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Eigenvalue Dependence on Problem Parameters for Stieltjes Sturm-Liouville Problems

Laurie Elizabeth Battle
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Laurie Elizabeth Battle entitled "Eigenvalue Dependence on Problem Parameters for Stieltjes Sturm-Liouville Problems." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Don Hinton, Major Professor

We have read this dissertation and recommend its acceptance:

Bo Guan, Suzanne Lenhart, Marianne Breinig

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Eigenvalue Dependence on Problem Parameters for Stieltjes Sturm-Liouville Problems

A Dissertation

Presented for the

Doctor of Philosophy Degree

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Laurie Elizabeth Battle

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Abstract

This work examines generalized Stieltjes Sturm-Liouville boundary value problems with particular consideration of self-adjoint problems. Of central importance is determining conditions under which the eigenvalues depend continuously and differentiably on the problem data. These results can be applied to various physical problems, such as constructing beams to maximize the fundamental frequency of vibration, or constructing columns to maximize the height without buckling. These problems involve maximizing the smallest eigenvalues of Sturm-Liouville equations, and the continuous dependence of the eigenvalues on the problem parameters can be used to accomplish this.

We first consider the generalized $2n$-dimensional initial value problem $dy = Aydt + dPz$, $dz = (dQ - \lambda dW)y + Dzdt$ on an interval $[a, b]$. In the proof of existence and uniqueness of a quasi-continuous solution, we establish some bounds and continuity properties of the solution that will be used throughout this work. Next we define a sequence of initial value problems and prove that the sequence of solutions converges to the solution of the limit problem.

We then consider the eigenvalue problem, adding general boundary conditions to the system of equations. The eigenvalues are shown to be the roots of an entire function. Taking a sequence of eigenvalue problems, we show that a sequence of eigenvalues converges. This result establishes conditions under which each eigenvalue depends continuously on the coefficients and on the boundary data. We find separate conditions for the continuous dependence on the endpoints of the interval.

We next turn to ascertaining conditions under which each eigenvalue depends differentiably on the problem data. For this topic, we consider the less general 2-

dimensional Stieltjes Sturm-Liouville problem $dy = dPz$, $dz = (dQ - \lambda dW)y$ with separated boundary conditions. Considering each eigenvalue as a function of the coefficients and of the boundary data, we conclude that these functions are differentiable under the same conditions we found for continuity. Separate conditions are found to guarantee the differentiability of each eigenvalue with respect to the endpoints. In all cases, we find expressions for the derivatives of the eigenvalues with respect to the problem parameters.

We conclude with an application to the problem of finding extremal values of an eigenvalue. For the fourth order problem $(ry'')'' + (py')' + qy = \lambda wy$ with boundary conditions $y(a) = y'(a) = y(b) = y'(b) = 0$, we consider the smallest eigenvalue λ_0 as a function of the coefficients. The continuous dependence of the eigenvalue on the coefficients is used to find a sequence of coefficients converging to a function that attains the supremum or infimum of λ_0 over a certain class of coefficient functions.

Contents

Chapter 1

Introduction

This work examines eigenvalues of generalized $2n$-dimensional Sturm-Liouville problems of the form

$$\begin{cases} dy = Aydt + dPz \\ dz = (dQ - \lambda dW)y + Dzdt \end{cases}$$

on an interval $[a, b]$ with general boundary conditions. One goal is to determine conditions under which each eigenvalue depends continuously and differentiably on the problem data, including the coefficients, boundary data, and endpoints of the interval.

One motivation for this work involves finding extremal values of an eigenvalue, which has applications in several types of physical problems. For vibrating elastic systems, a typical problem is to maximize the lowest frequency, i.e., to find the maximum of the smallest eigenvalue, under certain constraints. A simple example is the construction of a vibrating string with specified length L and total mass M to have the maximum or minimum lowest frequency possible. This problem is modeled by the second order differential equation $-y'' = \lambda\rho(x)y$ with boundary conditions $y(0) = y(L) = 0$ [14]. The mass constraint is $\int_0^L \rho(x)dx = M$. It is known that the minimum frequency is achieved by placing all of the mass at the center of the string. This leads to making $\rho(x)$ to be the delta function $\delta\left(x - \frac{L}{2}\right)$, and hence the problem is of the Stieltjes Sturm-Liouville type. The choice of admissible coefficients determines if the maximum problem has a solution. Here, one must place an upper bound on $\rho(x)$ in order for a maximum to exist.

