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The Stability Analysis of Linear Dynamical Systems with Time-Delays

Ajeet Ganesh Kamath
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Ajeet Ganesh Kamath entitled "The Stability Analysis of Linear Dynamical Systems with Time-Delays." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mechanical Engineering.

VijaySekhar Chellaboina, Major Professor

We have read this dissertation and recommend its acceptance:

William R. Hamel, Dongjun Lee, Seddik M. Djouadi

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Accepted for the Council:

Linda Painter
Interim Dean of Graduate Studies

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The Stability Analysis of Linear Dynamical Systems with Time-Delays

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Ajeet Ganesh Kamath
December 2006

Dedication

This dissertation is dedicated to the members of my family, who nurtured me, endured me and humored me all my life. My father, Ganesh Kamath, my mother Suman Kamath and my brother, Manju (Prashant) Kamath, thank you for all your love, support, guidance and prayers. To my fiancée, Aparna Kher, your presence makes me strong.

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In our research group, I am grateful to my colleagues and friends Jayanthy Ramakrishnan and Alex Melin for their help and input, and for putting up with me over the years. Also deserving a special mention are the other present and past members of the research group, Hancao Li, Mithun Ranga and Sushma Kalavagunta.

To all my teachers, present and past, I am deeply indebted to you for the role you have played in taking me to where I stand today. A big "thank you" to all of you. If I have omitted your names, it is only for a lack of space, and most definitely not for a lack of appreciation for your efforts.

Abstract

Time-delay systems, which are also sometimes known as hereditary systems or systems with memory, aftereffects or time-lag, represent a class of infinite-dimensional systems, and are used to describe, among other types of systems, propagation and transport phenomena, population dynamics, economic systems, communication networks and neural network models. A key method for the stability analysis of time-delay dynamical systems is the Lyapunov's second method, applied to functional differential equations. Specifically, stability of a given linear time-delay dynamical system is typically shown using a Lyapunov-Krasovskii functional, which involves a quadratic part and an integral part. The quadratic part is usually associated with the stability of the forward delay-independent part of the retarded dynamical system, but the integral part of the functional is less understood. We present a concrete method of arriving at the Lyapunov-Krasovskii functional using dissipativity theory. The stability analysis of time-delay systems has been mainly classified into two categories: *delay-dependent* and *delay-independent* analysis. Delay-independent stability criteria provide sufficient conditions for stability of time-delay dynamical systems independent of the amount of delay, whereas delay-dependent stability criteria provide sufficient conditions that are dependent on an upper bound of the delay. In systems where the time delay is known to be bounded, delay-dependent criteria usually give far less conservative stability predictions as compared to delay-independent results. Hence, for such systems it is of paramount importance to derive the sharpest possible delay-dependent stability margins. We show how the stability criteria may also be interpreted in the frequency domain in terms of a feedback interconnection of a matrix transfer function and a *phase* uncertainty block. We develop and present the general framework for a robust stability analysis method to account for phase uncertainties in linear systems. We present new robust stability results for time-delay systems based on pure phase information, and then, using this approach, we derive new and improved time-domain delay-dependent stability criteria for stability analysis of both retarded and neutral type time-delay systems, which we then compare with existing results in the literature.

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Notation

$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of imaginary numbers
$\mathbb{R}^{n\times m}$	set of $n \times m$ real matrices
$\mathbb{C}^{n\times m}$	set of $n \times m$ imaginary matrices
A^{T}	transpose of matrix A
A^*	complex conjugate transpose of matrix A
$\mathbb{H}^{n\times n}$	set of $n \times n$ Hermitian matrices
$\mathbb{S}^{n\times n}$	set of $n \times n$ real symmetric matrices
$\mathbb{P}^{n\times n}$	set of $n \times n$ positive-definite Hermitian matrices
$\mathbb{N}^{n\times n}$	set of $n \times n$ nonnegative-definite Hermitian matrices
$0_{n\times m}$	the $n \times m$ zero matrix
I_n	the $n \times n$ identity matrix
$M \geq 0$	denotes the fact that the Hermitian matrix M is nonnegative definite
$M > 0$	denotes the fact that the Hermitian matrix M is positive definite
$G(s) \sim \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	state space realization of transfer function $G(s) = C(sI - A)^{-1}B + D$
$\mathrm{spec}(M)$	spectrum of a square complex matrix M
$\rho(M)$	spectral radius of a square complex matrix M
$A \otimes B$	the Kronecker product of matrices A and B
$\mathcal{C}([a,b],\mathbb{R}^n)$	the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence

Chapter 1

Introduction

1.1. Motivation and Historical Overview

Time-delay systems, which are also known sometimes as hereditary systems or systems with memory, aftereffects or time-lag, represent a class of infinite-dimensional systems, and are used to describe propagation and transport phenomena or population dynamics (reproduction, development or extinction) [1]. In economic systems, delays appear naturally since decisions and effects are separated by some non-zero time interval. A non-zero time interval also accompanies initiation and delivery of data in communication networks. In some cases, the delay may arise due to a simplification of a model.

A feature that distinguishes such systems is that their dynamics can be described by differential equations that include information on the history of the system. There are several ways to mathematically represent such systems.

The presence of the time-delay elements in the system may induce complex behaviors, such as oscillations, instability and bad performance. In some cases, the presence of a "small" delay may destabilize the system, whereas in others, a "large" delay may lead to instability. In many cases, a sequence of stability *switches* (stability to instability) or *reversals* (instability to stability) may occur in a linear time-delay system, as the delay, seen as a parameter, is increased. *Chaotic* behavior may be seen if the delayed state is a nonlinear function, and in other cases the presence of a delayed output may stabilize a chaotic system. Because of the destabilizing nature of the delayed states in a system, stability analysis of time-delay dynamical systems remains a very important area of research (see [1–5] and references therein).

In neural networks, the propagation of a signal through the network requires some amount of time, that can be mathematically modeled using delays. In the 1950s some interest appeared in the use of delay models to describe the central nervous system in learning processes.

The aforementioned aspects motivate the study of delay effects on closed-loop dynamical systems to understand how the presence of the delay may deteriorate the behavior of the system and also to control their effects for better performance given an application.

Time-delay dynamical systems have been extensively studied in the literature (see [1–18] and the numerous references therein). Due to the emergence of recent applications such as complex networks [19] involving flow of information and biological systems involving material transfer from one subsystem to another where the transfer of information or material from one point in the system to another is not instantaneous, the modeling of the time delay in the overall system description has become very important. The study of time delay systems has received a strong and renewed interest in the recent years (see [1, 2, 4, 14, 19] and numerous references therein).

1.2. Applications and Examples of Time-Delay Systems

Many processes, both natural and man-made, including systems in biology and medicine, chemistry, engineering and economics involve time-delays. The following are some examples of time-delay systems from the literature of recent years. See

[1] and the references therein for a detailed overview on applications of time-delay systems.

1.2.1. Transport and Communication Delays

It is natural to expect that any interconnection of systems where material, energy or information are *transported* from one system to another involves some amount of delay in the loop with possibly some effect on the properties of the overall system.

Chemical engineering problems provide several examples of engineering mathematical models involving delays in their representations. Some examples include the modeling of *mixing tanks* where there is a delay in material transport and heat exchanger dynamics.

Combustion models are often described by delay dynamics. These typically arise in *continuous combustion* processes which occur in propulsion and power-generation. In many such examples, a time-delay mechanism is observed to play a key role and determines the nature of the stability properties of such a system.

Teleoperation systems, consisting of a *slave device* tracking a *master device* are used where human accessibility is limited due to the hazardous nature of an environment are modeled by delay systems.

In data transmission networks, the time-interval between the initiation time of a signal and the delivery time is not zero, and causes a communication delay in the network. It becomes important to study the effect of such a delay in order to maintain the efficiency and performance of such a network.

Some neural network models also involve transport delays in their dynamic representations. Such networks are encountered in associative memory analysis, parallel computation and signal processing optimization problems. Stability of the associated differential equations is an important requirement in these examples.

1.2.2. Biology and Population Dynamics

Engineering is not the only source for examples of delay systems. Recently, there has been an increased interest in the modeling of physiological, ecological, population dynamics and biomedical systems which use time-delays in their representation.

In biological systems, the delay is associated with processes that involve time-expenses such as reproduction, development and extinction.

1.2.3. Propagation Phenomena

Electrical circuit models and hydraulic engineering models are examples where a delay model is used to describe a *lossless propagation* phenomenon.

The two types of circuits which include elements with delay are transmission lines (TL) and partial element equivalent circuits (PEECs) [20]. Some of these problems result in neutral time-delay models.

As an example of an application where a time-delay system model is seen, we will describe a small test circuit example which consists of a PEEC circuit [20] shown in Figure 1.2. This circuit represents a full wave equivalent circuit for the small metal strip which is discretized in two cells as shown in Figure 1.1.

The PEEC model in Figure 1.2 includes new circuit elements which involve retarded mutual coupling between the partial inductances of the form $Lp_{ij}\dot{i}_j(t-\tau)$, and

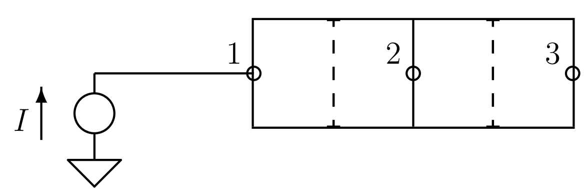

Figure 1.1: Metal strip with two L_p cells (three capacitive cells dashed)

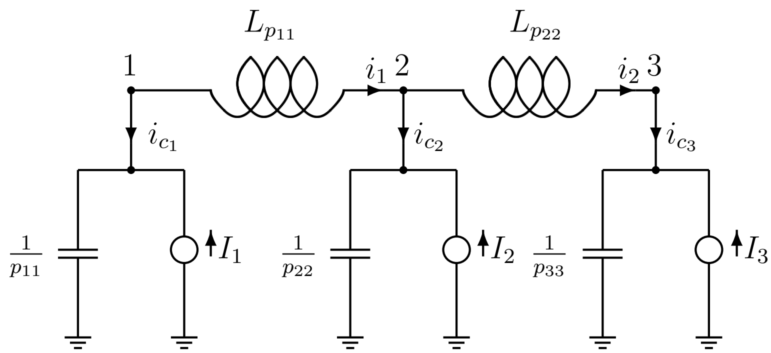

Figure 1.2: Small PEEC model for metal strip

retarded dependent current sources of the form $\frac{p_{ij}}{p_{ii}} i_{c_j}(t-\tau)$. The general form of the condensed neutral time-delay equations is given by

$$\dot{x}(t) - N\dot{x}(t-\tau) = Lx(t) + Mx(t-\tau), \quad x(\theta) = g(\theta),\ -\tau \leq \theta \leq t_0,\ t \geq t_0, \tag{1.1}$$

where all the matrices L, M, and N as well as the initial vector (or history) g are real-valued. The delay τ is a positive constant and t_0 is the initial time. The stability of this system was of interest to researchers in [20].

1.3. The Problem Statement

In control systems literature, mathematical models of physical or engineering systems and ordinary differential equations (ODEs) are practically synonymous. The most general form of the ODE used is

$$\dot{x}(t) = f(t, x(t)), \tag{1.2}$$

in which $x(t) \in \mathbb{R}^n$ represents the *state variables*, and the differential equations describe the evolution of the state variables over time. A fundamental assumption made on a system modeled using ODEs is that the future values of the state variables are completely determined by the current value of the state variables. Thus, in such a system model, all future values of the states can be evaluated, given an initial condition $x(t_0) = x_0$. ODEs in general, and their stability analysis and control aspects have been extensively studied and developed in the literature.

The functional differential equation (FDE) setting is used for the analysis of delay systems [1]:

$$\dot{x}(t) = f(t, x_t), \quad t > t_0, \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $x_t(\theta)$ denotes the delay operator acting on the trajectory, and $x_t(\theta) = x(t+\theta)$ for some (non-zero) interval $[-\tau, 0]$, that is, $\theta \in [-\tau, 0]$. Due to the form of (1.3), it is necessary to specify an initial condition given by some function ϕ defined on the delay interval $[-\tau, 0]$.

Since the construction of the evolution of the system (1.3) requires information on a non-zero interval, it follows that these systems belong to the class of *infinite-dimensional systems* [3]. Furthermore, the state of the system is not the vector $x(t)$ at the instant t; it is a function x_t corresponding to the past non-zero time interval $[-\tau, 0]$. In this research, we shall primarily be considering the linear version of the system (1.3), given by

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau), \quad x(\theta) = \phi(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \tag{1.4}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, $\tau \ge 0$, and $\phi(\cdot)$ is a continuous vector-valued function specifying the initial state of the system.

Both (1.3) and (1.4) belong to the class of *retarded* time-delay systems. We shall also consider the class of *neutral* time-delay systems, in which the evolution of the state of the system not only depends on the past state, but also on the time derivative of the past state. The general form of the neutral FDE is given by

$$\dot{x}(t) = f(t, x_t, \dot{x}_t). \tag{1.5}$$

Similar remarks as the retarded system may be assumed about the initial conditions and the delay operator in this case also. The neutral FDE is used to describe lossless propagation phenomena. The linear version of (1.5), which we shall consider, is given by

$$\dot{x}(t) + A_{\mathrm{n}}\dot{x}(t-\tau) = Ax(t) + A_{\mathrm{d}}x(t-\tau),\ x(\theta) = \eta(\theta),\ -\tau \leq \theta \leq 0,\ t \geq 0, \tag{1.6}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A, A_{\mathrm{d}}, A_{\mathrm{n}} \in \mathbb{R}^{n\times n}$, $\tau \geq 0$, $\eta(\cdot)$ is a continuously differentiable vector valued function specifying the initial state of the system.

A key method for the stability analysis of time-delay dynamical systems is the Lyapunov's second method, applied to functional differential equations. Specifically, stability analysis of a given linear time-delay dynamical system is typically shown using a Lyapunov-Krasovskii functional [3, 8]. A standard Lyapunov-Krasovskii functional typically involves a quadratic part and an integral part. The quadratic part is usually associated with the stability of the forward delay-independent part of the retarded dynamical system. However, the integral part of Lyapunov-Krasovskii functional is less understood. As part of this research, we shall present a concrete method of arriving at the Lyapunov-Krasovskii functional using dissipative systems theory [21].

Time-delay stability analysis has been mainly classified into two categories, namely, *delay-dependent* and *delay-independent* analysis [1, 4, 11–16]. Delay-independent stability criteria provide sufficient conditions for stability of time-delay dynamical systems independent of the amount of time delay, whereas delay-dependent stability criteria provide sufficient conditions that are dependent on an upper bound of the time delay. In systems where the time delay is known to be bounded, delay-dependent criteria usually give far less conservative stability predictions as compared to delay-independent results. Hence, for such systems it is of paramount importance to derive the sharpest possible delay-dependent stability margins.

1.4. Outline of the Dissertation

1.4.1. Dissipativity Approach to Time-Delay Systems

In Chapter 2, using the notions of dissipativity [21] and exponential dissipativity [22] theory, we present sufficient conditions for guaranteeing asymptotic stability of time delay dynamical systems. Specifically, representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional linear dynamical system and an infinite-dimensional time delay operator, we show that the time delay operator is dissipative with respect to a quadratic supply rate and with a storage functional involving an integral term which is identical to the integral term appearing in the LyapunovKrasovskii functional. Next, using stability of feedback interconnection results based on dissipativity of feedback systems, we develop sufficient conditions for asymptotic stability of time delay dynamical systems that are consistent with the results in the literature yet providing a system theoretic foundation for the Lyapunov Krasovskii functional forms. The overall approach provides a dissipativity theoretic interpretation of LyapunovKrasovskii functionals for asymptotically stable dynamical systems with arbitrary time delay. Analogous results are presented for the discrete-time linear time-delay systems also.

1.4.2. Dynamic Dissipativity Theory

In Chapter 3, we present dynamic extensions to the notions of dissipativity [21] and exponential dissipativity [22] theory, which we then use to derive new sufficient conditions for guaranteeing asymptotic stability of time delay dynamical systems.

1.4.3. Structured Phase Margin

In Chapter 4, we show how the stability criteria may also be interpreted in the frequency domain in terms of a feedback interconnection of a matrix transfer function and a *phase* uncertainty block [1]. We develop and present a robust stability analysis method to account for phase uncertainties. Specifically, we derive a general framework for stability analysis of linear systems with structured phase uncertainties. In addition, using this approach, we derive new and improved delay-dependent stability criteria for stability analysis of time-delay systems. Even though frequency-domain and integral quadratic constraints (IQCs) have been developed to address the time delay problem (see [1, 4, 5, 13, 16, 17] and references therein), with the notable exception of [13], all of these results rely on the scaled small gain theorem as applied to a *transformed* system. In contrast, we present new robust stability results for time-delay systems based on pure phase information.

1.4.4. Neutral Delay Systems

In this chapter, we shall extend the concepts from Chapter 4 to develop stability analysis results to neutral delay systems including LMI-based sufficient conditions for stability.

1.4.5. Conclusions and Future Work

Finally, in Chapter 6, we summarize the conclusions and contribution of the research presented in this dissertation, and propose future extensions that may be explored.

Chapter 2

A Dissipative Dynamical Systems Approach to the Stability Analysis of Time-Delay Systems

2.1. Introduction

In this chapter, using the concepts of dissipativity [21] and exponential dissipativity [22], we develop and present sufficient conditions for guaranteeing asymptotic stability of a linear time-delay dynamical system. Specifically, we first represent a time-delay dynamical system as a negative feedback interconnection of a (finite-dimensional) linear dynamical system and an (infinite-dimensional) time delay operator. Next, we show that the time delay operator is dissipative with respect to a quadratic supply rate and with a storage function involving an integral term which is identical to the integral term appearing in the Lyapunov-Krasovskii functional. Next, based on this result, we reprove a well known sufficient condition on the linear dynamical system that guarantees the stability of the negative feedback interconnection, or, equivalently, the original time delay dynamical system. The overall approach provides a concrete method to develop Lyapunov-Krasovskii functionals based on the dissipativity properties of the time delay operator. Finally, analogous results for discrete-time systems are also presented.

2.2. Mathematical Preliminaries

In this chapter, we represent dynamical systems $\mathcal{G}$ defined on the semi-infinite interval $[0, \infty)$ as a mapping between function spaces satisfying an appropriate set of axioms. For the following definition $\mathcal{U}$ is an input space and consists of bounded continuous U-valued functions on $[0, \infty)$. The set $U \subseteq \mathbb{R}^m$ contains the set of input values; that is, at any time t, $u(t) \in U$. The space $\mathcal{U}$ is assumed to be closed under the shift operator; that is, if $u \in \mathcal{U}$, then the function u_T defined by $u_T(t) = u(t+T)$ is contained in $\mathcal{U}$ for all $T \geq 0$. Furthermore, $\mathcal{Y}$ is an output space and consists of continuous Y-valued functions on $[0, \infty)$. The set $Y \subseteq \mathbb{R}^l$ contains the set of output values; that is, each value of $y(t) \in Y$, $t \geq 0$. The space $\mathcal{Y}$ is assumed to be closed under the shift operator; that is, if $y \in \mathcal{Y}$, then the function y_T defined by $y_T(t) = y(t+T)$ is contained in $\mathcal{Y}$ for all $T \geq 0$. Finally, $\mathcal{D}$ is a metric space with topology of uniform convergence and metric $\rho : \mathcal{D} \times \mathcal{D} \to [0, \infty)$. Hence, the notions of openness, convergence, continuity, and compactness that we use in the proceeding work refer to the topology generated on $\mathcal{D}$ by the metric $\rho(\cdot, \cdot)$.

Definition 2.2.1. [21] A *stationary dynamical system* on $\mathcal{D}$ is the octuple ($\mathcal{D}$, $\mathcal{U}$, U, $\mathcal{Y}$, Y, $[0, \infty)$, s, q), where $s : [0, \infty) \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$ and $q : \mathcal{D} \times U \to Y$ are such that the following axioms hold:

i) (Continuity): $s(\cdot, \cdot, u)$ is jointly continuous for all $u \in \mathcal{U}$.

ii) (Consistency): $s(0, x_0, u) = x_0$ for all $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$.

iii) (Determinism): $s(t, x_0, u_1) = s(t, x_0, u_2)$ for all $t \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \leq t$.

iv) (Semi-group property): $s(\tau, s(t, x_0, u), u_t) = s(t+\tau, x_0, u)$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $\tau, t \in [0, \infty)$.

v) There exists $y \in \mathcal{Y}$ such that $y(t) = q(s(t, x_0, u), u(t))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $t \geq 0$.

Henceforth, we denote the dynamical system $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, [0, \infty), s, q)$ by $\mathcal{G}$. Furthermore, we refer to $s(t, x_0, u)$, $t \geq 0$, as the *trajectory* or *state transition operator* of $\mathcal{G}$ corresponding to $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$. For a given trajectory $s(t, x_0, u)$, $t \geq 0$, we refer to $x_0 \in \mathcal{D}$ as the *initial condition* of $\mathcal{G}$. For the dynamical system $\mathcal{G}$ given by Definition 2.2.1, a function $r : U \times Y \to \mathbb{R}$ is called a *supply rate* [21] if it is locally integrable; that is, for all input-output pairs $u \in U$ and $y \in Y$, $r(\cdot, \cdot)$ satisfies $\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s < \infty$, $t_1, t_2 \geq 0$.

Definition 2.2.2 [21,22]. A dynamical system $\mathcal{G}$ is *exponentially dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$, called a *storage function* and a scalar $\varepsilon > 0$, such that the *dissipation inequality*

$$e^{\varepsilon t} V_\mathrm{s}(x(t)) \leq e^{\varepsilon t_1} V_\mathrm{s}(x(t_1)) + \int_{t_1}^{t} e^{\varepsilon s} r(u(s), y(s)) \mathrm{d}s, \tag{2.1}$$

is satisfied for all $t_1, t \geq 0$ and where $x(t) = s(t, x_0, u(t))$, $t \geq t_1$, with $x_0 \in \mathcal{D}$ and $u(t) \in U$. A dynamical system $\mathcal{G}$ is *dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ such that (2.1) is satisfied with $\varepsilon = 0$.

Remark 2.2.1. Recall that a (finite-dimensional) linear dynamical system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is exponentially dissipative with respect to the quadratic supply rate $r(u, y) = y^\mathrm{T} Q y + 2 y^\mathrm{T} S u + u^\mathrm{T} R u$ if and only if [22] there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} A^\mathrm{T} P + PA + \varepsilon P - C^\mathrm{T} Q C & PB - C^\mathrm{T}(S + QD) \\ B^\mathrm{T} P - (S + QD)^\mathrm{T} C & -R - S^\mathrm{T} D - D^\mathrm{T} S \end{bmatrix} \leq 0. \tag{2.2}$$

We begin by recalling a result on stability of feedback interconnections of dissipative dynamical systems. Specifically, in [22] using the notion of dissipative and exponentially dissipative dynamical systems, with appropriate storage functions and supply rates, Lyapunov functions were constructed for interconnected dynamical systems by appropriately combining storage functions of each subsystem. Here, we begin by considering the negative feedback interconnection of dynamical system $\mathcal{G}$ with a feedback system $\mathcal{G}_{\rm d}$ given by an octuple $(\mathcal{D}_{\rm d}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Y}_{\rm d}, Y_{\rm d}, s_{\rm d}, q_{\rm d})$. Note that with the feedback interconnection given in Figure 2.1, $u = -y_{\rm d}$ and $u_{\rm d} = y$.

Definition 2.2.3. A dynamical system $\mathcal{G}$ with input-output pair (u, y) is *zero-state observable* if $u(t) \equiv 0$, $y(t) \equiv 0$ implies $s(t, x_0, u) \equiv 0$.

Theorem 2.2.1 [22]. Consider the feedback system consisting of the stationary dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ with input-output pairs (u, y) and $(u_{\rm d}, y_{\rm d})$, respectively, and with $u_{\rm d} = y$ and $u = -y_{\rm d}$. Assume that $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ are dissipative with respect to the supply rates $r(u, y)$ and $r_{\rm d}(u_{\rm d}, y_{\rm d})$ and with C^0 positive definite, radially unbounded storage functions $V_{\rm s} : \mathcal{D} \to \mathbb{R}$ and $V_{\rm sd} : \mathcal{D}_{\rm d} \to \mathbb{R}$, respectively, such that

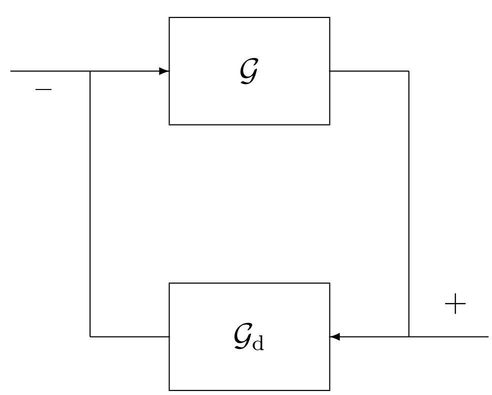

Figure 2.1: Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$

$V_{\rm s}(0)=0$, $V_{\rm sd}(0)=0$. Furthermore, assume that there exists a scalar $\sigma>0$ such that $r(u,y)+\sigma r_{\rm d}(u_{\rm d},y_{\rm d})\leq 0$. Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ is Lyapunov stable.

ii) If $\mathcal{G}$ is exponentially dissipative with respect to supply rate $r(u,y)$ and $\mathcal{G}_{\rm d}$ is zero-state observable then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ is asymptotically stable.

