_j \in D$, then

$$\begin{aligned}
I_r = & \int_{\mathbb{T}} | \int_{\mathbb{T}} \frac{\overline{re^{it}z}}{(1-\overline{re^{it}}z)^2} \sum_{j=1}^{\infty} (f_j(z)g_j(z) - f_j(re^{it})g_j(re^{it})) \cdot \\
& \qquad \overline{(b(z)-b(re^{it}))} \frac{|dz|}{2\pi} | \frac{dt}{2\pi} \\
= & \int_{\mathbb{T}} | \int_{\mathbb{T}} \frac{\overline{re^{it}z}}{(1-\overline{re^{it}}z)^2} \sum_{j=1}^{\infty} \Big[f_j(z)(g_j(z) - g_j(re^{it})) \cdot \\
& \qquad \overline{(b(z)-b(re^{it}))} + g_j(re^{it})(f_j(z) - f_j(re^{it})) \cdot \\
& \qquad \overline{(b(z)-b(re^{it}))} \Big] \frac{|dz|}{2\pi} | \frac{dt}{2\pi} \\
\leq & \sum_{j=1}^{\infty} \Big[\int_{\mathbb{T}} |f_j(z)| (\int_{\mathbb{T}} | \frac{g_j(z)-g_j(re^{it})}{z-re^{it}} |^2 \frac{dt}{2\pi})^{\frac{1}{2}} \\
& \qquad (\int_{\mathbb{T}} | \frac{b(z)-b(re^{it})}{z-re^{it}} |^2 \frac{dt}{2\pi})^{\frac{1}{2}} \frac{|dz|}{2\pi} \\
& + \int_{\mathbb{T}} |g_j(re^{it})| (\int_{\mathbb{T}} | \frac{f_j(z)-f_j(re^{it})}{z-re^{it}} |^2 \frac{|dz|}{2\pi})^{\frac{1}{2}} \\
& \qquad (\int_{\mathbb{T}} | \frac{b(z)-b(re^{it})}{z-re^{it}} |^2 \frac{|dz|}{2\pi})^{\frac{1}{2}} \frac{dt}{2\pi} \Big] \\
\leq & \sum_{j=1}^{\infty} \Big[(\int_{\mathbb{T}} |f_j(z)|^2 \int_{\mathbb{T}} | \frac{b(z)-b(re^{it})}{z-re^{it}} |^2 \frac{dt}{2\pi} \frac{|dz|}{2\pi})^{\frac{1}{2}} \\
& \qquad (\int_{\mathbb{T}} \int_{\mathbb{T}} | \frac{g_j(z)-g_j(re^{it})}{z-re^{it}} |^2 \frac{dt}{2\pi} \frac{|dz|}{2\pi})^{\frac{1}{2}} \\
& + (\int_{\mathbb{T}} |g_j(re^{it})|^2 \int_{\mathbb{T}} | \frac{b(z)-b(re^{it})}{z-re^{it}} |^2 \frac{|dz|}{2\pi} \frac{dt}{2\pi})^{\frac{1}{2}} \\
& \qquad (\int_{\mathbb{T}} \int_{\mathbb{T}} | \frac{f_j(z)-f_j(re^{it})}{z-re^{it}} |^2 \frac{|dz|}{2\pi} \frac{dt}{2\pi})^{\frac{1}{2}} \Big],
\end{aligned}$$

note that $b(z)$ exists a.e. on $\mathbb{T}$, and for a.e. $z \in \mathbb{T}$, $\frac{b(z)-b(w)}{z-w} \in H^2(\mathbb{D})$, thus

$$\int_{\mathbb{T}} | \frac{b(z)-b(re^{it})}{z-re^{it}} |^2 \frac{dt}{2\pi} \leq \int_{\mathbb{T}} | \frac{b(z)-b(\zeta)}{z-\zeta} |^2 \frac{|d\zeta|}{2\pi} = D_z(b).$$

Also by Lemma 3.3 in [52], for any $f \in H^2(\mathbb{D}), \lambda \in \mathbb{D}$,

$$\|\frac{f(z)-f(\lambda)}{z-\lambda}\|_{H^2}^2 \leq CD_\zeta(f), \quad \text{where } \zeta = \frac{\lambda}{|\lambda|}.$$

Thus

$$\begin{aligned} I_r &\leq \sum_{j=1}^{\infty} \Big[\Big(\int_{\mathbb{T}} |f_j(z)|^2 D_z(b) \frac{|dz|}{2\pi}\Big)^{\frac{1}{2}} \|g_j\|_D + \\ &\qquad\qquad C\Big(\int_{\mathbb{T}} |g_j(re^{it})|^2 D_{e^{it}}(b) \frac{dt}{2\pi}\Big)^{\frac{1}{2}} \|f_j\|_D \Big] \\ &\leq \sum_{j=1}^{\infty} \Big(C\|f_j\|_D \|g_j\|_D + C\|g_{j,r}\|_D \|f_j\|_D \Big) \\ &\leq \sum_{j=1}^{\infty} C\|f_j\|_D \|g_j\|_D, \end{aligned}$$

where in the second to the last inequality we used Lemma 2.3.11, and $g_{j,r}(z) = g_j(rz)$. Therefore $\sup_{0<r<1} I_r < \infty$. □

Using Lemma 2.3.12, we have the following corresponding Lemma as the Lemma 2.1 in [54].

Lemma 2.3.14. *Let $h \in D \odot D, b \in \mathcal{X}(D), \lambda \in \mathbb{D}$, and $\alpha, \beta \in \mathbb{C}$. Then*

$$\begin{aligned} \langle \frac{1}{1-\overline{\lambda}z} h, b \rangle_D = &\int_{\mathbb{T}} \frac{1}{1-\overline{\lambda}z} (h(z)\overline{b(z)} + D_z(h,b)) \frac{|dz|}{2\pi} \\ &+ \int_{\mathbb{T}} \frac{\overline{\lambda}z}{(1-\overline{\lambda}z)^2} h(z)\overline{b(z)} \frac{|dz|}{2\pi}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{T}} \frac{\overline{\lambda}z}{(1-\overline{\lambda}z)^2} h(z)\overline{b(z)} \frac{|dz|}{2\pi} = &\int_{\mathbb{T}} \frac{\overline{\lambda}z}{(1-\overline{\lambda}z)^2} (h(z)-\alpha)\overline{(b(z)-\beta)} \frac{|dz|}{2\pi} \\ &+ \alpha\overline{\lambda b'(\lambda)}. \end{aligned}$$

Now we can prove the following pseudocontinuation Theorem.

Theorem 2.3.15. *Let* $\mathcal{N} \in Lat(M_z, D \odot D), \mathcal{N} \neq 0$ *and* $b \in \mathcal{N}^{\perp} \subset \mathcal{X}(D)$, *then*

(a) $(zb)' \in N(\mathbb{D})$. *In fact,* $\forall h \in D \odot D, h(zb)' \in H^p(\mathbb{D})$ *for any* $p < 1$,

(b) $(zb)'$ *has a pseudocontinuation* B *across* $\mathbb{T}$, $B|_{\mathbb{D}_e} \in N(\mathbb{D}_e)$.

Proof. Note that by Theorem 3.1 in [55], $D \odot D \subseteq D_\alpha \subseteq H^2(\mathbb{D})$, and $\mathcal{X}(D) \subseteq D$. Apply Lemma 2.3.12 and Lemma 2.3.13, using the calculation in Theorem 2.2 of [54], the result follows. We sketch an outline here.

(a) Let $h \in \mathcal{N}$. Then $hb \in H^1$, thus it is enough to show that the function $h(\lambda)\lambda b'(\lambda), \lambda \in \mathbb{D}$, is in H^p for every $0 < p < 1$.

Since $b \in \mathcal{N}^{\perp}$, we have by Lemma 2.3.14, for $\lambda \in D$,

$$\begin{aligned} 0 &= \langle \frac{1}{1-\overline{\lambda} z} h, b \rangle_D \\ &= \int_{\mathbb{T}} \frac{1}{1-\overline{\lambda} z}(h(z)\overline{b(z)} + D_z(h,b)) \frac{|dz|}{2\pi} + \int_{\mathbb{T}} \frac{\overline{\lambda} z}{(1-\overline{\lambda} z)^2} h(z)\overline{b(z)} \frac{|dz|}{2\pi}. \end{aligned}$$

Thus for $\lambda \in \mathbb{D}$,

$$\begin{aligned} |h(\lambda)\lambda b'(\lambda)| \leq |h(\lambda)\overline{\lambda b'(\lambda)} - \int_{\mathbb{T}} \frac{\overline{\lambda} z}{(1-\overline{\lambda} z)^2} h(z)\overline{b(z)} \frac{|dz|}{2\pi}| \\ + |\int_{\mathbb{T}} \frac{1}{1-\overline{\lambda} z}(h(z)\overline{b(z)} + D_z(h,b)) \frac{|dz|}{2\pi}|. \end{aligned}$$

The function $h(z)\overline{b(z)}+D_z(h,b) \in L^1(\mathbb{T})$, hence the complex conjugate of $\int_{\mathbb{T}} \frac{1}{1-\overline{\lambda} z}(h(z)\overline{b(z)}+ D_z(h,b))\frac{|dz|}{2\pi}$ is in $H^p, 0 < p < 1$ (see P39 in [28]). Also, note that if we use

$\alpha = h(\lambda), \beta = b(\lambda)$ in Lemma 2.3.14 to obtain

$$
\begin{aligned}
&|h(\lambda)\overline{\lambda b'(\lambda)} - \int_{\mathbb{T}} \frac{\overline{\lambda} z}{(1-\overline{\lambda} z)^2} h(z)\overline{b(z)} \frac{|dz|}{2\pi}| \\
&= |\int_{\mathbb{T}} \frac{\overline{\lambda} z}{(1-\overline{\lambda} z)^2} (h(z) - h(\lambda))\overline{(b(z) - b(\lambda))} \frac{|dz|}{2\pi}|,
\end{aligned}
$$

apply Lemma 2.3.13, we conclude that $h(\lambda)\lambda b'(\lambda) \in H^p, 0 < p < 1$.

(b) Let $h \in \mathcal{N}, h \neq 0$. For $\lambda \in \mathbb{D}$ define

$$
H(\lambda) = \int_{\mathbb{T}} \frac{\overline{\lambda} z}{1-\overline{\lambda} z} (h(z)\overline{b(z)} + D_z(h,b)) \frac{|dz|}{2\pi}. \tag{2.3.7}
$$

Then H is the Cauchy transform of a finite measure, hence $H \in H^p, 0 < p < 1$. As inTheorem 2.2 of [54], H has nontangential limit $H(\zeta) = h(\zeta)\overline{\zeta b'(\zeta)}, a.e. \zeta \in \mathbb{T}$.

Thus the function $B(\lambda) := \overline{H(1/\overline{\lambda})/h(1/\overline{\lambda})}$ is the pseudocontinuation of $(zb)'$ to $\mathbb{D}_e$. □

2.3.4 Index

In this subsection we show that every nonezero M_z-invariant subspace in $D \odot D$ has index one.

For $\mathcal{M} \in Lat(L, \mathcal{Y})$, we write $\sigma(L|_{\mathcal{M}}), \sigma_p(L|_{\mathcal{M}})$ and $\sigma_{ap}(L|_{\mathcal{M}})$ for the spectrum of L on $\mathcal{M}$, point spectrum of L on $\mathcal{M}$ and the approximate point spectrum of L on $\mathcal{M}$.

In the following Lemma, we use the L^2_a pairing.

Lemma 2.3.16. *Let $\mathcal{M}$ be a weak* closed L-invariant subspace in $\mathcal{Y}$ with $\mathcal{M} \neq \mathcal{Y}$, if $|\lambda| > 1$, then for every $f \in \mathcal{M}$, the quantity*

$$
c_\lambda(f, \phi) = \langle \frac{zf}{z-\lambda}, \phi \rangle / \langle \frac{\lambda}{z-\lambda}, \phi \rangle
$$

is independent of the choice of $\phi \in \mathcal{A}$ *with* $\langle \frac{1}{z-\lambda}, \phi \rangle \neq 0$*, where* $\mathcal{A}$ *is the same as in (2.3.1).*

Proof. Method 1:

From the proof of Theorem 2.3.10, we have for $f \in \mathcal{M}$, $\phi \in \mathcal{A}$, $\frac{C_{\mu_1}(\lambda)}{C_{\mu_2}(\lambda)}|_{\mathbb{D}_e}$ is a pseudocontinuation of f, where $C_{\mu_1}(\lambda) = \int_{\mathbb{D}} \frac{f\overline{\phi}}{z-\lambda} dA$, $C_{\mu_2}(\lambda) = \int_{\mathbb{D}} \frac{\overline{\phi}}{z-\lambda} dA$. And by Privalov's uniqueness Theorem, the pseudocontinuation of f is unique, thus it is independent of $\phi \in \mathcal{A}$. Note that

$$\begin{aligned}
&\langle \frac{zf}{z-\lambda}, \phi \rangle / \langle \frac{\lambda}{z-\lambda}, \phi \rangle \\
&= \langle f, \phi \rangle / \langle \frac{\lambda}{z-\lambda}, \phi \rangle + \langle \frac{\lambda f}{z-\lambda}, \phi \rangle / \langle \frac{\lambda}{z-\lambda}, \phi \rangle \\
&= 0 + C_{\mu_1}(\lambda)/C_{\mu_2}(\lambda),
\end{aligned}$$

therefore $\langle \frac{zf}{z-\lambda}, \phi \rangle / \langle \frac{\lambda}{z-\lambda}, \phi \rangle$ is the evaluation of the pseudocontinuation function of f at λ, and so it is independent of $\phi \in \mathcal{A}$ with $\langle \frac{1}{z-\lambda}, \phi \rangle \neq 0$.

Method 2:

We verify directly that for any $f \in \mathcal{M}$, $c_\lambda(f, \phi) = F(\lambda)$, where F is a pseudocontinuation of f.

Suppose $f = (zb)' \in \mathcal{M}, b \in \mathcal{X}(D), \phi = (zh)' \in^\perp \mathcal{M}, h \in D \odot D$ with $\langle \frac{1}{z-\lambda}, \phi \rangle \neq 0$, then

$$\langle \frac{\lambda}{\lambda - z}, \phi \rangle_{L^2_a(\mathbb{D})} = \langle \frac{\lambda}{\lambda - z}, h \rangle_{(\mathcal{Y}(D), D \odot D)} = \overline{h(1/\overline{\lambda})},$$

and

$$\begin{aligned}\langle \frac{\lambda f}{\lambda - z}, \phi\rangle_{L^2_a(\mathbb{D})} &= \langle \frac{\lambda (zb)'}{\lambda - z}, h\rangle_{(\mathcal{Y}(D), D\odot D)} \\ &= \lambda \langle (\frac{zb}{z-\lambda})' - \frac{zb}{(z-\lambda)^2}, h\rangle_{(\mathcal{Y}(D), D\odot D)} \\ &= \langle \frac{\lambda b}{\lambda - z}, h\rangle_D - \int_{\mathbb{T}} \frac{\lambda z}{(\lambda - z)^2} b(z)\overline{h(z)} \frac{|dz|}{2\pi}.\end{aligned}$$

Note that if we replace λ by $1/\overline{\lambda}$ in Lemma 2.3.14, we have

$$\begin{aligned}\langle \frac{\lambda b}{\lambda - z}, h\rangle_D = \int_{\mathbb{T}} \frac{\lambda}{\lambda - z} (b(z)\overline{h(z)} + D_z(b,h)) \frac{|dz|}{2\pi} \\ + \int_{\mathbb{T}} \frac{\lambda z}{(\lambda - z)^2} b(z)\overline{h(z)} \frac{|dz|}{2\pi},\end{aligned}$$

thus

$$\langle \frac{\lambda f}{\lambda - z}, \phi\rangle_{L^2_a(\mathbb{D})} = \int_{\mathbb{T}} \frac{\lambda}{\lambda - z} (b(z)\overline{h(z)} + D_z(b,h)) \frac{|dz|}{2\pi}.$$

Replacing λ by $1/\overline{\lambda}$ in (2.3.7), we have $\langle \frac{\lambda f}{\lambda - z}, \phi\rangle_{L^2_a(\mathbb{D})} = \overline{H(1/\overline{\lambda})}$, therefore

$$\langle \frac{zf}{z-\lambda}, \phi\rangle / \langle \frac{\lambda}{z-\lambda}, \phi\rangle = \overline{H(1/\overline{\lambda})} / \overline{h(1/\overline{\lambda})}.$$

From the proof in Theorem 2.3.15, we have the right hand side is the evaluation at λ of the pseudocontinuation of $(zb)' = f$, and so $c_\lambda(f, \phi)$ is independent of the choice of $\phi \in^\perp \mathcal{M} \subseteq \mathcal{S}$ with $\langle \frac{1}{z-\lambda}, \phi\rangle \neq 0$. □

The following Theorem is basically the Proposition 2.8 in [4]. The difference is that in our case, we consider the functions $\phi \in {}^\perp\mathcal{M}$ instead of $M^\perp$, we include a proof here for completeness.

Theorem 2.3.17. *Let $\mathcal{M}$ be a weak* closed L-invariant subspace in $\mathcal{Y}$ with $\mathcal{M} \neq \mathcal{Y}$. Then $\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{D} = \sigma(L|_{\mathcal{M}}) \cap \mathbb{D}$.*

Proof. Let $|\lambda| > 1$ with $1/\lambda \notin \sigma_{ap}(L|_{\mathcal{M}})$, since $L(\frac{1}{z-\lambda}) = \frac{1}{\lambda}\frac{1}{z-\lambda}$, there exists $\phi \in^{\perp} \mathcal{M}$ with $\langle \frac{1}{z-\lambda}, \phi\rangle \neq 0$. For $f \in \mathcal{Y}(D)$, define $R_\lambda f = \frac{zf - \lambda c_\lambda(f,\phi)}{z-\lambda}$, where $c_\lambda(f,\phi)$ is the same as in Lemma 2.3.16.

Since $\mathcal{Y}(D)$ satisfies the five conditions (1.1)-(1.5) in [4], $R_\lambda \mathcal{Y}(D) \subseteq \mathcal{Y}(D)$, by the closed graph Theorem, R_λ is bounded. We show that $R_\lambda \mathcal{M} \subseteq \mathcal{M}$.

