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To the Graduate Council:

I am submitting herewith a dissertation written by Suphot Musiri entitled "Aspects of Black Hole Scattering." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

George Siopsis, Major Professor

We have read this dissertation and recommend its acceptance:

Alexandre Freire, Chia C. Shih, Ted Barnes

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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ASPECTS OF BLACK HOLE SCATTERING

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Suphot Musiri
May 2003

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Abstract

We discuss various aspects of black hole scattering. Firstly, we consider non-extremal rotating black branes. We solve the wave equation for a massless scalar field and calculate the absorption cross section. We obtain a function of two temperature parameters once we move away from extremality, which is similar to the case of Kerr-Newman black holes. We discuss the implications of this result to the AdS/CFT correspondence. Secondly, we study a system of maximally-charged slowly-moving black holes and take the limit of a continuous self-interacting matter distribution (black string). We quantize the system by using the path integral method. We show that a careful implementation of the Faddeev-Popov gauge-fixing procedure leads to a Hamiltonian possessing a well-defined vacuum. The Hamiltonian consists of a kinetic energy term and a potential which is the generator of special conformal transformations. We obtain an explicit expression for the Hamiltonian of a ring-shaped formation and show that it is equivalent to a harmonic oscillator in the non-relativistic limit. Thirdly, we investigate quasinormal modes. We develop a perturbative method of calculating quasinormal frequencies in the high temperature limit of AdS Schwarzschild spacetimes of varying dimensionality. In 2+1 dimensions, exact expressions involving hypergeometric functions have been obtained. We discuss the (4+1)-dimensional case in detail. In this case, the calculation of quasinormal modes amounts to solving Heun's equation. Higher dimensions are also considered. Our analytical results are in agreement with numerical results for the low-lying frequencies.

Contents

Chapter 1

Introduction

The concept of a black hole was proposed in 1784 by John Michell [1], but did not receive much attention until the arrival of general relativity in the early twentieth century when solutions to the Einstein field equations containing singularities were discovered. The term 'black hole' was first used in 1967 by Wheeler [2].

It was Bekenstein in 1972 [3] who first pointed out the similarity between the non-decreasing area theorem and the second law of thermodynamics. He proposed that the area of the black hole horizon should be proportional to its entropy. This idea contradicted the traditional idea that entropy should be proportional to the volume of the system. In 1973, Bardeen, Carter and Hawking [4] provided a rigorous proof of the first law of black holes,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J_H + \ldots \tag{1.1}$$

and the second law, the non-decreasing of the horizon area of a black hole, $\delta A \geq 0$, where M is the mass of the black hole, κ its surface gravity, Ω_H and J_H are its angular velocity and momentum, respectively, and the dots represent work done by other parameters. The third law is the unattainability of a zero-value of κ, stating that it takes an infinite amount of energy or time to reduce κ to zero.

In analogy to the laws of thermodynamics, κ should represent the temperature, T, and A should represent the entropy, S. However, in classical relativity, T is zero and S is infinite. In [5], Bekenstein suggested that the non-zero value of temperature and finite value of entropy should be the result of quantum effects. This implies that when $T \neq 0$, black holes should radiate particles out of the horizon, resulting in a decreasing entropy. In 1974, by applying second quantization to a boson field near the horizon, Hawking [6] proposed that the total entropy of a system that includes a black hole be given by

$$S = \frac{A}{4} + S' \tag{1.2}$$

where S' is the entropy in the exterior of the black hole. It should be emphasized that the constant multiplying the area of the horizon A ($1/4$ in units in which $G = c = \hbar = k = 1$, where G in Newton's constant, c is the speed of light, h is Planck's constant ($\hbar = h/2\pi$) and k is Boltzmann's constant) is fixed. The importance of S' is that it prevents the total entropy from decreasing by a decay process of the black hole or by an absorbing-entropy process into the black hole, such as very slowly dropping a box of entropy into the black hole, which causes the entropy of the box to disappear and the area of the horizon to barely increase [7]. The temperature at equilibrium is called the Hawking temperature

$$T_H = \frac{\kappa}{2\pi} \ . \tag{1.3}$$

Radiation from the black hole at this temperature is called Hawking radiation. On the other hand, superradiance occurs when $\omega < m\Omega$, where ω is the energy of the emitted particles from the black hole, m is the magnetic quantum number and Ω is the angular velocity of the black hole. In this case, an observer far away from the black hole sees a flux reflected from the black hole greater than the incident flux [8, 9].

The above semiclassical description has become the foundation of quantum gravity. To date, the research on this subject can be categorized into three groups [10]:

(I) classical calculation of the interior entropy which is proportional to the horizon area [11],

(II) the study of entanglement entropy resulting from quantum field correlations between the exterior and the interior of the black hole [12], and

(III) calculation of the entropy and scattering cross section (absorption coefficients) from the low energy effective action in (super)string theory.

String theory has been the most successful theory to provide an entropy value microscopically in agreement with the macroscopic value from general relativity ($A/4$). Moreover, scattering cross sections (absorption coefficients) from string theory have been shown to agree with classical results [13]. This correspondence is described in more detail in chapter 2. A review of string theory and the extended objects that arise as solutions (such as D-branes) can be found in numerous sources, such as [14, 15, 16, 17].

This dissertation is organized as follows.

In Chapter 2, we discuss the scattering cross section of scalar absorption by N D3-branes which are extended objects arising in (super)string theory. We divide the discussion into three sections. In section 2.1, we review the derivation of Dp-brane solutions in string theory and use them (in the $p = 3$ case) to calculate the entropy and absorption coefficients. They are found to be in agreement with results from corresponding conformal field theories, showing that a certain correspondence exists between supergravity models arising from string theory and conformal field theories living at the boundary of spacetime. This correspondence is further supported by the calculation of Green functions. Section 2.2 contains examples of the calculation of decay rates of a scalar field in 3+1 and 4+1 dimensional Kerr-Newman black holes, where a function of two temperature parameters, left and right, are obtained. The results in section 2.1 and 2.2 are described from literature as a motivation to calculate $D3$-brane absorption coefficients in section 2.3. We use the method in section 2.2 to calculate scattering cross sections of scalar fields off of non-extremal rotating black branes in section 2.3 and we also find the two temperature function. We comment on the implications of our results on the supergravity/field theory correspondence observed in the extremal limit.

In Chapter 3, we consider the quantization of a multi black hole system. Unlike chapter 2, where we study scattering off of a single extended object, here we discuss scattering of black holes by other black holes. We quantize the system in the extremal case, in which a non-relativistic expansion is possible, using the path-integral method. The Hamiltonian of this system appears to possess an ill-defined ground state. The problem can be fixed by the addition of a potential K. K turns out to be the generator of special conformal transformations. We show that the addition of K arises naturally from a careful implementation of the gauge-fixing procedure, following [18, 19]. We start by reviewing the quantization of a particle near a black hole, specializing to extremal Reissner-Nordström spacetimes, in Section 3.2. In Section 3.3, we extend the discussion to the quantization of slowly-moving maximally-charged black holes. The results in section 3.1, 3.2 and 3.3 are described from literature where we extend our calculation in section 3.4. In section 3.4, we consider a continuous self-interacting matter distribution (black string) and quantize it in the non-relativistic limit. We calculate the potential K explicitly for a ring-shaped formation and show that the Hamiltonian is equivalent to a harmonic oscillator Hamiltonian.

In Chapter 4, we calculate AdS Schwarzschild black hole quasinormal modes. In this chapter, we turn our attention back to scattering off of a single black hole in AdS Schwarzschild spacetime. Quasinormal modes are the solutions to the wave equation where the wave is ingoing at the horizon and outgoing at the boundary of spacetime (far away from the black hole). From the AdS/CFT correspondence, the poles of the Green functions (quasinormal frequencies) should provide information regarding perturbations of the corresponding conformal field theory residing at the boundary of AdS spacetime. The quasinormal frequencies are complex numbers whose imaginary part is negative, since the modes are decaying at the black hole horizon. The wave equation can be solved exactly in 2+1 dimensions. This is reviewed in section 4.2. The work in section 4.1 and 4.2 are from literature. In section 4.3, we tackle the (4+1)-dimensional case in which the wave equation turns into Heun's equation. We develop a perturbative method to calculate the quasinormal modes in the limit of high temperature, or large black hole. Our results are in agreement with results obtained through numerical methods for the low-lying frequencies. We then extend our method to higher dimensions and obtain explicit expressions in 6+1 and 3+1 dimensions in section 4.4.

Finally, Chapter 5 contains a discussion of our results. We summarize our conclusions, discuss related work that has appeared in the literature and present possible future directions.

Chapter 2

Absorption Coefficients of Branes

In January 1998, Maldacena [20] proposed a correspondence between (super)gravity models arising from (super)string theory and supersymmetric quantum field theories in one less dimension. For example, it was conjectured that $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions can be derived from type-IIB superstring theory in the presence of a large number of parallel D3-branes. In the strong coupling limit, the spacetime near the D3-branes is $\mathrm{AdS}_5 \times S^5$ and the super Yang-Mills theory, which is a superconformal field theory, lives on the boundary of the anti-de Sitter space AdS_5. This conjecture has led to a booming research on the subject and today, there is considerable evidence of the AdS/CFT correspondence, confirming Maldacena's conjecture. The correspondence has been extended to $M2$ or $M5$ branes (supergravity based on $\mathrm{AdS}_7 \times S^4$) and the (0,2) superconformal field theory in six dimensions, as well as $D1 + D5$ branes (supergravity based on M^4) and (4,4) superconformal field theory. In general, the supergravity model is defined in one more dimension than the corresponding field theory. Thus, we see the emergence of a holographic principle according to which all the degrees of freedom of the supergravity theory lie on a hypersurface (usually, the boundary) of the spacetime on which the theory is defined [21]. An extensive review on the AdS/CFT correspondence can be found in [22]. Our work in this chapter is part of on-going research on (super)gravity models related to D3-branes and the (super)conformal field theories they correspond to. Before going into the details of our work, we shall review the key developments in the subject. There is a vast literature on the subject, so only a few significant results will be discussed.

For completeness, we should mention that a similar idea was proposed back in 1992, suggesting that the 1+1-dimensional QCD theory is equivalent to string theory [23]. Also more recently, Strominger [24] proposed that a similar correspondence exists between D-dimensional de-Sitter space (dS) and $(D-1)$-dimensional conformal field theories living in the infinite past and future of the dS space. For details, see [25].

The structure of this chapter is the following. Section 2.1 contains the general properties of non-rotating D3-branes, a discussion of the Bekenstein-Hawking entropy, and extremal D3-brane absorption coefficients which we compare with those from conformal field theory providing a crucial piece of evidence of the AdS/CFT correspondence. In section 2.2, we present examples of absorption coefficients of Kerr-Newman black holes in 3+1 and 4+1 dimensions. The method used in section 2.2 is employed in section 2.3 to obtain absorption coefficients of rotating branes off extremality.

2.1 D3-Branes and the AdS/CFT correspondence

2.1.1 Black strings and p-branes

In this subsection we present the derivation of non-rotating D3-branes, following the work of Horowitz and Strominger [26] and also Gibbons [27] and Maeda [28]. We start from the low energy effective action from IIB super string theory [16, 14]

$$S = \int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left[R + 4(\nabla\phi)^2\right] - \frac{2e^{2\alpha\phi}}{(D-2)!}F^2\right] \tag{2.1}$$

where F is a $(D-2)$ form with $dF = 0$. The magnetic charge proportional to $\int F$ is carried by a spatially extended $(10-D)$-dimensional object, assuming $D \geq 4$, for example D3-branes whereas the electrical charge can be obtained from dualizing F, [29]. By assuming that the branes are flat, the solution can be written in the form

$$ds^2 = e^A d\hat{s}^2 + e^B dx_i dx^i, \tag{2.2}$$

where $d\hat{s}^2$ is a D-dimensional metric, x^i are p-dimensional cartesian coordinates with $p = 1, ..., 10-D$ and all fields are independent of x^i. Substituting the metric into the action (2.1), we obtain a D-dimensional action. In order to turn the action into the standard Einstein action, A, B and ϕ may be simplified into the linear combination of two scalar fields, ρ and σ, where F^2 does not couple to σ. The linear combinations of ρ and σ are

$$\begin{aligned}
\beta\phi &= \rho\frac{(4\alpha+7-D)}{2} - \sigma\frac{(D-3)}{2}\left[\frac{10-D}{D-2}\right]^{1/2} \\
\beta A &= \rho\left[\alpha - \frac{D-4}{D-2}\right] - \sigma(\alpha+1)\left[\frac{10-D}{D-2}\right]^{1/2} \\
\beta B &= \rho(\alpha+1) + \sigma\frac{(D-2)\alpha - D + 4}{[(10-D)(D-2)]^{1/2}},
\end{aligned} \tag{2.3}$$

where

$$\beta = -\left[4\alpha^2 + 2\alpha(7-D) + 2\frac{D-1}{D-2}\right]^{1/2}. \tag{2.4}$$

The D-dimensional action, after substituting these parameters, is

$$S = \int d^D x \sqrt{-\hat{g}} \left[\hat{R} - \frac{1}{2}(\nabla\rho)^2 - \frac{1}{2}(\nabla\sigma)^2 - e^{\beta\rho}\frac{2F^2}{(D-2)!}\right]. \tag{2.5}$$

The equations of motion from the action (2.5) are

$$\begin{aligned} 0 &= \nabla^{\mu_1}\left(e^{\beta\rho}F_{\mu_1\ldots\mu_{D-2}}\right), \\ \nabla^\mu\nabla_\mu\sigma &= 0, \qquad\qquad \nabla^\mu\nabla_\mu\rho = \frac{2\beta}{(D-2)!}e^{\beta\rho}F^2, \\ \hat{R}_{\mu\nu} &= \frac{1}{2}\nabla_\mu\rho\nabla_\nu\rho + \frac{1}{2}\nabla_\mu\sigma\nabla_\nu\sigma + \frac{2}{(D-3)!}e^{\beta\rho}F_{\mu\lambda_1\ldots\lambda_{D-3}}F_\nu^{\lambda_1\ldots\lambda_{D-3}} \\ &\quad -\hat{g}_{\mu\nu}\frac{2(D-3)}{(D-2)(D-2)!}e^{\beta\phi}F^2. \end{aligned} \tag{2.6}$$

To solve these equations, the metric, $d\hat{s}^2$, in D dimensions may be written in the form

$$d\hat{s}^2 = -\lambda^2 dt^2 + \lambda^{-2}d\hat{r}^2 + R^2 d\Omega_{D-2}^2, \tag{2.7}$$

where λ and R are function of $\hat{r}$. The solution can be simplified by letting the scalar fields vanish asymptotically. From the first equation of motion, F can be written as

$$F = Q\epsilon_{D-2}, \tag{2.8}$$

where Q is the charge of the black holes and ϵ_{D-2} is the volume of the unit D-dimensional sphere, or $\int_{S^D}\epsilon_D = 2\pi^{(D+1)/2}/\Gamma((D+1)/2)$. Then from this result, $(1/(D-2)!)F^2 = Q^2/R^{2D-4}$. As is known from general relativity only timelike $\hat{R}_{00}$, radial $\hat{R}_{11}$ and spherical $\hat{R}_{22}$ tensors give independent results and also one can find out that $\hat{R}_{00} = (D-3)\hat{R}_{22}$, or in terms of R and λ,

$$\frac{1}{2}R^{2-D}(R^{D-2}(\lambda^2)')' = -(D-3)R^{2-D}(R^{D-3}\lambda^2 R')' + (D-3)^2R^{-2}, \tag{2.9}$$

where a prime means differentiating with respect to $\hat{r}$. From the third line in the equation of motion (2.6) $\hat{R}_{00} = ((D-3)/\beta(D-2))\nabla^2\rho$. By setting $Z = \lambda^2 e^{-2(D-3)\rho/\beta(D-2)}$, this equation turns into

$$(R^{D-2}\lambda^2(\ln Z)')' = 0. \tag{2.10}$$

With the asymptotically flat and horizontally regular boundary conditions, these equations can be solved and the results are

$$\begin{aligned}
F &= Q\epsilon_{D-2}, \\
d\hat{s}^2 &= -\left[1-(r_+/r)^{D-3}\right]\left[1-(r_-/r)^{D-3}\right]^{1-\gamma(D-3)} dt^2 \\
&\quad +\left[1-(r_+/r)^{D-3}\right]^{-1}\left[1-(r_-/r)^{D-3}\right]^{\gamma-1} dr^2 \\
&\quad +r^2\left[1-(r_-/r)^{D-3}\right]^{\gamma} d\Omega_{D-2}^2, \\
e^{\beta\rho} &= \left[1-(r_-/r)^{D-3}\right]^{\gamma(D-3)}, \\
\sigma &= 0,
\end{aligned} \tag{2.11}$$

where

$$\gamma = \frac{2\beta^2(D-2)}{(D-3)(2(D-3)+\beta^2(D-2))}. \tag{2.12}$$

The coordinate r relates to $\hat{r}$ by $r^{D-4}dr = R^{D-4}d\hat{r}$. The parameters r_+ and r_- are related to Q and M by these two equations

$$Q = \left[\frac{\gamma(D-3)^3(r_+r_-)^{D-3}}{2\beta^2}\right]^{1/2}, \qquad M = [1-(D-3)\gamma]r_-^{D-3} + r_+^{D-3}. \tag{2.13}$$

For $r_- = 0$, then $F = 0$, $\rho = 0$ and the metric reduces to the D-dimensional Schwarzschild solution. For $r = r_+$, the timelike Killing field becomes null and there exists an event horizon, but for $r = r_-$, the horizon area shrinks to zero and there emerges a curvature singularity, therefore this solution describes black holes only for $r_+ > r_-$.

From (2.2), the solution in ten dimensions is

$$\begin{aligned}
F &= Q\epsilon_{D-2}, \\
ds^2 &= -\left[1-(r_+/r)^{D-3}\right]\left[1-(r_-/r)^{D-3}\right]^{\gamma_x-1} dt^2 \\
&\quad +\left[1-(r_+/r)^{D-3}\right]^{-1}\left[1-(r_-/r)^{D-3}\right]^{\gamma_r} dr^2 \\
&\quad +r^2\left[1-(r_-/r)^{D-3}\right]^{\gamma_r+1} d\Omega_{D-2}^2 + \left[1-(r_-/r)^{D-3}\right]^{\gamma_x} dx^i dx_i, \\
e^{-2\phi} &= \left[1-(r_-/r)^{D-3}\right]^{\gamma_\phi},
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
\gamma_r &= \frac{(\alpha-1)}{(2\alpha^2+(7-D)\alpha+2)} - \frac{D-5}{D-3} \\
\gamma_x &= \frac{(\alpha+1)}{(2\alpha^2+(7-D)\alpha+2)} \\
\gamma_\phi &= -\frac{(4\alpha+7-D)}{(2\alpha^2+(7-D)\alpha+2)}.
\end{aligned} \tag{2.15}$$

The solutions are invariant under the symmetry R×SO($D-1$)×E($10-D$), where E(n) is the n dimensional Euclidean group. Notice that under the extremal condition $r_+ = r_-$, the symmetry group is SO($D-1$)×P($11-D$) where P(n) is the n dimensional Poincaré group.

We are interested in the case of $p = 10 - D = 3$, D3-branes. From [29], a self-dual five-form is contained in the chiral IIB. This implies that black holes coupling to this F have to carry both electric and magnetic charges simultaneously and the solution can not be obtained directly from (2.14). In this case it is obvious that F is not a source of the dilaton, therefore one can let the dilaton be a constant. Then there is a only one equation of motion left

$$R_{\mu\nu} = F_{\mu\alpha_1\ldots\alpha_4} F_{\nu}^{\alpha_1\ldots\alpha_4}. \tag{2.16}$$

One can repeat the same step as before and obtain (or let $\alpha = 0$)

$$\begin{aligned} ds^2 &= -(1-\frac{r_+^4}{r^4})(1-\frac{r_-^4}{r^4})^{-1/2}dt^2 + \frac{dr^2}{(1-\frac{r_+^4}{r^4})(1-\frac{r_-^4}{r^4})} \\ &\quad + r^2 d\Omega_5^2 + (1-\frac{r_-^4}{r^4})^{1/2} dx_i dx^i, \\ F &= Q(\epsilon_5 + *\epsilon_5), \qquad \phi = \phi_0, \end{aligned} \tag{2.17}$$

where the charge Q is

$$Q = 2r_+^2 r_-^2. \tag{2.18}$$

2.1.2 Black 3-brane entropy

In this subsection we will describe the Bekenstein-Hawking entropy near the extremal limit and compare it with the entropy of field theories, following [30, 31, 32]. We start with the metric (2.17). The 8-dimensional area of the horizon is, substituting $r = r_+$,

$$A = \omega_5 r_+^5 L^3 (1 - \frac{r_-^4}{r_+^4})^{3/4}, \tag{2.19}$$

where $\omega_5 = \pi^3$ is the area of the unit 5-sphere and L is the radius of a large 3-torus, T^3, wrapped around the 3-branes. Notice that when $r_+ = r_-$(the extremal limit), the area vanishes. The Bekenstein-Hawking entropy of the black 3-branes is

$$S_{BH} = \frac{A}{4}. \tag{2.20}$$

A non-zero value of the entropy can be obtained by considering the metric slightly off the extremal limit by a small ADM mass, δM. The value of δM can be obtained in terms of a boosting parameter α. By boosting the metric (2.17) with constant momentum P along a spatial direction, x^1, we obtain on the branes,

$$\begin{aligned} ds^2 &= -(\cosh^2\alpha\,\triangle_+\,\triangle_-^{-1/2} - \sinh^2\alpha\triangle_-^{1/2})dt^2 \\ &\quad +(\cosh^2\alpha\,\triangle_-^{1/2} - \sinh^2\alpha\,\triangle_+\,\triangle_-^{-1/2})dx_1^2 \\ &\quad +\sinh(2\alpha)(\triangle_-^{1/2} - \triangle_+\triangle_-^{-1/2})dtdx_1^2 \\ &\quad +\triangle_-^{1/2}\,(dx_2^2 + dx_3^2) + \triangle_+^{-1}\,\triangle_-^{-1}\,dr^2 + r^2 d\Omega_5^2, \end{aligned} \tag{2.21}$$

where $\triangle_\pm = 1 - r_\pm/r$. The total ADM momentum is

$$\begin{aligned} P &= \frac{L^3\omega_5}{8\pi}\sinh(2\alpha)(r_+^4 - r_-^4) \\ &= \frac{2\pi n}{L}. \end{aligned} \tag{2.22}$$

Because α is an independent parameter, n is set in the above equation to be an integer. Assuming finite ADM momentum slightly away from the extremal limit, or $r_+^2 \approx r_-^2 + \delta M$, we obtain

$$P_{ADM} \sim L^3\omega_5 Q\left[e^{2\alpha'}\frac{\delta M}{M}\right], \tag{2.23}$$

where M and Q are from (2.13). Therefore, $\delta M/M \sim e^{-2\alpha}$. The entropy in the near extremal limit can be found from the area of the horizon,

$$\begin{aligned} S_{BH} &\sim \frac{\omega_5}{4}r_+^5 L^3\left[\triangle_-(r_+)^{3/4}\right]\cosh\alpha \\ &\sim \omega_5 L^3 r_0^5\left[\frac{\delta M}{M}\right]^{3/4} e^{\alpha}. \end{aligned} \tag{2.24}$$

The entropy of non extremal 3-branes from statistical mechanics can be obtained from the IIB partition function [30]

$$Z = \prod_{\vec{n}\in\mathbf{Z}^3}\left(\frac{1+q^{|\vec{n}|}}{1-q^{|\vec{n}|}}\right)^6, \tag{2.25}$$

where the momenta of the quantized string states are

$$\vec{p} = \frac{2\pi}{L}\vec{n}, \tag{2.26}$$

$q = e^{-2\pi/LT}$, at temperature T and the number 6 represents the transverse oscillation modes, instead of 8, as expected [30]. By using the standard thermodynamic relations

$$F = -T \log Z, \qquad E = T^2 \frac{\partial}{\partial T} \log Z, \qquad S = (E - F)/T,$$

one arrives at

$$E = \frac{3\pi^2}{8} L^3 T^4, \qquad S = \frac{\pi^2}{2} L^2 T^3. \tag{2.27}$$

The equation above holds for a single 3-brane, however if there are N 3-branes stacked together with no binding energy among them, the possibility strings connected between two 3-branes is N^2. Therefore for N 3-branes the energy and entropy are, respectively,

$$E = \frac{3\pi^2}{8} N^2 L^3 T^4, \qquad S = \frac{\pi^2}{2} N^2 L^2 T^3. \tag{2.28}$$

T can be eliminated and the entropy can be rewritten as

$$S = 2^{5/4} 3^{-3/4} \sqrt{\pi N} L^{3/4} E^{3/4}. \tag{2.29}$$

By setting $E = \delta M$, using the ground state mass

$$M_0 = \frac{\sqrt{\pi}}{\kappa} n L^3 \tag{2.30}$$

and the mass of the excited 3-branes [15],

$$M = M_0 + \delta M = \frac{\sqrt{\pi}}{\kappa} L^3 + \sum_{i=1}^{k} \frac{2\pi}{L} |\vec{n}_i| + O(g), \tag{2.31}$$

where $\kappa = \sqrt{8\pi G_N}$, G is the Newton's constant and k is the number of open strings, the entropy changes to

$$S = 2^{5/4} 3^{-3/4} \pi^{7/8} N^{5/4} \kappa^{-3/4} L^3 (\delta M / M_0)^{3/4}. \tag{2.32}$$

To be able to compare with S_{BH} let us consider the ADM mass of the metric (2.17) [33],

$$M_{ADM} = \frac{\omega_5 L^3}{2\kappa^2} (5 r_+^4 - r_-^4) \tag{2.33}$$

and let $r_+ = r_- = r_0$, where M_{ADM} becomes M_0. Then r_0 is

$$r_0^4 = \frac{\sqrt{\pi}}{2\omega_5} N\kappa. \tag{2.34}$$

However, we need the answer away from the extremal limit, so let $r_+ = r_0 + \epsilon$. Then from M_{ADM} in (2.33), $\delta M_0/M$ becomes

$$\frac{\delta M}{M} \sim 6\frac{\epsilon}{r_0}. \tag{2.35}$$

From the horizon area of the metric (2.17),

$$\begin{aligned} A_H &= \omega_5 r_+^5 L^3 \left(1 - \frac{r_-^4}{r_+^4}\right)^{3/4} \\ &= 2^{9/4}\omega_5 r_0^5 L^3 \left(\frac{\epsilon}{r_0}\right)^{3/4} \\ &= 2^{1/4}3^{-3/4}\pi^{-1/8}(N\kappa)^{5/4}L^3(\delta M/M_0)^{3/4}, \end{aligned} \tag{2.36}$$

the Bekenstein-Hawking entropy is

$$S_{BH} = \frac{2\pi A}{\kappa^2} = 2^{5/4}3^{-3/4}\pi^{7/8}N^{5/4}\kappa^{-3/4}L^3(\delta M/M_0)^{3/4}, \tag{2.37}$$

which is exactly the same as in (2.32).

The entropy of near extremal Dp-branes is calculated in [34] which gives the same leading contribution.

2.1.3 Extremal D3-brane absorption coefficients

In this subsection, we seek to gain further insight into the AdS/CFT correspondence by calculating and comparing absorption coefficients from N extremal non-dilatonic D3-branes and the corresponding conformal field theory. The non-dilatonic condition causes the metric to be considerably simplified, i.e., no singularity exists in the transverse part [35, 36]. From (2.17), changing the parameter r at the extremal limit $r_+ = r_- = R$ to $(1 - r_+^4/r^4)^{1/2} \to (1 + R^4/r^4)^{-1/2}$, we obtain

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2}(dr^2 + r^2 d\Omega_5^2). \tag{2.38}$$

We are interested in the case of N parallel D3-branes stacked together with no bindinding among them, which is also considered in subsection 2.1.2. The curvature of these N non-dilatonic D3-branes is of order, [37]

$$\frac{1}{\sqrt{N\kappa_{10}}} \sim \frac{1}{\alpha'\sqrt{Ng}}. \tag{2.39}$$

where $\kappa_{10} = \sqrt{8\pi G_{10}} = g\alpha'$. G_{10} is Newton's constant in ten dimensions, g is a coupling constant and α' is related to the tension of the string, T, by $T = 1/2\pi\alpha'$. To be able to control the order of expansion of $\frac{1}{Ng}$, consider the double scaling limit

$$Ng \to \infty, \qquad \omega^2\alpha' \to 0, \tag{2.40}$$

where ω is the frequency of the incident mode. We also keep

$$N\kappa_{10}\omega^4 \sim Ng\alpha'^2\omega^4 \tag{2.41}$$

small. The classical absorption cross section is normally of order $\omega^4/(\text{curvature})^2$, which is the same as the combination of parameters in the above equation.