In both cases of Theorem 2.2.1, $V(\cdot,\cdot)=V_{\rm s}(\cdot)+V_{\rm sd}(\cdot)$ is a Lyapunov function for the overall feedback system.

2.3. Stability Theory for Continuous-Time Time-Delay Dynamical Systems using Dissipativity Theory

We will be considering linear time-delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t)=Ax(t)+A_{\rm d}x(t-\tau),\quad x(\theta)=\phi(\theta),\quad -\tau\leq\theta\leq 0,\quad t\geq 0, \tag{2.3}$$

where $x(t)\in\mathbb{R}^n$, $t\geq 0$, $A\in\mathbb{R}^{n\times n}$, $A_{\rm d}\in\mathbb{R}^{n\times n}$, $\tau\geq 0$, and $\phi(\cdot)\in\mathcal{C}=\mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system. Note that the state of (2.3) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau,0]$ and taking values in $\mathbb{R}^n$; that is, $x_t\in\mathcal{C}([-\tau,0],\mathbb{R}^n)$. Hence, $x_t(\theta)=x(t+\theta)$, $\theta\in[-\tau,0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau,0]$, the operator norm $|||x_t|||=\sup_{\theta\in[-\tau,0]}\|x(t+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (2.3). For further details see [3, 8].

We consider a linear time delay dynamical system $\mathcal{G}$ of the form (2.3). Next, we rewrite (2.3) as

$$\dot{x}(t) = Ax(t) - A_{\rm d}u(t), \quad x(0) = \phi(0), \quad t \geq 0, \tag{2.4}$$

$$y(t) = x(t), \tag{2.5}$$

$$y_{\rm d}(t) = \mathcal{G}_{\rm d}(u_{\rm d}(t)), \tag{2.6}$$

where $u(t) = y_{\rm d}(t)$, $u_{\rm d}(t) = y(t)$, and $\mathcal{G}_{\rm d} : \mathcal{C}([-\tau, \infty), \mathbb{R}^n) \to \mathcal{C}([0, \infty), \mathbb{R}^n)$ denotes a delay operator defined by $\mathcal{G}_{\rm d}(u_{\rm d}(t))\Theta = u_{\rm d}(t - \tau)$. Note that (2.4)–(2.6) is a negative feedback interconnection of a linear (finite-dimensional) system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & -A_{\rm d} \\ \hline I_n & 0 \end{array}\right]$ and the (infinite-dimensional) delay operator $\mathcal{G}_{\rm d}$. Hence, stability of (2.3) is equivalent to stability of the negative feedback interconnection of $G(s)$ and $\mathcal{G}_{\rm d}$. Next, we present a key result that shows that the delay operator $\mathcal{G}_{\rm d}$ is dissipative with respect to a quadratic supply rate. First, however, we will show that the *input-output* operator $\mathcal{G}_{\rm d}$ can be characterized as a stationary dynamical system on $\mathcal{C}$. Specifically, let $\mathcal{U}_{\rm d} = \mathcal{C}([-\tau, \infty), \mathbb{R}^n)$, $\mathcal{Y}_{\rm d} = \mathcal{C}([0, \infty), \mathbb{R}^n)$, and $U_{\rm d} = Y_{\rm d} = \mathbb{R}^n$. Now, for every $\phi \in \mathcal{C}$, define $s_\theta : [0, \infty) \times \mathcal{C} \times \mathcal{U}_{\rm d} \to \mathcal{C}$ by

$$s_\theta(t, \phi, u_{\rm d}) = u_{\rm d}(t + \theta), \quad \theta \in [-\tau, 0], \quad t \geq 0, \tag{2.7}$$

where $u_{\rm d}(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$. Finally, define $q_{\rm d} : \mathcal{C} \times U_{\rm d} \to Y_{\rm d}$ by

$$q_{\rm d}(s_\theta(t, \phi, u_{\rm d}), u_{\rm d}(t)) = s_{-\tau}(t, \phi, u_{\rm d}) = u_{\rm d}(t - \tau) = \mathcal{G}_{\rm d}(u_{\rm d}(t)). \tag{2.8}$$

Note that the octuple $(\mathcal{C}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Y}_{\rm d}, Y_{\rm d}, [0, \infty), s_\theta, q_{\rm d})$ satisfies Axioms *i*)–*v*) which implies that the octuple $(\mathcal{C}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Y}_{\rm d}, Y_{\rm d}, [0, \infty), s_\theta, q_{\rm d})$ is a stationary dynamical system on $\mathcal{C}$. For notational convenience we refer to this dynamical system as $\mathcal{G}_{\rm d}$.

Theorem 2.3.1. Consider the dynamical system $\mathcal{G}_{\rm d}$ defined by the octuple ($\mathcal{C}$, $\mathcal{U}_{\rm d}$, $U_{\rm d}$, $\mathcal{Y}_{\rm d}$, $Y_{\rm d}$, $[0, \infty)$, s_θ, $q_{\rm d}$) where s_θ and $q_{\rm d}$ are given by (2.7), (2.8), respectively.

Then $\mathcal{G}_{\rm d}$ is dissipative with respect to the supply rate $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T}Qu_{\rm d} - y_{\rm d}^{\rm T}Qy_{\rm d}$, where $Q \in \mathbb{R}^{n\times n}$ is a positive-definite matrix. Furthermore,

$$V_{\rm sd}(\phi) = \int_{-\tau}^{0} \phi^{\rm T}(\theta)Q\phi((\theta){\rm d}\theta \tag{2.9}$$

is a storage function for $\mathcal{G}_{\rm d}$.

Proof. Note that the storage function $V_{\rm sd}(\cdot)$ evaluated on the trajectory $s_\theta(t, \phi, u_{\rm d})$ is given by

$$\begin{aligned} V_{\rm sd}(s_\theta) &= \int_{-\tau}^{0} s_\theta^{\rm T}(t, \phi, u_{\rm d})Qs_\theta(t, \phi, u_{\rm d}){\rm d}\theta \\ &= \int_{-\tau}^{0} u_{\rm d}^{\rm T}(t+\theta)Qu_{\rm d}(t+\theta){\rm d}t \\ &= \int_{t-\tau}^{t} u_{\rm d}^{\rm T}(\theta)Qu_{\rm d}(\theta){\rm d}\theta. \end{aligned}$$

Hence, the directional derivative of the storage function along the trajectory s_θ is given by

$$\begin{aligned} \dot{V}_{\rm sd}(s_\theta) &= u_{\rm d}(t)Qu_{\rm d}(t) - u_{\rm d}(t-\tau)Qu_{\rm d}(t-\tau) \\ &= u_{\rm d}(t)Qu_{\rm d}(t) - y_{\rm d}(t)Qy_{\rm d}(t), \end{aligned}$$

which is equivalent to (2.1) with $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T}Qu_{\rm d} - y_{\rm d}^{\rm T}Qy_{\rm d}$ and $\varepsilon = 0$. Thus, $\mathcal{G}_{\rm d}$ is dissipative with respect to $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T}Qu_{\rm d} - y_{\rm d}^{\rm T}Qy_{\rm d}$ with the storage function $V_{\rm sd}(\phi)$. □

Next, using Theorem 2.3.1, we present a sufficient condition on $G(s)$ that guarantees asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ or, equivalently, the stability of time delay dynamical system given by (2.3).

Theorem 2.3.2. Consider the linear dynamical system $\mathcal{G} = G(s) \sim \left[\begin{array}{c|c} A & -A_{\rm d} \\ \hline I_n & 0 \end{array}\right]$ with input-output pair (u, y), and the dynamical system $\mathcal{G}_{\rm d}$ given by (2.7), (2.8) with input-output pair $(u_{\rm d}, y_{\rm d})$. Assume that $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $r(u,y) = u^{\mathrm{T}}Qu - y^{\mathrm{T}}Qy$, where $Q \in \mathbb{R}^{n\times n}$ is a positive-definite matrix. Then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is asymptotically stable.

Proof. It follows from Theorem 2.3.1 that $\mathcal{G}_{\mathrm{d}}$ is dissipative with respect to the supply rate $r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) = u_{\mathrm{d}}^{\mathrm{T}}Qu_{\mathrm{d}} - y_{\mathrm{d}}^{\mathrm{T}}Qy_{\mathrm{d}}$ and with storage function $V_{\mathrm{sd}}(\phi) = \int_{-\tau}^{0} \phi^{\mathrm{T}}(\theta)Q\phi(\theta)\mathrm{d}\theta$. Next, it can be easily shown that $\mathcal{G}_{\mathrm{d}}$ is zero-state observable and $V_{\mathrm{sd}}(\cdot)$ is positive definite and radially unbounded. Furthermore, it follows from Remark 2.2.1 that if $\mathcal{G}$ is exponentially dissipative with respect to the quadratic supply rate $r(u,y) = u^{\mathrm{T}}Qu - y^{\mathrm{T}}Qy$, then there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$ is a storage function for $\mathcal{G}$. Now, the result follows as a direct consequence of Theorem 2.2.1. □

Remark 2.3.1. Note that it follows from Remark 2.2.1 that $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $r(u,y) = u^{\mathrm{T}}Qu - y^{\mathrm{T}}Qy$ if and only if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + \varepsilon P + Q & -PA_{\mathrm{d}} \\ -A_{\mathrm{d}}^{\mathrm{T}}P & -Q \end{bmatrix} \leq 0, \tag{2.10}$$

or, equivalently,

$$0 \geq A^{\mathrm{T}}P + PA + \varepsilon P + Q + PA_{\mathrm{d}}Q^{-1}A_{\mathrm{d}}^{\mathrm{T}}P. \tag{2.11}$$

Hence, asymptotic stability of a linear time delay dynamical system given by (2.3) can be checked by checking for feasibility of positive-definite solutions to the linear matrix inequality (2.10).

Remark 2.3.2. It follows from Theorem 2.2.1 that

$$V(x,\phi) = x^{\mathrm{T}}Px + \int_{-\tau}^{0} \phi^{\mathrm{T}}(\theta)Q\phi(\theta)\mathrm{d}\theta, \tag{2.12}$$

where P satisfies (2.10) is a Lyapunov function for the linear time delay dynamical system given by (2.3). Thus the results developed above provide a simple procedure for obtaining Lyapunov functions for time delay systems by exploring the dissipativity properties of the delay operator.

2.4. Stability Theory for Discrete-Time Time-Delay Dynamical Systems using Dissipativity Theory

In this section, we present results on discrete-time time delay systems, analogous to the results in Section 2.3. First, however, we introduce notation, several definitions, and some key results concerning discrete-time dynamical systems that are necessary for developing the main results of this section. Specifically, $\mathcal{N}$ denotes the nonnegative integers, let $\mathcal{C}(\{a,...,b\},\mathbb{R}^n)$ denote a Banach space of functions mapping $\{a,...,b\} \subset \mathcal{N}$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a given non-negative integer $\kappa \in \mathcal{N}$ if $\{a,...,b\} = \{-\kappa, -(\kappa-1), ..., 0\}$ we let $\mathcal{C} = \mathcal{C}(\{-\kappa, -(\kappa-1), ..., 0\}, \mathbb{R}^n)$ and designate the norm of an element ϕ in $\mathcal{C}$ by $|||\phi||| = \sup_{\theta\in\{-\kappa,-(\kappa-1),....,0\}} \|\phi(\theta)\|$. If $\alpha, \beta \in \mathcal{N}$ and $x \in \mathcal{C}(\{\alpha-\kappa, ..., \alpha+\beta\}, \mathbb{R}^n)$, then for every $k \in \{\alpha, ..., \alpha+\beta]$, we let $x_k \in \mathcal{C}$ be defined by $x_k(\theta) = x(k+\theta)$, $\theta \in \{-\kappa, -(\kappa-1), ..., 0\}$.

In this section we represent discrete-time dynamical systems $\mathcal{G}$ defined on $\mathcal{N}$ as a mapping between function spaces satisfying an appropriate set of axioms. For the following definition $\mathcal{U}$ is an input space and consists of bounded continuous U-valued functions on $\mathcal{N}$. The set $U \subseteq \mathbb{R}^m$ contains the set of input values; that is, at any time k, $u(k) \in U$. The space $\mathcal{U}$ is assumed to be closed under the shift operator; that is, if $u \in \mathcal{U}$, then the function u_K defined by $u_K(k) = u(k+K)$ is contained in $\mathcal{U}$ for all $K \in \mathcal{N}$. Furthermore, $\mathcal{Y}$ is an output space and consists of continuous Y-valued functions on $\mathcal{N}$. The set $Y \subseteq \mathbb{R}^l$ contains the set of output values; that is, each value of $y(k) \in Y$, $k \in \mathcal{N}$. The space $\mathcal{Y}$ is assumed to be closed under the shift operator; that is, if $y \in \mathcal{Y}$, then the function y_K defined by $y_K(t) = y(t+K)$ is contained in

$\mathcal{Y}$ for all $K \in \mathcal{N}$. Finally, $\mathcal{D}$ is a metric space with topology of uniform convergence and metric $\rho : \mathcal{D} \times \mathcal{D} \to [0, \infty)$.

Definition 2.4.1. A *stationary dynamical system* on $\mathcal{D}$ is the octuple ($\mathcal{D}$, $\mathcal{U}$, U, $\mathcal{Y}$, Y, $\mathcal{N}$, s, q), where $s : \mathcal{N} \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$ and $q : \mathcal{D} \times U \to Y$ are such that the following axioms hold:

i) (Continuity): $s(k, \cdot, u)$ is continuous for all $k \in \mathcal{N}$ and $u \in \mathcal{U}$.

ii) (Consistency): $s(0, x_0, u) = x_0$ for all $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$.

iii) (Determinism): $s(k, x_0, u_1) = s(k, x_0, u_2)$ for all $k \in \mathcal{N}$, $x_0 \in \mathcal{D}$, and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(\kappa) = u_2(\kappa)$, $\kappa \leq k$.

iv) (Semi-group property): $s(\kappa, s(k, x_0, u), u_k) = s(k + \kappa, x_0, u)$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $\kappa, k \in \mathcal{N}$.

v) There exists $y \in \mathcal{Y}$ such that $y(k) = q(s(k, x_0, u), u(k))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $k \in \mathcal{N}$.

Henceforth, we denote the dynamical system $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, \mathcal{N}, s, q)$ by $\mathcal{G}$. Furthermore, we refer to $s(k, x_0, u)$, $k \in \mathcal{N}$, as the trajectory of $\mathcal{G}$ corresponding to $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$. For the dynamical system $\mathcal{G}$ given by Definition 2.4.1, a function $r : U \times Y \to \mathbb{R}$ is called a *supply rate* if it is locally summable; that is, for all input-output pairs $u \in U$ and $y \in Y$, $r(\cdot, \cdot)$ satisfies $\sum_{k=k_1}^{k_2} |r(u(k), y(k))| < \infty$, $k_1, k_2 \in \mathcal{N}$.

Definition 2.4.2. A discrete-time dynamical system $\mathcal{G}$ is *geometrically dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$, called a *storage function* and a scalar $\rho > 0$, such that the *dissipation inequality*

$$\rho^k V_{\mathrm{s}}(x(k)) \leq \rho^{k_0} V_{\mathrm{s}}(x(k_0)) + \sum_{i=k_0}^{k-1} \rho^i r(u(i), y(i)) \tag{2.13}$$

is satisfied for all $k_0, k \in \mathcal{N}$ and where $x(k) = s(k, x_0, u(k)), k \geq k_0$, with $x_0 \in \mathcal{D}$ and $u(k) \in U$. A discrete-time dynamical system $\mathcal{G}$ is *dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$ such that (2.13) is satisfied with $\rho = 1$.

Remark 2.4.1. Recall that a discrete-time linear dynamical system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is geometrically dissipative with respect to the quadratic supply rate $r(u, y) = y^{\mathrm{T}} Q y + 2 y^{\mathrm{T}} S u + u^{\mathrm{T}} R u$ if and only if [22] there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ and a scalar $\rho > 1$ such that

$$\begin{bmatrix} A^{\mathrm{T}} P A - \frac{1}{\rho} P - C^{\mathrm{T}} Q C & A^{\mathrm{T}} P B - C^{\mathrm{T}} (S + QD) \\ B^{\mathrm{T}} P A - (S + QD)^{\mathrm{T}} C & -R - S^{\mathrm{T}} D - D^{\mathrm{T}} S - D^{\mathrm{T}} Q D + B^{\mathrm{T}} P B \end{bmatrix} \leq 0. \tag{2.14}$$

Next, note that Theorem 2.2.1 is also valid for the discrete-time dynamical systems with the assumption of geometric dissipativity instead of exponential Dissipativity.

Now, we consider a linear discrete-time time delay dynamical system $\mathcal{G}$ of the form

$$x(k+1) = A x(k) + A_{\mathrm{d}} x(k - \kappa), \quad x(\theta) = \phi(\theta), \quad \theta \in \{-\kappa, -(\kappa - 1), ..., 0\}, \quad k \in \mathcal{N}, \tag{2.15}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, $A \in \mathbb{R}^{n \times n}$, $A_{\mathrm{d}} \in \mathbb{R}^{n \times n}$, $\kappa \in \mathcal{N}$, and $\phi(\cdot) \in \mathcal{C} = \mathcal{C}(\{-\kappa, -(\kappa - 1), ..., 0\}, \mathbb{R}^n)$ is a vector valued function specifying the initial state of the system. Note that the state of (2.15) at time k is the piece of trajectories x between $k - \kappa$ and k, or, equivalently, the element x_k in the space of vector valued functions defined on $\{-\kappa, -(\kappa - 1), ..., 0\}$ and taking values in $\mathbb{R}^n$; that is, $x_k \in \mathcal{C}(\{-\kappa, -(\kappa - 1), ..., 0\}, \mathbb{R}^n)$. Hence, $x_k(\theta) = x(k+\theta)$, $\theta \in \{-\kappa, -(\kappa-1), ..., 0\}$. Furthermore, since for a given time k the piece of the trajectories x_k is defined on $\{-\kappa, -(\kappa - 1), ..., 0\}$, the operator norm $|||x_k||| = \sup_{\theta\in\{-\kappa,-(\kappa-1),...,0\}} \|x(k+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (2.15).

Next, we rewrite (2.15) as

$$x(k+1) = Ax(k) - A_{\mathrm{d}}u(k), \quad x(0) = \phi(0), \quad k \in \mathcal{N}, \tag{2.16}$$

$$y(k) = x(k), \tag{2.17}$$

$$y_{\mathrm{d}}(k) = \mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(k)), \tag{2.18}$$

where $u(k) = y_{\mathrm{d}}(k)$, $u_{\mathrm{d}}(k) = y(k)$, and $\mathcal{G}_{\mathrm{d}} : \mathcal{C}(\{-\kappa, -\kappa+1, ..., \infty\}, \mathbb{R}^n) \to \mathcal{C}(\mathcal{N}, \mathbb{R}^n)$ denotes a delay operator defined by $\mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(k))\Theta = u_{\mathrm{d}}(k-\kappa)$. Note that (2.16)–(2.18) is a negative feedback interconnection of a linear system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & -A_{\mathrm{d}} \\ \hline I_n & 0 \end{array}\right]$ and the delay operator $\mathcal{G}_{\mathrm{d}}$. Hence, stability of (2.15) is equivalent to stability of the negative feedback interconnection of $G(s)$ and $\mathcal{G}_{\mathrm{d}}$. Next, we present an analogous to Theorem 2.3.1 that shows that the discrete-time delay operator $\mathcal{G}_{\mathrm{d}}$ is dissipative with respect to a quadratic supply rate. First, however, we will show that the input-output operator $\mathcal{G}_{\mathrm{d}}$ can be characterized as a stationary dynamical system on $\mathcal{C}$. Specifically, let $\mathcal{U}_{\mathrm{d}} = \mathcal{C}(\{-\kappa, -\kappa+1, ...\}, \mathbb{R}^n)$, $\mathcal{Y}_{\mathrm{d}} = \mathcal{C}(\mathcal{N}, \mathbb{R}^n)$, and $U_{\mathrm{d}} = Y_{\mathrm{d}} = \mathbb{R}^n$. Now, for every $\phi \in \mathcal{C}$, define $s_\theta : \mathcal{N} \times \mathcal{C} \times \mathcal{U}_{\mathrm{d}} \to \mathcal{C}$ by

$$s_\theta(k, \phi, u_{\mathrm{d}}) = u_{\mathrm{d}}(k+\theta), \quad \theta \in \{-\kappa, -\kappa+1, ..., 0\}, \quad k \in \mathcal{N}, \tag{2.19}$$

where $u_{\mathrm{d}}(\theta) = \phi(\theta)$, $\theta \in \{-\kappa, -\kappa+1, ..., 0\}$. Finally, define $q_{\mathrm{d}} : \mathcal{C} \times U_{\mathrm{d}} \to Y_{\mathrm{d}}$ by

$$q_{\mathrm{d}}(s_\theta(k, \phi, u_{\mathrm{d}}), u_{\mathrm{d}}(k)) = s_{-\kappa}(k, \phi, u_{\mathrm{d}}) = u_{\mathrm{d}}(k-\kappa) = \mathcal{G}_{\mathrm{d}}(u_{\mathrm{d}}(k)). \tag{2.20}$$

Note that the octuple $(\mathcal{C}, \mathcal{U}_{\mathrm{d}}, U_{\mathrm{d}}, \mathcal{Y}_{\mathrm{d}}, Y_{\mathrm{d}}, \mathcal{N}, s_\theta, q_{\mathrm{d}})$ satisfies Axioms i)–v) which implies that the octuple $(\mathcal{C}, \mathcal{U}_{\mathrm{d}}, U_{\mathrm{d}}, \mathcal{Y}_{\mathrm{d}}, Y_{\mathrm{d}}, \mathcal{N}, s_\theta, q_{\mathrm{d}})$ is a stationary dynamical system on $\mathcal{C}$. For notational convenience we refer to this dynamical system as $\mathcal{G}_{\mathrm{d}}$.

Theorem 2.4.1. Consider the discrete-time dynamical system $\mathcal{G}_{\rm d}$ defined by the octuple $(\mathcal{C}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Y}_{\rm d}, Y_{\rm d}, \mathcal{N}, s_\theta, q_{\rm d})$ where s_θ and $q_{\rm d}$ are given by (2.19), (2.20), respectively. Then $\mathcal{G}_{\rm d}$ is dissipative with respect to the supply rate $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T} Q u_{\rm d} - y_{\rm d}^{\rm T} Q y_{\rm d}$, where $Q \in \mathbb{R}^{n\times n}$ is a positive-definite matrix. Furthermore,

$$V_{\rm sd}(\phi) = \sum_{i=-\kappa}^{-1} \phi^{\rm T}(i) Q \phi(i) \tag{2.21}$$

is a storage function for $\mathcal{G}_{\rm d}$.

Proof. Note that the storage function $V_{\rm sd}(\cdot)$ evaluated on the trajectory $s_\theta(k, \phi, u_{\rm d})$ is given by

$$\begin{aligned} V_{\rm sd}(s_i) &= \sum_{i=-\kappa}^{-1} s_i^{\rm T}(k, \phi, u_{\rm d}) Q s_i(k, \phi, u_{\rm d}) \\ &= \sum_{i=-\kappa}^{-1} u_{\rm d}^{\rm T}(k+i) Q u_{\rm d}(k+i) \\ &= \sum_{\theta=k-\kappa}^{k-1} u_{\rm d}^{\rm T}(i) Q u_{\rm d}(i). \end{aligned}$$

Hence, the difference in the storage function along the trajectory s_i is given by

$$\begin{aligned} \Delta V_{\rm sd}(x(k)) &= V_{\rm sd}(x_{\rm d}(k+1)) - V_{\rm sd}(x_{\rm d}(k)) \\ &= \sum_{i=k-\kappa+1}^{k} u_{\rm d}^{\rm T}(i) Q u_{\rm d}(i) - \sum_{i=k-\kappa}^{k-1} u_{\rm d}^{\rm T}(i) Q u_{\rm d}(i) \\ &= u_{\rm d}(k) Q u_{\rm d}(k) - u_{\rm d}^{\rm T}(k-\kappa) Q u_{\rm d}(k-\kappa) \\ &= u_{\rm d}(k) Q u_{\rm d}(k) - y_{\rm d}(k) Q y_{\rm d}(k), \end{aligned}$$

which implies that

$$V_{\rm sd}(x(k)) = V_{\rm sd}(x(k_0)) + \sum_{i=k_0}^{k-1} u_{\rm d}^{\rm T}(i) Q u_{\rm d}(i) - y_{\rm d}(k) Q y_{\rm d}(k)$$

which is equivalent to (2.13) with $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T} Q u_{\rm d} - y_{\rm d}^{\rm T} Q y_{\rm d}$ and $\rho = 1$. Thus, $\mathcal{G}_{\rm d}$ is dissipative with respect to $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T} Q u_{\rm d} - y_{\rm d}^{\rm T} Q y_{\rm d}$ with the storage function $V_{\rm sd}(\phi)$. □

Next, using Theorem 2.4.1, we present a sufficient condition on $G(s)$ that guarantees asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ or, equivalently, the stability of time delay dynamical system given by (2.15).