Let $\psi \in^{\perp} \mathcal{M}$, note that $\langle \frac{1}{z-\lambda}, \phi\rangle \neq 0$, there exists $a_n \to 0$ such that $(\psi - a_n\phi)|_{\mathcal{M}} = 0$ and $\langle \frac{1}{z-\lambda}, \psi - a_n\phi\rangle \neq 0$, then

$$
\begin{aligned}
&\langle R_\lambda f, \psi - a_n\phi\rangle = \langle \frac{zf}{z-\lambda}, \psi - a_n\phi\rangle - c_\lambda(f,\phi)\langle \frac{\lambda}{z-\lambda}, \psi - a_n\phi\rangle \\
&= \langle \frac{zf}{z-\lambda}, \psi - a_n\phi\rangle - c_\lambda(f,\psi - a_n\phi)\langle \frac{\lambda}{z-\lambda}, \psi - a_n\phi\rangle \\
&= 0,
\end{aligned}
$$

where in the last equlity we used Lemma 2.3.16. Now let $a_n \to 0$, we get that $\langle R_\lambda f, \psi\rangle = 0$, thus $R_\lambda \mathcal{M} \subseteq ({}^{\perp}\mathcal{M})^{\perp} = \mathcal{M}$.

By a calculation, we see that $R_\lambda(I - \lambda L)|_{\mathcal{M}} = (I - \lambda L)|_{\mathcal{M}} R_\lambda = I$, therefore $1/\lambda \notin \sigma(L|_{\mathcal{M}})$. □

Now we can prove that $\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{D} = \sigma(L|_{\mathcal{M}}) \cap \mathbb{D}$ is a Blaschke sequence.

Theorem 2.3.18. *Let $\mathcal{M}$ be a weak* closed L-invariant subspace in $\mathcal{Y}$ with $\mathcal{M} \neq \mathcal{Y}$. Then $\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{D} = \sigma_p(L|_{\mathcal{M}}) \cap \mathbb{D} = \sigma(L|_{\mathcal{M}}) \cap \mathbb{D}$ is a Blaschke sequence.*

Proof. By Proposition 2.1 in [4], $\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{D} = \sigma_p(L|_{\mathcal{M}}) \cap \mathbb{D} = \{\lambda \in \mathbb{D} : \frac{1}{1-\lambda z} \in \mathcal{M}\}$.

If $\mathcal{M} \neq \mathcal{Y}$, $\mathcal{M}$ is weak* closed, then there exists $f \neq 0, f \in D \odot D$, such that $f|_{\mathcal{M}} = 0$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$
g(\lambda) := \langle f, \frac{1}{1-\lambda z}\rangle_{(D\odot D,\mathcal{Y})} = \sum_{n=0}^{\infty} a_n \overline{\lambda}^n \in \overline{D \odot D},
$$

thus $\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{D} = \sigma_p(L|_{\mathcal{M}}) \cap \mathbb{D} = \{\lambda \in \mathbb{D} : \frac{1}{1-\lambda z} \in \mathcal{M}\} \subseteq \{\lambda \in \mathbb{D} : g(\lambda) = 0\}$ is a Blaschke sequence. □

Now we can prove the index one result in $D \odot D$.

Theorem 2.3.19. *Let $\mathcal{N} \neq (0)$ be an M_z-invariant subspace of $D \odot D$. Then $\dim \mathcal{N}/z\mathcal{N} = 1$.*

Proof. Let $\mathcal{N} \neq (0)$ be an M_z-invariant subspace of $D \odot D$. Let $\mathcal{M} = V\mathcal{N}^{\perp} = \{(zb)' : b \in \mathcal{N}^{\perp}\}$, where $V : \mathcal{X}(D) \to \mathcal{Y}$, $Vb = (zb)'$ is isometric. Then $\mathcal{M} \neq \mathcal{Y}$ and $\mathcal{M}$ is weak* closed.

Since $M_z^*|_{\mathcal{X}(D)}$ is isometrically isomorphic to $L|_{\mathcal{Y}}$, we have $\mathcal{M} \in Lat(L, \mathcal{Y})$ and $M_z^*|_{\mathcal{N}^{\perp}}$ is isometrically isomorphic to $L|_{\mathcal{M}}$, thus by Theorem 2.3.18, $\sigma(M_z^*|_{\mathcal{N}^{\perp}}) \cap \mathbb{D}$ is discrete. The conclusion follows from Theorem 4.5 in [48]. □

Remark 2.3.20. *The simplest proof of the index one result for D uses the fact that every non-zero M_z-invariant subspace contains a non-zero multiplier, but we don't know whether that is true for $D \odot D$, thus we can ask the following question:*

Question 2.3.21. *Does every non-zero M_z-invariant subspace of $D \odot D$ contain a non-zero multiplier of $D \odot D$?*

As in [4], for any weak* closed $\mathcal{M} \in Lat(L, \mathcal{Y}), \mathcal{M} \neq \mathcal{Y}$, we also have $\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{T} = \sigma(L|_{\mathcal{M}}) \cap \mathbb{T} = \mathbb{T} \backslash \{1/\zeta$: every $f \in \mathcal{M}$ extends to be analytic in a neighborhood of $\zeta\}$. In order to show this, we need one Lemma.

Lemma 2.3.22. *Let $f \in \mathcal{Y}$ be analytic in an neighborhood of a point $\zeta \in \mathbb{T}$, then*

$$\frac{zf - wf(w)}{z - w} \to \frac{zf - \zeta f(\zeta)}{z - \zeta} \quad \text{in the weak* topology of } \mathcal{Y} \tag{2.3.8}$$

as $w \to \zeta (w \in \mathbb{D})$.

Proof. Suppose that U is a ball with center at $\zeta \in \mathbb{T}$ and radius r such that f is analytic in U^-. Let $\phi \in \mathcal{S}$, then

$$\begin{aligned}
&|\langle \frac{zf - wf(w)}{z-w}, \phi\rangle_{L^2_a(\mathbb{D})} - \langle \frac{zf-\zeta f(\zeta)}{z-\zeta}, \phi\rangle_{L^2_a(\mathbb{D})}| \\
&= |\int_{\mathbb{D}} \frac{zf - wf(w)}{z-w}\overline{\phi(z)} - \frac{zf-\zeta f(\zeta)}{z-\zeta}\overline{\phi(z)} dA(z)| \\
&\leq \int_{\mathbb{D}\backslash U} |\frac{zf - wf(w)}{z-w}\overline{\phi(z)} - \frac{zf-\zeta f(\zeta)}{z-\zeta}\overline{\phi(z)}| dA(z) \\
&+ \int_{\mathbb{D}\cap U} |\frac{zf - wf(w)}{z-w}\overline{\phi(z)} - \frac{zf-\zeta f(\zeta)}{z-\zeta}\overline{\phi(z)}| dA(z) \\
&:= I_{w,1} + I_{w,2}
\end{aligned}$$

For $I_{w,1}$, note that if $|w-\zeta| < \frac{r}{2}$, then $\exists C$

$$\begin{aligned}
&|\frac{zf - wf(w)}{z-w}\overline{\phi(z)} - \frac{zf-\zeta f(\zeta)}{z-\zeta}\overline{\phi(z)}| \\
&\leq \frac{|f|+C}{r/2}|\phi(z)| + \frac{|f|+C}{r}|\phi(z)| \in L^1(\mathbb{D}),
\end{aligned}$$

thus $I_{w,1} \to 0$ as $w \to \zeta$.

For $I_{w,2}$, note that $\frac{zf-wf(w)}{z-w}$ is analytic in $U^- \times U^-$, by Dominated Convergence Theorem, $I_{w,2} \to 0$ as $w \to \zeta$. The conclusion follows. □

Using [4, Proposition 2.6] and Lemma 2.3.22, we conclude that:

$\sigma_{ap}(L|_{\mathcal{M}}) \cap \mathbb{T} = \sigma(L|_{\mathcal{M}}) \cap \mathbb{T} = \mathbb{T}\backslash\{1/\zeta\colon$ every $f \in \mathcal{M}$ extends to be analytic in a neighborhood of $\zeta\}$.

2.4 Some Calculations in $D \odot D$

In this section, we embed the space $D \odot D$ into some other function spaces, and calculate the exact norm of the monomials $z^n, n \in \mathbb{N}$.

For $\lambda \in \mathbb{D}$, let $\varphi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$.

Proposition 2.4.1. *Let $h \in D \odot D$, then*

$$\|h \circ \varphi_\lambda\|_* \leq \frac{2\sqrt{3}}{3}\|\varphi_\lambda h\|_*.$$

Proof. Let $\lambda = 0$, $\forall \varepsilon > 0$, then $\exists f_i \in D$, $zh = \sum_i f_i^2$ with $\|zh\|_* \geq \sum_i \|f_i\|^2 - \varepsilon$. We have

$$h = \sum_i \frac{f_i^2 - f_i^2(0)}{z} = \sum_i \frac{f_i - f_i(0)}{z}(f_i + f_i(0)).$$

Thus,

$$\begin{aligned}
\|h\|_* &\leq \sum_i \|\frac{f_i - f_i(0)}{z}\|\|f_i + f_i(0)\| \\
&\leq \|f_i - f_i(0)\|\|f_i + f_i(0)\| \\
&= \sum_i \sqrt{\|f_i\|^2 - |f_i(0)|^2}\sqrt{\|f_i\|^2 + 3|f_i(0)|^2} \\
&= \sum_i \|f_i\|^2\sqrt{1 - \frac{|f_i(0)|^2}{\|f_i\|^2}}\sqrt{1 + 3\frac{|f_i(0)|^2}{\|f_i\|^2}} \\
&\leq \sum_i \frac{2\sqrt{3}}{3}\|f_i\|^2 \leq \frac{2\sqrt{3}}{3}(\|zh\|_* + \varepsilon).
\end{aligned}$$

Hence $\|h\|_* \leq \frac{2\sqrt{3}}{3}\|zh\|_*$.

If $\lambda \neq 0$, let $\forall \varepsilon > 0$, then $\exists f_i \in D$, $\varphi_\lambda h = \sum_i f_i^2$ with $\|\varphi_\lambda h\|_* \geq \sum_i \|f_i\|^2 - \varepsilon$, then $z(h \circ \varphi_\lambda) = \sum_i (f_i \circ \varphi_\lambda)^2$, by the same calculation as above, we have

$$\begin{aligned}
\|h\circ\varphi_\lambda\|_* &\le \sum_i \|f_i\circ\varphi_\lambda - f_i\circ\varphi_\lambda(0)\|\|f_i\circ\varphi_\lambda + f_i\circ\varphi_\lambda(0)\| \\
&\le \frac{2\sqrt{3}}{3}\|f_i\circ\varphi_\lambda\|^2 \\
&= \sum_i \frac{2\sqrt{3}}{3}\|f_i\|^2 \le \frac{2\sqrt{3}}{3}(\|\varphi_\lambda h\|_* + \varepsilon).
\end{aligned}$$

Thus $\|h\circ\varphi_\lambda\|_* \le \frac{2\sqrt{3}}{3}\|\varphi_\lambda h\|_*$. □

This Proposition is not satisfying. For example, from the above Proposition, we only get the estimate of the norm of z in $D \odot D$: $\frac{\sqrt{3}}{2} \le \|z\|_* \le \sqrt{2}$. But we have the following estimate.

Theorem 2.4.2. *Let $h \in D$, then $\sqrt{2}|h(0)| \le \|zh\|_* \le \sqrt{2}\|h\|_D$. In particular, $\|z\|_* = \sqrt{2}$.*

Proof. $\forall\varepsilon > 0$, there exist $f_i \in D$, such that $zh = \sum_{i=1}^{\infty} f_i^2$, with $\sum_{i=1}^{\infty} \|f_i\|_D^2 \le \|zh\|_* + \varepsilon$. Then $h(0) = (zh)'(0) = \sum_{i=1}^{\infty} 2f_i(0)f_i'(0)$, this implies

$$|h(0)|^2 \le 4\sum_{i=1}^{\infty}|f_i(0)|^2 \sum_{i=1}^{\infty}|f_i'(0)|^2,$$

thus

$$\sum_{i=1}^{\infty}|f_i'(0)|^2 \ge \frac{|h(0)|^2}{2\sum_{i=1}^{\infty}|f_i(0)|^2},$$

and so

$$\begin{aligned}\|zh\|_* + \varepsilon &\geq \sum_{i=1}^{\infty} \|f_i\|_D^2 \geq \sum_{i=1}^{\infty} |f_i(0)|^2 + 2\sum_{i=1}^{\infty} |f_i'(0)|^2 \\ &\geq \sum_{i=1}^{\infty} |f_i(0)|^2 + \frac{|h(0)|^2}{4\sum\limits_{i=1}^{\infty} |f_i(0)|^2} \\ &\geq \sqrt{2}|h(0)|,\end{aligned}$$

the other inequality is clear. □

In [10], it was shown that for $n \in \mathbb{N}$, $\|z^n\|_* \sim \sqrt{n+1}$. We show in the following theorem that $\|z^n\|_* = \sqrt{n+1}$.

Theorem 2.4.3. *Let $h \in D \odot D$. Then for $n \in \mathbb{N}$, $\|z^n h\|_* \geq \sqrt{n+1}|h(0)|$. In particular, $\|z^n\|_* = \sqrt{n+1}$.*

Proof. Suppose $h(0) = 1$. $\forall \varepsilon > 0$, there exist $f_i \in D$, such that $z^n h = \sum\limits_{i=1}^{\infty} f_i^2$, with $\sum\limits_{i=1}^{\infty} \|f_i\|_D^2 \leq \|z^n h\|_* + \varepsilon$.

Suppose $f_i(z) = \sum\limits_{j=0}^{\infty} a_{ij} z^j$, then

$$\begin{aligned}z^n h = \sum_{i=1}^{\infty} f_i^2 &= \sum_{i=1}^{\infty} \sum_{j,k=0}^{\infty} a_{ij} z^j a_{ik} z^k \\ &= \sum_{k=0}^{\infty} (\sum_{j=0}^{k} \sum_{i=1}^{\infty} a_{ij} a_{i(k-j)}) z^k.\end{aligned}$$

Let $A_j = (a_{1j}, a_{2j}, \cdots) = (\frac{f_i^{(j)}(0)}{j!})_i$, $A_j \cdot A_{k-j} = \langle A_j, \overline{A_{k-j}} \rangle$, and $\|A_j\|^2 = \langle A_j, A_j \rangle$.

Then $z^n h = \sum\limits_{i=1}^{\infty} f_i^2 = \sum\limits_{k=0}^{\infty} \sum\limits_{j=0}^{k} (A_j \cdot A_{k-j}) z^k$, this implies $1 = 2\sum\limits_{j=0}^{n} A_j \cdot A_{n-j}$. When n is odd, we have $1 = 2\sum\limits_{j=0}^{\frac{n-1}{2}} A_j \cdot A_{n-j}$, when n is even, we have $1 = \sum\limits_{j=0}^{\frac{n-2}{2}} A_j \cdot A_{n-j} + A_{n/2} \cdot A_{n/2}$.

Note that $\sum\limits_{i=1}^{\infty}\|f_i\|_D^2 = \sum\limits_{i=1}^{\infty}\|\sum\limits_{j=0}^{\infty}a_{ij}z^j\|_D^2 = \sum\limits_{i=1}^{\infty}\sum\limits_{j=0}^{\infty}(j+1)|a_{ij}|^2 = \sum\limits_{j=0}^{\infty}(j+1)\|A_j\|^2$, we have $\|z^n h\|_* + \varepsilon \geq \sum\limits_{j=0}^{n}(j+1)\|A_j\|^2$.

When n is odd, we have $|1-B_n| := |1 - 2\sum\limits_{j=1}^{\frac{n-1}{2}} A_j \cdot A_{n-j}| = 2|A_0 \cdot A_n| \leq 2\|A_0\|\|A_n\|$, thus

$$\begin{aligned}
\|z^n h\|_* + \varepsilon &\geq \sum_{j=0}^{n}(j+1)\|A_j\|^2 \\
&\geq \sum_{j=0}^{n-1}(j+1)\|A_j\|^2 + (n+1)\frac{|1-B_n|^2}{4\|A_0\|^2} \\
&\geq \sqrt{n+1}|1-B_n| + \sum_{j=1}^{n-1}(j+1)\|A_j\|^2
\end{aligned}$$

note that $B_n = 2\sum\limits_{j=1}^{\frac{n-1}{2}} A_j \cdot A_{n-j}$, $|B_n| \leq \sum\limits_{j=1}^{\frac{n-1}{2}} \alpha_j\|A_j\|^2 + \beta_j\|A_{n-j}\|^2$, where $\alpha_j = \frac{j+1}{\sqrt{n+1}}$, $\beta_j = \frac{n-j+1}{\sqrt{n+1}}$, $j = 1, 2, ..., \frac{n-1}{2}$, therefore $\sqrt{n+1}|B_n| \leq \sum\limits_{j=1}^{n-1}(j+1)\|A_j\|^2$, and so

$$\|z^n h\|_* + \varepsilon \geq \sqrt{n+1}|1-B_n| + \sum_{j=1}^{n-1}(j+1)\|A_j\|^2 \geq \sqrt{n+1}.$$

In a similar way, for n even, we also have $\|z^n h\|_* + \varepsilon \geq \sqrt{n+1}$, since ε is arbitrary, we conclude that $\|z^n h\|_* \geq \sqrt{n+1}|h(0)|$. $\square$

From the calculation in the above Theorem, we also have $\|z^n + z^{n+1}h\|_* \geq \|z^n\|_*, h \in D \odot D$, and $\|z^n + z^k\|_* \geq \|z^n\|_*, k \neq n$.

Remark 2.4.4. *The above Theorem tells us the norm of z^n in $D \odot D$, but in general, we don't know how to calculate the norm for the simple functions like $1 + az, a \in \mathbb{C}$, using the same calculation as in Theorem 2.4.3, we only have the estimate: $\sqrt{2}|a| \leq \|1 + az\|_* \leq \sqrt{1 + 2|a|^2}$.*

Using the idea in Theorem 2.4.3, we have

Theorem 2.4.5. *Let $h \in D \odot D$. Then* $\|zh\|_* \geq \sqrt{n+1}|\frac{h^{(n-1)}(0)}{(n-1)!}|$, $\|h\|_* \geq \sqrt{n+1}|\frac{h^{(n)}(0)}{n!}|$.

Proof. We sketch an outline here.

As in Theorem 2.4.3, $\forall \varepsilon > 0$, there exist $f_i \in D$, such that $zh = \sum_{i=1}^{\infty} f_i^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (A_j \cdot A_{n-j}) z^n$, with $\sum_{i=1}^{\infty} \|f_i\|_D^2 \leq \|zh\|_* + \varepsilon$, where $A_j = (a_{1j}, a_{2j}, \cdots) = (\frac{f_i^{(j)}(0)}{j!})_i$.

