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The dS/CFT Correspondence and Quasinormal Modes of Black Holes

Scott Henry Ness
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Scott Henry Ness entitled "The dS/CFT Correspondence and Quasinormal Modes of Black Holes." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

George Siopsis, Major Professor

We have read this dissertation and recommend its acceptance:

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Vice Provost and Dean of the Graduate School

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The dS/CFT Correspondence and Quasinormal Modes of Black Holes

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Scott Henry Ness
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Abstract

In this thesis we discuss two aspects of quantum gravity and break it up in the following way.

In part I, we discuss a scalar field theory living in de Sitter space-time. We may describe the infinite past or future as being boundaries of this space-time and, on these boundaries we construct a field theory. It has been shown by Strominger that there exists a correspondence between the bulk de Sitter space-time and the field theory living in the infinite past [1]. This may be described as a holographic principle, where information in the bulk de Sitter space-time corresponds to information contained in the boundary field theory.

We discuss the correspondence in two dimensions where the field theory is represented by a quantum mechanical model with conformal symmetry. We build up the quantum mechanical model and construct its Hamiltonian along with its energy eigenstates.

Next, we study the correspondence for a three dimensional asymptotic de Sitter space. By approaching the boundary of the space-time the symmetry is enhanced for the corresponding field theory. These symmetries are generated by charges dictated by Noether's theorem. We explicitly calculate the generators of these symmetries and show they satisfy the Virasoro algebra with a central extension which helps to create a full picture of the correspondence.

In part II, we focus on the ramifications of perturbed black holes in asymptotically anti-de Sitter space-time. By perturbing a black hole, it vibrates in characteristic modes much like the ringing of a bell. These modes are known as quasi-normal modes. We will show that by applying the appropriate boundary conditions, the quasi-normal frequencies are quantized. We calculate the quasi-normal frequencies in four and five dimensions perturbatively for various types of perturbations.

Understanding these modes may help in understanding the holographic principle, and can give insight into the intrinsic parameters of the black holes. It is important to understand the characteristic modes and corresponding characteristic frequencies of these black holes in order to hopefully compare to experimental results from future gravitational wave detectors.

Contents

List of Figures

Part I

Introduction

This thesis is broken into two parts both dealing with aspects of quanutm gravity. In part II, we discuss the holographic principle for de Sitter space-times in two and three dimensions. We calculate various quantities on the boundary field theory, which help in explicitly showing the holographic principle. In part III, we follow by looking at particular aspects of black holes, namely quasinormal modes of black holes. Understanding these modes may help in understanding properties of the holographic principle in more detail.

Part II

The dS/CFT Correspondence

Chapter 1

Introduction

1.1 Motivation

One of the ultimate unsolved problems in physics today is the problem of unifying the four forces (strong, weak, electromagnetic and gravitational) into a single theory. There have been attempts to unify the forces that make up the Standard model. One attempt at unification was the SU(5) model. This theory contains all the forces of the Standard Model (strong, weak, electromagnetic) however, it predicts rapid proton decay at such a rate which is below current experimental limits and is rejected. Other grand unified theories have been proposed which include the Standard Model, and do not contradict experimental results. These theories however do not include the gravitational force. Presently, the best candidate for unification is superstring theory which unifies gravity along with the forces contained in the Standard model. String theory was originally invented to explain characteristics of hadron behavior. In particle-accelerator experiments, physicists observed that the angular momentum of a hadron is proportional to the square of its mass. No simple model of the hadron, such as picturing it as a set of constituent particles held together by spring-like forces, was able to explain these relationships. In order to account for these "Regge trajectories," physicists turned to a model where each hadron was in fact a rotating string, moving in accordance with Einstein's special theory of relativity. Although string theory could explain these relationships, a new theory known as quantum chromodynamics (QCD) moved to the forefront because it could explain the experimental data in the high energy regime and predicted asymptotic freedom. QCD was established as the theory that explains how quarks interact through the strong interaction.

Although QCD explains the experimental data, work continued in developing superstring theory, because it included gravity. In 1984-85 there was a series of theoretical developments [5] that convinced theorists that superstring theory is a very promising approach to unification of all forces. It wasn't long before the subject was transformed from an intellectual backwater to one of the most active areas of theoretical physics, which it has remained ever since. String theory abandons the notion of point particles and replaces them with one-dimensional strings. By the

1990s, work in this field led to the construction of five distinctly different string theories. This leads to the questions, why is there more than one theory and which is the ultimate theory? Evidence shows that these five string theories are in fact, particular limits of a larger theory known as M theory. These five string theories are related to each other via duality transformations. Let us look at an example.

The duality symmetry that obscures our ability to distinguish between large and small distance scales is called T-duality, and comes about from the compactification of extra space dimensions in a ten dimensional superstring theory. If we compactify one of the dimensions on a circle, we introduce a new parameter, known as the winding number. The energy of the string is now expressed in terms of the momentum modes and the winding modes. These momentum and winding modes may be interchanged, as long as we also interchange the radius R of the circle with the quantity ℓ_s^2/R, where ℓ_s is the string length. If R is very much smaller than the string length, then the quantity ℓ_s^2/R is going to be very large. Therefore, exchanging momentum and winding modes of the string exchanges a large distance scale with a small distance scale.

T-duality relates Type IIA superstring theory to Type IIB superstring theory, both of which are theories of closed strings that possess supersymmetry. That means if we take Type IIA and Type IIB theory and compactify them both on a circle, then switching the momentum and winding modes, and switching the distance scale, changes one theory into the other! The same is also true for the two heterotic theories, which also describe closed strings that possess supersymmetry. In addition to T-duality there is a symmetry called S-duality which relates the coupling strength of two theories. Understanding the weak(strong) coupling limit in a Type I superstring theory leads to understanding the strong(weak) coupling limit of the heterotic superstring theory, with gauge group $SO(32)$ through S-duality.

At present, physicists and mathematicians are studying various aspects of string theory. We can study the string theories in different limits and show how some are related via the dualities and it looks like they may combine into a single theory which we call M theory. We can probe various limits of M theory but its general description is presently not understood. We are hoping the fundamental principles of string theory will be found, which would explain many of the unanswered questions and lead us into the right direction. Presently there are many new aspects of string theory that can be studied. This thesis will cover only two aspects of quantum gravity: the de Sitter/ Conformal field theory (dS/CFT) correspondence and quasinormal modes of black holes. The dS/CFT correspondence is motived by the anti-de Sitter/ Conformal field theory (AdS/CFT) correspondence which was introduced by Maldacena, and because we live in a universe that appears to have a positive cosmological constant [17].

In 1998, Maldecena opened up a new era of exploration in string theory. He used the idea of 't Hooft and studied the large N (number of colors) limit of superconformal field theories and supergravity on the product of Anti-de Sitter space-times (appendix A), spheres and other compact manifolds [7]. The large N limit is presumably equivalent to the low energy limit. In either limit, the theory in the bulk, described by a type IIB string theory, decouples from the theory on the boundary, described by a $\mathcal{N} = 4$ super Yang Mills theory. In essence, Maldacena connected one particular superstring theory to a conformal field theory on the boundary of the space-time. This correspondence may connect string theory back to QCD. If a supersymmetry breaking mechanism is discovered, we would be able to explain QCD from string theory.

The conjecture was a possible breakthrough in relating string theory to the standard model and led to a surge of new research [8, 9, 10, 11, 12, 13, 14, 15]. Even though the AdS/CFT correspondence is by now well-understood, a similar correspondence for a de Sitter space has been quite a puzzle to establish. In contrast to anti-de Sitter space-time which has negative cosmological constant, de Sitter space-time has positive cosmological constant and may describe the vacuum state of our universe.

The AdS/CFT correspondence works, because anti-de Sitter space has a physical boundary on which the CFT can live. How can one construct a dS/CFT correspondence when de Sitter space-times have no physical boundary? A concrete proposal for a dS/CFT correspondence was recently put forth by Strominger [1] and attracted much attention [18]. According to this proposal, all observables in the bulk de Sitter space are generated by data specified on its asymptotic boundary which can be selected to be the Euclidean hypersurface $\mathcal{I}^-$ in the infinite past. The isometries of the de Sitter space are mapped onto generators of the conformal group of the theory defined on the boundary $\mathcal{I}^-$. This CFT is hard to construct in general, but various features, such as conformal weights and masses, are known. In three-dimensional de Sitter space, the conformal group on the boundary is infinite and the central charge is known [19].

Understanding the dS/CFT will give insight into our universe. We would like to extend the work in three dimensions to a higher dimensional correspondence, namely quantum gravity on a four-dimensional de Sitter space corresponding to the vacuum state of our universe.

1.2 The dS/CFT correspondence

In this section we will give an overview of what the dS/CFT correspondence is by following the work of [1]. We will choose a region of space-time comprising of the causal past of a timelike observer in space-time. We will calculate the Hadamard two-point function in this slice and on its boundary. The Hadamard two-point function may be expressed as the symmetrized product of two fields given by

$$G(\vec{X};\vec{X}') \sim \langle 0|\{\phi(\vec{X}),\phi(\vec{X}')\}|0\rangle$$

We will show that there exists a conformal field theory on the boundary and that its two-point function has the same behavior as the two-point function in the bulk of the region.

d-dimensional de Sitter space may be described by the hyperboloid

$$\sum_{i=1}^{d} X_i^2 - T^2 = \ell^2, \tag{1.1}$$

where $(\vec{X}, T)$ are expressed in terms of spherical coordinates $(\tau, \hat{n})$ defined by

$$T = \ell \sinh\tau, \quad \vec{X} = \ell \cosh\tau \hat{n},$$

where $\vec{X} = (X_1, X_2, ..., X_d)$ and $\hat{n}$ lives on the unit sphere S^{d-1}. The metric is given by

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2\tau d\Omega_{d-1}^2.$$

For additional properties on de Sitter and Anti-de Sitter spacetimes see appendix A.

We may calculate the Hadamard two-point function $G(\tau, \hat{n}; \tau', \hat{n}')$ for the space. Away from the singularities, it satisfies the equation

$$\nabla_d^2 G(\tau, \hat{n}; \tau', \hat{n}') = m^2 G(\tau, \hat{n}; \tau', \hat{n}'), \tag{1.2}$$

where ∇_d^2 acts on the coordinates $(\tau, \hat{n})$. By thinking of τ as an imaginary angle $i\theta$, the space becomes the surface of a sphere with radius ℓ. Without loss of generality we may fix $(\tau', \hat{n}')$ to lie along the T axis. By doing this, G depends on τ and not $\hat{n}$. This leads (1.2) to the differential equation

$$\frac{1}{\cosh^{d-1}\tau} \frac{d}{d\tau}\left(\cosh^{d-1}\tau \frac{dG}{d\tau}\right) + m^2\ell^2 G = 0$$

By making the substitution $P = \sinh\tau$, we have the differential equation

$$(P^2+1)\frac{d^2G}{dP^2} + dP\frac{dG}{dP} + m^2\ell^2 = 0, \tag{1.3}$$

whose solution is given by

$$G(P) = \text{Re } F(h_+, h_-; d/2; (1+P)/2), \tag{1.4}$$

where

$$h_\pm = \frac{1}{2}\left[(d-1) \pm \sqrt{(d-1)^2 - 4m^2\ell^2}\right]. \tag{1.5}$$

In an arbitrary frame, P is the geodesic distance between two points X and X' and is given by

$$\ell^2 P = X \cdot X',$$

where $X = (T, \vec{X})$ and $X' = (T', \vec{X}')$. Expressed in terms of the spherical coordinates

$$P = \cosh\tau \cosh\tau' \hat{n} \cdot \hat{n}' - \sinh\tau \sinh\tau'.$$

Let us focus on the case d=3. This case is special, because we may express gravity on $(A)dS_3$ as a Chern-Simons theory [8, 21], which we will show in chapter 3. The positive cosmological constant of de Sitter space acts like a negative pressure which drives the aceleration of the space-time. Due to this acceleration, there exist causally disconnected regions. We will choose a particular region $\mathcal{O}^-$ comprising of the causal past of a timelike observer in de Sitter space illustrated by the Penrose diagram in Figure 1.1.

We may define planar coordinates (z, t) for $\mathcal{O}^-$ as

$$t = -\ln \frac{Z-T}{\ell}, \quad z = \frac{X+iY}{Z-T}$$

$$Z - T = \ell e^{-t}, \quad Z + T = \ell e^{-t} - \ell z\bar{z}e^{-t}, \quad X + iY = \ell z e^{-t}. \tag{1.6}$$

The metric expressed in terms of planar coordinates is given by

$$\frac{ds^2}{\ell^2} = e^{-2t} dz d\bar{z} - dt^2 \tag{1.7}$$

It is shown in [10] that by relaxing the boundary conditions on the metric the symmetry of the space (isometries) is enhanced. This enhanced symmetry plays a fundamental role in both the (A)dS/CFT correspondences giving rise to a conformal group on the boundary. The relaxed boundary conditions for an asymptotic de Sitter space [1] are given by

$$g_{z\bar{z}} = \frac{1}{2}e^{-2t} + \mathcal{O}(1), \quad g_{tt} = -1 + \mathcal{O}(e^{2t}), \quad g_{zz} = \mathcal{O}(1), \quad g_{zt} = \mathcal{O}(e^{3t}), \tag{1.8}$$

and reduce to the normal boundary conditions in the limit as $t \to -\infty$. The asymptotic symmetries of dS_3 are diffeomorphisms which preserve (1.8). We may construct the Killing vector fields for the larger group of isometries

$$\zeta = U(z)\partial_z + \frac{1}{2}e^{2t}U''(z)\partial_{\bar{z}} + \frac{1}{2}U'(z)\partial_t, \tag{1.9}$$

where $U(z)$ is holomorphic. It should be noted that we have suppressed the anti-holomorphic contribution, and throughout this review we will only discuss the holomorphic side. By evaluating the Killing equations one can show that the metric is invariant under diffeomorphisms in the special case $U'''(z) = 0$. If we choose $U(z)$ to be a quadratic polynomial with complex coefficients, the asymptotic symmetry group for dS_3 is the conformal group of the complex plane, and the isometries form an $SL(2, C)$ subgroup. Therefore, by easing the restrictions on the boundary conditions we find an enhanced asymptotic symmetry group (conformal group) acting on $\mathcal{I}^-$, rather than just the isometry group $SL(2, C)$.

The asymptotic symmetry group near the boundary is conformal, and has a central charge. One may calculate this charge by constructing the stress tensor near

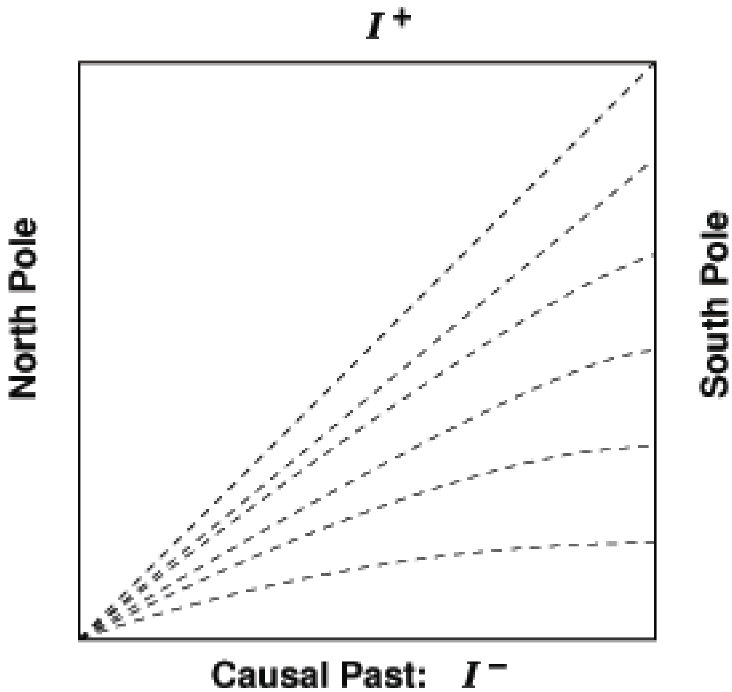

Figure 1.1: Penrose diagram for dS_3. A timelike observer in $\mathcal{O}^-$ is restricted to the lower left triangle in the diagram whose base represents the causal past.

$\mathcal{I}^-$. In general, the stress tensor is given by

$$T^{\mu\nu} = -\frac{4\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma_{\mu\nu}} \tag{1.10}$$

where γ is the induced metric on the boundary and the action is given by

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\sqrt{-g}(R - \frac{2}{\ell^2}) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{\gamma}K + \frac{1}{8\pi G\ell}\int_{\partial\mathcal{M}}\sqrt{\gamma},$$

where K is the extrinsic curvature given by $K_{\mu\nu} = -\nabla_{(\mu}n_{\nu)}$, with n^ν the outward pointing unit normal. The second integral is a surface term and is required for a well defined variational principle. The third integral is the counterterm action that we will add in order to obtain a finite stress tensor. By evaluating (1.10) we find

$$T^{\mu\nu} = \frac{1}{4G}\left[K^{\mu\nu} - (K + \frac{1}{\ell})\gamma^{\mu\nu}\right].$$

This vanishes for our choice of coordinates (1.7), however, for a more general set of coordinates we may express the stress tensor near the boundary as

$$T_{zz} = \frac{1}{4G}\left[K_{zz} + \frac{1}{\ell}\gamma_{zz}\right]. \tag{1.11}$$

Under the conformal transformations (1.9) one finds

$$\delta_U T_{zz} = -\frac{\ell}{8G} U'''.$$

This transformation identifies the central charge as

$$c = \frac{3\ell}{2G}.$$

There are various other ways to find the central charge now that the boundary stress tensor is known [8, 9, 16]. Next, let us look at the boundary field theory.

Consider a scalar field of mass m with the wave equation

$$-\partial_t^2 \phi + 2\partial_t \phi + 4e^{2t} \partial_z \partial_{\bar{z}} \phi = m^2 \ell^2 \phi,$$

whose solution near $\mathcal{I}^-$ behaves as

$$\phi \sim e^{h_\pm t}, \quad h_\pm = 1 \pm \sqrt{1 - m^2 \ell^2}.$$

We may restrict $m^2\ell^2$ such that $h_\pm \in \mathbb{R}$. As a boundary condition on $\mathcal{I}^-$ we demand

$$\lim_{t \to -\infty} \phi(z, \bar{z}, t) = e^{h_- t} \phi_-(z, \bar{z}). \tag{1.12}$$

The dS/CFT correspondence proposes $\phi_-(z, \bar{z})$ is dual to an operator $\mathcal{O}_\phi$ of dimension h_+ in the boundary CFT. The two point correlator of $\mathcal{O}_\phi$ (up to a normalization constant) is given by the quadratic coefficient of

$$\lim_{t \to -\infty} \int_{\mathcal{I}^-} d^2 z d^2 z' [e^{-2(t+t')} \phi(t, z, \bar{z}) \overleftrightarrow{\partial}_t G(t, z, \bar{z}; t', z', \bar{z}') \overleftrightarrow{\partial}_{t'} \phi(t', z', \bar{z}')]_{t=t'} \tag{1.13}$$

where G is the de Sitter invariant two-point function in three dimensions. By setting $d = 3$, the two-point function (1.4) reduces to

$$G(t, z, \bar{z}; t', z', \bar{z}') = \mathrm{Re} F(h_+, h_-, \frac{3}{2}; \frac{1+P}{2}), \tag{1.14}$$

where $h_\pm$ is given in (1.5), and

$$\lim_{t, t' \to -\infty} P(t, z, \bar{z}; t', z', \bar{z}') = -\frac{1}{2} e^{-t-t'} |z - z'|^2.$$

In this limit F diverges. We can make use of the transformation formula for the hypergeometric function

$$F(h_+, h_-; \frac{3}{2}; \frac{1+P}{2}) = \frac{\Gamma(\frac{3}{2})\Gamma(h_- - h_+)}{\Gamma(h_-)\Gamma(\frac{3}{2} - h_+)} (-z)^{-h_+} F(h_+, h_+ - \frac{1}{2}, h_+ + 1 - h_-; \frac{1}{z}) + (h_+ \leftrightarrow h_-), \tag{1.15}$$

and $F(\alpha, \beta; \gamma; 0) = 1$. One finds

$$\lim_{t,t' \to -\infty} G(t, z, \bar{z}; t', z', \bar{z}') = \frac{4^{h_+}\Gamma(\frac{3}{2})\Gamma(h_- - h_+)}{\Gamma(h_-)\Gamma(\frac{3}{2} - h_+)} \frac{e^{h_+(t+t')}}{|z - z'|^{2h_+}} \tag{1.16}$$

Near $\mathcal{I}^-$ the two-point function reduces to

$$\lim_{t,t' \to -\infty} G(t, z, \bar{z}; t', z', \bar{z}') = \frac{c_+ e^{h_+(t+t')}}{|z - z'|^{2h_+}} + \frac{c_- e^{h_-(t+t')}}{|z - z'|^{2h_-}}, \tag{1.17}$$

where $c_\pm$ are the coefficients given by

$$c_\pm = \frac{4^{h_+}\Gamma(\frac{3}{2})\Gamma(h_\mp - h_\pm)}{\Gamma(h_\mp)\Gamma(\frac{3}{2} - h_\pm)}.$$

By inserting (1.12) and (1.17) into (1.13), we find the quadratic coefficient of the two point correlator is given by

$$\int_{\mathcal{I}^-} d^2z d^2z' \, \frac{\phi_-(z, \bar{z})\phi_-(z', \bar{z}')}{|z - z'|^{2h_+}}.$$

which implies the two point correlator is proportional to

$$\langle \mathcal{O}_\phi(z, \bar{z})\mathcal{O}_\phi(z'\bar{z}')\rangle \sim \frac{1}{|z - z'|^{2h_+}},$$

which is correct for an operator of dimension h_+ and is in agreement with the two-point function in the bulk de Sitter space-time.