Theorem 2.4.2. Consider the discrete-time linear dynamical system defined by $\mathcal{G} = G(s) \sim \left[\begin{array}{c|c} A & -A_{\rm d} \\ \hline I_n & 0 \end{array}\right]$ with input-output pair (u, y), and the discrete-time dynamical system $\mathcal{G}_{\rm d}$ given by (2.19), (2.20) with input-output pair $(u_{\rm d}, y_{\rm d})$. Assume that $\mathcal{G}$ is geometrically dissipative with respect to the supply rate $r(u, y) = u^{\rm T} Q u - y^{\rm T} Q y$, where $Q \in \mathbb{R}^{n \times n}$ is a positive-definite matrix. Then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ is asymptotically stable.

Proof. It follows from Theorem 2.4.1 that $\mathcal{G}_{\rm d}$ is dissipative with respect to the supply rate $r_{\rm d}(u_{\rm d}, y_{\rm d}) = u_{\rm d}^{\rm T} Q u_{\rm d} - y_{\rm d}^{\rm T} Q y_{\rm d}$ and with storage function $V_{\rm sd}(\phi) = \sum_{i=-\kappa}^{-1} \phi^{\rm T}(i) Q \phi(i)$. Next, it can be easily shown that $\mathcal{G}_{\rm d}$ is zero-state observable and $V_{\rm sd}(\cdot)$ is positive definite and radially unbounded. Furthermore, it follows from Remark 2.4.1 that if $\mathcal{G}$ is geometrically dissipative with respect to the quadratic supply rate $r(u, y) = u^{\rm T} Q u - y^{\rm T} Q y$, then there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ such that $V_{\rm s}(x) = x^{\rm T} P x$ is a storage function for $\mathcal{G}$. Now, the result follows as a direct consequence of Theorem 2.2.1. □

Remark 2.4.2. Note that it follows from Remark 2.4.1 that $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $r(u, y) = u^{\rm T} Q u - y^{\rm T} Q y$ if and only if there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ and a scalar $\rho > 1$ such that

$$\begin{bmatrix} A^{\rm T} P A - \frac{1}{\rho} P - Q & -A^{\rm T} P A_{\rm d} \\ -A_{\rm d}^{\rm T} P A & -Q + A_{\rm d}^{\rm T} P A_{\rm d} \end{bmatrix} \leq 0, \quad (2.22)$$

or, equivalently,

$$0 \geq A^{\mathrm{T}}PA - \frac{1}{\rho}P - Q - A^{\mathrm{T}}PA_{\mathrm{d}}(A_{\mathrm{d}}^{\mathrm{T}}PA_{\mathrm{d}} - Q)^{-1}A_{\mathrm{d}}^{\mathrm{T}}PA \tag{2.23}$$

Hence, asymptotic stability of a discrete-time linear time delay dynamical system given by (2.15) can be checked by checking for feasibility of positive-definite solutions to the linear matrix inequality (2.22).

Remark 2.4.3. It follows from Theorem 3.1 that

$$V(x, \phi) = x^{\mathrm{T}}Px + \sum_{i=-\kappa}^{-1} \phi^{\mathrm{T}}(i)Q\phi(i), \tag{2.24}$$

where P satisfies (2.22) is a Lyapunov function for the linear time delay dynamical system given by (2.15). Thus the results developed above provide a simple procedure for obtaining Lyapunov functions for discrete-time time delay systems by exploring the dissipativity properties of the delay operator.

2.5. Conclusion

In this chapter, using the concepts of dissipativity and exponential dissipativity, we developed sufficient conditions to guarantee asymptotic stability of a time delay dynamical system. Specifically, we first represented a time delay dynamical system as a negative feedback interconnection of a (finite-dimensional) linear dynamical system and an (infinite-dimensional) time delay operator. Next, we showed that the time delay operator is dissipative with respect to a quadratic supply rate. Finally, based on this result, we developed a sufficient condition on the linear dynamical system that guarantees stability of the negative feedback interconnection. The overall approach provides a method for developing Lyapunov-Krasovskii functionals based on the dissipativity properties of the time delay operator. Here, we considered dissipative properties of the time delay operator that are independent of the amount of time delay. Future extensions of this work will involve dissipative properties of the time delay operator which will include the amount of time delay (i.e. *delay dependent* conditions) thus providing a mechanism for obtaining Lyapunov-Krasovskii functionals to prove stability of time delay dynamical systems that depend on the amount of time delay.

Chapter 3

Stability Analysis of Time Delay Systems using Dynamic Dissipativity Theory

3.1. Introduction

In this chapter, we extend the notions of dissipativity [21] and exponential dissipativity [22] theory to derive new sufficient conditions for guaranteeing asymptotic stability of time delay dynamical systems. Specifically, we introduce the notion of *dynamic dissipativity*; namely, $(\Sigma, \hat{Q})$-dissipativity, where Σ is a dynamical system and $\hat{Q}$ is a symmetric matrix. By choosing a certain dynamical system Σ and a symmetric matrix $\hat{Q}$ it can be shown that a system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if $\mathcal{G}$ is dissipative with respect to a quadratic supply rate. Thus, $(\Sigma, \hat{Q})$-dissipativity provides a nontrivial extension of dissipativity theory with respect to a quadratic supply rate. Based on $(\Sigma, \hat{Q})$-dissipativity theory, we then provide a result on stability of negative feedback interconnection of $(\Sigma, \hat{Q})$-dissipative systems. Next, representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional linear dynamical system and an infinite-dimensional time delay operator, we show that the time delay operator is $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipative. Furthermore, for a special choice of $\Sigma_{\rm d}$ and $\hat{Q}_{\rm d}$, we show that the storage functional of the time-delay operator involves an integral term which is identical to the integral term appearing in the Lyapunov-Krasovskii functional. Thus the overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator.

3.2. Mathematical Preliminaries

In this chapter, we shall continue to use the notion of dynamical systems as described in the previous chapter (see Section 2.2 for details). We shall begin by considering a dynamical system Σ given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}, U \times Y, \mathcal{Z}, Z, [0,\infty), \hat{s}, \hat{q})$, where $Z \subseteq \mathbb{R}^p$, $\mathcal{Z}$ is an output space which consists of continuous Z-valued functions on $[0,\infty)$, and consider the cascade interconnection of $\mathcal{G}$ and Σ as shown in Figure 3.1. We denote this interconnected dynamical system $(\mathcal{D}\times\hat{\mathcal{D}}, \mathcal{U}, U, \mathcal{Z}, Z, [0,\infty), [s^{\mathrm{T}}, \hat{s}^{\mathrm{T}}]^{\mathrm{T}}, \hat{q})$ by $\tilde{\mathcal{G}}$. For the following definition, let $\hat{Q} \in \mathbb{R}^{p\times p}$ and $\hat{Q} = \hat{Q}^{\mathrm{T}}$.

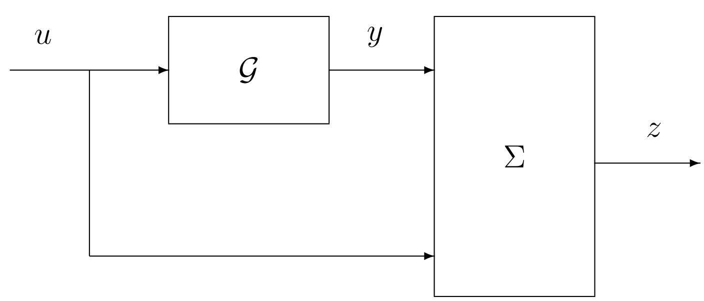

Figure 3.1: Interconnection of $\mathcal{G}$ and Σ

Definition 3.2.1. A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*exponentially dissipative* if there exists a C^0 nonnegative-definite function $\hat{V}_{\mathrm{s}} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$, called a $(\Sigma, \hat{Q})$-*storage function* and a scalar $\varepsilon > 0$, such that the $(\Sigma, \hat{Q})$-*dissipation inequality*

$$e^{\varepsilon t}\hat{V}_{\mathrm{s}}(x(t), \hat{x}(t)) \leq e^{\varepsilon t}\hat{V}_{\mathrm{s}}(x(t_1), \hat{x}(t_1)) + \int_{t_1}^{t} e^{\varepsilon s} z^{\mathrm{T}}(s)\hat{Q}z(s)\mathrm{d}s$$

is satisfied for all $t, t_1 \geq 0$ and where $x(t) = s(t, x_0, u(t))$, $\hat{x}(t) = \hat{s}(t, \hat{x}_0, u(t), y(t))$, $t \geq t_1$, with $x_0 \in \mathcal{D}$, $\hat{x}_0 \in \hat{\mathcal{D}}$, $\hat{x}_0 = 0$, $u(t) \in U$, and $y(t) = q(x(t), u(t))$. A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*dissipative* if there exists a C^0 nonnegative-definite function $\hat{V}_{\mathrm{s}} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$ such that (3.1) is satisfied with $\varepsilon = 0$.

Remark 3.2.1. If $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative, where Σ is a linear dynamical system given by the transfer function $\hat{G}(s)$, then

$$\int_{-\infty}^{\infty} \begin{bmatrix} U(j\omega) \\ Y(j\omega) \end{bmatrix}^{\star} \hat{G}^{\star}(j\omega)\hat{Q}\hat{G}(j\omega) \begin{bmatrix} U(j\omega) \\ Y(j\omega) \end{bmatrix} \mathrm{d}\omega \geq 0, \tag{3.1}$$

where $U(s)$ and $Y(s)$, $s \in \mathbb{C}$, are the Laplace transforms of $u(t)$ and $y(t)$, respectively. Hence, $(\Sigma, \hat{Q})$-dissipativity is a time-domain analog to Integral Quadratic Constraints (IQCs) [23].

Remark 3.2.2. Let $p = l + m$ and let the dynamical system Σ be such that $z = q(\hat{x}, u, y) = [u^{\mathrm{T}}\ y^{\mathrm{T}}]^{\mathrm{T}}$. Furthermore, let $\hat{Q} = \begin{bmatrix} R & S^{\mathrm{T}} \\ S & Q \end{bmatrix}$, where $Q = Q^{\mathrm{T}} \in \mathbb{R}^{l \times l}$, $S \in \mathbb{R}^{l \times m}$, and $R = R^{\mathrm{T}} \in \mathbb{R}^{m \times m}$. In this case, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$. Hence, $(\Sigma, \hat{Q})$-dissipativity provides a dynamic extension of dissipativity notions with respect to a quadratic supply rate.

The following result provides a sufficient condition for $(\Sigma, \hat{Q})$-dissipativity of $\mathcal{G}$ in the case where $\mathcal{G}$ and Σ are linear dynamical systems. Specifically, let $\mathcal{G}$ and Σ be given by transfer functions $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$, respectively, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{l\times m}$, $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times(l+m)}$, $\hat{C} \in \mathbb{R}^{p\times\hat{n}}$ and $\hat{D} \in \mathbb{R}^{p\times(l+m)}$. In this case, the interconnection of $\mathcal{G}$ and Σ as shown in Figure 3.1 is given by the transfer function $\tilde{G}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \hat{B}_y C & \hat{A} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ \hat{B}_y D + \hat{B}_u \end{bmatrix}, \tag{3.2}$$

$$\tilde{C} = [\hat{D}_y C \ \ \hat{C}], \qquad \tilde{D} = \hat{D}_u + \hat{D}_y D, \tag{3.3}$$

where $\hat{B}_u \in \mathbb{R}^{\hat{n}\times m}$, $\hat{B}_y \in \mathbb{R}^{\hat{n}\times l}$, $\hat{D}_u \in \mathbb{R}^{p\times m}$, and $\hat{D}_y \in \mathbb{R}^{p\times l}$ are such that $\hat{B} = [\hat{B}_u \ \hat{B}_y]$ and $\hat{D} = [\hat{D}_u \ \hat{D}_y]$.

Proposition 3.2.1. Consider the dynamical system $\mathcal{G}$ given by the transfer function $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, let $\hat{Q} \in \mathbb{R}^{p\times p}$, $\hat{Q} = \hat{Q}^{\mathrm{T}}$, and let Σ be a linear dynamical system given by the transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$. Then, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \varepsilon\tilde{P} & \tilde{P}\tilde{B} \\ \tilde{B}^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} \le \begin{bmatrix} \tilde{C}^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}} \end{bmatrix} \hat{Q} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix}. \tag{3.4}$$

Furthermore, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ such that (3.4) holds with $\varepsilon = 0$.

Proof. The proof is a direct consequence of the generalized Kalman-Yakubovich-Popov lemma [22]. □

Remark 3.2.3. It follows from Proposition 3.2.1 that if $\tilde{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, then $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a positive-definite matrix $\tilde{P}$ such that (3.4) holds.

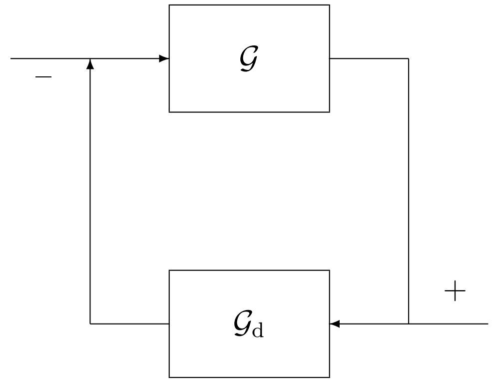

Figure 3.2: Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$

Next, we present a result on stability of feedback interconnection of dissipative dynamical systems. Specifically, consider the negative feedback interconnection of dynamical system $\mathcal{G}$ with a feedback system $\mathcal{G}_{\rm d}$ given by the octuple ($\mathcal{D}_{\rm d}$, $\mathcal{U}_{\rm d}$, $U_{\rm d}$, $\mathcal{Y}_{\rm d}$, $Y_{\rm d}$, $[0,\infty)$, $s_{\rm d}$, $q_{\rm d}$). Note that with the feedback interconnection given in Figure 3.2, $u = -y_{\rm d}$ and $u_{\rm d} = y$. Hence, $U = Y_{\rm d}$ and $Y = U_{\rm d}$. Furthermore, consider a dynamical system $\Sigma_{\rm d}$ given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}_{\rm d}, U_{\rm d} \times Y_{\rm d},\ Z_{\rm d}, Z,\ [0,\infty), \hat{s}_{\rm d}, \hat{q}_{\rm d})$, where $\hat{s}_{\rm d}(t,\hat{x},u_{\rm d},y_{\rm d}) = \hat{s}(t,\hat{x}_0,-y_{\rm d},u_{\rm d})$ and $\hat{q}_{\rm d}(\hat{x},u_{\rm d},y_{\rm d}) = \hat{q}(\hat{x},-y_{\rm d},u_{\rm d})$. In addition, consider the interconnected dynamical system $\tilde{\mathcal{G}}_{\rm d}$ given by the octuple $(\mathcal{D}_{\rm d} \times \hat{\mathcal{D}}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Z}_{\rm d}, Z,\ [0,\infty),\ [s_{\rm d}^{\rm T} \quad \hat{s}_{\rm d}^{\rm T}], \hat{q}_{\rm d})$ (see Figure 3.3). The following definition is needed for the statement of the next result.

For the statement of the next result let $\|\cdot\|_\sigma$ and $\|\cdot\|_\mu$ denote operator norms on $\mathcal{D}$ and $\mathcal{D}_{\rm d}$, respectively, and let $\gamma^+(x_0,x_{\rm d0}) = \cup_{t\geq 0}\{(s(t,x_0,u), s_{\rm d}(t,x_{\rm d0},u_{\rm d}))\}$, with $u = -y_{\rm d}$ and $u_{\rm d} = y$, denote the positive orbit of the feedback system $\mathcal{G}$ and $\mathcal{G}_{\rm d}$. Furthermore, recall that $\gamma^+(x_0,x_{\rm d0})$ is *precompact* if $\gamma^+(x_0,x_{\rm d0})$ can be enclosed in the union of a finite number of ε-balls around elements of $\gamma^+(x_0,x_{\rm d0})$.

Theorem 3.2.1. Let $\hat{Q}, \hat{Q}_{\rm d} \in \mathbb{R}^{p\times p}$ be such that $\hat{Q} = \hat{Q}^{\rm T}$ and $\hat{Q}_{\rm d} = \hat{Q}_{\rm d}^{\rm T}$. Consider the feedback system consisting of the stationary dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ with input-output pairs (u,y) and $(u_{\rm d},y_{\rm d})$, respectively, and with $u_{\rm d} = y$ and $u = -y_{\rm d}$.

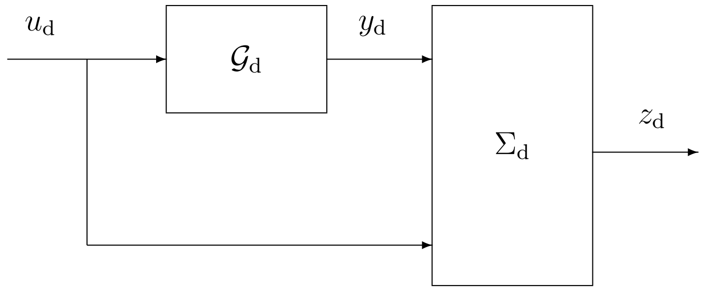

Figure 3.3: Interconnection of $\mathcal{G}_\mathrm{d}$ and Σ_d

Assume that $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ are $(\Sigma, \hat{Q})$-dissipative and $(\Sigma_\mathrm{d}, \hat{Q}_\mathrm{d})$-dissipative with C^0 storage functions $V_\mathrm{s} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$ and $V_\mathrm{sd} : \mathcal{D}_\mathrm{d} \times \hat{\mathcal{D}}_\mathrm{d} \to \mathbb{R}$, respectively, such that $V_\mathrm{s}(0,0) = 0$, $V_\mathrm{sd}(0,0) = 0$, and

$$\alpha(\|x\|_\sigma) \leq V_\mathrm{s}(x, \hat{x}), \ (x, \hat{x}) \in \mathcal{D} \times \hat{\mathcal{D}}, \tag{3.5}$$

$$\alpha_\mathrm{d}(\|x_\mathrm{d}\|_\mu) \leq V_\mathrm{sd}(x_\mathrm{d}, \hat{x}_\mathrm{d}), \ (x_\mathrm{d}, \hat{x}_\mathrm{d}) \in \mathcal{D}_\mathrm{d} \times \hat{\mathcal{D}}_\mathrm{d}, \tag{3.6}$$

where $\alpha, \alpha_\mathrm{d} : [0, \infty) \to [0, \infty)$ are class $\mathcal{K}_\infty$ functions. Furthermore, assume that for each initial condition $(x_0, x_{\mathrm{d}0}) \in \mathcal{D} \times \mathcal{D}_\mathrm{d}$, the positive orbit $\gamma^+(x_0, x_{\mathrm{d}0})$ of the feedback system $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is precompact. Finally, assume there exists a scalar $\sigma > 0$ such that $\hat{Q} + \sigma \hat{Q}_\mathrm{d} \leq 0$. Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is Lyapunov stable.

ii) If $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is Lyapunov stable and for every $x(0) \in \mathcal{D}$, $\|x(t)\|_\sigma \to 0$ as $t \to \infty$.

Proof. The proof follows from standard Lyapunov theory and invariant set arguments as applied to infinite-dimensional dynamical systems [3, 24]. Specifically, note that $u = -y_\mathrm{d}$, $u_\mathrm{d} = y$, and since $\hat{x}_0 = \hat{x}_{\mathrm{d}_0} = 0$, $\hat{x}(t) = \hat{x}_\mathrm{d}(t)$ and $z_\mathrm{d}(t) = z(t)$, $t \geq 0$. Hence, the state of the overall interconnection of $\mathcal{G}$, $\mathcal{G}_\mathrm{d}$, and Σ (see Figure 3.4) is given by $[x^\mathrm{T}, \ x_\mathrm{d}^\mathrm{T}, \ \hat{x}^\mathrm{T}]^\mathrm{T}$. Now, consider the Lyapunov function candidate

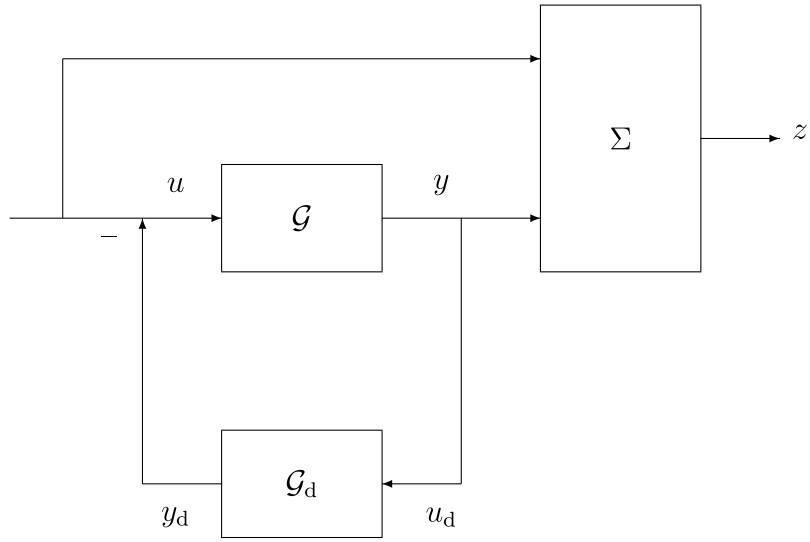

Figure 3.4: Interconnection of $\mathcal{G}$, $\mathcal{G}_\mathrm{d}$, and Σ

$V(x, x_\mathrm{d}, \hat{x}) = V_\mathrm{s}(x, \hat{x}) + \sigma V_\mathrm{sd}(x_\mathrm{d}, \hat{x})$ and since $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ are $(\Sigma, \hat{Q})$-dissipative and $(\Sigma_\mathrm{d}, \hat{Q}_\mathrm{d})$-dissipative, respectively, it follows that

$$\begin{aligned}\dot{V}(x(t), x_\mathrm{d}(t), \hat{x}(t)) &= \dot{V}_\mathrm{s}(x(t), \hat{x}(t)) + \sigma \dot{V}_\mathrm{s}(x_\mathrm{d}(t), \hat{x}(t)) \\ &\leq z^\mathrm{T}(t)\hat{Q}z(t) + \sigma z_\mathrm{d}^\mathrm{T}(t)\hat{Q}_\mathrm{d} z_\mathrm{d}(t) \\ &= z^\mathrm{T}(t)(\hat{Q} + \sigma \hat{Q}_\mathrm{d})z(t) \\ &\leq 0.\end{aligned}$$

Now, Lyapunov stability follows from standard arguments.

Next, if $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative, then it can be shown as above that for every $x_0 \in \mathbb{R}^n$ and $x_{\mathrm{d}0} \in \mathbb{R}^{n_\mathrm{d}}$,

$$\dot{V}(x(t), x_\mathrm{d}(t), \hat{x}(t)) \leq -\varepsilon V_\mathrm{s}(x, \hat{x}) \leq -\varepsilon \alpha(\|x\|) \leq 0,$$

where $\varepsilon > 0$. Hence, $V(x(t), x_\mathrm{d}(t), \hat{x}(t))$, $t \geq 0$, is a monotonically decreasing function and since $V(\cdot, \cdot, \cdot)$ is lower bounded it follows that $c \triangleq \lim_{t\to\infty} V(x(t),\, x_\mathrm{d}(t),\, \hat{x}(t)) \geq 0$

exists. Next it follows from LaSalle's invariant set theorem [25] that the positive limit set $\omega(x_0, x_{d0}, 0)$ is nonempty and invariant. Thus, $\dot{V}(x(t), x_d(t), \hat{x}(t)) = 0$, $t \geq 0$, $(x(0), x_d(0), 0) \in \omega(x_0, x_{d0}, 0)$, which further implies that $\|x(t)\| = 0$, $t \geq 0$, $(x(0), x_d(0), 0) \in \omega(x_0, x_{d0}, 0)$. Hence, $\|x(t)\| \to 0$ as $t \to \infty$. □

Remark 3.2.4. Note that (3.5) and (3.6) are only sufficient conditions needed to prove Lyapunov stability for the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_d$. In the case of stability analysis of time-delay systems, (3.5) and (3.6) may be replaced by a weaker condition. See Remark 3.3.2 below.

Remark 3.2.5. In the case where Σ and Σ_d are such that $z = [u^T\ y^T]^T$ and $z_d = [-y_d^T\ u_d^T]^T$, Theorem 3.2.1 specializes to Theorem 5.2 of [22].

3.3. Stability Theory for Time-Delay Dynamical Systems using Dissipativity Theory

In this section we consider linear time delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau), \quad x(\theta) = \phi(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0, \tag{3.7}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $\tau \geq 0$, and $\phi(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system. Note that the state of (3.7) at time t is the *piece of trajectories* x between $t - \tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau, 0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Hence, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau, 0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (3.7). For further details see [3, 8].