From the metric for N non-dilatonic extremal D3-branes (2.38), the radial part of the massless wave equation is

$$\left[\rho^{-5}\frac{d}{d\rho}\rho^5\frac{d}{d\rho} + 1 + \frac{(\omega R)^4}{\rho^4}\right]\phi(\rho) = 0, \tag{2.42}$$

where $\rho = \omega r$. The absorption coefficient can be obtained by matching the near solution (small ωr) to the far solution (large ωr) in the low energy limit (small ωR). For the near solution, let $z = (\omega R)^2/\rho$. Then (2.42) turns into

$$\left[\frac{d^2}{dz^2} - \frac{3}{z}\frac{d}{dz} + 1 + \frac{(\omega R)^4}{z^4}\right]\phi = 0. \tag{2.43}$$

Separating the singularity by letting $\phi = z^{3/2}f(z)$, the above equation becomes

$$\left[\frac{d^2}{dz^2} - \frac{15}{4z^2} + 1 + \frac{(\omega R)^4}{z^4}\right]f = 0. \tag{2.44}$$

The last term in the near region can be ignored, since $z \gg \omega R$. The equation becomes a Bessel equation. Thus, the incoming wave for small r can be written in terms of Bessel functions as

$$\phi = i(\omega R)^4\rho^{-2}\left[J_2\left(\frac{(\omega R)^2}{\rho}\right) + iN_2\left(\frac{(\omega R)^2}{\rho}\right)\right]. \tag{2.45}$$

In the far region, letting $\phi = \rho^{-5/2}\psi$ in (2.42), we obtain

$$\left[\frac{d}{d\rho^2} - \frac{15}{4\rho^2} + 1 + \frac{(\omega R)^4}{\rho^4}\right]\psi = 0, \tag{2.46}$$

where $\rho \gg (\omega R)^2$. Again, the solution is a Bessel function. By matching the two solutions in the overlapping region, the solution in the far region is determined to be

$$\phi = \frac{32}{\pi}\rho^{-2}J_2(\rho). \tag{2.47}$$

The absorption coefficient is defined as the ratio of the flux at the throat to the incoming flux from infinity,

$$\mathcal{F} = \frac{\pi}{16^2}(\omega R)^8. \tag{2.48}$$

The absorption cross section in D dimensions is [38]

$$\sigma = \frac{(2\pi)^{D-1}}{\omega^{D-1}\Omega_{D-1}}\mathcal{F}, \tag{2.49}$$

where $\Omega_D = \frac{2\pi^{(D+1)/2}}{\Gamma((D+1)/2)}$ is the volume of the unit D-dimensional sphere. Therefore, the D3-brane absorption cross section is (setting $D = 6$)

$$\sigma_{3-\text{brane}} = \frac{\pi^4}{8}\omega^3 R^8. \tag{2.50}$$

Next, we compare the above supergravity result to a field-theoretical calculation. We shall derive the absorption cross section of a scalar, e.g., the dilaton ϕ, using field theory. The relevant part of the D3-brane action is [39]

$$S = T_3 \int d^4x \left[\frac{1}{2}\sum_{i=4}^{9}\partial_\mu X^i \partial^\mu X^i - \frac{1}{4}e^{-\phi}F_{\mu\nu}^2\right], \tag{2.51}$$

where $F_{\mu\nu}$ is the field strength on the D3-brane and the six fields X^i $(i = 1, \ldots, 6)$ describe the transverse oscillations of the brane. $T_3 = \sqrt{\pi}/\kappa_{10}$ is the D3-brane tension [40]. By fixing the gauge for A_μ, we obtain two physical photons (two transverse polarizations). Then the dilaton coupling term to each photon polarization in the D3-brane action can be written as

$$-\frac{1}{2}\int d^4x \phi \partial_\mu \tilde{A} \partial^\mu \tilde{A},$$

where $\partial_\mu \tilde{A}$ is a canonically normalized physical field. The action in the 10-dimensional bulk space is

$$S_{\text{bulk}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{g}\left[R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \ldots\right] \tag{2.52}$$

and so the canonically normalized dilaton field is

$$\tilde{\phi} = \frac{\phi}{\sqrt{2}\kappa_{10}}.$$

Therefore, the dilaton coupling term in the D3-brane action can be written as

$$-\frac{\kappa_{10}}{\sqrt{2}}\int d^4x \tilde{\phi}\partial_\mu \tilde{A}\partial^\mu \tilde{A}. \tag{2.53}$$

By assuming a scalar incident on the brane at right angles splitting into two massless bosons which then move on the brane with momenta p_1, p_2, respectively, the scattering amplitude can be calculated using standard field-theoretical methods. The result is

$$\mathcal{A} = -\frac{\kappa_{10}}{\sqrt{2}} 2 \frac{p_1 \cdot p_2}{\sqrt{2}\omega^{3/2}} = -\frac{\kappa_{10}\sqrt{\omega}}{2}, \tag{2.54}$$

where we used $p_1^0 = p_2^0 = \omega/2$, $-\vec{p}_1 = \vec{p}_2$ and $p_1 \cdot p_2 = \omega^2/2$. The absorption cross section is

$$\sigma = 2\,\frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3}\int \frac{d^3p_2}{(2\pi)^3}(2\pi)^4\delta(E_1+E_2-\omega)\delta^3(\vec{p}_1+\vec{p}_2)|\mathcal{A}|^2, \tag{2.55}$$

where the factor of 2 is due to the two polarization states of the physical photon and the factor of 1/2 comes from the two identical particles. After integrating, we obtain

$$\sigma = \frac{\kappa_{10}^2\omega^3}{32\pi}$$

For a stack of N D3-branes, there are $2N^2$ possibilities for the photon states. Therefore the total absorption cross section for a stack of N D3-branes is

$$\sigma_{3-\text{brane}} = \frac{\kappa_{10}^2 N^2\omega^3}{32\pi}. \tag{2.56}$$

To compare with the supergravity result (2.50), note that the ADM mass per unit volume of the 3-brane, $\frac{2\pi^3 R^4}{\kappa_{10}^2}$, is equal to $\frac{\pi}{\kappa_{10}}N$ [40]. This gives

$$R^4 = \frac{\kappa_{10}N}{2\pi^{5/2}}. \tag{2.57}$$

Using this expression for R^4 in (2.50), it is seen immediately that the two expressions for the total absorption cross section (2.50) and (2.56) agree with each other. More details of the calculation of D3-brane absorption coefficients are presented in [41]. Other kinds of D-brane absorption coefficients have also been calculated, for example for D2-branes and D5-branes, in [37, 42].

2.1.4 Correlators

In previous sections, we calculated the entropy and absorption coefficients providing compelling evidence for the proposed AdS/CFT correspondence between N D3-branes in type-IIB superstring theory and $\mathcal{N} = 4$ supersymmetry Yang-Mills theory (conformal field theory). Moreover, note that the two theories share symmetry groups. The superconformal group in the super YM theory, $SO(2{,}4)$ [20], is also the isometry group of AdS_5, and the isometry group of the sphere S^5, $SO(6)$, is isomorphic to the supersymmetry group $SU(4)$ in the super YM theory [43, 44, 45]. This correspondence of supersymmetries is further investigated in [46].

Here, we show that correlators in $\mathrm{AdS}_5 \times S^5$ are also in agreement with their super-YM counterparts [35]. We start with the supergravity calculation. The metric of spacetime for a stack of N D3-branes in the large-N limit (zero Hawking temperature) is (cf. (2.38))

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2).$$

At large r (far away from the branes), the metric becomes flat ten-dimensional Minkowski metric. Near the throat, $r \sim R$, the metric can be simplified by changing parameters to $z = R^2/r$,

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2. \tag{2.58}$$

The range of z can be divided into two regions, the near region (between $z = \infty$ and $z = R$, or $r = R$ and $r = 0$) and the far region (between $z = R$ and $z = 0$, or $r = R$ and $r = \infty$). The near region is AdS space and the fields, e.g., a scalar field, may be viewed as entering from its boundary [35]. To obtain the generating functional of connected Green functions, the extremum of the supergravity action $S[\phi(x^\mu, z)]$, which is a classical action, is related to the generator of connected Green functions in the gauge field theory, $e^{-W[\phi(x^\mu)]}$, through

$$W[\phi(x^\mu)] = S[\phi(x^\mu, z)]\Big|_{\delta S = 0}. \tag{2.59}$$

A similar idea was presented in [47]. The simplest example is the action of a free scalar field ϕ in a fixed gravitational background $\mathrm{AdS}_5 \times S^5$,

$$S = \frac{1}{2\kappa^2} \int d^{10}x \, \sqrt{G}[\frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi]. \tag{2.60}$$

The action in the near region is further simplified by letting the angular momentum on the sphere S^5 vanish. We obtain

$$S = \frac{\pi^3 R^8}{4\kappa^2} \int d^4x \int_R^\infty \frac{dz}{z^3} \left[(\partial_z \phi)^2 + \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right], \tag{2.61}$$

where Greek indices represent the 4-dimensional spacetime (boundary of AdS_5). Varying ϕ in the action, we obtain the equation of motion

$$\left[z^3 \partial_z \frac{1}{z^3} \partial_z + \eta^{\mu\nu} \partial_\mu \partial_\nu\right] \phi = 0, \tag{2.62}$$

whose solution is

$$\phi_k(x^\mu, z) = \lambda_k e^{ik\cdot x}\tilde{\phi}_k(z), \qquad \tilde{\phi}_k(z) = \frac{z^2 K_2(kz)}{R^2 K_2(kR)}, \tag{2.63}$$

where $k^2 = \vec{k}^2 - \omega^2$. The modified Bessel function $K_2(kz)$ is chosen because it falls off exponentially at large z (near the horizon), whereas the other solution $I_2(kz)$ increases exponentially. Integrating by parts, we obtain the classical supergravity action

$$\begin{aligned} S\Big|_{\delta S=0} &= W[\phi(x^\mu)] \\ &= \frac{\pi^3 R^8}{4\kappa^2}\int d^4x \int_R^\infty \frac{dz}{z^3}\left[-\phi\left(z^3\partial_z\frac{1}{z^3}\partial_z + \eta^{\mu\nu}\partial_\mu\partial_\nu\right)\phi + z^3\partial_z\left(\phi\frac{1}{z^3}\partial_z\phi\right)\right] \end{aligned}$$

where $\phi(x^\mu)$ is the Fourier transform of λ_k,

$$\phi(x^\lambda) = \int d^4k \lambda_k e^{ik\cdot x}. \tag{2.64}$$

We may write

$$W = \frac{1}{2}\int d^4k d^4q \lambda_k\lambda_q (2\pi)^4\delta^4(k+q)\frac{N^2}{16\pi^2}\mathcal{F}, \tag{2.65}$$

where the flux factor $\mathcal{F}$ is

$$\mathcal{F} = \left[\tilde{\phi}_k\frac{1}{z^3}\partial_z\tilde{\phi}_k\right]_R^\infty. \tag{2.66}$$

By differentiating W twice with respect to the Fourier mode λ_k, we obtain a two-point function in the conformal field theory (super YM theory) residing at the boundary of AdS_5. Let $\mathcal{O}$ be the dual operator in the conformal field theory coupled to the source $\phi(x^\mu)$. We have

$$\begin{aligned} \langle\mathcal{O}(k)\mathcal{O}(q)\rangle &= \int d^4x d^4y e^{i(k\cdot x+q\cdot y)}\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \\ &= \frac{\partial^2 K}{\partial\lambda_k\partial\lambda_q} = (2\pi)^4\delta^4(k+q)\frac{N^2}{16\pi^2}\mathcal{F} \\ &= -(2\pi)^4\delta^4(k+q)\frac{N^2}{64\pi^2}k^4\ln(k^2R^2) + (\text{ analytic in } k^2). \end{aligned} \tag{2.67}$$

In position space,

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \sim \frac{N^2}{|x-y|^8}. \tag{2.68}$$

This result is in agreement with the result one obtains through a direct calculation in super YM theory, if $\mathcal{O} \sim F^2$, where $F^{\mu\nu}$ is the field strength of the YM potential. It should be noted that the field theory result is obtained in the small coupling ($g_{YM}^2 N \to 0$) limit, whereas the supergravity calculation produces results corresponding to the strong coupling regime ($g_{YM}^2 N \to \infty$).

The above procedure may also be used to calculate the conformal anomaly [48]. Indeed, noting that the operator coupled to the graviton field $h^{\mu\nu}$ is by definition the stress-energy tensor $T_{\mu\nu}$ [49], the classical value of the supergravity action should yield the two-point correlation function of two stress-energy tensors, which is proportional to the central charge in the conformal field theory. It suffices to consider, say, the T_{xy} component coupled to h_{xy}. A short calculation shows that the Einstein action for h_{xy} is of the same form as the scalar action (2.60). Therefore, we may repeat the above steps to arrive at the result

$$\langle T_{xy}(k) T_{xy}(q) \rangle = -(2\pi)^4 \delta^4(k+q) \frac{N^2}{64\pi^2} k^4 \ln(k^2 R^2) + (\text{ analytic in } k^2). \tag{2.69}$$

which is identical to (2.68). Therefore, we obtain the central charge as $c = N^2/4$. This can be compared to the result from super YM theory,

$$\langle T_{\alpha\beta}(x) T_{\gamma\delta}(0) \rangle = \frac{c}{48\pi^4} X_{\alpha\beta\gamma\delta}(1/x^4) \tag{2.70}$$

where $c = N^2/4$,

$$\begin{aligned} X_{\alpha\beta\gamma\delta} &= 2\Box^2 \eta_{\alpha\beta}\eta_{\gamma\delta} - 3\Box^2(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) - 4\partial_\alpha\partial_\beta\partial_\gamma\partial_\delta \\ &\quad -2\Box(\partial_\alpha\partial_\beta\eta_{\gamma\delta} + \partial_\alpha\partial_\gamma\eta_{\beta\delta} + \partial_\alpha\partial_\delta\eta_{\beta\gamma} + \partial_\beta\partial_\gamma\eta_{\alpha\delta} + \partial_\beta\partial_\delta\eta_{\alpha\gamma} + \partial_\gamma\partial_\delta\eta_{\alpha\beta}) \\ &, \end{aligned} \tag{2.71}$$

and $\Box$ is the four-dimensional Laplacian operator. Thus, we obtain agreement between the supergravity result (2.69) and the result from conformal field theory (2.70) on the central charge, providing one more piece of evidence in support of the AdS/CFT correspondence.

The above results have also been generalized to the massive scalar case as well as non-vanishing angular momentum in S^5 [35].

2.2 Absorption coefficients of black holes

Here, we extend the above results on the AdS/CFT correspondence for D-branes [20, 42, 36, 51, 52] to black holes. In doing so, we go from zero Hawking temperature to a system at finite temperature. In the next section, we shall extend the discussion further to rotating black branes.

2.2.1 3+1 Dimensions

Here, we calculate the absorption coefficients of (3+1)-dimensional Kerr-Newman black holes following [53]. The Kerr-Newman metric of a black hole with charge Q, mass M and angular momentum $J = Ma$ in 3+1 dimensions is

$$\begin{aligned} ds^2 &= -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \left(\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\right)dtd\phi \\ &+ \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 \\ &+ \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned} \tag{2.72}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 + Q^2 - 2Mr. \tag{2.73}$$

The inner and outer horizons, satisfying $\Delta = 0$, are

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}. \tag{2.74}$$

The horizon area $\mathcal{A}$, Hawking temperature T_H, angular velocity Ω and electric potential Φ are, respectively,

$$\begin{aligned} \mathcal{A} &= 4\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2-Q^2-a^2}\right) = 4\pi r_+^2, \\ T_H &= \frac{(r_+ - r_-)}{\mathcal{A}}, \\ \Omega &= \frac{4\pi a}{\mathcal{A}}, \\ \Phi &= \frac{4\pi Q r_+}{\mathcal{A}}. \end{aligned} \tag{2.75}$$

These quantities satisfy the first law of black hole mechanics

$$dM = T_H dS + \Omega dJ + \Phi dQ. \tag{2.76}$$

where the entropy is

$$S = \frac{\mathcal{A}}{4} \tag{2.77}$$

The wave equation for a massless scalar is

$$\partial_A g^{AB}\sqrt{-g}\partial_B\Phi = 0.$$

The solution can separated as [55]

$$\Phi = e^{im\phi - i\omega t} S_A^m(\theta; a\omega) R(r), \tag{2.78}$$

where $S_A^m(\theta; a\omega)$ satisfies

$$\left(\frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\sin^2\theta} + a^2\omega^2 \cos^2\theta \right) S_A^m(\theta; a\omega) = -A S_A^m(\theta; a\omega). \tag{2.79}$$

For small $a\omega$, we may expand

$$A = \ell(\ell+1) + \mathcal{O}(a^2\omega^2). \tag{2.80}$$

$R(r)$ obeys

$$\Delta \partial_r \Delta \partial_r R + K^2 R - \lambda \Delta R = 0, \tag{2.81}$$

where

$$K = \omega(r^2 + a^2) - ma, \qquad \lambda = A + a^2\omega^2 - 2m\omega a.$$

To solve the wave equation, we consider two separate but overlapping regions of spacetime outside the horizon, a near region, $(r - r_+)\omega \ll 1$, and a far region, $M \ll r - r_+$.

In the near region, we have $r \sim r_+$, so $K^2 - \lambda\Delta$ may be approximated by

$$K^2 - \lambda\Delta \simeq r_+^4 (\omega - m\Omega)^2 - \ell(\ell+1)\Delta, \tag{2.82}$$

where we neglected terms of order $(\omega a)^2$ and set $\lambda \simeq A \simeq \ell(\ell+1)$. The wave equation (2.81) becomes

$$\Delta \partial_r \Delta \partial_r R + r_+^4 (\omega - m\Omega)^2 - \ell(\ell+1)\Delta R = 0. \tag{2.83}$$

Changing parameters to $z = \frac{r - r_+}{r - r_-}$ (so that $0 \leq z \leq 1$), we obtain

$$z(1-z)\partial_z^2 R + (1-z)\partial_z R + \left(\frac{\omega - m\Omega}{4\pi T_H} \right)^2 (1 + \frac{1}{z}) R - \frac{\ell(\ell+1)}{1-z} R = 0. \tag{2.84}$$

The solution to the above equation can be separated as

$$R = A z^{i \frac{\omega - m\Omega}{4\pi T_H}} (1-z)^{\ell+1} F.$$

where A is a normalization constant. Thus, after removing the singularities, (2.84) reduces to

$$\begin{aligned} z(1-z)\partial_z^2 F \quad &+ \quad \left(1 + i\frac{\omega - m\Omega}{2\pi T_H} - (1 + 2(\ell+1) + i\frac{\omega - m\Omega}{2\pi T_H}) z \right) \partial_z F \\ &- \quad \left((\ell+1)^2 + i\frac{\omega - m\Omega}{2\pi T_H}(\ell+1) \right) F = 0. \end{aligned} \tag{2.85}$$

whose solution is the hypergeometric function $F(\alpha,\beta;\gamma;z)$, where

$$\alpha = \ell + 1 + i\frac{\omega - m\Omega}{2\pi T_H}, \qquad \beta = \ell + 1, \qquad \gamma = 1 + i\frac{\omega - m\Omega}{2\pi T_H}. \tag{2.86}$$

Next, we calculate the solution in the far region. From (2.81), letting $r \gg M$ removes the effects of the black hole. We obtain

$$\frac{1}{r^2}\partial_r r^2 \partial_r R + \omega^2 R - \frac{\ell(\ell+1)}{r^2} R = 0. \tag{2.87}$$

whose solution can be written in terms of Bessel functions,

$$R = \frac{1}{\sqrt{r}}\left[J_{\ell+\frac{1}{2}}(\omega r) + BJ_{-\ell-\frac{1}{2}}(\omega r)\right]. \tag{2.88}$$

where B is an arbitrary constant.

The coefficients are fixed by matching the expressions in the near and far regions on the overlapping region. From (2.85), letting $r \to \infty$ i.e., $1-z \to \frac{r_+ - r_-}{r} \to 0$, we obtain

$$\begin{aligned} R \ \sim \ & A\left(\frac{r}{r_+ - r_-}\right)^{-\ell-1} \Gamma\left(1 + i\frac{\omega - m\Omega}{2\pi T_H}\right) \times \\ & \left[\frac{\Gamma(-2\ell-1)}{\Gamma(-\ell)\Gamma(-\ell + i\frac{\omega - m\Omega}{2\pi T_H})} + \left(\frac{r}{r_+ - r_-}\right)^{2\ell+1} \frac{\Gamma(2\ell+1)}{\Gamma(\ell+1)\Gamma(\ell+1+i\frac{\omega - m\Omega}{2\pi T_H})}\right] \\ . \end{aligned} \tag{2.89}$$

This should be matched to (2.88) at small r. We obtain

$$A = N'\alpha \ , \quad N' = \frac{(r_+ - r_-)^\ell \omega^{\ell+\frac{1}{2}} \Gamma(\ell+1)\Gamma(\ell+1+i\frac{\omega - m\Omega}{2\pi T_H})}{2^{\ell+\frac{1}{2}}\Gamma(\ell+\frac{3}{2})\Gamma(2\ell+1)\Gamma(1+i\frac{\omega - m\Omega}{2\pi T_H})}. \tag{2.90}$$

The absorption cross section is a ratio of incoming fluxes, $\sigma = \frac{\mathcal{F}_{r\to r_+}}{\mathcal{F}_{r\to\infty}}$. The fluxes are

$$\begin{aligned} \mathcal{F} &= \frac{2\pi}{i}(R^*\Delta\partial_r R - R\Delta\partial_r R^*) \\ \mathcal{F}_{r\to r_+} &= 2|\alpha|^2 \\ \mathcal{F}_{r\to\infty} &= \frac{(\omega - m\Omega)}{T_H}(r_+ - r_-)|N'|^2|\alpha|^2, \end{aligned} \tag{2.91}$$

and the absorption cross section for the ℓ partial wave is

$$\sigma^\ell = \frac{(\omega - m\Omega)\mathcal{A}}{2}|N'|^2. \tag{2.92}$$

For $\ell = 0$, we have $\sigma^0 = \mathcal{A}$, which agrees with the general result [38] that the low energy total cross section of a massless scalar field incident on a black hole is proportional to the area of the horizon. Notice that σ^0 can be negative if $\omega < m\Omega$, which corresponds to superradiance. The decay rates can be written as, [6]

$$\Gamma^\ell = \frac{\sigma^\ell}{e^{\frac{\omega - m\Omega}{4\pi T_H}} - 1} = \frac{\pi\Gamma(\ell+1)^2\omega^{2\ell-1}T_H^{2\ell+1}\mathcal{A}^{2\ell+1}}{2^{2\ell+2}\Gamma(\ell+\frac{3}{2})^2\Gamma(2\ell+1)^2}\ e^{-\frac{\omega-m\Omega}{2\pi T_H}}|\Gamma(\ell+1+i\frac{(\omega-m\Omega)}{2\pi T_H})|^2. \tag{2.93}$$

In the extremal limit, $T_H \to 0$, Γ^ℓ vanishes for $\omega > m\Omega$ whereas $\Gamma^\ell \to |\sigma^\ell|$ for $\omega < m\Omega$.

2.2.2 4+1 Dimensions

The metric in five dimensions may be obtained from the low energy effective string action in ten dimensions [56],

$$\frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}e^{\phi}H^2\right] \tag{2.94}$$

where H is the three-form field strength, G_{10} is the ten-dimensional Newton's constant and ϕ is the dilaton which vanishes asymptotically. To reduce this to five dimensions, we shall compactify four of the extra dimensions on a torus T^4 and boost along the fifth dimension S^1 of radius R. By this Kaluza-Klein ansatz, the metric turns into the form

$$ds_{10}^2 = e^{2\chi}dx_i dx^i + e^{2\psi}(dx_5 + A_\mu dx^\mu)^2 + e^{-2(4\chi+\psi)/3}ds_5^2 \tag{2.95}$$

where $\mu = 0, 1, ..., 4$, $i = 6, ..., 9$ on the torus T^4, and x_5 is periodic with period $2\pi R$. All fields are assumed to depend on x^μ only. χ and ψ are assumed to vanish asymptotically. The five-field A_μ can be labeled by energy, three charges (obtained by boosting and therefore labeled by parameters (radii) r_1, r_5 and r_n) and R (see appendix A and [32, 56] for details). The five-dimensional metric is

$$ds_5^2 = -f^{-2/3}\left(1 - \frac{r_0^2}{r^2}\right)dt^2 + f^{1/3}\left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2\right] \tag{2.96}$$

where

$$f = \left(1 + \frac{r_1^2}{r^2}\right)\left(1 + \frac{r_5^2}{r^2}\right)\left(1 + \frac{r_n^2}{r^2}\right). \tag{2.97}$$

Therefore, in this spacetime the massless wave equation is

$$\frac{1}{r^3}(1 - \frac{r_0^2}{r^2})\frac{d}{dr}r^3(1 - \frac{r_0^2}{r^2})\frac{d\phi}{dr} + \frac{1}{r^2}(1 - \frac{r_0^2}{r^2})\nabla_\theta^2\phi + \omega^2 f\phi = 0, \tag{2.98}$$

where ∇_θ^2 is the angular Laplacian whose eigenvalues are $\ell(\ell+2)$ in five dimensions. The rotation group of this geometry is $SO(4)$. It can be decomposed as $SO(4) \sim SU(2)_L \times SU(2)_R$. We shall consider the diagonal representation labeled by $(\ell/2, \ell/2)$.

We are interested in the low energy limit $\omega \ll 1/r_1, 1/r_5$. Working as before, we define the far region $r \gg r_1, r_5$. Letting $\phi = \frac{1}{r}\psi$ and $\rho = \omega r$, the wave equation becomes

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{\rho}\frac{d\psi}{d\psi} + \left(1 - \frac{(\ell+1)^2}{\rho^2}\right)\psi = 0. \tag{2.99}$$

The solutions are Bessel functions. We obtain

$$\psi = \frac{1}{\rho}\left[AJ_{\ell+1}(\rho) + BJ_{-\ell-1}(\rho)\right]. \tag{2.100}$$

The incoming flux at large ρ is

$$\mathcal{F}_{\rho\to\infty} = \mathrm{Im}(\phi^* r^3 \partial_3 \phi) = \frac{1}{\pi\omega^2}\left|Ae^{i(\ell+1)\pi/2} + Be^{-i(\ell+1)\pi/2}\right|^2. \tag{2.101}$$

The small ρ behavior of ϕ is

$$\phi \sim \frac{1}{\rho}\left[A(\frac{\rho}{2})^{\ell+1}\left(\frac{1}{\Gamma(\ell+2)} - \mathcal{O}(\rho^2)\right) + B(\frac{\rho}{2})^{-\ell-1}\left(\frac{1}{\Gamma(-\ell)} - \mathcal{O}(\rho^2)\right)\right]. \tag{2.102}$$

The above equation has poles for integer ℓ. The divergence can be avoided by continuing to non-integer values of ℓ. We shall let ℓ approach an integer value at the end of the calculation.

In the near region (near the horizon), $r << 1/\omega$, we let $v = r_0^2/r^2$. The wave equation becomes

$$(1-v)^2\frac{d^2\phi}{dv^2} - (1-v)\frac{d\phi}{dv} + \left(C + \frac{D}{v} + \frac{E}{v^2}\right)\phi = 0, \tag{2.103}$$

where

$$C = \left(\frac{\omega r_n r_1 r_5}{2r_0^2}\right)^2, \qquad D = \frac{\omega^2 r_1^2 r_5^2}{4r_0^2} + \frac{\ell(\ell+2)}{4}, \qquad E = -\frac{\ell(\ell+2)}{4}. \tag{2.104}$$

The solution of (2.103) may be written in terms of a hypergeometric function as

$$\phi = Av^{-\ell/2}(1-v)^{-i\frac{\omega}{4\pi T_H}}F(-\ell/2 + q + i\sqrt{C},\ -\ell/2 + q - i\sqrt{C}; 1+2q; 1-v), \tag{2.105}$$

where $q = i\frac{\omega}{4\pi T_H}$. The large r behavior of ϕ is

$$\begin{aligned}\phi \;\sim\; & Av^{-\ell/2}\left\{\frac{\Gamma(1+2q)\Gamma(1+\ell)}{\Gamma(1+\ell/2+q-i\sqrt{C})\Gamma(1+\ell/2+q+i\sqrt{C})}(1+\mathcal{O}(v))\right. \\ & \left. + v^{1+\ell}\frac{\Gamma(1+2q)\Gamma(-1-\ell)}{\Gamma(-\ell/2+q-i\sqrt{C})\Gamma(-\ell/2+q+i\sqrt{C})}(1+\mathcal{O}(v))\right\}. \end{aligned} \tag{2.106}$$

Matching the solutions in the two regions on the overlapping region, we find that A satisfies

$$\begin{aligned} -\frac{\ell}{2} + q + i\sqrt{C} &= 2A(\omega r_0/2)^{-\ell}\Gamma(1+\ell)\Gamma(2+\ell) \times \\ &\qquad \frac{\Gamma(1+2q)}{\Gamma(1+\ell/2+q-i\sqrt{C})\Gamma(1+\ell/2+q+i\sqrt{C})}. \end{aligned} \tag{2.107}$$

The incoming flux at the horizon is

$$\mathcal{F}_{r\to r_0} = 2r_0^2 \mathrm{Im}(\phi^*(1-\frac{r_0^2}{r^2})\partial_v\phi) = 4r_0^2|q|\ |A|^2. \tag{2.108}$$

The cross section of a massless scalar field is obtained by multiplying the absorption coefficient (ratio of fluxes) by $4\pi/\omega^3$,

$$\sigma^\ell = \mathcal{A}_H(r_0\omega)^{2\ell}\left|\frac{2^\ell}{\Gamma(1+\ell)\Gamma(2+\ell)}\right|^2 \left|\frac{\Gamma(\frac{(\ell+2)}{2}+i\frac{\omega}{4\pi T_L})\Gamma(\frac{(\ell+2)}{2}+i\frac{\omega}{4\pi T_R})}{\Gamma(1+i\frac{\omega}{2\pi T_H})}\right|^2, \tag{2.109}$$

where

$$T_{L,R} = \frac{1}{T_H} \pm \frac{4\pi}{\omega}\sqrt{C} \tag{2.110}$$

and

$$\frac{1}{T_L} + \frac{1}{T_R} = \frac{2}{T_H}. \tag{2.111}$$

A similar result is obtained from calculation of of near-extremal nonrotating D5-branes, [13, 54]. The two temperature function, (2.111), is also predicted in [54] from considering the partition function of left and right moving open strings on the branes, which suggests an agreement between macroscopic calculation from string theory and microscopic calculation from conformal field theory, subsection 2.2.3. Finally, the decay rates can be obtained by multiplying the cross section by the Hawking thermal factor, similar to the (3+1)-dimensional case,

$$\Gamma_\ell = \frac{2^{4\ell+4}\pi^{2\ell+3}(r_1^2 r_5^2 T_L T_R)^{\ell+1}\omega^{2\ell-1}}{|\Gamma(\ell+1)\Gamma(\ell+2)|^2} e^{-\frac{\omega}{2T_H}} |\Gamma(1+\frac{\ell}{2}+i\frac{\omega}{4\pi T_L})\Gamma(1+\frac{\ell}{2}+i\frac{\omega}{4\pi T_R})|^2. \tag{2.112}$$

2.2.3 Microscopic calculation

Here, we show that the above classical results can be derived microscopically from superstring theory. We shall work in 3+1 dimensions and outline the changes needed in 4+1 dimensions at the end.

The microscopic decay rates can be deduced from the coupling of the scalar field ϕ to a chiral operator $\mathcal{O}$ in the effective string action,

$$S_{\text{int}} \sim \int dt d\sigma \partial^{\ell}\phi(0,t)\mathcal{O}(\sigma+t) \tag{2.113}$$

where σ is the spatial worldsheet coordinate. The amplitude $\mathcal{M}$ from this interaction is of the form

$$\mathcal{M} \sim \int d\sigma^{+}\langle f|\mathcal{O}(\sigma^{+})|i\rangle e^{-i\omega\sigma^{+}}, \tag{2.114}$$

where $\sigma^{+} = \sigma + t$ and ω is the energy of an emitted particle. After squaring and summing over final states, it becomes

$$\sum_{f} |\mathcal{M}|^2 \sim \int d\sigma^{+} d\sigma^{+'} \langle i|\mathcal{O}^{\dagger}(\sigma^{+})\mathcal{O}(\sigma^{+'})|i\rangle e^{-i\omega(\sigma^{+}-\sigma^{+'})}. \tag{2.115}$$

This amplitude should be calculated at thermal equilibrium, where the temperature is identified with the Hawking temperature. Introducing the weight $e^{-(\omega-m\Omega)/T_H}$, we need to calculate

$$\int d\sigma^{+}\langle \mathcal{O}^{\dagger}(0)\mathcal{O}(\sigma^{+})\rangle_{T_H} e^{-i(\omega-m\Omega)\sigma^{+}}.$$

where σ is periodic, $\sigma \sim \sigma + i2/T_H$. If $\mathcal{O}$ has conformal weight A', then the two point function of $\mathcal{O}$ is of the form

$$\langle \mathcal{O}^{\dagger}(0)\mathcal{O}(\sigma^{+})\rangle_{T_H} \sim \left[\frac{\pi T_H}{\sinh(\pi T_H \sigma^{+})}\right]^{2A'}. \tag{2.116}$$

To avoid the divergence from the pole in calculating the decay rates, we introduce a small parameter $i\epsilon$ in the exponent. We obtain

$$\int d\sigma^{+} e^{-i(\omega-m\Omega)(\sigma^{+}-i\epsilon)} \left[\frac{\pi T_H}{\sinh(\pi T_H \sigma^{+})}\right]^{2A'} \sim (T_H)^{2A'-1} e^{-\frac{\omega-m\Omega}{2T_H}} \left|\Gamma(A' + i\frac{\omega-m\Omega}{2\pi T_H})\right|^2.$$

For supersymmetric invariance, the weight of the conformal field is constrained by $A' \geq \ell+1$. In the low energy expansion, the leading contribution comes from $A' = \ell+1$ [53]. Therefore, the decay rate is

$$\Gamma^{\ell} \sim \omega^{2\ell-1} Q^{4\ell+2} (T_H)^{2\ell+1} e^{-\frac{\omega-m\Omega}{2\pi T_H}} \left|\Gamma(\ell+1+i(\frac{\omega-m\Omega}{2\pi T_H})\right|^2 \tag{2.117}$$

where $\omega^{2\ell}$ comes from an integration over energy, $1/\omega$ is the normalization factor of the outgoing state, and the presence of $Q^{4\ell+2}$ is justified on dimensional grounds.