The boundary stress tensor for our planar slice of space is zero. If we chose a more general metric, we may make use of a nonvanishing stress tensor to calculate the dimension of the operator $\mathcal{O}_\phi$ by evaluating the operator product expansion

$$T_{zz}(z)\mathcal{O}(0) \sim \frac{h_+}{z^2}\mathcal{O}(0) + \frac{1}{z}\partial\mathcal{O}(0) + ...$$

where T_{zz} is given in (1.11) and the coefficient h_+ denotes the dimension of the operator $\mathcal{O}(0)$.

For a planar slice $\mathcal{O}^-$ we see there is a correspondence between the bulk gravity and the boundary field theory. This is understood because the two-point function in the bulk space-time matches the two point correlator in the boundary field theory.

1.3 Outline of chapter 2

In chapter two we will investigate the dS/CFT in two dimensions [24]. We discuss the quantization of a scalar particle moving in two-dimensional de Sitter space. We construct the conformal quantum mechanical model on the asymptotic boundary of de Sitter space in the infinite past. We obtain explicit expressions for the generators of the conformal group and calculate the eigenvalues of the Hamiltonian. We also show that two-point correlators are in agreement with the two-point function one obtains from the wave equation in the bulk de Sitter space.

1.4 Outline of chapter 3

In chapter three we discuss the quantization of a scalar field in three-dimensional asymptotic de Sitter space. We obtain explicit expressions for the Noether currents generating the isometry group in terms of the modes of the scalar field and the Liouville gravitational field. We extend the SL(2,C) algebra of the Noether charges to a full Virasoro algebra by introducing non-local conserved charges. The Virasoro algebra has the expected central charge in the weak coupling limit (large central charge $c = 3l/2G$, where l is the dS radius and G is Newton's constant). We derive the action of the Virasoro charges on states in the boundary CFT thus elucidating the dS/CFT correspondence.

Chapter 2

Two dimensions

In this chapter we discuss the case of two-dimensional de Sitter space. The asymptotic boundary $\mathcal{I}^-$ is a circle, which upon a Wick rotation turns into time. The theory on the boundary is a conformal quantum mechanical model. We shall explicitly construct this model for the case of a scalar particle, obtain the generators of the conformal group, calculate the eigenvalues of the Hamiltonian and the Green functions. Our method of solution is similar to the one discussed by de Alfaro, Fubini and Furlan (DFF) [20], even though our Hamiltonian differs from theirs.

The two-dimensional de Sitter space (dS_2) may be parametrized as

$$\frac{ds^2}{l^2} = -d\tau^2 + \cosh^2\tau d\phi^2.$$

We will explicitly consider the case where ϕ spans the real axis (zero temperature limit) and then comment on what changes need to be made to turn ϕ periodic (finite temperature).

Consider a scalar field Φ of mass m. It obeys the wave equation in de Sitter space

$$l^2\nabla^2\Phi = -\frac{1}{\cosh\tau}\partial_\tau\left(\cosh\tau\partial_\tau\Phi\right) + \frac{1}{\cosh^2\tau}\partial_\phi^2\Phi = m^2l^2\Phi. \tag{2.1}$$

The operator ∇^2 is the Casimir operator in the $SL(2,\mathbb{R})$ algebra generated by

$$L_\pm = e^{\pm i\phi}\left(i\tanh\tau\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\tau}\right) \quad , \quad L_3 = -i\frac{\partial}{\partial\phi} \tag{2.2}$$

Indeed,

$$\ell^2\nabla^2 = \frac{1}{2}(L_+L_- + L_-L_+) - L_3^2$$

Focusing on the $\mathcal{I}^-$ boundary reduces the wave equation to

$$\left(-\partial_\tau^2 + \partial_\tau + 4e^{2\tau}\partial_\phi^2\right)\Phi = m^2l^2\Phi. \tag{2.3}$$

On this spacelike slice we see the third term is negligible and the wave equation becomes independent of ϕ. The solution to the wave equation (2.1) behaves asymptotically as

$$\Phi \sim e^{h_\pm \tau}, \qquad h_\pm = \frac{1}{2} \pm \nu, \qquad \nu = i\mu, \qquad \mu = \frac{1}{2}\sqrt{4m^2l^2 - 1}. \tag{2.4}$$

We will concentrate on the case of imaginary ν ($\mu \in \mathbb{R}$) in which $h_- = h_+^\star$. Introducing the coordinate $q = e^\tau$, we may write (2.3) as

$$f_k'' + 4k^2 f_k = -\frac{m^2\ell^2}{q^2}\, f_k \tag{2.5}$$

where $\Phi(\tau,\phi) = e^{ik\phi} f_k(q)$. The solutions are

$$f_k(q) = \sqrt{2kq}\; Z_\nu(2kq)$$

where Z_ν is a Bessel function. They form an orthogonal set under the inner product

$$(\Phi_1, \Phi_2) = i\int_{-\infty}^{\infty} \frac{d\phi}{2\pi}\, e^{i(k_1-k_2)\phi}\, (f_{k_1}^\star(q) f_{k_2}'(q) - f_{k_1}^{\star\,\prime}(q) f_{k_2}(q)).$$

The apparent q-dependence disappears after we apply the wave equation and the integral over ϕ leads to a δ-function, as expected. For the Euclidean choice

$$\Phi_k^E = C^E e^{ik\phi}\sqrt{2kq}\; H_\nu^{(1)}(2kq)$$

demanding orthonormality, fixes the normalization constant to $C^E = \frac{1}{\sqrt{8}}$ for real ν, where we used the Wronskian $H_\nu^{(1)}(x)H_\nu^{(2)\prime}(x) - H_\nu^{(1)\prime}(x)H_\nu^{(2)}(x) = \frac{-4i}{\pi x}$. Another interesting choice is

$$\Phi_k^+ = C^+ e^{ik\phi}\sqrt{2kq}\; J_\nu(2kq). \tag{2.6}$$

For an orthonormal set, we choose

$$C^+ = \frac{\sqrt{\nu}}{2}\,\frac{\Gamma(\nu)}{\Gamma(\frac{1}{2})}$$

Using the Bessel function identity

$$J_\nu(x) = \frac{1}{2}\left(H_\nu^{(1)}(x) - e^{i\pi\nu} H_\nu^{(1)}(-x)\right)$$

we obtain

$$\Phi_k^+(q,\phi) = \sqrt{\frac{\nu}{2\pi}}\Gamma(\nu)\left(\Phi_k^E(q,\phi) + e^{i\pi h_+}\Phi_k^E(-q,\phi)\right) \tag{2.7}$$

which differs from [19] by a phase. The Green function for the modes Φ_k^+ can be obtained from

$$G^+(q,\phi;q',\phi') = \int \frac{dk}{k} \Phi_k^+(q,-\phi)\Phi_k^+(q',\phi').$$

After some algebra, we arrive at

$$G^+(q,\phi;q',\phi') = (C^+)^2 \frac{\Gamma(h_+)}{\Gamma(h_+ + \frac{1}{2})\Gamma(\frac{1}{2})} \, (4P)^{-h_+} \, F(h_+, h_+; 2h_+; -1/P)$$

where P measures the distance between the two arguments of the Green function in the three-dimensional Minkowski space in which dS_2 is embedded,

$$P = \frac{4(q-q')^2 + (\phi-\phi')^2}{16qq'}.$$

When $P \to \infty$, the Green function behaves as

$$G^+(q,\phi;q',\phi') \sim (C^+)^2 \frac{\Gamma(h_+)}{\Gamma(h_+ + \frac{1}{2})\Gamma(\frac{1}{2})} \, (4P)^{-h_+}.$$

For the Euclidean modes, the Green function can be found in terms of the Green function obtained above using (2.7) [1]. Near the boundary, we deduce the two-point function of the dual conformal theory,

$$\langle \mathcal{O}^\dagger(\phi)\mathcal{O}(\phi')\rangle = 2\sqrt{\pi}\frac{\Gamma(h_+)}{\Gamma(\nu)} \, (\phi-\phi')^{-2h_+} \tag{2.8}$$

If ϕ is a periodic coordinate with period 2π, then k takes on discrete values and the integral defining the Green function turns into a sum. The form of the two-point function may be deduced from the periodicity condition and the singularity, which uniquely determine it. We obtain

$$\langle \mathcal{O}^\dagger(\phi)\mathcal{O}(\phi')\rangle = 2\sqrt{\pi}\frac{\Gamma(h_+)}{\Gamma(\nu)} \, \left(2\sin\frac{\phi-\phi'}{2}\right)^{-2h_+} \tag{2.9}$$

The boundary conformal theory can be defined in terms of the coordinates $q = e^\tau$, ϕ and their conjugate momenta $p = e^{-\tau}\partial_\tau$, H, respectively. Instead of the $SL(2,\mathbb{R})$ generators (2.2), it is more convenient to introduce the generators

$$\begin{aligned} D &= i\,(\phi\partial_\phi + \partial_\tau) \\ K &= -\phi^2\partial_\phi + 2i\phi\partial_\tau + 4e^{2\tau}\partial_\phi. \end{aligned} \tag{2.10}$$

in the vicinity of the boundary $\mathcal{I}^-$. The three operators H,D and K satisfy the $SL(2,\mathbb{R})$ algebra

$$[H,K] = 2iD, \qquad [H,D] = iH, \qquad [K,D] = -iK.$$

The wave equation is mapped onto the constraint

$$4q^2H^2 - q^2p^2 = g \quad , \quad g = m^2\ell^2 + A \tag{2.11}$$

where we have allowed for the possibility of a normal ordering ambiguity by introducing the constant A. Classically, we can solve the constraint (2.11) and express the conformal Hamiltonian as

$$H = \frac{1}{2}\sqrt{p^2 + \frac{g}{q^2}}.$$

The constraint thus reduces the system to a quantum mechanical model with a single degree of freedom q, with ϕ playing the role of imaginary time. Notice that our Hamiltonian differs from the one considered by DFF [20] (theirs is the square of ours), yet the results are similar.

In order to quantize this system, we promote (q,p) to operators (Q,P) and impose the standard commutation relation

$$[Q,P] = i$$

The coordinate ϕ is identified with time after a Wick rotation. D and K (eq. (2.10)) become the dilation and special conformal operators, respectively, given by

$$\begin{aligned} D &= \left(\phi H + \frac{1}{2}(QP+PQ)\right) \\ K &= -\phi^2 H + i\phi(QP+PQ) + 2(Q^2H + HQ^2). \end{aligned} \tag{2.12}$$

Following the procedure in [20], it is more convenient to start with the values of the $SL(2,\mathbb{R})$ generators at $\phi = 0$. Let $Q_0 = Q(0)$ and $P_0 = P(0)$. The three generators reduce to

$$\begin{aligned} H &= \frac{1}{2}\sqrt{P_0^2 + \frac{g}{Q_0^2}} \\ D &= \frac{1}{2}(Q_0P_0 + P_0Q_0) \\ K &= 2(Q_0^2H + HQ_0^2). \end{aligned}$$

It can be checked explicitly that they obey the $SL(2,\mathbb{R})$ algebra

$$[H,K] = -2iD, \qquad [H,D] = -iH, \qquad [K,D] = iK.$$

The Casimir operator of this algebra is given by

$$N = \frac{1}{2}(HK + KH) - D^2. \tag{2.13}$$

This commutes with all operators and its numerical value should equal $m^2\ell^2$ to match the wave equation. To calculate N, we first bring it into the form

$$N = [H,\ [Q_0^2, H]] + g + \frac{3}{4}$$

after some straightforward operator manipulations. The remaining double commutator is independent of g and can therefore be easily calculated by setting $g = 0$ in H. It may also be deduced from the corresponding Poisson brackets. The final result is

$$N = g - \frac{1}{4}$$

By comparing with the wave equation (2.3), we deduce that the normal ordering constant is $A = 1/4$ (*cf.* (2.11)) and

$$g = m^2\ell^2 + \frac{1}{4} \tag{2.14}$$

We follow the procedure of [20] to construct the states in the Hilbert space. Define new operators R, S:

$$\begin{aligned} R &= \frac{1}{2}\left(\frac{1}{a}K + aH\right), \\ S &= \frac{1}{2}\left(\frac{1}{a}K - aH\right). \end{aligned}$$

where a is a constant with dimension of length. We will let $a = 1$ for simplicity. R plays the role of a Hamiltonian for the quantum mechanical system on $\mathcal{I}^-$ whose eigenstates span the Hilbert space. By including the dilation operator, R, S and D form the closed algebra $O_{2,1}$,

$$[D, R] = iS \qquad [S, R] = -iD \qquad [S, D] = -iR.$$

Next, we define raising and lowering operators

$$L_\pm = S \mp iD.$$

These new operators, along with R form the closed algebra $SL(2, \mathbb{R})$,

$$[R, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = -2R.$$

The Casimir operator for this algebra (2.13) can be written in terms of R, L_+ and L_- as

$$N = \frac{1}{2}(HK + KH) - D^2 \ = R^2 - R - L_+L_-$$

The ground state energy can be found by operating on the ground state of R,

$$R|0\rangle = r_0|0\rangle,$$

$$N|0\rangle = r_0(r_0-1)|0\rangle = m^2l^2|0\rangle.$$

Thus, the ground state energy is

$$r_0 = \frac{1}{2}(1 \pm \sqrt{1-4m^2l^2}).$$

Therefore, $r_0 = h_\pm$ (eq. (2.4)). Without loss of generality, we choose $r_0 = h_+$. The other choice ($r_0 = h_- = h_+^\star$) leads to a dual Hilbert space related to the one we are about to study via complex conjugation. It should be noted that inner products ought to be defined in terms of both Hilbert spaces, leading to normalization conditions such as (2.16).

By successively acting with the raising operators, we deduce the eigenvalues of R,

$$R\psi_n(q) = (r_0+n)\psi_n(q)$$

In the position representation, we obtain the Schrödinger equation

$$\left((1+2q^2)\sqrt{-\frac{d^2}{dq^2}+\frac{g}{q^2}}+2\sqrt{-\frac{d^2}{dq^2}+\frac{g}{q^2}}\;q^2\right)\psi_n(q) = \sqrt{2}(r_0+n)\psi_n(q). \qquad (2.15)$$

To solve the Schrödinger equation, it is convenient to introduce the transition matrix $\mathcal{T}_{nk}$ between the two bases Φ_k^+ (2.6) and ψ_n, at time $\phi = 0$,

$$\psi_n(q) = \int_0^\infty dk \mathcal{T}_{nk}^\star \Phi_k^+(q),$$

where we impose orthonormality,

$$\int_0^\infty dq\ \psi_n(q)\psi_{n'}(q) = \delta_{nn'}. \qquad (2.16)$$

The action of the three $SL(2,\mathbb{R})$ generators on Φ_k^+ translates to an action on the matrix elements $\mathcal{T}_{nk}$,

$$\hat{H}\mathcal{T}_{nk} = k\mathcal{T}_{nk}, \quad D\mathcal{T}_{nk} = -i\left(k\frac{d}{dk}+\frac{1}{2}\right)\mathcal{T}_{nk}, \quad \hat{K}\mathcal{T}_{nk} = \left(-k\frac{d^2}{dk^2}-\frac{d}{dk}+\frac{\nu^2}{k}\right)\mathcal{T}_{nk}, \qquad (2.17)$$

respectively. In this representation, the Schrödinger equation (2.15) becomes

$$\frac{1}{2}(\hat{H}+\hat{K})\mathcal{T}_{nk} = (h_+ + n)\mathcal{T}_{nk}$$

which can be solved. The solution is given in terms of Laguerre polynomials [20],

$$\mathcal{T}_{nk} = \frac{2^{h_+}}{C^+}\sqrt{\frac{n!}{\Gamma(n+2h_+)}}k^\nu e^{-k} L_n^{2\nu}(2k)$$

where the normalization has been fixed as in [20] with the exception of the factor C^+ which is due to the different normalization condition we imposed on the wavefunctions Φ_k^+. It leads to a unit norm for the wavefunction ψ_n. It is convenient to introduce the Laplace transform of $\mathcal{T}_{nk}$ with respect to the energy k,

$$\widetilde{\mathcal{T}}_n(\phi) = 2^{-h_+}\int_0^\infty dk e^{-k\phi}\, k^\nu\, \mathcal{T}_{nk}$$

given in terms of the conjugate time variable ϕ. The additional energy factor k^ν ensures that the action of the $SL(2,\mathbb{R})$ generators is as expected. Explicitly,

$$\widetilde{\mathcal{T}}_n(\phi) = \frac{(-)^n}{(C^+)^2}\sqrt{\frac{\Gamma(n+2h_+)}{n!}}\ \left(\frac{1-\phi}{1+\phi}\right)^{n+h_+}\ (1-\phi^2)^{-h_+}$$

and the $SL(2,\mathbb{R})$ generators act as (*cf.* eq. (2.17))

$$\hat{H}\widetilde{\mathcal{T}}_n = -\frac{d}{d\phi}\widetilde{\mathcal{T}}_n \quad , \quad \hat{D}\widetilde{\mathcal{T}}_n = i\left(\phi\frac{d}{d\phi}+h_+\right)\widetilde{\mathcal{T}}_n \quad , \quad \hat{K}\widetilde{\mathcal{T}}_n = \left(\phi^2\frac{d}{d\phi}+2h_+\phi\right)\widetilde{\mathcal{T}}_n.$$

The two-point function is given by

$$G(\phi,\phi') = \sum_{n=0}^\infty \widetilde{\mathcal{T}}_n(\phi)\widetilde{\mathcal{T}}_n(-\phi')$$

Notice that we there is no complex conjugation on $\widetilde{\mathcal{T}}_n(-\phi')$, because it ought to be the complex conjugate of the dual of $\widetilde{\mathcal{T}}_n$ which is itself related to $\widetilde{\mathcal{T}}_n$ by complex conjugation (*cf.* also the inner product (2.16)). A short calculation reveals

$$G(\phi,\phi') = \frac{\Gamma(2h_+)}{2^{2h_+}(C^+)^2}\ (\phi-\phi')^{-2h_+} = 2\sqrt{\pi}\frac{\Gamma(h_+)}{\Gamma(\nu)}\ (\phi-\phi')^{-2h_+}$$

in agreement with the de Sitter space result (2.8). Higher-order correlators may also be obtained by using the standard methods developed in [20].