Next, we rewrite (3.7) as a feedback system so that

$$\dot{x}(t) = Ax(t) - A_{\rm d}u(t), \ x(0) = \phi(0), \ t \geq 0, \tag{3.8}$$

$$y(t) = x(t), \tag{3.9}$$

$$y_{\rm d}(t) = \mathcal{G}_{\rm d}(u_{\rm d}(t)), \tag{3.10}$$

where $u(t) = y_{\rm d}(t)$, $u_{\rm d}(t) = y(t)$, and $\mathcal{G}_{\rm d} : \mathcal{C}([-\tau, \infty), \mathbb{R}^n) \to \mathcal{C}([0, \infty), \mathbb{R}^n)$ denotes a delay operator defined by $\mathcal{G}_{\rm d}(u_{\rm d}(t)) \triangleq u_{\rm d}(t - \tau)$. Note that (3.8)–(3.10) is a negative feedback interconnection of a linear finite-dimensional system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & -A_{\rm d} \\ \hline I_n & 0 \end{array}\right]$ and the infinite-dimensional delay operator $\mathcal{G}_{\rm d}$. Hence, stability of (3.7) is equivalent to stability of the negative feedback interconnection of $G(s)$ and $\mathcal{G}_{\rm d}$. Next, we present a key result that shows that the delay operator $\mathcal{G}_{\rm d}$ is dissipative with respect to a quadratic supply rate. First, however, we show that the *input-output* operator $\mathcal{G}_{\rm d}$ can be characterized as a stationary dynamical system on $\mathcal{C}$. Specifically, let $\mathcal{U}_{\rm d} = \mathcal{C}([-\tau, \infty), \mathbb{R}^n)$, $\mathcal{Y}_{\rm d} = \mathcal{C}([0, \infty), \mathbb{R}^n)$, and $U_{\rm d} = Y_{\rm d} = \mathbb{R}^n$. Now, for every $\phi \in \mathcal{C}$, define $s_\theta : [0, \infty) \times \mathcal{C} \times \mathcal{U}_{\rm d} \to \mathcal{C}$ by

$$s_\theta(t, \phi, u_{\rm d}) = u_{\rm d}(t + \theta), \quad \theta \in [-\tau, 0], \quad t \geq 0, \tag{3.11}$$

where $u_{\rm d}(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$. Finally, define $q_{\rm d} : \mathcal{C} \times U_{\rm d} \to Y_{\rm d}$ by

$$q_{\rm d}(s_\theta(t, \phi, u_{\rm d}), u_{\rm d}(t)) = s_{-\tau}(t, \phi, u_{\rm d}) = u_{\rm d}(t - \tau) = \mathcal{G}_{\rm d}(u_{\rm d}(t)). \tag{3.12}$$

Note that the octuple $(\mathcal{C}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Y}_{\rm d}, Y_{\rm d}, [0, \infty), s_\theta, q_{\rm d})$ satisfies Axioms *i*)–*v*) of Definition 2.2.1 which implies that the octuple $(\mathcal{C}, \mathcal{U}_{\rm d}, U_{\rm d}, \mathcal{Y}_{\rm d}, Y_{\rm d}, [0, \infty), s_\theta, q_{\rm d})$ is a stationary dynamical system on $\mathcal{C}$. For notational convenience we refer to this dynamical system as $\mathcal{G}_{\rm d}$.

To show that $\mathcal{G}_{\rm d}$ is $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipative, let Σ denote a linear dynamical system given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}, \mathbb{R}^n \times \mathbb{R}^n, \mathcal{Z}, \mathbb{R}^{2\hat{p}}, [0, \infty), \hat{s}, \hat{q})$, where $\hat{\mathcal{D}} \subset \mathbb{R}^{2\hat{n}}$ and with transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$, where

$$\begin{aligned} \hat{A} &= \text{block–diag}[A_1, A_1],\ \hat{B} = \text{block–diag}[B_1, B_1], \\ \hat{C} &= \text{block–diag}[C_1, C_1],\ \hat{D} = I_{2n} \end{aligned} \tag{3.13}$$

and where $A_1 \in \mathbb{R}^{\hat{n}\times\hat{n}}$ is Hurwitz, $B_1 \in \mathbb{R}^{\hat{n}\times n}$, and $C_1 \in \mathbb{R}^{\hat{p}\times\hat{n}}$. In this case, the dynamical system $\Sigma_{\rm d}$ is given by the transfer function $\hat{G}_{\rm d}(s) \sim \left[\begin{array}{c|c} \hat{A}_{\rm d} & \hat{B}_{\rm d} \\ \hline \hat{C}_{\rm d} & \hat{D}_{\rm d} \end{array}\right]$, where

$$\hat{A}_{\rm d} = \hat{A},\ \hat{B}_{\rm d} = \begin{bmatrix} 0 & -B_1 \\ B_1 & 0 \end{bmatrix},\ \hat{C}_{\rm d} = \hat{C},\ \hat{D}_{\rm d} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}. \tag{3.14}$$

Hence, the state space representation of the interconnection shown in Figure 3.3 is given by

$$y_{\rm d}(t) = \mathcal{G}_{\rm d}(u_{\rm d}(t)), \quad s_\theta(0, \phi, u_{\rm d}) = \phi(\theta), \quad \theta \in [-\tau, 0],\ t \geq 0, \tag{3.15}$$

$$\dot{x}_{\rm d_1}(t) = A_1 x_{\rm d_1}(t) - B_1 y_{\rm d}(t), \quad x_{\rm d_1}(0) = 0, \tag{3.16}$$

$$\dot{x}_{\rm d_2}(t) = A_1 x_{\rm d_2}(t) + B_1 u_{\rm d}(t), \quad x_{\rm d_2}(0) = 0, \tag{3.17}$$

$$\hat{z}_{\rm d_1}(t) = C_1 x_{\rm d_1}(t) - D_1 y_{\rm d}(t), \tag{3.18}$$

$$\hat{z}_{\rm d_2}(t) = C_1 x_{\rm d_2}(t) + D_1 u_{\rm d}(t). \tag{3.19}$$

Lemma 3.3.1. Let $\hat{Q}_{\rm d} = \text{block-diag}[-Q, Q]$, where $Q \in \mathbb{R}^{\hat{p}\times\hat{p}}$. If $\phi(\theta) = 0$, $\theta \in [-\tau, 0]$, then for every $u_{\rm d}(\cdot) \in \mathcal{U}_{\rm d}$,

$$\int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t = \int_\theta^T \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \geq 0,\ T > 0, \tag{3.20}$$

where $\theta = 0$, $T \in [0, \tau]$, and $\theta = T - \tau$, $T > \tau$.

Proof. Note that

$$\begin{aligned} x_{\mathrm{d}_1}(t) &= -\int_0^t e^{A_1(t-s)} B_1 y_{\mathrm{d}}(s)\mathrm{d}s, \quad t \geq 0, \\ \text{and} \quad x_{\mathrm{d}_2}(t) &= \int_0^t e^{A_1(t-s)} B_1 u_{\mathrm{d}}(s)\mathrm{d}s, \quad t \geq 0. \end{aligned}$$

Since $y_{\mathrm{d}}(t) = u_{\mathrm{d}}(t-\tau)$, $t \geq 0$ and $u_{\mathrm{d}}(\theta) = \phi(\theta) = 0$, $\theta \in [-\tau, 0]$, it follows that $x_{\mathrm{d}_1}(t) = 0$, $t \in [0, \tau]$, and for all $t \geq \tau$,

$$x_{\mathrm{d}_1}(t) = -\int_\tau^t e^{A_1(t-s)} B_1 u_{\mathrm{d}}(s-\tau)\mathrm{d}s = -x_{\mathrm{d}_2}(t-\tau).$$

Hence, $\hat{z}_{\mathrm{d}_1}(t) = 0$, $t \in [0, \tau]$, and $\hat{z}_{\mathrm{d}_1}(t) = -\hat{z}_{\mathrm{d}_2}(t-\tau)$, $t > \tau$, which implies that

$$\begin{aligned} \int_0^T \hat{z}_{\mathrm{d}}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}}(t)\mathrm{d}t &= \int_0^T [\hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)Q\hat{z}_{\mathrm{d}_2}(t) - \hat{z}_{\mathrm{d}_1}^{\mathrm{T}}(t)Q\hat{z}_{\mathrm{d}_1}(t)]\mathrm{d}t \\ &= \int_{T-\tau}^T \hat{z}_{\mathrm{d}_2}^{\mathrm{T}}(t)Q\hat{z}_{\mathrm{d}_2}(t)\mathrm{d}t \geq 0, \quad T \geq \tau. \end{aligned}$$

The case where $T \in [0, \tau]$ follows in a similar manner. □

Theorem 3.3.1. Consider the dynamical system $\mathcal{G}_{\mathrm{d}}$ defined by the octuple ($\mathcal{C}$, $\mathcal{U}_{\mathrm{d}}$, U_{d}, $\mathcal{Y}_{\mathrm{d}}$, Y_{d}, $[0, \infty)$, s_θ, q_{d}), where s_θ and q_{d} are given by (3.11) and (3.12), respectively. Next, let Σ_{d} be a linear dynamical system with transfer function $\hat{G}_{\mathrm{d}}(s) \sim \left[\begin{array}{c|c} \hat{A}_{\mathrm{d}} & \hat{B}_{\mathrm{d}} \\ \hline \hat{C}_{\mathrm{d}} & \hat{D}_{\mathrm{d}} \end{array}\right]$, where $\hat{A}_{\mathrm{d}}$, $\hat{B}_{\mathrm{d}}$, $\hat{C}_{\mathrm{d}}$ and $\hat{D}_{\mathrm{d}}$ are given by (3.14), and let $\hat{Q}_{\mathrm{d}} =$ block-diag $[-Q,\, Q]$, where $Q \in \mathbb{R}^{\hat{p}\times\hat{p}}$, $Q > 0$. Then, $\mathcal{G}_{\mathrm{d}}$ is $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-dissipative. Furthermore,

$$V_{\mathrm{sd}}(\psi, \hat{x}_{\mathrm{d}_1}, \hat{x}_{\mathrm{d}_2}) = -\inf_{u_{\mathrm{d}}(\cdot)\in\mathcal{U}_{\mathrm{d}}, T\geq 0} \int_0^T \hat{z}_{\mathrm{d}}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}}(t)\mathrm{d}t \tag{3.21}$$

is a $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-storage function for $\mathcal{G}_{\mathrm{d}}$ where the infimum in (3.21) is performed over all trajectories of $\tilde{\mathcal{G}}_{\mathrm{d}}$ with initial conditions $\phi(\cdot) = \psi(\cdot)$, $x_{\mathrm{d}_1}(0) = \hat{x}_{\mathrm{d}_1}$, and $x_{\mathrm{d}_2}(0) = \hat{x}_{\mathrm{d}_2}$.

Proof. It follows from (3.21) that

$$V_{\mathrm{sd}}(\psi, \hat{x}_{\mathrm{d}_1}, \hat{x}_{\mathrm{d}_2}) = -\inf_{u_{\mathrm{d}}(\cdot)\in\mathcal{U}_{\mathrm{d}}, T\geq 0} \int_0^T \hat{z}_{\mathrm{d}}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}}(t)\mathrm{d}t$$

$$= \sup_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, T\geq 0} \int_0^T [\hat{z}_{\rm d_1}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_1}(t) - \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t)]{\rm d}t. \tag{3.22}$$

Hence, $V_{\rm sd}(\psi, \hat{x}_{\rm d_1}, \hat{x}_{\rm d_2}) \geq 0$, $\psi(\cdot) \in \mathcal{C}$, $\hat{x}_{\rm d_1}, \hat{x}_{\rm d_2} \in \mathbb{R}^n$. If $\psi(\theta) \equiv 0$, $\theta \in [-\tau, 0]$, $\hat{x}_{\rm d_1} = 0$, $\hat{x}_{\rm d_2} = 0$, then it follows from Lemma 3.3.1 that

$$\int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t = \int_0^T \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t, \quad T \geq 0,$$

i.e.,

$$V_{\rm sd}(0,0,0) = \sup_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, T\geq 0} -\int_0^T \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \leq 0.$$

Hence, since $V_{\rm sd}(0,0,0) \geq 0$, $V_{\rm sd}(0,0,0) = 0$. Next, note that for every $u_{\rm d}(t)$, $t \in [t_1, t_{\rm f}]$, and $T \in [t_1, t_{\rm f}]$,

$$\begin{aligned} &-V_{\rm sd}(s_\theta(t_1, \psi, u_{\rm d}), x_{\rm d_1}(t_1), x_{\rm d_2}(t_2)) \\ &\leq \int_{t_1}^{t_{\rm f}} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \\ &= \int_{t_1}^{T} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t + \int_{T}^{t_{\rm f}} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t. \end{aligned}$$

Hence,

$$\begin{aligned} &-V_{\rm sd}(s_\theta(t_1, \psi, u_{\rm d}), x_{\rm d_1}(t_1), x_{\rm d_2}(t_1)) - \int_{t_1}^{T} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \\ &\leq \int_{T}^{t_{\rm f}} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t, \end{aligned}$$

i.e.,

$$\begin{aligned} &-V_{\rm sd}(s_\theta(t_1, \psi, u_{\rm d}), x_{\rm d_1}(t_1), x_{\rm d_2}(t_2)) - \int_{t_1}^{T} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \\ &\leq \inf_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, t_{\rm f}\geq T} \int_{T}^{t_{\rm f}} \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t \\ &= -V_{\rm sd}(s_\theta(T, \psi, u_{\rm d}), x_{\rm d_1}(T), x_{\rm d_2}(T)), \end{aligned}$$

establishing the $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipativity of $\mathcal{G}_{\rm d}$. □

Remark 3.3.1. In the case where $A_1 = 0$, $B_1 = 0$, and $C_1 = 0$, it can be shown that

$$V_{\rm sd}(\psi, x_{\rm d1}, x_{\rm d2}) = V_{\rm sd}(\psi) = \int_{-\tau}^{0} \psi^{\rm T}(\theta) Q \psi(\theta) {\rm d}\theta. \tag{3.23}$$

Next, using Theorem 3.3.1, we present a sufficient condition on $G(s)$ that guarantees asymptotic stability of the negative feedback interconnection of the time delay dynamical system given by (3.7). For the following result we assume that $V_{\rm sd}(\cdot,\cdot,\cdot)$ given by (3.21) is continuously differentiable.

Theorem 3.3.2. Consider the linear time delay dynamical system given by (3.7). Let $\hat{Q} = \text{block-diag}[Q, -Q]$, where $Q \in \mathbb{R}^{\hat{p}\times\hat{p}}$, $Q > 0$. Assume there exists a nonnegative definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ and scalars $\varepsilon, \eta > 0$ such that (3.4) holds and $\tilde{P} \geq \text{block–diag}[\eta I_n,\ 0_{\hat{n}\times\hat{n}},\ 0_{\hat{n}\times\hat{n}}]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_1 & 0 \\ B_1 & 0 & A_1 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} -A_{\rm d} \\ B_1 \\ 0 \end{bmatrix},\ \tilde{C} = \begin{bmatrix} 0 & C_1 & 0 \\ I_n & 0 & C_1 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \tag{3.24}$$

Then the linear time delay dynamical system given by (3.7) is asymptotically stable for every $\tau \in [0, \infty)$.

Proof. It follows from Theorem 3.3.1 that $\mathcal{G}_{\rm d}$ is $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipative with $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-storage function $V_{\rm sd}(\psi, x_{{\rm d}_1}, x_{{\rm d}_2})$, $\psi \in \mathcal{C}$, $x_{{\rm d}_1}, x_{{\rm d}_2} \in \mathbb{R}^{\hat{n}}$, given by (3.21). Next, it follows from Proposition 3.2.1 that $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative with $(\Sigma, \hat{Q})$-storage function $V_{\rm s}(\tilde{x}) = \tilde{x}^{\rm T}\tilde{P}\tilde{x}$, where $\tilde{x} = [x^{\rm T}, x_1^{\rm T}, x_2^{\rm T}]^{\rm T}$. Furthermore, note that $x = \psi(0)$ and as in the proof of Theorem 3.2.1, it can be shown that $\hat{x}_1(t) = x_{{\rm d}_1}(t)$, $\hat{x}_2(t) = x_{{\rm d}_2}(t)$, $t \geq 0$, and hence the state of the overall interconnection of $\mathcal{G}$, $\mathcal{G}_{\rm d}$, and Σ (see Figure 3.4) is given by $[\psi^{\rm T},\ \hat{x}^{\rm T}]^{\rm T}$ where $\hat{x} = [\hat{x}_1^{\rm T},\ \hat{x}_2^{\rm T}]^{\rm T}$. Next, using the Lyapunov-Krasovskii functional candidate $V(\psi, \hat{x}_1, \hat{x}_2) = V_{\rm s}(\psi(0), \hat{x}_1, \hat{x}_2) + V_{\rm sd}(\psi, \hat{x}_1, \hat{x}_2)$, it follows that

$$\dot{V}(x_t, \hat{x}_1(t), \hat{x}_1(t)) \leq -\varepsilon\tilde{x}^{\rm T}(t)P\tilde{x}(t) \leq -\varepsilon\eta x^{\rm T}(t)x(t). \tag{3.25}$$

Now, Lyapunov stability follows from standard arguments as applied to time delay systems (see Theorem 2.1 of [3, p. 132] for a similar proof). The proof of asymptotic stability is similar to that of Theorem 3.2.1 and hence is omitted. □

Remark 3.3.2. Note that if $V_{\rm s}(\tilde{x})$ and $V_{\rm sd}(\psi, x_{\rm d_1}, x_{\rm d_2})$ satisfy (3.5) and (3.6), then Theorem 3.3.2 follows from Theorem 3.2.1. However, in the case of time delay dynamical systems (3.5) and (3.6) can be replaced by a weaker condition

$$\eta\psi^{\rm T}(0)\psi(0) \leq V(\psi, \hat{x}_1, \hat{x}_2), \quad \psi \in \mathcal{C}, \quad \hat{x}_1, \hat{x}_2 \in \mathbb{R}^{\hat{n}}. \tag{3.26}$$

In this case, Lyapunov and asymptotic stability can be shown using the fact that $\|x(t)\| \leq \varepsilon$, $t \geq 0$, if and only if $|||x_t||| \leq \varepsilon$, $t \geq 0$.

Remark 3.3.3. Recall that the linear time delay dynamical system given by (3.7) is stable for all $\tau \in [0, \infty)$ if and only if [26] there exists $N : \jmath\mathbb{R} \to \mathbb{C}^{n\times n}$ such that $N(\jmath\omega) > 0$, $\omega \in \mathbb{R}$, and

$$G^*(\jmath\omega)N(\jmath\omega)G(\jmath\omega) - N(\jmath\omega) < 0, \qquad \omega \in \mathbb{R}. \tag{3.27}$$

Thus, if there exists $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ such that (3.4) holds, then it follows from Proposition 3.2.1 that $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative which implies (3.27) (see Remark 3.2.1) with $N(\jmath\omega) = G_1^*(\jmath\omega)QG_1(\jmath\omega)$, where $G_1(\jmath\omega) = C_1(\jmath\omega I_{\hat{n}} - A_1)^{-1}B_1 + I_n$, $\omega \in \mathbb{R}$. Hence, (3.4) is a sufficient condition for satisfying (3.27) and $G_1^*(\jmath\omega)QG_1(\jmath\omega)$ is a real rational approximation to $N(\jmath\omega)$ in (3.27).

Remark 3.3.4. In the case where $A_1 = 0$, $B_1 = 0$, and $C_1 = 0$, it follows from Theorem 3.3.2 that if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^{\rm T}P + PA + \varepsilon P + Q & -PA_{\rm d} \\ -A_{\rm d}^{\rm T}P & -Q \end{bmatrix} \leq 0, \tag{3.28}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm d}$ is asymptotically stable. Furthermore, it follows from Remark 3.3.1 that $V_{\rm sd}(\psi) = \int_{-\tau}^{0} \psi^{\rm T}(\theta) Q \psi(\theta) {\rm d}\theta$ and hence $V(\psi) = \psi^{\rm T}(0) P \psi(0) + \int_{-\tau}^{0} \psi^{\rm T}(\theta) Q \psi(\theta) {\rm d}\theta$ is a Lyapunov-Krasovskii functional for the linear time delay dynamical system (3.7). Thus, Theorem 3.3.2 provides a generalization to the sufficient conditions for linear time delay dynamical systems given in [1, 4].

3.4. Illustrative Numerical Example

In this section, we provide a numerical example to illustrate the utility of the results developed above. Consider the linear time delay dynamical system given by (3.7) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix}, \quad A_{\rm d} = \begin{bmatrix} -0.0604 & 0.0060 & 0.3018 & 0 \\ 0.0060 & 0.0060 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.2074 & 0 & -0.6037 & 0 \end{bmatrix}. \tag{3.29}$$

Now, with $A_1 = -I_4$, $B_1 = I_4$, and $C_1 = [0_{4\times 4} \; I_4]^{\rm T}$, we can show that there exist positive definite matrices $\tilde{P}$ and Q such that (3.4) holds. Hence, it follows from Theorem 3.3.2 that the linear time-delay dynamical system given by (3.7) with A and $A_{\rm d}$ given by (3.29) is asymptotically stable for every $\tau \in [0, \infty)$. However, it can be shown that there does not exist positive-definite matrices P and Q such that (3.28) holds which shows that Theorem 3.3.2 provides less conservative sufficient conditions for stability analysis of time delay systems as compared to the standard sufficient conditions given in the literature (see, for example, [1, 4]).

3.5. Conclusion

In this chapter, we extended the concepts of dissipativity and exponential dissipativity to provide new sufficient conditions for guaranteeing asymptotic stability of a time delay dynamical system. Specifically, representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional linear dynamical system and an infinite-dimensional time delay operator, we showed that the time delay operator is dissipative. Finally, using stability of feedback interconnection results for dissipative systems, we developed new sufficient conditions for asymptotic stability of time delay dynamical systems. The overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator.

Chapter 4

Structured Phase Margin for Stability Analysis of Time-Delay Systems

4.1. Introduction

Phase information has largely been neglected in robust control theory, but is essential for maximizing achievable performance in controlling uncertain dynamical systems. Phase information, here, refers to the characterization of the phase of the modeling uncertainty in the frequency domain. The analysis and synthesis of robust feedback controllers entails a fundamental distinction between parametric and nonparametric uncertainty. Parametric uncertainty refers to plant uncertainty that is modeled as constant real parameters, whereas nonparametric uncertainty refers to uncertain transfer function gains that may be modeled as complex frequency-dependent quantities. Real parametric uncertainty in the time domain provides phase information in the frequency domain.

The distinction between parametric and nonparametric uncertainty is critical to the achievable performance of feedback control systems. This distinction can be illustrated by considering the central result of feedback control theory, namely, the small gain theorem, which guarantees robust stability by requiring that the loop gain (including desired weighting functions for loop shaping) be less than unity at all frequencies. The small gain theorem, however, does not make use of phase information in guaranteeing stability. In fact, the small gain theorem allows the loop transfer function to possess arbitrary phase at all frequencies, although in many applications at least some knowledge of phase is available [27]. Thus, small gain techniques such as H_∞ theory are generally conservative when phase information is available. More generally, since $|e^{\jmath\phi}| = 1$ regardless of the phase angle ϕ, it can be expected that any robustness theory based upon norm bounds will suffer from the same shortcoming. Of course, every real parameter can be viewed as a complex parameter with phase $\phi = 0°$ or $\phi = 180°$.

To some extent, phase information is accounted for by means of positivity theory [28–32]. In this theory, a positive real plant and a strictly positive real uncertainty are both assumed to have phase less than 90° so that the loop transfer function has less than 180° of phase shift, hence guaranteeing robust stability in spite of gain uncertainty. Both gain and phase properties can be simultaneously accounted for by means of the circle criterion [32–34] which yields the small gain and positivity theorems as special cases. It is important to note, however, that positivity theory and the circle criterion can be obtained from small gain conditions by means of suitable transformations, and hence, are equivalent results from a mathematical point of view.

The ability to address block-structured gain and phase uncertainty is essential for reducing conservatism in the analysis and synthesis of control systems involving robust stability and performance objectives. Accordingly, the structured singular value provides a generalization of the spectral (maximum singular value) norm to permit small-gain type analysis of systems involving block-structured complex, real, and mixed uncertainty [35–41]. Even though the structured singular value guarantees robust stability by means of bounds involving frequency-dependent scales and multipliers which account for the structure of the uncertainty as well as its real or complex nature [35–41], it does not directly capture phase uncertainty information.

Phase information for uncertain dynamical systems has been studied by a significant number of researchers. Concepts such as principal phases [42, 43], multivariable phase margin [44, 45], phase spread [46], phase envelope [47], phase matching [48–51], phase-sensitive structured singular value [52, 53], and plant uncertainty templates [54–56] are notable contributions. Principal phases are defined to be the phase angles associated with the eigenvalues of the unitary part of the polar decomposition of a complex matrix [42,43]. Exploiting transfer function phase information, the authors in [42] obtain a small phase theorem that provides less conservative stability results than the small gain theorem. Building on the results of [42, 43], the concept of multivariable phase margin is addressed in [44]. An alternative approach to capturing phase uncertainty is given in [46] in terms of the numerical range. In particular, the numerical range provides both gain and phase information, and hence, can be used to guarantee robust stability with respect to system uncertainties having phase-dependent gain variation. Phase-sensitive structured singular value results are obtained in [52, 53] that allow the incorporation of phase information with multiple-block uncertainty. An additional class of results involving phase matching for addressing system phase uncertainty is reported in [48–51]. Here, the goal is to obtain a reduced-order model of a power spectral density by approximating the phase of the spectral factor. An input-output description of system uncertainty is given in [47] in terms of gain and phase envelopes. Finally, gain and phase information is addressed in Quantitative Feedback Theory in the form of frequency domain uncertainty templates which account for both structured and unstructured uncertainty [54–56].