This expression of the decay rate is in agreement with the classical result (2.93) if we note that the area of the horizon, $\mathcal{A} \sim Q^2$.

The microscopic decay rate can be calculated in the same fashion in 4+1 dimensions. The modifications needed are due to the left-right symmetry. The coupling of the scalar field in the conformal field theory action is to two operators $\mathcal{O}_{L,R}$ of the form $\epsilon_{IJ}\mathcal{O}^I_{\ell/2,L}\mathcal{O}^J_{\ell/2,R}$ [53]. The dimensions of these operators are $A'_L = A'_R = 1 + \ell/2$. The decay rate is calculated in the same manner as in 3+1 dimensions [53],

$$\Gamma^\ell \sim (r_1^2 r_2^2 T_L T_R)^{\ell+1}\omega^{2\ell-1}e^{-\frac{\omega}{2T_H}}\left|\Gamma(1+\frac{\ell}{2}+i\frac{\omega}{4\pi T_L})\Gamma(1+\frac{\ell}{2}+i\frac{\omega}{4\pi T_R})\right|^2, \qquad (2.118)$$

which is in agreement with the classical result in 4+1 dimensions (2.112). Absorption coefficients of other kinds of black holes such as Schwarzschild, Kaluza-Klein, etc, have been calculated in [57].

2.3 Non-extremal rotating black 3-branes

In this section we explore the properties of rotating black branes away from the extremality and obtain the absorption coefficients. Extremal rotating charged D3-branes are studied in [58]. We first introduce the general properties of the branes and then write down, solve the wave equation for a scalar mode in the vicinity of the brane and find a similar result as in section 2.2.2, a function of left and right temperatures.

2.3.1 General properties

Details of the derivation of the metric of rotating charged black 3-branes in ten dimensions can be found in appendix A, see also [59, 60]. The metric is

$$\begin{aligned}
ds^2 &= \frac{1}{\sqrt{H}}\left(-(1-f\frac{r_0^4}{r^4})dt^4 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{H}f^{-1}\frac{dr^2}{\lambda - r_0^4/r^4} \\
&+\sqrt{H}r^2\left(\zeta d\theta^2 + \zeta'\cos^2\theta d\psi^2 - \frac{\ell_2^2-\ell_3^2}{2r^2}\sin(2\theta)\sin(2\psi)d\theta d\psi\right) \\
&-f\frac{2r_0^4\cosh\gamma}{r^4}\sqrt{H}\left(\ell_1\sin^2\theta d\phi_1 + \ell_2\cos^2\theta\sin^2\psi d\phi_2\right. \\
&\qquad\qquad \left.+\ \ell_3\cos^2\theta\cos^2\psi d\phi_3\right)dt \\
&+f\frac{r_0^4}{r^4}\sqrt{H}\left(\ell_1\sin^2\theta d\phi_1 + \ell_2\cos^2\theta\sin^2\psi d\phi_2 + \ell_3\cos^2\theta\cos^2\psi d\phi_3\right)^2 \\
&+\sqrt{H}r^2\left[\left(1+\frac{\ell_1^2}{r^2}\right)\sin^2\theta d\phi_1^2 + \left(1+\frac{\ell_2^2}{r^2}\right)\cos^2\theta\sin^2\psi d\phi_2^2\right.
\end{aligned}$$

$$+\left(1+\frac{\ell_3^2}{r^2}\right)\cos^2\theta\cos^2\psi d\phi_3^2\Bigg], \tag{2.119}$$

where

$$H = 1+f\frac{r_0^4\sinh^2\gamma}{r^4} \tag{2.120}$$

$$f^{-1} = \lambda\left(\frac{\sin^2\theta}{1+\frac{\ell_1^2}{r^2}}+\frac{\cos^2\theta\sin^2\psi}{1+\frac{\ell_2^2}{r^2}}+\frac{\cos^2\theta\cos^2\psi}{1+\frac{\ell_3^2}{r^2}}\right) \tag{2.121}$$

$$\lambda = \left(1+\frac{\ell_1^2}{r^2}\right)\left(1+\frac{\ell_2^2}{r^2}\right)\left(1+\frac{\ell_3^2}{r^2}\right) \tag{2.122}$$

$$\zeta = 1+\frac{\ell_1^2\cos^2\theta+\ell_2^2\sin^2\theta\sin^2\psi+\ell_3^2\sin^2\theta\cos^2\psi}{r^2} \tag{2.123}$$

$$\zeta' = 1+\frac{\ell_2^2\cos^2\psi+\ell_3^2\sin^2\psi}{r^2}. \tag{2.124}$$

To simplify the calculation, we limit ourselves to the case where

$$\ell_2=\ell_3=0. \tag{2.125}$$

The metric becomes

$$\begin{aligned} ds^2 = &\frac{1}{\sqrt{H}}\left(-(1-\frac{r_0^4}{\zeta r^4})dt^2+dx_1^2+dx_2^2+dx_3^2\right)+\sqrt{H}\zeta\frac{dr^2}{\lambda-\frac{r_0^4}{r^4}} \\ &-\frac{2r_0^4\cosh\gamma}{\zeta r^4\sqrt{H}}\ell_1\sin^2\theta d\phi_1+\frac{r_0^4}{\zeta r^4\sqrt{H}}\ell_1^2\sin^4\theta d\phi_1^2 \\ &+\sqrt{H}r^2\left[\zeta d\theta^2+\lambda\sin^2\theta d\phi_1^2+\cos^2\theta(d\psi^2+\sin^2\psi d\phi_2^2+\cos^2\psi d\phi_3^2)\right], \end{aligned} \tag{2.126}$$

where

$$H=1+\frac{r_0^4\sinh^2\gamma}{\zeta r^4}\quad \lambda=1+\frac{\ell_1^2}{r^2}\quad \zeta=1+\frac{\ell_1^2\cos^2\theta}{r^2}. \tag{2.127}$$

The horizon is at the positive root of $\lambda-r_0^4/r^4=0$,

$$r_H^2=\frac{1}{2}\left(\sqrt{\ell_1^4+4r_0^4}-\ell_1^2\right), \tag{2.128}$$

the other root being negative, $-r_+^2$, where

$$r_+^2=\frac{1}{2}\left(\sqrt{\ell_1^4+4r_0^4}+\ell_1^2\right). \tag{2.129}$$

It is convenient to introduce the dimensionless parameter $\triangle$,

$$\triangle = \frac{r_H^2}{r_+^2}. \tag{2.130}$$

As $r_0 \to 0$, we have $\triangle \to 0$ (extremal limit) and as $\ell_1 \to 0$, we have $\triangle \to 1$. Also in the extremal limit, as $r_0 \to 0$, the horizon shrinks to zero and $\gamma \to \infty$ so that R^4 remains finite, where

$$R^4 = \frac{1}{2} r_0^4 \sinh(2\gamma), \tag{2.131}$$

We assume R^4 is much larger than the other parameters, ℓ_1, r_0, r_H and r_+.

The energy, angular momentum, entropy density, Hawking temperature and angular velocity deduced from the metric are [61]

$$\epsilon = \frac{1}{4G} r_0^4 (4\cosh^2\gamma - 4\cosh\gamma\sinh\gamma + 1), \quad j = \frac{1}{2G} r_0^4 \ell_1 \cosh\gamma$$

$$s = \frac{\pi}{G} r_0^4 r_H \cosh\gamma, \qquad T_H = \frac{r_H}{2\pi r_0^4 \cosh\gamma}\sqrt{\ell_1^4 + 4r_0^4}, \qquad \Omega_H = \frac{\ell_1 r_H^2}{r_0^4 \cosh\gamma} \tag{2.132}$$

where G is Newton's constant. The above quantities obey the first law of thermodynamics,

$$T_H ds = d\epsilon - \Omega_H dj. \tag{2.133}$$

In the extremal limit, $r_0 \to 0$, the entropy and energy vanish, but the temperature remains finite, which implies a singularity in this limit. The metric in this case is

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{H}\zeta\frac{dr^2}{\lambda} \\ &= +\sqrt{H} r \left[\zeta d\theta^2 + \lambda \sin^2\theta d\phi_1^2 + \cos^2\theta(d\psi^2 \ + \sin^2\psi d\phi_2^2 + \cos^2\psi d\phi_3^2)\right], \end{aligned} \tag{2.134}$$

where

$$H = 1 + \frac{R^4}{\zeta r^4} = 1 + \frac{R^4}{(r^2 + \ell_1^2\cos^2\theta) r^2}. \tag{2.135}$$

Performing the transformation

$$\begin{aligned} y_1 &= \sqrt{r^2 + \ell_1^2}\sin\theta\cos\phi_1 \\ y_2 &= \sqrt{r^2 + \ell_1^2}\sin\theta\sin\phi_1 \\ y_3 &= \cos\theta\sin\psi\cos\phi_2 \\ y_4 &= \cos\theta\sin\psi\sin\phi_2 \\ y_5 &= \cos\theta\cos\psi\cos\phi_3 \\ y_6 &= \cos\theta\cos\psi\cos\phi_3, \end{aligned} \tag{2.136}$$

we can write the metric in a multi-center form,

$$ds^2 = \frac{1}{\sqrt{H}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{H}\left(dy_1^2 + dy_2^2 + dy_3^2 + dy_3^2 + dy_4^2 + dy_5^2\right). \tag{2.137}$$

2.3.2 The wave equation and absorption coefficients

The ten dimensional wave equation, using the metric (2.126),

$$\partial_A \sqrt{-g} g^{AB} \partial_B \Phi = 0,$$

can be separated by setting

$$\Phi(x^\mu; r, \theta) = e^{i\omega t}\Psi(r,\theta)\ , \tag{2.138}$$

assuming that there is no dependence on the three-dimensional flat space, $k^\mu = (\omega, \vec{0})$. The wave equation becomes

$$\frac{1}{r^3}\partial_r\left((\lambda - \frac{r_0^4}{r^4})r^5\partial_r\Psi\right) + \omega^2 r^2\Psi + \frac{\omega^2\lambda r_0^4\cosh^2\gamma}{r^2(\lambda - r_0^4/r^4)}\Psi - (\hat{L}^2 - \omega^2\ell_1^2\cos^2\theta)\Psi = 0, \tag{2.139}$$

We shall solve this equation in the the limit where the mass is small compared to the AdS curvature and the angular momentum is also small,

$$R, \ell_1 \ll 1/\omega. \tag{2.140}$$

In this limit, the term proportional to $\omega^2\ell_1^2$ may be ignored, since it is small compared to the angular momentum term, $\hat{L}^2$, whose eigenvalues are $j(j+4)$. The wave equation becomes

$$\frac{1}{r^3}\partial_r\left((\lambda - \frac{r_0^4}{r^4})r^5\partial_r\Psi\right) + \omega^2 r^2\Psi + \frac{\omega^2\lambda r_0^4\cosh^2\gamma}{r^2(\lambda - r_0^4/r^4)}\Psi - j(j+4)\Psi = 0. \tag{2.141}$$

We shall solve this equation in the two asymptotic regimes, $r \gg \omega R^2$ and $r \ll 1/\omega$ and then match the solutions in the overlapping region.

For $r \gg \omega R^2$, the wave equation becomes

$$\frac{1}{r^3}\partial_r(r^5\partial_r\Psi) + \omega^2\Psi - j(j+4)\Psi = 0 \tag{2.142}$$

and the solution is a Bessel function,

$$\Psi = \frac{1}{r^2}J_{j+2}(\omega r) \tag{2.143}$$

where we drop the other divergent solution. For small r, the solution behaves as

$$\Psi \sim \frac{\omega^2}{4(j+2)!}\left(\frac{\omega r}{2}\right)^j. \tag{2.144}$$

In the $r \ll 1/\omega$ regime, the wave equation becomes

$$\frac{1}{r^3}\partial_r\left((\lambda - \frac{r_0^4}{r^4})r^5\partial_r\Psi\right) + \frac{\omega^2\lambda r_0^4\cosh^2\gamma}{r^2(\lambda - r_0^4/r^4)}\Psi - j(j+4)\Psi = 0. \tag{2.145}$$

We shall solve this equation in the extremal case, $\triangle = 0$, near extremality, $\triangle \to 0$ and at the other end of the spectrum, $\triangle = 1$.

I. The end-point $\triangle = 0$.

In this limit, $r_0 \to 0$, the horizon shrinks to zero and the wave equation becomes

$$\frac{1}{r^3}\partial_r\left(\lambda r^5\partial_r\Psi\right) + \frac{\omega^2 R^4}{r^2}\Psi - j(j+4)\Psi = 0. \tag{2.146}$$

Changing variables to $u = 1/\lambda = 1/(1+\ell_1^2/r^2)$, we obtain

$$(1-u)u^2\frac{d^2\Psi}{du^2} + u(2-u)\frac{d\Psi}{du} + \frac{\omega^2R^4}{4\ell_1^2}\Psi - \frac{j(j+4)}{4(1-u)}\Psi = 0. \tag{2.147}$$

The singularity at $u \to 0$ is of the form $\Psi \sim u^a$, where

$$a = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\omega^2R^4}{\ell_1^2}} = -\frac{1}{2} + i\kappa, \quad \kappa = \frac{1}{2}\sqrt{\frac{\omega^2R^4}{\ell_1^2} - 1} \approx \frac{\omega R^2}{2\ell_1} = \frac{\omega}{4\pi T_H}, \tag{2.148}$$

and $T_H = \frac{\ell_1}{2\pi R^2}$ is the Hawking temperature. Notice that when we approach the extremal limit ($r_0 \to 0$), the Hawking temperature does not vanish even though both the energy and the entropy do.

The other singularity at $u \to 1$ is of the form $\Psi \sim (1-u)^b$, where

$$b = \frac{j+4}{2}. \tag{2.149}$$

Isolating the singularities,

$$\Psi = u^{-1/2+i\kappa}(1-u)^{j/2+2}f(u) \tag{2.150}$$

the wave equation becomes

$$(1-u)u\frac{d^2f}{du^2} + [1+2i\kappa - (j+4+2i\kappa)u]\frac{df}{du} - \frac{(j+3+2i\kappa)^2}{4}f = 0. \tag{2.151}$$

whose solution is a hypergeometric function,

$$f(u) = A\ F\left(\frac{j+3}{2} + i\kappa, \frac{j+3}{2} + i\kappa; 1 + 2i\kappa; u\right). \tag{2.152}$$

where A is a coefficient to be determined. To obtain the behavior of Ψ at large r, i.e., $u \to 1$, note the behavior of the hypergeometric function,

$$F\left(\frac{j+3}{2} + i\kappa, \frac{j+3}{2} + i\kappa; 1 + 2i\kappa; u\right) = \frac{\Gamma(1+2i\kappa)\Gamma(j+2)}{(\Gamma((j+3)/2 + i\kappa))^2} \frac{1}{(1-u)^{j+2}} + \ldots \tag{2.153}$$

We obtain

$$\Psi \approx A \frac{\Gamma(1+2i\kappa)\Gamma(j+2)}{(\Gamma(\frac{j+3}{2} + i\kappa))^2} \left(\frac{r}{\ell_1}\right)^j. \tag{2.154}$$

Comparing with the asymptotic form (2.144), we arrive at

$$A = \frac{(\Gamma(\frac{j+3}{2} + i\kappa))^2}{\Gamma(1+2i\kappa)} \frac{\omega^{j+2}\ell_1^j}{2^{j+2}(j+1)!(j+2)!}. \tag{2.155}$$

In the small r limit, we have $u \approx r^2/\ell_1^2$. Therefore, the wavefunction Ψ, behaves as

$$\Psi \approx A \left(\frac{r}{\ell_1}\right)^{-1+2i\kappa}. \tag{2.156}$$

The absorption coefficient (ratio of incoming flux at $r \to 0$ to incoming flux at $r \to \infty$) is

$$\begin{aligned} \mathcal{P} &= \frac{\mathcal{F}_{r\to 0}}{\mathcal{F}_{r\to\infty}} \\ &= \frac{(\lambda r^5 \Psi^* \partial_r \Psi)_{r\to 0}}{(\lambda r^5 \Psi^* \partial_r \Psi)_{r\to\infty}} \\ &= 4\pi\kappa \frac{|\Gamma((j+3)/2 + i\kappa)|^4}{|\Gamma(1+2i\kappa)|^2} \frac{\omega^{2j+4}\ell_1^{2j+4}}{4^{j+2}((j+1)!(j+2)!)^2}. \end{aligned} \tag{2.157}$$

This absorption coefficient has the same form as the grey-body factors obtained for black holes in section 2.2 (see also [62]) for large j. Indeed,

$$\mathcal{P} \sim \left|\Gamma\left(\frac{j+3}{2} + i\kappa\right)\right|^4 = \left|\Gamma\left(\frac{j+3}{2} + i\frac{\omega}{4\pi T_H}\right)\right|^4, \tag{2.158}$$

to be compared with the grey-body factor [62]

$$\mathcal{P}_{\text{black hole}} \sim \left|\Gamma\left(\frac{j+2}{2} + \frac{i\omega}{4\pi T_+}\right)\right|^2 \left|\Gamma\left(\frac{j+2}{2} + \frac{i\omega}{4\pi T_-}\right)\right|^2. \tag{2.159}$$

Thus, we obtain agreement provided we identify

$$T_+ = T_- = T_H. \tag{2.160}$$

This implies the equilibrium of the system at Hawking temperature. It suggests the existence of the AdS/CFT correspondence in the system. In the small temperature limit, $\kappa \to \infty$, the constant A in (2.155) becomes

$$A \approx \sqrt{\pi}\frac{i^{j+2}R^{2j+4}\omega^{2j+4}}{4^{j+2+i\kappa/2}\ell_1^2(j+1)!(j+2)!}\frac{\Gamma(\frac{1}{2}+i\kappa)}{\Gamma(1+i\kappa)} \tag{2.161}$$

$$|A|^2 \approx \frac{\pi R^{4j+8}\omega^{4j+8}}{4^{2j+4}\kappa\ell_1^4((j+1)!(j+2)!)^2}, \tag{2.162}$$

where we used the Gamma function identities

$$\begin{aligned} \Gamma(2x) &= \tfrac{1}{2\pi}2^{2x-1/2}\Gamma(x)\Gamma(x+1/2), & \Gamma(x+1) &= x\Gamma(x) \\ |\Gamma(1/2+i\kappa)|^2 &= \tfrac{\pi}{\cosh(\pi\kappa)}, & |\Gamma(1+i\kappa)|^2 &= \tfrac{\pi\kappa}{\sinh(\pi\kappa)} \end{aligned}$$

The absorption coefficient (2.157) becomes

$$\mathcal{P} = 4\pi\kappa\ell_1^4|A|^2 \sim \frac{\pi R^{4j+8}\omega^{4j+8}}{4^{2j+3}((j+1)!(j+2)!)^2}, \tag{2.163}$$

which is in agreement with our earlier result [58, 42].

II. Near the limit $\triangle \to 0$

To study the behavior near the horizon, we isolate the singularity there,

$$\Psi \sim \left(1 - \frac{r_H^2}{r^2}\right)^b, \tag{2.164}$$

Substituting back into the wave equation (2.145), we obtain

$$b = \frac{i\omega\cosh\gamma r_0^4}{2r_H\sqrt{\ell_1^4+4r_0^4}} = \frac{i\omega}{4\pi T_H} \tag{2.165}$$

Let us then rewrite the wavefunction as

$$\Psi = \left(1 - \frac{r_H^2}{r^2}\right)^b f(r). \tag{2.166}$$

where the function $f(r)$ is regular at the horizon, $r = r_H$. Substituting $f(r)$ back into the wave equation, we obtain

$$\begin{aligned} r^2(\lambda - \frac{r_0^4}{r^4})f'' &+ \left[2(2b+1)\frac{r_H^2}{r^2}\left(1+\frac{r_+^2}{r^2}\right) + \left(5+3\frac{r_+^2}{r^2}\right)\left(1-\frac{r_H^2}{r^2}\right)\right]rf' \\ &+ \left[4\frac{r_H^2}{r^2} - \frac{4b^2}{r^2}\left(\frac{(r_+^2+r_H^2)r_H^2}{r_+^2(1+r_+^2/r^2)} + r_+^2 + r_H^2(1+r_+^2/r^2)\right) - j(j+4)\right]f = 0. \end{aligned} \tag{2.167}$$

In the limit $\triangle \to 0$, we have $r_+ \gg r_H$. We shall solve (2.167) in the asymptotic regime $r \gg r_H$ and then take the limit $r \to r_H$. This will give us an approximate expression for the behavior of the wavefunction near the horizon. Letting $r \gg r_H$ in (2.167), we obtain

$$r^2\left(1+\frac{r_+^2}{r^2}\right)f'' + \left(5+3\frac{r_+^2}{r^2}\right)rf' + \left[-\frac{4b^2r_+^2}{r^2} - j(j+4)\right]f = 0. \tag{2.168}$$

This equation is of the same form as (2.146) and its solution is

$$f(u) = Au^{-1/2+b}(1-u)^{j/2+2}F(\frac{j+3}{2}+b, \frac{j+3}{2}+b; 1+2b; u), \tag{2.169}$$

$$u = \left(1+\frac{r_+^2}{r^2}\right)^{-1}, \qquad A = \frac{(\Gamma(j+3)/2+b))^2}{\Gamma(1+2b)}\frac{\omega^{j+2}r_+^j}{2^{j+2}(j+1)!(j+2)!}. \tag{2.170}$$

At the horizon, $r = r_H$, we have $u = u_H = \triangle/(1+\triangle)$ and also

$$\begin{aligned} f(u_H) &= Au_H^{-1/2+b}(1-u_H)^{j/2+2}F(\frac{j+3}{2}+b, \frac{j+3}{2}+b; 1+2b; u_H) \\ &= A\triangle^{-1/2+b}\left(1+\frac{((j+3)/2+b)((j+1)/2-b)}{1+2b}\triangle + o(\triangle^2)\right). \end{aligned} \tag{2.171}$$

If j is large and $|b| \gg j$, we can use the Gamma functions identities

$$\Gamma(2x) = \frac{1}{\sqrt{2\pi}}2^{2x-1/2}\Gamma(x)\Gamma(x+1/2), \qquad \Gamma(x+1) = x\Gamma(x) \tag{2.172}$$

to bring (2.171) into the asymptotic form

$$\begin{aligned} f(u_H) &= \frac{\omega^{j+2}r_+^j\triangle^{-1/2+b}(1+\triangle)^{(j-b)/2}}{2^{j+2}(j+1)!(j+2)!} \times \\ &\qquad \frac{(\Gamma(1+b))^2\Gamma((j+3)/2+b_+)\Gamma((j+3)/2+b_-)}{\Gamma(1+b_+)\Gamma(1+b_-)\Gamma(1+2b)}, \end{aligned} \tag{2.173}$$

where

$$b_\pm = b\left(1 \pm \sqrt{\frac{\triangle}{2}}\right), \tag{2.174}$$

Let us introduce temperature parameters

$$T_\pm = \frac{T_H}{1 \pm \sqrt{\triangle/2}} \tag{2.175}$$

satisfying

$$\frac{1}{T_+} + \frac{1}{T_-} = \frac{2}{T_H}. \tag{2.176}$$

This is in agreement with (2.111) and the result from non-rotating $D5$-branes in [54]. Notice that $T_\pm \to T_H$ as $\triangle \to 0$, as expected in the extremal limit. The absorption coefficient can be written as

$$\mathcal{P} \sim \left| \Gamma(\frac{j+3}{2} + \frac{i\omega}{4\pi T_+})\Gamma(\frac{j+3}{2} + \frac{i\omega}{4\pi T_-}) \right|^2. \tag{2.177}$$

This result suggests that fields may not be in thermal equilibrium away from extremality similar to the case of black holes where two distinct temperature parameters satisfying (2.176) are obtained [62].

III. The other end-point, $\triangle = 1$.

The limit $\triangle = 1$ corresponds to $r_H = r_0$ and $\ell_1 = 0$ (no rotation). In this limit, the wave equation (2.145) becomes

$$\frac{1}{r^3}\partial_r \left((1 - \frac{r_0^4}{r^4}) r^5 \partial_r \Psi \right) + \frac{\omega^2 \lambda r_0^4 \cosh^2 \gamma}{r^2(1 - r_0^4/r^4)} \Psi - j(j+4)\Psi = 0. \tag{2.178}$$

Separating the horizon singularity, $(1 - r_0^4/r^4)^{i\kappa}$, we write

$$\Psi = A \left(1 - \frac{r_0^4}{r^4} \right)^{i\kappa} f(r), \qquad \kappa = \frac{\omega r_0 \cosh \gamma}{4} = \frac{\omega}{4\pi T_H}, \tag{2.179}$$

where $T_H = \frac{1}{\pi r_0^2 \cosh \gamma}$ is the Hawking temperature. The wave equation (2.178) becomes

$$r^2 \left(1 - \frac{r_0^4}{r^4} \right) f'' + r \left[5 - (1 - 2i\kappa) \frac{r_0^4}{r^4} \right] f' - j(j+4) f = - \frac{4\omega^2 r_0^4 \cosh^2 \gamma}{r^2} \frac{1 + \frac{r_0^2}{r^2} + \frac{r_0^4}{r^4}}{1 + \frac{r_0^2}{r^2}} f. \tag{2.180}$$

In the asymptotic regime $r \gg r_0$, it reduces further to

$$r^2 f'' + 5 r f' + \frac{\omega^2 r_0^4 \cosh^2 \gamma}{r^2} f - j(j+4) f = 0 \tag{2.181}$$

whose solution is

$$f(r) = \frac{1}{r^2} H_{j+2}^{(1)} \left(\frac{\omega r_0^2 \cosh \gamma}{r} \right). \tag{2.182}$$

In the large r limit, we have

$$\Psi \approx -iA \frac{2^{j+2}(j+1)!}{r_0^{2j+2} \omega^{j+2} \cosh^{j+2} \gamma} r^j. \tag{2.183}$$

Matching this asymptotic form to (2.144), we arrive at

$$A = i\frac{\omega^{2j+4} r_0^{2j+4} \cosh^{j+2}\gamma}{4^{j+2}(j+1)!(j+2)!}. \tag{2.184}$$

The absorption coefficient is

$$\mathcal{P} = 8\pi\kappa r_0^4 |A|^2 |f(r_0)|^2 \approx 8\pi\kappa |A|^2 \left| H_{j+2}^{(1)}(4\kappa) \right|^2. \tag{2.185}$$

For large κ, we may expand the Bessel function as

$$H_{j+2}^{(1)}(4\kappa) = \frac{(-i)^{j+5/2}}{\sqrt{2\pi\kappa}} \left(1 + \frac{i}{8\kappa}(j+5/2)(j+3/2) + \dots \right) \tag{2.186}$$

Using the Gamma function identity (2.172), the Bessel function becomes

$$H_{j+2}^{(1)}(4\kappa) = \frac{-i}{2^{2i\kappa}(2\kappa)^{j+2}} e^{4i\kappa} \frac{\Gamma(j/2+5/4-2i\kappa)\Gamma(j/2+7/4-2i\kappa)}{\Gamma(1-4i\kappa)} + \dots \quad . \tag{2.187}$$

Finally the absorption coefficient (2.185) can be written as

$$\mathcal{P} \approx \frac{8\pi\kappa}{((j+1)!(j+2)!)^2} \frac{|\Gamma(j/2+5/4+2i\kappa)\Gamma(j/2+7/4+2i\kappa)|^2}{|\Gamma(1+4i\kappa)|^2} \left(\frac{\omega r_0}{2}\right)^{2j+4}. \tag{2.188}$$

This is of the same form as the result in the extremal case (2.158), except the effective temperature is *half* the Hawking temperature. It is not clear what the implications of this result are to the AdS/CFT correspondence. It is worth looking further into the fate of the correspondence once the system is heated to a small but finite temperature.

Chapter 3

Quantization of Maximally Charged Black Holes

In the previous chapter we discussed the connection between field theory and string theory in the presence of a single black hole. In this chapter, we discuss the quantization of a particle near a black hole as well as a multi-black hole system. We also discuss the quantization of a continuous distribution of matter (black string) and obtain explicit results in the case of a ring-shaped formation. This chapter is organized as follows. In section 3.1, we review a simple example of a conformal mechanical system. In section 3.2, we discuss the quantization of a particle near a black hole. In section 3.3, we extend the discussion to the quantization of a system of slowly-moving maximally-charged black holes in four and five dimensions. Finally, in section 3.4, we quantize this system of black holes in the limit of a continuous distribution of matter (black string).

3.1 A conformal quantum mechanical system

The simplest example of a quantum mechanical system with conformal invariance is given by the Hamiltonian [73, 74]

$$H = \frac{p^2}{2} + \frac{g}{2x^2}. \tag{3.1}$$

where g is a coupling constant [73]. H has no well-defined ground state because the spectrum is continuous down to zero energy. The generator of dilations D and special conformal transformations K are

$$D = \frac{1}{2}(px + xp), \qquad K = \frac{1}{2}x^2. \tag{3.2}$$

These operators obey the $SL(2,\mathbb{R})$ algebra,

$$[D,H] = 2iH, \qquad [D,K] = -2iK, \qquad [H,K] = -iD. \tag{3.3}$$

Consider the linear combinations of H, D and K,

$$L_{\pm 1} = \frac{1}{2}\left(aH - \frac{K}{a} \mp iD\right) \tag{3.4}$$

$$L_0 = \frac{1}{2}\left(aH + \frac{K}{a}\right), \tag{3.5}$$

where a is a constant with a length-squared dimension. These new operators obey the Virasoro form of the $SL(2,\mathbb{R})$ algebra,

$$[L_1, L_{-1}] = 2L_0, \qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1}. \tag{3.6}$$

For a=1, L_0 can be written as

$$L_0 = \frac{1}{2}(H+K) = \frac{p^2}{4} + \frac{g}{4x^2} + \frac{x^2}{4} \tag{3.7}$$

which has a well-defined ground state and a discrete spectrum. The problem of an ill-defined Hamiltonian can thus be fixed by considering a generalized operator L_0 proportional to $H+K$. This leads to a well-defined Hilbert space for the quantum mechanical system.

3.2 Quantization of a particle near a black hole

In this section we follow the discussion in [18, 19] and quantize a particle moving in the vicinity of an extremal black hole (zero temperature). We use the standard Faddeev-Popov procedure and show that in the naive gauge which leads to an obstruction at the boundary of spacetime (see subsection 3.2.3), one obtains an ill-defined ground state. We show how the gauge can be fixed properly so that the Hilbert space is well-defined.

3.2.1 Neutral particle

We start by discussing the quantization of a particle moving in a fixed spacetime background. We consider its path integral and apply the Faddeev-Popov procedure to fix the gauge.