Suppose $h(z) = \sum_{n=0}^{\infty} a_n z^n, a_n = \frac{h^{(n)}(0)}{n!}$, then $a_{n-1} = \sum_{j=0}^{n} A_j \cdot A_{n-j}$, then we can use the same estimates as we did in Theorem 2.4.3. □

Let $l_a^\infty(\sqrt{n+1}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D}) : \sup_n \sqrt{n+1}|a_n| < \infty\}$, for $f \in l_a^\infty(\sqrt{n+1}), f = \sum_{n=0}^{\infty} a_n z^n$, the norm of f is defined by $\|f\|_{l_a^\infty(\sqrt{n+1})} := \sup_n \sqrt{n+1}|a_n|$.

By Theorem 2.4.5, if $h \in D \odot D$, then $h \in l_a^\infty(\sqrt{n+1})$, and the embedding is a contraction, i.e., $\|h\|_* \geq \|h\|_{l_a^\infty(\sqrt{n+1})}$. Thus $D \odot D \subseteq l_a^\infty(\sqrt{n+1})$.

In fact, more is true. Let

$$c_{0,a}(\sqrt{n+1}) = \{f \in l_a^\infty(\sqrt{n+1}) : \lim_{n\to\infty} \sqrt{n+1} a_n = 0\}.$$

$\forall h \in D \odot D, \forall \varepsilon > 0$, there exists a polynomial p_N with degree N, such that $\|p_N - h\|_* < \varepsilon$. If $f \in Hol(\mathbb{D})$, let $a_n(f)$ be the n-th coefficient of f, then

$$\sup_n \sqrt{n+1}|a_n(h - P_N)| \leq \|p_N - h\|_* < \varepsilon,$$

note that when $n > N$, $a_n(h - p_N) = a_n(h)$, therefore $h \in c_{0,a}(\sqrt{n+1})$. Thus we have

Theorem 2.4.6. *$D \odot D \subseteq c_{0,a}(\sqrt{n+1})$ with*

$$\|h\|_{c_{0,a}(\sqrt{n+1})} \le \|h\|_*, h \in D \odot D.$$

By Theorem 3.1 in [55], $\mathcal{H} \odot \mathcal{H} \subseteq \mathcal{H}(k^2)$ with $\|h\|_{\mathcal{H}(k^2)} \le \|h\|_*, h \in \mathcal{H} \odot \mathcal{H}$, where $\mathcal{H} = \mathcal{H}(k)$ has reproducing kernel k.

If $\mathcal{H} = D$, then $k_\lambda(z) = \sum\limits_{n=0}^{\infty} \frac{1}{n+1} \overline{\lambda}^n z^n$, so $k_\lambda^2(z) = \sum\limits_{n=0}^{\infty} \frac{\alpha_n}{n+1} \overline{\lambda}^n z^n$ with $\alpha_n = \sum\limits_{k=0}^{n} \frac{2}{k+1} \sim \log(n+1)$. Then $\forall f \in \mathcal{H}(k_D^2), f(z) = \sum\limits_{n=0}^{\infty} b_n z^n$, $\|f\|_{\mathcal{H}(k_D^2)} \sim \sum\limits_{n=0}^{\infty} \frac{n+1}{\log(n+1)} |b_n|^2$.

Proposition 2.4.7. *$\mathcal{H}(k_D^2) \not\subseteq c_{0,a}(\sqrt{n+1})$ and $c_{0,a}(\sqrt{n+1}) \not\subseteq \mathcal{H}(k_D^2)$.*

Proof. Let $h = \sum\limits_{n=0}^{\infty} b_n z^n \in Hol(\mathbb{D})$ satisfy

$$b_n = \begin{cases} \frac{1}{\sqrt{n+1}}, & : \quad n = 2^{k^2}, k \in \mathbb{N}, \\ 0, & : \quad \text{otherwise.} \end{cases}$$

Then $h \in \mathcal{H}(k_D^2)$, but $h \notin c_{0,a}(\sqrt{n+1})$.

On the other side, for any $K \in \mathbb{N}$, let $h = \sum\limits_{n=0}^{\infty} b_n z^n \in Hol(\mathbb{D})$ satisfy

$$b_n = \begin{cases} \frac{1}{\sqrt{n+1}}, & : \quad n \le K, \\ 0, & : \quad \text{otherwise.} \end{cases}$$

Then $\|h\|_{c_{0,a}(\sqrt{n+1})} = 1$, but $\|h\|_{\mathcal{H}(k_D^2)} \sim \sum\limits_{n=0}^{K} \frac{n+1}{\log(n+1)} |b_n|^2 = \sum\limits_{n=0}^{K} \frac{1}{\log(n+1)}$ diverges, thus $c_{0,a}(\sqrt{n+1}) \not\subseteq \mathcal{H}(k_D^2)$. □

The following Proposition is in Page 33 of [77].

Proposition 2.4.8. *If X, Y are Banach spaces, then the space $X \cap Y$ is a Banach space with norm*

$$\|z\|_{X \cap Y} = \max\{\|z\|_X, \|z\|_Y\},$$

and the space $X + Y$ is a Banach space with norm

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y, x \in X, y \in Y\}.$$

By the above Proposition, we have

Theorem 2.4.9. *$D \odot D \subseteq \mathcal{H}(k_D^2) \cap c_{0,a}(\sqrt{n+1})$ with*

$$\|h\|_{\mathcal{H}(k_D^2) \cap c_{0,a}(\sqrt{n+1})} \leq \|h\|_*, h \in D \odot D.$$

Let $l_a^1(\sqrt{n+1}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D}) : \sum_{n=0}^{\infty} \sqrt{n+1}|a_n| < \infty\}$, and its norm is defined by $\|f\|_{l_a^1(\sqrt{n+1})} := \sum_{n=0}^{\infty} \sqrt{n+1}|a_n|$.

Theorem 2.4.10. *$c_{0,a}(\sqrt{n+1})^{*,D} = l_a^1(\sqrt{n+1})$ under the pairing*

$$\langle f, g \rangle_D = \sum_{n=0}^{\infty} (n+1) a_n \overline{b_n},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in c_{0,a}(\sqrt{n+1}), g(z) = \sum_{n=0}^{\infty} b_n z^n \in l_a^1(\sqrt{n+1})$.

Proof. Let $L \in c_{0,a}(\sqrt{n+1})^{*,D}$, then for $f \in c_{0,a}(\sqrt{n+1})$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ with } \sqrt{n+1}|a_n| \to 0, \text{ as } n \to \infty,$$

and

$$L(f) = \sum_{n=0}^{\infty} a_n L(z^n),$$

thus

$$\begin{aligned} |L(f)| &\leq \sum_{n=0}^{\infty} \sqrt{n+1}|a_n| \frac{1}{\sqrt{n+1}} |L(z^n)| \\ &\leq \|f\|_{c_{0,a}(\sqrt{n+1})} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |L(z^n)|, \end{aligned}$$

and so $\|L\| \leq \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |L(z^n)|$.

Fix $i \geq 0$, let

$$\xi_n = \begin{cases} \frac{\text{sgn} L(z^n)}{\sqrt{n+1}}, & : \quad 0 \leq n \leq i, \\ 0, & : \quad n > i, \end{cases}$$

where

$$\text{sgn} w = \begin{cases} \frac{w}{|w|}, & : \quad w \neq 0, \\ 0, & : \quad w = 0. \end{cases}$$

Then $f_0 = \sum_{n=0}^{i} \xi_n z^n \in c_{0,a}(\sqrt{n+1})$ with $\|f_0\|_{c_{0,a}(\sqrt{n+1})} = 1$, and

$$|L(f_0)| = \sum_{n=0}^{i} \frac{1}{\sqrt{n+1}} |L(z^n)| \leq \|L\|,$$

therefore $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |L(z^n)| \leq \|L\|$ and so $\|L\| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |L(z^n)|$.

Let $(n+1)\overline{b_n} = L(z^n), g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\begin{aligned}\|g\|_{l_a^1(\sqrt{n+1})} &= \sum_{n=0}^{\infty} \sqrt{n+1}|b_n| \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}|L(z^n)| \\ &= \|L\|,\end{aligned}$$

thus

$$\begin{aligned}L(f) &= \sum_{n=0}^{\infty} a_n L(z^n) \\ &= \sum_{n=0}^{\infty} (n+1) a_n \overline{b_n} \\ &:= \langle f, g\rangle_D.\end{aligned}$$

The other part is clear. □

It is shown in [9] that $(D \odot D)^* = \mathcal{X}(D)$ (see also [20]), where $\mathcal{X}(D) = \{b \in Hol(\mathbb{D}) : |b'|^2 dA$ is a Carleson measure for $D\}$, and the norm in $\mathcal{X}(D)$ is $\|b\|^2_{\mathcal{X}(D)} = |b(0)|^2 + \||b'|^2 dA\|_{CM(D)}$. By Theorem 2.4.10, we see

$$\mathcal{X}(D) = (D \odot D)^* \supseteq c_{0,a}(\sqrt{n+1})^{*,D} = l_a^1(\sqrt{n+1}).$$

Let $\mathcal{H}(k_D^2)^{*,D} = \{f \in Hol(\mathbb{D}) : |\langle f, g\rangle_D| \le C\|g\|_{\mathcal{H}(k_D^2)}, \forall g \in \mathcal{H}(k_D^2)\}$, then $\mathcal{H}(k_D^2)^{*,D} = \{f = \sum\limits_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D}) : \sum\limits_{n=0}^{\infty} (n+1)\log(n+1)|a_n|^2 < \infty\}$, and so $\mathcal{H}(k_D^2)^{*,D} \subseteq (D \odot D)^* = \mathcal{X}(D)$ (see Theorem 4 of [10], also Proposition 18 of [16]).

Proposition 2.4.11. *$\mathcal{H}(k_D^2)^{*,D} \not\subseteq l_1^a(\sqrt{n+1})$ and $l_1^a(\sqrt{n+1}) \not\subseteq \mathcal{H}(k_D^2)^{*,D}$.*

Proof. Let $f = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D})$ satisfy $a_n = \frac{1}{(n+1)\log(n+1)^{3/2}}$, then $f \in \mathcal{H}(k_D^2)^{*,D}$, but $f \notin l_1^a(\sqrt{n+1})$.

On the other side, let $f = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D})$ satisfy

$$a_n = \begin{cases} \frac{1}{\sqrt{(n+1)\log(n+1)}}, & : \quad n = 2^{k^3}, k \in \mathbb{N}, \\ 0, & : \quad \text{otherwise.} \end{cases}$$

Then $f \in l_1^a(\sqrt{n+1})$, but $f \notin \mathcal{H}(k_D^2)^{*,D}$. $\square$

If we use an equivalent norm in $\mathcal{X}(D)$:

$$\|f\|_{\mathcal{X}(D)} = \sup_{\|h\|_* \leq 1} |\langle h, f\rangle_D|, f \in \mathcal{X}(D), \tag{2.4.1}$$

then we have

Theorem 2.4.12. *$\mathcal{H}(k_D^2)^{*,D} + l_a^1(\sqrt{n+1}) \subseteq \mathcal{X}(D)$ with*

$$\|f\|_{\mathcal{X}(D)} \leq \|f\|_{\mathcal{H}(k_D^2)^{*,D} + l_a^1(\sqrt{n+1})}, f \in \mathcal{X}(D).$$

It is shown in Theorem 2 of [10] that for $n \in \mathbb{N}, \|z^n\|_{\mathcal{X}(D)} \sim \sqrt{n}$. If we use the equivalent norm in $\mathcal{X}(D)$ defined by (2.4.1), then indeed the norm of z^n in $\mathcal{X}(D)$ is $\sqrt{n+1}$.

Theorem 2.4.13. *For $n \in \mathbb{N}, \|z^n\|_{\mathcal{X}(D)} = \sqrt{n+1}$.*

Proof.

$$\begin{aligned} \|z^n\|_{\mathcal{X}(D)} &= \sup_{\|h\|_* \leq 1} |\langle h, z^n\rangle_D| = \sup_{\|h\|_* \leq 1} |\hat{h}(n)|(n+1) \\ &\leq \sqrt{n+1}, \end{aligned}$$

in the last inequality, we used $\sqrt{n+1}|\hat{h}(n)| \leq \|h\|_* \leq 1$.

Let $h = \frac{z^n}{\sqrt{n+1}}$, then by Theorem 2.4.3, $\|h\|_* = 1$, therefore from the above equations we get $\|z^n\|_{\mathcal{X}(D)} = \sqrt{n+1}$. □

We can also use the observation: $n+1 = \langle z^n, z^n \rangle_D \leq \|z^n\|_* \|z^n\|_{\mathcal{X}(D)}$, and $\|z^n\|_{\mathcal{X}(D)} \leq \|z^n\|_{l^1_a(\sqrt{n+1})} = \sqrt{n+1}$ to conclude that $\|z^n\|_{\mathcal{X}(D)} = \sqrt{n+1}$.

2.5 $\mathcal{H}(k_D^2)$ space

2.5.1 $\mathcal{H}(k_D^2)$ space

In this subsection, we study the index of the M_z-invariant subspaces in $\mathcal{H}(k_D^2)$, where k_D is the reproducing kernel for the Dirichlet space D and $\mathcal{H}(k_D^2)$ is a Hilbert space with reproducing kernel k_D^2.

Recall that D is the Dirichlet space with reproducing kernel $k_\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{n+1}\overline{\lambda}^n z^n$, so $k_\lambda^2(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n+2}\overline{\lambda}^n z^n$ with $\alpha_n = \sum_{k=0}^{n} \frac{2}{k+1}$. Then $\forall f \in \mathcal{H}(k_D^2), f(z) = \sum_{n=0}^{\infty} b_n z^n$, $\|f\|_{\mathcal{H}(k_D^2)} = \sum_{n=0}^{\infty} \frac{n+2}{\alpha_n}|b_n|^2$. We can calculate the first few α_n's: $\alpha_0 = 2, \alpha_1 = 3, \alpha_2 = \frac{11}{3}, \alpha_3 = \frac{25}{6}, \cdots$. By checking the conditions in Theorem 7.33 of [2, Page 88], we see that $k_\lambda^2(z)$ is not a complete Nevanlinna-Pick kernel.

It is clear that $\mathcal{H}(k_D^2)$ is a Hilbert space of analytic functions such that point evaluations at points of $\mathbb{D}$ are continuous, and it is clear that $\mathcal{H}(k_D^2)$ is M_z-invariant. Now we show that $\mathcal{H}(k_D^2)$ has the following property:

(∗) If $f \in \mathcal{H}(k_D^2)$ and $f(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, then there is a function $g \in \mathcal{H}(k_D^2)$ such that $(z-\lambda)g = f$.

First, let's show that M_z on $\mathcal{H}(k_D^2)$ is expansive, i.e., $\|zf\|_{D(k^2)} \geq \|f\|_{\mathcal{H}(k_D^2)}$, $f \in \mathcal{H}(k_D^2)$.

Lemma 2.5.1. *For any $n \in \mathbb{N}$,*

$$\frac{n+3}{\alpha_{n+1}} - \frac{n+2}{\alpha_n} \geq 0,$$

and when $n \geq 4$,

$$\frac{n+3}{\alpha_{n+1}} - \frac{n+2}{\alpha_n} \geq \frac{n+4}{\alpha_{n+2}} - \frac{n+3}{\alpha_{n+1}}.$$

Proof. When $n = 0$, then $\frac{3}{\alpha_1} - \frac{2}{\alpha_0} = \frac{3}{3} - \frac{2}{2} = 0$. When $n \geq 1$,

$$\begin{aligned}(n+3)\alpha_n - (n+2)\alpha_{n+1} &= (n+3)\sum_{k=0}^{n} \frac{2}{k+1} - (n+2)\sum_{k=0}^{n+1} \frac{2}{k+1} \\ &= \sum_{k=1}^{n} \frac{2}{k+1} \\ &= \alpha_n - 2 \geq 0.\end{aligned}$$

$$\begin{aligned}&\left(\frac{n+3}{\alpha_{n+1}} - \frac{n+2}{\alpha_n}\right) - \left(\frac{n+4}{\alpha_{n+2}} - \frac{n+3}{\alpha_{n+1}}\right) = \frac{\alpha_n - 2}{\alpha_n\alpha_{n+1}} - \frac{\alpha_{n+1} - 2}{\alpha_{n+1}\alpha_{n+2}} \\ &= \frac{(\alpha_n - 2)\alpha_{n+2} - (\alpha_{n+1} - 2)\alpha_n}{\alpha_n\alpha_{n+1}\alpha_{n+2}} \\ &= \frac{(\alpha_n - 2)(\alpha_{n+2} - \alpha_n) - \frac{2}{n+2}\alpha_n}{\alpha_n\alpha_{n+1}\alpha_{n+2}} \\ &= \frac{\frac{2}{n+3}(\alpha_n - 2) - \frac{4}{n+2}}{\alpha_n\alpha_{n+1}\alpha_{n+2}},\end{aligned}$$

and the last expression is nonnegative when $n \geq 4$. □

Note that if $f \in \mathcal{H}(k_D^2)$, $f = \sum_{n=0}^{\infty} b_n z^n$, then $zf = \sum_{n=0}^{\infty} b_n z^{n+1}$ with norm $\|zf\|_{\mathcal{H}(k_D^2)} = \sum_{n=0}^{\infty} \frac{n+3}{\alpha_{n+1}} |b_n|^2$, thus from the above Lemma, we conclude that M_z on $\mathcal{H}(k_D^2)$ is expansive, and so for every $\lambda \in \mathbb{D}$, $M_z - \lambda$ is bounded below. If $f \in \mathcal{H}(k_D^2)$ with $f(\lambda) = 0$, let $g = f/(z - \lambda)$, then

$$\|f\|_{\mathcal{H}(k_D^2)} = \|(z-\lambda)g\|_{\mathcal{H}(k_D^2)} \geq C(\lambda)\|g\|_{\mathcal{H}(k_D^2)},$$

therefore the space $\mathcal{H}(k_D^2)$ satisfies the property $(*)$.

Also note that by the above Lemma when $n \geq 4$, let $\mathcal{H}_n = \{f \in \mathcal{H}(k_D^2) : f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0\}$, then M_z on $\mathcal{H}_n$ satisfies,

$$\bigcap_{k>0} M_z^k \mathcal{H}_n = (0),$$

and

$$\|f\|^2 + \|M_z^2 f\|^2 \leq 2\|M_z f\|^2, \quad \forall f \in \mathcal{H}_n.$$

Thus, by Theorem 2.2.4, if $\mathcal{M} \in Lat(M_z, \mathcal{H}_n)$, then $[\ker(M_z|_{\mathcal{M}})^*]_{M_z} = \mathcal{M}$. But we can verify that M_z on $\mathcal{H}_n$ is not Dirichlet type in the sense of Aleman ([3]).