If we demand periodicity of ϕ ($\phi \equiv \phi + 2\pi$), the above calculations become cumbersome. The energy k takes on discrete values and the differential equations determining the transition matrix $\mathcal{T}_{nk}$ turn into difference equations. Nevertheless, correlators may be deduced without explicit calculations, because they are uniquely determined by the periodicity requirements and their singularities.

To summarize, we have discussed the conformal quantum mechanical model which resides on the boundary $\mathcal{I}^-$ of two-dimensional de Sitter space in the infinite past. We calculated the eigenvalues and corresponding eigenfunctions of the Hamiltonian and deduced correlation functions. We showed that the Green functions agree with the propagators one obtains from the de Sitter space wave equation. Most of our results are similar to those of ref. [20], even though our Hamiltonian is different and the Schrödinger equation (2.15) cannot be solved explicitly. What saves the day is the $SL(2,\mathbb{R})$ symmetry which determines much of the structure of the conformal theory as was shown in [20]. It would be interesting to extend our results to de Sitter spaces of dimension higher than two and shed some light on the behavior of the conformal field theories on the boundary.

Chapter 3

Three dimensions

3.1 Introduction

In this chapter we extend the earlier work to the dS/CFT correspondence for massive scalar fields in three dimensions. Extending the results of ref. [19], we obtain explicit expressions for all Noether charges generating the dS_3 isometry group in terms of the modes of the scalar field and the Liouville gravitational field. These are semi-classical expressions valid in the weak coupling limit (large central charge). Quantum corrections may then be calculated using perturbation theory. We then extend the algebra of isometries to a full Virasoro algebra by introducing non-local conserved charges for the massive field (their Liouville gravitational field counterparts have already been discussed in [21]). We discuss the action of the Virasoro generators on the boundary CFT states.

Our discussion is organized as follows. We start in section 3.2 by sketching the derivation of the salient features of pure three-dimensional gravity [26, 27] whose asymptotic dynamics is described by a Euclidean Liouville theory [21, 22, 23]. In section 3.3, we introduce a massive scalar field. Using static coordinates in dS_3, we expand it in modes in the southern and northern diamonds, respectively. We also obtain its asymptotic form on the dS boundary, $\mathcal{I}^-$. In section 3.4, we express the Noether charges in terms of the modes of the massive field. We find expressions which may be easily extended to yield a full Virasoro algebra. The new charges are conserved and non-local. We show that their central charge vanishes classically. We also obtain the action of the Virasoro generators on the boundary fields. In section 3.5, we discuss the quantization of the system. We obtain semi-classical expressions of the Virasoro operators which form an algebra of central charge given by

$$c = \frac{3\ell}{2G} \tag{3.1}$$

in the weak coupling limit ($c \gg 1$). We discuss their action on states built from the Euclidean vacuum which is dual to the $SL(2,\mathbb{C})$ invariant CFT vacuum on the boundary. Finally, in section 3.6, we briefly summarize our conclusions.

3.2 Asymptotic de Sitter space

In this section we discuss the salient features of pure three-dimensional gravity with a positive cosmological constant. Our discussion is based mostly on refs. [21, 22, 23]. de Sitter space emerges as a classical solution. We shall parametrize the metric using planar coordinates

$$ds^2_{\mathrm{dS}} = -\frac{d\tau^2}{\tau^2} + \tau^2 dz d\bar{z} \tag{3.2}$$

henceforth setting the dS radius

$$l = 1 \ . \tag{3.3}$$

Three-dimensional gravity is derivable from the Chern-Simons action [26, 27]

$$S = -\frac{i}{16\pi G}\int \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \frac{i}{16\pi G}\int \mathrm{Tr}\left(\bar{A}\wedge d\bar{A} + \frac{2}{3}\bar{A}\wedge\bar{A}\wedge\bar{A}\right) \tag{3.4}$$

The metric can be obtained from the $SL(2,\mathbb{C})$ vector potentials

$$A_\mu = A^a_\mu \tau_a \quad , \quad \bar{A}_\mu = \bar{A}^a_\mu \tau_a, \tag{3.5}$$

where τ_a are the $SL(2,\mathbb{C})$ generators

$$\tau_0 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad , \quad \tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.6}$$

with the normalization condition

$$\mathrm{tr}(\tau_a\tau_b) = \frac{1}{2}\eta_{ab}. \tag{3.7}$$

It is advantageous to switch to the new basis

$$\tau_\pm = \frac{1}{2}(\tau_1 \mp i\tau_2) \quad , \quad \tau_+ = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \tau_- = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.8}$$

We can express the vector potential in the new basis explicitly as

$$A_\mu = \frac{1}{2}\begin{pmatrix} -iA^0_\mu & A^-_\mu \\ A^+_\mu & iA^0_\mu \end{pmatrix}, \tag{3.9}$$

and similarly for $\bar{A}_\mu$.

The metric expressed in terms of the vector potential is given as

$$g_{\mu\nu} = -\frac{1}{2}\mathrm{tr}(A_\mu - \bar{A}_\mu)(A_\nu - \bar{A}_\nu). \tag{3.10}$$

For asymptotic de Sitter space, we need

$$g_{z\bar{z}} = \frac{\tau^2}{2} + o(1) \ \ , \ \ g_{zz} = o(1) \ \ , \ \ g_{\tau\tau} = -\frac{1}{\tau^2} + o(\tau^{-4}) \ \ , \ \ g_{z\tau} = o(\tau^{-3}). \tag{3.11}$$

We can solve the equations of motion for the vector potential, plug the solutions into (3.10), and get explicit corrections to the asymptotically de Sitter metric. The equations of motion from the action (3.4) yield

$$A_\mu = G^{-1}\partial_\mu G \ \ , \ \ \bar{A}_\mu = \bar{G}^{-1}\partial_\mu \bar{G} \tag{3.12}$$

where

$$G = g(z)M \ \ , \ \ \bar{G} = \bar{g}(\bar{z})M^{-1} \ \ , \ \ M = \begin{pmatrix} \sqrt{\tau} & 0 \\ 0 & 1/\sqrt{\tau} \end{pmatrix}. \tag{3.13}$$

The only dynamical variable is $\hat{g} = g^{-1}\bar{g}$. We obtain the components

$$A_\tau = -\bar{A}_\tau = \frac{1}{2\tau}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ \ , \ \ A_{\bar{z}} = \bar{A}_z = 0 \ , \tag{3.14}$$

and

$$A_z = -M^{-1}\partial_z\hat{g}\hat{g}^{-1}M \ \ , \ \ \bar{A}_{\bar{z}} = M\hat{g}^{-1}\partial_{\bar{z}}\hat{g}M^{-1}. \tag{3.15}$$

Decomposing $\hat{g}$ *à la* Gauss (including an appropriate normalization factor),

$$\hat{g} = e^{-\sqrt{2G}\Omega}\begin{pmatrix} e^{2\sqrt{2G}\Omega} + XY & X \\ Y & 1 \end{pmatrix} \ . \tag{3.16}$$

we obtain

$$A_z = M^{-1}\begin{pmatrix} \sqrt{2G}\partial_z\Omega + e^{-2\sqrt{2G}\Omega}X\partial_z Y & \partial_z X - 2\sqrt{2G}X\partial_z\Omega - e^{-2\sqrt{2G}\Omega}X^2\partial_z Y \\ e^{-2\sqrt{2G}\Omega}\partial_z Y & -\sqrt{2G}\partial_z\Omega - e^{-2\sqrt{2G}\Omega}X\partial_z Y \end{pmatrix}M. \tag{3.17}$$

and similarly for $\bar{A}_{\bar{z}}$. All three fields X, Y and Ω are independent of the τ coordinate. Demanding asymptotic de Sitter space leads to constraints on X and Y of the form

$$X = i\sqrt{2G}\partial_z\Omega \ \ , \ \ Y = -i\sqrt{2G}\partial_{\bar{z}}\Omega$$

$$e^{-2\sqrt{2G}\Omega}\partial_{\bar{z}}X = i \ \ , \ \ e^{-2\sqrt{2G}\Omega}\partial_z Y = -i \ . \tag{3.18}$$

Using these constraints we arrive at explicit expressions for the $SL(2,\mathbb{C})$ fields solely in terms of the field Ω

$$A_z = \begin{pmatrix} 0 & \frac{2iG}{\tau}\Theta_{zz} \\ -i\tau & 0 \end{pmatrix} \ \ , \ \ \bar{A}_{\bar{z}} = \begin{pmatrix} 0 & i\tau \\ -\frac{2iG}{\tau}\Theta_{\bar{z}\bar{z}} & 0 \end{pmatrix}, \tag{3.19}$$

where off-diagonal elements are proportional to the stress-energy tensor of a linear dilaton theory given by

$$\Theta_{zz} = (\partial_z\Omega)^2 - \frac{1}{\sqrt{2G}}\partial_z^2\Omega \ , \quad \Theta_{\bar{z}\bar{z}} = (\partial_{\bar{z}}\Omega)^2 - \frac{1}{\sqrt{2G}}\partial_{\bar{z}}^2\Omega \ , \tag{3.20}$$

The constraints (3.18) imply that the field Ω obeys the Liouville equation

$$\partial_z\partial_{\bar{z}}\Omega = \frac{1}{\sqrt{2G}}e^{2\sqrt{2G}\Omega}.$$

Noether charges for the linear dilaton theory are given by

$$L_n = \oint_{\mathcal{C}} \frac{dz}{2\pi i} z^{n+1}\Theta_{zz} \quad , \quad \bar{L}_n = -\oint_{\mathcal{C}} \frac{d\bar{z}}{2\pi i}\bar{z}^{n+1}\Theta_{\bar{z}\bar{z}} \tag{3.21}$$

These charges form Virasoro algebras with central charge

$$c = \frac{1}{4} + \frac{3}{2G} \approx \frac{3}{2G} \ . \tag{3.22}$$

in agreement with (3.1) for $l = 1$ (in the limit $G \ll 1$). The contour $\mathcal{C}$ may be viewed as residing on the boundary of dS_3. For $n = -1, 0, 1$, they form a $SL(2, \mathbb{C})$ algebra. In the latter case, they are related to bulk expressions involving the stress-energy tensor $\mathcal{T}_{\mu\nu}$ [10] by a conservation law [19]

$$L_n = \int_{\Sigma_{\mathcal{C}}} d^2\Sigma^\mu \mathcal{T}_{\mu\nu}\zeta_n^\nu \quad , \quad n = -1, 0, 1 \tag{3.23}$$

where ζ_n^ν is the corresponding Killing vector and $\Sigma_{\mathcal{C}}$ is a two-dimensional surface in dS_3 whose boundary is the curve $\mathcal{C}$ (similarly for $\bar{L}_n$).

Using the explicit form of the vector potential and eq. (3.10), we arrive at an explicit expression for the metric on an asymptotic de Sitter space,

$$g_{z\bar{z}} = \frac{\tau^2}{2} + \frac{2G^2}{\tau^2}\Theta_{zz}\Theta_{\bar{z}\bar{z}} \ , \ g_{zz} = -2G\Theta_{zz} \ , \ g_{\bar{z}\bar{z}} = -2G\Theta_{\bar{z}\bar{z}} \ , \ g_{\tau\tau} = -\frac{1}{\tau^2} \ , \ g_{z\tau} = g_{\bar{z}\tau} = 0 \ , \tag{3.24}$$

clearly satisfying the requirement (3.11).

3.3 Scalar field

Next, we consider a massive scalar field Φ of mass m living in three-dimensional de Sitter space (dS_3). It is advantageous to parametrize dS_3 using static coordinates,

$$ds^2_{\mathrm{dS}} = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + r^2 d\tau^2 \tag{3.25}$$

instead of the planar coordinates (3.2) used in the pure gravitational case. The two coordinate systems are related by a conformal transformation near the boundary $\mathcal{I}^-$ (infinite past, $r, \tau \to \infty$) [19],

$$z = e^{-iw} \quad , \quad w = \theta + it \tag{3.26}$$

The field Φ obeys the wave equation

$$\frac{1}{r}\partial_r \left(r(1-r^2)\partial_r \Phi \right) - \frac{1}{1-r^2}\partial_t^2 \Phi + \frac{1}{r^2}\partial_\tau^2 \Phi = m^2 \Phi \ . \tag{3.27}$$

Following ref. [19], we shall solve this equation in the northern and southern diamonds corresponding to the range $0 \le r < 1$ and then analytically continue the solution beyond these domains aiming at reaching the boundary $\mathcal{I}^-$ which belongs to the past triangle. We shall present details for the southern diamond; the discussion in the northern diamond is similar.

A complete set of positive frequency solutions in the southern diamond is provided by the wavefunctions

$$\phi^S_{\omega j} = A_{\omega j} f_{\omega j}(r) e^{-i\omega t + ij\tau} \ , \tag{3.28}$$

where $f_{\omega j}$ obeys the radial equation

$$(1-r^2) f''_{\omega j} + \left(\frac{1}{r} - 3r \right) f'_{\omega j} + \left(\frac{\omega^2}{1-r^2} - \frac{j^2}{r^2} - m^2 \right) f_{\omega j} = 0. \tag{3.29}$$

and we have included a normalization constant $A_{\omega j}$, to be determined. A solution of (3.29) which is regular at $r = 0$ is given by

$$f_{\omega j} = r^{|j|}(1-r^2)^{i\omega/2} \ F(a_+, a_-; c; r^2) \quad , \quad a_\pm = \frac{1}{2}(i\omega + |j| + h_\pm) \quad , \quad c = 1 + |j| \ , \tag{3.30}$$

where $h\pm$ for d dimensions is given by eq.(1.5) and for this case is

$$h_\pm = 1 \pm i\mu \quad , \quad \mu = \sqrt{m^2 - 1} \quad , \quad m^2 > 1 \ . \tag{3.31}$$

The choice of normalization constant

$$A_{\omega j} = e^{i\pi |j|/2} \, \frac{\Gamma(\frac{1}{2}(i\omega + |j| + h_+))\Gamma(\frac{1}{2}(-i\omega + |j| + h_+))}{\Gamma(1 + |j|)} \tag{3.32}$$

yields an orthonormal set of wavefunctions,

$$\langle \phi^S_{\omega j} | \phi^S_{\omega' j'} \rangle = \frac{1}{2 \sinh \pi\omega} \delta(\omega - \omega') \delta_{j,j'} \ , \tag{3.33}$$

under the inner product

$$\langle \phi | \chi \rangle = \frac{1}{2} \int_{\Sigma^S} \frac{r dr d\tau}{1-r^2} \, \phi^* \overleftrightarrow{\partial_t} \, \chi, \tag{3.34}$$

where we are integrating over a spacelike hypersurface Σ^S $(t = \text{const.})$ in the southern diamond $(0 \leq r < 1,\, 0 \leq \theta < 2\pi)$.

It is useful to find expressions of the negative frequency eigenfunctions in terms of the positive frequency ones (3.28). Using the transformation properties of hypergeometric functions, one can show that

$$\phi^{S*}_{\omega j} = e^{2i\vartheta_{\omega j}} \phi^S_{-\omega -j} \quad , \quad e^{2i\vartheta_{\omega j}} = (-)^{|j|} \, \frac{\Gamma(\frac{1}{2}(i\omega + |j| + h_-))\Gamma(\frac{1}{2}(-i\omega + |j| + h_-))}{\Gamma(\frac{1}{2}(i\omega + |j| + h_+))\Gamma(\frac{1}{2}(-i\omega + |j| + h_+))} \, . \tag{3.35}$$

This is a phase factor ($\vartheta_{\omega j} \in \mathbb{R}$) for frequencies ω along the real and imaginary axes. Notice also that $\vartheta_{-\omega\, j} = \vartheta_{\omega j} = \vartheta_{\omega\, -j}$. It is of interest to note that $\vartheta_{\omega j}$ may also be expressed in terms of j as

$$e^{2i\vartheta_{\omega j}} = (-)^j \, \frac{\Gamma(\frac{1}{2}(i\omega + j + h_-))\Gamma(\frac{1}{2}(-i\omega + j + h_-))}{\Gamma(\frac{1}{2}(i\omega + j + h_+))\Gamma(\frac{1}{2}(-i\omega + j + h_+))} \, . \tag{3.36}$$

This is obvious for $j > 0$. For $j < 0$, it follows from standard Gamma function identities.

A general solution of the wave equation (3.27) in the southern diamond may be expanded as

$$\Phi^S(t, r, \tau) = \sum_{j=-\infty}^{\infty} \int_0^{\infty} d\omega \left(\alpha^S_{\omega j} \phi^S_{\omega j} + \alpha^{S\dagger}_{\omega j} \phi^{S*}_{\omega j} \right) \ , \tag{3.37}$$

This may also be written as a single integral over the entire real ω-axis as

$$\Phi^S(t, r, \tau) = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \ \alpha^S_{\omega j} \phi^S_{\omega j} \ , \tag{3.38}$$

where the negative frequency modes are given by

$$\alpha^S_{-\omega\, -j} = e^{2i\vartheta_{\omega j}} \alpha^{S\dagger}_{\omega j} \tag{3.39}$$

on account of (3.35). These modes are related to the annihilation operators in [19] by

$$\alpha^S_{\omega j} = \sqrt{\sinh \pi\omega} \ a^S_{\omega j} \quad , \quad \omega > 0 \tag{3.40}$$

due to our choice of normalization condition (3.33). They obey the commutation relations

$$[\alpha^S_{\omega j}, \alpha^S_{\omega' j'}] = \sinh \pi\omega \ e^{2i\vartheta_{\omega j}} \ \delta(\omega + \omega')\delta_{j+j',0} \tag{3.41}$$

Strictly speaking, since we are discussing the classical theory, the above should be interpreted in terms of Poisson brackets,

$$[\mathcal{A},\mathcal{B}] \equiv i\{\mathcal{A},\mathcal{B}\}_{\rm P.B.} \tag{3.42}$$

Eq. (3.41) also holds quantum mechanically, after the modes $\alpha^S_{\omega j}$ are promoted to operators. At slight risk of confusion, we shall use the notation (3.42) invariably, pointing out the potential changes under quantization, as needed.

We may use Kruskal coordinates to go beyond the horizon ($r = 1$) and into the past triangle whose boundary is $\mathcal{I}^-$ ($r \to \infty$). The eigenfunctions in the southern diamond may then be written as linear combinations of eigenfunctions in the past triangle [19]. After some algebra, one obtains for the general wavefunction (3.38)

$$\Phi^S(r,t,\theta) = \Phi^{S+}(r,t,\theta) + \Phi^{S-}(r,t,\theta) \tag{3.43}$$

where

$$\begin{aligned} \Phi^{S-}(r,t,\theta) &= \sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{\sinh\pi\omega}\ \alpha^S_{\omega j}\phi^-_{\omega j}\ , \\ \Phi^{S+}(t,\theta) &= \Phi^{S-\dagger}(t,\theta)\ , \end{aligned} \tag{3.44}$$

and

$$\phi^-_{\omega j} = B_{\omega j}\, r^{-h_-}\left(1-\frac{1}{r^2}\right)^{i\omega/2} F(b_{\omega j}, b_{\omega\, -j}; h_-; \frac{1}{r^2})e^{-i\omega t+ij\theta}, \tag{3.45}$$

$$B_{\omega j} = i\Gamma(i\mu)\sin\frac{\pi}{2}(-i\omega+j+h_-)\ \ ,\ \ b_{\omega j} = \frac{1}{2}(i\omega+j+h_-)\ . \tag{3.46}$$

In deriving the above, we used $i^{|j|}\sin\left(\frac{\pi}{2}|j|+\lambda\right) = \sin\left(\frac{\pi}{2}j+\lambda\right)$ in order to express the wavefunction entirely in terms of j rather than its absolute value, $|j|$.