I. Flat spacetime

First, let us review the simple case of a particle of mass m moving in flat spacetime. The action and the Lagrangian, respectively, are

$$\begin{aligned} S &= \int d\tau L, \\ L &= \frac{1}{2\eta}\dot{x}^\mu \dot{x}_\mu - \frac{1}{2}\eta m^2, \end{aligned} \tag{3.8}$$

and we adopt the mostly positive signature. Varying η, we obtain the constraint

$$\eta^2 = -\dot{x}^\mu \dot{x}_\mu / m^2. \tag{3.9}$$

The conjugate momenta are

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\dot{x}_\mu}{\eta}, \qquad P_\eta = 0. \tag{3.10}$$

The Hamiltonian is

$$H = \dot{x}^\mu P_\mu - L = m\eta\chi, \tag{3.11}$$

where χ is given by

$$\chi = \frac{1}{2m} P_\mu P^\mu + \frac{1}{2} m. \tag{3.12}$$

The action in this case is

$$S = \int d\tau (\dot{x}^\mu P_\mu - m\eta\chi). \tag{3.13}$$

η is a Lagrange multiplier enforcing the constraint

$$\chi \equiv \frac{1}{2m} P_\mu P^\mu + \frac{1}{2} m = 0, \tag{3.14}$$

which is the mass-shell condition. This constraint (analogous to Gauss's Law in electrodynamics) generates parameterizations of τ through Poisson brackets,

$$\delta x^\mu = \{x^\mu, \chi\}_P \delta\tau = \frac{1}{m} P^\mu \delta\tau \quad , \qquad \delta P_\mu = \{P_\mu, \chi\}_P \delta\tau = 0. \tag{3.15}$$

The solutions of these differential equations are the orbits of these transformations, in this case straight lines,

$$x^\mu = \frac{P_0^\mu}{m}\tau + x_0^\mu, \tag{3.16}$$

with constant vectors P_0^μ and x_0^μ. The family of orbits in the same direction P_0^μ fills spacetime. We can obtain all other families by coordinate transformations (rotations).

To quantize the system, consider the path integral,

$$\begin{aligned} Z &= \mathcal{N}\int \mathcal{D}x\mathcal{D}P\mathcal{D}\eta e^{iS} \\ &= \mathcal{N}\int \mathcal{D}x\mathcal{D}P\delta(\chi)e^{i\int d\tau \dot{x}^\mu P_\mu}. \end{aligned} \quad (3.17)$$

A choice of gauge fixing condition is

$$h(x^\mu) = \tau, \quad (3.18)$$

which defines a hyper-surface that cuts each orbit precisely once. Identifying the function $h(x^\mu)$ with the time coordinate means that we choose its conjugate momentum to be the Hamiltonian $\mathcal{H}$ of the reduced system. From the standard Faddeev-Popov procedure, we write

$$1 = \det\{h, \chi\}\int \mathcal{D}\epsilon\ \delta(h - \{h, \chi\}\epsilon - \tau), \quad (3.19)$$

and insert the above expression into the path integral. After performing a reparametrization, we obtain

$$Z = \mathcal{N}\int \mathcal{D}x\mathcal{D}P\ \det\{h, \chi\}\delta(h - \tau)\delta(\chi)e^{i\int d\tau \dot{x}^\mu P_\mu}. \quad (3.20)$$

The dimension is reduced and the Faddeev-Popov determinant is canceled by an integration over the δ-function. The reduced system can be written in terms of new coordinates $\bar{x}^i$ and conjugate momenta $\bar{P}_i$. The Hamiltonian $\mathcal{H}$ of the reduced system is chosen to be a conjugate momentum to h. The path integral becomes

$$Z = \mathcal{N}\int \mathcal{D}\bar{x}\mathcal{D}\bar{P}e^{i\int d\tau \dot{\bar{x}}^i \bar{P}_i - \mathcal{H}}. \quad (3.21)$$

A different way to quantize the system in the operator formalism is through Dirac brackets,

$$\{A, B\}_D = \{A, B\}_P - \{A, \chi_i\}_P\{\chi_i, \chi_j\}_P^{-1}\{\chi_j, B\}_P \quad (3.22)$$

where i, $j = 1$, 2, $\chi_1 = \chi$ and $\chi_2 = h$.

For example, consider the case $h(x^\mu) = x^0$. The Hamiltonian of the reduced system is

$$\mathcal{H} = -P_0 = \sqrt{P_i P^i + m^2}, \quad (3.23)$$

with the coordinate $\bar{x}^i = x^i$. The commutation relations from Dirac brackets are

$$[P_i, x^j] = -i\delta_i^j \quad , \quad [\mathcal{H}, x^i] = -i\frac{P^i}{\mathcal{H}}, \quad (3.24)$$

which are appropriate for $\mathcal{H}$ given by (3.23). Next, we consider the case of a particle moving in a curved spacetime background.

II. Curved spacetime

The action in the curved spacetime has the same form as in the flat case. The difference is the background metric $g^{\mu\nu}$ is introduced and the gauge transformation (3.15) changes to

$$\delta x^\mu = \frac{1}{m}P^\mu \delta\tau, \qquad \delta P_\mu = \frac{1}{m}\Gamma_{\nu\lambda\mu}P^\mu P^\nu \delta\tau, \tag{3.25}$$

where $\Gamma_{\nu\lambda\mu}$ are the Christoffel symbols. We obtain the equation of geodesic orbits

$$D\frac{dx^\mu}{d\tau} \equiv \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0,$$

which may also be written in terms of the conjugate momenta,

$$\frac{dP_\mu}{d\tau} + \Gamma_{\nu\lambda\mu}P^\nu P^\lambda = 0. \tag{3.26}$$

The same procedure of quantizing the system in flat space can be performed by using the path integral and choosing a gauge fixing condition.

3.2.2 Charged particle

Consider a particle of charge q interacting with an external electromagnetic field. The action is

$$\begin{aligned} S &= \int d\tau L, \\ L &= \frac{1}{2\eta}\dot{x}^\mu \dot{x}_\mu - \frac{1}{2}\eta m^2 + q\dot{x}^\mu A_\mu. \end{aligned} \tag{3.27}$$

When varying τ, we obtain the constraint

$$\eta^2 = -\dot{x}^\mu \dot{x}_\mu / m^2. \tag{3.28}$$

The conjugate momenta are

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{\tau}\dot{x}_\mu + qA_\mu \quad , P_\eta = 0. \tag{3.29}$$

The Hamiltonian is

$$\begin{aligned} H &= \dot{x}^\mu P_\mu - L = m\tau\chi, \\ \chi &= \frac{1}{2m}\pi_\mu \pi^\mu + \frac{1}{2}m, \\ \pi_\mu &= P_\mu - qA_\mu. \end{aligned} \tag{3.30}$$

Then the action, in the canonical formalism, changes to

$$S = \int d\tau(\dot{x}^\mu P_\mu - m\tau\chi), \tag{3.31}$$

which is of the same form as in the non-interactive case (3.8). This implies that η is a Lagrange multiplier enforcing the constraint

$$\chi \equiv \frac{1}{2m}\pi_\mu \pi^\mu + \frac{1}{2}m = 0, \tag{3.32}$$

which is the mass-shell condition in the presence of an external vector potential.

The orbits of the gauge transformations (τ reparametrizations) are the trajectories of the equations of motion (Lorentz force law in curved spacetime)

$$\begin{aligned} \dot{x}^\mu &= \frac{1}{m}\pi^\mu, \\ \dot{\pi}_\mu &+ \frac{1}{m}\Gamma_{\nu\lambda\mu}\pi^\nu \pi^\mu = \frac{q}{m}\pi^\nu F_{\mu\nu}, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{3.33}$$

The equations of motion can also be written in terms of the coordinates x^μ

$$\ddot{x}_\mu + \Gamma_{\mu\nu\lambda}\dot{x}^\nu \dot{x}^\lambda = \frac{q}{m}\dot{x}^\nu F_{\mu\nu}. \tag{3.34}$$

The quantization of this system follows the same steps as in the free particle case.

3.2.3 Extreme Reissner-Nordström black hole

Next, we discuss the quantization of a particle moving near an extreme Reissner-Nordström black hole, $M = Q$, [75, 76, 77, 78] in four and five dimensions. We consider only the extremal case for both the black hole and the particle (charge equal to mass in units in which $G = 1$, where G is Newton's constant). The metric in five dimensions is

$$ds^2 = -\frac{1}{\psi}dt^2 + \psi d\vec{x}^2 \quad , \quad \psi = 1 + \frac{Q}{\vec{x}^2}, \tag{3.35}$$

with the vector potential

$$A_0 = \frac{1}{\psi} \quad , \vec{A} = 0. \tag{3.36}$$

The constant Q is the total charge (mass) of the black hole. Near the horizon, ψ can be written as

$$\psi = \frac{Q}{\vec{x}^2}, \tag{3.37}$$

which is the Coulomb potential in five dimensions. To simplify the metric, we change coordinates to polar coordinates and use (3.37) to write the metric in terms of ψ, i.e.,

$$ds^2 = -\frac{1}{\psi^2}(dt^2 - \frac{Q}{4}d\psi^2) + 4Q^2 d\Omega_3^2. \tag{3.38}$$

Changing parameters again to

$$x^{\pm} = t \pm \frac{\sqrt{Q}}{2}\psi, \tag{3.39}$$

$$t = \frac{x^+ + x^-}{2} \quad , \quad \psi = \frac{x^+ - x^-}{\sqrt{Q}}, \tag{3.40}$$

the metric and vector potential become

$$ds^2 = -\frac{1}{\psi^2}dx^+ dx^- + Q d\Omega_3^2, \tag{3.41}$$

$$A_+ = A_- = \frac{1}{2\psi}. \tag{3.42}$$

In four dimensions, the metric and vector potential are, respectively,

$$ds^2 = -\frac{1}{\psi^2}dt^2 + \psi^2 d\vec{x}^2 \; , \quad \psi = 1 + \frac{Q}{|\vec{x}|}, \tag{3.43}$$

$$A_t = \psi^{-1} \; , \quad \vec{A} = 0. \tag{3.44}$$

Near the horizon, ψ becomes

$$\psi = \frac{Q}{|\vec{x}|}, \tag{3.45}$$

which is the Coulomb potential in four dimensions. After changing the metric to polar coordinate and switching variables to ψ, we obtain

$$ds^2 = -\frac{1}{\psi^2}(dt^2 - Q^2 d\psi^2) + Q^2 d\Omega_2^2. \tag{3.46}$$

Change parameters again to

$$x^{\pm} = t \pm Q\psi, \tag{3.47}$$

the metric becomes

$$ds^2 = -\frac{1}{\psi^2}dx^+dx^- + Q^2 d\Omega_2^2, \tag{3.48}$$

which has the same form as in five dimensions (3.41). Both four and five dimensional metrics take the form of the product $AdS_2 \times S^n$, $n = 2$ and 3. Therefore, we may consider the AdS_2 part only to tackle the problem. The only non-vanishing connection coefficients are $\Gamma^{\pm}_{\pm\pm} = \partial_{\pm} \ln |g_{+-}|$. The geodesic equations for $x^{\pm}$ are

$$\ddot{x}^{\pm} \pm (\ln |g_{+-}|)'(\dot{x}^{\pm})^2 = \pm \dot{x}^{\pm} F_{+-}, \tag{3.49}$$

where $A= A_+ =A_-$, and $(\ln |g_{+-}|)' = \partial_+ \ln |g_{+-}| = -\partial_- \ln |g_{+-}|$ and we used $q = m$ for the particle. Let us impose the simplest gauge-fixing condition

$$h(x^+, x^-) = \frac{1}{2}(x^+ + x^-) = \tau. \tag{3.50}$$

The Hamiltonian, which is the conjugate momentum, generates motion along the geodesics,

$$\mathcal{H} = -H = P_+ + P_-. \tag{3.51}$$

By using $\psi \frac{dA}{d\psi} = -A$, $\psi \frac{g_{+-}}{d\psi} = -2g_{+-}$ and $F_{+-} = 2\partial_+ A$, the following quantities can be straightforwardly shown to be gauge-invariant (constant along geodesics)

$$H = -P_+ - P_- \quad , \quad D = 2x^+P_+ + 2x^-P_- \quad , \quad K = -(x^+)^2 P_+ - (x^-)^2 P_- + \frac{1}{2} mQ\psi. \tag{3.52}$$

In AdS_2 symmetry, they obey an $SL(2, \mathbb{R})$ algebra

$$\{H \ , \ D\} = -2H \quad , \quad \{H \ , \ K\} = -D \quad , \quad \{K \ , \ D\} = 2K, \tag{3.53}$$

which reflect the symmetry in the AdS_2 spacetime. H, D and K generate time translations, dilatations and special conformal transformations, respectively. The brackets can be defined as Poisson or Dirac, which means this is also an algebra of the gauge-fixed system. The constraint, generator of gauge transformations, $\chi \equiv \frac{1}{2m}\pi^{\mu}\pi_{\mu} + \frac{1}{2}m = 0$, becomes

$$-4\psi^2 P_+ P_- + 2m\psi(P_+ + P_-) + \frac{L^2}{Q} = 0, \tag{3.54}$$

where $L^2 = \hat{g}^{ij} P_i P_j$ is the square of the angular momentum operator. If the constraint is applied, the Hamiltonian changes to

$$\mathcal{H} = \frac{1}{\psi}(-m + \sqrt{m^2 + 4(\psi^2 P_{\psi}^2 + L^2)/Q}). \tag{3.55}$$

P_ψ is the conjugate momentum to ψ. The other two operators in the $SL(2,\mathbb{R})$ algebra are

$$D = -2\tau H + 2\psi P_\psi \quad , \quad K = \frac{1}{2}\tau^2 H - \frac{1}{4}\tau D + \frac{1}{4}Q\psi^2 H + \frac{1}{2}mQ\psi. \tag{3.56}$$

In the non-relativistic limit and for large ψ (near the horizon),

$$\mathcal{H} = \frac{2\psi P_\psi^2}{mQ} \quad , \quad D = 2\psi P_\psi \quad , \quad K = \frac{1}{2}mQ\psi, \tag{3.57}$$

we express ψ in terms of x as

$$\psi = \frac{x^2}{Q}. \tag{3.58}$$

H, D and K become

$$\mathcal{H} = \frac{P^2}{2m} \quad , \quad D = xP \quad , \quad K = \frac{1}{2}mx^2, \tag{3.59}$$

where P is the conjugate momentum to x. H in this case has no well-defined vacuum, as shown before. This implies that the gauge we choose $h = \tau$ is not a good one. However, the theory is gauge invariant, therefore it should be possible to find a gauge that gives a well-defined ground state. Also note that if K is added to H, the problem is fixed [73]. This suggests that adding K to H might be equivalent to choosing a different gauge-fixing condition.

Next we introduce a gauge that gives a well-defined vacuum. Define the gauge-fixing condition by

$$h(x^+, x^-) = \arctan\left(\frac{\omega x^+ + \omega x^-}{1 - \omega^2 x^+ x^-}\right) = \tau, \tag{3.60}$$

where ω is an arbitrary scale. A derivative with respect to τ gives

$$\partial_+ h\dot{x}^+ + \partial_- h\dot{x}^- = 1 \quad , \quad \partial_\pm h = \frac{\omega}{1 + \omega^2 (x^\pm)^2}. \tag{3.61}$$

The Hamiltonian can be obtained from the Lagrangian,

$$L = \dot{x}^+ P_+ + \dot{x}^- P_- + \dot{\Lambda}, \tag{3.62}$$

where we added the time derivative of a function Λ, which does not alter the dynamics. Also we can view it as a gauge transformation ($A \to A + dA$). The Lagrangian can be rewritten in terms of the new coordinate

$$\zeta = \arctan\left(\frac{\omega x^+ - \omega x^-}{1 + \omega^2 x^+ x^-}\right), \tag{3.63}$$

and the Lagrangian in this new coordinate is

$$L = \dot{\zeta} P_\zeta - \dot{h}\mathcal{H}, \tag{3.64}$$

where

$$\begin{aligned} \mathcal{H} &= -\frac{1}{2}\left(\frac{P+}{\partial_+ h} + \frac{P_-}{\partial_- h}\right) - \partial_h \Lambda = \frac{1}{2\omega}(H + \omega^2 K'), \\ K' &= -(x^+)^2 P_+ - (x^-)^2 P_- - \frac{2}{\omega}\partial_h \Lambda. \end{aligned} \tag{3.65}$$

$\mathcal{H}$ is the momentum conjugate to h and P_ζ is the momentum conjugate to ζ. Because $\dot{h} = 1$, $\mathcal{H}$ can be considered as the Hamiltonian. It is convenient to choose Λ so that $K' = K$ to ensure that the constraint χ and the Hamiltonian have no explicit time dependence, because

$$\partial_h K = \{\chi, K\} = 0, \tag{3.66}$$

from the conservation of the charge K. Λ is obtained from solving (3.66),

$$\Lambda = \frac{m\sqrt{Q}}{4} \ln \frac{\partial_+ h}{\partial_- h} = -\frac{m\sqrt{Q}}{4} \ln \frac{1 + \omega^2 (x^+)^2}{1 + \omega^2 (x_-)^2}. \tag{3.67}$$

The Hamiltonian can be obtained by solving the constraint (3.54),

$$\mathcal{H} = \frac{\sqrt{Q}}{\sin\zeta}\left(-m\cos\zeta + \sqrt{m^2\cos^2\zeta + (4\sin^2\zeta P_\zeta^2/Q + \frac{1}{2}m^2 Q \sin^2\zeta + 4L^2)/Q}\right). \tag{3.68}$$

In the non-relativistic limit, $\mathcal{H}$ can be obtained by letting $\zeta \to 0$, or $\omega x^\pm \to 0$,

$$\mathcal{H} = \frac{1}{2\omega}\left(\frac{P^2}{2m} + \frac{1}{2}m\omega^2 u^2\right), \tag{3.69}$$

where $u^2 = M\zeta \approx M\omega(x^+ - x^-)$, and P is the momentum conjugate to u and the system under this gauge has a well-defined vacuum. Notice that the spectrum in the non-relativistic limit is independent of ω as expected.

Notice that the non-relativistic Hamiltonian (3.69) can be written in terms of the Hamiltonian in the naive gauge (3.50) corrected by the addition of the potential term K ,

$$\mathcal{H} = \frac{1}{2\omega}(H + \omega^2 K) , \qquad K \approx \frac{2}{\omega}\partial_h \Lambda = \frac{1}{2} m M \psi. \tag{3.70}$$

The question arises as to how these two gauges lead to physically different vacua. To gain further insight, let us reconsider the Faddeev-Popov gauge-fixing procedure in the gauge (3.50). Inserting (3.19) into the path integral and then performing an inverse gauge transformation to eliminate the gauge parameter, we are led to an obstruction at the boundary of spacetime. Varying the action (3.8), we obtain

$$\delta S = \int d\tau \frac{d}{d\tau}(\delta x^\mu P_\mu) - \int d\tau \epsilon \dot{\chi}, \tag{3.71}$$

using $\delta\chi = \epsilon\{\chi, \chi\} = 0$. The action changes by a total derivative

$$\delta S = \int d\tau \frac{d}{d\tau}(\delta x^\mu P_\mu). \tag{3.72}$$

Since

$$\delta x^\mu = \{x^\mu, \chi\}\epsilon = \frac{\partial \chi}{\partial P_\mu}\epsilon, \tag{3.73}$$

we have

$$\delta S = P_\mu \frac{\partial \chi}{\partial P_\mu}\epsilon \bigg|_\partial . \tag{3.74}$$

Note that if the generator of the gauge transformations is quadratic, as is the case for the free particle in (3.12), after substituting χ in (3.74) and imposing the condition $\chi = 0$, we deduce $\delta S = 0$. However, when we use the naive gauge (3.50), the generator of the gauge transformation, χ (3.32), is not quadratic due to the presence of the vector potential. Therefore, $\delta S \neq 0$ and the contribution from the boundary of spacetime cannot be ignored. One has to perform the Faddeev-Popov procedure in the presence of a boundary [79]. The boundary terms can be avoided if, instead, we use the gauge condition (3.60). In this case, there is no boundary contribution, because the Faddeev-Popov determinant

$$\{h, \chi\} \propto \frac{P_+}{1 + \omega^2 (x^+)^2} + \frac{P_-}{1 + \omega^2 (x^-)^2}, \tag{3.75}$$

vanishes at the boundary (as $x^\pm \to \infty$). The boundary contribution to the path integral is

$$\int_\partial d\epsilon d^D x d^D P \{h, \chi\} \delta(h - \tau) \delta(\chi) \exp\left(i P_\mu \frac{\partial \chi}{\partial P_\mu} \epsilon \right). \tag{3.76}$$

It is absent when $\{h, \chi\} = 0$ at the boundary. The condition implies invariance under gauge transformations generated by h, which is the time coordinate after gauge-fixing ($h = \tau$). Therefore, the boundary ought to be invariant under time translations.

3.3 Slowly-moving maximally charged black holes

In this section we discuss the quantization of a system of slow-moving maximally charged black holes. First, we review the general features of the classical system in five dimensions following [74] (four dimensions is similar [80, 81, 82, 83, 84, 19]).Then we quantize the system showing that a proper gauge-fixing procedure amounts to the DFF trick for a conformal quantum mechanical system.

3.3.1 The classical system

The equations of motion for multiple black holes can be solved analytically if the black holes are maximally charged (each black hole's mass is equal to its charge, $M = Q$) and the speed of each black hole is small (we shall keep only the first-order terms assuming $v \ll 1$ in units in which $G = c = 1$). The metric and the vector potential for N static maximally charged black holes, respectively, are [80, 82]

$$\begin{aligned} ds^2 &= -\psi^{-2}dt^2 + \psi d\vec{x}^2, \\ A &= \psi^{-1}dt, \end{aligned}$$

where

$$\psi = 1 + \sum_a \frac{Q_a}{|\vec{x} - \vec{x}_a|^2},$$

The metric near a black hole (singularity) becomes AdS_2, leading to a system whose Hamiltonian has no well-defined ground state (similar to the system discussed in subsection 3.2.3).

In the non-relativistic limit, we may perturb around the static metric and vector potential (3.77) and write

$$\begin{aligned} ds^2 &= -\psi^{-2}dt^2 + \psi d\vec{x}^2 + 2\psi^{-2}\vec{R} \cdot d\vec{x}dt, \\ A &= \psi^{-1}dt + (\vec{P} - \psi^{-1}\vec{R}) \cdot d\vec{x}\ . \end{aligned} \tag{3.77}$$

The gravitational and electromagnetic field contributions to the action are [85]

$$S_{\text{field}} = \frac{1}{2}\int d^5x\sqrt{-g}[R - \frac{3}{4}F^2] + \frac{1}{2}\int A \wedge F \wedge F, \tag{3.78}$$

whereas the source (matter) contribution is

$$S_{\text{source}} = \int d^5x\sqrt{-g}A_\mu \rho u^\mu - \int d^5x\sqrt{-g}\rho, \tag{3.79}$$

where ρ is the matter density and u^μ is the matter four-velocity. Gauss's Law reads

$$\nabla^2\psi = -\frac{2}{3}\psi^2\rho \to -4\pi^2\sum_a Q_a\delta^{(4)}(\vec{x}-\vec{x}_a-\vec{v}_a t), \tag{3.80}$$

where $v^\mu = \frac{x^\mu}{dt}$ and the last term is appropriate for a discrete distribution (black holes). The integral over the mass (charge) density also turns into a sum in the discrete case turning (3.79) into

$$S_{\text{source}} = -6\pi^2\sum_a Q_a\int ds_a + 6\pi^2\sum_a Q_a\int A_\mu dx^\mu. \tag{3.81}$$

Using (3.77) for the metric and vector potential, we obtain

$$\begin{aligned} S_{\text{source}} &= -6\pi^2\sum_a Q_a\int dt\psi^{-1}\left(1-\vec{R}\cdot\vec{v}_a-\frac{1}{2}\psi^3\vec{v}_a^2\right) \\ &\quad +6\pi^2\sum_a Q_a\int dt\left(\psi^{-1}+\vec{P}\cdot\vec{v}_a-\psi^{-1}\vec{R}\cdot\vec{v}_a\right) \\ &= \frac{1}{2}\sum_a Q_a\int dt\left(6\pi^2\psi^2\vec{v}_a^2+12\pi^2\vec{v}_a^2\cdot\vec{P}\right). \end{aligned} \tag{3.82}$$

The total action to second order in the velocities is

$$\begin{aligned} S &= \frac{1}{2}\int d^5x\{6\pi^2\psi^2\sum_a Q_a\delta^{(4)}(\vec{x}-(\vec{x}_a+\vec{v}_a t))\vec{v}_a^2 \\ &\quad +12\pi^2\sum_a Q_a\delta^{(4)}(\vec{x}-(\vec{x}_a+\vec{v}_a t))\vec{v}_a\cdot\vec{P}+3\partial_0P_i\partial_i\psi-\frac{3}{4}\psi^{-1}(\partial_{[i}P_{j]})^2 \\ &\quad +\frac{3}{2}\psi^{-2}\partial_{[i}P_{j]}\partial_{[i}R_{j]}-\frac{1}{2}\psi^{-3}(\partial_{[i}R_{j]})^2-3\psi(\partial_0\psi)^2-3\psi^{-1}\epsilon^{ijkl}\partial_iP_k\partial_jP_l \\ &\quad +3\psi^{-2}\epsilon^{ijkl}\partial_iP_k\partial_jR_l-3\psi^{-3}\epsilon^{ijkl}\partial_iR_k\partial_jR_l+\text{t.d.}\}, \end{aligned} \tag{3.83}$$

where i and j run from 1 to 4 and t.d. stands for total derivative. The equations of motion are obtained by varying P_i and R_i,

$$\begin{aligned} dR &= -3\psi^2\sum_a d\frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2}\wedge v_a, \\ dP &= 2\psi\sum_a d\frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2}\wedge v_a. \end{aligned} \tag{3.84}$$

We may use these equations to eliminate P and R from the action. This is straightforward, except for the terms which do not contain derivatives of P and R. They may be massaged as follows:

$$\begin{aligned}
&12\pi^2 \sum_a Q_a \delta^{(4)}(\vec{x}-(\vec{x}_a+\vec{v}_a t))\vec{v}_a\cdot\vec{P} + 3\partial_0 P_i \partial_i \psi \\
= \; & -3\vec{P}\cdot\vec{\partial}\partial_0\psi - 3\sum_a \vec{\partial}^2 \frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2}\vec{v}_a\cdot\vec{P} + \text{t.d.} \\
= \; & 3\vec{P}\cdot\vec{\partial}\sum_a \vec{v}_a\cdot\vec{\partial}\frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2} - 3\sum_a \vec{\partial}^2 \frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2}\vec{v}_a\cdot\vec{P} + \text{t.d.} \\
= \; & -3\sum_a \vec{v}_a\cdot\vec{\partial}\vec{P}\cdot\vec{\partial}\frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2} + 3\sum_a \vec{\partial}\frac{Q_a}{|\vec{x}-(\vec{x}_a+\vec{v}_a t)|^2}\cdot\vec{\partial}\vec{P}\cdot\vec{v}_a + \text{t.d.} \\
= \; & -3\sum_a v_{ai}(\partial_{[i}P_{j]})\partial_{aj}\psi + \text{t.d.}
\end{aligned} \tag{3.85}$$

Then the action becomes, on account of (3.84),

$$\begin{aligned}
S \;=\; & \frac{1}{2}\int d^5x\{-\frac{3}{2}\psi^2\sum_a \vec{\partial}_a^2\psi^2\vec{v}_a^2 - 3\psi\sum_{a,b}\vec{v}_a\cdot\vec{v}_b\vec{\partial}_a\psi\cdot\vec{\partial}_b\psi \\
& +3\psi\sum_{a,b}\vec{v}_a\cdot\vec{\partial}_b\psi\vec{v}_b\cdot\vec{\partial}_a\psi - 3\psi\sum_{a,b}\vec{v}_a\cdot\vec{\partial}_a\psi\vec{v}_b\cdot\vec{\partial}_b\psi \\
& -3\psi\epsilon^{ijkl}\sum_{a,b}\partial_{ai}\psi v_{aj}\partial_{bk}\psi v_{bl} + \text{t.d.}\}.
\end{aligned} \tag{3.86}$$

It can be rewritten as

$$S = \frac{1}{4}\int dt \sum_{a,b}(\delta^{ij}\delta_{kl} + \delta^i_k\delta^j_l - \delta^i_l\delta^j_k + \epsilon^{ij}{}_{kl})\partial_{ai}\partial_{bj}L \; v^{ak}v^{bl}, \tag{3.87}$$

where

$$L = -\int d^4x\psi^3. \tag{3.88}$$

To perform the integration in L, we shall only consider the near horizon limit in which the metric gets simplified. We have

$$\psi \to \sum_a \frac{Q_a}{|\vec{x}-\vec{x}_a|^2}.$$

In the near horizon limit, distances between black holes are much smaller than the Planck length, $L_P = 1$. The metric in this limit simplifies to

$$ds^2 = -\psi^{-2}dt^2 + \psi d\vec{x}^2,$$

which leads to a Hamiltonian with no well-defined ground state, as was discussed before (subsection 3.2.3). The metric of the moduli space of the multi-black hole system is obtained by differentiating the action with the velocity [74], i.e.,

$$ds^2 = \frac{1}{4}(\delta^{ij}\delta_{kl} + \delta^i_k\delta^j_l - \delta^i_l\delta^j_k + \epsilon^{ij}{}_{kl})\partial_{ai}\partial_{bj}L \; dx^{ak}dx^{bl}, \tag{3.89}$$

where a summation over both black hole and space repeated indices is implied. L splits into three pieces, $L = L_1 + L_2 + L_3$, representing 1-body, 2-body and 3-body interactions, respectively,

$$L_1 = -\sum_a \int d^4x \frac{Q_a}{|\vec{x} - \vec{x}_a|^6},$$

$$L_2 = -3\sum_{a\neq b} \int d^4x \frac{Q_a^2 Q_b}{|\vec{x} - \vec{x}_a|^4 |\vec{x} - \vec{x}_b|^2},$$

$$L_3 = -\sum_{a\neq b\neq c} \int d^4x \frac{Q_a Q_b Q_c}{|\vec{x} - \vec{x}_a|^2 |\vec{x} - \vec{x}_b|^2 |\vec{x} - \vec{x}_c|^2}. \tag{3.90}$$

Note that there are no higher than 3-body interactions. L_1 may be ignored because the variable $\vec{x}$ can be shifted to $\vec{x} \to \vec{x} - \vec{x}_a$. L_2 gives divergent terms but they may be eliminated after differentiation with respect to x^a and x^b. After introducing a cutoff δ, we obtain (see Appendix B.1)

$$L_2 = -6\pi^2 \sum_{a\neq b} Q_a^2 Q_b \frac{[\ln|\vec{x}_a - \vec{x}_b| - \ln\delta]}{|\vec{x}_a - \vec{x}_b|^2}. \tag{3.91}$$

To obtain a Hamiltonian with a well-defined ground state, we shall apply the DFF trick (section 3.1). To this end, we need to derive an operator that will play the role of K in the $SL(2,\mathbb{R})$ algebra.