Phase information is critical in capturing system time delays which play an important role in modern engineering systems. In particular, many complex engineering network systems involve power transfers between interconnected system components that are not instantaneous, and hence, realistic models for capturing the dynamics of such systems should account for information in transit [19]. Such models lead to delay dynamical systems. Time-delay dynamical systems have been extensively studied in the literature (see [1–18] and the numerous references therein). Since time delay can severely degrade system performance and in many cases drive the system to instability, stability analysis of time-delay dynamical systems remains a very important area of research [1–5]. Time-delay stability analysis has been mainly classified into two categories, namely, *delay-dependent* and *delay-independent* analysis [1, 4, 11–16]. Delay-independent stability criteria provide sufficient conditions for stability of time-delay dynamical systems independent of the amount of time delay, whereas delay-dependent stability criteria provide sufficient conditions that are dependent on an upper bound of the time delay. In systems where the time delay is known to be bounded, delay-dependent criteria usually give far less conservative stability predictions as compared to delay-independent results. Hence, for such systems it is of paramount importance to derive the sharpest possible delay-dependent stability margins.

A key method for analyzing stability of time-delay dynamical systems is Lyapunov's second method as applied to functional differential equations. Specifically, stability analysis of a given linear time-delay dynamical system is typically shown using a Lyapunov-Krasovskii functional [3, 8]. These stability criteria may also be interpreted in the frequency domain in terms of a feedback interconnection of a matrix transfer function and a *phase* uncertainty block [1]. Since phase uncertainties have unit gain, delay-independent stability criteria may be derived using the classical small gain theorem or, more generally, the scaled small gain theorem [1,16]. However, in order to derive delay-dependent stability criteria using the (scaled) small gain approach, one has to perform certain model transformations and then apply the scaled small gain theorem [1, 16]. The necessity for such model transformations lies in the fact that delay-dependent stability criteria may be derived only if we can characterize the phase of the uncertainty in addition to the gain uncertainty.

In this paper, we present a robust stability analysis method to account for phase uncertainties. Specifically, we develop a general framework for stability analysis of linear systems with structured phase uncertainties. In particular, we introduce the notion of the *structured phase margin* for characterizing stability margins for a dynamical system with block-structured phase uncertainty. In the special case where the uncertainty has no internal structure, the structured phase margin is shown to specialize to the multivariable phase margin given in [44]. Furthermore, since the structured phase margin may be, in general, difficult to compute, we derive an easily computable lower bound in terms of a generalized eigenvalue problem. This bound is constructed by choosing stability multipliers that are tailored to the structure of the phase uncertainty. In addition, using the structured phase margin, we derive new and improved delay-dependent stability criteria for stability analysis of time-delay systems. Even though frequency-domain and integral quadratic constraints (IQCs) have been developed to address the time delay problem (see [1, 4, 5, 13, 16, 17] and references therein), with the notable exception of [13, 57, 58], all of these results rely on the scaled small gain theorem as applied to a *transformed* system. In contrast, we present new robust stability results for time-delay systems based on pure phase information.

4.2. Mathematical Preliminaries

The following results are needed for the main results of this paper.

Lemma 4.2.1. Let $M \in \mathbb{C}^{n\times n}$. Assume there exist matrices $Q \in \mathbb{H}^{n\times n}$, $R \in \mathbb{H}^{n\times n}$, and $S \in \mathbb{C}^{n\times n}$ such that

$$M^*RM - M^*S - S^*M + Q < 0. \tag{4.1}$$

Then, $\det(I + M\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$, where

$$\mathbf{\Delta} \triangleq \{\Delta \in \mathbb{C}^{n\times n} : \Delta^*Q\Delta + \Delta^*S^* + S\Delta + R \geq 0\}. \tag{4.2}$$

Proof. Let $\Delta \in \mathbf{\Delta}$, and suppose, *ad absurdum*, $\det(I + M\Delta) = 0$. Then, there exists $x \in \mathbb{C}^n$, $x \neq 0$, such that $x = -M\Delta x$. Since $\Delta \in \mathbf{\Delta}$, it follows that

$$\begin{aligned}
0 &\leq x^*[\Delta^*Q\Delta + \Delta^*S^* + S\Delta + R]x \\
&= x^*\Delta^*Q\Delta x + x^*\Delta^*S^*x + x^*S\Delta x + x^*Rx \\
&= x^*\Delta^*Q\Delta x - x^*\Delta^*S^*M\Delta x - x^*\Delta^*M^*S\Delta x + x^*\Delta^*M^*RM\Delta x \\
&= x^*\Delta^*[M^*RM - M^*S - S^*M + Q]\Delta x,
\end{aligned}$$

which contradicts (4.1). □

The following result is a generalization of the Kalman-Yakubovich-Popov (KYP) lemma, and establishes the equivalence between a generalized frequency domain inequality and a linear matrix inequality.

Proposition 4.2.1 [59]. Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times m}$, and $D \in \mathbb{R}^{p\times m}$. Furthermore, let $Q \in \mathbb{S}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$, and $R \in \mathbb{S}^{m\times m}$. Then,

$$G^*(\jmath\omega)QG(\jmath\omega) + G^*(\jmath\omega)S + S^{\mathrm{T}}G(\jmath\omega) + R < 0, \; \omega \in [0, \infty), \tag{4.3}$$

if and only if there exists $P \in \mathbb{S}^{n\times n}$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + C^{\mathrm{T}}QC & PB + C^{\mathrm{T}}(QD+S) \\ B^{\mathrm{T}}P + (QD+S)^{\mathrm{T}}C & R + S^{\mathrm{T}}C + C^{\mathrm{T}}S + D^{\mathrm{T}}QD \end{bmatrix} < 0. \tag{4.4}$$

For the statement of the next theorem, let $Q \in \mathbb{H}^{n\times n}$ and $R \in \mathbb{N}^{n\times n}$ such that $\lambda_{\min}(R) \le 1 \le \lambda_{\max}(R)$, and define the optimization problems:

i) $\mathcal{O}_1 \triangleq \min_{x\in\mathcal{X}_1\cap\mathcal{X}_2} x^*Qx$, where $\mathcal{X}_1 \triangleq \{x \in \mathbb{C}^n : x^*x = 1\}$ and $\mathcal{X}_2 \triangleq \{x \in \mathbb{C}^n : x^*Rx = 1\}$.

ii) $\mathcal{O}_2 \triangleq \sup_{\lambda\in\mathbb{R}} \phi(\lambda)$, where $\phi(\lambda) \triangleq \min_{x\in\mathcal{X}_1}[x^*Qx + \lambda(1 - x^*Rx)]$.

iii) $\mathcal{O}_3 \triangleq \sup_{\lambda,\mu\in\mathbb{R}} \psi(\lambda,\mu)$, where $\psi(\lambda,\mu) \triangleq \inf_{x\in\mathbb{C}^n}[x^*Qx+\lambda(1-x^*Rx)+\mu(1-x^*x)]$.

Theorem 4.2.1. Let $Q \in \mathbb{H}^{n\times n}$ and $R \in \mathbb{N}^{n\times n}$ such that $\lambda_{\min}(R) \le 1 \le \lambda_{\max}(R)$. Then

$$\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \sup\{\lambda + \mu : Q - \lambda R - \mu I_n \ge 0,\ \lambda,\ \mu \in \mathbb{R}\}. \tag{4.5}$$

Proof. Since $\lambda_{\min}(R) \le 1 \le \lambda_{\max}(R)$, it follows that $\mathcal{X}_1 \cap \mathcal{X}_2$ is nonempty and compact and since x^*Qx is continuous, it follows that there exists $\bar{x} \in \mathcal{X}_1 \cap \mathcal{X}_2$ such that $-\infty < \mathcal{O}_1 = \bar{x}^*Q\bar{x} < \infty$. Next, let $x \in \mathcal{X}_1 \cap \mathcal{X}_2$ and note that

$$\phi(\lambda) \le x^*Qx + \lambda(1 - x^*Rx) = x^*Qx,\ \lambda \in \mathbb{R}.$$

Hence, $\mathcal{O}_2 \le x^*Qx$, $x \in \mathcal{X}_1 \cap \mathcal{X}_2$, which further implies that $\mathcal{O}_2 \le \mathcal{O}_1$. Now, let $\lambda,\mu \in \mathbb{R}$ be such that $Q - \lambda R - \mu I_n \ge 0$. The existence of $\lambda,\mu \in \mathbb{R}$ such that this inequality holds can be easily established. In this case, it follows that

$$\psi(\lambda,\mu) = \lambda + \mu + \inf_{x\in\mathbb{C}^n} x^*(Q - \lambda R - \mu I_n)x = \lambda + \mu.$$

Next, for all $\lambda, \mu \in \mathbb{R}$ such that $Q - \lambda R - \mu I_n$ has a negative eigenvalue, it follows that $\psi(\lambda, \mu) = -\infty$. Hence,

$$\mathcal{O}_3 = \sup\{\lambda + \mu : Q - \lambda R - \mu I_n \geq 0,\ \lambda,\ \mu \in \mathbb{R}\}.$$

Furthermore, for all $\lambda, \mu \in \mathbb{R}$,

$$\psi(\lambda, \mu) \leq x^* Q x + \lambda(1 - x^* R x),\ x \in \mathcal{X}_1,$$

which implies that $\psi(\lambda, \mu) \leq \phi(\lambda) \leq \mathcal{O}_2$, and hence, $\mathcal{O}_3 \leq \mathcal{O}_2$. Thus,

$$\mathcal{O}_1 \geq \mathcal{O}_2 \geq \mathcal{O}_3 = \sup\{\lambda + \mu : Q - \lambda R - \mu I_n \geq 0,\ \lambda,\ \mu \in \mathbb{R}\}.$$

Next, consider the optimization problem $i)$ and assume that $\bar{x} \in \mathcal{X}_1 \cap \mathcal{X}_2$ (a global maximizer) is a regular point, that is, $R\bar{x}$ and $\bar{x}$ are linearly independent vectors. In this case, it follows from the first- and second-order necessary conditions for optimality [60] that there exist $\bar{\lambda}, \bar{\mu} \in \mathbb{R}$ satisfying

$$(Q - \bar{\lambda} R - \bar{\mu} I_n)\bar{x} = 0 \quad \text{and} \quad Q - \bar{\lambda} R - \bar{\mu} I_n \geq 0. \tag{4.6}$$

Thus, $\bar{x}^*(Q - \bar{\lambda} R - \bar{\mu} I_n)\bar{x} = 0$, and hence,

$$\begin{aligned}
\mathcal{O}_1 &= \bar{x}^* Q \bar{x} \\
&= \bar{x}^* Q \bar{x} + \lambda(1 - \bar{x}^* R \bar{x}) + \mu(1 - \bar{x}^* \bar{x}) \\
&= \bar{x}^*(Q - \lambda R - \mu I_n)\bar{x} + \lambda + \mu \\
&= \bar{\lambda} + \bar{\mu} \\
&\leq \sup\{\lambda + \mu : Q - \lambda R - \mu I_n \geq 0,\ \lambda, \mu \in \mathbb{R}\} \\
&= \mathcal{O}_3,
\end{aligned}$$

establishing $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3$. The proof of the result in the case where $\bar{x}$ is not a regular point is considerably longer and, hence, it is omitted. □

4.3. The Structured Phase Margin of a Complex Matrix

In this section, we introduce the notion of the structured phase margin of a complex matrix which is essential for characterizing phase information in dynamical systems with block-structured uncertainty and deriving delay-dependent stability criteria for time-delay systems.

Definition 4.3.1. Let $M \in \mathbb{C}^{n\times n}$. The *structured phase margin* $\phi(M)$ is defined by

$$\phi(M) \triangleq \begin{cases} \infty, & \text{if } \det(I_n + Me^{\jmath\Theta}) \neq 0,\ \Theta \in \mathbf{\Theta}, \\ \min\{\rho(\Theta) : \det(I_n + Me^{\jmath\Theta}) = 0,\ \Theta \in \mathbf{\Theta}\}, & \text{otherwise}, \end{cases}$$

where $\mathbf{\Theta} \subseteq \mathbb{C}^{n\times n}$ is a set of block-diagonal phase uncertainty matrices defined by

$$\begin{aligned} \mathbf{\Theta} \triangleq \{\ \Theta \in \mathbb{H}^{n\times n} &: -\pi I_n < \Theta \leq \pi I_n,\ \Theta = \text{block–diag}(I_{l_1} \otimes \Theta_1, I_{l_2} \otimes \Theta_2, \ldots, I_{l_r} \otimes \Theta_r), \\ &\Theta_i \in \mathbb{H}^{n_i\times n_i}, i = 1, \ldots, r\}, \end{aligned} \tag{4.7}$$

where the dimension n_i and the number of repetitions l_i of each block are such that $\sum_{i=1}^{r} l_i n_i = n$ and $r \geq 1$.

In the case where $r = 1$, $l_1 = 1$, and $n_1 = n$, $\phi(M)$ is specialized to the *multivariable phase margin* $\underline{\phi}(M)$ given in [44]. Furthermore, in the case where $r = 1$, $n_1 = 1$, and $l_1 = n$, $\phi(M)$ collapses to the classical (*scalar*) *phase margin* of M and is denoted by $\overline{\phi}(M)$.

Remark 4.3.1. In the case where $r = 1$, $n_1 = n = 1$, and $l_1 = 1$, $\phi(M)$ corresponds to the smallest angle by which the complex number M needs to be rotated (either clockwise or counterclockwise) in the complex plane before intersecting the

$-1+\jmath 0$ point. Specifically, if $|M| \neq 1$, then $\phi(M) = \infty$ since no amount of rotation of M in the complex plane will intersect $-1+\jmath 0$. Alternatively, if $M = e^{\jmath\alpha}$, where $\alpha \in [-\pi, \pi]$, then the angle of rotation of M in the complex plane needed to intersect $-1+\jmath 0$ is simply $|\pi - \alpha|$, that is, $\phi(M) = |\pi - \alpha|$. More generally, let $G(s)$ denote a single-input, single-output transfer function. In this case, $\inf_{\omega\in\mathbb{R}} \phi(G(\jmath\omega))$ is the phase margin of $G(s)$.

Remark 4.3.2. In the case where $\mathbf{\Theta} = \mathbb{H}^{n\times n}$, that is, the set of phase perturbations has no internal structure, $\phi(M) = \underline{\phi}(M)$ and is identical to the multivariable phase margin defined in [44].

Next, in order to account for the phase structure of $\mathbf{\Theta}$ we introduce the following scaling matrix set $\mathcal{T}$ defined by

$$\mathcal{T} \triangleq \{T \in \mathbb{H}^{n\times n} : \ T\Theta = \Theta T, \quad \Theta \in \mathbf{\Theta}\}. \tag{4.8}$$

Note that in light of the definition of $\mathbf{\Theta}$, $\mathcal{T}$ is the set of Hermitian matrices given by

$$\begin{aligned} \mathcal{T} = \{\ T \in \mathbb{H}^{n\times n} :\ & T = \text{block--diag}(T_1 \otimes I_{n_1}, T_2 \otimes I_{n_2}, \ldots, T_r \otimes I_{n_r}), \\ & T_i \in \mathbb{H}^{l_i\times l_i},\ i = 1, \ldots, r\}. \end{aligned} \tag{4.9}$$

Proposition 4.3.1. Let $M \in \mathbb{C}^{n\times n}$. Then the following statements hold:

i) Let $\alpha > 0$. Then $\alpha < \phi(M)$ (resp., $\alpha \leq \phi(M)$) if and only if $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, $\rho(\Theta) \leq \alpha$ (resp., $\rho(\Theta) < \alpha$).

ii) Let $T \in \mathcal{T}$ be nonsingular. Then $\phi(M) = \phi(T^{-1}MT)$.

iii) $\underline{\phi}(M) \leq \phi(M) \leq \overline{\phi}(M)$.

iv) $\phi(M) \in [0, \pi] \cup \{\infty\}$.

v) $\overline{\phi}(M) \leq \pi$ if and only if there exists $\lambda \in \operatorname{spec}(M)$ such that $|\lambda| = 1$.

Proof. The proof is a direct consequence of the definitions of $\phi(M)$, $\underline{\phi}(M)$, and $\overline{\phi}(M)$.

□

4.4. A Computable Lower Bound for the Structured Phase Margin

Since the computation of the structured phase margin $\phi(M)$ is in general difficult, in this section we derive a lower bound for the structured phase margin. This lower bound is presented in the form of a generalized eigenvalue problem, and hence, can be computed using linear matrix inequalities [61]. Specifically, let $M \in \mathbb{C}^{n\times n}$ and define

$$\gamma_{\mathrm{lb}}(M) \triangleq \inf\{\ \gamma \in \mathbb{R} : \text{there exist } R \in \mathcal{T} \text{ and } S \in \mathcal{T} \text{ such that } S \geq 0, \text{ and}$$
$$M^*RM - R - M^*S - SM < 2\gamma S\}. \tag{4.10}$$

Furthermore, define

$$\phi_{\mathrm{lb}}(M) \triangleq \begin{cases} \alpha, & \text{if } \gamma_{\mathrm{lb}}(M) \in [-1,1], \\ \infty, & \text{if } \gamma_{\mathrm{lb}}(M) < -1, \end{cases} \tag{4.11}$$

where $\alpha \in [0,\pi]$ is such that $\cos(\alpha) = \gamma_{\mathrm{lb}}(M)$.

Theorem 4.4.1. $\phi_{\mathrm{lb}}(\cdot)$ is well defined and, for every $M \in \mathbb{C}^{n\times n}$, $\phi_{\mathrm{lb}}(M) \leq \phi(M)$.

Proof. Let $\gamma = 1 + \varepsilon$, where $\varepsilon > 0$, and let $S = I_n$ and $R = -I_n$. In this case,

$$\begin{aligned} 2\gamma S + M^*S + SM - M^*RM + R &= (M + I_n)^*(M + I_n) + 2\varepsilon I_n \\ &\geq 2\varepsilon I_n \\ &> 0, \end{aligned}$$

which implies that for every $\varepsilon > 0$, $\gamma_{\mathrm{lb}}(M) \leq 1+\varepsilon$. Hence, $\gamma_{\mathrm{lb}}(M) \leq 1$ which implies that $\phi_{\mathrm{lb}}(\cdot)$ is well defined.

Next, assume $\phi_{\mathrm{lb}}(M) = \infty$ or, equivalently, $\gamma_{\mathrm{lb}}(M) < -1$, and let $\gamma < -1$ be such that $\gamma_{\mathrm{lb}}(M) \leq \gamma$. Hence, there exists $S \in \mathcal{T}$ and $R \in \mathcal{T}$, $S \geq 0$, such that

$$2\gamma S + M^*S + SM - M^*RM + R > 0. \tag{4.12}$$

Now, with $Q \triangleq -2\gamma S - R$, it follows from Lemma 4.2.1 that $\det(I_n + M\Delta) \neq 0$, $\Delta \in \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is given by (4.2). Next, define

$$\boldsymbol{\Theta}_\gamma \triangleq \{\Theta \in \boldsymbol{\Theta} : \cos(\rho(\Theta)) \geq \gamma\}.$$

Now, for all $\Theta \in \boldsymbol{\Theta}_\gamma$, let $\Delta = e^{\jmath\Theta}$ and note that since $S \in \mathcal{T}$, $R \in \mathcal{T}$, $S \geq 0$, $S\Delta = \Delta S$, and $R\Delta = \Delta R$, it follows that

$$\begin{aligned} \Delta^*Q\Delta + \Delta^*S + S\Delta + R &= \text{block–diag}[S_i \otimes (-2\gamma I_{n_i} + e^{\jmath\Theta_i} + e^{-\jmath\Theta_i})] \\ &\geq \text{block–diag}[S_i \otimes (-2\gamma + 2\cos(\rho(\Theta)))I_{n_i}] \\ &\geq 0. \end{aligned}$$

Hence, for all $\Theta \in \boldsymbol{\Theta}_\gamma$, $\det(I_n + Me^{\jmath\Theta}) \neq 0$. Next, since $\gamma < -1$, it follows that $\boldsymbol{\Theta}_\gamma = \boldsymbol{\Theta}$, and hence, by definition, $\phi(M) = \infty$.

Finally, consider the case where $\phi_{\mathrm{lb}}(M) < \infty$ or, equivalently, $\gamma_{\mathrm{lb}}(M) \in [-1, 1]$. Let $\gamma \geq \gamma_{\mathrm{lb}}(M)$ and, using identical arguments as in the proof above, note that for all $\Theta \in \boldsymbol{\Theta}_\gamma$, $\det(I_n + Me^{\jmath\Theta}) \neq 0$. Next, since for all $\Theta \in \boldsymbol{\Theta}_\gamma$, $\cos(\rho(\Theta)) \geq \gamma$ if and only if $\rho(\Theta) \leq \alpha$, where $\alpha \in [0, \pi]$ is such that $\cos(\alpha) = \gamma$, it follows that $\boldsymbol{\Theta}_\gamma = \{\Theta \in \boldsymbol{\Theta} : \rho(\Theta) \leq \alpha\}$. Hence, it follows that $\det(I_n + Me^{\jmath\Theta}) \neq 0$, $\Theta \in \boldsymbol{\Theta}_\gamma$, and it follows from *i*) of Proposition 4.3.1 that $\alpha < \phi(M)$. Finally, since $\gamma_{\mathrm{lb}}(M) = \inf\{\gamma \in \mathbb{R} :$ there exists $S \in \mathcal{T}$ and $R \in \mathcal{T}$, $S \geq 0$, such that (4.12) holds$\}$, it follows that $\phi_{\mathrm{lb}}(M) \leq \phi(M)$. □

The following result shows that the lower bound $\phi_{\rm lb}(M)$ equals the phase margin $\overline{\phi}(M)$ in the case where $r = 1$, $n_1 = 1$, and $l_1 = n$. First, however, the following lemma is required.

Lemma 4.4.1. Let $\mathcal{T} = \mathbb{H}^{n\times n}$, let $k, \alpha \in \mathbb{R}$, $k \geq 0$, $\alpha \in (-\pi, \pi]$, and let

$$J \triangleq \begin{bmatrix} ke^{\jmath\alpha} & 1 & 0 & \cdots \\ 0 & ke^{\jmath\alpha} & 1 & \cdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 0 & ke^{\jmath\alpha} \end{bmatrix}. \tag{4.13}$$

Then the following statements hold:

i) If $k = 1$, then $\phi_{\rm lb}(J) = \overline{\phi}(J) = |\pi - \alpha|$.

ii) If $k \neq 1$, then $\phi_{\rm lb}(J) = \overline{\phi}(J) = \infty$.

Proof. The fact that $\overline{\phi}(J) = |\alpha|$ (resp., $\overline{\phi}(J) = \infty$) if $k = 1$ (resp., $k \neq 1$) follows as a direct consequence of the definition of $\overline{\phi}(J)$.

i) Let $\gamma = -\cos(\alpha) + \varepsilon/2$, where $\varepsilon > 0$, let $S = \mathrm{diag}(1, k_1/\varepsilon^2, k_2/\varepsilon^4, \cdots)$, where for $i = 1, 2, \ldots, n$, $k_{i+1} = k_i^2 + 1$ with $k_0 = 1$, and let $R = 0$. In this case,

$$2\gamma S + J^*S + SJ - J^*RJ + R = 2\gamma S + J^*S + SJ.$$

Now, using mathematical induction, it can be easily shown that $2\gamma S + J^*S + SJ > 0$ for all $\varepsilon > 0$. Hence, $\gamma_{\rm lb}(J) = -\cos(\alpha)$ or, equivalently, $\phi_{\rm lb}(J) = |\pi - \alpha|$.

ii) Assume $k < 1$. Then it follows that $\rho(J) < 1$, and hence, there exists $R > 0$ such that $J^*RJ - R < 0$, which implies that for every $\gamma \in \mathbb{R}$, with $S = 0$,

$$2\gamma S + J^*S + SJ - J^*RJ + R = R - J^*RJ > 0.$$

Hence, $\gamma_{\rm lb}(J) = -\infty$ or, equivalently, $\phi_{\rm lb}(J) = \infty$. In the case where $k > 1$, the result can be shown in a similar manner. □

Theorem 4.4.2. Let $\mathcal{T} = \mathbb{H}^{n\times n}$ and let $M \in \mathbb{C}^{n\times n}$. Then $\overline{\phi}(M) = \phi_{\rm lb}(M)$.

Proof. Let $T \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $M = TJT^{-1}$, where J is the Jordan matrix of M given by $J = \text{block–diag}(J_1, J_2, \cdots, J_r)$, where J_i, $i = 1,\ldots,r$, is of the form (4.13). Now, it can be easily shown that $\gamma_{\rm lb}(M) = \gamma_{\rm lb}(J)$ and $\overline{\phi}(M) = \overline{\phi}(J)$. Furthermore, it can also be shown that $\gamma_{\rm lb}(J) = \max\{\gamma_{\rm lb}(J_i) : i = 1,\ldots,r\}$ and $\overline{\phi}(J) = \min\{\overline{\phi}(J_i) : \ i = 1,\ldots,r\}$. Now, the result is a direct consequence of Lemma 4.4.1. □

Finally, the following result shows that the lower bound $\phi_{\rm lb}(M)$ equals the multivariable phase margin $\underline{\phi}(M)$ in the case where $r = 1$, $l_1 = 1$, and $n_1 = n$.

Theorem 4.4.3. Let $\mathcal{T} = \{T \in \mathbb{R}^{n\times n} : \ T = tI_n, \ t \in \mathbb{R}\}$ and let $M \in \mathbb{C}^{n\times n}$. Then $\underline{\phi}(M) = \phi_{\rm lb}(M)$.