Now we can study the index for M_z invariant subspaces in $\mathcal{H}(k_D^2)$. For $\mathcal{M} \in Lat(M_z, \mathcal{H}(k_D^2))$, the index of $\mathcal{M}$ is defined by

$$\operatorname{ind}(\mathcal{M}) = \dim(\mathcal{M}/z\mathcal{M}) = \dim(\mathcal{M} \cap (z\mathcal{M})^{\perp}).$$

We use the Cauchy duality to study the index for any M_z invariant subspace in $\mathcal{H}(k_D^2)$.

If $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, then the Cauchy dual $\mathcal{H}'$ of $\mathcal{H}$ consists of analytic function of the form

$$f(\lambda) = \langle g, (1 - \overline{\lambda} z)^{-1} \rangle_{\mathcal{H}}, \quad \lambda \in \mathbb{D},$$

where $g \in \mathcal{H}$ and $\|f\|_{\mathcal{H}'} = \|g\|_{\mathcal{H}}$ (see [6]).

In our case, the Cauchy dual of $\mathcal{H}(k_D^2)$ is $\mathcal{H}(\omega) = \{g \in Hol(\mathbb{D}) : \|g\|^2_{\mathcal{H}(\omega)} = \sum_{n=0}^{\infty} |\hat{g}(n)|^2 \frac{\alpha_n}{n+2} < \infty\}$, where $\omega = (\frac{\alpha_n}{n+2})_{n \geq 0}$ is nonincreasing by Lemma 2.5.1.

Let $\omega_n = \frac{\alpha_n}{n+2}$, it is pointed out in [1] that if $\frac{\omega_{n+1}}{\omega_n}$ is nondecreasing with n then the norm on $\mathcal{H}(\omega)$ is equivalent to a Bergman norm with a radial weight and the norm satisfies (1.9) in [6].

In the Cauchy dual $\mathcal{H}(\omega)$ of $\mathcal{H}(k_D^2)$, $\frac{\omega_{n+1}}{\omega_n} = \frac{(n+2)\alpha_{n+1}}{(n+3)\alpha_n}$ and

$$\begin{aligned}\frac{\omega_{n+2}}{\omega_{n+1}} - \frac{\omega_{n+1}}{\omega_n} &= \frac{(n+3)\alpha_{n+2}}{(n+4)\alpha_{n+1}} - \frac{(n+2)\alpha_{n+1}}{(n+3)\alpha_n} \\ &= \frac{(n+3)^2\alpha_n\alpha_{n+2} - (n+2)(n+4)\alpha_{n+1}^2}{(n+3)(n+4)\alpha_n\alpha_{n+1}} \\ &= \frac{-2\frac{n+4}{n+3}\alpha_{n+1} - 4\frac{n+4}{n+3} + \alpha_n\alpha_{n+2}}{(n+3)(n+4)\alpha_n\alpha_{n+1}},\end{aligned}$$

when $n = 0$, the above expression is negative, thus $\frac{\omega_{n+1}}{\omega_n}$ is not nonincreasing with n, but we show that $\mathcal{H}(\omega)$ satisfies (1.6) in [6], which is the following condition:

($\star$) There is a $c > 0$ such that $\|\frac{z-\lambda}{1-\bar{\lambda}z}F\|_{\mathcal{H}(\omega)} \geq c\|F\|_{\mathcal{H}(\omega)}$ for all $F \in \mathcal{H}(\omega)$ and all $\lambda \in \mathbb{D}$,

and this will imply the index of any invariant subspace in $\mathcal{H}(k_D^2)$ is 1.

Note that $\omega = (\frac{\alpha_n}{n+2})_{n\geq 0}$ is nonincreasing, we have $\|zf\|_{\mathcal{H}(\omega)} \leq \|f\|_{\mathcal{H}(\omega)}$ for each $f \in \mathcal{H}(\omega)$. This also follows from a general fact of the Cauchy dual (see [6]):

Lemma 2.5.2. *Let $\mathcal{H}$ be a Hilbert function space, $\mathcal{H}'$ is the Cauchy dual of $\mathcal{H}$. Let $U : \mathcal{H} \to \mathcal{H}', UG = g$, where $g(\lambda) = \langle G, \frac{1}{1-\bar{\lambda}z}\rangle_{\mathcal{H}}$ with $\|g\|_{\mathcal{H}'} = \|G\|_{\mathcal{H}}$. Then*

(i) *$L|_{\mathcal{H}'}$ is unitary equivalent to $M_z^*|_{\mathcal{H}}$, where L is the backward shift, $Lf = \frac{f-f(0)}{z}$.*

(ii) *If M_z is an bounded expansive operator on $\mathcal{H}$, then M_z is a contractive operator on $\mathcal{H}'$.*

Proof. (i) Note that if $F \in \mathcal{H}, g \in \mathcal{H}'$, then

$$\langle zF, g\rangle_{(\mathcal{H},\mathcal{H}')} = \lim_{r\to 1^-}\int_{\mathbb{T}}(zF)(re^{it})\overline{g(re^{it})}\frac{dt}{2\pi} = \langle F, Lg\rangle_{(\mathcal{H},\mathcal{H}')}.$$

Thus

$$\langle F, LUG\rangle_{(\mathcal{H},\mathcal{H}')} = \langle zF, g\rangle_{(\mathcal{H},\mathcal{H}')} = \langle zF, G\rangle_{\mathcal{H}} = \langle F, M_z^* G\rangle_{\mathcal{H}}$$
$$= \langle F, UM_z^* G\rangle_{(\mathcal{H},\mathcal{H}')},$$

this implies $LU = UM_z^*$, and so $L|_{\mathcal{H}'}$ is unitary equivalent to $M_z^*|_{\mathcal{H}}$.

(ii) Since $\|g\|_{\mathcal{H}'} = \sup_{F\in\mathcal{H},\|F\|_{\mathcal{H}}\leq 1} |\langle F, g\rangle_{(\mathcal{H},\mathcal{H}')}|$, we have

$$\begin{aligned}\|zg\|_{\mathcal{H}'} &= \sup_{F\in\mathcal{H},\|F\|_{\mathcal{H}}\leq 1} |\langle F, zg\rangle_{(\mathcal{H},\mathcal{H}')}| \\ &\geq \sup_{F\in\mathcal{H},\|F\|_{\mathcal{H}}\leq 1} |\langle zF, zg\rangle_{(\mathcal{H},\mathcal{H}')}| \\ &= \sup_{F\in\mathcal{H},\|F\|_{\mathcal{H}}\leq 1} |\langle F, g\rangle_{(\mathcal{H},\mathcal{H}')}| \\ &= \|g\|_{\mathcal{H}'},\end{aligned}$$

where in the second inequality, we used that M_z is an bounded expansive operator on $\mathcal{H}$. □

Now we verify that for each $\lambda \in \mathbb{D}$, $M_z - \lambda$ is bounded below in $\mathcal{H}(\omega)$. Note that $\alpha_n \sim \ln n$, we have the following Lemma follows from [65, Theorem 4, Page 66]. We include a different proof here.

Lemma 2.5.3. $\sigma(M_z|_{\mathcal{H}(k_D^2)}) = \mathbb{D}^-$.

Proof. Note that for $\lambda \in \mathbb{D}$, $M_z - \lambda$ is not onto, thus $\sigma(M_z|_{\mathcal{H}(k_D^2)}) \supseteq \mathbb{D}^-$.

For the converse inclusion, we will use the idea in [42].

If $f \in \mathcal{H}(k_D^2), |\lambda| > 1$, then $\frac{1}{1-\lambda z} f = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \hat{f}(k)\lambda^{n-k} z^n$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{n+2}{\alpha_n} \Big| \sum_{k=0}^{n} \hat{f}(k)\lambda^{n-k} \Big|^2 \\
&\le \sum_{n=0}^{\infty} \frac{n+2}{\alpha_n} \sum_{k=0}^{n} \frac{k+2}{\alpha_k} |\hat{f}(k)|^2 (n-k)|\lambda|^{n-k} \sum_{k=0}^{n} \frac{\alpha_k}{k+2} \frac{1}{n-k} |\lambda|^{n-k} \\
&\le \sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \frac{k+2}{\alpha_k} |\hat{f}(k)|^2 (n-k)|\lambda|^{n-k} \\
&\le \sup_n \{C_n\} \|f\|_{\mathcal{H}(k_D^2)} \sum_{j=0}^{\infty} j|\lambda|^j,
\end{aligned}$$

where $C_n = \frac{n+2}{\alpha_n} \sum_{k=0}^{n} \frac{\alpha_k}{k+2} \frac{1}{n-k} |\lambda|^{n-k} \le \sum_{k=0}^{n} |\lambda|^k \le \frac{1}{1-|\lambda|}$.

Thus $\frac{1}{1-\lambda z} f \in \mathcal{H}(k_D^2)$ and so $\sigma(M_z|_{\mathcal{H}(k_D^2)}) = \mathbb{D}^-$. □

Lemma 2.5.4. *If $F \in \mathcal{H}(\omega)$, and $F(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{D}$, then there is a function $G \in \mathcal{H}(\omega)$ such that $(z - \lambda_0)G = F$.*

Proof. Suppose $f \in \mathcal{H}(k_D^2)$ such that $F(\lambda) = \langle f, (1 - \overline{\lambda} z)^{-1} \rangle_{\mathcal{H}(k_D^2)}$, then

$$\begin{aligned}
\frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0} &= \langle f, \frac{\frac{1}{1-\overline{\lambda} z} - \frac{1}{1-\overline{\lambda_0} z}}{\overline{\lambda} - \overline{\lambda_0}} \rangle_{\mathcal{H}(k_D^2)} \\
&= \langle f, \frac{z}{(1-\overline{\lambda} z)(1-\overline{\lambda_0} z)} \rangle_{\mathcal{H}(k_D^2)} \\
&= \langle (I - \lambda_0 M_z^*)^{-1} M_z^* f, \frac{1}{1-\overline{\lambda} z} \rangle_{\mathcal{H}(k_D^2)} \in \mathcal{H}(\omega),
\end{aligned}$$

where in the last equality, we used Lemma 2.5.3.

Let $G = \frac{F(\lambda)}{\lambda - \lambda_0}$, then $G \in \mathcal{H}(\omega)$ and $(z - \lambda_0)G = F$. □

Since for any $\lambda \in \mathbb{D}$, $M_z - \lambda$ is one to one, $M_z - \lambda$ is bounded below if and if $M_z - \lambda$ has closed range. By the above Lemma,

$$\text{ran } (M_z - \lambda) = \{f \in \mathcal{H}(\omega) : f(\lambda) = 0\},$$

therefore $M_z - \lambda$ is bounded below on $\mathcal{H}(\omega)$ (see also Lemma 2.1 of [48]).

As in [6], let $\alpha_-(\omega) = \lim\inf_{n\to\infty}(n+1)(1 - \frac{\omega_{n+1}}{\omega_n})$, $\alpha_+(\omega) = \lim\sup_{n\to\infty}(n+1)(1 - \frac{\omega_{n+1}}{\omega_n})$, then

Lemma 2.5.5. $\alpha_-(\omega) = \alpha_+(\omega) = 1$.

Proof.

$$\begin{aligned} 1 - \frac{\omega_{n+1}}{\omega_n} &= 1 - \frac{(n+2)\alpha_{n+1}}{(n+3)\alpha_n} \\ &= \frac{(n+3)\alpha_n - (n+2)\alpha_{n+1}}{(n+3)\alpha_n} \\ &= \frac{\alpha_n - 2}{(n+3)\alpha_n}, \end{aligned}$$

hence $\lim_{n\to\infty}(n+1)(1 - \frac{\omega_{n+1}}{\omega_n}) = 1$. □

Thus $\mathcal{H}(\omega)$ satisfies the conditions in Corollary 4.3 in [6], and so it satisfies (1.6) in [6], which is the condition ($\star$).

Let L be the backward shift on $\mathcal{H}(\omega)$, i.e., $Lf = \frac{f(z)-f(0)}{z}, f \in \mathcal{H}(\omega)$. Then by Theorem 2.2 in [6], for $\mathcal{N} \in Lat(L, \mathcal{H}(\omega))$, $\sigma(L|_{\mathcal{N}}) \cap \mathbb{D}$ is discrete. Note that $L|_{\mathcal{H}(\omega)}$ is unitarily equivalent to $M_z^*|_{\mathcal{H}(k_D^2)}$, we conclude that

Theorem 2.5.6. *Let* $\mathcal{M} \in Lat(M_z, \mathcal{H}(k_D^2))$, *then* $\text{ind}\mathcal{M} = 1$.

Proof. This follows from Theorem 4.5 in [48]. □

2.5.2 Other spaces

In this subsection, we study the index of M_z-invariant subspaces in $D_\alpha \odot D_\beta$ and $\mathcal{H}(k^\alpha \cdot k^\beta)$, where for $\alpha \in \mathbb{R}$, $D_\alpha = \{f \in Hol(\mathbb{D}) : \|f\|^2_{D_\alpha} = \sum_{n=0}^\infty (n+1)^\alpha |\hat{f}(n)|^2 < \infty\}$, D_α has reproducing kernel k^α, and $\mathcal{H}(k^\alpha \cdot k^\beta)$ is the Hilbert space with reproducing kernel $k^\alpha \cdot k^\beta$.

$D_\alpha \odot D_\beta$ space

For $\alpha \in \mathbb{R}$, let

$$D_\alpha = \{f \in Hol(\mathbb{D}) : \|f\|^2_{D_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha |\hat{f}(n)|^2 < \infty\},$$

then D_α is a Hilbert space of analytic functions such that point evaluations in $\mathbb{D}$ are bounded, we denote the reproducing kernel of D_α by k^α.

For $\alpha = -1, 0, 1$, we have $D_{-1} = L^2_a(\mathbb{D})$ is the Bergman space, $D_0 = H^2(\mathbb{D})$ is the Hardy space, $D_1 = D$ is the Dirichlet space.

For $\alpha < 0$, the norm in D_α is equivalent to

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{-1-\alpha} dA(z)$$

see [68, Lemma 2], therefore $D_\alpha = L^2_a((1-|z|^2)^{-1-\alpha})$, where

$$\begin{aligned} &L^2_a((1-|z|^2)^{-1-\alpha}) \\ &= \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{-1-\alpha} dA(z) < \infty\}. \end{aligned}$$

For $\alpha > 1$, D_α is an algebra (see [42, Theorem 3]), thus $M(D_\alpha) = D_\alpha$, where $M(D_\alpha)$ is the multiplier algebra of D_α, i.e.

$$M(D_\alpha) = \{\varphi \in D_\alpha : \varphi f \in D_\alpha, \forall f \in D_\alpha\}.$$

For $\alpha \geq \beta$, $D_\alpha \subseteq D_\beta$ and $M(D_\alpha) \subseteq M(D_\beta)$ (see [68, Page 233]).

Recall that $D_\alpha \odot D_\beta$ is the space of weak products of function in D_α and D_β, i.e.,

$$D_\alpha \odot D_\beta = \{h = \sum_{i=1}^{\infty} f_i g_i : f_i \in D_\alpha, g_i \in D_\beta, \sum_{i=1}^{\infty} \|f_i\|_{D_\alpha} \|g_i\|_{D_\beta} < \infty\}.$$

Therefore if $\alpha > 1, \beta \leq \alpha$, we have $D_\alpha \odot D_\beta = D_\beta$, in this case the index of M_z-invariant subspace in D_β is well-known (see [4], [8], [35] and [36]).

Definition 2.5.7. *A set $\Lambda \in \mathbb{D}$ is called dominating for $\mathbb{T}$ if* $\sup_{\lambda \in \Lambda} |f(\lambda)| = \|f\|_{H^\infty(\mathbb{D})}$ *for every $f \in H^\infty(\mathbb{D})$.*

It was shown in [17] that Λ is dominating for $\mathbb{T}$ if and only if a.e. point ζ of $\mathbb{T}$ is the limit of a sequence of points from Λ that approach ζ nontangentially.

To study the index for M_z-invariant subspaces in $D_\alpha \odot D_\beta$, we need to consider the interpolating sequences (see [43] for the background on interpolating sequences).

Definition 2.5.8. (1) *A sequence $\{\lambda_n\}_{n\geq 1} \subseteq \mathbb{D}$ is called interpolating for D_α if the linear transformation defined by*

$$T_\alpha f = \{\frac{f(\lambda_n)}{\|k^\alpha\|_{D_\alpha}}\}$$

maps D_α into and onto l^2.

(2) *A sequence* $\{\lambda_n\}_{n\geq 1} \subseteq \mathbb{D}$ *is called interpolating for* $D_\alpha \odot D_\beta$ *if the linear transformation defined by*

$$T_{\alpha\beta} f = \{\frac{f(\lambda_n)}{\|k^\alpha\|_{D_\alpha}\|k^\beta\|_{D_\beta}}\}$$

maps $D_\alpha \odot D_\beta$ *into and onto* l^1.

For a sequence $\Lambda = \{\lambda_n\}_{n\geq 1} \subseteq \mathbb{D}$, write

$$I(\Lambda) = \{f \in D_\alpha \odot D_\beta : f(\lambda_n) = 0 \text{ for all } n\},$$

where we mean that if λ occurs in $\{\lambda_n\}_{n\geq 1}$ n times, then f has a zero at λ of order at least n. If $\mathcal{M}$ is an M_z-invariant subspace in $D_\alpha \odot D_\beta$, write $Z(\mathcal{M}) = \{z_n \in \mathbb{D} : f(z_n) = 0, \forall f \in \mathcal{M}\} = \bigcap_{f\in\mathcal{M}} Z(f)$.

The following Proposition can be derived from [5, Proposition 7.3].

Proposition 2.5.9. *Let* $\Lambda = \{\lambda_n\}_{n\geq 1} \subseteq \mathbb{D}$ *be interpolating for* $D_\alpha \odot D_\beta$. *Then the following are equivalent:*

(a). Λ *is dominating for* $\mathbb{T}$;

(b). *there is an invariant subspace* $\mathcal{M}$ *of* $(M_z, D_\alpha \odot D_\beta)$ *such that* $I(\Lambda) \subseteq \mathcal{M}$ *and* $\text{ind}\mathcal{M} > 1$.

Theorem 2.5.10. *If* $\alpha, \beta < 0$, *then there is an* M_z*-invariant subspace* $\mathcal{M} \subseteq D_\alpha \odot D_\beta$ *with* $\text{ind}\mathcal{M} > 1$.