3.3.2 An $SL(2,\mathbb{R})$ algebra

Our goal in this subsection is this subsection is to derive the generators of the $SL(2,\mathbb{R})$ algebra

$$[D,H] = 2iH, \quad , \quad [D,K] = -2iK, \quad , \quad [H,K] = -iD.$$

We shall do this for a general system described by the Hamiltonian [86]

$$H = \frac{1}{2}P_a^\dagger g^{ab} P_b + V(X), \tag{3.92}$$

where a and b are black hole indices. By letting

$$P_a = g_{ab}\dot{X}^b = -i\partial_a,$$

P_a and X^a satisfy $[P_a, X^b] = -i\delta_a^b$ and $[P_a, P_b] = 0$, where

$$P_a^\dagger = \frac{1}{\sqrt{-g}} P_a \sqrt{-g}.$$

D is the generator of dilation operator

$$D = \frac{1}{2} D^a P_a + \text{h.c.}, \tag{3.93}$$

and X^a transforms as

$$\delta_D X^a = D^a(X).$$

From equation (3.92), the algebra $[D, H]$ becomes

$$[D, H] = -\frac{i}{2} P_a^\dagger (\mathcal{L}_D g^{ab}) P_b - i\mathcal{L}_D V - \frac{i}{4}\nabla^2\nabla_a D^a, \tag{3.94}$$

where $\mathcal{L}_D$ is the Lie derivative which is, on g_{ab},

$$\mathcal{L}_D g_{ab} = D^c g_{ab,c} + D^c{}_{,a} g_{cb} + D^c{}_{,b} g_{ac}. \tag{3.95}$$

Comparing the result to the general form of H, one can find out that

$$\mathcal{L}_D g_{ab} = 2g_{ab}, \tag{3.96}$$

where setting $\nabla^2\nabla_a D^a = 0$, and

$$\mathcal{L}_D V = -2V. \tag{3.97}$$

From $[D, K] = -2iK$, the algebra can be written in the form of operator $\mathcal{L}$ as

$$\mathcal{L}_D K = 2K. \tag{3.98}$$

Also from $[H, K] = -iD$, the algebra is in the form

$$D = D_a dX^a = dK. \tag{3.99}$$

Then D is a one-form. If D is also exact, K could be written as

$$K = \frac{1}{2} g_{ab} D^a D^b. \tag{3.100}$$

In general D might not be exact, causing K not in the the form of (3.100).

To find an explicit expression for K in our case, note that

$$dK = D_{ai}dx^{ai} = -g_{ai\ bj}x^{bj}dx^{ai}, \tag{3.101}$$

L_3 does not contribute to the operator K because its contribution to D vanishes,

$$D_{3ak} = -g_{3ak\ bl}x^{bl} = \frac{1}{4}(\delta^{ij}\delta_{kl} + \delta^i_k\delta^j_l - \delta^i_l\delta^j_k + \epsilon^{ij}{}_{kl})\partial_{ai}\partial_{bj}L_3\ x^{bl} = 0 \tag{3.102}$$

(appendix B.1). Therefore, only L_2 contributes,

$$D_{ai}dx^{ai} = -g_{2ai\ bj}x^{bj}dx^{ai} = d\left[\frac{3\pi^2}{4}\sum_{a\neq b}\frac{Q_a^2Q_b}{|\vec{x}_a - \vec{x}_b|^2}\right] \tag{3.103}$$

leading to the expression

$$K = \frac{3\pi^2}{4}\sum_{a\neq b}\frac{Q_a^2Q_b}{|\vec{x}_a - \vec{x}_b|^2}, \tag{3.104}$$

for the generator of special conformal transformations K.

3.3.3 Quantization

We shall now quantize the multi-black hole system using the path integral method of section 3.2. Our discussion will follow [19, 18]. We start by re-writing the source part of the action in terms of coordinates and their conjugate momenta,

$$S_{\text{source}} = 6\pi^2\sum_a\int dx_a^\mu\ P_{a\mu}. \tag{3.105}$$

The constraint equations are obtained in a similar fashion,

$$\chi_a \equiv \frac{1}{2M_a}\pi_{a\mu}\pi^{a\mu} + \frac{1}{2}M_a = 0\quad,\quad \pi_{a\mu} = P_{a\mu} - Q_aA_\mu\quad,\quad Q_a = M_a. \tag{3.106}$$

The electromagnetic and gravitational field parts remain the same. Solving the field equations (Einstein and Maxwell equations), we may express the fields in terms of the vector potential $\mathcal{A}_\mu$ as

$$\psi = \mathcal{A}_0\quad,\quad F_{ij} = 2\psi\mathcal{F}_{ij}\quad,\quad G_{ij} = 3\psi^3\mathcal{F}_{ij}, \tag{3.107}$$

where we have defined

$$F_{ij} = \partial_iPj - \partial_jP_i\quad,\quad G_{ij} = \partial_iR_j - \partial_jR_i \tag{3.108}$$

We also need their duals

$$\tilde{F}^{ij} = \epsilon^{ijkl} F_{kl} \quad , \quad \tilde{G}^{ij} = \epsilon^{ijkl} G_{kl}.$$

The vector potential $\mathcal{A}_\mu$ is generated by the source current j^μ in flat spacetime

$$\begin{aligned} \partial_\mu \mathcal{F}^{\mu\nu} &= 2\pi^2 j^\nu, \\ \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \\ j^\mu &= \sum_a Q_a \int dX_a^\mu \delta^5(x - x_a). \end{aligned} \tag{3.109}$$

For the sources, we obtain the Lorentz force equation,

$$\ddot{x}_a^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}_a^\nu \dot{x}_a^\lambda = \dot{x}_a^\nu F^\mu_\nu. \tag{3.110}$$

To quantize the system, we start with the path integral

$$Z = \mathcal{N} \int \mathcal{D}g \mathcal{D}A \prod_a \mathcal{D}x_a \mathcal{D}P_a \delta(\chi_a) e^{iS}. \tag{3.111}$$

and fix the gauge. An obvious gauge choice is

$$x_a^0 = t, \tag{3.112}$$

for all black holes. Then the current becomes

$$j^\mu = \sum_a Q_a v_a^\mu \delta^4(\vec{x} - \vec{x}_a) \quad , \quad v_a^\mu = (1, \vec{v}_a) \quad , \quad \vec{v}_a = \frac{d\vec{x}_a}{dt}, \tag{3.113}$$

and the vector potentials are

$$\mathcal{A}_0 = \psi = \sum_a \frac{Q_a}{(\vec{x} - \vec{x}_a(t))^2} \quad , \quad \vec{\mathcal{A}} = \sum_a \frac{Q_a \vec{v}_a}{(\vec{x} - \vec{x}_a(t))^2}. \tag{3.114}$$

The conjugate momentum is obtained from solving the constraint $\chi_a = 0$ in the non-relativistic limit as

$$P_{a0} = \frac{g^{ij}\pi_i \pi_j}{2Q_a} + Q_a A_0, \tag{3.115}$$

and the action for the source becomes

$$S_{\text{source}} = \sum_a 6\pi^2 \int dt (\dot{x}_a^i P_{ai} - H_a) \quad , \quad H_a = -P_{a0}. \tag{3.116}$$

Plugging the source action back into the path integral (3.111) and integrating over the momentum $\vec{P}_a$, we obtain

$$Z = \mathcal{N} \int \prod_a \mathcal{D}x_a e^{iS}. \tag{3.117}$$

where the action may be written as [74]

$$S = S_{\text{field}} + S_{\text{source}} = \int dt \sum_{a \neq b} G_{ab} (\vec{v}_a - \vec{v}_b)^2 + \dots \ , \tag{3.118}$$

where

$$G_{ab} = \frac{3\pi^2 Q_a Q_b (Q_a + Q_b)}{8(\vec{x}_a - \vec{x}_b)^4}, \qquad \vec{v}_a = \frac{d\vec{x}_a}{dt}. \tag{3.119}$$

This leads to a Hamiltonian with no well-defined ground state. This pathology is due to the fact that the gauge choice (3.112) is not good, which is similar to the problem we encountered in subsection 3.2.3. We need to make a good gauge choice. Denote the gauge-fixing condition for the black hole labeled by the index b by

$$h_b(x_b^\mu) = t. \tag{3.120}$$

This black hole interacts with other black holes by gravitational and electromagnetic forces. In the case where our b^{th} black hole approaches another black hole, say, the a^{th}, the influence of the rest of the black holes is negligible. The problem is then reduced to one that we have already tackled in subsection 3.2.3. The net effect of a good gauge choice was the addition of a potential to the Hamiltonian. Similarly, here we expect that a non-relativistic potential will be added our b^{th} black hole Hamiltonian, of the form

$$K_a^{(b)} = \frac{3\pi^2 Q_b Q_a^2}{4(\vec{x}_b - \vec{x}_a)^2}. \tag{3.121}$$

By applying this argument to the rest of the black holes, we obtain the total potential energy of our b^{th} black hole in the non-relativistic limit as

$$K^{(b)} = \sum_{a \neq b} K_{(a)}^{(b)} = \frac{3\pi^2}{4} Q_b \sum_{a \neq b} \frac{Q_a^2}{(\vec{x}_b - \vec{x}_b)^2}. \tag{3.122}$$

It is convenient to introduce the coordinates

$$X_a^{(b)\pm} = x_b^0 \pm \frac{Q_a^{3/2}}{2(\vec{x}_b - \vec{x}_a)^2}, \tag{3.123}$$

similar to (3.47). We shall choose the gauge-fixing condition (similar to (3.60))

$$h_b(x_b^\mu) = x_b^0 + \sum_{a\neq b}\left(\arctan\left\{\frac{x_b^0}{1+\frac{1}{4}X_a^{(b)+}X_a^{(b)-}}\right\} - x_b^0\right), \tag{3.124}$$

where we set $\omega = 1/2$, for simplicity. Notice that when $(\vec{x}_b - \vec{x}_a)^2 \to 0$ for a fixed a, the above expression reduces to (3.60). This ensures that there will be no boundary contributions, because $\dot{h}_b \to 0$, near the boundary of moduli space. We added a total time derivative to the Lagrangian to ensure that there is no explicit dependence on h_b, resulting in a time-dependent Hamiltonian. Therefore, the action can be written as

$$\begin{aligned} S_b &= 6\pi^2 \int dt(\dot{x}_b^\mu P_{b\mu} + \dot{\Lambda}^{(b)}), && (3.125)\\ \Lambda^{(b)} &= \sum_{a\neq b}\Lambda_a^{(b)}, \\ \Lambda_a^{(b)} &= -\frac{Q_b\sqrt{Q_a}}{4}\ln\left\{\frac{1+\frac{1}{4}(X^{(b)+})^2}{1+\frac{1}{4}(X^{(b)-})^2}\right\}. && (3.126)\end{aligned}$$

In the non-relativistic limit, $h_b = t \approx x_b^0$ and

$$\Lambda_a^{(b)} \approx x_b^0 \frac{Q_b Q_a^2}{8(\vec{x}_b - \vec{x}_a)^2} = \frac{1}{2} t \frac{K_a^{(b)}}{3\pi^2}. \tag{3.127}$$

Thus, the additional term is

$$\dot{\Lambda}^{(b)} \approx \frac{1}{2}\sum_{a\neq b}\frac{K_a^{(b)}}{3\pi^2} = \frac{1}{2}\frac{K^{(b)}}{3\pi^2}, \tag{3.128}$$

as expected. By repeating the above procedure with the rest of the black holes, we may sum over the index b. Then the action for the source is written as

$$S_{\text{source}} = \sum_b S_b = \sum_b \int dt(6\pi^2)\left(\dot{x}_b^\mu P_{b\mu} + \dot{\Lambda}^{(b)}\right). \tag{3.129}$$

The net effect of the gauge (3.124) in the non-relativistic limit is the addition of the potential

$$\begin{aligned} K &= \sum_a K^{(a)} = \sum_a\sum_{b\neq a}\frac{3\pi^2 Q_a Q_b^2}{4(\vec{x}_a - \vec{x}_b)^2}, \\ &= \sum_{a<b}\frac{3\pi^2 Q_a Q_b(Q_b + Q_a)}{8(\vec{x}_a - \vec{x}_b)^2}. && (3.130)\end{aligned}$$

Therefore, by fixing the gauge in a way that eliminates the boundary contribution, we obtained a potential K (3.130) which needs to be added to the Hamiltonian one obtains in the naive gauge (3.112). The resulting Hamiltonian has a well-defined Hilbert space.

3.4 A continuous distribution (Black string)

In this section we extend the previous discussion to a continuous mass distribution. Only a one-dimensional distribution gives a finite potential K. We shall consider explicitly the case of a ring-shaped black string. Details of the calculation can be found in appendix B.2.

3.4.1 The classical system

The metric of a continuous system is [82]

$$ds^2 = -\psi^{-2}dt^2 + \psi d\vec{x}^2 + 2\psi^2 \vec{R} \cdot d\vec{x}dt,$$

where

$$\begin{aligned}
\psi &= 1 + \int d^4x' \sqrt{g} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|^2}, \\
A &= \psi^{-1}dt + (\vec{P} - \psi^{-1}\vec{R}) \cdot d\vec{x}.
\end{aligned}$$

The field and source parts of the action, respectively, are [74]

$$\begin{aligned}
S_{\text{field}} &= \frac{1}{2}\int dx^5 \sqrt{-g}\left[R - \frac{3}{4}F^2\right] + \frac{1}{2}\int A \wedge F \wedge F, \\
S_{\text{source}} &= \int d^5x\sqrt{-g}A_\mu \rho u^\mu - \int d^5x\sqrt{-g}\rho.
\end{aligned}$$

The source contribution may be written the terms of the velocity field, $\vec{v}$, in the non-relativistic limit,

$$\begin{aligned}
S_{\text{source}} &= \int d^5x\sqrt{-g}\rho\left[A_\mu u^\mu \frac{dt}{d\tau} - 1\right] \\
&= \int d^5x \psi^2 \rho \left\{\frac{\psi^2 v^2}{2} + \vec{P} \cdot \vec{v}\right\}.
\end{aligned}$$

Then the total action can be written in terms of the potentials P and R, as before,

$$\begin{aligned} S &= S_{\text{source}} + S_{\text{field}} \\ &= \frac{1}{2}\int d^5x \Big\{ \psi^2\rho(\psi^2 v^2 + 2\vec{P}\cdot\vec{v}) \\ &\qquad +3\partial_0 P_i\partial_i\psi - \frac{3}{4}\psi^{-1}(\partial_{[i}P_{j]})^2 + \frac{3}{2}\psi^{-2}\partial_{[i}P_{j]}\partial_{[i}R_{j]} - \frac{1}{2}\psi^{-3}(\partial_{[i}R_{j]})^2 \\ &\qquad -3\psi(\partial_0\psi)^2 - 3\psi^{-1}\epsilon^{ijkl}\partial_i P_k\partial_j P_l + 3\psi^{-2}\epsilon^{ijkl}\partial_i P_k\partial_j R_l \\ &\qquad -\psi^{-3}\epsilon^{ijkl}\partial_i R_k\partial_j R_l + \text{t.d.}\Big\}. \end{aligned}$$

Varying P and R we obtain the equations of motion,

$$\begin{aligned} 0 &= -\partial_0\partial_i\psi - \partial_j(\psi^{-1}\partial_{[i}P_{j]}) + \partial_j(\psi^{-2}\partial_{[i}R_{j]}) - 2\psi^{-1}\epsilon^{ijkl}\partial_k\psi\partial_j P_l \\ &\qquad +2\psi^{-3}\epsilon^{ijkl}\partial_k\psi\partial_j R_l + \frac{2}{3}\psi^2\rho v_i, \\ 0 &= \partial_j(\psi^{-2}\partial_{[i}P_{j]}) - \frac{2}{3}\partial_j(\psi^{-3}\partial_{[i}R_{j]}) - 2\psi^{-3}\epsilon^{ljki}\partial_j\psi\partial_l P_k \\ &\qquad 2\psi^{-4}\epsilon^{ljki}\partial_j\psi\partial_l R_k. \end{aligned} \tag{3.131}$$

By eliminating the ϵ^{ijkl} terms, we obtain

$$\partial_j\left(\psi^{-2}\partial_{[i}R_{j]}\right) = 3\partial_0\partial_i\psi - 2\psi\rho v_i. \tag{3.132}$$

Introduce a potential $\vec{K}$ obeying

$$\partial_{[i}K_{j]} = \psi^{-2}\partial_{[i}R_{j]} = \frac{3}{2}\psi^{-1}\partial_{[i}P_{j]} \tag{3.133}$$

Then

$$\partial_j\partial_{[i}K_{j]} = 3\partial_0\partial_i\psi - 2\psi^2\rho v_i. \tag{3.134}$$

Let us choose

$$\nabla^2 K_i = 2\psi^2\rho v_i. \tag{3.135}$$

We have

$$\begin{aligned} \partial_i\partial_j K_j &= 3\partial_0\partial_i\psi, \\ \partial_j K_j &= 3\partial_0\psi, \\ \partial_j\nabla^2 K_j &= 3\nabla^2\psi, \\ \partial_j(2\psi^2\rho v_j) &= 3\nabla^2\psi. \end{aligned} \tag{3.136}$$

From the continuity equation for the current,

$$\partial_j(\bar{\rho}v_j) + \partial_0\bar{\rho} = 0, \tag{3.137}$$

where $\bar{\rho} = \psi^2 \rho$, we deduce

$$\nabla^2 \psi = -\frac{2}{3}\bar{\rho}. \tag{3.138}$$

Using the Green function in four dimensions,

$$G(\vec{r}_1, \vec{r}_2) = -\frac{1}{4\pi^2}\frac{1}{\mid \vec{r}_1 - \vec{r}_2 \mid^2}, \tag{3.139}$$

we obtain

$$\psi = \frac{1}{6\pi^2}\int d^4 r_2 \frac{\bar{\rho}(\vec{r}_2)}{\mid \vec{r}_1 - \vec{r}_2 \mid^2}. \tag{3.140}$$

Then $\vec{K}$ can be calculated using (3.135), and hence the field strengths of the potentials P and R are obtained using (3.133),

$$\psi^{-2}\partial_{[i}R_{j]} = -\frac{2}{4\pi^2}\int d^4 r_2 \frac{\partial_{[i}\bar{\rho}v_{j]}}{|\vec{r}_1 - \vec{r}_2|^2}, \tag{3.141}$$

$$\psi^{-1}\partial_{[i}P_{j]} = -\frac{1}{3\pi^2}\int d^4 r_2 \frac{\partial_{[i}\bar{\rho}v_{j]}}{|\vec{r}_1 - \vec{r}_2|^2}. \tag{3.142}$$

The first term in the action can be manipulated as follows:

$$\begin{aligned}
\bar{\rho}\psi^2 v^2 &= \int d^4 z \delta(\vec{U} - \vec{z})\bar{\rho}_z \psi^2 v_z^2 \\
&= \int d^4 z \bar{\rho}_z \psi^2 v_z^2 (-\frac{1}{4\pi})\partial^2_{Uj}\frac{1}{|\vec{U} - \vec{z}|^2} \\
&= -\frac{3}{2}(\frac{2}{3\cdot 4\pi^2})^3 \int d^4x d^4y d^4z \bar{\rho}_x \bar{\rho}_y \bar{\rho}_z \frac{v_z^2}{|\vec{U} - \vec{x}|^2 |\vec{U} - \vec{y}|^2}\partial^2_{zj}\frac{1}{|\vec{U} - \vec{z}|^2} \\
&= 3(\frac{2}{3\cdot 4\pi^2})^3 \int d^4x d^4y d^4z \bar{\rho}_x \bar{\rho}_y \bar{\rho}_z \frac{v_z^2}{|\vec{U} - \vec{x}|^2}\partial_{yi}\frac{1}{|\vec{U} - \vec{y}|^2}\partial_{zi}\frac{1}{|\vec{U} - \vec{z}|^2} \\
&\quad + \text{t.d.}.
\end{aligned} \tag{3.143}$$

The second and third terms are combined together:

$$\begin{aligned}
& 2\bar{\rho}p_i v_i + 3\partial_0 P_i \partial_i \psi = -3\nabla^2\psi P_i v_i + 3\partial_0 P_i \partial_i \psi \\
=\ & 3v_i(\partial_j P_i)(\partial_j \psi) + 3P_i(\partial_j v_i)\partial_j \psi - 3P_i \partial_i \partial_0 \psi + \text{ t.d.} \\
=\ & 3v_i(\partial_j P_i)(\partial_j \psi) + 3P_i(\partial_j v_i)\partial_j \psi - P_i \partial_i \nabla^{-2}\partial_j(2\bar{\rho}v_j) + \text{ t.d.} \\
=\ & 3(\partial_j P_i)\partial_U(\frac{2}{3\cdot 4\pi^2})\int d^4 z \frac{\bar{\rho}v_j}{|\vec{U} - \vec{z}|^2} + \partial_j P_i \partial_i \nabla^{-2}(2\bar{\rho}v_j) + \text{ t.d.} \\
=\ & -\partial_{[i}P_{j]}\partial_j \frac{2}{4\pi^2}\int d^4 z \frac{\bar{\rho}v_i}{|\vec{U} - \vec{z}|^2} + \text{ t.d.}
\end{aligned}$$

$$
\begin{aligned}
&= 3\cdot 2(\frac{2}{3\cdot 4\pi^2})^2\psi\int d^4y\frac{\partial_{[i}\bar{\rho}v_{j]}}{|\vec{U}-\vec{y}|^2}\int d^4z\bar{\rho}v_i\partial_{Uj}\frac{1}{|\vec{U}-\vec{z}|^2}+\text{ t.d.}\\
&= 3\cdot 2(\frac{2}{3\cdot 4\pi^2})^2\int d^4xd^4yd^4z\frac{\bar{\rho}_x\bar{\rho}_y\bar{\rho}_z}{|\vec{U}-\vec{x}|^2}v_{[yj}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{y}|^2}v_{zi}\partial_{Uj}\frac{1}{|\vec{U}-\vec{z}|^2}\\
&\quad +\text{t.d. .}
\end{aligned}
\tag{3.144}
$$

The fourth, fifth and sixth terms may be combined into

$$
\begin{aligned}
-\frac{3}{4}\psi^{-1}(\partial_{[i}P_{j]})^2 &+ \frac{3}{2}\psi^{-2}\partial_{[i}P_{j]}\partial_{[i}R_{j]}-\frac{1}{2}\psi^{-3}(\partial_{[i}R_{j]})^2\\
&= \frac{3}{2}(\frac{2}{3\cdot 4\pi^2})^2\psi\int d^4y\bar{\rho}_yv_{[yj}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{y}|^2}\int d^4z\bar{\rho}_zv_{[zj}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{z}|^2}\\
&\quad +\text{t.d. .}
\end{aligned}
\tag{3.145}
$$

The seventh term becomes

$$
-3\psi(\partial_0\psi)^2=-3(\frac{2}{3\cdot 4\pi^2})^2\psi\int d^4y\bar{\rho}_yv_{yi}\partial_{Ui}\frac{1}{|\vec{U}-\vec{y}|^2}\int d^4z\bar{\rho}_zv_{zj}\partial_{Uj}\frac{1}{|\vec{U}-\vec{z}|^2}+\text{ t.d. .}
\tag{3.146}
$$

The eighth, ninth and tenth terms are

$$
\begin{aligned}
&- 3\psi^{-1}\epsilon^{ijkl}\partial_iP_k\partial_jP_l+3\psi^{-2}\epsilon^{ijkl}\partial_iP_k\partial_jR_l-\psi^{-3}\epsilon^{ijkl}\partial_iR_k\partial_jR_l\\
&= -\frac{3}{4}(\frac{2}{3\cdot 4\pi^2})^2\psi\epsilon^{ijkl}\int d^4y\bar{\rho}_yv_{[yk}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{y}|^2}\psi\int d^4z\bar{\rho}_zv_{[zl}\partial_{Uj]}\frac{1}{|\vec{U}-\vec{z}|^2}\\
&\quad +\text{ t.d. .}
\end{aligned}
\tag{3.147}
$$

Then the action may be written as

$$
\begin{aligned}
S &= \frac{1}{2}\int d^5U(\frac{2}{3\cdot 4\pi^2})^3\int d^4xd^4yd^4z\frac{\bar{\rho}_x\bar{\rho}_y\bar{\rho}_z}{|\vec{U}-\vec{x}|^2}\\
&\quad 3\Bigg\{v_z^2\partial_{yi}\frac{1}{|\vec{U}-\vec{y}|^2}\partial_{zi}\frac{1}{|\vec{U}-\vec{z}|^2}+2v_{[yj}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{y}|^2}v_{zi}\partial_{Uj}\frac{1}{|\vec{U}-\vec{z}|^2}\\
&\qquad +\frac{1}{2}v_{[yj}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{y}|^2}v_{[zj}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{z}|^2}-v_{yi}\partial_{Ui}\frac{1}{|\vec{U}-\vec{y}|^2}v_{zj}\partial_{Uj}\frac{1}{|\vec{U}-\vec{z}|^2}\\
&\qquad -\frac{1}{4}\epsilon^{ijkl}v_{[yk}\partial_{Ui]}\frac{1}{|\vec{U}-\vec{y}|^2}v_{[zl}\partial_{Uj]}\frac{1}{|\vec{U}-\vec{z}|^2}\Bigg\}\\
&= \frac{1}{2}\int d^5U(\frac{2}{3\cdot 4\pi^2})^3\int d^4xd^4yd^4z\frac{\bar{\rho}_x\bar{\rho}_y\bar{\rho}_z}{|\vec{U}-\vec{x}|^2}\times\\
&\quad 3\left\{(v_z^2-v_{yi}v_{zi})\partial_{yj}\partial_{zj}+v_{yi}v_{zj}\left(\partial_{yj}\partial_{zi}-\partial_{yi}\partial_{zj}+\epsilon^{ijkl}\partial_{yk}\partial_{zl}\right)\right\}
\end{aligned}
$$

$$\frac{1}{|\vec{U}-\vec{y}|^2|\vec{U}-\vec{z}|^2}. \tag{3.148}$$

Following [74], we introduce the metric of the moduli space of the continuous distribution case as

$$\begin{aligned} g_{a\alpha b\beta} &= \frac{\delta S}{\delta v_{a\alpha}\delta v_{b\beta}} \\ &= \frac{1}{2}\int d^5U \left(\frac{2}{3\cdot 4\pi^2}\right)^3 \int d^4x d^4y \bar{\rho}_x\bar{\rho}_y 3\cdot 2\times \\ &\quad \left\{(\bar{\rho}_a\delta_{ab}\delta_{\alpha\beta}\partial_{yj}\partial_{aj}) - \delta_{\alpha\beta}\partial_{aj}\partial_{bj} + \partial_{a\beta}\partial_{b\alpha} - \partial_{a\alpha}\partial_{b\beta} - \epsilon^{\alpha\beta kl}\partial_{ak}\partial_{bl}\right\} \\ &\quad \frac{1}{|\vec{U}-\vec{x}|^2|\vec{U}-\vec{a}|^2|\vec{U}-\vec{b}|^2}. \end{aligned} \tag{3.149}$$

We are interested in finding the generators of dilatation D and special conformal transformations K. We expect that K will play the role of a potential that will lead to a well-defined ground state for the Hamiltonian, as in the discrete case. From section 3.3.1 and 3.3.2, we know that if D is exact, then K is related to D by $dK = D_{ai}dx^{ai} = -g_{ai\ bj}b^j da^i$. Because we now have a continuous distribution, we have to integrate over the continuous index b,

$$\begin{aligned} \int g_{a\alpha b\beta} b^\beta d^4b &= \frac{1}{2}\int d^5U \left(\frac{2}{3\cdot 4\pi^2}\right)^3 \int d^4x d^4b \bar{\rho}_x\bar{\rho}_a\bar{\rho}_b 3\cdot 2 \\ &\quad \left\{(a^\alpha - b^\alpha)\partial_{aj}\partial_{bj} + b^\beta(\partial_{a\beta}\partial_{b\alpha} - \partial_{a\alpha}\partial_{b\beta} - \epsilon^{\alpha\beta kl}\partial_{ak}\partial_{bl}\right\} \\ &\quad \frac{1}{|\vec{U}-\vec{x}|^2|\vec{U}-\vec{a}|^2|\vec{U}-\vec{b}|^2}. \end{aligned} \tag{3.150}$$

To integrate over $\vec{U}$, we introduce a cutoff δ

$$\begin{aligned} |\vec{U}-\vec{a}|^2 &\to |\vec{U}-\vec{a}|^2 + \delta^2, \\ |\vec{U}-\vec{b}|^2 &\to |\vec{U}-\vec{b}|^2 + \delta^2, \\ |\vec{U}-\vec{c}|^2 &\to |\vec{U}-\vec{c}|^2 + \delta^2, \end{aligned} \tag{3.151}$$

We shall let $\delta \to 0$ at the end. To perform the integral, we introduce Feynman parameters. Using the formula

$$\begin{aligned} \frac{1}{D_1D_2D_3} &= 2\int \frac{[dx]}{(D_1x_1 + D_2x_2 + D_3x_3)^3} \quad , \\ \int [dx] &= \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1 - x_1 - x_2 - x_3) \end{aligned} \tag{3.152}$$

we obtain

$$\begin{aligned}
\int g_{a\alpha b\beta} b^\beta d^4 b &= \frac{1}{2}\left(\frac{2}{3\cdot 4\pi^2}\right)^3 (3\cdot 2\cdot 4)\int d^4 b d^4 c \bar{\rho}_a \bar{\rho}_b \bar{\rho}_c (2\pi^2)\int [dx] \\
&\quad \frac{x_1 x_2 (a^\alpha - b^\alpha)\delta^2}{\left[(\vec{a}-\vec{b})^2 x_1 x_2 + (\vec{a}-\vec{c})^2 x_1 x_2 + (\vec{b}-\vec{c})^2 x_1 x_3 + \delta^2\right]^3} \\
&= \frac{1}{2}\left(\frac{2}{3\cdot 4\pi^2}\right)^3 (3\cdot 2\cdot 4)\int d^4 b d^4 c \bar{\rho}_a \bar{\rho}_b \bar{\rho}_c (2\pi^2)\int [dx] \\
&\quad (-\frac{1}{8})\frac{\partial}{\partial a^\alpha}\frac{1}{\left[(\vec{a}-\vec{b})^2 x_1 x_2 + (\vec{a}-\vec{c})^2 x_1 x_2 + (\vec{b}-\vec{c})^2 x_1 x_3 + \delta^2\right]^2}.
\end{aligned} \tag{3.153}$$

As an example, consider a ring-shaped formation of uniform density $\bar{\rho}$ and radius R. The total mass (charge) of the ring is

$$M = Q = 2\pi R\bar{\rho}. \tag{3.154}$$

From (3.153), we obtain (see appendix B.2)

$$\int g_{a\alpha b\beta} b^\beta d^4 b = 4\left(\frac{2}{3\cdot 4\pi^2}\right)^3 \left(\frac{M}{2\pi}\right)^3 \pi^3 \frac{a^\alpha}{R^4}.$$

We also have

$$\begin{aligned}
D_{a\alpha} dx^{a\alpha} &= -g_{2a\alpha b\beta} x^{b\beta} dx^{a\alpha}, \\
dK &= d\left[3\cdot 4(\frac{1}{6\pi^2})^3 (\frac{M}{2\pi})^3 \frac{\pi^3}{6R^2}\right], \\
K &= (\frac{1}{6\pi^2})^3 \frac{M^3}{4R^2},
\end{aligned} \tag{3.155}$$

The Lagrangian in (3.148) reduces to (see appendix B.2)

$$S = (\frac{1}{6\pi^2})^3 3M^3 \frac{\dot{R}^2}{R^4}. \tag{3.156}$$

This leads to a Hamiltonian with no well-defined ground state. With a proper gauge choice, we expect that this Lagrangian will be the kinetic part T of the Hamiltonian which will include a potential $V = K$. Thus, we expect the correct form of the Lagrangian to be of the form

$$\begin{aligned}
L &= T - V \\
&= (\frac{1}{6\pi^2})^3 \left[\frac{3M^3\dot{R}^2}{R^4} - \frac{M^3}{4R^2}\right],
\end{aligned} \tag{3.157}$$

leading to the Hamiltonian

$$H = \frac{(6\pi^2)^3}{12}\frac{R^4}{M^3}P^2 + \frac{1}{4}(\frac{1}{6\pi^2})^3\frac{M^3}{R^2}, \tag{3.158}$$

where $P = (1/6\pi^2)^3 6M^3\dot{R}/(R^4)$ is the momentum conjugate to R. The three quantities, H, $D = -RP$ and K obey an $SL(2,\mathbb{R})$ algebra. By changing variables to

$$\begin{aligned} u &= \frac{3^{1/4}}{(6\pi^2)^{3/2}}\frac{M}{R}, \\ P_u &= M\dot{u}, \end{aligned} \tag{3.159}$$

where P_u is the momentum conjugate to u, the Hamiltonian turns into the Hamiltonian for a simple harmonic oscillator,

$$H = \frac{1}{\sqrt{3}}\left(\frac{P_u^2}{2M} + \frac{1}{2}Mu^2\right). \tag{3.160}$$

Thus, the ground state is well-defined and all the eigenvalues are known explicitly. This is the expected result once a good gauge choice is made, which we proceed to discuss next.