Proof. It follows from [51] that $\underline{\phi}(M) = \infty$ if $M^*M < I_n$ or $M^*M > I_n$, and

$$2\cos\underline{\phi}(M) = -\min_{x\in\mathcal{X}_1\cap\mathcal{X}_2} x^*(M + M^*)x, \ \lambda_{\min}(M^*M) \leq 1 \leq \lambda_{\max}(M^*M), \tag{4.14}$$

where $\mathcal{X}_1 \triangleq \{x \in \mathbb{C}^n : x^*x = 1\}$ and $\mathcal{X}_2 \triangleq \{x \in \mathbb{C}^n : x^*M^*Mx = 1\}$. Furthermore, note that $\phi_{\rm lb}(M) = \infty$ if $M^*M < I_n$ or $M^*M > I_n$, and

$$\begin{aligned} 2\cos\phi_{\rm lb}(M) &= \inf\{\gamma \in \mathbb{R} : \text{there exist } r, s \in \mathbb{R}, \ s \geq 0, \ \text{such that} \\ &\qquad -s(M + M^*) - r(I_n - M^*M) < \gamma s I_n\} \\ &= -\sup_{r\in\mathbb{R}} \lambda_{\min}(M + M^* - r(I_n - M^*M)), \ \lambda_{\min}(M^*M) \leq 1 \leq \lambda_{\max}(M^*M). \end{aligned}$$

Now, the result follows directly from Theorem 4.2.1. □

Remark 4.4.1. Note that if the phase uncertainties in the set $\boldsymbol{\Theta}$ are constrained by the condition $0 \leq \Theta < 2\pi I_n$ instead of $-\pi I_n < \Theta \leq \pi I_n$, then $\phi(M)$ denotes the smallest destabilizing phase uncertainty whose eigenvalues are restricted to $[0, 2\pi]$. In this case, using identical arguments for deriving the lower bound given by (4.11), it can be shown that

$$\phi_{\rm lb}(M) \triangleq 2\cot^{-1}(\gamma_{\rm lb}(M)) \leq \phi(M), \tag{4.15}$$

where

$$\begin{aligned}\gamma_{\rm lb}(M) \triangleq \inf\{\ \gamma \in \mathbb{R} : \text{there exist } R \in \mathcal{T} \text{ and } S \in \mathcal{T} \text{ such that } S \geq 0 \text{ and} \\ -(\gamma - \jmath)M^*S - (\gamma + \jmath)SM + M^*RM - R < 2\gamma S\}.\end{aligned} \tag{4.16}$$

When applied to the time delay problem, the lower bound $\phi_{\rm lb}(M)$ given by (4.15) generalizes the stability results given in [13]. However, we will not pursue such an extension here since (4.15) is not amenable to deriving stability conditions in terms of a state space formulation. The application of the lower bound $\phi_{\rm lb}(M)$ given by (4.11) for computing the the maximum allowable delay amount for the time delay problem is discussed in Sections 4.7 and 4.8.

4.5. Connections between the Structured Phase Margin and the Structured Singular Value

In this section, we show that the structured phase margin may be obtained through the structured singular value [35–38, 40].

Definition 4.5.1 [40]. Let $M \in \mathbb{C}^{n\times n}$. The *Hermitian structured singular value* $\mu_{\mathbb{H}}(M)$ is defined by

$$\mu_{\mathbb{H}}(M) \triangleq \begin{cases} 0, \quad \text{if } \det(I_n + M\Delta) \neq 0,\ \Delta \in \boldsymbol{\Delta}, \\ (\min\{\sigma_{\max}(\Delta):\ \det(I_n + M\Delta) = 0,\ \Delta \in \boldsymbol{\Delta}\})^{-1}, \quad \text{otherwise,}\end{cases}$$

where $\mathbf{\Delta} \subseteq \mathbb{H}^{n\times n}$ is a set of block-diagonal uncertainty matrices defined by

$$\begin{aligned}\mathbf{\Delta} \triangleq\ & \{\Delta \in \mathbb{H}^{n\times n} :\ \Delta = \text{block–diag}(I_{l_1} \otimes \Delta_1, I_{l_2} \otimes \Delta_2, \ldots, I_{l_r} \otimes \Delta_r),\\ & \Delta_i \in \mathbb{H}^{n_i\times n_i},\ i = 1, \ldots, r\},\end{aligned}$$

where the dimension n_i and the number of repetitions l_i of each block are such that $\sum_{i=1}^{r} l_i n_i = n$ and $r \geq 1$.

Note that $\mu_{\mathbb{H}}(M)$ is an extension to the classical structured singular value [35–38] and specializes to the real structured singular value [37, 40] in the case of scalar blocks. It is well known that the structured singular value is computationally difficult and hence a significant effort was made in the literature to obtain computable upper bounds to the structured singular value [37]. The problem of Hermitian block uncertainties was considered in [40]. Specifically, let

$$\mathcal{T} \triangleq \{T \in \mathbb{H}^{n\times n}\ :\ T\Delta = \Delta T,\ \Delta \in \mathbf{\Delta}\}$$

and define

$$\begin{aligned}\mu_{\rm ub} \triangleq\ & \inf\{\gamma \geq 0\ :\ \text{ there exist } D \in \mathcal{T} \text{ and } N \in \mathcal{T} \text{ such that } D \geq 0 \text{ and}\\ & M^*DM + \jmath(NM - M^*N) < \gamma^2 D\}.\end{aligned}$$

It has been shown in [40] that $\mu_{\mathbb{H}}(M) \leq \mu_{\rm ub}(M)$. The following result relates the structured phase margin with the structured singular value.

Proposition 4.5.1. Let $M \in \mathbb{C}^{n\times n}$ be such that $\det(I + M) \neq 0$. Then,

$$\phi(M) = 2\cot^{-1}(\mu_{\mathbb{H}}(\jmath(I + M)^{-1}(M - I)))$$

Proof. First, for every $\Theta \in \mathbf{\Theta}$, define

$$\Delta \triangleq \jmath(I + e^{\jmath\Theta})^{-1}(I - e^{\jmath\Theta}) \tag{4.17}$$

and it can be shown that $\Delta \in \mathbf{\Delta}$. Alternatively, let $\Delta \in \mathbf{\Delta}$ and define $X = (I + \jmath\Delta)(I - \jmath\Delta)^{-1}$. Now, it can be easily shown that there exists $\Theta \in \mathbf{\Theta}$ such that $e^{\jmath\Theta} = X$. Hence, there exists a one-to-one mapping from $\mathbf{\Theta}$ to $\mathbf{\Delta}$ given by (4.17). Let $\Theta \in \mathbf{\Theta}$ and $\Delta \in \mathbf{\Delta}$ such that (4.17) holds. In this case,

$$\begin{aligned}\det(I + Me^{\jmath\Theta}) &= \det(I + M(I + \jmath\Delta)(I - \jmath\Delta)^{-1}) \\ &= \det(I + M)\det(I - \jmath(I + M)^{-1}(I - M)\Delta)\det(I - \jmath\Delta)^{-1}.\end{aligned}$$

Hence, $\det(I + Me^{\jmath\Theta}) = 0$ if and only if $\det(I - \jmath(I + M)^{-1}(I - M)\Delta) = 0$. Now the result follows immediately by noting that $\rho(\Theta) = 2\tan^{-1}(\sigma_{\max}(\Delta))$. □

The following result shows that $\gamma_{\mathrm{lb}}(M)$ can be written in terms of $\mu_{\mathrm{ub}}(M)$.

Proposition 4.5.2. Let $M \in \mathbb{C}^{n\times n}$ be such that $\det(I + M) \neq 0$. Then,

$$\gamma_{\mathrm{lb}}(M) = \frac{\eta^2 + 1}{\eta^2 - 1},$$

where $\eta \triangleq \mu_{\mathrm{ub}}(\jmath(I + M)^{-1}(M - I))$.

Proof. Let $G = \jmath(I + M)^{-1}(M - I) = \jmath(M - I)(M + I)^{-1}$, and let $D, N \in \mathcal{T}$ be such that $D \geq 0$ and for some $\alpha \neq 0$,

$$G^*DG + \jmath(NG - G^*N) < \alpha^2 D. \tag{4.18}$$

Now, post- and pre-multiplying (4.18) with $(I + M^*)$ and $(I + M)$, respectively, yields

$$(1 - \gamma)[(I - M^*)D(I - M) + 2N - M^*NM] < (\gamma + 1)(I + M^*)D(I + M). \tag{4.19}$$

Next, with $S \triangleq D$ and $R \triangleq -\gamma D - (1 - \gamma)N$, (4.19) may be rewritten as

$$M^*RM - R - M^*S - SM < 2\gamma S. \tag{4.20}$$

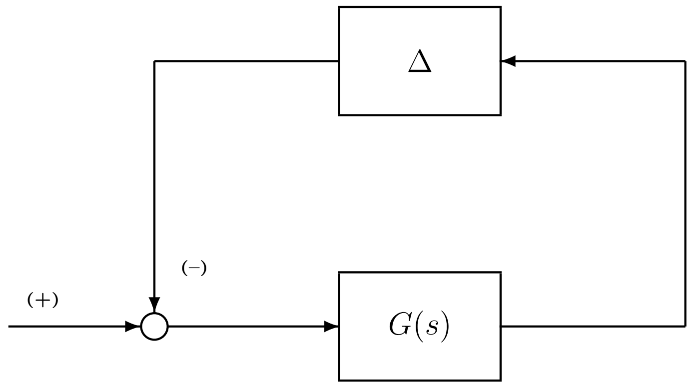

Figure 4.1: Interconnection of transfer function $G(s)$ with uncertain matrix Δ

Hence, there exist $D, N \in \mathcal{T}$ such that $D \geq 0$ and (4.18) holds if and only if there exist $S, R \in \mathcal{T}$ such that $S \geq 0$ and (4.20) holds, where $\gamma = \frac{\alpha^2-1}{\alpha^2+1}$. Now, the result is immediate from the definitions of $\gamma_{\rm lb}(\cdot)$ and $\mu_{\rm ub}(\cdot)$. □

4.6. Stability of Linear Dynamical Systems with Structured Phase Uncertainties

In this section we state and prove a stability criterion for multivariable systems involving generalized frequency domain inequalities with frequency-dependent multipliers [62]. This criterion involves a square nominal transfer function $G(s)$ in a negative feedback interconnection with a complex, square, uncertain matrix Δ as shown in Figure 4.1. For this result, define the set $\mathbf{\Delta}_\alpha$ consisting of unitary matrices given by

$$\mathbf{\Delta}_\alpha \triangleq \{\Delta \in \mathbb{C}^{n\times n} : \Delta = e^{\jmath\Theta}, \quad \Theta \in \mathbf{\Theta}, \quad \rho(\Theta) < \alpha\},$$

where $\alpha \in (-\pi, \pi] \cup \{\infty\}$.

The following result is a direct consequence of the multivariable Nyquist criterion [63].

Lemma 4.6.1. Let $\alpha \in (-\pi, \pi]$. Assume the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\alpha$ if and only if $\det(I_n+G(\jmath\omega)\Delta) \neq 0$, $\Delta \in \mathbf{\Delta}_\alpha$, $\omega \in \mathbb{R}$.

The following result presents a necessary and sufficient condition for asymptotic stability of the negative feedback interconnection of $G(s)$ and Δ.

Theorem 4.6.1. Let $\alpha \in (-\pi, \pi]$. Assume the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\alpha$ if and only if $\alpha \leq \phi(G(\jmath\omega))$, $\omega \in \mathbb{R}$.

Proof. Let $\omega \in \mathbb{R}$. It follows from *i)* of Proposition 4.3.1 that $\alpha \leq \phi(G(\jmath\omega))$ if and only if $\det(I_n + G(\jmath\omega)\Delta) \neq 0$, $\Delta \in \mathbf{\Delta}_\alpha$. Now, the result is a direct consequence of Lemma 4.6.1. □

Next, using the lower bound $\phi_{\mathrm{lb}}(\cdot)$ developed in Section 4, we present a sufficient condition for asymptotic stability of the negative feedback interconnection of $G(s)$ and Δ.

Corollary 4.6.1. Let $\alpha \in (-\pi, \pi]$. Assume the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Furthermore, assume there exist $R : \jmath\mathbb{R} \to \mathcal{T}$ and $S : \jmath\mathbb{R} \to \mathcal{T}$ such that for every $\omega \in \mathbb{R}$, $S(\jmath\omega) \geq 0$, and

$$2\cos\alpha S(\jmath\omega) > G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) - G^*(\jmath\omega)S(\jmath\omega) - S(\jmath\omega)G(\jmath\omega) - R(\jmath\omega). \quad (4.21)$$

Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\alpha$.

Proof. Note that $\gamma_{\mathrm{lb}}(G(\jmath\omega)) \leq \cos\alpha$, $\omega \in \mathbb{R}$, and hence, $\phi(G(\jmath\omega)) \geq \phi_{\mathrm{lb}}(G(\jmath\omega)) \geq \alpha$, $\omega \in \mathbb{R}$. Now, the result is a direct consequence of Theorem 4.6.1. □

Finally, we present a sufficient condition for asymptotic stability of the negative feedback interconnection of $G(s)$ and Δ, where $\Delta \in \mathbf{\Delta}_\infty$.

Corollary 4.6.2. Assume that the negative feedback interconnection of $G(s)$ and $\Delta = I_n$ is asymptotically stable. Furthermore, assume there exists $R : \jmath\mathbb{R} \to \mathcal{T}$ such that

$$R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0, \ \omega \in \mathbb{R}. \tag{4.22}$$

Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\infty$.

Proof. The proof is a direct consequence of Corollary 4.6.1. Specifically, it follows from (4.22) that $\gamma_{\text{lb}}(G(\jmath\omega)) < -1$, $\omega \in \mathbb{R}$, which implies that $\phi(G(\jmath\omega)) = \phi_{\text{lb}}(G(\jmath\omega)) = \infty$. $\square$

Remark 4.6.1. Corollaries 4.6.1 and 4.6.2 provide sufficient conditions for robust stability of linear dynamical systems with block-structured phase uncertainties in terms of generalized frequency domain inequalities involving frequency-dependent multipliers. Hence, using Proposition 4.2.1, one can obtain sufficient conditions for robust stability using linear matrix inequalities involving the state space realizations of $G(s)$, $R(s)$, and $S(s)$.

4.7. Stability Theory for Time Delay Dynamical Systems

In this section, we consider the problem of stability analysis of linear dynamical systems in the presence of unknown (finite or infinite) time delay. Specifically, we transform the time delay stability analysis problem to a robust stability analysis problem involving phase uncertainty. Then, using the results developed in Sections 4.3 and 4.4, we present new sufficient conditions for stability analysis of time-delay dynamical systems.

Consider the linear time-delay dynamical systems $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_{\mathrm{d}i}x(t-\tau_i), \quad x(\theta) = \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \tag{4.23}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\mathrm{d}i} \in \mathbb{R}^{n\times n}$, $\tau_i \ge 0$, $i = 1,\ldots,m$, $\tau = \max_{i=1,\ldots,m} \tau_i$, $\eta(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system, and $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Note that the state of (4.23) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau,0]$ and taking values in $\mathbb{R}^n$, that is, $x_t \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t+\theta)$, $\theta \in [-\tau,0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau,0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definition of asymptotic stability of (4.23) where $\|\cdot\|$ is a vector norm defined on $\mathbb{R}^n$. For further details see [3, 8].

With $\eta(\theta) \equiv 0$, the Laplace transform of (4.23) yields

$$X(s) = -G(s)\Delta(s)X(s), \tag{4.24}$$

where $X(s)$ is the Laplace transform of $x(t)$, $G(s) = C(sI_n - A)^{-1}B$, $\Delta(s) =$ block–diag $(e^{-s\tau_1}I_n, \ldots, e^{-s\tau_m}I_n)$, and where $C = [-I_n,\ldots,-I_n]^{\mathrm{T}}$ and $B = [A_{\mathrm{d}1}, \cdots, A_{\mathrm{d}m}]$. The following result is standard.

Proposition 4.7.1 [5, p. 80]. Assume $A + \sum_{i=1}^{m} A_{\mathrm{d}i}$ is Hurwitz. Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0,\tau)$ if and only if $\det(I_{nm} + G(\jmath\omega)\Delta(\jmath\omega)) \neq 0$ for all $\omega \in (0,\infty)$ and $\tau_i \in [0,\tau)$.

The following result presents a lower bound to the maximum allowable time delay using the structured phase margin. For this result, we consider the special structure for $\mathbf{\Theta}$ given by

$$\mathbf{\Theta} = \{\Theta \in \mathbb{R}^{nm\times nm} : \ \Theta = \text{block–diag}(\theta_1 I_n, \cdots, \theta_m I_n), \ \theta_i \in (-\pi, \pi], \ i = 1, \ldots, m\}.$$

Theorem 4.7.1. Assume $A + \sum_{i=1}^{m} A_{\mathrm{d}i}$ is Hurwitz and let $\tau_{\mathrm{lb}} \triangleq \inf_{\omega>0} \frac{\phi_{\mathrm{lb}}(G(\jmath\omega))}{\omega}$. Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0, \tau_{\mathrm{lb}})$, $i = 1, \ldots, m$.

Proof. If $\tau_{\mathrm{lb}} = \infty$, then for all $\omega \in (0, \infty)$, $\phi_{\mathrm{lb}}(G(\jmath\omega)) = \infty$. Now, it follows from Theorem 4.4.1 that $\phi(G(\jmath\omega)) = \infty$, $\omega \in (0, \infty)$. Hence, by definition, $\det(I_{nm} + G(\jmath\omega)e^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, which implies that $\det(I_{nm} + G(\jmath\omega)\Delta(\jmath\omega)) \neq 0$, $\omega \in (0, \infty)$, $\tau_i \in [0, \tau_{\mathrm{lb}})$, $i = 1, \ldots, m$.

Next, assume $\tau_{\mathrm{lb}} < \infty$. Since for every $\omega \in (0, \infty)$, $\phi_{\mathrm{lb}}(G(\jmath\omega)) \leq \phi(G(\jmath\omega))$, it follows from $i)$ of Proposition 4.3.1 that $\det(I_{nm} + G(\jmath\omega)e^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, $-\phi_{\mathrm{lb}}(G(\jmath\omega))I_{nm} < \Theta < \phi_{\mathrm{lb}}(G(\jmath\omega))I_{nm}$. Now, for all $\tau_i \in [0, \tau_{\mathrm{lb}})$, it follows that $-\phi_{\mathrm{lb}}(G(\jmath\omega)) < -\omega\tau_i$ which implies that $\det(I_{nm} + G(\jmath\omega)\Delta(\jmath\omega)) \neq 0$, $\tau_i \in [0, \tau_{\mathrm{lb}})$, $i = 1, \ldots, m$. Thus, $\det(I_{nm}+G(\jmath\omega)\Delta(\jmath\omega)) \neq 0$, $\omega \in (0, \infty)$, $\tau_i \in [0, \tau_{\mathrm{lb}})$, $i = 1, \ldots, m$, and hence, it follows from Proposition 4.7.1 that $\mathcal{G}$ is asymptotically stable for all $\tau_i \in [0, \tau_{\mathrm{lb}})$, $i = 1, \ldots, m$. □

Remark 4.7.1. Note that, in general, τ_{lb} is less than the maximum allowable destabilizing delay. However, in certain cases, it can be easily shown that τ_{lb} is the maximum allowable delay before instability occurs. Specifically, if $m = 1$ and for every $\omega \in (0, \infty)$, the phase of all of the eigenvalues of $G(\jmath\omega)$ is negative, then τ_{lb} is the maximum allowable delay before instability. Alternatively, if $m = 1$ and for every $\omega \in \mathbb{R}$, there exists at most one eigenvalue of $G(\jmath\omega)$ with unit magnitude, then τ_{lb} is the maximum allowable delay before instability occurs.

The following corollaries present several different sufficient conditions for stability of time-delay dynamical systems using the lower bounds to the structured phase margin.

Corollary 4.7.1. Let $\bar{\tau} > 0$ and assume $A + \sum_{i=1}^{m} A_{\mathrm{d}i}$ is Hurwitz. Assume there exist functions $M_{\bar{\tau}} : \jmath\mathbb{R} \to \mathbb{C}^{nm \times nm}$, $R : \jmath\mathbb{R} \to \mathcal{T}$, and $S : \jmath\mathbb{R} \to \mathcal{T}$, such that $M_{\bar{\tau}}^*(\jmath\omega)M_{\bar{\tau}}(\jmath\omega) \in \mathcal{T}$, $S(\jmath\omega) \geq 0$, $\omega \in (0, \infty)$,

$$\sigma_{\min}(M_{\bar{\tau}}(\jmath\omega)) \geq \begin{cases} 2\sin(\frac{\omega\bar{\tau}}{2}), & \omega \in (0, \frac{\pi}{\bar{\tau}}] \\ 2, & \omega > \frac{\pi}{\bar{\tau}}, \end{cases} \tag{4.25}$$

and

$$\begin{aligned} 2S(\jmath\omega) - M_{\bar{\tau}}^*(\jmath\omega)S(\jmath\omega)M_{\bar{\tau}}(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) \\ +G^*(\jmath\omega)S(\jmath\omega) + S(\jmath\omega)G(\jmath\omega) + R(\jmath\omega) > 0, \ \omega \in (0, \infty). \end{aligned} \tag{4.26}$$

Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0, \bar{\tau})$, $i = 1, \ldots, m$.

Proof. It follows from (4.26) that for every $\omega \in (0, \infty)$, there exists $\varepsilon > 0$ such that

$$\begin{aligned} [(2-\varepsilon)I_{nm} \ - \ \lambda_{\min}(M_{\bar{\tau}}^*(\jmath\omega)M_{\bar{\tau}}(\jmath\omega)]S(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) \\ + \ G^*(\jmath\omega)S(\jmath\omega) + S(\jmath\omega)G(\jmath\omega) + R(\jmath\omega) > 0, \end{aligned}$$

which implies that

$$\gamma_{\mathrm{lb}}(G(\jmath\omega)) < 1 - \frac{1}{2}\lambda_{\min}(M_{\bar{\tau}}^*(\jmath\omega)M_{\bar{\tau}}(\jmath\omega)).$$

Now, it follows from (4.25) that

$$\gamma_{\mathrm{lb}}(G(\jmath\omega)) < 1 - \frac{1}{2}\sigma_{\min}^2(M_{\bar{\tau}}(\jmath\omega)) \leq 1 - 2\sin^2(\frac{\omega\bar{\tau}}{2}) = \cos(\omega\bar{\tau}), \ \omega \in (0, \frac{\pi}{\bar{\tau}}],$$

and $\gamma_{\rm lb}(G(\jmath\omega)) < -1$, $\omega > \frac{\pi}{\bar\tau}$, or, equivalently, $\phi_{\rm lb}(G(\jmath\omega)) > \omega\bar\tau$, $\omega \in (0, \frac{\pi}{\bar\tau}]$, and $\phi_{\rm lb}(G(\jmath\omega)) = \infty$, $\omega > \frac{\pi}{\bar\tau}$. Hence, it follows that $\inf_{\omega>0} \frac{\phi_{\rm lb}(G(\jmath\omega))}{\omega} \geq \bar\tau$ which further implies that $\bar\tau \leq \tau_{\rm lb}$. Now, the result is a direct consequence of Theorem 4.7.1. □

Remark 4.7.2. An obvious choice for $M_{\bar\tau}(\jmath\omega)$ is $M_{\bar\tau}(\jmath\omega) = \jmath\omega\bar\tau I_{nm}$, since, for $\omega\bar\tau \in (0,\pi]$, we have $\omega\bar\tau > 2\sin(\frac{\omega\bar\tau}{2})$, and $\omega\bar\tau > 2$ otherwise. In Section 7, we utilize this choice of $M_{\bar\tau}(\cdot)$ to derive several linear matrix inequality conditions for the stability analysis of the time-delay system $\mathcal{G}$.

Remark 4.7.3. If (4.26) holds with $M_{\bar\tau}(\jmath\omega) = 2I_{nm}$ ((4.25) is trivially satisfied in this case), then it follows from Corollary 4.7.1 that $\mathcal{G}$ is asymptotically stable for all $\tau_i \in [0,\infty)$.

Corollary 4.7.2. Let $\bar\tau > 0$ and assume $A + \sum_{i=1}^{m} A_{{\rm d}i}$ is Hurwitz. Assume there exist functions $M_{\bar\tau} : \jmath\mathbb{R} \to \mathbb{C}^{nm\times nm}$, $R : \jmath\mathbb{R} \to \mathcal{T}$, and $S : \jmath\mathbb{R} \to \mathcal{T}$ such that $M_{\bar\tau}^*(\jmath\omega)M_{\bar\tau}(\jmath\omega) \in \mathcal{T}$, $S(\jmath\omega) \geq 0$, $\omega \in (0,\infty)$,

$$\sigma_{\min}(M_{\bar\tau}(\jmath\omega)) \geq \left\{ \begin{array}{ll} 2\sin(\frac{\omega\bar\tau}{2}), & \omega \in (0, \frac{\pi}{\bar\tau}] \\ 2, & \omega > \frac{\pi}{\bar\tau}, \end{array} \right. \tag{4.27}$$

and

$$\begin{aligned}[I_{nm} + G^*(\jmath\omega)]S(\jmath\omega)[I_{nm} + G(\jmath\omega)] \; &- \; G^*(\jmath\omega)M_{\bar\tau}^*(\jmath\omega)S(\jmath\omega)M_{\bar\tau}(\jmath\omega)G(\jmath\omega) + R(\jmath\omega) \\ &- \; G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0, \;\; \omega \in (0,\infty). \end{aligned} \tag{4.28}$$

Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0,\bar\tau)$, $i = 1,\ldots,m$.