Near the boundary $\mathcal{I}^-$, we have

$$\Phi^{S\pm}(r,t,\theta) \sim \frac{\Psi^{S+}(t,\theta)}{r^{h_+}} + \frac{\Psi^{S-}(t,\theta)}{r^{h_-}}\,, \tag{3.47}$$

where the boundary wavefunctions are

$$\begin{aligned} \Psi^{S-}(t,\theta) &= \sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{\sinh\pi\omega}\ B_{\omega j}\ \alpha^S_{\omega j}\ e^{-i\omega t+ij\theta}\ , \\ \Psi^{S+}(t,\theta) &= \Psi^{S-\dagger}(t,\theta)\ . \end{aligned} \tag{3.48}$$

Similar expressions hold for boundary wavefunctions $\Psi^{N\pm}(t,\theta)$ derived from the northern diamond.

3.4 Symmetries and corresponding charges

The group of isometries of de Sitter space in three dimensions is generated by the Killing vectors

$$\begin{aligned}
\zeta_0 &= \frac{1}{2}(\partial_t + i\partial_\theta), \\
\zeta_{\pm 1} &= \frac{1}{2}e^{\pm(t-i\theta)}\left[\frac{\mp r}{\sqrt{1-r^2}}\partial_t - \sqrt{1-r^2}\left(\partial_r \mp \frac{i}{r}\partial_\theta\right)\right], \\
\bar{\zeta}_0 &= \frac{1}{2}(\partial_t - i\partial_\theta) \\
\bar{\zeta}_{\pm 1} &= \frac{1}{2}e^{\pm(t+i\theta)}\left[\frac{\mp r}{\sqrt{1-r^2}}\partial_t - \sqrt{1-r^2}\left(\partial_r \pm \frac{i}{r}\partial_\theta\right)\right],
\end{aligned} \tag{3.49}$$

forming a $SL(2,\mathbb{C})$ algebra,

$$[\zeta_n, \zeta_m] = (n-m)\zeta_{n+m} , \quad [\bar{\zeta}_n, \bar{\zeta}_m] = (n-m)\bar{\zeta}_{n+m} , \quad n,m = -1,0,1 \tag{3.50}$$

We can easily construct the corresponding Noether currents and charges,

$$Q_n = \int_\Sigma d^2\Sigma \; j_n^0 , \quad j_n^\mu = g^{\mu\nu}T_{\nu\alpha}\zeta_n^\alpha , \tag{3.51}$$

where

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\mathcal{L} , \quad \mathcal{L} = \frac{1}{2}g^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi - \frac{1}{2}m^2\Phi^2 \tag{3.52}$$

and similarly for $\bar{Q}_n$. Henceforth, we shall concentrate on the southern diamond and find the corresponding charge Q_n^S by choosing $\Sigma = \Sigma^S$, as in eq. (3.34). The analysis of Q_n^N on the northern diamond is similar. The total charge is

$$Q_n = Q_n^S + Q_n^N \tag{3.53}$$

If we integrate by parts and use the wave equation (3.27), we can massage (3.51) into the form

$$Q_n^S = \langle \Phi^S | \zeta_n \Phi^S \rangle , \tag{3.54}$$

where the inner product is defined in (3.34). The Killing vectors ζ_n act on the eigenstates $\phi_{\omega j}^S$ (eq. (3.28)) in a particularly simple manner,

$$\zeta_n \phi_{\omega j}^S = \frac{1}{2}(i\omega + j - nh_-)\phi_{\omega+in,j-n}^S , \quad n = -1,0,1 . \tag{3.55}$$

This is partly due to our judicious choice of normalization constant (3.32) and is shown using standard hypergeometric function identities. Remarkably, eq. (3.55) is independent of the sign of j, even though its derivation follows different paths in the two cases, $j > 0$ and $j < 0$, respectively. We may use it, together with the orthogonality conditions (3.33), to express the Noether charges (3.54) in terms of creation and annihilation operators. To arrive at such an expression for Q_n^S, let us first write it in the form

$$Q_n^S = \frac{1}{2} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \; \alpha_{\omega j}^S \; \langle \Phi^S | \zeta_n \phi_{\omega j}^S \rangle \tag{3.56}$$

Using (3.55), this can be written as

$$Q_n^S = \frac{1}{2} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \; (i\omega + j - nh_-) \alpha_{\omega j}^S \; \langle \Phi^S | \phi_{\omega+ni\; j-n}^S \rangle \tag{3.57}$$

Stifting $\omega \to \omega - ni$, $j \to j + n$, yields

$$Q_n^S = \frac{1}{2} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \; (i\omega + j + nh_+) \alpha_{\omega-ni\; j+n}^S \; \langle \Phi^S | \phi_{\omega j}^S \rangle \tag{3.58}$$

The inner products may be easily evaluated by expanding Φ in modes (eq. (3.38)). Using the orthogonality relation (3.33), we obtain

$$Q_n^S = \frac{1}{4} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{\sinh \pi\omega} \; (i\omega + j + nh_+) \alpha_{\omega-ni\; j+n}^S \; \alpha_{\omega j}^{S\dagger} \tag{3.59}$$

Shifting back $\omega \to \omega + ni$, $j \to j - n$, we obtain the equivalent expression

$$Q_n^S = \frac{1}{4} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{\sinh \pi\omega} \; (i\omega + j - nh_-) \alpha_{\omega j}^S \; \alpha_{\omega+ni\; j-n}^{S\dagger} \tag{3.60}$$

The above expressions are deceptively simple; their complexities are revealed in eqs. (3.39) and (3.36). Remarkably, even though they have been derived for $n = -1, 0, 1$ as generators of isometries, they are well-defined for all $n \in \mathbb{Z}$. One may be tempted to use this fact to extend the $SL(2, \mathbb{C})$ algebra of isometries to an infinite-dimensional algebra of the conserved charges Q_n^S ($n \in \mathbb{Z}$). Unfortunately, these charges do not form a closed algebra.

To remedy this, let us split the integral (3.60) into positive and negative frequencies,

$$Q_n^S = Q_n^{S+} + Q_n^{S-} \tag{3.61}$$

Using eq. (3.39), we may write Q_n^{S-} in terms of positive frequencies as

$$Q_n^{S-} = \frac{1}{4} \sum_{j=-\infty}^{\infty} \int_0^{\infty} \frac{d\omega}{\sinh \pi\omega} \; (i\omega + j + nh_-) e^{2i(\vartheta_{\omega j} - \vartheta_{\omega-ni\; j+n})} \alpha_{\omega j}^{S\dagger} \; \alpha_{\omega-ni\; j+n}^S \tag{3.62}$$

It is straightforward to deduce from the definition (3.36),

$$(i\omega + j + nh_-)e^{2i(\vartheta_{\omega j} - \vartheta_{\omega - ni\, j+n})} = (-)^n (i\omega + j + nh_+) \quad , \quad n = -1, 0, 1 \tag{3.63}$$

It is worth mentioning that eq. (3.63) does not hold for any other $n \in \mathbb{Z}$. It follows that

$$Q_n^{S-} = \frac{1}{4} \sum_{j=-\infty}^{\infty} \int_0^\infty \frac{d\omega}{\sinh \pi\omega} \, (i\omega + j + nh_+) \, \alpha_{\omega j}^{S\dagger} \, \alpha_{\omega - ni\, j+n}^{S} \tag{3.64}$$

for $n = -1, 0, 1$. Shifting $\omega \to \omega + ni$, $j \to j - n$ yields an expression which matches the one for Q_n^{S+}, as is evident from eq. (3.60). However, we need to exercise care in this shift, since the integral is over the positive real ω-axis. The shift amounts to a contour deformation in the complex ω-plane and yields an additional contribution (see Figure 3.1 for $n = 1$; $n = -1$ is similar; for $n = 0$ we have $\delta Q_0^S = 0$, trivially),

$$\delta Q_n^S = \frac{1}{4} \sum_{j=-\infty}^{\infty} \int_0^n \frac{d\lambda}{\sin \pi\lambda} \, (-\lambda + j + nh_+) \, \alpha_{i\lambda j}^{S\dagger} \, \alpha_{i(\lambda - n)\, j+n}^{S} \tag{3.65}$$

This can be seen to vanish by reflecting $\lambda \to n - \lambda$, $j \to -j - n$. Using eqs. (3.39) and (3.63), we deduce $\delta Q_n^S = -\delta Q_n^S$, and therefore,

$$\delta Q_n^S = 0 \tag{3.66}$$

Thus, we arrive at a modified expression

$$Q_n^S = \frac{1}{2} \sum_{j=-\infty}^{\infty} \int_0^\infty \frac{d\omega}{\sinh \pi\omega} \, (i\omega + j - nh_-) \, \alpha_{\omega + ni\, j-n}^{S\dagger} \alpha_{\omega j}^{S} \tag{3.67}$$

which agrees with (3.60) for $n = -1, 0, 1$. We have been cavalier about ordering of modes, since we are discussing classical expressions. We shall use eq. (3.67) for all $n \in \mathbb{Z}$. These conserved charges form a closed algebra. It is a straightforward exercise to show that they form a Virasoro algebra,

$$[Q_m^S, Q_n^S] = (m - n) Q_{m+n}^S \,, \tag{3.68}$$

with vanishing classical central charge.

Next, we investigate how the charges act on a scalar field in three-dimensional de Sitter space. In the bulk (southern diamond) a short calculation shows that the charges act on the scalar field as

$$[Q_n^S, \Phi^S] = \zeta_n \Phi^S \quad , \quad n = -1, 0, 1, \tag{3.69}$$

where ζ_n are the Killing vectors defined in (3.49), as expected. For all other n, we obtain non-local expressions, since the corresponding charges do not generate local symmetries.

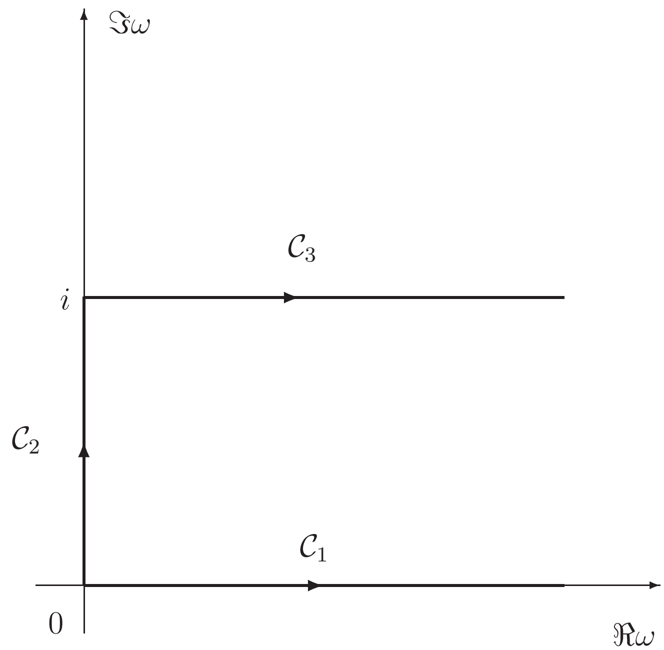

Figure 3.1: Contour deformation for Q_n^{S-} (eq. (3.64)) for $n = 1$. δQ_n^S (eq. (3.65)) corresponds to an integral over the segment $\mathcal{C}_2$.

Near the boundary $\mathcal{I}^-$,

$$[Q_n^S, \Psi^-] = \xi_n \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{\sinh \pi\omega} \mathcal{F}_{n\omega j} B_{\omega j} \, \alpha_{\omega j}^S \, e^{-i\omega t + ij\theta} , \tag{3.70}$$

where

$$\xi_n = \frac{1}{2} e^{-niw}(\partial_w - nh_-) \quad , \quad \forall n \in \mathbb{Z} \tag{3.71}$$

generate conformal transformations on the boundary with $w = \theta + it$ (eq. (3.26)), obeying the algebra

$$[\xi_m, \xi_n] = (m - n)\xi_{m+n} \tag{3.72}$$

and

$$\mathcal{F}_{n\omega j} = \begin{cases} 1 & , \quad \omega \geq 0 \\ \frac{i\omega + j - nh_-}{i\omega + j - nh_+} e^{2i(\vartheta_{\omega j} - \vartheta_{\omega + in \, j-n})} & , \quad \omega < 0 \end{cases} \tag{3.73}$$

It is easy to see from eq. (3.63) that

$$\mathcal{F}_{n\omega j} = 1 \quad , \quad \forall \omega \in \mathbb{R} \quad , \quad n = -1, 0, 1 \tag{3.74}$$

therefore,

$$[Q_n^S, \Psi^+] = \xi_n \Psi^+ \quad , \quad n = -1, 0, 1 \tag{3.75}$$

For other n, we may perform a high-frequency expansion. Using eq. (3.36), we obtain

$$\mathcal{F}_{n\omega j} = 1 + o\left((i\omega + j)^{-1}\right) \ , \tag{3.76}$$

therefore

$$[Q_n^S, \Psi^+] = \xi_n \Psi^+ + \dots \ , \tag{3.77}$$

where the corrections vanish at short distances.

3.5 Quantization

The full quantum theory includes interactions between the scalar field (in general, matter fields) and the gravitational field. In the weak coupling limit (in which we are working), interactions may be ignored. It is often stated that matter fields do not contribute to the central charge, the latter being given by (3.1). However, the matter fields renormalize Newton's constant, hence also alter the central charge of pure gravity through (infinite) renormalization. For example, the component T_{zz} of the scalar field stress-energy tensor in de Sitter space with metric (3.2) is modified to

$$T_{zz} = (\partial_z \Phi)^2 + G\Theta_{zz}\mathcal{L} \ \ , \ \ \mathcal{L} = \frac{1}{2} g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m^2 \Phi^2 \tag{3.78}$$

when the metric is perturbed as in (3.24), and similarly for $T_{\bar{z}\bar{z}}$. The vacuum expectation value of the modification is proportional to $\langle \mathcal{L} \rangle$ and contributes an infinite renormalization to the gravitational field Ω. Thus, in the quantum theory, the stress-energy tensor of the matter fields cannot be separated in a meaningful way from the stress-energy tensor of the gravitational field. Consequently, the Noether charges Q_n ($n = -1, 0, 1$) cannot be independently defined. The physical quantities are

$$\mathcal{Q}_n = Q_n + L_n = Q_n^S + Q_n^N + L_n \tag{3.79}$$

where L_n is given in bulk form (3.23), or by its equivalent boundary expression (3.21). When extended to all $n \in \mathbb{Z}$, the above conserved charges form a Virasoro algebra of central charge given by (3.22).

On the other hand, because of symmetry, the infinities arising in $Q_n^{S,N}$ in the southern and northern diamonds, respectively, match each other. Consequently, the charges

$$\mathcal{R}_n = Q_n^S - Q_n^N \tag{3.80}$$

are well-defined and form a Virasoro algebra of vanishing central charge,

$$[\mathcal{R}_m, \mathcal{R}_n] = (m - n)\mathcal{R}_{m+n} \tag{3.81}$$

in the semiclassical approximation we are working in.

We can build the Hilbert space by acting with creation operators on the product of the three vacuum states corresponding to the southern and northern diamonds

($|0\rangle_{S,N}$) and the Liouville field Ω ($|0\rangle_\Omega$), defined respectively by

$$\alpha^S_{\omega j}|0\rangle_S = 0 \ \ , \ \ \alpha^N_{-\omega\, j}|0\rangle_N = 0 \ \ , \ \ \omega > 0 \tag{3.82}$$

Let us define the gravitational vacuum state $|0\rangle_\Omega$ as the $SL(2,\mathbb{C})$ vacuum state satisfying

$$L_n|0\rangle_\Omega = \bar{L}_n|0\rangle_\Omega = 0 \ \ , \ \ n \geq -1 \tag{3.83}$$

On the matter side, the $SL(2,\mathbb{C})$ invariant vacuum state is the Euclidean vacuum state defined by

$$|E\rangle = C_E \ \exp\left\{\sum_{j=-\infty}^{\infty}\int_0^\infty \frac{d\omega}{e^{2\pi\omega}-1}\ e^{-2i\vartheta_{\omega j}}\ \alpha^S_{-\omega\ -j}\alpha^N_{\omega j}\right\}\ |0\rangle_S \otimes |0\rangle_N \tag{3.84}$$

where we have included an (infinite) normalization constant, C_E. Both $\alpha^S_{-\omega j}$ and $\alpha^N_{\omega j}$ are creation operators for $\omega > 0$, acting on the vacuum states $|0\rangle_{S,N}$, respectively. This state is annihilated by the appropriately normalized annihilation operators

$$\beta_{\omega j} = \frac{1}{\sqrt{2\sinh\pi\omega}}(e^{\pi\omega/2}\alpha^S_{\omega j} - e^{-\pi\omega/2}\alpha^N_{\omega j}) \ \ , \ \ \omega \in \mathbb{R} \tag{3.85}$$

It should be emphasized that these are annihilation operators, for both positive and negative frequencies. They satisfy the commutation relations (*cf.* eq. (3.41))

$$[\beta_{\omega j}, \beta^\dagger_{\omega' j'}] = \sinh\pi\omega\ \delta(\omega-\omega')\delta_{jj'} \tag{3.86}$$

We may express the 'bulk' α-modes in terms of the Euclidean β-modes by inverting (3.85). We obtain

$$\begin{aligned}\alpha^S_{\omega j} &= \frac{1}{\sqrt{2\sinh\pi\omega}}(e^{\pi\omega/2}\beta_{\omega j} - ie^{-\pi\omega/2}e^{2i\vartheta_{\omega j}}\beta^\dagger_{-\omega\ -j})\\ \alpha^N_{\omega j} &= -\frac{1}{\sqrt{2\sinh\pi\omega}}(e^{-\pi\omega/2}\beta_{\omega j} - ie^{\pi\omega/2}e^{2i\vartheta_{\omega j}}\beta^\dagger_{-\omega\ -j})\end{aligned} \tag{3.87}$$

We may express the charges in terms of α-modes,

$$\begin{aligned}Q_n &= Q^S_n + Q^N_n\\ &= \frac{1}{2}\sum_{j=-\infty}^{\infty}\int_0^\infty \frac{d\omega}{\sinh\pi\omega}\ (i\omega+j-nh_-)\left\{\alpha^{S\dagger}_{\omega+in\ j-n}\alpha^S_{\omega j} - \alpha^N_{\omega+in\ j-n}\alpha^{N\dagger}_{\omega j}\right\}\end{aligned} \tag{3.88}$$

or equivalently in terms of the Euclidean creation and annihilation β-modes,

$$Q_n = \frac{1}{2}\sum_{j=-\infty}^{\infty}\int_0^\infty \frac{d\omega}{\sinh\pi\omega}\ \left\{(i\omega+j-nh_-)\ \beta^\dagger_{\omega+in\ j-n}\beta_{\omega j}\right.$$

$$-(i\omega + j + nh_+)\, e^{2i(\vartheta_{\omega-ni\, j+n} - \vartheta_{\omega j})}\, \beta^\dagger_{-\omega+in\, -j-n}\beta_{-\omega\, -j}\Big\} \quad (3.89)$$

after normal ordering. In particular, the generators of the $SL(2,\mathbb{C})$ subalgebra take on a simple form,

$$Q_n = \frac{1}{2}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{\sinh \pi\omega}\,(i\omega + j - nh_-)\,\beta^\dagger_{\omega+in\, j-n}\beta_{\omega j} \quad , \quad n = -1, 0, 1 \qquad (3.90)$$

on account of eq. (3.63), which however does not hold for any other n. Evidently, all charges annihilate the Euclidean vacuum,

$$Q_n|E\rangle = 0 \quad , \quad \forall n \in \mathbb{Z} \qquad (3.91)$$

This implies (from eqs. (3.79) and (3.83))

$$\mathcal{Q}_n\, |E\rangle \otimes |0\rangle_\Omega = 0 \quad , \quad n \geq -1 \qquad (3.92)$$

confirming that this is a $SL(2,\mathbb{C})$ invariant vacuum state for the entire system (matter + gravity). The other Virasoro generators $\mathcal{R}_n$ (eq. (3.80)) may also be expressed in terms of the β-modes. They contain terms quadratic in creation operators $\beta^\dagger$ and therefore do not annihilate the Euclidean vacuum.