3.4.2 Fixing the gauge

Here we show that the modified Hamiltonian obtained in the previous section (leading to a well-defined vacuum) is the result of a gauge-fixing procedure. Our discussion is an extension of the calculation in the discrete case considered above [87].

Let σ label the points on the black string. Consider the motion of a small segment of the string of length $2L\delta$ situated at $\sigma = \bar{\sigma}$ under the influence of the rest of the string. L is the physical length of the string ($L = 2\pi R$ in the case of a circular ring of radius R). The string segment experiences a potential

$$\bar{\psi} = \psi(\vec{x}(\bar{\sigma})) = \frac{1}{6\pi^2}\int d\sigma\sqrt{g}\,\frac{\bar{\rho}}{|\vec{x}(\bar{\sigma}) - \vec{x}(\sigma)|^2 + L^2\delta^2},$$

where we regulated the integral by introducing the cutoff δ. This is slghtly different from the cutoff introduced in the previous section - it is dimensionless. Due to the singularity, the leading contribution to this integral comes from the neighborhood of the segment (around $\sigma = \bar{\sigma}$). We obtain

$$\bar{\psi} = \int dl\frac{\bar{\rho}}{l^2 + L^2\delta^2} + \ldots = \frac{2\bar{\rho}}{L\delta} + \ldots = \frac{m}{L^2\delta^2} + \ldots\ , \tag{3.161}$$

where $l = \sqrt{g}|\sigma - \bar{\sigma}|$ is the distance along the string as measured from $\bar{\sigma}$, $m = 2\bar{\rho}L\delta$ is the mass of the string segment, and the dots represent higher-order terms in δ. If we place the origin at $\vec{x}(\bar{\sigma})$ and approximate the segment by a point particle of mass m at distance $r = L\delta$ from the origin, we can have radial motion under which the length of the segment changes, as well as angular motion leaving its length unchanged. Therefore, the line element along its trajectory can be written as

$$ds^2 = -\frac{L^2\delta^2}{m}(dx^0)^2 + \frac{m}{L^2}dL^2 + md\Omega_3^2, \tag{3.162}$$

which is a line element in $AdS_2 \times S^3$. Switching variables to $\bar{\psi}$, we may write

$$ds^2 = -\frac{1}{\bar{\psi}^2}\left((dx^0)^2 - \frac{m}{4}d\bar{\psi}^2\right)^2 + md\Omega_3^2.$$

which describes the motion of a particle in the vicinity of a Reissner-Nordström black hole of mass m, provided $\bar{\psi} \gg 1$. This is satisfied if $m \gg L^2\delta^2$. Since $m = 2\bar{\rho}L\delta$, we need $\bar{\rho}/L \gg \delta$, which is certainly true, since $\delta \to 0$. We may therefore apply our earlier results in subsection 3.2.3 to quantize the system at hand.

Introducing new variables

$$x^{\pm} = x^0 \pm \frac{\sqrt{m}}{2}\psi, \tag{3.163}$$

and their conjugate momenta $p_{\pm}$, we may write the constraint (generator of reparametrizations) as

$$2m\chi = -\psi^2 p_+ p_- + \frac{1}{2}m\psi(p_+ + p_-) + \frac{L^2}{m} = 0, \tag{3.164}$$

where $\vec{L}$ is the angular momentum operator. χ commutes with

$$h = -p_+ - p_- \ , \quad d = 2x^+p_+ + 2x^-p_- \ , \quad k = -(x^+)^2p_+ - (x^-)^2p_- + \frac{1}{2}m^2\psi, \tag{3.165}$$

which form an $SL(2,\mathbb{R})$ algebra.

The simplest and obvious gauge choice is

$$x^0 = \tau. \tag{3.166}$$

leading to a non-relativistic Hamiltonian

$$h \approx \frac{2\bar{\psi}p^2}{m^2},$$

where p is the momentum conjugate to $\bar{\psi}$. The action in this gauge is

$$S = \int d\tau \left(p \frac{d\bar{\psi}}{d\tau} - h \right). \tag{3.167}$$

After integrating over the momentum in the path integral, the action becomes

$$S = \int d\tau \, \frac{1}{2} m \bar{\psi}^2 v^2, \tag{3.168}$$

where $v = \dot{r}$ is the velocity of the center of mass of the segment. Integrating over the entire length of the string, we obtain the total action

$$S_{\text{matter}} = \frac{1}{2} \int d\tau \int dl \bar{\rho} \bar{\psi}^2 v^2. \tag{3.169}$$

This action describes a system without a well-defined vacuum. As explained in section 3.2.3, this is due to the fact that (3.166) is not a good gauge choice. A good gauge-fixing condition is given by

$$\tau(x^+, x^-) \equiv \arctan\left(\frac{\omega x^+ + \omega x^-}{1 - \omega^2 x^+ x^-} \right) = \tau. \tag{3.170}$$

The conjugate momentum to τ in this case is

$$p_\tau = -\frac{1}{2} \left(\frac{p_+}{\partial_+ \tau} + \frac{p_-}{\partial_- \tau} \right) = \frac{1}{2\omega} \left(h + \omega^2 k' \right) \quad , \quad k' = -(x^+)^2 p_+ - (x^-)^2 p_-. \tag{3.171}$$

This momentum, p_τ, is not a good candidate for the Hamiltonian of the system because it is not a conserved quantity (p_τ does not commute with χ: $\{p_\tau, \chi\} \neq 0$). The problem can be fixed, as before, by a gauge transformation, $A \to A + d\Lambda$. From (3.126) and (3.66), we obtain

$$\Lambda = -\frac{m^{3/2}}{4} \ln \frac{1 + \omega^2 (x^+)^2}{1 + \omega^2 (x^-)^2} \tag{3.172}$$

leading to a new conjugate momentum

$$h' = p_\tau - \partial_\tau \Lambda = \frac{1}{2\omega} (h + \omega^2 k). \tag{3.173}$$

Since both h and k commute with χ (conserved quantities), so does h'. It follows that h' is also a conserved quantity. In the non-relativistic limit, we obtain

$$h' \approx \frac{1}{2\omega} \left(\frac{2\bar{\psi}^2 p^2}{m^2} + \frac{1}{2} m^2 \omega^2 \bar{\psi}^2 \right). \tag{3.174}$$

This Hamiltonian has a well-defined ground state thanks to the second term (potential). This is easily seen by the change of variables, $\psi = Cx^2$, under which h' turns into the Hamiltonian of a harmonic oscillator. Note that ω is arbitrary in the gauge-fixing condition (3.170), but no physical quantities, such as the eigenvalues of the Hamiltonian (energy levels), depend on it. The action in this gauge is

$$S_{\text{matter}} = \int dt \int dl \left(\frac{1}{2}\bar{\rho}\psi^2 v^2 - \frac{12\pi^2}{(6\pi^2)^3}\bar{\rho}^3 \right). \tag{3.175}$$

Adding the contribution of the electromagnetic and gravitational fields, we arrive at the action

$$S = \int dt(T - V), \tag{3.176}$$

where

$$V = (\frac{1}{6\pi^2})^3 12\pi^2 \int dl\bar{\rho}^3 \tag{3.177}$$

and T is the Lagrangian in (3.148). This modified action describes a well-defined quantum mechanical system and is the result of a good gauge choice.

As an example, consider a ring-shaped formation of radius R. Its mass is $M = 2\pi R\bar{\rho}$ and its charge is $Q = M$. The kinetic energy part T is obtained in appendix B.2,

$$T = (\frac{1}{6\pi^2})^3 \frac{M^3}{4}\frac{\dot{R}^2}{R^4}. \tag{3.178}$$

The potential is found to be

$$V = (\frac{1}{6\pi^2})^3 12\pi^2 \int dl\bar{\rho}^3 = (\frac{1}{6\pi^2})^3 \frac{3M^3}{R^2}. \tag{3.179}$$

Therefore, $V = K$, on account of (3.155), where K is the generator of special conformal transformations, in agreement with our expectations.

Chapter 4

Quasinormal Modes of Black Holes

To probe the connection between supergravity on anti-de Sitter spacetime and conformal field theories on the boundary [20] further, we shall study quasinormal modes for an AdS black hole. Quasinormal modes describe small perturbations around equilibrium and are expected to correspond to perturbations of the corresponding conformal field theory. This chapter is organized as follows. In section 4.1, we review general properties of AdS Schwarzschild black holes and their quasinormal modes. In section 4.2, we concentrate on 2+1 dimensions, where the radial part of the wave equation can be reduced to a hypergeometric equation whose solution is known. We obtain an exact expression for quasinormal modes. In section 4.3, we investigate the (4+1)-dimensional case, in which quasinormal modes are obtained from solutions to Heun's equation. Unfortunately, the solution cannot be expressed in terms of known functions in a closed form [88]. We develop a perturbative method to obtain the quasinormal modes at high temperature and compare our results to numerical calculations [89, 90]. In sections 4.4, we extend the discussion to higher dimensions. This perturbative method can be used as a future direction to analytically calculate the correct pattern of poles on the complex plane, which contains important information of the correspondence Green function in the conformal field theory, because the quasinormal frequencies turn out to be the poles of the Green function in subsection 2.1.3 and [89, 90].

4.1 Introduction

The metric of an AdS Schwarzschild black hole with mass M, and therefore non-zero Hawking temperature, in $n+1$ dimensions [91] is

$$ds^2 = \left(\frac{r^2}{b^2} + 1 - \frac{\omega_n M}{r^{n-2}}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2} + 1 - \frac{\omega_n M}{r^{n-2}}\right)} + r^2 d\Omega^2. \tag{4.1}$$

The constant ω_n is

$$\omega_n = \frac{16\pi G_n}{(n-1)\mathrm{Vol}(\mathbf{S}^{n-1})}, \tag{4.2}$$

where G_n is the $n+1$ dimensional Newton's constant and $\mathrm{Vol}(\mathbf{S}^{n-1})$ is the volume of a unit $n-1$ sphere. This metric has a horizon at r_H, where

$$\frac{r_H^2}{b^2} + 1 - \frac{\omega_n M}{r_H^{n-2}} = 0. \tag{4.3}$$

The entropy of the black hole is [91]

$$S = \frac{A_H}{4},$$

and the Hawking temperature is

$$T_H = \frac{1}{4\pi}\frac{n r_H^2 + (n-2)b^2}{b^2 r_H}. \tag{4.4}$$

High temperature corresponds to $r_H \to 0$ or $r_H \to \infty$, but only $r_H \to \infty$ is acceptable [91] because of the existence of the black hole. At high temperature, the angular part of the metric $d\Omega^2$ becomes asymptotically flat at infinity, $r^2 d\Omega^2 \to \sum_i x_i^2$. The horizon at high temperature is at

$$r_H = b\left[\frac{\omega_n M}{b^{n-2}}\right]^{1/n}, \tag{4.5}$$

and the Hawking temperature can be written as

$$T_H = \frac{n}{4\pi b}\left[\frac{\omega_n M}{b^{n-2}}\right]^{1/n}. \tag{4.6}$$

The method of analyzing quasinormal modes of an *AdS* Schwarzschild black hole is discussed in [89]. The wavefunction has to be zero at infinity, because the potential diverges in this region. In the massless case, the wavefunction at infinity becomes a constant, which depends on the frequency ω of the mode. Once the constant is set to zero, only a certain set of complex values of ω are allowed. These values of ω are the quasinormal frequencies. In 2+1 dimensions, they appear as poles of the retarded Green's function [92].

For an $(n+1)$-dimensional AdS Schwarzschild metric (4.1), the wave equation is

$$\frac{1}{\sqrt{-g}}\partial_A g^{AB}\sqrt{-g}\partial_B \Phi = m^2 \Phi.$$

In the massless case, Φ is written in the form,

$$\Phi = r^{\frac{1-n}{2}}\Psi(r)Y(\text{angles})e^{-i\omega t}, \tag{4.7}$$

where Y is a harmonic function on the sphere S^{n-1}. Changing the parameter r to

$$dr_* = \frac{dr}{f(r)} \tag{4.8}$$

where $f(r) = \frac{r^2}{b^2} + 1 - \frac{\omega_n M}{r^{n-2}}$, the metric becomes

$$ds^2 = f(r)(-dt^2 + dr_*^2) + r^2 d\Omega_{n-1}^2. \tag{4.9}$$

The wave equation then is written as

$$[\partial_{r_*}^2 + \omega^2 - \tilde{V}(r_*)]\Psi = 0 \tag{4.10}$$

where

$$\tilde{V}(r_*) = \frac{(n-1)(n-3)}{4r^2}f^2 + \frac{c}{r^2}f + m^2 f, \tag{4.11}$$

and $c = l(l+n-1)$ is the total angular momentum on S^{n-1}. The potential $\tilde{V}$ is positive, vanishes at the horizon and diverges at infinity. Near the horizon, Φ behaves like $e^{-i\omega(t\pm r_*)}$. In general, quasinormal modes are defined as solutions which are purely ingoing at the horizon, $\Phi \sim e^{-i\omega(t-r_*)}$ and purely outgoing at infinity, $\Phi \sim e^{-i\omega(t+r_*)}$, so no outside wave is incoming at infinity. This restricts the frequencies to a discrete set of complex values, called quasinormal frequencies. For the massless AdS case, Φ at infinity has to be zero because $\tilde{V}$ diverges there.

Because we are interested only in ingoing modes near the horizon, we change the parameter t to $v = t + r_*$ (ingoing Eddington coordinate), in order to extract some useful information from the wave equation. The metric reads

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega_{n-1}^2. \tag{4.12}$$

The wavefunction can be written in the form

$$\Phi = r^{\frac{1-n}{2}}\Psi(r)Y(\text{angles})e^{-i\omega v}, \tag{4.13}$$

and the radial part of the wave equation becomes

$$f(r)\frac{d^2}{dr^2}\Psi(r) + [f'(r) - 2i\omega]\frac{d}{dr}\Psi(r) - V(r)\Psi(r) = 0, \tag{4.14}$$

where

$$V(r) = \frac{(n^2-1)}{4} + \frac{(n-1)(n-3)+4c}{4r^2} + \frac{(n-1)\omega_n M}{4r^n}. \tag{4.15}$$

V is positive for $n \geq 3$. Let us consider the variable r_*,

$$
\begin{aligned}
r_* &= \int \frac{dr}{f(r)} \approx \int \frac{dr}{\frac{r^2}{b^2} - \frac{\omega_n M}{r^{n-2}}} \qquad \text{at high temperature} \\
&= \frac{b^2}{(\omega_n M b^2)^{1/n}} \int \frac{dy_*}{y_*^n - 1}, \qquad y_* = \frac{(\omega_n M b^2)^{1/n}}{r} \\
&= \frac{b^2}{(\omega_n M b^2)^{1/n}} \sum_{i=1}^{n} \frac{\ln(y_* - a_i)}{\prod_{j \neq i}(a_i - a_j)},
\end{aligned} \tag{4.16}
$$

where a_i is one of the n solutions to $y^n - 1 = 0$. Then the outgoing modes near the horizon, which should be set to zero, behave like

$$
e^{-i\omega(t - r_*)} \approx e^{-i\omega v}(y_* - 1)^{\frac{2i\omega}{4\pi T_H}} \prod_{i=2}^{n} (y_* - a_i)^{\frac{2i\omega}{4\pi T_H} \frac{n}{\prod_{i \neq j}(a_i - a_j)}}, \tag{4.17}
$$

where we used $T_H = \frac{n}{4\pi b}[\frac{\omega_n M}{b^{n-2}}]^{1/n}$, $y_* = 1$ at the horizon and 0 at infinity. From the above equation, the imaginary part of ω has to be negative, in order to make the outgoing modes vanish in this region. This can be shown by multiplying (4.14) with $\bar{\Psi}$ and integrating r from the horizon to infinity,

$$
\int_{r_H}^{\infty} dr \left[\bar{\Psi} \frac{d}{dr}\left(f \frac{d\Psi}{dr} \right) - 2i\omega \bar{\Psi} \frac{d\Psi}{dr} - V|\Psi|^2 \right] = 0. \tag{4.18}
$$

Integrating by parts and throwing away the boundary terms (using $f(r_H) = 0$ and $\bar{\Psi}(r = \infty) = 0$), the above equation becomes

$$
\int_{r_H}^{\infty} dr \left[f|\Psi'|^2 + 2i\omega \bar{\Psi}\Psi' + V|\Psi|^2 \right] = 0. \tag{4.19}
$$

The imaginary part is

$$
\int_{r_H}^{\infty} dr \left[\omega \bar{\Psi}\Psi' + \bar{\omega}\Psi\bar{\Psi}' \right] = 0. \tag{4.20}
$$

Integrating by parts the second term

$$
(\omega - \bar{\omega}) \int_{r_H}^{\infty} dr \bar{\Psi}\Psi' = \bar{\omega}|\Psi(r_H)|^2, \tag{4.21}
$$

and substituting this result back into (4.19), we obtain

$$
\int_{r_H}^{\infty} dr \left[f|\Psi'|^2 + V|\Psi|^2 \right] = -\frac{|\omega|^2 |\Psi(r_H)|^2}{\mathrm{Im}(\omega)}. \tag{4.22}
$$

The left-hand side is positive, therefore Im ω has to be negative, as expected for purely ingoing modes near the horizon. There is no solution for Im $\omega \geq 0$. Only the solution Im $\omega < 0$ decays in time.

4.2 2+1 Dimensions

In this subsection the quasinormal modes and frequencies in 2+1 dimensions are obtained. The wave equation in this case is

$$\frac{1}{r}\partial_r r(\frac{r^2}{b^2} - \omega_2 M + 1)\partial_r\Phi - \frac{1}{\frac{r^2}{b^2} - \omega_2 M + 1}\partial_t^2\Phi + \frac{1}{r^2}\partial_\theta^2\Phi - m^2\Phi = 0. \tag{4.23}$$

Let

$$\Phi = e^{i\omega' t}e^{il'\theta}\Psi(y), \qquad y = (\omega_2 M - 1)\frac{b^2}{r^2}, \tag{4.24}$$

we obtain

$$y^2(y-1)\partial_y(y-1)\partial_y\Psi + \frac{(\omega b)^2}{4}y\Psi + \frac{l^2}{4}y(y-1)\Psi + \frac{m^2b^2}{4}(y-1)\Psi = 0, \tag{4.25}$$

where $0 < y < 1$ and

$$\omega^2 = \frac{\omega'^2}{\omega_2 M - 1} \sim \frac{\omega'^2}{(2\pi b T_H)^2}, \quad l^2 = \frac{l'^2}{\omega_2 M - 1}. \tag{4.26}$$

Let us factor out the singularities at $y = 0, 1$ by setting

$$\Psi = y^n(y-1)^p f(y), \tag{4.27}$$

where

$$n = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + m^2b^2} \quad p = -\frac{i\omega b}{2} \sim \frac{1}{T_H}. \tag{4.28}$$

Note that we choose the minus sign for p which corresponds to the ingoing mode at the horizon. In the massless case, $m = 0$, n is an integer, $n = 0, 1$. Then the wave equation becomes

$$y(y-1)\partial_y^2 f + \{(1+2p)y + 2n(y-1)\}\partial_y f + \left((n+p)^2 + \frac{l^2}{4}\right)f = 0. \tag{4.29}$$

The ingoing solution is the hypergeometric function

$$\begin{aligned} f(y) &= F(n+p+i\frac{l}{2}, n+p-i\frac{l}{2}; 1+2p; 1-y) \\ &= \frac{\Gamma(1+2p)}{\Gamma(n+p+i\frac{l}{2})\Gamma(n+p-i\frac{l}{2})} + o(y). \end{aligned} \tag{4.30}$$

As $y \to 0$, Ψ becomes a constant. To make $\Psi(y=0) = 0$ [93, 94], we need to set the argument of the Gamma function in the denominator to zero or a negative integer,

$$n + p \pm i\frac{l}{2} = -n', \quad n' = 0, 1, 2, \ldots \tag{4.31}$$

Solving for $p = -i(R + iI)$, we obtain

$$R = \pm\frac{l}{2}, \qquad I = -n' - n = -1, -2, \ldots \tag{4.32}$$

From [89], it is suggested that the quasinormal frequencies appear as poles in the retarded Green function, [89, 92, 95, 96]. This is because the Green function is proportional to the absorption coefficient, which is the ingoing flux at the horizon divided by the ingoing flux at infinity, and the flux in 2+1 dimensions is $\mathcal{F} \sim (1 - y)\Psi^*\partial_y\Psi$.

4.3 4+1 Dimensions

In this subsection we develop a perturbative method to calculate the quasinormal modes and the lowest lying quasinormal frequency at high temperature. The wave equation at high temperature is

$$\frac{1}{r^3}\partial_r r^3\left(\frac{r^2}{b^2} - \frac{\omega_n M}{r^2}\right)\partial_r\Phi - \frac{1}{\left(\frac{r^2}{b^2} - \frac{\omega_n M}{r^2}\right)}\partial_t^2\Phi - \frac{l^2}{r^2}\Phi - m^2\Phi = 0. \tag{4.33}$$

Let

$$\Phi = e^{i\omega t}Y(\text{angles})\Psi(r), \tag{4.34}$$

and change the parameter r to y

$$y = \frac{b^2}{y_+ r^2}, \tag{4.35}$$

where $y_+ = b^2/r_H^2$. The radial part of the wave equation then turns to

$$y^3(y^2-1)\partial_y\frac{1}{y}(y^2-1)\partial_y\Psi + W^2 y\Psi + L^2 y(y^2-1)\Psi + m'^2(y^2-1)\Psi = 0, \tag{4.36}$$

where

$$W^2 = \frac{(\omega b)^2/4}{y_+^3(\omega_4 M/b^2)^2} \sim \frac{(\omega b)^2/4}{(\pi b T_H)^2}, \qquad L^2 = \frac{l^2/4}{y_+\omega_4 M/b^2} \sim \frac{l^2/4}{(\pi b T_H)^2}, \tag{4.37}$$

$$m'^2 = \frac{m^2/4}{y_+^2\omega_4 M/b^4} \sim \frac{m^2 b^2}{4}. \tag{4.38}$$

Next, let us factor out the singularities

$$\Psi = y^n(y-1)^p(y+1)^h f, \tag{4.39}$$

where

$$n = 1 \pm \sqrt{1 + \frac{m^2 b^2}{4}}, \qquad p = -i\frac{\omega}{2\pi T_H}, \qquad h = \pm\frac{\omega}{2\pi T_H}. \tag{4.40}$$

Again, we chose the minus sign for p because we are interested in an ingoing mode at the horizon. In the massless case, $m = 0$, we have $n = 0, 2$. The wave equation becomes Heun's equation,

$$\begin{aligned} y(y^2-1)\partial_y^2 f + \left[(-1+2n)(y^2-1) + (1+2p)y(y+1) + (1+2h)y(y-1)\right]\partial_y f \\ + \left[(n+p+h)^2 y - q\right] f = 0, \end{aligned} \tag{4.41}$$

where

$$q = (2n-1)(h-p) - L^2 + 4h^2. \tag{4.42}$$

The solutions to Heun's equation can be written as

$$f(y) = Af_1(y) + Bf_2(y), \tag{4.43}$$

where $f_1(y)$ is power series of y

$$f_1(y) = \sum_r b_r y^r,$$

and $f_2(y)$ is, where we can set $m = 0$

$$\begin{aligned} f_2(y) &= f_1 \int^y \frac{dy}{y^{-1+2n}(y-1)^{1+2p}(y-a)^{1+2h}(f_1)^2} \\ &\sim \frac{d_0}{y^2} + \frac{d_1}{y} + d_2 \ln y + d_3 y + \dots . \end{aligned} \tag{4.44}$$

At infinity ($y \to 0$), the wavefunction must vanish. Therefore, we ought to discard $f_2(y)$. Then we need to study the behavior of the acceptable solution $f_1(y)$ near the horizon ($y \to 1$). It can be written as a superposition

$$f_1(y) = \mathcal{A} f_{\rm in}(1-y) + \mathcal{B} f_{\rm out}(1-y), \tag{4.45}$$

where $f_{\rm in}$ is ingoing and $f_{\rm out}$ is outgoing at the horizon. To obtain quasinormal frequencies, we set $\mathcal{B} = 0$ and solve for the frequencies. Unfortunately, the coefficients $\mathcal{A}$ and $\mathcal{B}$ are hard to calculate [88]. Numerical results have been obtained [89, 90]. Here, we develop a perturbative method of solving the wave equation at high temperatures (or large black hole mass, $\omega_4 M/b^2 \gg 1$). Changing the parameter y to $x = y^2$, the wave equation becomes

$$\begin{aligned} x(1-x)\partial_x^2 f &+ [n - (1+n+p+h)x]\partial_x f - \left(\frac{n+p+h}{2}\right)^2 f \\ &= (p-h)\sqrt{x}\partial_x f - \frac{q}{4\sqrt{x}} f. \end{aligned} \tag{4.46}$$

We shall treat the parameters p and h as small at high temperature, noting that $p, h \sim \frac{1}{T_H}$. To control the perturbative expansion, we need to add and subtract $(p-h)\partial_x f + \frac{q}{4} f$,

$$\begin{aligned} x(1-x)\partial_x^2 f &+ [n+h-p-(1+n+p+h)x]\partial_x f - \left[(\frac{n+p+h}{2})^2 - \frac{q}{4}\right] f \\ &= (p-h)(\sqrt{x}-1)\partial_x f - \frac{q}{4\sqrt{x}}(1-\sqrt{x})f, \end{aligned} \tag{4.47}$$

and treat the right-hand side as perturbation. We calculate the constant $\mathcal{B}$ perturbatively, by expanding f_1 and then looking at its behavior near the horizon,

$$\begin{aligned} f_1 &= F_0 + F_1 + \ldots \\ &= (\mathcal{A}_0 + \mathcal{A}_1 + \ldots) f_{\text{in}} + (\mathcal{B}_0 + \mathcal{B}_1 + \ldots) f_{\text{out}}. \end{aligned}$$

Let

$$H_0 = x(1-x)\partial_x^2 + [n+h-p-(1+n+p+h)x]\partial_x - \left[(\frac{n+p+h}{2})^2 - \frac{q}{4}\right], \tag{4.48}$$

$$H_1 = -(p-h)(\sqrt{x}-1)\partial_x + \frac{q}{4\sqrt{x}}(1-\sqrt{x}), \tag{4.49}$$

The wave equation turns into the set of equations

$$H_0 F_0 = 0, \qquad H_0 F_1 = -H_1 F_0, \quad \ldots .$$

The zeroth order equation reads

$$x(1-x)\partial_x^2 F_0 + [n+h-p-(1+n+p+h)x]\partial_x F_0 - \left[(\frac{n+p+h}{2})^2 - \frac{q}{4}\right] F_0 = 0, \tag{4.50}$$

whose solution is the hypergeometric function

$$F_0 = F(1 + \frac{p+h+\sqrt{q}}{2}, 1 + \frac{p+h-\sqrt{q}}{2}, 2+h-p, x). \tag{4.51}$$

Its behavior near the horizon is easily deduced from standard hypergeometric function identities. We have

$$F_0(x) = \mathcal{A}_0 f_{\text{in}}^{(0)}(x) + \mathcal{B}_0 f_{\text{out}}^{(0)}(x) \tag{4.52}$$

where

$$\mathcal{A}_0 = \frac{\Gamma(2+h-p)\Gamma(-2p)}{\Gamma(1+\frac{h-3p+\sqrt{q}}{2})\Gamma(1+\frac{h-3p-\sqrt{q}}{2})}, \qquad \mathcal{B}_0 = \frac{\Gamma(2+h-p)\Gamma(2p)}{\Gamma(1+\frac{h+p+\sqrt{q}}{2})\Gamma(1+\frac{h+p-\sqrt{q}}{2})} \tag{4.53}$$

$$
\begin{aligned}
f_{\rm in}^{(0)}(x) &= F(1+\frac{h+p+\sqrt{q}}{2},1+\frac{h+p-\sqrt{q}}{2},1+2p,1-x), \\
f_{\rm out}^{(0)}(x) &= (1-x)^{-2p}F(1+\frac{h-3p+\sqrt{q}}{2},1+\frac{h-3p-\sqrt{q}}{2},1-2p,1-x) \quad (4.54)
\end{aligned}
$$

At first order, we have

$$
\begin{aligned}
x(1-x)\partial_x^2 F_1 &+ [n+h-p-(1+n+p+h)x]\partial_x F_1 - \left[(\frac{n+p+h}{2})^2-\frac{q}{4}\right]F_1 \\
&= (p-h)(\sqrt{x}-1)\partial_x F_0 - \frac{q}{4\sqrt{x}}(1-\sqrt{x})F_0,
\end{aligned}
\tag{4.55}
$$

Near the horizon ($x\to 1$), the solution behaves as

$$
F_1(x) \approx \frac{(1-(1-x)^{-2p})}{(-2p)x}\,(p-h)\left[1+\ln 2-\frac{\pi^2}{8}\right]+\dots \tag{4.56}
$$

We deduce

$$
\mathcal{B}_1 = \frac{1}{2p}(p-h)\left[1+\ln 2-\frac{\pi^2}{8}\right]+\dots, \tag{4.57}
$$

where the dots represent second-order contributions. Expanding $\mathcal{B}_0$,

$$
\begin{aligned}
\mathcal{B}_0 &= \frac{\Gamma(2+h-p)\Gamma(2p)}{\Gamma(1+\frac{h+p+\sqrt{q}}{2})\Gamma(1+\frac{h+p-\sqrt{q}}{2})} \\
&= \frac{1}{2p}\left\{1+(-1+\frac{3}{4}\frac{\pi^2}{6})(p-h)+\dots\right\},
\end{aligned}
\tag{4.58}
$$

where we used

$$
\Gamma(a+\delta)=\Gamma(a)+\delta\Gamma(a)\psi(a)+\frac{\delta^2}{2}\Gamma(a)(\psi^2(a)+\psi'(a))+\dots\ ,
$$

and $\psi'(1)=\frac{\pi^2}{6}$, $q=-3p+3h-L^2+4h^2$, we obtain the coefficient $\mathcal{B}=\mathcal{B}_0+\mathcal{B}_1+\dots$ at first-order,

$$
\mathcal{B} = \frac{1}{2p}\left\{1+\ln 2\,(p-h)+\dots\right\}. \tag{4.59}
$$

Then setting $\mathcal{B}=0$ and letting $p=-i(R+iI)$, $h=\pm(R+iI)$, we obtain

$$
4R=\pm 2.89, \qquad 4I=-2.89, \tag{4.60}
$$

to be compared with the numerical results [89, 90],

$$
4R=\pm 3.119452, \qquad 4I=-2.746676. \tag{4.61}
$$

Thus, we obtain good agreement for the lowest lying quasinormal frequency.