Proof. Without loss of generality, suppose $\beta \leq \alpha < 0$, then $D_\alpha \subseteq D_\beta$. Let $\mathcal{N} \subseteq \text{Lat}(M_z, D_\alpha)$ with $Z(\mathcal{N})$ dominating for $\mathbb{T}$. Let $\Lambda = Z(\mathcal{N})$, as in [5, Corollary 7.4], we can choose a subsequence Λ' such that Λ' is dominating for $\mathbb{T}$ and for all $\lambda, \mu \in \Lambda'$, $\rho(\lambda, \mu) = \left|\frac{\lambda-\mu}{1-\bar{\lambda}\mu}\right| \geq r$ for some $r \in (0,1)$, where r depends on α and β so that Λ' is interpolating for D_α and D_β (see [57]). Thus Λ' is interpolating for $D_\alpha \odot D_\beta$

and dominating for $\mathbb{T}$, and so by Proposition 2.5.9, we conclude that there is an M_z-invariant subspace $\mathcal{M} \subseteq D_\alpha \odot D_\beta$ with $\text{ind}\mathcal{M} > 1$. $\square$

If $\alpha = \beta = 0$, then $D_0 \odot D_0 = H^2(\mathbb{D}) \odot H^2(\mathbb{D}) = H^1(\mathbb{D})$, in this case every nonzero M_z-invariant subspace in $H^1(\mathbb{D})$ has index one.

If $\alpha = \beta = 1$, then $D_1 \odot D_1 = D \odot D$, and we have proved in section 2.3 that every nonzero M_z-invariant subspace in $D \odot D$ has index one.

Question 2.5.11. (1) *If $\alpha < 0 \leq \beta \leq 1$, what is the index for M_z-invariant subspace in $\subseteq D_\alpha \odot D_\beta$? Especially, what is the index for M_z-invariant subspace in $\subseteq L_a^2(\mathbb{D}) \odot D$?*

(2) *If $\alpha, \beta \in [0, 1]$ except $\alpha = \beta = 0$ and $\alpha = \beta = 1$, what is the index for M_z-invariant subspace in $\subseteq D_\alpha \odot D_\beta$?*

$\mathcal{H}(k^\alpha \cdot k^\beta)$ space

Note that if $\alpha < 1$, then D_α has reproducing kernle $k^\alpha(z) = \frac{1}{(1-z)^{1-\alpha}}$. For $\alpha, \beta < 1$, let $\gamma = \alpha + \beta - 1$, then

$$\begin{aligned} k^\alpha(z) \cdot k^\beta(z) &= \frac{1}{(1-z)^{1-\alpha}} \cdot \frac{1}{(1-z)^{1-\beta}} \\ &= \frac{1}{(1-z)^{2-\alpha-\beta}} \\ &= \frac{1}{(1-z)^{1-\gamma}}, \end{aligned}$$

in this case $\mathcal{H}(k^\alpha \cdot k^\beta) = D_\gamma$ and the index of M_z-invariant subspace in D_β is well-known (see [4], [8], [35] and [36]).

Theorem 2.5.12. *If $\alpha = 1, \beta = 0$, then there is an M_z-invariant subspace $\mathcal{M} \subseteq \mathcal{H}(k^\alpha \cdot k^\beta)$ with* $\text{ind}\mathcal{M} > 1$.

Proof. If $\alpha = 1, \beta = 0$, then $k^1(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n = \frac{1}{z}\log(\frac{1}{1-z})$ is the kernel for the Dirichlet space D, $k^0(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ is the kernel for the Hardy space $H^2(\mathbb{D})$, thus

$$\begin{aligned} k^1(z) \cdot k^0(z) &= \frac{1}{z}\log(\frac{1}{1-z}) \cdot \frac{1}{1-z} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k+1} z^n. \end{aligned}$$

Let $a_n = \sum_{k=0}^{n} \frac{1}{k+1}$, then a_n is increasing. For $f \in \mathcal{H}(k^1 \cdot k^0)$, we have $\|f\|^2_{\mathcal{H}(k^1 \cdot k^0)} = \sum_{n=0}^{\infty} \frac{1}{a_n} |\hat{f}(n)|^2$, therefore $\|zf\|_{\mathcal{H}(k^1 \cdot k^0)} \leq \|f\|_{\mathcal{H}(k^1 \cdot k^0)}$.

Note that $\frac{k^1_\lambda(z) k^0_\lambda(z)}{k^0_\lambda(z)} = k^1_\lambda(z)$ is a positive definite kernel, thus

$$H^\infty(\mathbb{D}) = M(H^2(\mathbb{D})) \subseteq M(\mathcal{H}(k^1 \cdot k^0)) \subseteq \mathcal{H}(k^1 \cdot k^0).$$

Also $\lim_{|\lambda| \to 1}(1 - |\lambda|^2) k^1_\lambda(\lambda) k^0_\lambda(\lambda) = \lim_{|\lambda| \to 1} k^1_\lambda(\lambda) = \infty$, therefore by [7, Theorem 4.1], there is a sequence $\Lambda \subseteq \mathbb{D}$ that is interpolating for $H(k^1 \cdot k^0)$ and dominating for $\mathbb{T}$, and so by [5, Proposition 7.3] (also see Proposition 2.5.9), we have there is an M_z-invariant subspace $\mathcal{M} \subseteq \mathcal{H}(k^\alpha \cdot k^\beta)$ with $\text{ind}\mathcal{M} > 1$. □

By a similar argument, we have

Theorem 2.5.13. *If $\alpha > 1, \beta < 0$, or $\alpha = 1, \beta \leq 0$, then there is an M_z-invariant subspace $\mathcal{M} \subseteq \mathcal{H}(k^\alpha \cdot k^\beta)$ with* $\text{ind}\mathcal{M} > 1$.

Theorem 2.5.14. *If $\alpha, \beta > 1$, then every nonzero M_z-invariant subspace $\mathcal{M} \subseteq \mathcal{H}(k^\alpha \cdot k^\beta)$ has index 1.*

Proof. We show that $\mathcal{H}(k^\alpha \cdot k^\beta)$ is an algebra, and then the conclusion follows.

Note that $k^\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha} z^n$, we have

$$k^\alpha(z) \cdot k^\alpha(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(k+1)^\alpha} \frac{1}{(n-k+1)^\alpha} z^n.$$

Let $a_n = \sum_{k=0}^{n} \frac{1}{(k+1)^\alpha} \frac{1}{(n-k+1)^\alpha}$, then

$$\begin{aligned}
1 \leq (n+2)^\alpha \cdot a_n &= \sum_{k=0}^{n} \frac{(n+2)^\alpha}{(k+1)^\alpha (n-k+1)^\alpha} \\
&= \sum_{k=0}^{n} \Big(\frac{1}{k+1} + \frac{1}{n-k+1}\Big)^\alpha \\
&\leq 2^{1+\alpha} \sum_{k=0}^{n} \frac{1}{(k+1)^\alpha} \leq 2^{1+\alpha} \sum_{k=0}^{\infty} \frac{1}{(k+1)^\alpha} \\
&:= c_\alpha < \infty.
\end{aligned}$$

Similarly, let $b_n = \sum_{k=0}^{n} \frac{1}{(k+1)^\beta} \frac{1}{(n-k+1)^\beta}$, then

$$k^\beta(z) \cdot k^\beta(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(k+1)^\beta} \frac{1}{(n-k+1)^\beta} z^n = \sum_{n=0}^{\infty} b_n z^n,$$

and $1 \leq (n+2)^\beta b_n \leq c_\beta$ for some constant c_β depending only on β.

For $f \in \mathcal{H}(k^\alpha \cdot k^\beta)$, we have

$$\begin{aligned}
\|f\|^2_{\mathcal{H}(k^\alpha \cdot k^\beta)} &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \frac{1}{(k+1)^\alpha} \frac{1}{(n-k+1)^\beta}\Big)^{-1} |\hat{f}(n)|^2 \\
&\asymp \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)^{-1} |\hat{f}(n)|^2,
\end{aligned}$$

thus $f \in \mathcal{H}(k^\alpha \cdot k^\beta)$ if and only if $f \in \mathcal{H}((k^\alpha \cdot k^\beta)^2)$.

Note that if $f, g \in \mathcal{H}(k^\alpha \cdot k^\beta)$ with $\|f\|_{\mathcal{H}(k^\alpha \cdot k^\beta)}, \|g\|_{\mathcal{H}(k^\alpha \cdot k^\beta)} \leq 1$, then $fg \in \mathcal{H}((k^\alpha \cdot k^\beta)^2)$ with $\|fg\|_{\mathcal{H}((k^\alpha \cdot k^\beta)^2)} \leq 1$, thus $\mathcal{H}(k^\alpha \cdot k^\beta)$ is an algebra. $\square$

If $\alpha = \beta = 1$, then $k^1(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n = \frac{1}{z} \log(\frac{1}{1-z})$ is the kernel for the Dirichlet space D, therefore $\mathcal{H}(k^1 \cdot k^1) = \mathcal{H}(k_D^2)$ and we have proved in section 2.5 that every nonzero M_z-invariant subspace in $\mathcal{H}(k_D^2)$ has index one.

Question 2.5.15. (1) *If* $\alpha > 1, 0 \leq \beta \leq 1$, *what is the index for* M_z*-invariant subspace in* $\mathcal{H}(k^\alpha \cdot k^\beta)$*?*

(2) *If* $\alpha = 1, \beta > 0$ *except* $\alpha = \beta = 1$, *what is the index for* M_z*-invariant subspace in* $\subseteq D_\alpha \odot D_\beta$*?*

Chapter 3

Corona Theorem and Bass Stable Rank for $M(D(\mu))$

3.1 Corona theorem for $M(D(\sum_{i=1}^{k} a_i\delta_{\zeta_i}))$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. Let μ be a nonnegative Borel measure on the boundary $\mathbb{T}$ of the unit disc. Let φ_μ be the harmonic function

$$\varphi_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The Dirichlet type space $D(\mu)$ is defined as the space of all analytic functions on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$$

is finite. For any $f \in D(\mu)$, $\|f\|^2_{D(\mu)} := \|f\|^2_{H^2(\mathbb{D})} + \int_{\mathbb{D}} |f'(z)|^2 \varphi_\mu(z) dA(z)$. When $\mu = \frac{dt}{2\pi}$, $D(\frac{dt}{2\pi})$ is the Dirichlet space D.

Dirichlet type spaces were introduced by Richter in [50] as he was studying analytic two-isometries. Let $D_\zeta(f) = \|\frac{f - f(\zeta)}{z - \zeta}\|^2_{H^2(\mathbb{D})}$ be the local Dirichlet integral of f at ζ.

In [52], Richter and Sundberg showed that if $f \in D(\delta_\zeta)$, then

$$D_\zeta(f) = \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|\zeta - z|^2} dA(z), \quad \zeta \in \mathbb{T}.$$

Thus for any $f \in D(\mu)$, $\|f\|^2_{D(\mu)} = \|f\|^2_{H^2(\mathbb{D})} + \int_{\mathbb{T}} D_\zeta(f) d\mu(\zeta) = \|f\|^2_{H^2(\mathbb{D})} + \int_{\mathbb{T}} \|\frac{f-f(\zeta)}{z-\zeta}\|^2_{H^2(\mathbb{D})} d\mu(\zeta)$. $D_\zeta(f)$ turns out to be a convenient tool in studying these spaces.

In this section, we prove the corona theorem for $M(D(\mu))$ when $\mu = \sum_{i=1}^k a_i \delta_{\zeta_i} := \mu_k$, where a_i's are positive numbers, ζ_i's are in $\mathbb{T}$. Let $M(D(\mu_k))$ be the space of multipliers of $D(\mu_k)$, that is

$$M(D(\mu_k)) = \{\phi \in D(\mu_k) : \phi f \in D(\mu_k), \forall f \in D(\mu_k)\}.$$

First, we consider that $k = 1$ and $\mu_1 = \delta_1$, the unit point mass at 1. To prove the corona theorem for $M(D(\delta_1))$, we need the following two Lemmas (see [52]).

Lemma 3.1.1. *Let $f \in D(\delta_1)$. Then*

(i) $f = f(1) + (z-1)g$ *for some* $g \in H^2(\mathbb{D})$ *and* $D_1(f) = \|g\|^2_{H(\mathbb{D})}$.

(ii) $\lim_{r \to 1^-} f(r) = f(1)$.

(iii) $|f(1)| \le C\|f\|_{D(\delta_1)}$ *(see [63]).*

Lemma 3.1.2. *Let $\varphi \in H^\infty(\mathbb{D})$ and $f \in D(\delta_\zeta)$. Then $\varphi f \in D(\delta_\zeta)$ if and only if $f(\zeta) = 0$ or $\varphi \in D(\delta_\zeta)$. Furthermore,*

$$D_\zeta(\varphi f) \le 2(||\varphi||^2_\infty D_\zeta(f) + |f(\zeta)|^2 D_\zeta(\varphi))$$

and

$$|f(\zeta)|^2 D_\zeta(\varphi) \le 2(||\varphi||^2_\infty D_\zeta(f) + D_\zeta(\varphi f)).$$

If $f(\zeta) = 0$ then one even has $D_\zeta(\varphi f) \le ||\varphi||^2_\infty D_\zeta(f)$, while the second inequality can be replaced with the trivial observation that the right-hand side is nonnegative.

Thus, by Lemma 3.1.2, we have $M(D(\mu_k)) = D(\mu_k)\cap H^\infty(\mathbb{D})$, where $\mu_k = \sum_{i=1}^k a_i\delta_{\zeta_i}$. The norm in $D(\mu_k)\cap H^\infty(\mathbb{D})$ is defined by

$$||f||_{D(\mu_k)\cap H^\infty(\mathbb{D})} = ||f'||_{L^2(\varphi_{\mu_k}dA)} + ||f||_\infty, \quad f \in D(\mu_k)\cap H^\infty(\mathbb{D}),$$

where $||f'||^2_{L^2(\varphi_{\mu_k}dA)} = \int_{\mathbb{D}} |f'(z)|^2\varphi_{\mu_k}(z)dA(z)$.

We will use an idea which is similar to Lemma 2.1 of [47] to prove the corona theorem for $M(D(\delta_1))$.

For ease of notation, we let $K := M(D(\delta_1)) = D(\delta_1)\cap H^\infty(\mathbb{D})$, and $K_0 := \{f \in K, f(1) = 0\}$. Note that $K_0 \subset K$, and K_0 is a Banach algebra without identity.

Note that evaluation at $z \in \mathbb{D}\cup\{1\}$ is a multiplicative linear functional on K_0 (if $z = 1$, then it is a trivial one). We have the following lemma.

Lemma 3.1.3. *The set of multiplicative linear functionals consisting of evaluations at points of* $\mathbb{D}$ *is dense in the set of all multiplicative linear functionals on* K_0.

Proof. Let m be a non-zero multiplicative linear functional on K_0, then there exists a function $g_0 \in K_0$, such that $m(g_0) \neq 0$.

If $f \in H^\infty(\mathbb{D})$, define $M(f) := \frac{m(fg_0)}{m(g_0)}$.

Claim: M is well-defined, and M is a non-zero multiplicative linear functional on $H^\infty(\mathbb{D})$.

If we assume that the claim holds, then by Carleson's corona Theorem, there exists a net $(\beta_i)_{i\in I}$ of point evaluations in $\mathbb{D}$ that converges to M in the weak* topology of the maximal ideal space of $H^\infty(\mathbb{D})$. Note that m is the restriction of M to K_0:

$$M(f) = \frac{m(fg_0)}{m(g_0)} = \frac{m(f)m(g_0)}{m(g_0)} = m(f), f \in K_0.$$

Also the restriction of $(\beta_i)_{i\in I}$ gives a net of point evaluations in $\mathbb{D}$ that converges to m in the weak* topology of the dual space of K_0.

We are left to prove the claim: $f \in H^\infty(\mathbb{D})$, $g_0 \in K_0$, so $fg_0 \in K$ by Lemma 3.1.2. Also $(fg_0)(1) = 0$, so $fg_0 \in K_0$, which implies M is well-defined.

Clearly M is linear, when $f \in H^\infty(\mathbb{D})$,

$$\begin{aligned}
|M(f)| &= |\frac{m(fg_0)}{m(g_0)}| \leq \frac{\|fg_0\|_K}{|m(g_0)|} \\
&= \frac{\|fg_0\|_\infty + \|fg_0\|_{D(\delta_1)}}{|m(g_0)|} \leq \frac{\|f\|_\infty \|g_0\|_\infty + \|f\|_\infty \|g_0\|_{D(\delta_1)}}{|m(g_0)|} \\
&= \frac{\|g_0\|_K}{|m(g_0)|} \|f\|_\infty,
\end{aligned}$$

so M is a bounded functional on $H^\infty(\mathbb{D})$.

When $f, h \in H^\infty(\mathbb{D})$, $m(fhg_0)m(g_0) = m(fhg_0g_0) = m(fg_0)m(hg_0)$, thus we get

$$\begin{aligned}
M(fh) &= \frac{m(fhg_0)}{m(g_0)} \\
&= \frac{[m(fg_0)m(hg_0)]/m(g_0)}{m(g_0)} \\
&= M(f)M(h).
\end{aligned}$$

Therefore the claim is proved. □

Now, we can prove the following Theorem.

Theorem 3.1.4. *The set of multiplicative linear functionals consisting of evaluations at points of* $\mathbb{D} \cup \{1\}$ *is dense in the maximal ideal space of* K.

Proof. Suppose M is a non-zero multiplicative linear functional on K.

Let $m = M|_{K_0}$, then m is a multiplicative linear functional on K_0. If $f \in K$, then $f - f(1) \in K_0$, so $M(f) = f(1) + m(f - f(1))$.

<u>*Case* 1.</u> If $m = 0$, then $M(f) = f(1)$, so M is the point evaluation at 1.

<u>*Case* 2.</u> If $m \neq 0$, the by Lemma 3.1.3, there exists a net $(\beta_i)_{i \in I}$ of point evaluations in $\mathbb{D}$ that converges to m in the weak* topology on the dual space of K_0. Therefore,

for all $f \in K$,

$$\begin{aligned} M(f) &= f(1) + m(f - f(1)) = f(1) + (\lim_{i \in I} \beta_i)(f - f(1)) \\ &= f(1) + \lim_{i \in I}(f(\beta_i) - f(1)) \\ &= \lim_{i \in I} f(\beta_i) = (\lim_{i \in I} \beta_i)(f). \end{aligned}$$

Thus $M = \lim_{i \in I} \beta_i$, and this completes the proof. □

Remark 3.1.5. *For any $f \in K, 0 < r < 1$, let $E_r(f) = f(r)$, then from Lemma 3.1.1 we have $f(r) \to f(1)$ as $r \to 1$. Thus $E_r \to E_1$ in the weak star topology of K as $r \to 1$, which implies the set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D}$ is dense in the maximal ideal space of K.*

Now we consider general $k \geq 1$. Let $\zeta \in \mathbb{T}$, μ be a Borel measure in $\mathbb{T}$ with $\mu(\zeta) = 0$, and suppose that $\mathbb{D}$ is dense in the maximal ideal space of $M(D(\mu))$. Let $H := M(D(\mu)) \cap D(\delta_\zeta)$ and $H_0 := \{f \in H, f(\zeta) = 0\}$. The norm in H is defined by

$$\|f\|_* = \|f\|_{M(D(\mu))} + \|f'\|_{L^2(\varphi_{\delta_\zeta} dA)}, \quad f \in H,$$

then we have:

Lemma 3.1.6. *H is a Banach algebra, $H_0 \subset H$ and H_0 is a Banach algebra without identity.*

Proof. We only need to verify that H is an algebra. Suppose $f, g \in H = M(D(\mu)) \cap D(\delta_\zeta)$, then $fg \in M(D(\mu))$. Also $f - f(\zeta) \in H$ implies $\frac{f - f(\zeta)}{z - \zeta} g \in H^2(\mathbb{D})$, thus

$$fg = (z - \zeta)\Big(\frac{f - f(\zeta)}{z - \zeta} g\Big) + f(\zeta) g \in D(\delta_\zeta),$$

and so $fg \in H$. □

Lemma 3.1.7. *The set of multiplicative linear functionals consisting of evaluations at points of* $\mathbb{D}$ *is dense in the maximal ideal space of* H_0.