The general wavefunction on the boundary is a linear combination of $\Psi^{S\pm}$ (eq. (3.48)) and its northern counterpart, $\Psi^{N\pm}$, and can therefore be expressed in terms of the α-modes or, equivalently, the β-modes. Therefore, all Virasoro generators are expressible in terms of the modes of boundary wavefunctions, encoding the symmetries of the boundary CFT. No reference to the southern and northern diamonds in the bulk de Sitter space is needed. Observables in the bulk may be obtained in terms of the β-modes. It would be desirable to study them in detail and develop a physical intuition in terms of the boundary CFT dynamics.

3.6 Conclusions

We studied the dS/CFT correspondence in three dimensions for a massive scalar field expanding on the resuls of ref. [19]. We obtained explicit expressions for the generators of isometries of dS_3 and brought them to a form that suggested an extension of the algebra to an infinite-dimensional Virasoro algebra.

We studied the action of the Virasoro generators on the boundary wavefunctions and discussed quantization based on the Euclidean vacuum. We showed that the Virasoro algebra could be defined entirely in terms of the dynamics of the boundary CFT which provided a dictionary for the dS/CFT correspondence. Thus, dS observables may be understood in terms of corresponding quantities in the boundary CFT. Further work on the CFT side is needed to develop physical intuition for and better understand the dS/CFT correspondence.

Part III

Quasinormal modes of black holes

Chapter 4

Introduction

4.1 Quasinormal modes

We are familiar with normal modes and their characteristic frequencies. If we strum a guitar, we hear a "characteristic sound", related to a natural set of real (normal) frequencies. The response is given as a superposition of stationary modes, the normal modes. Almost thirty years ago, it was shown [28, 29, 30] that black holes also have a characteristic sounds and modes. These characteristic oscillations have been termed "quasinormal modes", along with their characteristic frequencies, "quasinormal frequencies". The part "normal" comes from the similarities to normal modes, whereas "quasi" comes from the decay of the system which is entirely fixed by the black hole. After a black hole is perturbed, it radiates energy out to infinity, and the corresponding modes decay, giving rise to complex frequencies. The radiation associated with these modes is expected to be seen with future gravitational wave detectors.

Recently there has been a lot of research studying quasinormal modes of black holes in asymptotically AdS space-times [36], and we will extend their research to calculate these modes for in various dimensions.

Understanding these modes is important, because they may give some insight into the conformal field theory which lies on the boundary of AdS. According to the AdS/CFT correspondence, a large black hole in AdS corresponds to a thermal state in the boundary CFT. By perturbing the black hole the corresponding thermal state is perturbed. The perturbed thermal state will eventually decay back to thermal equilibrium. The quasinormal modes can predict the time-scale for the thermal state to reach equilibrium [34, 35]. This time-scale is extremely difficult to calculate in the corresponding CFT.

4.2 Asymptotically flat space-time

In this section we will review a separate method for calculating quasinormal modes of black holes introduced in [2]. In chapter four we started with the wave equation for a massive scalar. and could solve it pertubatively in the high frequency limit. Now we will look at massless scalar fields. We can start our analysis with the wave equation in a particular geometry. By transforming our coordinates to the so called Tortoise coordinates we may express the wave equation in a Schrödinger-like form. Solving the differential equation yields a continuous set of complex quasinormal frequencies. The real and imaginary portions represent the oscillations and damping respectively. By imposing "out going" boundary conditions

$$\begin{aligned} \Phi_\omega(x) &\sim e^{i\omega x} \quad \text{as} \quad x \to -\infty \\ \Phi_\omega(x) &\sim e^{-i\omega x} \quad \text{as} \quad x \to \infty. \end{aligned} \tag{4.1}$$

we introduce a discrete contribution to the quasinormal frequencies [31, 30]. The out going boundary conditions state that nothing arrives from infinity nor from within the black hole horizon. However, since the frequencies are complex, understanding the boundary conditions from an operational standpoint is somewhat complicated. As long as $x \in \mathbb{R}$, the out going boundary conditions above amount to distinguishing between exponentially increasing and decreasing terms. If we analytically continue r or x to the complex plane, we can define the boundary conditions in a new way [2].

$$\begin{aligned} \Phi_\omega(x) &\sim e^{i\omega x} \quad \text{as} \quad \omega x \to -\infty \\ \Phi_\omega(x) &\sim e^{-i\omega x} \quad \text{as} \quad \omega x \to \infty. \end{aligned} \tag{4.2}$$

If one picks the contour $\text{Im}(\omega x) = 0$ in the complex plane, then the asymptotic behavior of $e^{\pm i\omega x}$ is always oscillatory, and we no longer have to worry about exponentially increasing or decreasing terms. Therefore we will restrict our analysis to the Stokes line, which are lines in the complex plane such that $\text{Im}(\omega x) = 0$. We will apply this new method to an example of a Schwarzschild black hole in five dimensional Minkowski space-time.

Consider a massless scalar field given by the wave equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0,$$

where $g_{\mu\nu}$ is the metric, and the line element is

$$ds^2 - f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \quad f(r) = 1 - \frac{2\mu}{r^2},$$

where μ is related the the horizon via

$$R_H = (2\mu)^{\frac{1}{2}}.$$

We may expand the ϕ in terms of three-dimensional spherical harmonics.

$$\phi = \sum_{\ell,m} r^{-\frac{3}{2}}\psi_\ell(r,t)Y_{\ell m}(\theta_i),$$

where θ_i are the angles of the three sphere. Then, if

$$\psi_\ell(r,t) = \Phi_\omega(r)e^{i\omega t}$$

is the Fourier decomposition of the scalar field. The wave equation may be expressed in a Schrödinger-like form as

$$-\frac{d^2\Phi_\omega}{dx^2} + V[r(x)]\Phi_\omega(x) = \omega^2\Phi_\omega(x) \tag{4.3}$$

where $V[r(x)]$ is the potential and x is known as the tortoise coordinate defined by

$$dx = \frac{dr}{f(r)}.$$

The tortoise coordinate x maps the event horizon $r = R_H$ to $x = -\infty$ and keeps infinity at $x = \infty$. For this choice of coordinates, $f(r) > 0$ for $x \in \mathbb{R}$.

The potential $V(r)$ is given by [39]

$$V(r) = f(r)\left(\frac{\ell(\ell+2)}{r^2} + \frac{3f(r)}{4r^2} + \frac{3f'(r)}{2r}\right). \tag{4.4}$$

where $\ell(\ell+2)$ are eigenvalues of the Laplacian on the $\mathbb{S}^3$ sphere. We must express the potential in terms of the Tortoise coordinate in order to evaluate the Schrödinger-like differential equation (4.3). By solving (4.3) near the horizon and near $r \sim \infty$ we may match the solutions and calculate the frequencies. Therefore we need to calculate the Tortoise coordinates in these two regions and can plug them into eq.(4.4) to find the potential in terms of x.

The horizons for our space are defined by $f(R_H) = 0$ giving

$$r^2 - 2\mu = 0, \tag{4.5}$$

and the roots are given by $r = \pm\sqrt{2\mu}$. Given the roots, one can factorize $f(r)$ and the tortoise coordinate can be found easily.

$$x[r] = \int \frac{dr}{f(r)} = r + \sum_{n=0}^{1} \frac{1}{2k_n} \log\left(1 - \frac{r}{R_n}\right), \tag{4.6}$$

where $k_n = \frac{1}{2}f'(R_n)$ is the surface gravity for the horizon R_n. There are two regions of interest. Near $r = 0$ we find

$$x[r] \sim -\frac{1}{2\mu}\int dr\, r^2 = -\frac{r^3}{6\mu}, \tag{4.7}$$

while as $r \sim \infty$ we find

$$x[r] \sim \int dr = r. \tag{4.8}$$

With x understood in terms of r, we may look at the Schrödinger-like equation for the two regions. We will start by looking at the region near $x \sim \infty$ which leads to $r \sim \infty$. In this region, the potential vanishes, and solutions are given by

$$\Phi(x) \sim A_+ e^{i\omega x} + A_- e^{-i\omega x}.$$

Applying the out going boundary condition (4.2) for the quasinormal mode at infinity we find

$$A_+ = 0.$$

Next, let us look in the region near $x = 0$. The potential for scalar and tensor type perturbations is given by

$$V[r(x)] = \frac{j^2 - 1}{4x^2},$$

where j is a parameter which describes the type of perturbation. For scalar, tensor and electromagnetic type perturbations $j = 0, 1, 2$ respectively. Plugging this potential into the Schrödinger-like equation, we find

$$-\frac{d^2\Phi_\omega}{dx^2} + \frac{j^2 - 1}{4x^2}\Phi_\omega(x) = \omega^2 \Phi_\omega(x).$$

whose solution is given by

$$\Phi(x) \sim B_+ \sqrt{2\pi\omega x} J_{\frac{1}{2}}(\omega x) + B_- \sqrt{2\pi\omega x} J_{-\frac{1}{2}}(\omega x) \tag{4.9}$$

We would like to link the solutions near the origin and at infinity along a Stokes line. Can this be done? Asymptotic quasinormal modes for a Schwarzschild black hole can be written as $\omega = \omega_R + in\omega_I$, with $\omega_R, \omega_I \in \mathbb{R}$, $\omega_i > 0$ and satisfy the condition $\mathrm{Im}(\omega) \gg \mathrm{Re}(\omega)$. This implies $n \to \infty$ for asymptotic quasinormal modes. Since $\mathrm{Im}(\omega) \gg \mathrm{Re}(\omega)$, ω is very large and approximately purely imaginary. Near the origin, one has $\omega x \in \mathbb{R}$ for $x \in i\mathbb{R}$. From (4.7) we see this happens when

$$r = \rho\, e^{\frac{i\pi}{6} + \frac{in\pi}{3}}$$

with $\rho > 0$ and $n = 0, 1, ..., 5$.

Because we are interested in the asymptotic quasinormal modes we may consider the asymptotic expansion of the solution (4.9)

$$J_\nu \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{4} - \frac{\pi}{4}\right), \quad \bar{z} \gg 1,$$

allowing eq.(eq-4-6) to be approximated by

$$\begin{aligned} \Phi(x) &\sim 2B_+ \cos(\omega x - \alpha_+) + 2B_- \cos(\omega x - \alpha_-), \\ &= (B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-}) e^{i\omega x} + (B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}) e^{-i\omega x}, \end{aligned} \tag{4.10}$$

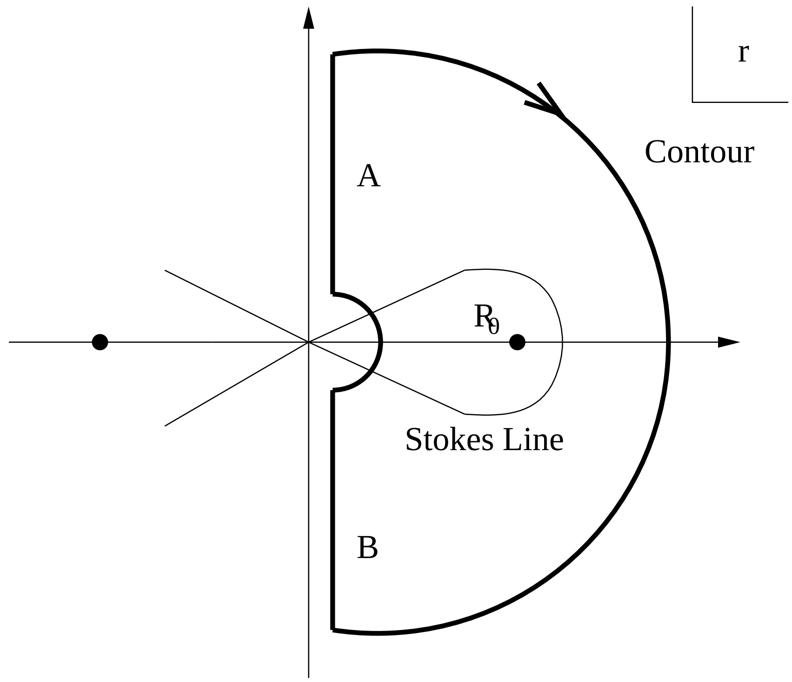

Figure 4.1: Stokes lines for the Schwarzschild black hole in five dimensions, along with the chosen contour for monodromy matching for $d = 5$.

where we make the definition

$$\alpha_{\pm} = \frac{\pi}{4}(1 \pm j). \tag{4.11}$$

Rather than looking at $\omega \in \mathbb{C}$, let us look at $r, x \in \mathbb{C}$ via analytic continuation. This allows us to match along a Stokes line where $\mathrm{Im}(\omega x) = 0$. By choosing the complex contour $\mathrm{Im}(\omega x) = 0$, the asymptotic behavior of $e^{\pm i\omega x}$ is always oscillatory. The Stokes line for a five dimensional Schwarzschild black hole is given in Figure 4.1.

For the Schwarzschild case the Stokes lines are defined by $\mathrm{Re}(x) = 0$. The origin ($r = 0$) is the only singular point of the Stokes curve. We understand the behavior near this singularity from the expansion given in eq.(4.7). x is a multivalued function with the horizons given in (4.5). Near the branch points the tortoise coordinate is given by

$$x \sim \frac{1}{f'(R_H)} \ln(r - R_H).$$

The $\mathrm{Re}(x)$ is well defined at infinity and we see from eq.(4.8) that the Stokes line will be radial at infinity. It can be shown that two of the six Stokes lines must be unbounded and are represented by lines along the positive and negative imaginary axis. The remaining four can either connect to another branch or end up in a branch cut. Let us now consider the contour in Figure 4.1, by closing the Stokes line near $r \sim \infty$. At point A we have $\omega x \gg 0$, which allows us to apply the boundary conditions

(4.2) to the solution (4.10). This give us a constraint equation on $B_\pm$

$$(B_+e^{-i\alpha_+} + B_-e^{-i\alpha_-}) = 0. \tag{4.12}$$

Near $z \sim 0$ we may expand the Bessel function in terms of an even holomorphic function, w, given by

$$J_\nu(z) = z^\nu w(z).$$

We may rotate in the complex r plane from the branch containing point A to the branch containing point B. In doing so, we rotate r though an angle of π. In terms of the tortoise coordinate, x gets rotated through an angle of 3π. Our solution along the branch containing point B becomes

$$\begin{aligned}\Phi(x) &\sim B_+e^{6i\alpha_+}\cos(-\omega x - \alpha_+) + 2B_-e^{6i\alpha_+}\cos(-\omega x - \alpha_-) \\ &= (B_+e^{7i\alpha_+} + B_-e^{7i\alpha_-})e^{i\omega x} + (B_+e^{5i\alpha_+} + B_-e^{5i\alpha_-})e^{-i\omega x} \end{aligned} \tag{4.13}$$

where $\alpha_\pm$ are given in eq.(4.11) and we used the transformation of the Bessel function given by

$$\sqrt{2\pi e^{3\pi i}\omega x}J_{\pm\frac{1}{2}}(e^{3\pi i}\omega x) = e^{\frac{3\pi i}{2}(1\pm j)}\sqrt{2\pi\omega x}J_{\pm\frac{1}{2}}(\omega x) \sim 2e^{6i\alpha_\pm}\cos(\omega x - \alpha_\pm).$$

As we close the contour near $r \sim \infty$, $e^{i\omega x}$ is very small since $\mathrm{Im}(\omega) \gg 0$, therefore only the coefficient of $e^{-i\omega x}$ should be trusted. As we match the solutions on the two regions the coefficient of $e^{-i\omega x}$ must be multiplied by the factor

$$\frac{B_+e^{5i\alpha_+} + B_-e^{5i\alpha_-}}{B_+e^{i\alpha_+} + B_-e^{i\alpha_-}}.$$

However, as we rotate clockwise through the contour, we need to find the monodromy of $e^{-i\omega x}$. Near the horizon (R_0), the Tortoise coordinate behaves as

$$x \sim \frac{1}{f'(R_0)}\log(r - R_0), \tag{4.14}$$

where $f'(R_0)$ is related to the surface gravity at the horizon through

$$k = \frac{1}{2}f'(R_0).$$

As we rotate clockwise through the contour x in eq.(4.14) will increase by $-\frac{2\pi i}{f'(R_0)} = -\frac{\pi i}{k}$ and $e^{-i\omega x}$ will be multiplied by the factor

$$e^{-i\omega\left(-\frac{2\pi i}{f'(R_0)}\right)} = e^{-\frac{\pi\omega}{k}}$$

Therefore the clockwise monodromy of Φ around the contour in Figure 4.1 is given

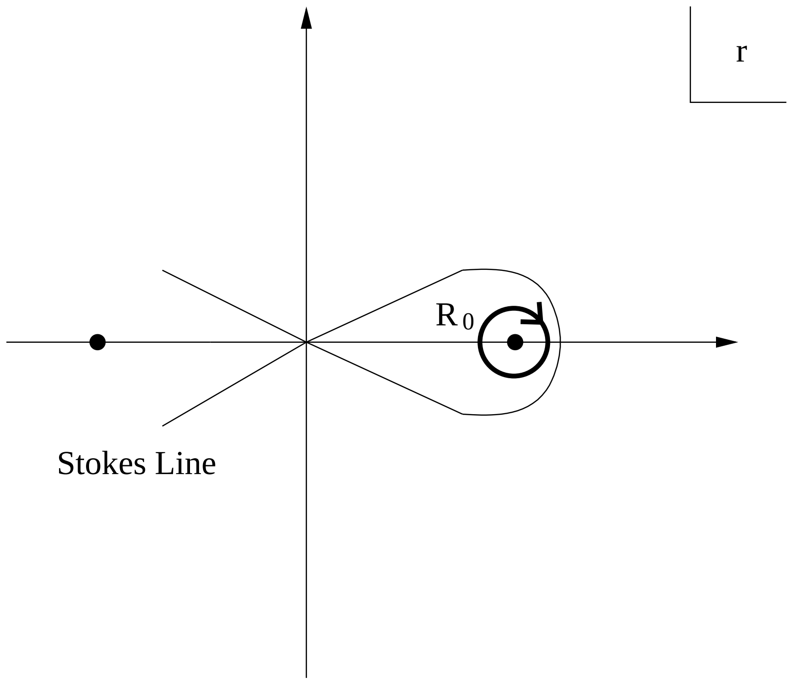

Figure 4.2: Deformed contour for monodromy matching.

by

$$\frac{B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}}{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}} e^{-\frac{\pi\omega}{k}}. \tag{4.15}$$

Since we are in the complex plane, we may deform our contour as long as we do not cross a singularity. We can deform the contour around $r = R_0$ as shown in Figure 4.2.