4.4 Higher dimensions

In this subsection we apply the method in the previous subsection to the higher dimensions, first any odd and later any even dimensions and then specialize to the (6+1)-dimensional case ($j = 3$). We start with an arbitrary odd dimension, $d = 2j+1$. The wave equation at high temperature is

$$\frac{1}{r^{2j-1}}\partial_r \left(\frac{r^2}{b^2} - \frac{\omega_n M}{r^{2j-2}} \right) r^{2j-1}\partial_r \Phi - \frac{1}{\frac{r^2}{b^2} - \frac{\omega_n M}{r^{2j-2}}} \partial_t^2 \Phi + \frac{1}{\sqrt{g}}\partial_{\theta_i} g^{\theta_i\theta_j}\sqrt{g}\partial_{\theta_j}\Phi = m^2\Phi. \quad (4.62)$$

Let

$$\Phi = e^{i\omega t} Y(\text{angles})\Psi(y) \ , \ y = (\frac{\omega_n M}{b^{2j-2}})^{1/j}\frac{b^2}{r^2}. \quad (4.63)$$

We have $0 < y < 1$, the horizon (infinity) being at $y = 1$ ($y = 0$). By changing the parameter r to y, the wave equation becomes

$$\begin{aligned} 0 &= y^{j+1}(y^j - 1)\partial_y \frac{(y^j-1)}{y^{j-1}}\partial_y\Psi + W^2 y\Psi + L^2 y(y^j-1)\Psi + \frac{m^2b^2}{4}(y^j-1)\Psi \\ &= y^2(y^j-1)^2\partial_y^2\Psi + y[j(y^j-1) + (y^j-1)^2]\partial_y\Psi + W^2y\Psi + L^2y(y^j-1)\Psi \\ &\quad + \frac{m^2b^2}{4}(y^j-1)\Psi \end{aligned} \quad (4.64)$$

where

$$W^2 = \frac{\omega^2 b^2}{4}(\frac{b^{2j-2}}{\omega_n M})^{1/j} = \left(\frac{wb/2}{2\pi b T_H/j}\right)^2, \quad L^2 = \frac{l^2}{4}(\frac{b^{2j-2}}{\omega_n M})^{1/j} = \left(\frac{l/2}{2\pi b T_H/j}\right)^2, \quad (4.65)$$

and l^2 is the total angular momentum in $2j+1$ dimensions. We have singularities at $y = 0, a_k$, where a_k are the roots of $y^j = 1$ ($k = 1, \ldots, j$). We may write

$$y^j - 1 = (y - a_1)\cdots(y - a_j), \qquad a_k = e^{i(k-1)\cdot 2\pi/j}. \quad (4.66)$$

To factor out the singularities, we let

$$\Psi = y^n(y - a_1)^{p_1}(y - a_2)^{p_2}\cdots(y - a_j)^{p_j} f, \quad (4.67)$$

where n and p_i satisfy the equations

$$n^2 - jn - \frac{m^2b^2}{4} = 0, \qquad n = \frac{j}{2} \pm \frac{j}{2}\sqrt{1 + \frac{m^2b^2}{j^2}}, \quad (4.68)$$

$$a_i[\prod_{k\neq i}^{j}(a_i - a_k)^2](p_i^2 - p_i) + j[\prod_{k\neq i}^{j}(a_i - a_k)]p_i + W^2 = 0, \quad (4.69)$$

respectively. By using the equation

$$\frac{y^j-1}{y-a_i}=\prod_{k\neq i}^{j}(y-a_k)=y^{j-1}+a_iy^{j-2}+\ldots+a_i^{j-2}y+a_i^{j-1}, \tag{4.70}$$

and

$$\prod_{k\neq i}^{j}(a_i-a_k)=ja_i^{j-1}, \tag{4.71}$$

then

$$p_k=\pm\frac{W}{j}i\sqrt{a_k}=\pm\frac{W}{j}e^{\frac{(k-1)}{j}i\pi+i\frac{\pi}{2}}. \tag{4.72}$$

We choose the minus sign for p_1 corresponding to the ingoing mode at the horizon, $y=a_1=1$. After using the identities

$$a_1a_2...a_j=1, \qquad \sum_i^j a_i=a_1+a_2+...+a_j=0$$

$$\sum_{i<k}^{j}a_ia_k=a_1a_2+a_1a_3+...+a_1a_j+a_2a_3+a_2a_4+...+a_2a_j+....+a_{j-1}a_j=0$$

$$\sum_{i<k<r}^{r}a_ia_ka_r=0, \qquad \sum_{i_1<i_2<...<i_{j-1}}a_{i_1}a_{i_2}...\ a_{j-1}=0, \tag{4.73}$$

and factoring out the singularities, the wave equation becomes

$$\begin{aligned}
&y(y^j-1)\partial_y^2f+\left(j+(1+2n)(y^j-1)+2y\sum_i p_i\frac{y^j-1}{y-a_i}\right)\partial_yf\\
&+2y\sum_{i<k}p_ip_k\frac{y^j-1}{(y-a_i)(y-a_k)}f+\sum_i p_i^2\frac{y^j-1}{y-a_i}f\\
&+\sum_i a_i\frac{p_i^2-p_i}{y-a_i}\left[\frac{y^j-1}{y-a_i}-\prod_{k\neq i}(a_i-a_k)\right]f\\
&+L^2f+n^2y^{j-1}f+2n\sum_i p_i\frac{y^j-1}{y-a_i}f=0.
\end{aligned} \tag{4.74}$$

By using the equations

$$\frac{y^j-1}{y-a_i} \;=\; \prod_{k\neq i}^{j}(y-a_k)=y^{j-1}+a_iy^{j-2}+\ldots+a_i^{j-2}y+a_i^{j-1}$$

$$\frac{y^j-1}{(y-a_i)(y-a_k)} = y^{j-2}+(a_i+a_k)y^{j-3}+(a_i^2+a_ia_k+a_k^2)y^{j-4}$$
$$+(a_i^3+a_i^2a_k+a_ia_k^2+a_k^3)y^{j-5}+\dots$$
$$+\left(a_i^{j-3}+a_i^{j-4}a_k+\dots+a_ia_k^{j-4}+a_k^{j-3}\right)y$$
$$+\left(a_i^{j-2}+a_i^{j-3}a_k+\dots+a_ia_k^{j-3}+a_k^{j-2}\right)$$

$$\frac{1}{y-a_i}\left\{\frac{y^j-1}{y-a_i}-\prod_{k\neq i}(a_i-a_k)\right\} = y^{j-2}+2a_iy^{j-3}+3a_i^2y^{j-4}+\dots$$
$$+(j-3)a_i^{j-4}y^2+(j-2)a_i^{j-3}y+(j-1)a_i^{j-2}, \tag{4.75}$$

the wave equation can be written as

$$\begin{aligned} 0 = \; & y(y^j-1)\partial_y^2 f \\ & + \left\{ j+(1+2n)(y^j-1)+2\sum_i p_i[y^j+a_iy^{j-1}+\dots+a_i^{j-1}y] \right\}\partial_y f \\ & + \left(n+\sum_i p_i\right)^2 y^{j-1}f+2y\sum_{i\neq k}A_{ik}p_ip_kf \\ & + \sum_i \left\{B_i(p_i^2+2np_i)+C_i(p_i^2-p_i)+L^2\right\} f\,, \end{aligned} \tag{4.76}$$

where

$$\begin{aligned} A_{ik} &= (a_i+a_k)y^{j-3}+(a_i^2+a_ia_k+a_k^2)y^{j-4}+\dots \\ & \quad +(a_i^{j-2}+a_i^{j-3}a_k+\dots+a_ia_k^{j-3}+a_k^{j-2}) \\ B_i &= a_iy^{j-2}+a_i^2y^{j-3}+\dots+a_i^{j-2}y+a_i^{j-1} \\ C_i &= a_iy^{j-2}+2a_i^2y^{j-3}+\dots+(j-2)a_i^{j-2}y+(j-1)a_i^{j-1}, \end{aligned} \tag{4.77}$$

Next, we let $x=y^j$, and obtain

$$\begin{aligned} 0 = \; & x(1-x)\partial_x^2 f+[\frac{2n}{j}-(1+\frac{2n+\sum_i 2p}{j})x]\partial_x f-(\frac{n+\sum_i p_i}{j})^2 f \\ - \; & 2\sum_i \frac{p_i}{j}[a_iy^{j-1}+a_i^2y^{j-2}+\dots+a_i^{j-2}y^2+a_i^{j-1}y\;]\partial_y f \\ - \; & \frac{2}{j^2y^{j-1}}\sum_{i\neq k}p_ip_k\{(a_i+a_k)y^{j-2}+(a_i^2+a_ia_k+a_k^2)y^{j-3}+\dots \\ & \dots+(a_i^{j-2}+a_i^{j-3}a_k+\dots+a_ia_k^{j-3}+a_k^{j-2})y\;\}f \end{aligned}$$

$$
\begin{aligned}
- & \frac{1}{j^2 y^{j-1}} \sum_i (p_i^2 + 2np_i)\{a_i y^{j-2} + a_i^2 y^{j-3} + \ldots + a_i^{j-2} y + a_i^{j-1}\} f \\
- & \frac{1}{j^2 y^{j-1}} \sum_i (p_i^2 - p_i)\{a_i y^{j-2} + 2a_i^2 y^{j-3} + \ldots + (j-2) a_i^{j-2} y + (j-1) a_i^{j-1}\} f \\
- & \frac{L^2}{j^2 y^{j-1}} f.
\end{aligned} \tag{4.78}
$$

There are two solutions to this equation,

$$
\begin{aligned}
f_1 &= \sum_r b_r y^{r+j} \\
f_2 &= \sum_{n \neq j} \frac{d_n}{n-j} f_1 + d_j f_1 \ln y = \sum_r c_r y^r + d_j f_1 \ln y.
\end{aligned} \tag{4.79}
$$

In 6+1 dimensions ($j = 3$), the wave equation becomes

$$
\begin{aligned}
x(1-x)\partial_x^2 f &+ [\frac{2n}{3} - (1 + \frac{2n + 2p_k}{3})x]\partial_x f - (\frac{n + p_k}{3})^2 f \\
&= \frac{2}{3}[(p_k a_k) y^2 + (p_k a_k^2) y]\partial_x f + \frac{1}{9y}\left[(-1 + 2n)(p_k a_k) - 2P_2\right] f \\
&\quad + \frac{1}{9y^2}\left[(-2 + 2n)(p_k a_k^2) - W^2 + L^2\right] f,
\end{aligned}
$$

where

$$
P_2 = p_1 p_2 a_3 + p_2 p_3 a_1 + p_3 p_1 a_2,
$$

and a summation over $k = 1, 2, 3$ is implied.

Quasinormal modes in 6+1 dimensions have been calculated numerically [89, 90]. We shall calculate them analytically by extending our perturbative method in 4+1 dimensions that we presented in the previous section. To control the perturbative expansion, we need to add and subtract the term

$$
-P_1 \partial_x f + \frac{1}{9}\left[(-1 + 2n)(p_k a_k) - 2P_2\right] f + \frac{1}{9}\left[(-2 + 2n)(p_k a_k^2) - W^2 + L^2\right] f,
$$

where

$$
P_1 = -\frac{2}{3}[(p_k a_k) + (p_k a_k^2)]
$$

and write the wave equation in the form

$$
(H_0 + H_1) f = 0, \tag{4.80}
$$

where

$$H_0 = x(1-x)\partial_x^2 + \left[n + P_1 - \left(1 + \frac{2n+2p_k}{3}\right)x\right]\partial_x - \left[\left(\frac{n+p_k}{3}\right)^2 - q\right], \quad (4.81)$$

and the perturbation is

$$-H_1 = \left[A(y^2-1) + B(y-1)\right]\partial_x + \left[C(y^{-1}-1) + D(y^{-2}-1)\right], \quad (4.82)$$

where

$$q = -\frac{1}{9}\left[(-1+2n)(p_k a_k) + (-2+2n)(p_k a_k^2) - 2P_2 - W^2 + L^2\right],$$

$$A = \frac{2}{3}(p_k a_k), \qquad B = \frac{2}{3}(p_k a_k^2),$$

$$C = \frac{1}{9}\left[(-1+2n)(p_k a_k) - 2P_2\right], \quad D = \frac{1}{9}\left[(-2+2n)(p_k a_k^2) - W^2 + L^2\right]. \quad (4.83)$$

The zeroth order equation, $H_0 F_0 = 0$, is a hypergeometric equation whose solutions in the massless case (in which $n = j = 3$) are

$$\begin{aligned} F_0 &= F(1 + \frac{p_k}{3} + \sqrt{q}, 1 + \frac{p_k}{3} - \sqrt{q}, 3 + P_1, x), \\ G_0 &= F_0 \int^x \frac{W}{F_0^2}, \end{aligned} \quad (4.84)$$

where W is the Wronskian

$$W(F_0, G_0) \sim x^{-(2+P_1)}(1-x)^{-(1+2p_1)}. \quad (4.85)$$

The first-order equation $H_0 F_1 = -H_1 F_0$ may be solved by using

$$F_1 = -G_0 \int_0^x dx' \frac{F_0}{x'(1-x')W} H_1 F_0 + F_0 \int_0^x dx' \frac{G_0}{x'(1-x')W} H_1 F_0. \quad (4.86)$$

After some algebra, we obtain

$$F_1 = \mathcal{C}\, F_0 + \dots\ , \quad (4.87)$$

where

$$\begin{aligned} \mathcal{C} &= \left(-3\ln 3 + \frac{\pi}{\sqrt{3}} + 1\right)A + \left(-\frac{3}{4}\ln 3 - \frac{\pi}{4\sqrt{3}} + 1\right)B \\ &\quad + \left(\frac{9}{2}\ln 3 - \frac{\sqrt{3}}{2}\pi - \frac{\pi^2}{6}\right)C + \left(\frac{9}{4}\ln 3 + \frac{\sqrt{3}}{4}\pi - \frac{\pi^2}{6}\right)D. \end{aligned} \quad (4.88)$$

After expressing it as a superposition of ingoing and outgoing waves at the horizon, we obtain the first-order correction to the coefficient $\mathcal{B}$,

$$\mathcal{B}_1 = \frac{1}{2p_1}\,\mathcal{C}$$

Therefore,

$$\begin{aligned}
\mathcal{B} &= \frac{\Gamma(2+P_1)\Gamma(2p_1)}{\Gamma(1+\frac{p_k}{j}-\sqrt{q})\Gamma(1+\frac{p_k}{j}+\sqrt{q})} + \mathcal{B}_1 + \dots \\
&= \frac{1}{2p_1}\left\{1 + P_1 - q + \frac{\pi^2}{6} + \mathcal{C} + \dots\right\} \\
&= \frac{1}{2p_1}\left\{1 - i\left(\frac{1}{2}\ln 3 - \frac{\pi}{6\sqrt{3}}\right)\frac{W}{3} + \left(\frac{1}{2}\ln 3 + \frac{\pi}{6\sqrt{3}}\right)\left(-1 \pm \frac{\sqrt{3}}{2}\right)\frac{W}{3} + \dots\right\}
\end{aligned} \tag{4.89}$$

where A, B, C, D, P_1 and q are from (4.83), i.e., where the two sign choices correspond to the sign choices in (4.72). We have to choose the same sign in p_2 and p_3 in order to obtain non-vanishing real part of the frequency. Writing $2W = R + iI$, and setting $\mathcal{B} = 0$, we obtain the quasinormal modes at first order. The lowest lying frequency is

$$\begin{aligned}
R &= \pm\frac{6\sqrt{3}(\frac{1}{2}\ln 3+\frac{\pi}{6\sqrt{3}})}{(\ln 3)^2+3[\frac{1}{2}\ln 3+\frac{\pi}{6\sqrt{3}}]^2}, \qquad & I &= -\frac{6\ln 3}{(\ln 3)^2+3(\frac{1}{2}\ln 3+\frac{\pi}{6\sqrt{3}})^2} \\
&= \pm 2.616381 & &= -1.948673,
\end{aligned} \tag{4.90}$$

to be compared with the numerical result [89]

$$R = 5.008, \qquad I = -2.612. \tag{4.91}$$

Evidently, we need to include higher orders in perturbation theory to obtain a better agreement.

The above procedure can also be applied to even dimensions with minor modifications. In 3+1 dimensions, we arrive at the following value of the lowest-lying quasinormal frequency, $W^{(3+1)} = R^{(3+1)} + iI^{(3+1)}$,

$$\begin{aligned}
R^{(3+1)} &\simeq \pm\frac{\sqrt{3}[\frac{1}{2}\ln 3-\frac{\pi}{6\sqrt{3}}]}{(\ln 3)^2+3[\frac{1}{2}\ln 3-\frac{\pi}{6\sqrt{3}}]^2}, \qquad & I^{(3+1)} &\simeq -\frac{\ln 3}{(\ln 3)^2+3[\frac{1}{2}\ln 3-\frac{\pi}{6\sqrt{3}}]^2} \\
&= \pm 0.923378, & &= -2.371139,
\end{aligned} \tag{4.92}$$

to be compared with the numerical result [89]

$$R^{(3+1)} = 1.849534, \qquad I^{(3+1)} = -2.663856. \tag{4.93}$$

Again, we see that we need to include higher orders in perturbation theory to obtain a better agreement, or improve on the zeroth-order approximation.

Chapter 5

Discussion

In chapter 2, we calculated absorption coefficients for non-extremal rotating D3-branes and found that they were functions of two temperature parameters. This was similar to the case of Kerr-Newman black holes, reviewed in section 2.2, even though the geometry of rotating D3-branes, $O(4, 20)$, is considerably more complicated than the geometry of Kerr-Newman black holes, which is based on $SO(4) \sim SU_L(2) \times SU_R(2)$. The two different temperature parameters suggest the existence of two distinct ensembles in the system off extremality. It is possible that away from the extremality and at low temperatures, supersymmetry is broken and the duality between supergravity and conformal field theory is destroyed. It would be interesting to examine how the symmetry of D3-branes affects the thermodynamic properties of the system away from extremality. One could then explore the underlying physics based on the symmetry group $O(11 - D, 27 - D)$ for a general dimension D (we considered the case $D = 7$). In [98], our method discussed in section 2.3 was used to study scattering of other fields, such as a scalar and a vector arising from a two-form field, an antisymmetric tensor from a four-form field and a two-form from an antisymmetric tensor in the near extremal limit of non-rotating branes. In [99], the radial part of the scalar wave equation was simplified by a transformation which was obtained by considering the singularities of the system. The new reduced equation was then solved perturbatively and the zeroth order solution was shown to be a Hankel function. It would be interesting to apply the method of [99] to the system of rotating 3-branes discussed in section 2.3, also extending the results to other fields, such as the ones discussed in [98]. This should deepen our understanding of the physics giving rise to the thermodynamic properties of non-extremal rotating branes and the attendant AdS/CFT correspondence. This is also expected to shed light on the emergence of extended black objects (black holes, branes, etc) in superstring theory.

In chapter 3, we extended previous results on the quantization of a (discrete) system of maximally-charged black holes to the continuous case. We showed that a careful implementation of the gauge-fixing procedure on path integrals leads to a modification of the naive Hamiltonian by the addition of the potential K (generator of special conformal transformations). We obtained an explicit expression for K in the case of a ring-shaped black string and showed that the resulting Hamiltonian was equivalent to the Hamiltonian of a harmonic oscillator. Our calculations were performed in the extremal case where a static solution to the field equations exists, permitting the study of the non-relativistic limit. It would be interesting to go away from extremality, and extend our results to possibly more realistic cases, such as a Schwarzschild black hole.

In chapter 4, we offered a perturbative method to calculate the quasinormal modes and frequencies in an AdS Schwarzschild spacetime. Our results in 4+1 dimensions were in agreement with numerical results for the low-lying frequencies. In higher dimensions, the convergence of the perturbative expansion appears to be slower. As a possible extension of our work, one should consider higher orders in perturbation theory. It would also be interesting to explore the possibility of improving on the choice of the zeroth order wave equation. Another direction of interest is the application of our method to spacetimes of more immediate physical relevance (e.g., Schwarzschild black holes, based on the approximation to the wave equation discussed in [57]). Finally, a more complete analysis should include a study of the dependence of the quasinormal modes on the other parameters of the scattered field (e.g., mass, charge, angular momentum).

The work that we have carried out amounts to a small part of the current intensive research on black hole scattering. There is a lot of mystery surrounding the physics of black holes, which arise as solutions to the classical field equations of General Relativity, because it is not clear how the Heisenberg Uncertainty Principle applies to their strong gravitational field. Advances have been made recently in the quantitative understanding of the microscopic origin of some of their properties (such as entropy) from superstring theory. Even so, basic properties are barely understood, such as entanglement information, the information loss paradox and the interior entropy. Today, black holes are undoubtedly the ultimate arena to test our frontier knowledge. They challenge our imagination and may change our view of Nature by revolutionizing our understanding of basic physical principles in the years to come.

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Appendix

Appendix A

Derivation of Metric Tensors of Rotating Charged Branes

We summarize the procedure of obtaining the metric of D-dimensionally compactified rotating charged branes in ten dimensions following [59, 60, 63, 64, 65, 66, 67, 68]. From the NS-NS sector of type-IIA superstring theory compactified on a $(10-D)$-torus, the low-energy effective massless field action is obtained as [16, 65, 66]

$$\begin{aligned} S &= \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-\hat{G}}e^{-\hat{\Phi}}\left[R_{\hat{G}}+\hat{G}^{MN}\partial_M\hat{\Phi}\partial_N\hat{\Phi}-\frac{1}{12}\hat{H}_{MNP}\hat{H}^{MNP}\right. \\ &\qquad\left.-\frac{1}{4}\hat{F}^I_{MN}\hat{F}^{I\ MN}\right]. \end{aligned} \tag{A.1}$$

G_{10} is the ten-dimensional Newton's constant, $\hat{G}=\det\hat{G}_{MN}$ $(M,N=0,1,\ldots,9)$, $R_{\hat{G}}$ is the Ricci scalar of the metric $\hat{G}_{MN}$, $\hat{\Phi}$ is the dilaton field, and $\hat{H}_{MN}$, $\hat{F}_{MN}$ are

$$\begin{aligned} \hat{H}_{MN} &= \partial_M\hat{B}_{NP}-\frac{1}{2}A^I_MF^I_{NP}+\text{ cyclic permutations in } M,\ N,\ P \\ F^I_{MN} &= \partial_MA^I_N-\partial_NA^I_M, \end{aligned} \tag{A.2}$$

where $\hat{B}_{MN}$ is an anti-symmetric tensor field and A^I_M are $U(1)$ gauge potentials $(I=1,\ldots,16)$.

To compactify the extra $10-D$ dimensions, we employ an Abelian Kaluza-Klein *ansatz* in ten dimensions [63, 65],

$$\hat{G}_{MN}=\begin{pmatrix} e^{a\varphi}g_{\mu\nu}+G_{mn}A^{(1)m}_\mu A^{(1)n}_\nu & A^{(1)m}_\mu G_{mn} \\ A^{(1)n}_\nu G_{mn} & G_{mn}\end{pmatrix}, \tag{A.3}$$

where $A^{(1)m}_\mu$ are D-dimensional Kaluza-Klein $U(1)$ gauge potentials ($\mu=0,1,\ldots,D-1$ and $m=1,\ldots,10-D$), φ is the dilaton field $\varphi=\hat{\Phi}-\frac{1}{2}\ln\det G_{mn}$, and $a=\frac{2}{D-2}$. It is convenient to define a new set of $(36-2D)$ $U(1)$ gauge potentials $\mathcal{A}^i_\mu$ by

$$\begin{aligned}
\mathcal{A}^i_\mu &= (A^{(1)m}_\mu, A^{(2)}_{\mu\ m}, A^{(3)I}_\mu) \\
A^{(2)}_{\mu\ m} &= \hat{B}_{\mu m} + \hat{B}_{mn}A^{(1)n}_\mu + \frac{1}{2}\hat{A}^I_m A^{(3)I}_\mu \\
A^{(3)I}_\mu &= \hat{A}^I_\mu - \hat{A}^I_m A^{(1)I}_\mu
\end{aligned} \tag{A.4}$$

and a new anti-symmetric tensor $B_{\mu\nu}$ by

$$B_{\mu\nu} = \hat{B}_{\mu\nu} - \hat{B}_{\mu\nu}A^{(1)m}_\mu A^{(1)n}_\nu - \frac{1}{2}(A^{(1)m}_\mu A^{(2)}_{\nu m} - A^{(1)m}_\nu A^{(2)}_{\mu m}). \tag{A.5}$$

The theory possesses an $O(10-D, 26-D)$ symmetry. Introducing the matrix

$$M = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}a^T \\ -C^T G^{-1} & G + C^T G^{-1} C + a^T a & C^T G^{-1} a^T + a^T \\ -aG^{-1} & aG^{-1}C + a & I + aG^{-1}a^T \end{pmatrix}. \tag{A.6}$$

where $G = [\hat{G}_{mn}]$, $C = [\frac{1}{2}\hat{A}^I_m A^I_n + \hat{B}_{mn}]$ and $a = [\hat{A}^I_m]$, we may write the D-dimensionally reduced action (A.1) as

$$\begin{aligned}
S \sim \frac{1}{16\pi G_D}\int d^D x\sqrt{-g}\Big[R_g &- \frac{1}{D-2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{8}g^{\mu\nu}\mathrm{Tr}(\partial_\mu ML\partial_\nu ML) \\
&- \frac{1}{12}e^{-2a\varphi}g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'} - \frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}\mathcal{F}^i_{\mu\nu}(LML)_{ij}\mathcal{F}^j_{\mu'\nu'}\Big],
\end{aligned} \tag{A.7}$$

where $G_D = (2\pi\sqrt{\alpha'})^{D-10}G_{10}$ is the Newton's constant in D dimensions, α' is related to the string tension (we let the radius of each compactified dimension equal $\sqrt{\alpha'}$), $g = \det g_{\mu\nu}$, R_g is the Ricci scalar of the metric $g_{\mu\nu}$, $\mathcal{F}^i_{\mu\nu} = \partial_\mu\mathcal{A}^i_\nu - \partial_\nu\mathcal{A}^i_\mu$ is the field strength, and

$$H_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} - \frac{1}{2}\mathcal{A}^i_\mu L_{ij}\mathcal{F}^j_{\nu\rho}) + \text{ cyclic permutations in } \mu,\ \nu,\ \rho.$$

$L \in O(10-D, 26-D)$ is the matrix

$$L = \begin{pmatrix} 0 & I_{10-D} & 0 \\ I_{10-D} & 0 & 0 \\ 0 & 0 & I_{26-D} \end{pmatrix}. \tag{A.8}$$

The matrices M and L satisfy

$$MLM^T = L, \qquad M^T = M. \tag{A.9}$$

The following $O(10-D, 26-D)$ transformation leaves the action (A.7) invariant,

$$M \to \Omega M \Omega^T, \qquad \mathcal{A}^i_\mu \to \Omega_{ij}\mathcal{A}^j_\mu, \qquad g_{\mu\nu} \to g_{\mu\nu}, \qquad B_{\mu\nu} \to B_{\mu\nu}, \tag{A.10}$$

where Ω satisfies

$$\Omega^T L \Omega = L. \tag{A.11}$$

By varying $H_{\mu\nu}$ in the action, we obtain a simple equation of motion

$$D_\mu(e^{-2a\varphi}H^{\mu\nu\rho}) = 0, \tag{A.12}$$

where D_μ is the covariant derivative of $g_{\mu\nu}$. The solution to the above equation can be written in the form

$$H^{\mu\nu\rho} = -\frac{e^{2a\varphi}}{2!\sqrt{-g}}\epsilon^{\mu\nu\rho\lambda\sigma}F_{\lambda\sigma}, \tag{A.13}$$

where $F_{\mu\nu}$ is the field strength of the gauge potential A_μ. The rest of the equations of motion can be obtained from the action (A.7) by varying the fields $g_{\mu\nu}$, $\mathcal{A}^i_\mu$ and φ. They are much more difficult to solve [65]. A general solution can be constructed by taking a known special solution, in this case the uncharged D-dimensional rotating metric [67], and boosting it by a certain matrix. One arrives at the D-dimensional rotating (Kerr) metric

$$\begin{aligned} ds^2 &= -\frac{(\Delta - 2N)}{\Delta}dt^2 + \frac{\Delta}{\prod_{i=1}^{[\frac{D-1}{2}]}(r^2+l_i^2) - 2N}dr^2 \\ &+ (r^2 + l_1^2\cos^2\theta + K_1\sin^2\theta)d\theta^2 \\ &+ (r^2 + l_{i+1}^2\cos^2\psi_i + K_{i+1}\sin^2\psi_i)\cos^2\theta\cos^2\psi_1 \ ...\cos^2\psi_{i-1}d\psi_i^2 \\ &- 2(l_{i+1}^2 - K_{i+1})\cos\theta\sin\theta\cos^2\psi_1 \ ...\cos^2\psi_{i-1}\cos\psi_i\sin\psi_i d\theta d\psi_i \\ &- 2\sum_{i<j}(l_j^2 - K_j)\cos^2\theta\cos^2\psi_1 \ ...\cos^2\psi_{i-1}\cos\psi_i \ \sin\psi_i ... \\ &\qquad \cos^2\psi_{j-1}\cos\psi_j\sin\psi_j d\psi_i\phi_j \\ &+ \frac{\mu_i^2}{\Delta}[(r^2+l_i^2)\Delta + 2l_i^2\mu_i^2N]d\phi_i^2 - \frac{4l_i\mu_i^2N}{\Delta}dtd\phi_i + \sum_{i<j}\frac{4l_il_j\mu_i^2\mu_j^2N}{\Delta}d\phi_id\phi_j. \end{aligned} \tag{A.14}$$

In even dimensions,

$$\begin{aligned} \Delta &= \alpha^2\prod_{i=1}^{\frac{D-2}{2}}(r^2+l_i^2) + r^2\sum_{i=1}^{\frac{D-2}{2}}\mu_i^2(r^2+l_1^2)...(r^2+l_{i-1}^2)(r^2+l_{i+1}^2)...(r^2+l_{\frac{D-2}{2}}^2), \\ K_i &= l_{i+1}^2\sin^2\psi_i + ... + l_{\frac{D-2}{2}}^2\cos^2\psi_i...\cos^2\psi_{\frac{D-6}{2}}\sin^2\psi_{\frac{D-4}{2}}, \qquad N = mr, \\ \mu_1 &= \sin\theta, \quad \mu_2 = \cos\theta\sin\psi_1, \quad ..., \quad \mu_{\frac{D-2}{2}} = \cos\theta\cos\psi_1...\cos\psi_{\frac{D-6}{2}}\sin\psi_{\frac{D-4}{2}}\ , \\ \alpha &= \cos\theta\cos\psi_1...\cos\psi_{\frac{D-4}{2}}, \end{aligned} \tag{A.15}$$

where the indices in ϕ and ψ, $i, j = 1, \ldots, \frac{D-2}{2}$, whereas in odd dimensions,

$$\begin{aligned}
\Delta &= r^2 \sum_{i=1}^{\frac{D-1}{2}} \mu_i^2 (r^2+l_1^2)...(r^2+l_{i-1}^2)(r^2+l_{i+1}^2)...(r^2+l_{\frac{D-1}{2}}^2), \qquad N = mr^2 \\
K_i &= l_{i+1}^2 \sin^2\psi_i + ... + l_{\frac{D-3}{2}}^2 \cos^2\psi_i ... \cos^2\psi_{\frac{D-7}{2}} \sin^2\psi_{\frac{D-5}{2}} \\
&\qquad + l_{\frac{D-1}{2}}^2 \cos^2\psi_i ... \cos^2\psi_{\frac{D-5}{2}}, \\
\mu_1 &= \sin\theta, \quad \mu_2 = \cos\theta\sin\psi_1, \quad ..., \quad \mu_{\frac{D-3}{2}} = \cos\theta\cos\psi_1 ... \cos\psi_{\frac{D-7}{2}} \sin\psi_{\frac{D-5}{2}} \;, \\
\mu_{\frac{D-1}{2}} &= \cos\theta\cos\psi_1 ... \cos\psi_{\frac{D-5}{2}}. \qquad \text{(A.16)}
\end{aligned}$$

where $i, j = 1, \ldots \frac{D-1}{2}$. The D-dimensional Kerr solution is parameterized by the ADM mass m and angular momenta l_i. It is a solution to the Einstein equations similar to the four-dimensional Kerr solution [69, 70, 71]. In $D = 4$ and with vanishing angular momenta, $l_i = 0$, it reduces to the Schwarzschild metric.