Proof. The proof is a direct consequence of Corollary 4.7.1 by replacing $R(\jmath\omega)$ with $R(\jmath\omega) - S(\jmath\omega) + M_{\bar\tau}^*(\jmath\omega)S(\jmath\omega)M_{\bar\tau}(\jmath\omega)$. □

Corollary 4.7.3. Let $\bar{\tau} > 0$ and assume $m = 1$ and $A + A_{\rm d1}$ is Hurwitz. Furthermore let $M_{\bar{\tau}} : \jmath\mathbb{R} \to \mathbb{C}^{n\times n}$ be such that

$$\sigma_{\min}(M_{\bar{\tau}}(\jmath\omega)) \geq \begin{cases} 2\sin(\frac{\omega\bar{\tau}}{2}), & \omega \in (0, \frac{\pi}{\bar{\tau}}] \\ 2, & \omega > \frac{\pi}{\bar{\tau}}, \end{cases} \tag{4.29}$$

and $\rho(H(\jmath\omega)) < 1$, $\omega \in (0, \infty)$, where $H(\jmath\omega) \triangleq M_{\bar{\tau}}(\jmath\omega)G(\jmath\omega)(I_n + G(\jmath\omega))^{-1}$. Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof. The proof is a direct consequence of Corollary 4.7.2 with $R(\jmath\omega) = 0$. □

Remark 4.7.4. If $M_{\bar{\tau}}(\jmath\omega) = \jmath\omega\bar{\tau}I_{nm}$, then it follows from Corollary 4.7.3 that $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \bar{\tau})$, where $\bar{\tau} = \inf_{\omega\in(0,\infty)} \frac{1}{\rho(H(\jmath\omega))}$ and $H(s) \triangleq sG(s)(I_n + G(s))^{-1}$.

Corollary 4.7.4. Let $\bar{\tau} > 0$ and assume $A + \sum_{i=1}^{m} A_{{\rm d}i}$ is Hurwitz. Let $\hat{\gamma} \in \mathbb{R}$ and $\hat{\omega} \in [0, \infty)$ be defined by

$$\begin{aligned} \hat{\gamma} \triangleq \inf_{\omega>0}\{&\gamma \in \mathbb{R} : \text{there exist } R : \jmath\mathbb{R} \to \mathcal{T} \text{ and } S : \jmath\mathbb{R} \to \mathcal{T} \text{ such that for every} \\ &\omega \in (0, \infty), \quad S(\jmath\omega) \geq 0 \text{ and } 2\gamma S(\jmath\omega) + G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) + G^*(\jmath\omega)S(\jmath\omega) \\ &\quad +S(\jmath\omega)G(\jmath\omega) - R(\jmath\omega) > 0\}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} \hat{\omega} \triangleq \inf\{&\bar{\omega} \in (0, \infty) : \text{there exists } R : \jmath\mathbb{R} \to \mathcal{T} \text{ such that for every } \omega \geq \bar{\omega}, \\ &R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0\}. \end{aligned} \tag{4.31}$$

Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0, \hat{\tau})$, $i = 1, \ldots, m$, where $\hat{\tau} = \frac{\cos^{-1}(\hat{\gamma})}{\hat{\omega}}$.

Proof. Note that $\gamma_{\rm lb}(G(\jmath\omega)) \leq \hat{\gamma}$, $\omega \in (0, \infty)$, and hence, $\phi_{\rm lb}(G(\jmath\omega)) \geq \cos^{-1}(\hat{\gamma})$. Now, for all $\omega \in (0, \hat{\omega}]$, $\frac{\phi_{\rm lb}(G(\jmath\omega))}{\omega} \geq \frac{\cos^{-1}(\hat{\gamma})}{\hat{\omega}} = \hat{\tau}$. Next, for every $\omega > \hat{\omega}$, since there exists $R(\jmath\omega)$ such that $G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) - R(\jmath\omega) < 0$, it follows that $\gamma_{\rm lb}(G(\jmath\omega)) = -\infty$ or, equivalently, $\phi_{\rm lb}(G(\jmath\omega)) = \infty$. Hence, $\hat{\tau} \leq \tau_{\rm lb}$. Now, the result is a direct consequence of Theorem 4.7.1. $\square$

Corollary 4.7.5. Assume $A + \sum_{i=1}^{m} A_{{\rm d}i}$ is Hurwitz. Furthermore, assume there exists $R : \jmath\mathbb{R} \to \mathcal{T}$ such that

$$R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0, \qquad \omega \in (0, \infty). \tag{4.32}$$

Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0, \infty)$, $i = 1, \ldots, m$.

Proof. The proof is a direct consequence of Corollary 4.7.4. Specifically, it follows from (4.32) that $\hat{\omega} = 0$ which implies that $\hat{\tau} = \infty$. $\square$

Corollary 4.7.6. Let $m = 1$ and assume $A + A_{\rm d1}$ is Hurwitz. Then the linear time-delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\tau_1 \in [0, \infty)$ if and only if there exist $R : \jmath\mathbb{R} \to \mathbb{H}^{n\times n}$ such that (4.32) holds.

Proof. Sufficiency follows from Corollary 4.7.5. To show necessity, assume that $\mathcal{G}$ is asymptotically stable for all $\tau_1 \in [0, \infty)$. Hence, it follows from Proposition 4.7.1 that $\det(I_{nm} + G(\jmath\omega)e^{-\jmath\omega\tau_1}) \neq 0$, where $\omega \in (0, \infty)$ and $\tau_1 \in [0, \infty)$, which implies that $\det(I_{nm} + G(\jmath\omega)e^{\jmath\theta}) \neq 0$, $\theta \in \mathbb{R}$. Thus, it follows from Theorem 4.4.2 that $\overline{\phi}(G(\jmath\omega)) = \phi_{\rm lb}(G(\jmath\omega)) = \infty$, which proves the result. $\square$

4.8. Time-Domain Conditions for Stability Analysis of Time-Delay Systems

In this section we apply Corollary 4.7.2 to derive new time-domain conditions for stability analysis of time-delay dynamical systems.

Theorem 4.8.1. Let $\bar{\tau} > 0$ and assume $A + \sum_{i=1}^{m} A_{\mathrm{d}i}$ is Hurwitz. Furthermore, let $M_{\bar{\tau}} : \mathbb{C} \to \mathbb{C}^{nm\times nm}$ be such that $M_{\bar{\tau}}^*(\jmath\omega)M_{\bar{\tau}}(\jmath\omega) \in \mathcal{T}$ and (4.29) holds, and let

$$H(s) = \begin{bmatrix} I_{mn} \\ M_{\bar{\tau}}(s)G(s)(I_{mn} + G(s))^{-1} \\ (I_{mn} + G(s))^{-1} \\ G(s)(I_{mn} + G(s))^{-1} \end{bmatrix} \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right],$$

where $\tilde{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbb{R}^{\tilde{n}\times n}$, $\tilde{C} \in \mathbb{R}^{4n\times\tilde{n}}$, $\tilde{D} \in \mathbb{R}^{4n\times n}$, and where $\tilde{n} \geq n$. Finally, assume there exists $P \in \mathbb{S}^{\tilde{n}\times\tilde{n}}$, $R \in \mathcal{T}$, and $S \in \mathcal{T}$, $S \geq 0$, such that

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}P + P\tilde{A} + \tilde{C}^{\mathrm{T}}\tilde{Q}\tilde{C} & P\tilde{B} + \tilde{C}^{\mathrm{T}}\tilde{Q}\tilde{D} \\ \tilde{B}^{\mathrm{T}}P + \tilde{D}^{\mathrm{T}}\tilde{Q}\tilde{C} & \tilde{D}^{\mathrm{T}}\tilde{Q}\tilde{D} \end{bmatrix} < 0, \tag{4.33}$$

where $\tilde{Q} = \text{block–diag}(-S, S, -R, R)$. Then the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0, \bar{\tau})$, $i = 1, \ldots, m$.

Proof. Note that (4.28) is equivalent to

$$H^*(\jmath\omega)\tilde{Q}H(\jmath\omega) < 0, \ \omega \in (0, \infty).$$

Now, the result is a direct consequence of Corollary 4.7.2 and Proposition 4.2.1. □

Corollary 4.8.1. Let $\bar{\tau} > 0$ and assume $A + \sum_{i=1}^{m} A_{\mathrm{d}i}$ is Hurwitz. Assume there exist $R \in \mathcal{T}$, $S \in \mathcal{T}$, $S \geq 0$, and $P \in \mathbb{S}^{n\times n}$ such that (4.33) holds, where

$$\tilde{A} = A - BC,\ \tilde{B} = B,\ \tilde{C} = [0_{mn\times mn}\ \ \bar{\tau}\tilde{A}^{\mathrm{T}}C^{\mathrm{T}}\ \ -C^{\mathrm{T}}\ \ C^{\mathrm{T}}]^{\mathrm{T}},\ \tilde{D} = [I_{mn}\ \ \bar{\tau}B^{\mathrm{T}}C^{\mathrm{T}}\ \ I_{mn}\ \ 0]^{\mathrm{T}}, \tag{4.34}$$

and where $B = [A_{\mathrm{d}1}, \ldots A_{\mathrm{d}m}]$ and $C = [-I_{mn}, \ldots, -I_{mn}]^{\mathrm{T}}$. Then, the linear time-delay dynamical system $\mathcal{G}$ given by (4.23) is asymptotically stable for all $\tau_i \in [0, \bar{\tau})$, $i = 1, \ldots, m$.

Proof. The proof is a direct consequence of Theorem 4.8.1 with $M_{\bar{\tau}}(s) = \bar{\tau}sI_{mn}$ and noting that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ in $H(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ are given by (4.34). □

Remark 4.8.1. In the case of a single delay system, that is, $m = 1$, Corollary 4.8.1 is an extension of the delay-dependent stability condition given in [17]. Specifically, if $R = 0$, then (4.33) specializes to the stability condition given in [17].

Finally, consider the time-delay dynamical system $\mathcal{G}$ given by (4.23) where $m = 1$ so that $\mathcal{G}$ is given by

$$\dot{x}(t) = Ax(t) + A_{\rm d}x(t-\tau),\ x(\theta) = \eta(\theta),\ -\tau \le \theta \le 0,\ t \ge 0.$$

Now, let $B_i \in \mathbb{R}^{n\times p}$ and $C_i \in \mathbb{R}^{p\times n}$, $i = 1, 2$, be such that $A_{\rm d} = \sum_{i=1}^{2} B_i C_i$. In this case, (4.35) becomes

$$\dot{x} = Ax(t) + \sum_{i=1}^{2} B_i C_i x(t-\tau), \tag{4.35}$$

or, equivalently, in the frequency domain,

$$X(s) = -G(s)\Delta(s)X(s), \tag{4.36}$$

where $X(s)$ is the Laplace transform of $x(t)$, $G(s) = C(sI - A)^{-1}B$, $\Delta(s) = e^{-\tau s}I_{2n}$, and where $C = [-C_1^{\rm T}\ \ -C_2^{\rm T}]^{\rm T}$ and $B = [B_1\ \ B_2]$. Next, it follows from Corollary 4.8.1 that if there exist $R \in \mathbb{S}^{2n\times 2n}$, $S \in \mathbb{S}^{2n\times 2n}$, $S > 0$, and $P \in \mathbb{S}^{n\times n}$ such that (4.33) holds, where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are given by (4.34), then the linear time-delay system given by (4.35) is asymptotically stable for all $\tau \in [0, \bar{\tau})$. With $B = [A_{\rm d}\ \ A_{\rm d}]$ and $C = \begin{bmatrix} -A(A+A_{\rm d})^{-1} \\ -A_{\rm d}(A+A_{\rm d})^{-1} \end{bmatrix}$, (4.33) is similar to the delay-dependent condition of [12] (see (8) of [16]). Alternatively, if $B = [A_{\rm d}A\ \ A_{\rm d}A_{\rm d}]$ and $C = \begin{bmatrix} -(A+A_{\rm d})^{-1} \\ -(A+A_{\rm d})^{-1} \end{bmatrix}$, then (4.33) is similar to the stability result given in [11] (see (7) of [16]). Finally, if $B = [M\ \ (I-M)A_{\rm d}]$ and $C = \begin{bmatrix} -A_{\rm d} \\ -I \end{bmatrix}$, where $M \in \mathbb{R}^{n\times n}$, then (4.33) is similar to the stability result given in [15] (see (9) of [16]).

4.9. Illustrative Numerical Examples

In this section we consider several numerical examples to demonstrate the utility of the proposed robust stability theory. Specifically, Example 4.9.1 considers stability analysis of a linear dynamical system with phase uncertainty. In Examples 4.9.2–4.9.5, we demonstrate the utility of the time-delay analysis results developed in Sections 4.7 and 4.8 and we compare our results to those given in [15]. In Examples 4.9.2–4.9.5, for notational convenience, let τ_{exact} denote the maximum allowable time delay, let τ_{lb} denote the lower bound of τ_{exact} obtained via Theorem 4.7.1, let τ_{LMI} be the lower bound of τ_{exact} obtained by the Corollary 4.8.1, and let τ_{LK} denote the lower bound given in [15] (see also (9) of [16]). In all of these examples, we apply the generalized Nyquist criterion [63] to obtain the value of τ_{exact}.

Example 4.9.1. Consider a linear dynamic model of a two-body spacecraft with non-collocated sensors and actuators [44], whose transfer function is

$$G(s) \sim \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -0.75 & 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & -4 & -1 & 0.25 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]. \tag{4.37}$$

Here, we consider the stability of the negative feedback interconnection of $G(s)$ and Δ (See Figure 4.1) where $\Delta \in \boldsymbol{\Delta}_\alpha$, and where $\boldsymbol{\Theta} = \mathbb{H}^{2\times 2}$, or

$$\boldsymbol{\Theta} = \{\Theta \in \mathbb{R}^{2\times 2} : \Theta = \operatorname{diag}(\theta_1, \theta_2),\ \theta_1, \theta_2 \in [-\pi, \pi]\}, \tag{4.38}$$

or

$$\boldsymbol{\Theta} = \{\Theta \in \mathbb{R}^{2\times 2} : \Theta = \theta I_2,\ \theta \in \mathbb{R}\}. \tag{4.39}$$

For $G(s)$ given by (4.37), with $\boldsymbol{\Theta} = \mathbb{H}^{2\times 2}$, $\inf_{\omega\in\mathbb{R}} \phi(G(\jmath\omega)) = \inf_{\omega\in\mathbb{R}} \underline{\phi}(G(\jmath\omega)) = \inf_{\omega\in\mathbb{R}} \phi_{\text{lb}}(G(\jmath\omega)) = 15.4689°$. Hence, it follows from Theorem 4.6.1 that the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Theta \in \mathbb{H}^{2\times 2}$ such that $-15.4689^\circ \le \Theta \le 15.4689^\circ$. However, with $\boldsymbol{\Theta}$ given by (4.39), $\inf_{\omega\in\mathbb{R}} \phi(G(\jmath\omega))$ $= \inf_{\omega\in\mathbb{R}} \overline{\phi}(G(\jmath\omega)) = \inf_{\omega\in\mathbb{R}} \phi_{\mathrm{lb}}(G(\jmath\omega)) = 22.3532^\circ$. Hence, the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Theta \in \boldsymbol{\Theta}$ given by (4.39) such that $-22.3532^\circ \le \Theta \le 22.3532^\circ$. Finally, with $\boldsymbol{\Theta}$ given by (4.38), $\inf_{\omega\in\mathbb{R}} \phi(G(\jmath\omega)) \ge \inf_{\omega\in\mathbb{R}} \phi_{\mathrm{lb}}(G(\jmath\omega)) = 17.7924^\circ$. Hence, the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Theta \in \boldsymbol{\Theta}$ given by (4.38) such that $-17.7924^\circ \le \Theta \le 17.7924^\circ$. Note that this example verifies the fact that the phase margin increases with the block structure of the allowable phase uncertainty. Finally, it should be noted that the small gain condition as well as the small $\mu-$condition fail to establish stability of the negative feedback interconnection of $G(s)$ and Δ if Δ is treated as a gain bounded uncertainty (with or without block structure).

Example 4.9.2. Our second example considers a two-dimensional, linear time-delay dynamical system of the form (4.23) with a single time delay, that is, $m = 1$, and with system matrices [15]

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad A_{\mathrm{d}} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

For this example, $\alpha_{\mathrm{lb}}(G(\jmath\omega)) = \infty$, $\omega \ne \sqrt{0.19}$, and $\alpha_{\mathrm{lb}}(G(\jmath\omega)) = \pi - \sin^{-1}(\sqrt{0.19})$, $\omega = \sqrt{0.19}$. Hence, $\tau_{\mathrm{lb}} = 6.1726$. Furthermore, using the multivariable Nyquist criterion, $\tau_{\mathrm{exact}} = \tau_{\mathrm{lb}} = 6.1726$. Finally, $\tau_{\mathrm{LK}} = 4.3589$, while $\tau_{\mathrm{LMI}} = 4.4721$.

Example 4.9.3. Our third example considers a three-dimensional, linear time-delay dynamical system of the form (4.23) with a single delay, that is, $m = 1$, and with system matrices

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad A_{\mathrm{d}} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix}.$$

For this example, $\alpha_{\rm lb}(G(\jmath\omega)) = \infty$, $\omega \notin \{1.25, 2.137\}$, $\alpha_{\rm lb}(G(\jmath\omega)) = 2.232$, $\omega = 2.137$, and $\alpha_{\rm lb}(G(\jmath\omega)) = 0.999$, $\omega = 1.25$. Hence, $\tau_{\rm lb} = 0.798$. Furthermore, using the multivariable Nyquist criterion, $\tau_{\rm exact} = \tau_{\rm lb} = 0.798$. Finally, $\tau_{\rm LK} = 0.590$, while $\tau_{\rm LMI} = 0.7652$.

Example 4.9.4. This example is adapted from [16, 64, 65] and models the dynamics of chatter during a machining process. Specifically, the time-delay dynamical system is given by

$$\begin{aligned}
\ddot{x}_1(t) = \frac{1}{m_1}[-k_1x_1(t) + k_1x_2(t) \; &- \; k\sin(\phi+\beta)\sin(\phi)x_1(t) \\
&+ \; k\sin(\phi+\beta)\sin(\phi)x_1(t-\tau)], \qquad (4.40) \\
\ddot{x}_2(t) = \frac{1}{m_2}[-k_1x_1(t) - k_1x_2(t) \; &- \; k_2x_2(t) - c\dot{x}_2(t)], \qquad (4.41) \\
x_1(\theta) = \eta_1(\theta),\ \dot{x}_1(t) = \dot{\eta}_1(\theta),\ x_2(\theta) \; &= \; \eta_2(\theta),\ \dot{x}_2(\theta) = \dot{\eta}_2(\theta),\ \theta \in [-\tau, 0],\ t \geq 0,
\end{aligned}$$

where $x_1(t)$ and $x_2(t)$, $t \geq 0$, denote the blade and tool displacements respectively, m_1 is the mass of the cutter, m_2 is the mass of the spindle, k_1 and k_2 are the stiffness of the cutter and spindle, respectively, ϕ denotes the angular position of the blade, k denotes the cutting stiffness, and β denotes a parameter that depends on the tool and the material used. Next, let $x = [x_1 \;\; x_2 \;\; \dot{x}_1 \;\; \dot{x}_2]^{\rm T}$ so that (4.40)–(4.41) can be written in the state space form given by (4.35), where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} + K & \frac{k_1}{m_1} & 0 & 0 \\ \frac{k_1}{m_2} & \frac{-(k_1+k_2)}{m_2} & 0 & \frac{-c}{m_2} \end{bmatrix}, \quad A_{\rm d} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and where $K = \frac{k\sin(\phi)\sin(\phi+\beta)}{m_1}$. For this example, we choose $m_1 = 1$, $m_2 = 2$, $k_1 = 10$, $k_2 = 20$, and $c = 0.5$ [16]. Figure 4.2 shows the comparison of $\tau_{\rm exact}$, $\tau_{\rm lb}$, $\tau_{\rm LK}$, and $\tau_{\rm LMI}$ for a range of values of K. As shown in Figure 4.2, $\tau_{\rm exact} = \tau_{\rm lb}$ and $\tau_{\rm LK} = \tau_{\rm LMI}$ for all values of K.

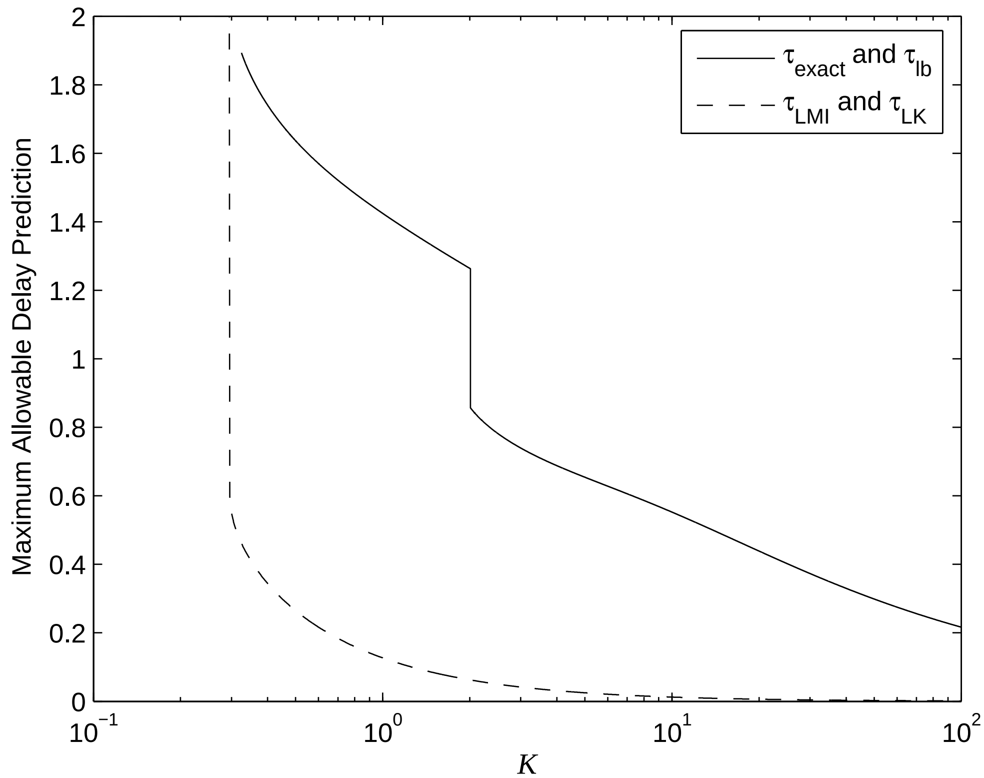

Figure 4.2: Maximum allowable delay prediction versus K for Example 4.9.4

Example 4.9.5. Our final example considers a scalar, linear time-delay dynamical system of the form (4.23) with two delays, that is, $m = 2$, and with system matrices $A = -1$, $A_{\mathrm{d1}} = -3$, and $A_{\mathrm{d2}} = 1$. For this example, $\alpha_{\mathrm{lb}}(G(\jmath\omega)) = \phi_{\mathrm{lb}}(G(\jmath\omega)) = \infty$, $\omega \neq 1.732$, and $\alpha_{\mathrm{lb}}(G(\jmath\omega)) = \phi_{\mathrm{lb}}(G(\jmath\omega)) = 2.094$, $\omega = 1.732$. Hence, $\tau_{\mathrm{lb}} = 1.209$. In addition, $\overline{\phi}(G(\jmath\omega)) = \infty = \phi_{\mathrm{lb}}(G(\jmath\omega))$, $\omega \neq 1.732$, and $\overline{\phi}(G(\jmath\omega)) = 2.094 = \phi_{\mathrm{lb}}(G(\jmath\omega))$, $\omega = 1.732$. Since $\phi_{\mathrm{lb}}(G(\jmath\omega)) \leq \phi(G(\jmath\omega)) \leq \overline{\phi}(G(\jmath\omega))$, $\omega \in [0, \infty)$, it follows that $\phi_{\mathrm{lb}}(G(\jmath\omega)) = \phi(G(\jmath\omega)) = \overline{\phi}(G(\jmath\omega))$. Hence, it follows from Remark 4.7.1 that $\tau_{\mathrm{exact}} = \tau_{\mathrm{lb}} = 1.209$. For this example, $\tau_{\mathrm{LMI}} = 1.000$.

4.10. Conclusion

In this chapter, we introduced the notion of the structured phase margin for characterizing stability margins for dynamical systems with block-structured phase uncertainty. Furthermore, an easily computable lower bound was derived in terms of a generalized eigenvalue problem. This bound is constructed by choosing stability multipliers that are tailored to the structure of the phase uncertainty. Next, using the structured phase margin, we presented new and improved delay-dependent stability criteria for stability analysis of time-delay systems. Finally, we demonstrated the newly developed stability analysis tests on several numerical examples and showed that our results are less conservative compared to the other results in the literature for capturing phase information uncertainty.

Chapter 5

Sufficient Conditions for Stability of Neutral Time-Delay Systems using the Structured Phase Margin

5.1. Introduction

In this chapter, we consider the problem of stability analysis of linear neutral time-delay dynamical systems. Specifically, we will transform the neutral time-delay stability analysis problem to a robust stability analysis problem with phase perturbations as discussed in Chapter 4. Then using the results developed in Section 4.4 we present several sufficient conditions for stability analysis of dynamical systems with neutral time-delay. Although the results presented below are restricted to the case of single time-delay it should be noted that all the sufficient conditions can be trivially generalized to the case of multiple time-delays (see previous chapter for the results on multiple time-delays in the case of retarded time-delay systems).