We may solve the Schrödinger-like equation near the horizon. Near this horizon the tortoise coordinate may be written as

$$x \sim \frac{1}{f'(R_0)} \log(r - R_0).$$

This leads to a vanishing potential and the solution may be easily written down as

$$\Phi \sim C_+ e^{i\omega x} + C_- e^{-i\omega x}.$$

By applying the out going boundary conditions (4.2) we find $C_+ = 0$. We can restate the boundary condition as a monodromy condition of Φ. The monodromy of ϕ going clockwise around the contour gives

$$e^{i\omega\left(-\frac{2\pi i}{f'(R_0)}\right)} = e^{\frac{\pi\omega}{k}}. \tag{4.16}$$

Since the monodromy is invariant under deformations of the contour, eq(4.15) is equal to eq.(4.16), leading to the second constraint equation on the coefficients $B_\pm$

$$\frac{B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}}{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}} e^{-\frac{\pi\omega}{k}} = e^{\frac{\pi\omega}{k}}. \tag{4.17}$$

We can solve the two equations of constraint (4.12) and (4.17) and calculate the quasinormal frequencies

$$\begin{vmatrix} e^{-i\alpha_+} & e^{-i\alpha_-} \\ e^{5i\alpha_+} e^{-\frac{\pi\omega}{k}} - e^{i\alpha_+} e^{\frac{\pi\omega}{k}} & e^{5i\alpha_-} e^{-\frac{\pi\omega}{k}} - e^{i\alpha_-} e^{\frac{\pi\omega}{k}} \end{vmatrix} = 0,$$

which can be re-expressed as

$$\begin{vmatrix} e^{-i\alpha_+} & e^{-i\alpha_-} \\ \sin(2\alpha_+ + \frac{i\pi\omega}{k}) & \sin(2\alpha_- + \frac{i\pi\omega}{k}) \end{vmatrix} = 0. \tag{4.18}$$

We can solve this equation for ω for a general value of j. After the equation is solved we may take the limit as $j \to 0$. Our choice of solutions ($J_{\pm j/2}$) to the Shrodinger-like wave equations does not form a complete basis of solutions. However we can still take this limit which amounts to writing the equation as a power series in j and equating to zero the first nonvanishing coefficient. Therefore we just have to require the derivative of the determinate above for j be zero for $j = 0$.

$$\begin{vmatrix} -i\pi e^{-i\pi} & i\pi e^{-i\pi} \\ \sin(\frac{\pi}{2} + \frac{i\pi\omega}{k}) & \sin(\frac{\pi}{2} - + \frac{i\pi\omega}{k}) \end{vmatrix} + \begin{vmatrix} e^{-i\pi} & e^{-i\pi} \\ \frac{\pi}{2}\sin(\frac{\pi}{2} + \frac{i\pi\omega}{k}) & -\frac{\pi}{2}\sin(\frac{\pi}{2} + \frac{i\pi\omega}{k}) \end{vmatrix} = 0. \tag{4.19}$$

We can solve for ω and we find

$$\frac{2\pi\omega}{k} = \log 3 + i(2n+1)\pi. \tag{4.20}$$

This result is in agreement with [32, 2, 33]. This method can be used to calculate quasinormal modes for massless perturbations of various types of black holes in different dimensions [38]. We will employ this method in the next chapter for anti-de Sitter black holes.

Chapter 5

Massless perturbations

5.1 Introduction

In this chapter we will employ the method described in chapter 4 for massless perturbations of an AdS black hole. We will see that in the asymptotically AdS case, there is no need to do a monodromy calculation. We keep an additional term in the approximation of the tortoise coordinate, thus adding a small correction to the Schrödinger-like wave equation. We can then solve the Schrödinger-like wave equation perturbatively in powers of ω for gravitational and electromagnetic type perturbations. To zeroth order, we review the results of [37, 38]. In the case of electromagnetic type perturbations first order correction behaves as $\ln n$ in the large frequency limit.

Understanding the quasinormal modes for black holes will give insight into various areas of physics. Since the modes are dependent on intrinsic parameters of the black hole itself, we may use this information to estimate the parameters of a black hole for some given frequency. These modes may also help in estimating the thermalization timescales in connection with the AdS/CFT.

5.2 Gravitational perturbations

In this section we discuss gravitational perturbations. Here we present a fairly comprehensive study of quasinormal modes of AdS Schwarzschild black holes with a metric in d dimensions given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \quad , \quad f(r) = \frac{r^2}{R^2} + 1 - \frac{2\mu}{r^{d-3}} \ . \tag{5.1}$$

and derive analytical expressions including first-order corrections. The results are in good agreement with results of numerical analysis.

The radial wave equation for gravitational perturbations in the black-hole background (5.1) can be cast into a Schrödinger-like form,

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \ , \tag{5.2}$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \ . \tag{5.3}$$

The potential V is determined by the type of perturbation and may be deduced from the Master Equation derived in [41]. For tensor, vector and scalar perturbations, we obtain, respectively, [38]

$$V_{\mathsf{T}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right\} \tag{5.4}$$

$$V_{\mathsf{V}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d-3)}\right\} \tag{5.5}$$

$$\begin{aligned} V_{\mathsf{S}}(r) \;=\; & \frac{f(r)}{4r^2}\left[\ell(\ell+d-3)-(d-2)+\frac{(d-1)(d-2)\mu}{r^{d-3}}\right]^{-2} \\ \times & \left\{\frac{d(d-1)^2(d-2)^3\mu^2}{R^2r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{R^2r^{d-5}}\right. \\ & + \frac{(d-4)(d-6)[\ell(\ell+d-3)-(d-2)]^2r^2}{R^2} + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\ & + \frac{4(d-1)(d-2)(2d^2-11d+18)[\ell(\ell+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\ & + \frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} - \frac{6(d-2)(d-6)[\ell(\ell+d-3)-(d-2)]^2\mu}{r^{d-3}} \\ & - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-3}} \\ & \left. +4[\ell(\ell+d-3)-(d-2)]^3 + d(d-2)[\ell(\ell+d-3)-(d-2)]^2\right\} \end{aligned} \tag{5.6}$$

Evidently, the potential always vanishes at the horizon ($V(r_H) = 0$, since $f(r_H) = 0$) regardless of the type of perturbation.

Near the black hole singularity ($r \sim 0$), the tortoise coordinate (5.3) may be expanded as

$$r_* = -\frac{1}{(d-2)}\frac{r^{d-2}}{2\mu} - \frac{1}{(2d-5)}\frac{r^{2d-5}}{(2\mu)^2} + \ldots \tag{5.7}$$

where we have kept the second term in the expansion of r and have chosen the integration constant so that $r_* = 0$ at $r = 0$. Using (5.7), we may expand the potential near the black hole singularity in the three different cases (eqs. (5.4), (5.5) and (5.6)), respectively as

$$V_{\mathsf{T}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{T}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \ldots\,, \quad \mathcal{A}_{\mathsf{T}} = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2}, \tag{5.8}$$

$$V_{\mathsf{V}} = \frac{3}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{V}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \ldots\quad , \quad \mathcal{A}_{\mathsf{V}} = \frac{d^2-8d+13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2} \tag{5.9}$$

and

$$V_{\mathsf{S}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{S}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \ldots\,, \tag{5.10}$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(2d^3-24d^2+94d-116)}{4(2d-5)(d-2)} + \frac{(d^2-7d+14)[\ell(\ell+d-3)-(d-2)]}{(d-1)(d-2)^2} \tag{5.11}$$

We have included only the terms which contribute to the order we are interested in. We may summarize the behavior of the potential near the origin by

$$V = \frac{j^2-1}{4r_*^2} + \mathcal{A}\, r_*^{-\frac{d-1}{d-2}} + \ldots \tag{5.12}$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations and the constant coefficient $\mathcal{A}$ can be found from eqs.(5.8), (5.9), (5.10) and (5.11) in the various cases. Throughout the calculation, we shall pretend that j is not an integer. At the end of the calculation, we shall let $j \to 0, 2$, as appropriate.

After rescaling the tortoise coordinate ($z = \omega r_*$), the Schrödinger-like wave equation (5.2) with the potential (5.12) becomes

$$-\frac{d^2\Psi}{dz^2} + \left[\frac{j^2-1}{4z^2} - 1\right]\Psi = -\mathcal{A}\,\omega^{-\frac{d-3}{d-2}}\, z^{-\frac{d-1}{d-2}}\Psi\,, \tag{5.13}$$

In the large frequency limit, we may treat the right-hand side of (5.13) as a correction. This will allow us to to solve the equation perturbatively. We may re-express (5.13) as

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}}\,\mathcal{H}_1\right)\Psi = 0, \tag{5.14}$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2-1}{4z^2} - 1\right] \quad , \quad \mathcal{H}_1 = -\mathcal{A}\, z^{-\frac{d-1}{d-2}}. \tag{5.15}$$

By treating $\mathcal{H}_1$ as a perturbation, we may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}}\,\Psi_1(z) + \ldots \tag{5.16}$$

and solve (5.14) perturbatively. The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0, \tag{5.17}$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z}\, J_{\frac{j}{2}}(z) + A_2 \sqrt{z}\, N_{\frac{j}{2}}(z). \tag{5.18}$$

For large z, it behaves as

$$\begin{aligned}\Psi_0(z) &\sim \sqrt{\frac{2}{\pi}} \left[A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+)\right], \\ &= \frac{1}{\sqrt{2\pi}}(A_1 - iA_2)e^{-i\alpha_+}e^{iz} + \frac{1}{\sqrt{2\pi}}(A_1 + iA_2)e^{+i\alpha_+}e^{-iz}. \end{aligned} \tag{5.19}$$

where $\alpha_\pm = \frac{\pi}{4}(1 \pm j)$.

Next, we study the behavior of the wavefunction at large r. In this region, the tortoise coordinate (5.3) may be expanded as

$$r_* - \bar{r}_* = -\frac{R^2}{r} + \frac{1}{3}\frac{R^4}{r^3} + \dots \tag{5.20}$$

The integration constant is readily deduced from the definition (5.3) of the tortoise coordinate,

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)} \tag{5.21}$$

The potential (eqs. (5.4), (5.5) and (5.6)) for large r may be expanded as

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots \tag{5.22}$$

where $j_\infty = d-1$, $d-3$ and $d-5$ for tensor, vector and scalar perturbations, respectively. The Schrödinger-like wave equation (5.2) in the region of large r becomes

$$-\frac{d^2\Psi}{dr_*^2} + \left[\frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} - \omega^2\right]\Psi = 0 \tag{5.23}$$

Since the potential does not vanish as $r \to \infty$, the wavefunction ought to vanish there. Imposing this boundary condition yields the acceptable solution to eq. (5.23),

$$\Psi(r_*) = B\sqrt{\omega(r_* - \bar{r}_*)}\; J_{\frac{j_\infty}{2}}(\omega(r_* - \bar{r}_*))\ . \tag{5.24}$$

Notice that $\Psi \to 0$ as $r_* \to \bar{r}_*$, as desired. Asymptotically, it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}}\, B \cos\left[\omega(r_* - \bar{r}_*) + \beta\right] \ , \qquad \beta = \frac{\pi}{4}(1 + j_\infty) \tag{5.25}$$

By matching this expression to the asymptotic behavior (5.19) of the solution in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$, we find a constraint on the coefficients $A_1,\ A_2$,

$$A_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2 = 0. \tag{5.26}$$

A second constraint is obtained by imposing the boundary condition

$$\Psi(z) \sim e^{iz} \quad , \qquad z \to -\infty \ , \tag{5.27}$$

at the horizon. To this end, we need to analytically continue the wavefunction near the origin to negative values of z. A rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r-plane, on account of (5.7). Since near the origin, $J_\nu(z) \sim z^\nu$ (multiplied by an even holographic function of z) and using the identity

$$N_\nu(z) = \cot \pi\nu \, J_\nu(z) - \csc \pi\nu \, J_{-\nu}(z) \ , \tag{5.28}$$

we deduce

$$J_\nu(e^{-i\pi} z) = e^{-i\pi\nu} J_\nu(z) \ , \qquad N_\nu(e^{-i\pi} z) = e^{i\pi\nu} N_\nu - 2i \cos \pi\nu \, J_\nu(z) \tag{5.29}$$

Thus for $z < 0$, the wavefunction (5.18) changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2} \sqrt{-z} \left\{ \left[A_1 - i(1 + e^{i\pi j}) A_2\right] \, J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j} \, N_{\frac{j}{2}}(-z) \right\} \ . \tag{5.30}$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} \left[A_1 - i(1 + 2e^{j\pi i}) A_2\right] \, e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} \left[A_1 - iA_2\right] \, e^{iz} \tag{5.31}$$

Imposing the boundary condition (5.27) at the horizon, we deduce the constraint

$$A_1 - i(1 + 2e^{j\pi i}) A_2 = 0. \tag{5.32}$$

The two constraints (5.26) and (5.32) are compatible provided

$$\begin{vmatrix} 1 & -i(1 + 2e^{j\pi i}) \\ \tan(\omega \bar{r}_* - \beta - \alpha_+) & -1 \end{vmatrix} = 0, \tag{5.33}$$

which yields the quasinormal frequencies [38]

$$\omega \bar{r}_* = \frac{\pi}{4}(2+j+j_\infty) - \tan^{-1}\frac{i}{1+2e^{j\pi i}} + n\pi \ . \tag{5.34}$$

These are zeroth-order expressions deduced from the zeroth-order wave equation (5.17).

Next, we calculate the first-order correction to the asymptotic expressions (5.34) for quasinormal frequencies. We begin by focusing on the region near the black-hole singularity ($r \sim 0$). To first order, the wave equation (5.14) becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0 \ , \tag{5.35}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are given in eq. (5.15). The solution is

$$\Psi_1(z) = \sqrt{z}\, N_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, J_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} - \sqrt{z}\, J_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, N_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}}, \tag{5.36}$$

written in terms of the two linearly independent solutions (5.18) of the zeroth-order eq. (5.17). $\mathcal{W} = 2/\pi$ is their Wronskian. Using (5.18) and (5.36), we may express the solution to the wave equation (5.14) up to first order (eq. (5.16)) explicitly as

$$\Psi(z) = \{A_1[1-b(z)] - A_2 a_2(z)\}\sqrt{z} J_{\frac{j}{2}}(z) + \{A_2[1+b(z)] + A_1 a_1(z)\}\sqrt{z} N_{\frac{j}{2}}(z) \tag{5.37}$$

where the functions $a_1(z)$, $a_2(z)$ and $b(z)$ are given by

$$a_1(z) = \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\ z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z'), \tag{5.38}$$

$$a_2(z) = \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\ z'^{-\frac{1}{d-2}} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z'), \tag{5.39}$$

$$b(z) = \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\ z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \ , \tag{5.40}$$

respectively. The coefficient $\mathcal{A}$ is defined in eq. (5.15) and depends on the type of perturbation. The wavefunction (5.37) behaves asymptotically as

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}}\,[A_1' \cos(z-\alpha_+) + A_2' \sin(z-\alpha_+)] \ , \tag{5.41}$$

where

$$A_1' = [1-\bar{b}]A_1 - \bar{a}_2 A_2 \ , \quad A_2' = [1+\bar{b}]A_2 + \bar{a}_1 A_1 \tag{5.42}$$

and we introduced the notation

$$\bar{a}_1 = a_1(\infty) \ , \quad \bar{a}_2 = a_2(\infty) \ , \quad \bar{b} = b(\infty) \ . \tag{5.43}$$

By matching this to the asymptotic expression (5.25), we obtain

$$A_1' \tan(\omega\bar{r}_* - \beta - \alpha_+) - A_2' = 0 \tag{5.44}$$

correcting the zeroth-order constraint (5.26). Using (5.42), the first-order constraint (5.44) in terms of A_1 and A_2 reads

$$[(1-\bar{b})\tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1]A_1 - [1 + \bar{b} + \bar{a}_2 \tan(\omega\bar{r}_* - \beta - \alpha_+)]A_2 = 0 \tag{5.45}$$

To find the first-order correction to the second constraint (5.32), we need to approach the horizon. This entails a rotation by $-\pi$ in the z-plane. From the small-z behavior of a Bessel function, $J_\nu(z) \sim z^\nu$, and using the identity (5.28), we deduce after some algebra

$$\begin{aligned} a_1(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j} a_1(z) \ , \\ a_2(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} \left[e^{i\pi j} a_2(z) - 4\cos^2\frac{\pi j}{2} a_1(z) - 2i(1+e^{i\pi j}) b(z) \right] \ , \\ b(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}} \left[b(z) - i(1+e^{-i\pi j}) a_1(z) \right] \end{aligned} \tag{5.46}$$

From these expressions and eq. (5.29), we arrive at a modified expression for the wavefunction (5.37) valid for $z < 0$. In the limit $z \to -\infty$, we obtain

$$\Psi(z) \sim -ie^{-ij\pi/2} B_1 \cos(-z - \alpha_+) - ie^{ij\pi/2} B_2 \sin(-z - \alpha_+) \tag{5.47}$$

where

$$\begin{aligned} B_1 &= A_1 - A_1 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] \\ &\quad - A_2 e^{-i\pi\frac{d-3}{d-2}} \left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1+e^{+i\pi j})\bar{b} \right] \\ &\quad - i(1+e^{i\pi j}) \left[A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j}\bar{a}_1 \right] \\ B_2 &= A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}} [\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}} e^{-i\pi j}\bar{a}_1 \end{aligned} \tag{5.48}$$

By imposing the boundary condition (5.27) at the horizon, we obtain

$$[1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b})]A_1 - [i(1+2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b})]A_2 = 0 \tag{5.49}$$

correcting the zeroth-order constraint (5.32). For compatibility of the two first-order constraints, (5.45) and (5.49), we need

$$\begin{vmatrix} 1 + \bar{b} + \bar{a}_2\tan(\omega\bar{r}_* - \beta - \alpha_+) & i(1+2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b}) \\ (1-\bar{b})\tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1 & 1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b}) \end{vmatrix} = 0 \tag{5.50}$$

Solving (5.50), we arrive at the first-order expression for quasinormal frequencies,

$$\begin{aligned}\omega\bar{r}_* &= \frac{\pi}{4}(2+j+j_\infty)+\frac{1}{2i}\ln 2+n\pi \\ &-\frac{1}{8}\left\{6i\bar{b}-2ie^{-i\pi\frac{d-3}{d-2}}\bar{b}-9\bar{a}_1+e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1+\bar{a}_2-e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2\right\} \end{aligned} \tag{5.51}$$

where we took the limit of interest $j \to 0, 2$ wherever it was unambiguous, in order to simplify the notation. Using

$$\int_0^\infty dx\, x^{-\lambda} J_\mu(x) J_\nu(x) = \frac{\Gamma(\lambda)\Gamma(\frac{\nu+\mu+1-\lambda}{2})}{2^\lambda \Gamma(\frac{-\nu+\mu+1+\lambda}{2})\Gamma(\frac{\nu-\mu+1+\lambda}{2})\Gamma(\frac{\nu+\mu+1+\lambda}{2})}\ , \tag{5.52}$$

we obtain explicit expressions for the first-order coefficients,

$$\begin{aligned}\bar{a}_1 &= \frac{\pi\mathcal{A}}{4}\left(\frac{n\pi}{2\bar{r}_*}\right)^{-\frac{d-3}{d-2}} \frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2}+\frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2}+\frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1+2\cot\frac{\pi(d-3)}{2(d-2)}\cot\frac{\pi}{2}\left(-j+\frac{d-3}{d-2}\right)\right]\bar{a}_1 \\ \bar{b} &= -\cot\frac{\pi(d-3)}{2(d-2)}\ \bar{a}_1 \end{aligned} \tag{5.53}$$

where we used the identity $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$. We also set $\omega = n\pi/\bar{r}_*$, since corrections contribute to higher than first order. Notice that these expressions are well-defined when j becomes an integer. Thus, the first-order correction is $\sim o(n^{-\frac{d-3}{d-2}})$.

Next, we compare with numerical results in four dimensions [39]. It is convenient to set the AdS radius $R = 1$. From (5.1), the radius of the horizon r_H is related to the black hole parameter μ by

$$2\mu = r_H^3 + r_H \tag{5.54}$$

for $d = 4$. $f(r)$ has two more (complex) roots, r_- and its complex conjugate, where

$$r_- = e^{i\pi/3}\left(\sqrt{\mu^2+\frac{1}{27}}-\mu\right)^{1/3} - e^{-i\pi/3}\left(\sqrt{\mu^2+\frac{1}{27}}+\mu\right)^{1/3} \tag{5.55}$$

The integration constant in the tortoise coordinate (5.21) is

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)} = -\frac{r_-}{3r_-^2+1}\ln\frac{r_-}{r_H} - \frac{r_-^*}{3r_-^{*2}+1}\ln\frac{r_-^*}{r_H} \tag{5.56}$$

Despite appearances, this is not a real number, because we ought to define arguments as $0 \le \arg r < 2\pi$.