Next, we introduce charges by transforming the above uncharged metric to a charged one. This is done by boosting with a matrix carrying two electric charges parameterized by parameters δ_1 and δ_2 [64, 59],

$$\begin{aligned}
\Omega &= \Omega_1 \Omega_2 \;, \\
\Omega_1 &= \begin{pmatrix} \cosh\delta_1 & \cdot & \cdot & \cdot & -\sinh\delta_1 & \cdot \\ \cdot & I_{9-D} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cosh\delta_1 & \cdot & \cdot & \sinh\delta_1 \\ \cdot & \cdot & \cdot & I_{25-D} & \cdot & \cdot \\ -\sinh\delta_1 & \cdot & \cdot & \cdot & \cosh\delta_1 & \cdot \\ \cdot & \cdot & \sinh\delta_1 & \cdot & \cdot & \cosh\delta_1 \end{pmatrix} \\
\Omega_2 &= \begin{pmatrix} \cosh\delta_2 & \cdot & \cdot & \cdot & \cdot & \sinh\delta_2 \\ \cdot & I_{9-D} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cosh\delta_2 & \cdot & -\sinh\delta_2 & \cdot \\ \cdot & \cdot & \cdot & I_{25-D} & \cdot & \cdot \\ \cdot & \cdot & -\sinh\delta_2 & \cdot & \cosh\delta_2 & \cdot \\ \sinh\delta_2 & \cdot & \cdot & \cdot & \cdot & \cosh\delta_2 \end{pmatrix}. \qquad \text{(A.17)}
\end{aligned}$$

This is an $SO(1,1)$ matrix and we have $SO(1,1) \subset O(11-D, 27-D)$. Applying the transformation (A.10) with the above matrix Ω, we obtain

$$\begin{aligned}
ds^2 &= \Delta^{\frac{D-4}{D-2}} W^{\frac{1}{D-2}} \Bigg[-\frac{(\Delta - 2N)}{W} dt^2 + \frac{dr^2}{\prod_{i=1}^{[\frac{D-1}{2}]} (r^2 + l_i^2) - 2N} \\
&+ \frac{r^2 + l_1^2 \cos^2\theta + K_1 \sin^2\theta}{\Delta} d\theta^2
\end{aligned}$$

$$
\begin{aligned}
+ & \frac{\cos^2\theta\cos^2\psi_1\ ...\cos^2\psi_{i-1}}{\Delta}(r^2+l_{i+1}^2\cos^2\psi_i+K_{i+1}\sin^2\psi_i)d\psi_i^2 \\
- & 2\sum_{i<j}\frac{l_j^2-K_j}{\Delta}\cos^2\theta\cos^2\psi_1\ ...\cos^2\psi_{i-1}\cos\psi_i\ \sin\psi_i ... \\
& \qquad\qquad \cos^2\psi_{j-1}\cos\psi_j\sin\psi_j d\psi_i d\psi_j \\
- & 2\frac{l_{i+1}^2-K_{i+1}}{\Delta}\cos\theta\sin\theta\cos^2\psi_1\ ...\cos^2\psi_{i-1}\cos\psi_i\sin\psi_i d\theta d\psi_i \\
+ & \frac{\mu_i^2}{\Delta W}\left[(r^2+l_i^2)\Delta^2+2Nl_i^2\mu_i^2+4N^2\sinh^2\delta_1\sinh^2\delta_2(r^2+l_i^2(1-\mu_i^2))\right. \\
& \qquad \left. +2N\Delta(\sinh^2\delta_1+\sinh^2\delta_2)(r^2+l_i^2)\right]d\phi_i^2 \\
- & \frac{4Nl_i\mu_i^2\cosh\delta_1\cosh\delta_2}{W}dtd\phi_i \\
+ & \left.\sum_{i<j}\frac{4Nl_il_j\mu_i^2\mu_j^2(\Delta-2N\sinh^2\delta_1\sinh^2\delta_2)}{\Delta W}d\phi_i d\phi_j\right], \qquad \text{(A.18)}
\end{aligned}
$$

where $W=(2N\sinh^2\delta_1+\Delta)(2N\sinh^2\delta_2+\Delta)$. The other fields can be extracted from the matrix M. The ADM mass, angular momentum and $U(1)$ electric charges are, respectively,

$$
\begin{aligned}
M &= \frac{\Omega_{D-2}m}{8\pi G_D}[(D-3)(\cosh^2\delta_1+\cosh^2\delta_2)-(D-4)], \\
J_i &= \frac{\Omega_{D-2}}{4\pi G_D}ml_i\cosh\delta_1\cosh\delta_2 \\
Q_1^{(1)} &= \frac{\Omega_{D-2}}{8\pi G_D}(D-3)m\cosh\delta_1\sinh\delta_1 \\
Q_1^{(2)} &= \frac{\Omega_{D-2}}{8\pi G_D}(D-3)m\cosh\delta_2\sinh\delta_2, \qquad \text{(A.19)}
\end{aligned}
$$

where $\Omega=\frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}$ is the area of the unit $(D-2)$ sphere.

In $D=7$, the above metric describes a 3-brane in ten dimensions [59, 72]. To simplify the metric further, let one of the charges be zero. We obtain [59, 58]

$$
\begin{aligned}
ds^2 &= \frac{1}{\sqrt{H}}\left(-(1-f\frac{r_0^4}{r^4})dt^4+dx_1^2+dx_2^2+dx_3^2\right)+\sqrt{H}f^{-1}\frac{dr^2}{\lambda-r_0^4/r^4} \\
& +\sqrt{H}r^2\left(\zeta d\theta^2+\zeta'\cos^2\theta d\psi^2-\frac{\ell_2^2-\ell_3^2}{2r^2}\sin(2\theta)\sin(2\psi)d\theta d\psi\right) \\
& -f\frac{2r_0^4\cosh\gamma}{r^4}\sqrt{H}(\ell_1\sin^2\theta d\phi_1+\ell_2\cos^2\theta\sin^2\psi d\phi_2
\end{aligned}
$$

$$+\ell_3 \cos^2\theta \cos^2\psi d\phi_3)dt$$
$$+f\frac{r_0^4}{r^4}\sqrt{H}\left(\ell_1 \sin^2\theta d\phi_1 + \ell_2 \cos^2\theta \sin^2\psi d\phi_2 + \ell_3 \cos^2\theta \cos^2\psi d\phi_3\right)^2$$
$$+\sqrt{H}r^2\left[\left(1+\frac{\ell_1^2}{r^2}\right)\sin^2\theta d\phi_1^2 + \left(1+\frac{\ell_2^2}{r^2}\right)\cos^2\theta \sin^2\psi d\phi_2^2 + \left(1+\frac{\ell_3^2}{r^2}\right)\cos^2\theta\cos^2\psi d\phi_3^2\right], \tag{A.20}$$

where

$$H = 1 + f\frac{r_0^4 \sinh^2\gamma}{r^4} \tag{A.21}$$
$$f^{-1} = \lambda\left(\frac{\sin^2\theta}{1+\frac{\ell_1^2}{r^2}} + \frac{\cos^2\theta\sin^2\psi}{1+\frac{\ell_2^2}{r^2}} + \frac{\cos^2\theta\cos^2\psi}{1+\frac{\ell_3^2}{r^2}}\right) \tag{A.22}$$
$$\lambda = \left(1+\frac{\ell_1^2}{r^2}\right)\left(1+\frac{\ell_2^2}{r^2}\right)\left(1+\frac{\ell_3^2}{r^2}\right) \tag{A.23}$$
$$\zeta = 1 + \frac{\ell_1^2\cos^2\theta + \ell_2^2\sin^2\theta\sin^2\psi + \ell_3^2\sin^2\theta\cos^2\psi}{r^2} \tag{A.24}$$
$$\zeta' = 1 + \frac{\ell_2^2\cos^2\psi + \ell_3^2\sin^2\psi}{r^2}, \tag{A.25}$$

and $r_0^4 = 2m$, $\gamma = \delta_1$. The thermodynamic properties are discussed in section 2.3. The extremal symmetric case (asymptotically AdS) is obtained by letting $l_1 = l_2 = l_3 = 0$, and also the mass $m \to 0$ (so that the horizon shrinks to zero) and $\gamma \to \infty$, keeping $r_0^2 \sinh^2\gamma = R^4$ fixed. Reparametrizing,

$$\begin{aligned} y_1 &= r\sin\theta\cos\phi_1 \\ y_2 &= r\sin\theta\sin\phi_1 \\ y_3 &= r\cos\theta\sin\psi\cos\phi_2 \\ y_4 &= r\cos\theta\sin\psi\sin\phi_2 \\ y_5 &= r\cos\theta\cos\psi\cos\phi_3 \\ y_6 &= r\cos\theta\cos\psi\cos\phi_3, \end{aligned} \tag{A.26}$$

the metric reduces to

$$ds^2 = \frac{1}{\sqrt{H}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{H}\left(dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 + dy_5^2 + dy_6^2\right) \tag{A.27}$$

with $H = 1 + \frac{R^4}{r^4}$.

Appendix B

Multi-Black Hole Moduli Space

B.1 The discrete case

In subsection 3.3.1, we found that the metric in moduli space of a multi-black hole system was given in terms of the quantity (sum over black hole positions (moduli))

$$L_2 = -3 \sum_{a \neq b} \int d^4x \frac{Q_a^2 Q_b}{|\vec{x} - \vec{x}_a|^4 |\vec{x} - \vec{x}_b|^2},$$

This expression needs to be regulated by introducing a cutoff δ. No physical quantities should depend on δ as $\delta \to 0$. We define

$$L_2 = -3 \sum_{a \neq b} \int d^4x \frac{Q_a^2 Q_b}{((\vec{x} - \vec{x}_a)^2 + \delta^2)^2 ((\vec{x} - \vec{x}_b)^2 + \delta^2)},$$

Introducing a Feynman parameter y, we obtain

$$L_2 = -6 \sum_{a \neq b} \int d^4x \int_0^1 dy \frac{Q_a^2 Q_b \; y}{[(\vec{x} - \vec{x}_a)^2 y + (\vec{x} - \vec{x}_b)^2 (1-y) + \delta^2]^3}.$$

To integrate over $\vec{x}$, we use the general formula

$$\int d^N U \frac{1}{(U^2 + 2U \cdot p + M^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(M^2 - p^2)^{A - N/2}}. \tag{B.1}$$

We finally obtain

$$L_2 = -6\pi^2 \sum_{a \neq b} Q_a^2 Q_b \frac{[\ln |\vec{x}_a - \vec{x}_b| - \ln \delta]}{|\vec{x}_a - \vec{x}_b|^2}. \tag{B.2}$$

After differentiating L_2 twice, we obtain

$$\partial_{ai}\partial_{bj}L_2 = 3\pi^2 \sum_{c\neq b} \frac{Q_bQ_c(Q_b+Q_c)}{|\vec{x}_c-\vec{x}_b|^4}[\delta_{ca}-\delta_{ba}]\left\{\delta_{ij}-6\frac{[x_{ci}-x_{bi}][x_{cj}-x_{bj}]}{|\vec{x}_c-\vec{x}_b|^2}\right.$$
$$\left.+\left(\delta_{ij}-4\frac{[x_{ci}-x_{bi}][x_{cj}-x_{bj}]}{|\vec{x}_c-\vec{x}_b|^2}\right)\left(\ln|\vec{x}_c-\vec{x}_b|^{-2}+4\ln(\frac{\delta}{2})\right)\right\},$$

where we are summing over the index c only. The contribution of L_2 to the generator of dilatations, D, is

$$\begin{aligned} D_{2ak}dx^{ak} &= -g_{2ak\ bl}x^{bl}dx^{ak} = \frac{1}{4}(\delta^{ij}\delta_{kl}+\delta^i_k\delta^j_l-\delta^i_l\delta^j_k+\epsilon^{ij}{}_{kl})\partial_{ai}\partial_{bj}L_2\ x^{bl}dx^{ak} \\ &= \frac{3\pi^2}{4}\sum_{b\neq c}\frac{Q_bQ_c(Q_b+Q_c)}{|\vec{x}_c-\vec{x}_b|^2}[\delta_{ca}-\delta_{ba}][-2x^{bk}dx^{ak}] \\ &= \frac{3\pi^2}{4}d\sum_{a\neq b}\frac{Q_a^2Q_b}{|\vec{x}_a-\vec{x}_b|^2}. \end{aligned} \tag{B.3}$$

Next, we show that the three-point term

$$L_3 = -\sum_{a\neq b\neq c}\int d^4x\frac{Q_aQ_bQ_c}{|\vec{x}-\vec{x}_a|^2|\vec{x}-\vec{x}_b|^2|\vec{x}-\vec{x}_c|^2}$$

does not contribute to D. To this end, we shall first prove

$$x^{ai}\partial_{ai}L_3 = -2L_3 \tag{B.4}$$
$$x^{bl}I^{ri}_kI^{rj}_l\partial_{bj}L_3 = 0, \tag{B.5}$$

where I^r are the natural triplet of self-dual complex structures on $\mathbb{R}^4$ [74], where

$$\begin{aligned} I^{ri}_kI^{rj}_l &= \delta^{ij}\delta_{kl}-\delta^i_l\delta^j_k+\epsilon^{ij}{}_{kl}, \\ \delta^l_iI^{ri}_kI^{rj}_l &= -3\delta^j_k. \end{aligned} \tag{B.6}$$

We shall prove (B.4) in detail; the proof of (B.5) is similar. Applying Feynman parametrization, we obtain

$$\begin{aligned} L_3 &= -\sum_{a\neq b\neq c}\int d^4x\int_0^1 dy_1\int_0^{1-y_1}dy_2\ \times \\ &\qquad \frac{Q_aQ_bQ_c}{[(\vec{x}-\vec{x}_a)^2y_1+(\vec{x}-\vec{x}_b)^2y_2+(\vec{x}-\vec{x}_c)^2(1-y_1-y_2)]^3} \end{aligned} \tag{B.7}$$

Integrating over $\vec{x}$,

$$L_3 = -\sum_{a\neq b\neq c} \pi^2 \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \frac{Q_a Q_b Q_c}{\mathcal{D}}$$

$$\mathcal{D} = (\vec{x}_c - \vec{x}_a)^2(-y_1^2 + y_1) + (\vec{x}_c - \vec{x}_b)^2(-y_2^2 + y_2) - 2(\vec{x}_c - \vec{x}_a)\cdot(\vec{x}_c - \vec{x}_b)y_1 y_2.$$

Differentiating, we obtain the desired result (B.4). We are now ready to show that

$$g_{3ak\ bl}x^{bl} = 0. \tag{B.8}$$

We have

$$\begin{aligned} 4g_{3ak\ bl}x^{bl} &= (\delta^{ij}\delta_{kl} + \delta^i_k\delta^j_l - \delta^i_l\delta^j_k + \epsilon^{ij}{}_{kl})\partial_{ai}\partial_{bj}L_3 x^{bl} \\ &= (\delta^i_k\delta^j_l + I^{ri}_k I^{rj}_l)\partial_{ai}\partial_{bj}L_3 x^{bl} \\ &= \partial_{ak}\partial_{bl}L_3 x^{bl} + I^{ri}_k I^{rj}_l \partial_{ai}\partial_{bj}L_3 x^{bl}. \end{aligned}$$

Integrating by parts,

$$4g_{3ak\ bl}x^{bl} = \partial_{ak}(x^{bl}\partial_{bl}L_3) - \delta_{kl}\partial_{al}L_3 + \partial_{ai}(x^{bl}I^{ri}_k I^{rj}_l\partial_{bj}L_3) - \delta^l_i I^{ri}_k I^{rj}_l \partial_{aj}L_3$$

and applying (B.4), (B.5) and (B.6), we obtain

$$\begin{aligned} 4g_{3ak\ bl}x^{bl} &= -2\partial_{ak}L_3 - \partial_{ak}L_3 - \delta^l_i I^{ri}_k I^{rj}_l \partial_{aj}L_3, \\ &= 0. \end{aligned}$$

B.2 A black string

In the continuous case, the generator D is given in terms of an integral,

$$\int g_{a\alpha b\beta}b^\beta d^4b = -\frac{\pi^2}{8}\left(\frac{2}{3\cdot 4\pi^2}\right)^3 (3\cdot 2\cdot 4)\int d^4b \int d^4c\,\bar{\rho}_a\bar{\rho}_b\bar{\rho}_c \int [dx]\,\frac{\partial}{\partial a^\alpha}\frac{1}{\mathcal{D}^2}\ ,$$

$$\mathcal{D} = (\vec{a}-\vec{b})^2 x_1 x_2 + (\vec{a}-\vec{c})^2 x_1 x_2 + (\vec{b}-\vec{c})^2 x_1 x_3 + \delta^2\ ,$$

where we are integrating over the position vectors $\vec{b}$ and $\vec{c}$, spanning the continuous matter distribution, and the Feynman parameters with measure $[dx] = dx_1 dx_2 dx_3\,\delta(1-x_1-x_2-x_3)$. Concentrating on the case of a circular ring of radius R, we shall parametrize the position vectors by

$$\begin{aligned} \vec{a} &= R\hat{e}_x\cos\phi_a + R\hat{e}_y\sin\phi_a, \\ \vec{b} &= R\hat{e}_x\cos\phi_b + R\hat{e}_y\sin\phi_b, \\ \vec{c} &= R\hat{e}_x\cos\phi_c + R\hat{e}_y\sin\phi_c. \end{aligned} \tag{B.9}$$

Then the denominator becomes

$$\mathcal{D} = -\frac{1}{z}\,\left[\beta z^2 - (\alpha+\delta^2)z + \gamma\right]\ ,\ z = e^{i\phi_c}\ , \tag{B.10}$$

where

$$\begin{aligned}
\alpha &= (a-b)^2 x_1 x_2 + a^2 x_1 x_2 + R^2 x_1 x_3 + 2R^2 x_3 x_2,\\
\beta &= aRx_1 x_3 e^{-i\phi_a} + R^2 x_2 x_3 e^{-i\phi_b},\\
\gamma &= aRx_1 x_3 e^{+i\phi_a} + R^2 x_2 x_3 e^{+i\phi_b}.
\end{aligned} \tag{B.11}$$

Next, we need to integrate over z along the unit circle $|z| = 1$. The poles (roots of $\mathcal{D} = 0$) are at

$$z_\pm = \frac{1}{2}\frac{\alpha+\delta^2}{\beta} \pm \frac{1}{2}\left[\frac{(\alpha+\delta^2)^2}{\beta^2} - 4\frac{\gamma}{\beta}\right]^{1/2}. \tag{B.12}$$

Only one pole (z_-) lies inside the circle. Therefore,

$$\begin{aligned}
\frac{1}{2\pi i}\oint \frac{dz}{z}\frac{1}{\mathcal{D}^2} &= \frac{d}{dz}\left\{(z-z_-)^2 \frac{z}{\left[\beta z^2 - (\alpha+\delta^2)z + \gamma\right]^2}\right\}_{z\to z_-}\\
&= \frac{(\alpha+\delta^2)}{\left((\alpha+\delta^2)^2 - 4\beta\gamma\right)^{3/2}}\\
&= -\frac{\partial}{\partial\delta^2}\frac{1}{\left((\alpha+\delta^2)^2 - 4\beta\gamma\right)^{1/2}}.
\end{aligned} \tag{B.13}$$

The metric becomes

$$\begin{aligned}
g_{a\alpha b\beta} b^\beta db &= (3\cdot 4)\left(\frac{2}{3\cdot 4\pi^2}\right)^3 \int db\bar{\rho}_a \lambda^2 (2\pi^2)\ \times\\
&\qquad \delta^2(-\frac{1}{8}\frac{\partial}{\partial a^\alpha})(2\pi)(-\frac{\partial}{\partial\delta^2})\int \frac{[dx]}{\left[(\alpha+\delta^2)^2 - 4\beta\gamma\right]^{1/2}},
\end{aligned} \tag{B.14}$$

where $\lambda = \frac{M}{2\pi}$ is the mass density, M being the mass of the string. The denominator can be written in the from

$$(\alpha+\delta^2)^2 - 4\beta\gamma = 4a^2R^2x_1^2x_2^2(1-\cos(\phi_b-\phi_a))^2 + f(1-\cos(\phi_b-\phi_a)) + g, \tag{B.15}$$

where

$$\begin{aligned}
f &= (a^2 - 2aR + R^2)x_1x_2 + x_1x_3(a^2+R^2) + 2R^2x_3x_2 + 2R^2x_3^2 + \delta^2\\
g &= \left[(a^2 - 2aR + R^2)x_1x_2 + x_1x_3(a^2+R^2) + 2R^2x_3x_2 + \delta^2\right]^2\\
&\quad -4r^2x_3^2(ax_1 + Rx_2)^2.
\end{aligned}$$

Because of the explicit factor of δ^2 in (B.14), only the divergent terms contribute as $\delta \to 0$. We get divergences in the small $(\phi_b - \phi_a)$ region. Notice also that if we let $a = R$, then we get

$$\begin{aligned} f &= 2R^2 x_1 x_2 [2R^2 x_3 (x_1 + x_2) + 2R^2 x_3 + \delta^2] \\ g &= [4R^2 x_3 (x_1 + x_2) + \delta^2]\delta^2. \end{aligned} \tag{B.16}$$

As $\delta \to 0$, we have $g \to 0$, and the integral takes the form $\int d\phi/\sqrt{1 - \cos\phi}$. Therefore, we need to keep g in the limit $\delta \to 0$ in order to regulate the integral.

Next, we integrate over ϕ_b by changing parameters to $y = 2\sin^2 \frac{(\phi_b - \phi_a)}{2}$. The integral takes the form

$$\begin{aligned} \int d\phi_b \frac{1}{[f(1-\cos\phi_{ba}) + g]^{1/2}} &= 2\sqrt{2}\int_0^{y_0} dy \frac{1}{[fy^2 + g]^{1/2}} + \ldots \\ &= 2\sqrt{\frac{2}{f}} \ln\left[\sqrt{1 + \frac{f}{g}y_0} + \sqrt{\frac{f}{g}y_0}\right] + \ldots, \end{aligned} \tag{B.17}$$

where the limit of integration y_0 is not important (a change in y_0 does not affect the singular part) and the dots represent finite terms. The metric (B.14) becomes

$$\begin{aligned} g_{a\alpha b\beta} b^\beta db &= (3\cdot 4)\left(\frac{2}{3\cdot 4\pi^2}\right)^3 \lambda^3 (2\pi^2)\delta^2 (-\frac{1}{8}\frac{\partial}{\partial a^\alpha})(2\pi) \\ &\quad \times(-\frac{\partial}{\partial \delta^2})\int [dx] 2\sqrt{\frac{2}{f}} \ln\left[\sqrt{1 + \frac{f}{g}y_0} + \sqrt{\frac{f}{g}y_0}\right] \\ &= (3\cdot 4)\left(\frac{2}{3\cdot 4\pi^2}\right)^3 \lambda^3 (2\pi^2)\delta^2 (-\frac{1}{8})(-\frac{\partial}{\partial \delta^2})\int [dx](2\pi)(2\sqrt{2}) \times \\ &\quad \left\{-\frac{f'_\alpha}{f^{3/2}} \ln\left[\sqrt{1 + \frac{f}{g}y_0} + \sqrt{\frac{f}{g}y_0}\right] + \frac{(f'_\alpha g - f g'_\alpha)\sqrt{y_0}}{gf\sqrt{g + f y_0}}\right\}, \end{aligned} \tag{B.18}$$

where $f'_\alpha = \frac{\partial f}{\partial a^\alpha}$ and similarly for g'_α. Setting $a = R$, after some algebra, one can show that the second term in (B.18) vanishes as $\delta \to 0$. Thus, only the first term contributes. Using (B.16), we obtain

$$g_{a\alpha b\beta} b^\beta db = (3\cdot 4)\left(\frac{2}{3\cdot 4\pi^2}\right)^3 \lambda^3 (2\pi^2)\delta^2 (-\frac{1}{8})(-\frac{\partial}{\partial \delta^2})\int [dx](2\pi)(2\sqrt{2})\left\{-\frac{f'_\alpha}{f^{3/2}}\ln\delta\right\}$$

To perform the integral over the Feynman parameters, observe

$$\delta^2 \int [dx] \frac{\partial}{\partial \delta^2}(\frac{f'}{f^{3/2}}\frac{1}{2}\ln\delta^2)$$

$$
\begin{aligned}
&= \delta^2 \int [dx] \frac{a^\alpha}{4R^3\sqrt{x_1 x_2}} \times \\
&\left[\frac{\ln \delta^2}{(2R^2 x_3 + \delta^2)^{3/2}} - \frac{3}{2} \frac{[2R^2 x_3 + R^2 x_3 (1-x_3) + \delta^2]}{[2R^2 x_3 + \delta^2]^{5/2}} \ln \delta^2 \right. \\
&\qquad \left. + \frac{1}{\delta^2} \frac{[2R^2 x_3 + R^2 x_3(1-x_3) + \delta^2]}{(2R^2 x_3 + \delta^2)^{3/2}} \right] \\
&= \delta^2 \frac{a^\alpha}{4R^3} \int_{-1}^{1} dz \int_0^1 dx_3 \frac{1}{\sqrt{1-z^2}} \times \\
&\left[\frac{[2R^2 x_3 + \delta^2 - 3R^2 x_3 - \frac{3}{2} R^2 x_3 (1-x_3) - \frac{3}{2}\delta^2]}{[2R^2 x_3 + \delta^2]^{5/2}} \ln \delta^2 \right. \\
&\qquad \left. + \frac{1}{\delta^2} \frac{2R^2 x_3 + R^2 x_3(1-x_3) + \delta^2}{(2R^2 x_3 + \delta^2)^{3/2}} \right] \\
&= \frac{\sqrt{2}\pi}{3} \frac{a^\alpha}{R^4}
\end{aligned}
\tag{B.19}
$$

where we used the substitution

$$
z = \frac{x_2 - \frac{1}{2}(1-x_3)}{\frac{1}{2}(1-x_3)} \tag{B.20}
$$

The range of the new parameter is $z \in [-1, 1]$. Therefore, we obtain

$$
\int g_{a\alpha b\beta} b^\beta db = (3 \cdot 4) \left(\frac{2}{3 \cdot 4\pi^2} \right)^3 \lambda^3 \frac{\pi^3}{3} \frac{a^\alpha}{R^4}. \tag{B.21}
$$

The action

$$
\begin{aligned}
S &= \frac{1}{2} \int d^5 U (\frac{2}{3 \cdot 4\pi^2})^3 \int dl_x dl_y dl_z \frac{\bar{\rho}_x \bar{\rho}_y \bar{\rho}_z}{|\vec{U} - \vec{x}|^2} \times \\
&\quad 3 \left\{ (v_z^2 - v_{yi} v_{zi}) \partial_{yj} \partial_{zj} + v_{yi} v_{zj} \left(\partial_{yj} \partial_{zi} - \partial_{yi} \partial_{zj} + \epsilon^{ijkl} \partial_{yk} \partial_{zl} \right) \right\} \\
&\quad \frac{1}{|\vec{U} - \vec{y}|^2 |\vec{U} - \vec{z}|^2},
\end{aligned}
$$

can be manipulated in a manner similar to the discrete case. After introducing the regulator δ and integrating over $\vec{U}$, we obtain

$$
\begin{aligned}
S &= \frac{1}{2} (\frac{1}{6\pi^2})^3 \int dt (\bar{\rho}_x \delta) \int d^4 y d^4 z \bar{\rho}_y \bar{\rho}_z \times \\
&\left[(v_z^2 - v_{yi} v_{zi}) \partial_{yj} \partial_{zj} + v_{yi} v_{zj} \left(\partial_{yj} \partial_{zi} - \partial_{yi} \partial_{zj} \right) \right] \pi^2 \left[\frac{\ln((y-z)^2/\delta^2)}{(\vec{y} - \vec{z})^2 + \delta^2} \right]
\end{aligned}
\tag{B.22}
$$

where we used the fact that the only contributions to the integral come from the region where $\vec{x}$ approaches either $\vec{y}$ or $\vec{z}$. Because of translation invariance, we could set $\vec{x} = \vec{0}$ in the integrand, and then $\int \bar{\rho}_x dx \approx \bar{\rho}_x \delta$. In the case of a circular ring of radius R and mass M uniformly distributed, we have $\bar{\rho} = \frac{M}{2\pi R}$. Differentiating with respect to y_i and z_i, we obtain

$$
\begin{aligned}
S &= \frac{1}{2}(\frac{2}{3 \cdot 4\pi^2})^3 \int dt(\bar{\rho}_x \delta)(\frac{M}{2\pi})^2 \int_0^{2\pi} d\phi_y d\phi_z (v_z^2 - \vec{v}_y \cdot \vec{v}_z) \frac{4\pi^2}{((\vec{y} - \vec{z})^2 + \delta^2)^2} \\
&= \frac{1}{2}(\frac{2}{3 \cdot 4\pi^2})^3 \int dt(\bar{\rho}_x \delta)(\frac{M}{2\pi})^2 (-4\pi^3) \frac{\dot{R}^2}{R} \int_0^{2\pi} \frac{d\phi}{4\sin^2(\phi/2) + \delta^2} \\
&= \int dt(\frac{2}{3 \cdot 4\pi^2})^3 3M^3 \frac{\dot{R}^2}{R^4} \ ,
\end{aligned}
\tag{B.23}
$$

where we used $v^2 = \dot{R}^2$.

Vita

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