Proof. Let m be a non-zero multiplicative linear functional on H_0, then there exists a function $g_0 \in H_0$, such that $m(g_0) \neq 0$.

If $f \in M(D(\mu))$, define $M(f) := \frac{m(fg_0)}{m(g_0)}$.

Claim: M is well-defined, and M is a non-zero multiplicative linear functional on $M(D(\mu))$.

The proof of the claim is similar to the argument in Lemma 3.1.3. Then there exists a net $(\beta_i)_{i\in I}$ of point evaluations in $\mathbb{D}$ that converges to M in the Gelfand topology of the maximal ideal space of $M(D(\mu))$. Note that m is the restriction of M to H_0. Also the restriction of $(\beta_i)_{i\in I}$ gives a net of point evaluations in $\mathbb{D}$ that converges to m in the weak* topology on the dual of H_0.

□

By the same argument as in Theorem 3.1.4 we have the following Proposition:

Proposition 3.1.8. *The set of multiplicative linear functionals consisting of evaluations at points of* $\mathbb{D}$ *is dense in the maximal ideal space of* H.

Now we can prove the corona theorem for $M(D(\mu_k))$.

Theorem 3.1.9. *The set of multiplicative linear functionals consisting of evaluations at points of* $\mathbb{D}$ *is dense in the maximal ideal space of* $M(D(\mu_k))$.

Proof. This clearly follows from Proposition 3.1.8 and induction. □

Remark 3.1.10. *If we let* $d\mu = \frac{dt}{2\pi}$, *then* $D(\frac{dt}{2\pi})$ *is the Dirichlet space* D. *By Tolokonnikov [69], Xiao [76] we have the corona theorem holds in* $M(D)$, *then by Proposition 3.1.8 we also have the corona theorem holds in* $M(D) \cap D(\delta_\zeta)$ *for any* $\zeta \in \mathbb{T}$.

By the standard Gelfand theory of Banach algebras Theorem 3.1.9 implies:

Corollary 3.1.11. *The following are equivalent:*

(i) $\varphi_1, ..., \varphi_n \in M(D(\mu_k))$ *and there exists a* $\eta > 0$ *such that*

$$\sum_{j=1}^{n} |\varphi_j(z)|^2 \geq \eta > 0, \quad z \in \mathbb{D}.$$

(ii) *There are functions* $b_1, ..., b_n \in M(D(\mu_k))$ *such that*

$$\sum_{j=1}^{n} \varphi_j(z) b_j(z) = 1, \quad z \in \mathbb{D}.$$

We can generalize the proof of the corona theorem for $M(D(\mu_k))$ to the following theorem which is due to Carl Sundberg.

Theorem 3.1.12 (Sundberg)**.** *Let* $\mathcal{A} \subseteq H^\infty(\mathbb{D})$ *be a Banach algebra, then a corona theorem holds for* $\mathcal{A}$ *if and only if every* $\varphi \in \mathcal{M}_\mathcal{A}$ *extends to some* $\psi \in \mathcal{M}_{H^\infty}$*.*

Proof. Suppose every $\varphi \in \mathcal{M}_\mathcal{A}$ extends to some $\psi \in \mathcal{M}_{H^\infty}$. By Carleson's corona theorem, there exists a net z_α, such that $\lim_\alpha f(z_\alpha) = \psi(f), \forall f \in H^\infty$. Note that $\psi|_\mathcal{A} = \varphi$, thus $\lim_\alpha f(z_\alpha) = \varphi(f), \forall f \in \mathcal{A}$.

On the other hand, suppose a corona theorem holds for $\mathcal{A}$. Let $\varphi \in \mathcal{M}_\mathcal{A}$, then there exists a net z_α, such that $\lim_\alpha f(z_\alpha) = \varphi(f), \forall f \in \mathcal{A}$. Also z_α has a subnet converging to some $\psi \in \mathcal{M}_{H^\infty}$, thus $\varphi(f) = \psi(f), \forall f \in \mathcal{A}$. □

The following results are due to Stefan Richter.

Lemma 3.1.13. *Let* $\mathcal{A} \subseteq H^\infty(\mathbb{D})$ *be a Banach algebra. Suppose* $\|f\|_\infty \leq \|f\|_\mathcal{A}$ *for all* $f \in \mathcal{A}$*, then* $\mathcal{A} \cap D(\delta_\lambda)$ *is also a Banach algebra with norm* $\|f\|_* = \|f\|_\mathcal{A} + \|f'\|_{L^2(\varphi_{\delta_\zeta} dA)}$*, where* $\lambda \in \overline{\mathbb{D}}$*.*

Theorem 3.1.14 (Richter)**.** *Let* $\mathcal{A} \subseteq H^\infty(\mathbb{D})$ *be a Banach algebra that contains 1. Suppose* $\|f\|_\infty \leq \|f\|_\mathcal{A}$ *for all* $f \in \mathcal{A}$*. If* $\varphi \in \mathcal{M}_{\mathcal{A} \cap D(\delta_\lambda)}$*, then* φ *extends to some* $\psi \in \mathcal{M}_\mathcal{A}$*.*

Proof. Let $\mathcal{B} = \{f \in \mathcal{A} \cap D(\delta_\lambda) : f(\lambda) = 0\}$.

<u>*Case* 1.</u> There exists $g \in \mathcal{B}$ with $\varphi(g) \neq 0$. If $f \in \mathcal{A}$, then $fg = (z-\lambda)f\frac{g}{z-\lambda} \in \mathcal{A} \cap D(\delta_\lambda)$, thus $fg \in \mathcal{B}$.

Set $\psi(f) = \frac{\varphi(fg)}{\varphi(g)}$, then $\psi \in \mathcal{M}_\mathcal{A}$ and $\psi|_\mathcal{B} = \varphi|_\mathcal{B}$.

If $f \in \mathcal{A} \cap D(\delta_\lambda)$, then $f - f(\lambda) \in \mathcal{B}$, therefore

$$\begin{aligned}\psi(f) &= f(\lambda) + \psi(f - f(\lambda)) \\ &= f(\lambda) + \varphi(f - f(\lambda)) = \varphi(f),\end{aligned}$$

and so $\psi|_{\mathcal{A}\cap D(\delta_\lambda)} = \varphi$.

<u>*Case* 2.</u> If $\varphi|_\mathcal{B} = 0$. Note that for any $g \in \mathcal{A} \cap D(\delta_\lambda)$, we have $g = a + (z-\lambda)f$ for some $a \in \mathbb{C}$, $f \in H^2(\mathbb{D})$, then $\varphi(g) = a = g(\lambda)$.

Let $r_k \to 1^-, \mathcal{I} = \{f \in \mathcal{A} : f(r_k\lambda) \to 0 \text{ as } k \to \infty\}$, then $\mathcal{I} \subsetneq \mathcal{A}$ is an ideal. Thus $\exists \psi \in \mathcal{M}_\mathcal{A}$, such that $\mathcal{I} \subseteq \ker \psi$. If $g \in \mathcal{A} \cap D(\delta_\lambda)$, then $g - g(\lambda) \in \mathcal{I} \subseteq \ker \psi$, therefore $\psi(g) = g(\lambda) = \varphi(g)$ and so $\psi|_{\mathcal{A}\cap D(\delta_\lambda)} = \varphi$. □

3.2 Infinite version for $M(D(\sum_{i=1}^k a_i\delta_{\zeta_i}))$

The corona theorem for $H^\infty(\mathbb{D})$ and $M(D)$ has been generalized to infinitely many functions (see Rosenblum [58], Tolokonnikov [69] and Trent [73]). The infinite version, given by Rosenblum [58] and Tolokonnikov [69], can be formulated as follows (see also Uchiyama [74], Trent [72]):

Theorem 3.2.1. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq H^\infty(\mathbb{D})$. Suppose that*

$$0 < \epsilon^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1, \quad \textit{for all } z \in \mathbb{D}.$$

Then there exists $\{e_j\}_{j=1}^{\infty} \subseteq H^{\infty}(\mathbb{D})$ *such that* $\sum_{j=1}^{\infty} \varphi_j e_j = 1$ *and* $\sup_{z\in\mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \leq \frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2}$, *where* C_0 *is a constant.*

In this section, we consider $D_{l^2}(\mu_k)$, or $\oplus_1^{\infty} D(\mu_k)$, which can be considered as l^2-valued $D(\mu_k)$ space, where $\mu_k = \sum_{i=1}^{k} a_i \delta_{\zeta_i}$. If $F = (f_1, f_2, \ldots) \in \oplus_1^{\infty} D(\mu_k)$, then the norm is defined by

$$\begin{aligned}
&\|F\|^2_{\oplus_1^{\infty} D(\mu_k)} \\
&= \int_0^{2\pi} \|F(e^{it})\|^2_{l^2} \frac{dt}{2\pi} + \int_{\mathbb{T}} \int_0^{2\pi} \frac{\|F(e^{it}) - F(\zeta)\|^2_{l^2}}{|e^{it} - \zeta|^2} \frac{dt}{2\pi} d\mu_k(\zeta) \\
&= \sum_{j=1}^{\infty} \|f_j\|^2_{D(\mu_k)}.
\end{aligned}$$

Given $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\mu_k))$, we let $\Phi(z) = (\varphi_1(z), \varphi_2(z), \ldots)$. We use M_Φ to denote the (column) operator from $D(\mu_k)$ to $\oplus_1^{\infty} D(\mu_k)$ defined by

$$M_\Phi(f) = \{\varphi_j f\}_{j=1}^{\infty} \quad \text{for } f \in D(\mu_k).$$

Note that the pointwise hypothesis $\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ in Theorem 3.2.1 implies that the operator T_Φ defined on $H^2(\mathbb{D})$ in analogy to that of M_Φ is bounded and $\|T_\Phi\| = \sup_{z\in\mathbb{D}} (\sum_{j=1}^{\infty} |\varphi_j(z)|^2)^{\frac{1}{2}}$. Since $M(D(\mu_k)) = D(\mu_k) \cap H^{\infty}(\mathbb{D})$, the pointwise upper bound hypothesis will not be sufficient to conclude that M_Φ is bounded from $D(\mu_k)$ to $\oplus_1^{\infty} D(\mu_k)$. Thus, we will replace the assumption $\sum_{j=1}^{\infty} |\varphi_j(z)|^2 \leq 1$ for $z \in \mathbb{D}$ by the condition $\|M_\Phi\| \leq 1$.

First, we consider $M(D(\delta_1))$.

The following Lemma can be derived from [73, Lemma 6] (see also [60]), we include a proof here for completeness.

Lemma 3.2.2. *Let $\{a_j\}_{j=1}^{\infty} \in l^2$ and $A = (a_1, a_2, \ldots) \in B(l^2, \mathbb{C})$. Then there exists an $\infty \times \infty$ matrix Q_A, such that the entries of Q_A belong to the set $\{0, \pm a_j : j = 1, 2, \ldots\}$ and Q_A satisfies*

(a) *range of $Q_A \subseteq$ kernel of A.*

(b) $(AA^*)I - A^*A = Q_A Q_A^*$.

(c) *If $\{d_j\}_{j=1}^{\infty} \in l^2$ and $D = (d_1, d_2, \ldots)$, then*

$$(AD^\top)I - D^\top A = Q_A Q_D^\top.$$

Proof. For $k = 1, 2, \cdots$, let $C_k = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ a_{k+1} & a_{k+2} & a_{k+3} & \cdots \\ -a_k & 0 & 0 & \cdots \\ 0 & -a_k & 0 & \cdots \\ 0 & 0 & -a_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

where the first $k-1$ rows of C_k have only 0 entries.

Then

$$C_k C_k^* = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \sum_{j=k+1}^{\infty} |a_j|^2 & -\overline{a_k} a_{k+1} & -\overline{a_k} a_{k+2} & \cdots \\ 0 & \cdots & 0 & -a_k \overline{a_{k+1}} & |a_k|^2 & 0 & \cdots \\ 0 & \cdots & 0 & -a_k \overline{a_{k+2}} & 0 & |a_k|^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

thus

$$\sum_{k=1}^{\infty} C_k C_k^* = \begin{pmatrix} \sum_{k\neq 1} |a_k|^2 & -\overline{a_1}a_2 & -\overline{a_1}a_3 & \cdots \\ -\overline{a_2}a_1 & \sum_{k\neq 2} |a_k|^2 & -\overline{a_2}a_3 & \cdots \\ -\overline{a_3}a_1 & -\overline{a_3}a_2 & \sum_{k\neq 3} |a_k|^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= AA^*I - A^*A.$$

Let $Q_A = [C_1, C_2, \cdots] \in B(\oplus_1^\infty l^2, l^2)$, then $AA^*I - A^*A = Q_A Q_A^*$. □

We need one lemma before we prove the corona theorem for infinitely many functions in $M(D(\delta_1))$.

Lemma 3.2.3. *Let* $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$. *Then*

(i) M_Φ *is a bounded operator if and only if* $\sum_{j=1}^\infty \|\varphi_j\|^2_{D(\delta_1)}$ *and* $\sup_{z\in\mathbb{D}} \sum_{j=1}^\infty |\varphi_j(z)|^2$ *are finite.*

(ii) *If* $\|M_\Phi\| \leq 1$ *and* $0 < \epsilon^2 \leq \sum_{j=1}^\infty |\varphi_j(z)|^2$ *for all* $z \in \mathbb{D}$, *then*

$$\Phi(1) = (\varphi_1(1), \varphi_2(1), \ldots) \neq 0.$$

(iii) *If* $\|M_\Phi\| \leq 1$ *and* $f = \sum_{i=1}^\infty [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}$, *then* $f \in M(D(\delta_1))$ *and* $f(1) = 0$.

Proof. (i): Suppose that M_Φ is bounded from $D(\delta_1)$ to $\oplus_1^\infty D(\delta_1)$ with $\|M_\Phi\| \leq 1$, then $\sup_{z\in\mathbb{D}} \sum_{j=1}^\infty |\varphi_j(z)|^2 \leq 1$ (see [73]). Let $f = 1 \in D(\delta_1)$, then

$$\sum_{j=1}^\infty \|\varphi_j\|^2_{D(\delta_1)} = \|M_\Phi f\|^2_{\oplus_1^\infty D(\delta_1)}$$

$$\leq \|M_\Phi\|^2 \|1\|_{D(\delta_1)} \leq 1.$$

Conversely suppose $\sup_{z\in\mathbb{D}} \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \le 1$ and $\sum_{j=1}^{\infty} \|\varphi_j\|^2_{D(\delta_1)} \le 1$. Let $f \in D(\delta_1)$, suppose $f = f(1) + (z-1)g$ for some $g \in H^2(\mathbb{D})$, then $D_1(f) = \|g\|^2_{H^2(\mathbb{D})}$ and

$$
\begin{aligned}
&\|M_\Phi f\|^2_{\oplus_1^\infty D(\delta_1)} = \sum_{j=1}^{\infty} \|\varphi_j f\|^2_{D(\delta_1)} \\
&= \sum_{j=1}^{\infty} \|\varphi_j f\|^2_{H^2(\mathbb{D})} + \sum_{j=1}^{\infty} \|\frac{\varphi_j f - (\varphi_j f)(1)}{z-1}\|^2_{H^2(\mathbb{D})} \\
&\le \|f\|^2_{H^2(\mathbb{D})} + \sum_{j=1}^{\infty} \Big[2\|\frac{\varphi_j f(1) - (\varphi_j f)(1)}{z-1}\|^2_{H^2(\mathbb{D})} + 2\|\frac{\varphi_j g(z-1)}{z-1}\|^2_{H^2(\mathbb{D})}\Big] \\
&\le \|f\|^2_{H^2(\mathbb{D})} + 2|f(1)|^2 \sum_{j=1}^{\infty} D_1(\varphi_j) + 2\|g\|^2_{H^2(\mathbb{D})} \\
&\le 2\|f\|_{D(\delta_1)} + 2|f(1)|^2.
\end{aligned}
$$

Since $|f(1)| \le C\|f\|_{D(\delta_1)}$ (see [63]), we conclude that M_Φ is bounded from $D(\delta_1)$ to $\oplus_1^\infty D(\delta_1)$.

(ii): Suppose $\{g_j\}_{j=1}^{\infty} \subseteq H^2(\mathbb{D})$ such that

$$
\varphi_j(z) = \varphi_j(1) + (z-1)g_j(z), \quad \text{and} \quad D_1(\varphi_j) = \|g_j\|^2_{H^2(\mathbb{D})}, j = 1, 2, \cdots .
$$

Note that

$$
\begin{aligned}
|\varphi_j(z)|^2 &\le |\varphi_j(1)|^2 + |z-1|^2|g_j(z)|^2 + 2|\varphi_j(1)||z-1||g_j(z)| \\
&\le (1+\eta)|\varphi_j(1)|^2 + (1+\frac{1}{\eta})|z-1|^2|g_j(z)|^2,
\end{aligned}
$$

where η is any positive number. Then we have

$$\begin{aligned}\epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 &\leq \sum_{j=1}^{\infty} (1+\eta)|\varphi_j(1)|^2 + (1+\frac{1}{\eta})|z-1|^2|g_j(z)|^2 \\ &\leq \sum_{j=1}^{\infty} (1+\eta)|\varphi_j(1)|^2 + (1+\frac{1}{\eta})\frac{|z-1|^2}{1-|z|^2}\sum_{j=1}^{\infty}\|\varphi_j\|^2_{D(\delta_1)} \\ &\leq \sum_{j=1}^{\infty} (1+\eta)|\varphi_j(1)|^2 + (1+\frac{1}{\eta})\frac{|z-1|^2}{1-|z|^2} \quad \text{for all } z \in \mathbb{D},\end{aligned}$$

where in the last inequality we used part (i). Let $z = r \to 1^-$ we get

$$\epsilon^2 \leq \sum_{j=1}^{\infty} (1+\eta)|\varphi_j(1)|^2 := (1+\eta)|\Phi(1)|^2.$$

Let $\eta \to 0$, we have $|\Phi(1)|^2 = \sum_{j=1}^{\infty} |\varphi_j(1)|^2 \geq \epsilon^2$, thus $\Phi(1) = (\varphi_1(1), \varphi_2(1), \dots) \neq 0$.