For scalar perturbations, we find from eqs. (5.51), (5.53), together with (5.10), (5.11) and (5.12),

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\mathsf{S}}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}} \ , \qquad \mathcal{A}_{\mathsf{S}} = \frac{\ell(\ell+1)-1}{6} \tag{5.57}$$

Notice that only the first-order correction is ℓ-dependent. In the limit of large horizon radius ($r_H \approx (2\mu)^{1/3} \gg 1$), we have from (5.56)

$$\bar{r}_* \approx \frac{\pi(1+i\sqrt{3})}{3\sqrt{3}r_H} \tag{5.58}$$

Numerically for $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.508 + 0.293i}{r_H^2\sqrt{n}} \tag{5.59}$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.654 + 0.458i}{\sqrt{n}} \tag{5.60}$$

In Figure 5.1 we compare this analytical result with numerical results [39]. We plot the gap

$$\Delta\omega_n = \omega_n - \omega_{n-1} \tag{5.61}$$

because the offset does not always agree with numerical results [38]. We show both zeroth-order and first-order analytical results.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.093 + 0.561i}{\sqrt{n}} \tag{5.62}$$

For tensor perturbations, we find from eqs. (5.51), (5.53), together with (5.8) and (5.12),

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\mathsf{T}}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}} \ , \qquad \mathcal{A}_{\mathsf{T}} = \frac{3\ell(\ell+1)+1}{6} \tag{5.63}$$

Again, only the first-order correction is ℓ-dependent. Numerically for large r_H and $\ell = 0$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.102 + 0.0586i}{r_H^2\sqrt{n}} \tag{5.64}$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.131 + 0.0916i}{\sqrt{n}} \tag{5.65}$$

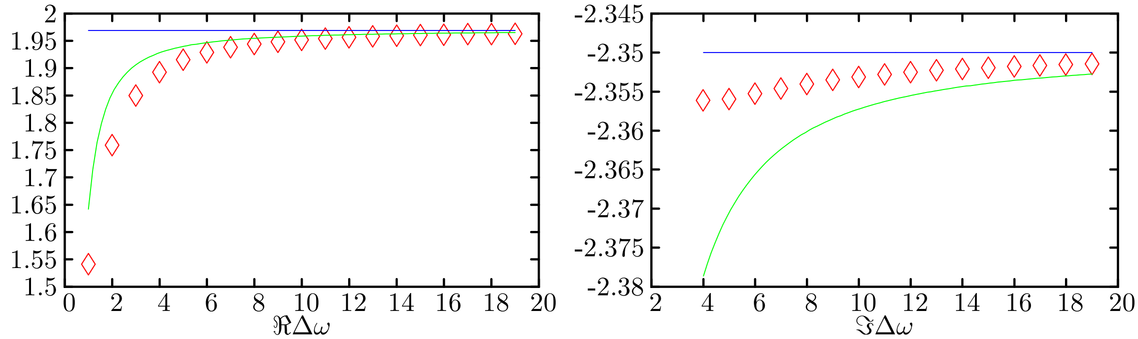

Figure 5.1: The frequency gap (5.61) for scalar perturbations in $d = 4$ for $r_H = 1$ and $\ell = 2$: zeroth and first order analytical (eq. (5.60)) compared with numerical data [39].

In Figure 5.2, we plot the gap (5.61), including both zeroth and first order and compare with numerical results [39].

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{0.489 + 0.251i}{\sqrt{n}} \tag{5.66}$$

and compare the gap with numerical results in Figure 5.3.

Finally, for vector perturbations, we find from eqs. (5.51), (5.53), together with (5.9) and (5.12),

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\mathsf{V}}\Gamma^4(\frac{1}{4})}{48\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\ , \qquad \mathcal{A}_{\mathsf{V}} = \frac{\ell(\ell+1)}{2} + \frac{3}{14} \tag{5.67}$$

Numerically for large r_H and $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{8.19 + 6.29i}{r_H^2\sqrt{n}} \tag{5.68}$$

For an intermediate black hole, $r_H = 1$, we obtain (see Figure 5.4)

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.741 + 0.519i}{\sqrt{n}} \tag{5.69}$$

and for a small black hole, $r_H = 0.2$, we obtain (see Figure 5.5)

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.239 + 0.6357i}{\sqrt{n}} \tag{5.70}$$

In all cases of gravitational perturbations, regardless of the size of the black hole, our analytical results are in good agreement with numerical results [39].

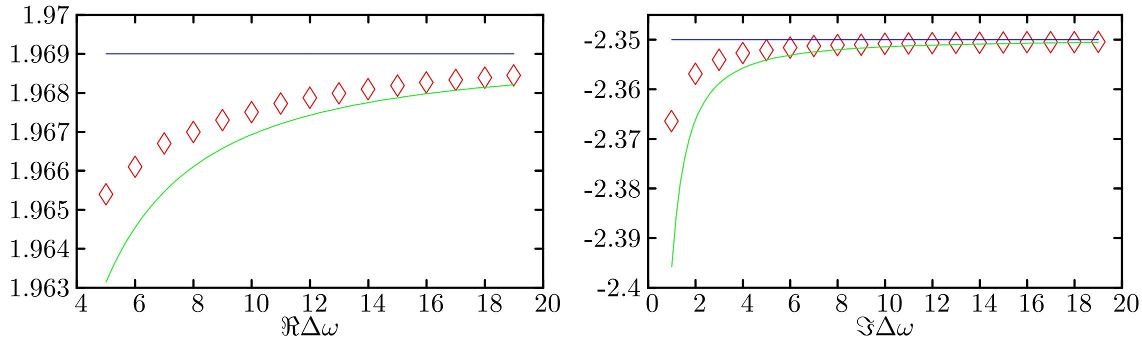

Figure 5.2: The frequency gap (5.61) for tensor perturbations in $d = 4$ for $r_H = 0.2$ and $\ell = 0$: zeroth and first order analytical (eq. (5.66)) compared with numerical data [39].

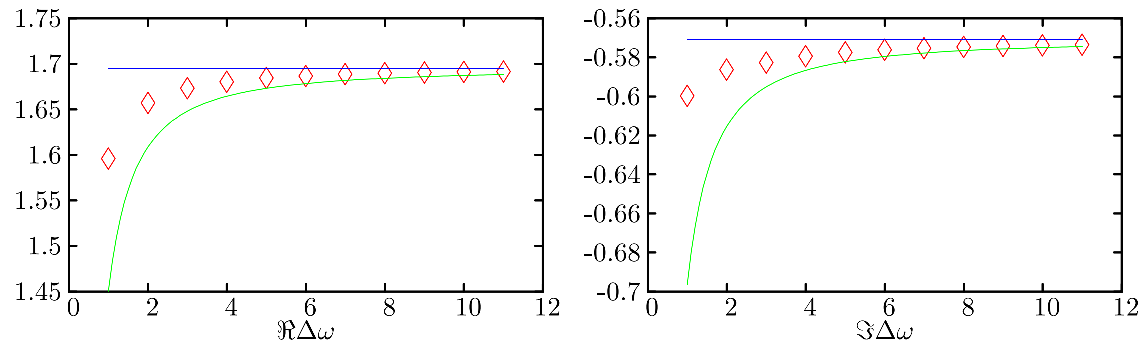

Figure 5.3: The frequency gap (5.61) for vector perturbations in $d = 4$ for $r_H = 1$ and $\ell = 2$: zeroth and first order analytical (eq. (5.69)) compared with numerical data [39].

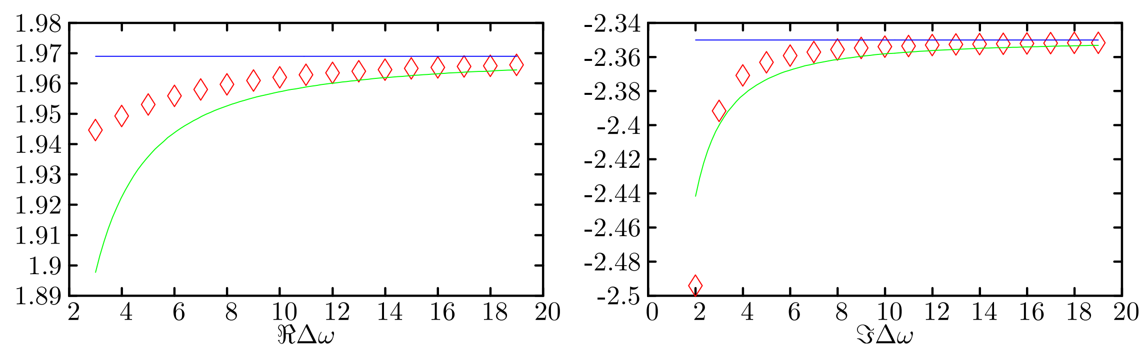

Figure 5.4: The frequency gap (5.61) for tensor perturbations in $d = 4$ for $r_H = 1$ and $\ell = 0$: zeroth and first order analytical (eq. (5.65)) compared with numerical data [39].

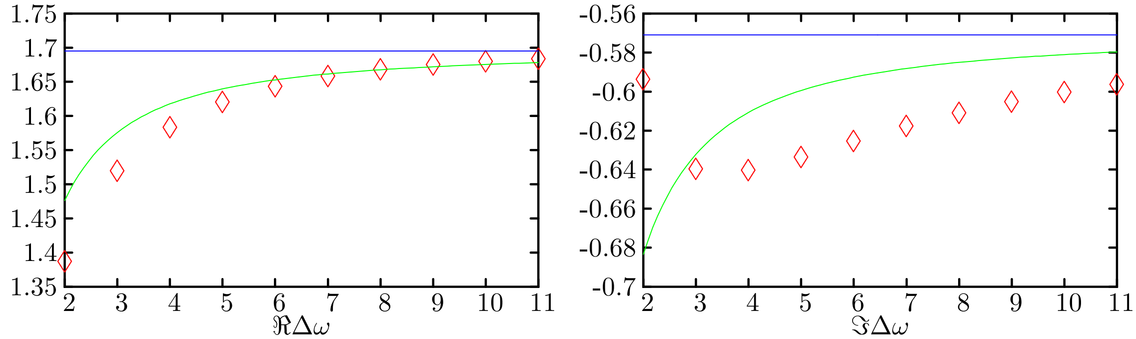

Figure 5.5: The frequency gap (5.61) for vector perturbations in $d = 4$ for $r_H = 0.2$ and $\ell = 2$: zeroth and first order analytical (eq. (5.70)) compared with numerical data [39].

5.3 Electromagnetic perturbations

In this section we extend the discussion to electromagnetic perturbations. This is a singular case because the potential vanishes at zeroth order. Consequently, the compatibility condition (5.33) discussed in the previous section has no solutions and no asymptotic expression for quasinormal frequencies may be deduced [38]. Nevertheless, the numerical results are similar to the ones we discussed in the case of gravitational perturbations [39]. We shall show that including first-order corrections leads to analytical asymptotic expressions for quasinormal frequencies in agreement with numerical results. Unlike with gravitational perturbations, where first-order corrections were a power of n (eqs. (5.51) and (5.53)), for electromagnetic perturbations first-order corrections are $o(\ln n)$.

We shall concentrate on the four-dimensional case for definiteness. Generalization to higher dimensions is straightforward. The wave equation reduces to (5.2) with electromagnetic potential

$$V_{\mathsf{EM}} = \frac{\ell(\ell+1)}{r^2} f(r). \tag{5.71}$$

where $f(r)$ is given in (5.1) with $d = 4$. Near the origin, this potential may be expanded in terms of the tortoise coordinate. Using eq. (5.7), we obtain

$$V_{\mathsf{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1)r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots\ , \tag{5.72}$$

where $j = 1$. This leads to a vanishing potential to zeroth order. Consequently, no analytic expression for quasinormal frequencies is deduced. This is easily seen by substituting $j = 1$ in the zeroth-order expression (5.34); we obtain a divergent result because $\tan^{-1} i$ is not finite.

This is remedied by including first-order corrections. The compatibility condition (5.50) of the two first-order constraints (5.45) and (5.49) reads

$$\begin{vmatrix} 1+\bar{b}+\bar{a}_2\tan\omega\bar{r}_* & -i-\bar{b}+i\bar{a}_2 \\ (1-\bar{b})\tan\omega\bar{r}_*-\bar{a}_1 & 1-\bar{a}_1+i\bar{b} \end{vmatrix} = 0 \tag{5.73}$$

where we used $d = 4$, $j = 1$, $j_\infty = d-3 = 1$, and $\alpha_+ = \beta = \frac{\pi}{2}$. At zeroth order (setting $\bar{a}_1 = \bar{a}_2 = \bar{b} = 0$), we obtain

$$\tan\omega\bar{r}_* = i \tag{5.74}$$

which has no solution, as expected [38]. At first order, we obtain

$$\tan\omega\bar{r}_* = i + (1-i)(\bar{a}_1 - \bar{a}_2 - 2\bar{b}) \tag{5.75}$$

whose first-order solution is

$$\omega\bar{r}_* = n\pi + \frac{1}{2i}\ln\frac{(1+i)(\bar{a}_1-\bar{a}_2-2\bar{b})}{2} \tag{5.76}$$

Using (5.53) and (5.72), we deduce explicit expressions for the first-order coefficients,

$$\bar{a}_1 = \mathcal{A}\sqrt{\frac{\bar{r}_*}{n}}\ , \qquad \bar{a}_2 = \bar{b} = -\bar{a}_1\ , \qquad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}} \tag{5.77}$$

and eq. (5.76) reads explicitly

$$\omega\bar{r}_* = n\pi - \frac{i}{4}\ln n + \frac{1}{2i}\ln\left(2(1+i)\mathcal{A}\sqrt{\bar{r}_*}\right) \tag{5.78}$$

Therefore, the correction to the quasinormal frequencies behaves as $\ln n$ in the large ω limit.

To compare with numerical results, set $R = 1$. As with gravitational perturbations, we shall compare the gap, because the offset is not reliable. For the gap, we have from (5.78)

$$\Delta\omega_n \equiv \omega_n - \omega_{n-1} = \frac{\pi}{\bar{r}_*}\left(1 - \frac{i}{4\pi n} + \dots\right) \tag{5.79}$$

Both leading and sub-leading terms are independent of ℓ.

For a large black hole, using (5.58), we obtain the spectrum

$$\frac{\Delta\omega_n}{r_H} \approx \frac{3\sqrt{3}(1-i\sqrt{3})}{4}\left(1-\frac{i}{4\pi n}+\dots\right) = 1.299 - 2.25i - \frac{0.179+0.103i}{n} + \dots \tag{5.80}$$

This analytical result is compared with numerical results [39] for $r_H = 100$ in Figure 5.6.

Using eqs. (5.54), (5.55), (5.56) and (5.78), we obtain the spectrum of an

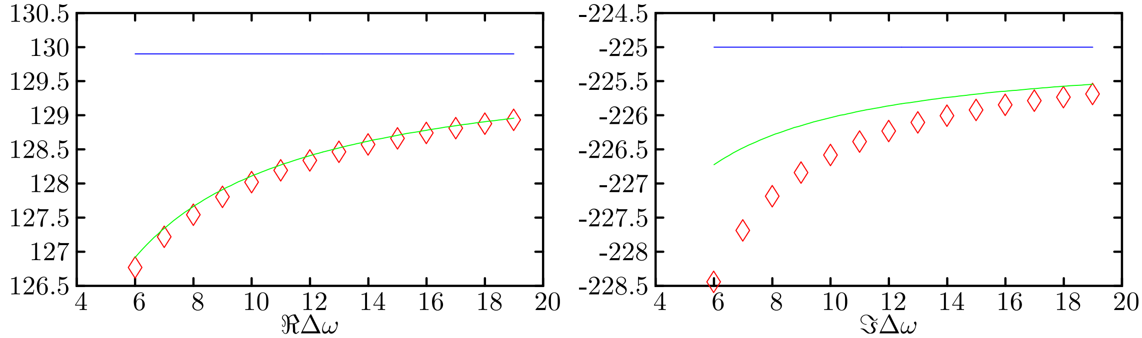

Figure 5.6: The frequency gap (5.61) for electromagnetic perturbations in $d = 4$ for $r_H = 100$ and $\ell = 1$: zeroth and first order analytical (eq. (5.80)) compared with numerical data [39].

intermediate black hole, $r_H = 1$, (see Figure 5.7)

$$\omega_n = (1.969 - 2.350i)n - (0.187 + 0.1567i)\ln n + \dots \tag{5.81}$$

and for a small black hole, $r_H = 0.2$, (see Figure 5.8)

$$\omega_n = (1.695 - 0.571i)n - (0.045 + 0.135i)\ln n + \dots \tag{5.82}$$

All first-order analytical results are in good agreement with numerical results [39].

5.4 Conclusions

By extending the the expansion of the Tortoise coordinate by an additional term, we solved the Schrödinger-like equation perturbatively in orders of $\hat{\omega}$. We showed the zeroth order contribution to the gravitational frequencies were in agreement with [39]. In addition, we found the first order corrections to the frequencies and found their behavior in the large frequency limit. We also found the first order frequencies for electromagnetic type perturbations and showed the gaps are in agreement with numerical results. It would be interesting to extend this work to more generalized black holes.

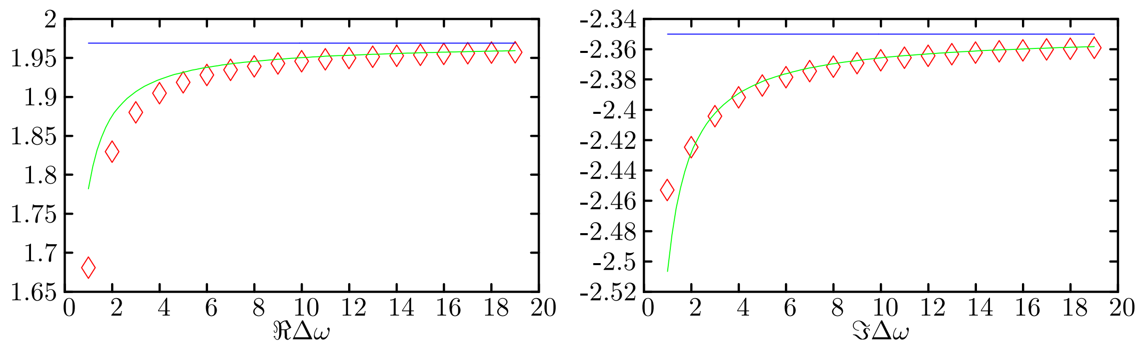

Figure 5.7: The frequency gap (5.61) for electromagnetic perturbations in $d = 4$ for $r_H = 1$ and $\ell = 1$: zeroth and first order analytical (eq. (5.81)) compared with numerical data [39].

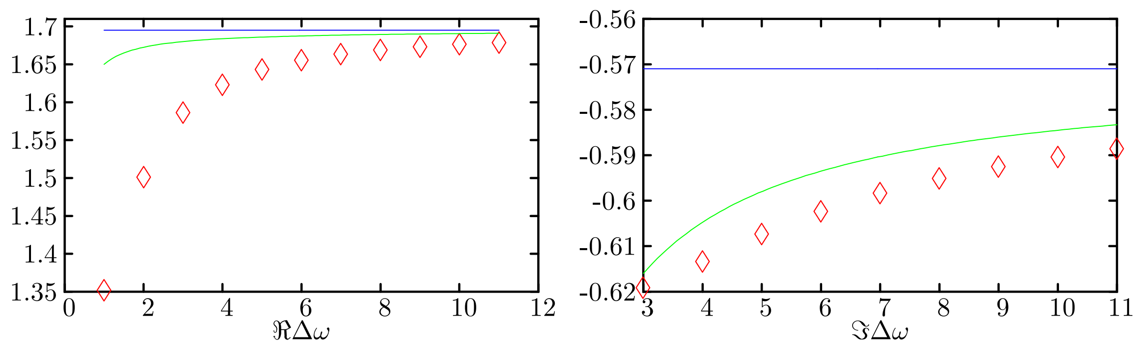

Figure 5.8: The frequency gap (5.61) for electromagnetic perturbations in $d = 4$ for $r_H = 0.2$ and $\ell = 1$: zeroth and first order analytical (eq. (5.82)) compared with numerical data [39].