So far, in this dissertation, we have considered retarded time-delay systems, where the time-derivative of the state depends on current state as well as past (delayed) state. In a neutral delay system, the time-derivative of the state not only depends on the current and delayed stated but also the past (delayed) derivative [1–5].

Neutral time-delay systems arise in many engineering systems and have been studied extensively in the literature (see [1, 3, 5, 66, 67] and references within). In general, stability of the neutral time-delay systems proves to be a more complex issue because of the presence of the derivative of the delayed state. Recently, there has been a strong renewed interest in the study of neutral time-delay systems. They arise in applications to electric networks involving lossless transmission lines. Such networks are utilized in high speed computers where lossless transmission lines are used to make connections between switching circuits. Neutral systems are also encountered in problems involving vibrating masses attached to elastic beams, and have applications involving vibrations in heat exchanger tubes and aircraft dynamics.

As described in Chapter 4 in the case of retarded time-delay systems, the basic idea relies on the fact that the stability characteristics of a linear neutral time-delay system can be studied in terms of a feedback interconnection of a matrix transfer function and a *phase* uncertainty block [1, 5]. Since phase uncertainties have unit gain, many delay-independent stability criteria were derived in the literature using the classical small gain theorem or, more generally, the scaled small gain theorem [1,5,16]. Furthermore, many delay-dependent stability criteria were also derived by applying the (scaled) small gain approach on a *transformed* time-delayed system [1, 5, 16]. Using the results on structured phase margin in Chapter 4, we will derive several new frequency-domain sufficient conditions for stability of linear neutral time-delay systems. We will provide both delay-independent as well as delay-dependent sufficient conditions for stability. Since the lower bounds derived in Section 4.4 are given in terms of a minimization problem involving linear matrix inequalities all the sufficient conditions developed will present themselves to be solved as generalized eigenvalue problems [68]. Next, using ideas analogous to the results presented in Sections 4.7

and 4.8, we will also derive an LMI-based delay-dependent sufficient condition for the stability of a linear neutral time-delay system.

5.2. Frequency-Domain Stability Conditions for Neutral Time-Delay Dynamical Systems

Consider the linear neutral time-delay dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) + A_{\mathrm{n}}\dot{x}(t-\tau) = Ax(t) + A_{\mathrm{d}}x(t-\tau),$$
$$x(\theta) = \eta(\theta),\ -\tau \leq \theta \leq 0,\ t \geq 0, \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $A, A_{\mathrm{d}}, A_{\mathrm{n}} \in \mathbb{R}^{n\times n}$, $\tau \geq 0$, $\eta(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system, and $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Note that the state of (5.1) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau, 0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t+\theta)$, $\theta \in [-\tau, 0]$. Here, we assume that $\dot{\eta}(0) + A_{\mathrm{n}}\dot{\eta}(-\tau) = A\eta(0) + A_{\mathrm{d}}\eta(-\tau)$, so that $x_t \in \mathcal{C}$ for all $t \geq 0$ [3]. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau, 0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definition of asymptotic stability of (5.1) where $\|\cdot\|$ is a vector norm defined on $\mathbb{R}^n$. For further details see [3, 8].

In this chapter, we derive sufficient conditions for stability of neutral time-delay systems. Two necessary conditions for the stability of a linear neutral time-delay system of the form (5.1) are that $\rho(A_{\mathrm{n}}) < 1$ and the linear system given by (5.1) is stable with $\tau = 0$, that is $(I + A_{\mathrm{n}})^{-1}(A + A_{\mathrm{d}})$ is Hurwitz. Hence, in what follows, we assume that $\rho(A_\mathrm{n}) < 1$ and $(I + A_\mathrm{n})^{-1}(A + A_\mathrm{d})$ is Hurwitz. The following lemma is a direct consequence of Theorems 3.19 and 3.20 of [5, p. 109].

Lemma 5.2.1. Let $\bar{\tau} \in [0, \infty]$ Assume that $\rho(A_\mathrm{n}) < 1$ and $(I + A_\mathrm{n})^{-1}(A + A_\mathrm{d})$ is Hurwitz. Then, the neutral time-delay system $\mathcal{G}$ given by (5.1) is asymptotically stable for all $\tau \in [0, \bar{\tau})$ if and only if

$$\det[I + G(j\omega)\Delta(j\omega)] \neq 0,\ \omega \in (0, \infty),\ \tau \in [0, \bar{\tau}), \tag{5.2}$$

where $G(s) \sim \left[\begin{array}{c|c} A & A_\mathrm{d} - AA_\mathrm{n} \\ \hline I & -A_\mathrm{n} \end{array}\right]$ and $\Delta(s) = e^{-\tau s} I_n$.

In this section, we consider a special structure of $\mathbf{\Theta}$ given by

$$\mathbf{\Theta} = \{\Theta \in \mathbb{R}^{n \times n} :\ \Theta = \theta I_n,\ \theta \geq 0,\ \}, \tag{5.3}$$

so that $\mathcal{T} = \mathbb{C}^{n \times n}$. Note that if $\mathbf{\Theta}$ is given by (5.3) then it follows from Theorem 4.4.2 that $\phi(M) = \phi_\mathrm{lb}(M) = \overline{\phi}(M)$, $M \in \mathbb{C}^{n \times n}$.

Theorem 5.2.1. Assume $\rho(A_\mathrm{n}) < 1$ and $(I + A_\mathrm{n})^{-1}(A + A_\mathrm{d})$ is Hurwitz and let $\tau_\mathrm{lb} \triangleq \inf_{\omega > 0} \frac{\phi_\mathrm{lb}(G(\jmath\omega))}{\omega}$. Then the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for $\tau \in [0, \tau_\mathrm{lb})$.

Proof. If $\tau_\mathrm{lb} = \infty$, then for all $\omega \in (0, \infty)$, $\alpha_\mathrm{lb}(G(\jmath\omega)) = \phi_\mathrm{lb}(G(\jmath\omega)) = \infty$. Now, it follows from Theorem 4.4.1 that $\phi(G(\jmath\omega)) = \infty$, $\omega \in (0, \infty)$. Hence, by definition, $\det(I - G(\jmath\omega)e^{j\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, which implies that $\det(I + G(\jmath\omega)\Delta(\jmath\omega)) \neq 0$, $\omega \in (0, \infty)$, $\tau \in [0, \tau_\mathrm{lb})$.

Next, assume $\tau_\mathrm{lb} < \infty$ and let $\omega \in (0, \infty)$ be such that $\alpha_\mathrm{lb}(G(\jmath\omega)) = \phi_\mathrm{lb}(G(\jmath\omega))$. Now, since $\phi_\mathrm{lb}(G(\jmath\omega)) = \phi(G(\jmath\omega))$, it follows from i) of Proposition 4.3.1 that $\det(I - G(\jmath\omega)e^{\jmath\Theta}) \neq 0$, $\Theta \in \mathbf{\Theta}$, $-\alpha_\mathrm{lb}(G(\jmath\omega))I_n < \Theta < \alpha_\mathrm{lb}(G(\jmath\omega))I_n$. Now, for $\tau \in [0, \tau_\mathrm{lb})$, it follows that $-\alpha_\mathrm{lb}(G(\jmath\omega)) < -\omega\tau$ which implies that $\det(I + G(\jmath\omega)\Delta(\jmath\omega)) \neq 0$, $\tau \in [0, \tau_\mathrm{lb})$. □

Corollary 5.2.1. Let $\bar{\tau} > 0$ and assume $\rho(A_{\rm n}) < 1$ and $(I + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Assume there exist functions $p_{\bar{\tau}} : \mathbb{R} \to \mathbb{R}$, $R, S : \jmath\mathbb{R} \to \mathbb{C}^{nm\times nm}$ such that $p_{\bar{\tau}}(\omega) = p_{\bar{\tau}}(-\omega)$, $R(\jmath\omega) = R^*(\jmath\omega)$, $S(\jmath\omega) \geq 0$, $\omega \in \mathbb{R}$, and

$$p_{\bar{\tau}}(\omega) < \begin{cases} \cos(\omega\bar{\tau}), & \text{if } \omega \leq \frac{\pi}{\bar{\tau}}, \\ -1, & \text{if } \omega > \frac{\pi}{\bar{\tau}}, \end{cases} \tag{5.4}$$

and

$$\begin{aligned} &G^*(\jmath\omega)(2p_{\bar{\tau}}(\omega)S(\jmath\omega) - R(\jmath\omega))G(\jmath\omega) + R(\jmath\omega) \\ &-G^*(\jmath\omega)S(\jmath\omega) - S(\jmath\omega)G(\jmath\omega) > 0, \quad \omega \in (0, \infty). \end{aligned} \tag{5.5}$$

Then the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof. It follows from (5.5) that $\gamma_{\rm lb}(G(\jmath\omega)) \leq p_{\bar{\tau}}(\omega)$, $\omega \in (0, \infty)$. Now, it follows from (5.4) that $\gamma_{\rm lb}(G(\jmath\omega)) < \cos(\omega\bar{\tau})$ if $\omega \leq \frac{\pi}{\bar{\tau}}$ and $\gamma_{\rm lb}(G(\jmath\omega)) < -1$, otherwise; or, equivalently, $\phi_{\rm lb}(G(\jmath\omega)) > \omega\bar{\tau}$ if $\omega \leq \frac{\pi}{\bar{\tau}}$ and $\phi_{\rm lb}(G(\jmath\omega)) = \infty$, otherwise. Hence, it follows that $\inf_{\omega>0} \phi_{\rm lb}(G(\jmath\omega))/\omega \geq \bar{\tau}$ which further implies that $\bar{\tau} \leq \tau_{\rm lb}$. Now, the result is a direct consequence of Theorem 5.2.1. □

Corollary 5.2.2. Let $\bar{\tau} > 0$ and assume $\rho(A_{\rm n}) < 1$ and $(I + A_{\rm n})^{-1}(A + A_{\rm d})$ is Hurwitz. Let $\hat{\gamma} \in \mathbb{R}$ and $\hat{\omega} \in [0, \infty)$ be defined by

$$\begin{aligned} \hat{\gamma} \triangleq \inf_{\omega>0} \{ &\gamma \in \mathbb{R} : \text{there exist } R, S : \jmath\mathbb{R} \to \mathbb{C}^{n\times n} \\ &\text{such that for every } \omega \in \mathbb{R},\ R(\jmath\omega) = R^*(\jmath\omega), \\ &S(\jmath\omega) \geq 0, \text{ and } G^*(\jmath\omega)(2\gamma S(\jmath\omega) - R(\jmath\omega))G(\jmath\omega) \\ &-G^*(\jmath\omega)S(\jmath\omega) - S(\jmath\omega)G(\jmath\omega) + R(\jmath\omega) > 0\}, \end{aligned} \tag{5.6}$$

$$\begin{aligned}\hat{\omega} \triangleq \inf\{&\bar{\omega} \in (0,\infty) : \text{there exist } R : \jmath\mathbb{R} \to \mathbb{C}^{n\times n} \\ &\text{such that for every } \omega \geq \bar{\omega},\ R(\jmath\omega) = R^*(\jmath\omega) \text{ and} \\ &R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0\}. \end{aligned} \tag{5.7}$$

Then the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for $\tau \in [0, \hat{\tau})$, where $\hat{\tau} = \frac{\cos^{-1}(\hat{\gamma})}{\hat{\omega}}$.

Proof. Note that $\gamma_{\text{lb}}(G(\jmath\omega)) \leq \hat{\gamma}$, $\omega \in (0,\infty)$, and hence $\phi_{\text{lb}}(G(\jmath\omega)) \geq \cos^{-1}(\hat{\gamma})$. Now, for all $\omega \in (0, \hat{\omega}]$, $\frac{\phi_{\text{lb}}(G(\jmath\omega))}{\omega} \geq \frac{\cos^{-1}(\hat{\gamma})}{\hat{\omega}} = \hat{\tau}$. Next, for every $\omega > \hat{\omega}$, since there exists $R(\jmath\omega)$ such that $G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) - R(\jmath\omega) < 0$, it follows that $\gamma_{\text{lb}}(G(\jmath\omega)) = -\infty$ or, equivalently, $\phi_{\text{lb}}(G(\jmath\omega)) = \infty$. Hence, $\hat{\tau} \leq \tau_{\text{lb}}$. Now, the result is a direct consequence of Theorem 5.2.1. □

Remark 5.2.1. Since $G(j\infty) = A_{\text{n}}$ and $\rho(A_{\text{n}}) < 1$, it follows that

$$\hat{\omega} = \inf\{\overline{\omega} \in (0,\infty) : \text{for every } \omega \geq \overline{\omega},\ \rho(A_{\text{n}}) < 1\}. \tag{5.8}$$

Corollary 5.2.3. Assume $\rho(A_{\text{n}}) < 1$ and $(I + A_{\text{n}})^{-1}(A + A_{\text{d}})$ is Hurwitz. The neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for $\tau \in [0,\infty)$, if and only if there exists $R : \jmath\mathbb{R} \to \mathbb{C}^{n\times n}$ such that $R(j\omega) = R^*(j\omega)$ and

$$R(\jmath\omega) - G^*(\jmath\omega)R(\jmath\omega)G(\jmath\omega) > 0, \quad \omega \in (0,\infty). \tag{5.9}$$

Proof. Sufficiency is a direct consequence of Corollary 5.2.2. Specifically, it follows from (5.9) that $\hat{\omega} = 0$ which implies that $\hat{\tau} = \infty$. Next, assume that $\mathcal{G}$ is asymptotically stable for $\tau \in [0,\infty)$. Hence, it follows from Lemma 5.2.1 that $\det(I - G(\jmath\omega)e^{-\jmath\omega\tau}) \neq 0$, $\omega \in \mathbb{R}$ and $\tau \in [0,\infty)$ which implies that $\det(I - G(\jmath\omega)e^{-\jmath\theta}) \neq 0$, $\theta \in \mathbb{R}$. Thus, it follows that $\overline{\phi}(G(\jmath\omega)) = \phi_{\text{lb}}(G(\jmath\omega)) = \infty$ which proves the result. □

Remark 5.2.2. As in Remark 5.2.1, since $G(j\infty) = A_{\text{n}}$ and $\rho(A_{\text{n}}) < 1$, it follows that (5.9) holds for all $\omega \in (0,\infty)$ if and only if $\rho(G(j\omega)) < 1$. Hence, the neutral time-delay dynamical system $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$ if and only if $\rho(G(j\omega)) < 1$.

Remark 5.2.3. In the case where $A, A_{\mathrm{n}}, A_{\mathrm{d}} \in \mathbb{R}$, that is $\mathcal{G}$ is a scalar neutral time-delay system, it follows from Corollary 5.2.3 that $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$ if and only if $|G(j\omega)| < 1$, $\omega \in (0, \infty)$, or, equivalently, $|A_{\mathrm{n}}| < 1$ and $|A_{\mathrm{d}}| < |A|$.

5.3. Time-Domain Test for Stability Analysis of Linear Neutral Time-Delay Systems

In this section we apply Theorem 4.7.2 to derive a new LMI test for stability analysis of linear neutral time-delay dynamical systems.

Theorem 5.3.1. Let $\bar{\tau} > 0$ and assume $\rho(A_{\mathrm{n}}) < 1$ and $(I_n + A_{\mathrm{n}})^{-1}(A + A_{\mathrm{d}})$ is Hurwitz. Now, let

$$H(s) = \begin{bmatrix} \left(\frac{1}{s+1}\right) I_n \\ \bar{\tau}\left(\frac{s}{s+1}\right) I_n G(s)(I_n + G(s))^{-1} \\ (I_n + G(s))^{-1} \\ G(s)(I_n + G(s))^{-1} \end{bmatrix} \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right],$$

where

$$\tilde{A} = \begin{bmatrix} -I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & -I_n & (A_{\mathrm{n}} + I_n)^{-1} \\ 0_{n\times n} & 0_{n\times n} & A + (A_{\mathrm{d}} - AA_{\mathrm{n}})(A_{\mathrm{n}} + I_n)^{-1} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} I_n \\ I_n - (A_{\mathrm{n}} + I_n)^{-1} \\ (A_{\mathrm{d}} - AA_{\mathrm{n}})(A_{\mathrm{n}} + I_n)^{-1} \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & -\bar{\tau} I_n & -\bar{\tau}(A_{\mathrm{n}} + I_n)^{-1} \\ 0_{n\times n} & 0_{n\times n} & (A_{\mathrm{n}} + I_n)^{-1} \\ 0_{n\times n} & 0_{n\times n} & -(A_{\mathrm{n}} + I_n)^{-1} \end{bmatrix}, \quad \text{and}$$

$$\tilde{D} = \begin{bmatrix} 0_{n\times n} \\ \bar{\tau}(I_n - (A_{\mathrm{n}} + I_n)^{-1}) \\ (A_{\mathrm{n}} + I_n)^{-1} \\ I_n - (A_{\mathrm{n}} + I_n)^{-1} \end{bmatrix}.$$

Finally, assume there exists $P \in \mathbb{S}^{3n\times 3n}$, $R \in \mathcal{T}$, and $S \in \mathcal{T}$, $S \geq 0$, such that

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}P + P\tilde{A} + \tilde{C}^{\mathrm{T}}\tilde{Q}\tilde{C} & P\tilde{B} + \tilde{C}^{\mathrm{T}}\tilde{Q}\tilde{D} \\ \tilde{B}^{\mathrm{T}}P + \tilde{D}^{\mathrm{T}}\tilde{Q}\tilde{C} & \tilde{D}^{\mathrm{T}}\tilde{Q}\tilde{D} \end{bmatrix} < 0, \tag{5.10}$$

where $\tilde{Q} = \text{block–diag}(-S, S, -R, R)$. Then the linear neutral time-delay dynamical system $\mathcal{G}$ given by (5.1) is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof. Note that by choosing $M_{\bar{\tau}}(\jmath\omega) = \jmath\omega\bar{\tau}I_n$ and $S(\jmath\omega) = \left(\frac{1}{\jmath\omega+1}I_n\right)^* S \left(\frac{1}{\jmath\omega+1}I_n\right)$, where $S \geq 0$, (4.28) is equivalent to

$$H^*(\jmath\omega)\tilde{Q}H(\jmath\omega) < 0, \ \omega \in (0, \infty).$$

Now, the result is a direct consequence of Theorem 4.7.2 and the Kalman-Yakubovich-Popov lemma [59]. □

Remark 5.3.1. Note that the choice of $M_{\bar{\tau}}(\jmath\omega)$, $S(\jmath\omega)$ and $R(\jmath\omega)$ is not unique, and there may be other choices that lead to useful LMI tests. The specific choice used in Theorem 5.3.1 ensures that $H(s)$ is realizable, which is essential for developing the LMI test for stability.

5.4. Illustrative Numerical Examples

In this section we consider several illustrative numerical examples to demonstrate the utility of the proposed theory and the LMI test developed in the previous section. We will compare the results from the LMI test developed in this paper with tests from [67, 69]. In all the following examples, we shall use the notation τ_{LMI} to denote the delay margin predicted using Theorem 5.3.1, and τ_{H} and τ_{F} to denote the delay margins predicted using Theorem 1 from [67] and Theorem 1 from [69], respectively.

Example 5.4.1. Consider a two-dimensional, linear neutral time-delay dynamical system of the form (5.1) given by

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}, \ A_{\rm n} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and}$$

$$A_{\rm d} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}.$$

For this example, we obtained $\alpha_{\rm lb}(G(\jmath\omega)) = \infty$, $\omega \neq 0.7063$ and $\alpha_{\rm lb}(G(\jmath\omega)) = 2.6932$, $\omega = 0.7063$. Hence, $\tau_{\rm lb} = 3.8133$. Furthermore, using multivariable Nyquist criterion it can be shown that $\tau = \tau_{\rm lb} = 3.8133$.

For this numerical example, the LMI test from Theorem 5.3.1 predicted a maximum allowable delay of $\tau_{\rm LMI} = 2.7412$ while we obtained $\tau_{\rm H} = \tau_{\rm F} = 0.1754$. The LMI in Theorem 5.3.1 provides a much better allowable delay margin in this case.

Example 5.4.2. In this example, we adopt the linear neutral time-delay system of the form (5.1) with system matrices [67]

$$\dot{x}(t) - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \dot{x}(t-\tau) = \begin{bmatrix} -0.8 & 0.2 \\ -0.2 & -0.8 \end{bmatrix} x(t-\tau), \tag{5.11}$$

where $0 < c < 1$.

For this example, we compute $\tau_{\rm lb}$ using Theorem 5.1 from [70] and $\tau_{\rm LMI}$ using Theorem 5.3.1 for several values of $c \in [0, 1)$. We then compare our predictions with those of the results given in [67]. Table 5.1 shows that our LMI test gives better predictions than those of [67], and the predicted delay margins are closer to the values of $\tau_{\rm lb}$ predicted using the frequency-domain results.

Example 5.4.3. In this example, we adopt the linear neutral time-delay system of the form (5.1) with system matrices [69]

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \ A_{\rm n} = \begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.1 \end{bmatrix}, \text{ and } A_{\rm d} = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}.$$

Table 5.1: Maximum allowable delay prediction for Example 5.4.2

c	0.0	0.1	0.2	0.3	0.4
τ_{lb} (Theorem 5.1 [70])	1.6078	1.4468	1.2784	1.1046	0.9273
τ_{LMI} (Theorem 5.3.1)	1.4926	1.3575	1.2114	1.0564	0.8944
Theorem 1 [67]	1.176	1.055	0.933	0.812	0.691

c	0.5	0.6	0.7	0.8	0.9
τ_{lb} (Theorem 5.1 [70])	0.7484	0.5704	0.3964	0.2315	0.0863
τ_{LMI} (Theorem 5.3.1)	0.7277	0.5587	0.3909	0.2297	0.0860
Theorem 1 [67]	0.570	0.448	0.327	0.206	0.085

For this example, using Theorem 5.1 from [70], we obtained a value of $\tau_{\text{lb}} = 2.2254$ while the LMI from Theorem 5.3.1 gave a value of $\tau_{\text{LMI}} = 1.7891$. We also computed $\tau_{\text{H}} = \tau_{\text{F}} = 0.7436$, using the LMIs from [67,69]. In this example also, our LMI predicts a sharper alowable delay margin.

Example 5.4.4. This example is adopted from [71]. The linear neutral time-delay system is of the form 5.1 with the system matrices

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.9 \end{bmatrix}, \quad A_{\text{n}} = \begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.2 \end{bmatrix}, \text{ and}$$

$$A_{\text{d}} = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}.$$

For this example, we computed $\tau_{\text{lb}} = 2.4617$, and $\tau_{\text{LMI}} = 1.9413$. The results from [71] predict an allowable delay margin of $\bar{\tau} = 1.5687$. It is evident that for this numerical example our LMI result predicts a better delay margin than that in [71].

Chapter 6

Conclusions and Future Research

6.1. Contributions

In the preceding chapters, we described how we have used ideas from dissipativity theory and developed the notion of the structured phase margin to arrive at sufficient conditions to guarantee the asymptotic stability of linear time-delay dynamical systems. Listed below are the important contributions of this research work:

- Provided a concrete method for developing Lyapunov-Krasovskii functionals for linear time-delay systems based on the dissipativity properties of the time delay operator

- Developed dynamic extensions to the concepts of dissipativity and exponential dissipativity. The new approach provides a time-domain analog to the notion of integral quadratic constraints (IQCs)

- Dynamic dissipativity was used to develop sufficient conditions for stability of linear time-delay systems

- Introduced the notion of the structured phase margin for characterizing stability margins for dynamical systems with block-structured phase uncertainty

- An easily computable lower bound was derived in terms of a generalized eigenvalue problem for the structured phase margin

- Delay-dependent stability criteria for stability analysis of linear time-delay systems were developed using the notion of the structured phase margin

- New time-domain conditions (in terms of LMIs) for stability analysis of linear time-delay dynamical systems were developed

- Illustrative examples were presented to demonstrate the utility and superiority of the newly developed results over existing results

- In the presented research, a solid theoretical basis was used to develop the stability analysis results

In Chapter 2, using the concepts of dissipativity and exponential dissipativity, we developed sufficient conditions to guarantee asymptotic stability of a time delay dynamical system. We considered dissipative properties of the time delay operator that are independent of the amount of time delay. Future extensions of this work will involve dissipative properties of the time delay operator which will include the amount of time delay (i.e. *delay dependent* conditions) thus providing a mechanism for obtaining Lyapunov-Krasovskii functionals to prove stability of time delay dynamical systems that depend on the amount of time delay.

In Chapter 3, we extended the concepts of dissipativity and exponential dissipativity to provide new sufficient conditions for guaranteeing asymptotic stability of a time delay dynamical system. The overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator.

In Chapter 4, we introduced the notion of the structured phase margin for characterizing stability margins for dynamical systems with block-structured phase uncertainty. Furthermore, an easily computable lower bound was derived in terms of a generalized eigenvalue problem. We also demonstrated the newly developed stability analysis tests on several numerical examples and showed that our results are less conservative compared to the other results in the literature for capturing phase information uncertainty.

In Chapter 5, we extended the results from the structured phase margin for the analysis of neutral time-delay systems, and specifically we derived a new LMI-based delay-dependent stability condition for linear neutral time-delay systems. We also demonstrated that the new test provides less conservative results in many examples.

In conclusion, the research in this dissertation will provide a new understanding and approach to the stability analysis of linear time-delay systems.

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