(iii) Suppose $\|M_\Phi\| \leq 1$ and $f = \sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}$, then $f \in H^\infty(\mathbb{D})$ and

$$\begin{aligned}\|f\|^2_{D(\delta_1)} &= \|\sum_{i=1}^{\infty} [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}\|^2_{D(\delta_1)} \\ &\leq \sum_{i=1}^{\infty} \|\varphi_i - \varphi_i(1)\|^2_{D(\delta_1)} \sum_{i=1}^{\infty} |\varphi_i(1)|^2 \\ &\leq 2\Big[\sum_{i=1}^{\infty} \|\varphi_i\|^2_{D(\delta_1)} + \sum_{i=1}^{\infty} |\varphi_i(1)|^2\Big] \sum_{i=1}^{\infty} |\varphi_i(1)|^2 \\ &\leq 4,\end{aligned}$$

where in the last inequality we used part (i).

For any $k \in \mathbb{N}$, let $f_k = \sum_{i=1}^{k} [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}$. Then $f_k \to f \in D(\delta_1)$, note that $f_k(1) = 0$ and point evaluation at 1 is continuous, we conclude that $f(1) = 0$. $\square$

Now we can prove the corona theorem for $M(D(\delta_1))$.

Theorem 3.2.4. *Let $\{\varphi_j\}_{j=1}^{\infty} \subseteq M(D(\delta_1))$. Suppose that $\|M_\Phi\| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2$ for all $z \in \mathbb{D}$. Then there exists $\{b_j\}_{j=1}^{\infty} \subseteq M(D(\delta_1))$ such that*

(i) $\Phi(z)B(z)^\top = 1$ *for all* $z \in \mathbb{D}$, *and*

(ii) $\|M_B\| \leq \frac{1}{\varepsilon}(2 + 8\frac{C_0}{\epsilon^2}\ln\frac{1}{\epsilon^2})^{1/2}$.

Proof. (i): By Theorem 3.2.1, there exists an $E \in H^\infty_{l^2}(\mathbb{D})$ such that

$$\Phi(z)E(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$

and

$$\|E\|^2_{H^\infty_{l^2}(\mathbb{D})} := \sup_{z\in\mathbb{D}} \sum_{j=1}^{\infty} |e_j(z)|^2 \leq \frac{C_0}{\epsilon^2}\ln\frac{1}{\epsilon^2}.$$

Let $A = \Phi(z), D = E(z)$ in Lemma 3.2.2, then

$$I - E(z)^\top\Phi(z) = Q_{\Phi(z)}Q^\top_{E(z)},$$

thus

$$I = E(z)^\top\Phi(1) + E(z)^\top(\Phi(z) - \Phi(1)) + Q_{\Phi(z)}Q^\top_{E(z)}. \tag{3.2.1}$$

Let $\Phi(1)^* = (\overline{\varphi_1(1)}, \overline{\varphi_2(1)}, \ldots)^\top$, then $|\Phi(1)|^2 = \Phi(1)\Phi(1)^*$ and

$$\begin{aligned}\Phi(1)^* = E(z)^\top|\Phi(1)|^2 + E(z)^\top[\Phi(z) - \Phi(1)]\Phi(1)^* \\ + Q_{\Phi(z)}Q^\top_{E(z)}\Phi(1)^*.\end{aligned} \tag{3.2.2}$$

By Lemma 3.2.3 we have $\Phi(1) = (\varphi_1(1), \varphi_2(1), \ldots) \neq 0$, then from (3.2.2) we have

$$\frac{\Phi(1)^*}{|\Phi(1)|^2} = E(z)^\top + E(z)^\top\frac{[\Phi(z) - \Phi(1)]\Phi(1)^*}{|\Phi(1)|^2} + Q_{\Phi(z)}Q^\top_{E(z)}\frac{\Phi(1)^*}{|\Phi(1)|^2},$$

therefore,

$$E(z)^\top + Q_{\Phi(z)}Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2} = \frac{\Phi(1)^*}{|\Phi(1)|^2} - \frac{[\Phi(z)-\Phi(1)]\Phi(1)^*}{|\Phi(1)|^2}E(z)^\top$$
$$= \frac{\Phi(1)^*}{|\Phi(1)|^2} - \frac{\sum_{i=1}^\infty [\varphi_i(z)-\varphi_i(1)]\overline{\varphi_i(1)}}{|\Phi(1)|^2}E(z)^\top.$$

Let $B(z)^\top = E(z)^\top + Q_{\Phi(z)}Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2}$. From Lemma 3.2.2, we have

$$\Phi(z)B(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$

and

$$b_j(z) = \frac{\overline{\varphi_j(1)}}{|\Phi(1)|^2} - \frac{\sum_{i=1}^\infty [\varphi_i(z)-\varphi_i(1)]\overline{\varphi_i(1)}}{|\Phi(1)|^2} e_j(z), j = 1,2,3,\cdots.$$

By Lemma 3.2.3 we have $f := \sum_{i=1}^\infty [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)} \in M(D(\delta_1))$ and $f(1) = 0$. Thus from Lemma 3.1.2 we have $b_j \in H^\infty(\mathbb{D}) \cap D(\delta_1) = M(D(\delta_1)), j = 1,2,\cdots$.

(ii): Let $f \in D(\delta_1)$, then

$$\sum_{j=1}^\infty \|b_j f\|^2_{D(\delta_1)}$$
$$\le \frac{2}{|\Phi(1)|^4}\Big[\sum_{j=1}^\infty \|\overline{\varphi_j(1)}f\|^2_{D(\delta_1)} + \sum_{j=1}^\infty \|\sum_{i=1}^\infty [\varphi_i - \varphi_i(1)]\overline{\varphi_i(1)}e_j f\|^2_{D(\delta_1)}\Big]$$
$$\le \frac{2}{|\Phi(1)|^4}\Big[|\Phi(1)|^2\|f\|^2_{D(\delta_1)} + \sum_{j=1}^\infty\sum_{i=1}^\infty \|[\varphi_i - \varphi_i(1)]e_j f\|^2_{D(\delta_1)}|\Phi(1)|^2\Big]$$
$$= \frac{2}{|\Phi(1)|^2}\Big[\|f\|^2_{D(\delta_1)} + \sum_{j=1}^\infty\sum_{i=1}^\infty \|[\varphi_i - \varphi_i(1)]e_j f\|^2_{D(\delta_1)}\Big]$$

Note that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\|[\varphi_i-\varphi_i(1)]e_jf\|^2_{D(\delta_1)} \qquad (3.2.3)\\
&=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\|[\varphi_i-\varphi_i(1)]e_jf\|^2_{H^2(\mathbb{D})}+\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\|\frac{\varphi_i-\varphi_i(1)}{z-1}e_jf\|^2_{H^2(\mathbb{D})}\\
&\le \|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\sum_{i=1}^{\infty}\Big[\|(\varphi_i-\varphi_i(1))f\|^2_{H^2(\mathbb{D})}+\|\frac{\varphi_i-\varphi_i(1)}{z-1}f\|^2_{H^2(\mathbb{D})}\Big]\\
&=\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\sum_{i=1}^{\infty}\|(\varphi_i-\varphi_i(1))f\|^2_{D(\delta_1)}\\
&\le 2\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\Big[\sum_{i=1}^{\infty}\|\varphi_if\|^2_{D(\delta_1)}+\sum_{i=1}^{\infty}\|\varphi_i(1)f\|^2_{D(\delta_1)}\Big]\\
&\le 2\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\Big[\|M_\Phi\|^2+|\Phi(1)|^2\Big]\|f\|^2_{D(\delta_1)}\\
&\le 4\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\|f\|^2_{D(\delta_1)}.
\end{aligned}$$

Thus

$$\sum_{j=1}^{\infty}\|b_jf\|^2_{D(\delta_1)}\le\frac{2}{|\Phi(1)|^2}\Big[\|f\|^2_{D(\delta_1)}+4\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\|f\|^2_{D(\delta_1)}\Big],$$

therefore

$$\begin{aligned}
\|M_B\| &\le\Big[\frac{2}{|\Phi(1)|^2}(1+4\|E\|^2_{H^\infty_{l^2}(\mathbb{D})})\Big]^{1/2}\\
&\le\frac{1}{\varepsilon}(2+8\frac{C_0}{\epsilon^2}\ln\frac{1}{\epsilon^2})^{1/2},
\end{aligned}$$

where in the last inequality we used $|\Phi(1)|\ge\varepsilon$ in the proof of Lemma 3.2.3. □

Remark 3.2.5. *From equation (3.2.1), we can get another corona solution* $D(z)=(d_1(z),d_2(z),\cdots,)$ *such that*

$$\sum_{j=1}^{\infty}\varphi_j(z)d_j(z)=1,\quad z\in\mathbb{D}. \qquad (3.2.4)$$

Suppose $|\varphi_1(1)| = \max_{\{j=1,2,\ldots\}} |\varphi_j(1)|$*, let* $d_1(z) = \frac{1}{\varphi_1(1)} - \frac{\varphi_1(z)-\varphi_1(1)}{\varphi_1(1)} e_1(z)$*,* $d_j(z) = -\frac{\varphi_1(z)-\varphi_1(1)}{\varphi_1(1)} e_j(z), j = 2, 3, \cdots$*. Then (3.2.4) is satisfied and we have*

$$\|M_D\| \leq \Big[\frac{2}{|\varphi_1(1)|^2} + 4\big(\frac{\|\varphi_1\|^2_{M(D(\delta_1))}}{|\varphi_1(1)|^2} + 1\big)\frac{C_0}{\epsilon^2} \ln \frac{1}{\epsilon^2}\Big]^{1/2},$$

but in this case the bound of the corona solution depends on the chosen φ_1*. It would be of interest to determine the best possible bound for the solution* B *in terms of* $\|M_\Phi\|$ *and* ε*.*

For general k, we have the following theorem.

Theorem 3.2.6. *Let* $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$*. Suppose that*

$$\|M_\Phi\| \leq 1 \quad \text{and} \quad 0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $\{b_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$ *such that*

(i) $\Phi(z)B(z)^\top = 1$ *for all* $z \in \mathbb{D}$*, and*

(ii) $\|M_B\| \leq \frac{1}{\varepsilon}\Big(2 + 16\|M_{B_{k-1}}\|^2\Big)^{1/2}$*, where* B_{k-1} *is the solution for the corona theorem in* $M(D(\mu_{k-1}))$*.*

Proof. The idea is the same as in Theorem 3.2.4. We sketch a proof here.

If $k = 1$, then by Theorem 3.2.4, it is true.

Suppose $k = l \geq 1$, it is true.

If $k = l + 1$, note that $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_{l+1})) \subseteq M(D(\mu_l))$, by induction, there exists $\{e_j\}_{j=1}^\infty \subseteq M(D(\mu_l))$ such that

$$\Phi(z)E(z)^\top = 1 \quad \text{for } z \in \mathbb{D},$$

and

$$\|M_E\| \leq \frac{1}{\varepsilon}\Big(2 + 16\|M_{B_{l-1}}\|^2\Big)^{1/2},$$

Following the same argument as in Lemma 3.2.3, we have $\Phi(\zeta_{l+1}) = (\varphi_1(\zeta_{l+1}), \varphi_2(\zeta_{l+1}), \ldots) \neq 0$ and

$$I = E(z)^\top \Phi(\zeta_{l+1}) + E(z)^\top (\Phi(z) - \Phi(\zeta_{l+1})) + Q_{\Phi(z)} Q_{E(z)}^\top. \tag{3.2.5}$$

Thus

$$b_j(z) = \frac{\overline{\varphi_j(\zeta_{l+1})}}{|\Phi(\zeta_{l+1})|^2} - \frac{\sum_{i=1}^\infty [\varphi_i(z) - \varphi_i(\zeta_{l+1})]\overline{\varphi_i(\zeta_{l+1})}}{|\Phi(\zeta_{l+1})|^2} e_j(z) \in M(D(\mu_l)),$$

and $\Phi(z)B(z)^\top = 1$ for all $z \in \mathbb{D}$.

Now we estimate $\|M_B\|$. Let $f \in D(\mu_{l+1})$, then

$$\begin{aligned}
&\sum_{j=1}^\infty \|b_j f\|^2_{D(\mu_{l+1})} \\
&\leq \frac{2}{|\Phi(\zeta_{l+1}))|^2}\Big[\|f\|^2_{D(\mu_{l+1})} + \sum_{j=1}^\infty \sum_{i=1}^\infty \|[\varphi_i - \varphi_i(\zeta_{l+1}))] e_j f\|^2_{D(\mu_{l+1})}\Big].
\end{aligned}$$

Suppose $\mu_{l+1} = \mu_l + \delta_{\zeta_{l+1}}$, note that using inequality (3.2.3) we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1}))]e_j f\|^2_{D(\mu_{l+1})} \\
&\leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1}))]e_j f\|^2_{D(\mu_l)} \\
&\qquad\qquad + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \|[\varphi_i - \varphi_i(\zeta_{l+1}))]e_j f\|^2_{D(\delta_{\zeta_{l+1}})} \\
&\leq \sum_{i=1}^{\infty} \|M_E\|^2 \|[\varphi_i - \varphi_i(\zeta_{l+1}))]f\|^2_{D(\mu_l)} + 4\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\|f\|^2_{D(\delta_{\zeta_{l+1}})} \\
&\leq \|M_E\|^2 2\Big[\|M_\Phi\| + |\Phi(\zeta_{l+1})|^2)\Big]\|f\|^2_{D(\mu_{l+1})} + 4\|E\|^2_{H^\infty_{l^2}(\mathbb{D})}\|f\|^2_{D(\delta_{\zeta_{l+1}})} \\
&\leq 4\|M_E\|^2\|f\|^2_{D(\mu_{l+1})} + 4\|M_E\|^2\|f\|^2_{D(\mu_{l+1})} \\
&= 8\|M_E\|^2\|f\|^2_{D(\mu_{l+1})}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{\infty} \|b_j f\|^2_{D(\mu_{l+1})} &\leq \frac{2}{|\Phi(\zeta_{l+1}))|^2}\Big[\|f\|^2_{D(\mu_{l+1})} + 8\|M_E\|^2\|f\|^2_{D(\mu_{l+1})}\Big] \\
&\leq \frac{1}{\varepsilon^2}\Big(2 + 16\|M_E\|^2\Big)\|f\|^2_{D(\mu_{l+1})},
\end{aligned}$$

and so $\|M_B\| \leq \frac{1}{\varepsilon}\Big(2 + 16\|M_E\|^2\Big)^{1/2}$. □

3.3 Wolff's ideal theorem for $M(D(\sum_{i=1}^{k} a_i\delta_{\zeta_i}))$

Carleson's corona theorem states that the ideal generated by finitely many functions $\{\varphi_j\}_{j=1}^n \subseteq H^\infty(\mathbb{D})$ is the entire space $H^\infty(\mathbb{D})$ given that $\sum_{j=1}^n |\varphi_j(z)|^2 \geq \eta > 0$ for all $z \in \mathbb{D}$ and some $\eta > 0$. In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(\mathbb{D})$, Wolff ([32]) proved the following theorem.

Theorem 3.3.1. *If* $\{\varphi_j\}_{j=1}^n \subseteq H^\infty(\mathbb{D})$*,* $H \in H^\infty(\mathbb{D})$ *and*

$$|H(z)| \leq \Big(\sum_{j=1}^{n} |\varphi_j(z)|^2\Big)^{1/2} \quad \text{for all } z \in \mathbb{D}. \tag{3.3.1}$$

Then $H^3 \in \mathcal{I}(\{\varphi_j\}_{j=1}^n)$*, the ideal generated by* $\{\varphi_j\}_{j=1}^n$ *in* $H^\infty(\mathbb{D})$*.*

It is shown by Rao ([32]) that condition (3.3.1) is not sufficient for H to be in $\mathcal{I}(\{\varphi_j\}_{j=1}^n)$, also Treil ([71]) has shown that condition (3.3.1) is not sufficient for H^2 to be in $\mathcal{I}(\{\varphi_j\}_{j=1}^n)$.

Recently, Banjade and Trent [12], [13] proved the Wolff's ideal theorem for the multiplier algebra of the Dirichlet space and the multiplier algebra of the weighted Dirichlet space. Also Banjade, Holloway and Trent [14] proved the Wolff's ideal theorem on certain subalgebras of $H^\infty(\mathbb{D})$.

If we consider the radical of the ideal $\mathcal{I}(\{\varphi_j\}_{j=1}^n)$,

$$\text{Rad}(\{\varphi_j\}_{j=1}^n) = \{G \in H^\infty(\mathbb{D}) : \exists m \in \mathbb{N} \text{ with } G^m \in \mathcal{I}(\{\varphi_j\}_{j=1}^n)\},$$

then (3.3.1) gives a characterization of radical ideal membership. That is, $G \in \text{Rad}(\{\varphi_j\}_{j=1}^n)$ if and only if there exists $m \in \mathbb{N}$ such that $|G^m(z)| \leq \Big(\sum_{j=1}^n |\varphi_j(z)|^2\Big)^{1/2}$ for all $z \in \mathbb{D}$.

In this section, we consider Wolff's ideal theorem for $M(D(\mu_k))$, where $\mu_k = \sum_{i=1}^k a_i \delta_{\zeta_i}$.

First, we introduce the *harmonic Dirichlet - type spaces.*

Definition 3.3.2. *The harmonic Dirichlet - type space* $HD(\mu)$ *is the set of all functions* $f \in L^2(\mathbb{T})$ *such that* $D_\zeta(f)$ *is integrable with respect to* μ*, where* $D_\zeta(f) := \int_0^{2\pi} |\frac{f(e^{it})-f(\zeta)}{e^{it}-\zeta}|^2 \frac{dt}{2\pi}$ *is the local Dirichlet integral of* f *at* ζ*. Define the norm by* $\|f\|^2_{HD(\mu)} = \|f\|^2_{L^2(\mathbb{T})} + \int_{\mathbb{T}} D_\zeta(f) d\mu(\zeta)$*.*

The following Proposition is in [23].