Chapter 6

Massive perturbations

6.1 Introduction

In this chapter we extend the work done in the previous chapter by studying massive perturbations. For simplicity, we will study massive scalar perturbations however, one can extend this to include massive vector or tensor perturbations. We may apply a different method in calculating these quasinormal modes introduced by [3]. We will find the quasinormal modes of a finite AdS black hole in five dimensions and show they are quantized. We will calculate the behavior of the modes in the large frequency limit and show they match earlier results [3] in the large black hole limit.

6.2 Massive scalar perturbations

In this section we calculate quasinormal frequencies for massive scalar perturbations of *finite* black holes in AdS generalizing a procedure introduced in [3]. For massive perturbations, the method discussed in section 5.2 is not directly applicable. We consider explicitly the five-dimensional case in which the wave equation reduces to a Heun equation. Generalizing to higher dimensions is straightforward albeit tedious due to the increase in singular points.

Using the line element (5.1) in $d = 5$, we obtain the horizon radius

$$\frac{r_H^2}{R^2} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu}{R^2}} \tag{6.1}$$

The wave equation for a massive scalar of mass m is

$$\frac{1}{r^3}\partial_r \left(r^3 f(r)\partial_r \Phi\right) - \frac{1}{f(r)}\partial_t^2\Phi + \frac{1}{r^2}\nabla_\Omega^2\Phi = m^2\Phi \ . \tag{6.2}$$

It is convenient to transform to a dimensionless coordinate

$$y = s\left(\frac{2r^2}{R^2}+1\right) \quad , \qquad s = \frac{1}{\frac{2r_H^2}{R^2}+1} \ , \tag{6.3}$$

in terms of which the factor $f(r)$ (eq. (5.1) with $d = 5$) reads

$$f[r(y)] = \frac{y^2-1}{2s(y-s)} \ . \tag{6.4}$$

We see that s is a parameter describing the size of the black hole. When $s \to 0$, we approach the large black hole limit ($r_H \to \infty$) and expect to arrive at the results of [3].

Separating variables,

$$\Phi = e^{-i\omega t} Y_{\ell\vec{m}}(\Omega)\Psi(y) \ , \tag{6.5}$$

we obtain the radial wave equation expressed in terms of y,

$$(y-s)(y^2-1)\Psi'' + \left(3y^2-1-2sy\right)\Psi' + \left[\frac{\hat{\omega}^2}{4}\frac{(y-s)^2}{y^2-1} - \frac{\hat{L}^2}{4} - (y-s)\hat{m}^2\right]\Psi = 0 \tag{6.6}$$

where we introduced the dimensionless parameters

$$\hat{\omega}^2 = 2s\omega^2 R^2 \ , \quad \hat{L}^2 = 2s\ell(\ell+2) \ , \quad \hat{m} = \frac{mR}{2} \ . \tag{6.7}$$

The singularities of the wave equation are given by

$$y = \pm 1, s \ , \tag{6.8}$$

where $y = 1$ is the horizon, $y = s$ is the black hole singularity and $y = -1$ is an unphysical singularity. In order to bring (6.6) into a manageable form, we need to study the behavior of the wavefunction near the singularities. Two independent solutions of (6.6) are obtained by examining the behavior near the horizon ($y \to 1$),

$$\Psi_\pm \sim (y-1)^{\pm i\frac{\hat{\omega}}{4}\sqrt{1-s}} \ . \tag{6.9}$$

where Ψ_+, Ψ_- represent outgoing and ingoing waves, respectively. We will choose Ψ_- for quasinormal modes.

Near the singularity $y \to -1$ we obtain a different set of independent solutions

$$\Psi \sim (y+1)^{\pm\frac{\hat{\omega}}{4}\sqrt{1+s}} \ . \tag{6.10}$$

Since this is an unphysical singularity there is no physical choice. By studying the behavior at large r ($y \to \infty$), we find another set of independent solutions which determine the scaling behavior and are given by

$$\Psi \sim y^{-h_\pm} \quad , \quad h_\pm = 1 \pm \sqrt{1+\hat{m}^2}. \tag{6.11}$$

For quasinormal modes we want the solution to vanish for large r ($y \to \infty$), leading us to choose

$$\Psi \sim y^{-h_+} \ . \tag{6.12}$$

We may write the solution of (6.6) in the form

$$\Psi = (y-1)^{-i\frac{\hat{\omega}}{4}\sqrt{1-s}}(y+1)^{-\frac{\hat{\omega}}{4}\sqrt{1+s}}F(y) \ . \tag{6.13}$$

Substituting this expression into the wave equation (6.6), we obtain an equation for $F(y)$,

$$\begin{aligned}(y^2-1)F'' &+ \left\{ \left(3 - (\sqrt{1+s} + i\sqrt{1-s})\frac{\hat{\omega}}{2}\right) y + s + (\sqrt{1+s} - i\sqrt{1-s})\frac{\hat{\omega}}{2} \right\} F' \\ &+ \left\{ \frac{\hat{\omega}}{2}\left[\left(s + i\sqrt{1-s^2}\right)\frac{\hat{\omega}}{4} - (\sqrt{1+s}+i\sqrt{1-s}) \right] - \hat{m}^2 \right\} F \\ &+ \frac{1}{y-s}\left\{ (s^2-1)F' - \frac{\hat{L}^2}{4}F + \left[(1-s)\sqrt{1+s} - i(1+s)\sqrt{1-s}\right]\frac{\hat{\omega}}{4}F \right\} \\ &= 0\end{aligned} \tag{6.14}$$

If we are interested in the limit of large frequencies $\hat{\omega}$, we may focus on the region of large y [3]. In this case, the last term on the left-hand side of (6.14) is negligible compared with the other terms and the wave equation simplifies to a hypergeometric equation,

$$\begin{aligned}(y^2-1)F'' &+ \left\{ \left(3 - (\sqrt{1+s} + i\sqrt{1-s})\frac{\hat{\omega}}{2}\right) y + s + (\sqrt{1+s} - i\sqrt{1-s})\frac{\hat{\omega}}{2} \right\} F' \\ &+ \left\{ \frac{\hat{\omega}}{2}\left[\left(s + i\sqrt{1-s^2}\right)\frac{\hat{\omega}}{4} - (\sqrt{1+s}+i\sqrt{1-s}) \right] - \hat{m}^2 \right\} F \\ &= 0\end{aligned} \tag{6.15}$$

Two linearly independent solutions of (6.15) are

$$\mathcal{F}_1 = F(a_+, a_-; c; -x) \ , \quad \mathcal{F}_2 = x^{1-c}F(1+a_+-c, 1+a_--c; 2-c; -x) \ , \quad x = \frac{y-1}{2} \ , \tag{6.16}$$

where

$$\begin{aligned} a_\pm &= h_\pm - \left(\sqrt{1+s} + i\sqrt{1-s}\right)\frac{\hat{\omega}}{4} \ , &(6.17)\\ c &= \frac{3}{2} + \frac{1}{2}(s - i\sqrt{1-s}\,\hat{\omega}) \ . &(6.18)\end{aligned}$$

Using the transformation properties of hypergeometric functions, we may re-express the solutions (6.16) in terms of a new set of independent solutions which match the scaling behavior (6.11) for large r ($x \to \infty$),

$$\mathcal{K}_\pm = (x+1)^{-a_\pm} F(a_\pm, c-a_\mp; a_\pm - a_\mp + 1; 1/(x+1)) \tag{6.19}$$

We ought to choose $\mathcal{K}_+$, since it leads to $\Psi \to 0$ as $x \to \infty$. $\mathcal{K}_+$ may be expressed as a linear combination of $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$\mathcal{K}_+ = \mathcal{A}_0 \mathcal{F}_1 + \mathcal{B}_0 \mathcal{F}_2, \tag{6.20}$$

where

$$\mathcal{A}_0 = \frac{\Gamma(1-c)\Gamma(1-a_- + a_+)}{\Gamma(1-a_-)\Gamma(1-c+a_+)} \quad , \quad \mathcal{B}_0 = \frac{\Gamma(c-1)\Gamma(1+a_+ - a_-)}{\Gamma(a_+)\Gamma(c-a_-)}. \tag{6.21}$$

For the correct behavior at the horizon, we demand

$$\mathcal{B}_0 = 0,$$

which leads to two conditions

$$c - a_- = 1 - n \quad , \quad n = 1, 2, 3, \ldots \tag{6.22}$$

or

$$a_+ = 1 - n \quad , \quad n = 1, 2, 3, \ldots \tag{6.23}$$

Eq. (6.22) leads to the zeroth-order frequencies,

$$\hat{\omega}_n = -2(\sqrt{1+s} + i\sqrt{1-s}) \left[n + h_+ - \frac{3}{2} + \frac{s}{2} \right] \tag{6.24}$$

Notice that the phase approaches $\pi/4$ in the large black-hole limit ($r_H \to \infty$ or $s \to 0$), as expected [3].

Using (6.23), we find a second set of frequencies given by

$$\hat{\omega}_n = 2(\sqrt{1+s} - i\sqrt{1-s})(n + h_+ - 1) \ . \tag{6.25}$$

Both sets of frequencies, (6.24) and (6.25), at leading order agree on the imaginary part and have opposite real parts. We shall work with (6.24) without loss of generality. Notice also that the two sets of quasinormal frequencies match the results of [3] in the large black hole limit ($s \to 0$).

To find the first-order correction to the zeroth-order expression for quasinormal frequencies (6.24), we shall solve the Heun equation (6.14) perturbatively. To this end, let us bring it to the form

$$(\mathcal{H}_0 + \mathcal{H}_1) F = 0, \tag{6.26}$$

where (*cf.* eq. (6.15))

$$\mathcal{H}_0 = \partial_y^2 + \frac{1}{y^2-1}\left\{\left(3-(\sqrt{1+s}+i\sqrt{1-s})\frac{\hat{\omega}}{2}\right)y+(s+(\sqrt{1+s}-i\sqrt{1-s})\frac{\hat{\omega}}{2}\right\}\partial_y$$
$$+\frac{1}{y^2-1}\left\{\frac{\hat{\omega}}{2}\left[(s+i\sqrt{1-s^2})\frac{\hat{\omega}}{4}-(\sqrt{1+s}+i\sqrt{1-s})\right]-\hat{m}^2\right\}, \quad (6.27)$$

and the correction (to be treated as a perturbation) is given by

$$\mathcal{H}_1 = \frac{1}{(y^2-1)(y-s)}\left[(s^2-1)\partial_y+\left((1-s)\sqrt{1+s}-i(1+s)\sqrt{1-s}\right)\frac{\hat{\omega}}{4}\right]. \quad (6.28)$$

We have neglected the angular momentum contribution for simplicity. We may expand the wave function as

$$F = F_0 + F_1 + \dots \quad (6.29)$$

where F_0 obeys the zeroth-order equation (eq. (6.15))

$$\mathcal{H}_0 F_0 = 0. \quad (6.30)$$

Solving this equation leads to the zeroth-order expressions for quasinormal frequencies (6.24). The first-order equation is

$$\mathcal{H}_1 F_0 + \mathcal{H}_0 F_1 = 0 \quad (6.31)$$

We may solve for F_1 by using variation of parameters,

$$F_1 = \mathcal{K}_- \int_x^\infty \frac{\mathcal{K}_+\mathcal{H}_1 F_0}{\mathcal{W}} - \mathcal{K}_+ \int_x^\infty \frac{\mathcal{K}_-\mathcal{H}_1 F_0}{\mathcal{W}} \quad (6.32)$$

where $\mathcal{K}_\pm$ are the two linearly independent solutions (6.19) of eq. (6.15) and $\mathcal{W}$ is their Wronskian given by

$$\mathcal{W} = (a_+ - a_-)x^{-c}(1+x)^{c-a_+-a_--1}. \quad (6.33)$$

To study the behavior near the horizon ($x \to 0$), we may analytically continue the parameters in (6.32) without affecting the singularity. For $x \sim 0$, we obtain

$$F_1 \sim \mathcal{A}_1 + x^{1-c}\mathcal{B}_1 \ , \quad (6.34)$$

where

$$\mathcal{B}_1 = \beta_- \int_0^\infty \frac{\mathcal{K}_+\mathcal{H}_1 F_0}{\mathcal{W}} - \beta_+ \int_0^\infty \frac{\mathcal{K}_-\mathcal{H}_1 F_0}{\mathcal{W}}, \quad (6.35)$$

and

$$\beta_\pm = \frac{\Gamma(c-1)\Gamma(1+a_\pm-a_\mp)}{\Gamma(a_\pm)\Gamma(c-a_\mp)}. \quad (6.36)$$

With our choice (6.22), we find

$$\mathcal{B}_1 = \beta_- \int_0^\infty \frac{\mathcal{K}_+ \mathcal{H}_1 F_0}{\mathcal{W}}.$$

Therefore the quasinormal frequencies, to first order, are found as solutions of

$$\mathcal{B}_0 + \mathcal{B}_1 = 0, \tag{6.37}$$

where $\mathcal{B}_0$ is given by (6.21).

We can now find explicit expressions for the first-order correction to quasinormal frequencies. Writing to first order

$$\hat{\omega}_n = -2(\sqrt{1+s} + i\sqrt{1-s}) \left[n + h_+ - \frac{3}{2} + \frac{s}{2} - \epsilon_n \right] \tag{6.38}$$

we aim at calculating ϵ_n. Let us start with the case of $n = 1$. Our quantization condition (6.22) becomes $c = a_-$. This truncates the expansion of the hypergeometric solution (6.19) to

$$F_0 = \mathcal{K}_+ = (1+x)^{-a_+}. \tag{6.39}$$

After some algebra, we find

$$\mathcal{B}_1 = \frac{\beta_-}{2(a_+ - a_-)} \sum_{k=0}^{1} \alpha_k \int_0^\infty dx \frac{x^c (1+x)^{-(c+a_+-a_--k)}}{2x+1-s} , \tag{6.40}$$

where the coefficients, α_k $(k = 0, 1)$, are given by

$$\alpha_0 = -a_+(s^2-1) \quad , \quad \alpha_1 = \left[(s^2-1) + is\sqrt{1-s^2} \right] [a_+ - a_- + 1 + s] . \tag{6.41}$$

Using

$$\int_0^\infty dx \frac{x^\lambda (1+x)^{-\mu}}{1+\delta x} = B(\lambda+1, \mu-\lambda) F(1, \lambda+1; \mu+1; 1-\delta), \tag{6.42}$$

we find

$$\begin{aligned} \mathcal{B}_1 \quad = \quad & \frac{B(a_- - 1, a_+ - a_- + 1)}{1-s} \left(\frac{a_-}{a_+ - a_-} \right) \left[-\frac{\alpha_0}{2a_+} F(1, a_- + 1; a_+ + 1; \frac{s+1}{s-1}) \right. \\ & \left. - \frac{\alpha_1}{2(a_+ - a_- - 1)} F(1, a_- + 1, a_+; \frac{s+1}{s-1}) \right] . \end{aligned} \tag{6.43}$$

Expanding in $1/h_+$ (large mass expansion), we obtain

$$\mathcal{B}_1 = B(a_- - 1, a_+ - a_- + 1) \left[\frac{s}{4} \left(s^2 - 1 + is\sqrt{1-s^2} \right) - \frac{i}{8h_+} (1 + o(s)) \right] , \tag{6.44}$$

where we made use of the expansion of a hypergeometric function

$$F(1,\alpha;\beta;z) = \left(1-\frac{\alpha}{\beta}\,z\right)^{-1} + \left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{\alpha^2 z^2}{\beta^2}\left(1-\frac{\alpha}{\beta}\,z\right)^{-3} + \dots \tag{6.45}$$

which is valid for large α and β ($\sim o(h_+)$). We obtain from (6.21) and (6.38)

$$\mathcal{B}_0 = \epsilon_1 B(a_- - 1, a_+ - a_- + 1) + \dots \tag{6.46}$$

By using (6.37) we find the first-order correction for $n = 1$,

$$\epsilon_1 = -\frac{s}{4}\left(s^2 - 1 + is\sqrt{1-s^2}\right) + \frac{i}{8h_+}\,(1 + o(s)) \tag{6.47}$$

For a finite-size black hole ($s \neq 0$), this is a $o(h_+^0)$ correction to $n = 1$ quasinormal frequencies. The correction is $o(1/h_+)$ for an infinite-size black hole ($s = 0$) [3]. It should be pointed out that the calculation of ϵ_1 involved cancellation of $o(h_+)$ terms. For a general n, one obtains expressions $o(h_+^n)$. Non-trivial cancellations occur between various terms involving hypergeometric functions and after the dust settles, one arrives at the general expression

$$\epsilon_n = -\frac{s}{4n}\left(s^2 - 1 + is\sqrt{1-s^2}\right) + \frac{i(2-1/n)}{8h_+}\,(1 + o(s)) \quad , \quad n = 1, 2, 3, \dots \tag{6.48}$$

which is $o(h_+^0)$ for finite-size black holes and $o(1/h_+)$ for infinite-size black holes,

$$\epsilon_n = \frac{i}{4}\left(1 - \frac{1}{2n}\right)\frac{1}{h_+}\,. \tag{6.49}$$

We have been unable to provide an analytical proof of the above results for general n but have verified them for several n using Mathematica.

6.3 Conclusions

By applying the method of [3], we calculated the quasinormal modes of a finite AdS black hole in five dimensions for a massive scalar perturbation. We solved the problem perturbatively and calculated the first order correction to the quasinormal frequency. By looking in the large black hole limit, we showed that our result is in agreement with [3]. Studying massive perturbations may also help to give insight into various aspects of quantum gravity. One could extend this work by studying massive vector or tensor perturbations of black holes.

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Appendix

A: Maximally symmetric space-times

In general relativity, we arrive at the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ by asking that the field equations be written in terms of the unique symmetric, rank two covariantly conserved object constructed out of the metric and its derivatives which has Minkowski space as a vacuum solution. By relaxing the final condition we may generalize the Einstein tensor by adding a term proportional to the metric. The constant of proportionality, Λ, is known as the Cosmological constant. We may add this term since it is covariantly a constant. This leads to a more general Einstein tensor given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu}. \tag{A1}$$

When expressing the full set of equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{A2}$$

where

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{A3}$$

We see the cosmological constant, Λ acts like an intrinsic pressure. $\Lambda > 0$ gives a repulsion, where as $\Lambda < 0$ gives an attraction. The type of solutions of (A2) we want to study are called "maximally symmetric" solutions of the Einstein field equations. There are three solutions that are maximally symmetric and they depend on the sign of Λ. If $\Lambda > 0$, the space-time is de Sitter. For $\Lambda < 0$, the space-time is known as an anti-de Sitter space-time, and if $\Lambda = 0$, the space-time is known as a Minkowski space-time.

A maximally symmetric solution will satisfy the condition

$$R_{\mu\nu\rho\sigma} = \mp\frac{2}{\ell^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{A4}$$

where $\mp$ represents de Sitter and Anti-de Sitter respectively and

$$\ell^2 = -\frac{(D-1)(D-2)}{2\Lambda}.$$

Also, by taking the trace

$$R_{\mu\nu} = \pm \frac{2\Lambda}{(D-2)} g_{\mu\nu}. \tag{A5}$$

For the signature $(- + + + ...)$ the spaces of interest here may be written as

$$ds^2 = -\left(1 \pm \frac{r^2}{\ell^2}\right) dt^2 + \left(1 \pm \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2 \tag{A6}$$

where $d\Omega_{D-2}^2$ is the metric on a unit round $D-2$ sphere. The cosmological constant sets the scale of ℓ and if we let $\ell \gg r$ we arrive at Minkowski space.

For a more detailed introduction to de Sitter and Anti-de Sitter space-times see [42, 43].

Vita

Scott Henry Ness was born in Tacoma, WA to Hank and Diane on February 26, 1977. After graduating from Stadium High School in 1995, he pursued a Bachelor's degree in Physics from Eastern Illinois University in Charleston, IL. He joined the University of Tennessee in the fall of 1999 to pursue the Doctor of Philosophy degree in Physics and graduated in December 2005.