Proposition 3.3.3. *Let f be a harmonic function on $\mathbb{D}$ of the form $f = f_+ + f_-$, where $f_+, \overline{f_-} \in D(\mu)$ and $f_-(0) = 0$. Then*

$$\int_{\mathbb{T}} D_\zeta(f)d\mu(\zeta) = \int_{\mathbb{D}}(|\frac{\partial f}{\partial z}|^2 + |\frac{\partial f}{\partial \overline{z}}|^2)\varphi_\mu dA = \int_{\mathbb{D}}(|f_+'|^2 + |\overline{f_-}'|^2)\varphi_\mu dA. \tag{3.3.2}$$

Remark 3.3.4. *For any $f \in HD(\mu)$, let $f(z) = (Pf)(z)$ be the harmonic extension of f to $\mathbb{D}$. Then f satisfies equation (3.3.2), and in the usual way, elements of $HD(\mu)$ can be regarded as functions on $\mathbb{T}$. As pointed out in [23], $HD(\mu)$ is a reproducing - kernel Hilbert space containing $D(\mu)$ as a closed subspace.*

First we consider $k = 1$, and we have the following theorem.

Theorem 3.3.5. *If $\{\varphi_j\}_{j=1}^n \subseteq M(D(\delta_1))$, $H \in M(D(\delta_1))$ and*

$$|H(z)| \leq \Big(\sum_{j=1}^n |\varphi_j(z)|^2\Big)^{1/2} \quad \textit{for all } z \in \mathbb{D}. \tag{3.3.3}$$

Then $H^3 \in \mathcal{I}(\{\varphi_j\}_{j=1}^n)$.

Proof. Let $\psi_j(z) = H(z)\overline{\varphi_j(z)}/\sum_{k=1}^n |\varphi_k(z)|^2, j = 1, \cdots, n$, then

$$|\psi_j(z)| \leq 1, \quad \text{and} \quad \sum_{j=1}^n \varphi_j(z)\psi_j(z) = H(z),\ z \in \mathbb{D}.$$

Also there exist $a_{jk}(z), j, k = 1, \cdots, n$ (see Garnett [32, P320]). with

$$|a_{jk}(z)| \leq C(n), \quad \text{and} \quad \frac{\partial a_{jk}(z)}{\partial \overline{z}} = H(z)\psi_j(z)\frac{\partial \psi_k(z)}{\partial \overline{z}},\ z \in \mathbb{D}.$$

Let

$$e_j(z) = H^2(z)\psi_j(z) + \sum_{k=1}^n \big(a_{jk}(z) - a_{kj}(z)\big)\varphi_k(z), j = 1, \cdots, n, \tag{3.3.4}$$

then

$$e_j \in H^\infty(\mathbb{D}), \quad \text{and} \quad \sum_{j=1}^{n} \varphi_j(z) e_j(z) = H^3(z),\ z \in \mathbb{D}.$$

<u>*Case* 1.</u> If $\Phi(1) = (\varphi_1(1), \cdots, \varphi_n(1)) = (0, \cdots, 0)$, then from (3.3.3) we have $H(1) = 0$. Thus by Proposition 3.3.3 and Lemma 3.1.2 we have $e_j \in H^\infty(\mathbb{D}) \cap D(\delta_1), j = 1, \cdots, n$. Let $b_j(z) = e_j(z), j = 1, \cdots, n$, then

$$e_j \in H^\infty(\mathbb{D}) \cap D(\delta_1), \quad \text{and} \quad \sum_{j=1}^{n} \varphi_j(z) e_j(z) = H^3(z),\ z \in \mathbb{D}.$$

<u>*Case* 2.</u> If $\Phi(1) = (\varphi_1(1), \cdots, \varphi_n(1)) \neq (0, \cdots, 0)$, note that there exist $Q_{\Phi(z)}$ and $Q_{E(z)}$ such that

$$\Phi(z) E(z)^\top I - E(z)^\top \Phi(z) = Q_{\Phi(z)} Q_{E(z)}^\top,$$

thus

$$H^3(z) = E(z)^\top \Phi(1) + E(z)^\top (\Phi(z) - \Phi(1)) + Q_{\Phi(z)} Q_{E(z)}^\top.$$

Let $\Phi(1)^* = (\overline{\varphi_1(1)}, \overline{\varphi_2(1)}, \ldots)^\top$, then $|\Phi(1)|^2 = \Phi(1)\Phi(1)^*$ and

$$\begin{aligned} H^3(z)\Phi(1)^* = E(z)^\top |\Phi(1)|^2 + E(z)^\top [\Phi(z) - \Phi(1)] \Phi(1)^* \\ + Q_{\Phi(z)} Q_{E(z)}^\top \Phi(1)^*, \end{aligned}$$

therefore,

$$\begin{aligned} E(z)^\top + Q_{\Phi(z)} Q_{E(z)}^\top \frac{\Phi(1)^*}{|\Phi(1)|^2} = H^3(z) \frac{\Phi(1)^*}{|\Phi(1)|^2} \\ - E(z)^\top \frac{[\Phi(z) - \Phi(1)]\Phi(1)^*}{|\Phi(1)|^2}. \end{aligned}$$

Let $B(z)^\top = H^3(z)\frac{\Phi(1)^*}{|\Phi(1)|^2} - E(z)^\top\frac{[\Phi(z)-\Phi(1)]\Phi(1)^*}{|\Phi(1)|^2}$, then

$$\begin{aligned} b_j &= H^3\frac{\overline{\varphi_j(1)}}{|\Phi(1)|^2} - \frac{\sum_{k=1}^n \big(\varphi_k - \varphi_k(1)\big)\overline{\varphi_k(1)}}{|\Phi(1)|^2}e_j \\ &\in M(D(\delta_1)), j = 1, 2, \cdots, n, \end{aligned}$$

and $\Phi(z)B(z)^\top = H^3(z)$.

□

For general k, let $\mu_k = \sum_{i=1}^k a_i\delta_{\zeta_i}$, where a_i's are positive numbers, ζ_i's are in $\mathbb{T}$. We have

Theorem 3.3.6. *If* $\{\varphi_j\}_{j=1}^n \subseteq M(D(\mu_k))$, $H \in M(D(\mu_k))$ *and*

$$|H(z)| \leq \Big(\sum_{j=1}^n |\varphi_j(z)|^2\Big)^{1/2} \quad \textit{for all } z \in \mathbb{D}.$$

Then $H^3 \in \mathcal{I}(\{\varphi_j\}_{j=1}^n)$.

Proof. For simplicity, we consider $k = 2$ and $\mu_2 = \delta_{\zeta_1} + \delta_{\zeta_2}$.

Case 1. If $\Phi(\zeta_2) = (\varphi_1(\zeta_2), \cdots, \varphi_n(\zeta_2)) = (0, \cdots, 0)$, we define e_j as in (3.3.4), then we have $e_j \in M(D(\delta_{\zeta_2})), j = 1, 2, \cdots, n$.

If $\Phi(\zeta_1) = (0, \cdots, 0)$, then e_j is also in $M(D(\delta_{\zeta_1}))$, thus $e_j \in M(D(\mu_2))$. Let $b_j = e_j$, then $\sum_{j=1}^n \varphi_j(z)e_j(z) = H^3(z), z \in \mathbb{D}$.

If $\Phi(\zeta_1) \neq (0, \cdots, 0)$, let

$$\begin{aligned} b_j &= H^3\frac{\overline{\varphi_j(\zeta_1)}}{|\Phi(\zeta_1)|^2} - \frac{\sum_{k=1}^n \big(\varphi_k - \varphi_k(\zeta_1)\big)\overline{\varphi_k(\zeta_1)}}{|\Phi(\zeta_1)|^2}e_j \\ &\in M(D(\delta_{\zeta_1})), j = 1, 2, \cdots, n, \end{aligned}$$

then $b_j \in M(D(\mu_2))$ and $\sum_{j=1}^n \varphi_j(z)b_j(z) = H^3(z), z \in \mathbb{D}$.

Case 2. If $\Phi(\zeta_2) \neq (0, \cdots, 0)$, let

$$b_j = H^3 \frac{\overline{\varphi_j(\zeta_2)}}{|\Phi(\zeta_2)|^2} - \frac{\sum_{k=1}^n \big(\varphi_k - \varphi_k(\zeta_2)\big)\overline{\varphi_k(\zeta_2)}}{|\Phi(\zeta_2)|^2} f_j, j = 1, 2, \cdots, n,$$

where $f_j \in M(D(\delta_{\zeta_1}))$ such that $\sum_{j=1}^n \varphi_j(z) f_j(z) = H^3(z), z \in \mathbb{D}$, then we have $b_j \in M(D(\mu_2))$ and $\sum_{j=1}^n \varphi_j(z) b_j(z) = H^3(z), z \in \mathbb{D}$. □

Corollary 3.3.7. *If $\{\varphi_j\}_{j=1}^n \subseteq M(D(\mu_k))$, $H \in M(D(\mu_k))$. Suppose that there exists $m \in \mathbb{N}$ such that*

$$|H^m(z)| \leq \Big(\sum_{j=1}^n |\varphi_j(z)|^2\Big)^{1/2} \quad \textit{for all } z \in \mathbb{D}.$$

Then $H \in Rad(\{\varphi_j\}_{j=1}^n)$.

Now we consider infinitely many functions.

Theorem 3.3.8. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$, $H \in M(D(\delta_1))$. Suppose that $\|M_\Phi\| \leq 1$ and*

$$|H(z)| \leq \Big(\sum_{j=1}^\infty |\varphi_j(z)|^2\Big)^{1/2} \quad \textit{for all } z \in \mathbb{D}.$$

Then there exist $\{b_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$ such that $\sum_{j=1}^\infty \varphi_j(z) b_j(z) = H^4(z)$ for all $z \in \mathbb{D}$.

Proof. *Case 1.* If $\Phi(1) \neq (0, 0, \cdots)$, let

$$f_j = H^3 \frac{\overline{\varphi_j(1)}}{|\Phi(1)|^2} - \frac{\sum_{k=1}^n \big(\varphi_k - \varphi_k(1)\big)\overline{\varphi_k(1)}}{|\Phi(1)|^2} e_j, j = 1, 2, \cdots,$$

where $e_j \in H^\infty(\mathbb{D})$ such that $\sum_{j=1}^n \varphi_j(z) e_j(z) = H^3(z), z \in \mathbb{D}$. Then we have $f_j \in M(D(\delta_1))$ and $\sum_{j=1}^n \varphi_j(z) f_j(z) = H^3(z), z \in \mathbb{D}$. Let $b_j = H f_j$, then $b_j \in M(D(\delta_1))$ and $\sum_{j=1}^n \varphi_j(z) b_j(z) = H^4(z), z \in \mathbb{D}$.

Case 2. If $\Phi(1) = (0, 0, \cdots)$, then by $\|M_\Phi\| \le 1$ we have $H(1) = 0$. Let $b_j = He_j$, where $e_j \in H^\infty(\mathbb{D})$ such that $\sum_{j=1}^n \varphi_j(z)e_j(z) = H^3(z), z \in \mathbb{D}$. Then we have $b_j \in M(D(\delta_1))$ and $\sum_{j=1}^n \varphi_j(z)b_j(z) = H^4(z), z \in \mathbb{D}$. □

Question 3.3.9. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$, $H \in M(D(\delta_1))$. Suppose that $\|M_\Phi\| \le 1$ and*

$$|H(z)| \le \Big(\sum_{j=1}^\infty |\varphi_j(z)|^2\Big)^{1/2} \quad \text{for all } z \in \mathbb{D}.$$

Does there exist $\{b_j\}_{j=1}^\infty \subseteq M(D(\delta_1))$ such that $\sum_{j=1}^\infty \varphi_j(z)b_j(z) = H^3(z)$ for all $z \in \mathbb{D}$?

By induction we have

Theorem 3.3.10. *Let $\{\varphi_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$, $H \in M(D(\mu_k))$. Suppose that $\|M_\Phi\| \le 1$ and*

$$|H(z)| \le \Big(\sum_{j=1}^\infty |\varphi_j(z)|^2\Big)^{1/2} \quad \text{for all } z \in \mathbb{D}.$$

Then there exist $\{b_j\}_{j=1}^\infty \subseteq M(D(\mu_k))$ such that $\sum_{j=1}^\infty \varphi_j(z)b_j(z) = H^{3+k}(z)$ for all $z \in \mathbb{D}$.

3.4 Bass stable rank for $M(D(\sum_{i=1}^k a_i\delta_{\zeta_i}))$

The notion of stable rank of a ring was introduced by Bass [15] to facilitate computations in algebraic K-theory.

Definition 3.4.1. *Let $\mathcal{A}$ be any ring with identity 1. An n-tuple $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$ is called unimodular or invertible, if there exists an n-tuple $b = (b_1, \ldots, b_n) \in \mathcal{A}^n$ such that $\sum_{i=1}^n a_ib_i = 1$. The set of all invertible n-tuples is denoted by $U_n(\mathcal{A})$. An (n+1)-tuple $x = (x_1, \ldots, x_{n+1}) \in \mathcal{A}^{n+1}$ is called reducible, if there exists an n-tuple*

$y = (y_1, \ldots, y_n)$ *such that* $(x_1 + y_1 x_{n+1}, \ldots, x_n + y_n x_{n+1})$ *is invertible. The Bass stable rank of* $\mathcal{A}$ *is the least* n *such that every invertible (n+1)-tuple is reducible.*

In recent years, the Bass stable rank has been studied by many authors in the setting of Banach algebras. Jones, Marshall and Wolff [40] showed that the Bass stable rank of the disc algebra $A(\mathbb{D})$ is one; Treil [70] proved that the Bass stable rank of $H^\infty(\mathbb{D})$ is one; and in [47], Mortini, Sasane and Wick showed that the Bass stable rank of $\mathbb{C} + BH^\infty$ and A_B are one as well. In this subsection, we show that the Bass stable rank of $M(D(\mu_k))$ is also one, where $\mu_k = \sum_{i=1}^k a_i \delta_{\zeta_i}$.

First, we prove that the Bass stable rank of $M(D(\delta_1)) = D(\delta_1) \cap H^\infty(\mathbb{D})$ is one.

Lemma 3.4.2. *The Bass stable rank of* $D(\delta_1) \cap H^\infty(\mathbb{D})$ *is one.*

Proof. Let (f, h) be a unimodular pair in $(D(\delta_1) \cap H^\infty(\mathbb{D}))^2$, i.e., there exists $(g_1, g_2) \in (D(\delta_1) \cap H^\infty(\mathbb{D}))^2$ such that $fg_1 + hg_2 = 1$. Then $\inf_{z\in\mathbb{D}} |f(z)| + |h(z)| := \eta > 0$.

Case 1. If $f(1) \neq 0$, then we claim $(f, (f - f(1))h)$ is unimodular.

In fact, if $z \in \mathbb{D}$ is such that $|f(z) - f(1)| \geq \frac{|f(1)|}{2}$, then $|f(z)| + |(f(z) - f(1)h(z)| \geq |f(z)| + \frac{|f(1)|}{2}|h(z)| \geq \min\{1, \frac{|f(1)|}{2}\}\eta$.

If $z \in \mathbb{D}$ is such that $|f(z) - f(1)| \leq \frac{|f(1)|}{2}$, then $|f(z)| = |f(z) - f(1) + f(1)| \geq |f(1)| - |f(z) - f(1)| \geq |\frac{|f(1)|}{2}|$, and so $|f(z)| + |(f(z) - f(1)h(z)| \geq |f(z)| \geq |\frac{|f(1)|}{2}|$.

Thus, $(f, (f - f(1))h)$ is unimodular. By Theorem 1 in [70], there is some element $g \in H^\infty(\mathbb{D})$ such that $f + g[(f - f(1))h]$ is invertible in $H^\infty(\mathbb{D})$. Note that $g(f - f(1)) \in D(\delta_1) \cap H^\infty(\mathbb{D})$, by the corona theorem for $M(D(\delta_1))$, we get that $f + g[(f - f(1))h]$ is also invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$.

Case 2. If $f(1) = 0$, then $h(1) \neq 0$, since $\inf_{z\in\mathbb{D}} |f(z)| + |h(z)| := \eta > 0$. We claim the pair $(f + h, h)$ is unimodular: By the corona theorem for $M(D(\delta_1))$, there exists $(g_1, g_2) \in (D(\delta_1) \cap H^\infty(\mathbb{D}))^2$ such that $fg_1 + hg_2 = 1$, so $(f + h)g_1 + h(g_2 - g_1) = 1$, which implies $(f + h, h)$ is unimodular.

By Case 1, there exists some $g \in D(\delta_1) \cap H^\infty(\mathbb{D})$, such that $(f+h)+gh$ is invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$. Note that $(f+h)+gh = f+(1+g)h$, and $1+g \in D(\delta_1) \cap H^\infty(\mathbb{D})$, we are done.

□

Now we show the Bass stable rank of $M(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2})) = D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.

Lemma 3.4.3. *The Bass stable rank of $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one.*

Proof. Let (f, h) be a unimodular pair in $(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D}))^2$.

Case 1. $f(\zeta_2) \neq 0$. As in Lemma 3.4.2 we conclude that $(f, (f - f(\zeta_2))h)$ is unimodular. Then by Lemma 3.4.2, there exists some $g \in D(\delta_1) \cap H^\infty(\mathbb{D})$ such that $f + g[(f - f(\zeta_2))h]$ is invertible in $D(\delta_1) \cap H^\infty(\mathbb{D})$. Note that $g(f - f(\zeta_2)) \in D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$, by the corona theorem for $M(D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}))$, we get $f + g[(f - f(1))h]$ is also invertible in $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$.

Case 2. $f(\zeta_2) = 0$. As in Lemma 3.4.2, we consider the pair $(f + h, h)$ and conclude that the Bass stable rank of $D(\delta_{\zeta_1}) \cap D(\delta_{\zeta_2}) \cap H^\infty(\mathbb{D})$ is one. □

For general k, by induction we obtain that the Bass stable rank of $M(D(\mu_k))$ is one.

Theorem 3.4.4. *The Bass stable rank of $M(D(\mu_k))$ is one.*

Topological Stable Rank for $M(D(\sum_{i=1}^k a_i\delta_{\zeta_i}))$

Definition 3.4.5. *Let $\mathcal{A}$ be a Banach algebra with identity 1. The topological stable rank of $\mathcal{A}$ (denoted by $tsr(\mathcal{A})$) is the least n such that $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$.*

It is shown in [56] that $bsr(\mathcal{A}) \leq tsr(\mathcal{A})$.

Note that $z \in M(D(\mu_k))$, where $\mu_k = \sum_{i=1}^{k} a_i \delta_{\zeta_i}$, by the maximum modulus Theorem or Hurwitz Theorem, z can not be uniformly approximated by invertible elements in $D(\mu_k)$. Thus $tsr(M(D(\mu_k)) \geq 2$.

Question 3.4.6. *Is the topological stable rank of $M(D(\mu_k))$ two?*

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