For vibrating beams (transverse vibrations) or plates, the model is a fourth order problem such as $(a^p y'')'' - \lambda a y = 0$, where $p \geq 1$. The cases where $p = 1, 2, 3$ are especially important because they correspond to beams with rectangular cross sections of given uniform width. This problem can have various boundary conditions. For example, the boundary conditions with the right end clamped and the left end free are given by $y(1) = y'(1) = 0$, $a^p y''(0) = 0$, $(a^p y'')'(0) = \lambda q y(0)$. The constraint here is $a(t) > 0$ and $\int_0^1 a(t)dt = 1$ [19]. For the beam problem, the objective is to determine the mass distribution that maximizes the fundamental frequency under the constraint of fixed total mass, which again involves determining the maximum of the smallest eigenvalue. In chapter 7, we show how the results of our work can be applied to a problem of fourth order.

Another type of problem involving extremal values of an eigenvalue is the problem of constructing the tallest column that will not buckle, under constraints such as a specified volume. One such problem is modeled by the eigenvalue problem $-(a(x)^2 y')' = \lambda\left(\int_x^1 a(t)dt\right)y$, $0 < x < 1$, with boundary conditions $y(0) = a(1)^2 y'(1) = 0$, with the constraint that $a(t) > 0$ and $\int_0^1 a(t)dt = 1$ [3], [8], [15].

Another motivation for this work is to be able to rely on calculations of eigenvalues when the problem data are approximated. Software packages for computing eigenvalues often approximate the problem parameters, under the assumption that the eigenvalues of the approximate problem will be close to those of the given problem [13]. This work establishes some conditions under which this assumption is valid, i.e., under which the eigenvalues depend continuously on the problem parameters.

Previous work on this problem dealt with less general conditions. Kong and Zettl [10], [13] and Kong, Wu, and Zettle [11], [12], consider eigenvalue dependence for scalar, self-adjoint equations, whereas this work allows for Stieltjes Hamiltonian systems. They take sequences of eigenvalue problems to examine eigenvalue dependence on the problem data, and they require L^1 convergence of the coefficients. In the equation $-(py')' + qy = \lambda wy,$ the approximate equation $-(p_1y')' + q_1y = \lambda w_1y$ is considered to be close to the original equation if $\int_a^b \left|\frac{1}{p} - \frac{1}{p_1}\right| + \int_a^b |q - q_1| + \int_a^b |w - w_1|$ is small. We take the same approach of using sequences of eigenvalue problems, but we allow for more general modes of convergence on the coefficients, with two sequences of coefficients converging weakly in $L^1,$ one converging uniformly, and two converging pointwise.

Reid [16] addresses this problem for Hamiltonian systems, but we relax his hypotheses on the data and on the modes of convergence. Further, the issue of differentiability of the eigenvalues with respect to the problem parameters is not addressed by Reid.

Knotts-Zides [9] extends Reid's results to more general conditions, but her problem is only 2-dimensional and contains fewer coefficients. We extend Knotts-Zides' results concerning the initial value problem and the continuous dependence of the eigenvalues to a more general Sturm-Liouville problem. Then we go a step further by proving the differential dependence of the eigenvalues. We also give a brief discussion of eigenvalue extremal problems, while Knotts-Zides gives a more detailed analysis. Our analysis addresses the problem $(ry'')'' + (py')' + qy = \lambda wy$ with boundary conditions $y(a) = y(b) = y'(a) = y'(b) = 0$, under the constraints $\int_a^b p \leq M_1$, $\int_a^b q \leq M_2$, $\int_a^b r \leq M_3$, and $\int_a^b w \leq M_4$. Knotts-Zides' analysis addresses the problem $-y'' + qy = \lambda y$ with separated boundary conditions and the constraint $\int_a^b |q| \leq M$. We prove the existence of extremizing functions for our problem, wheras Knotts-Zides actually characterizes the extremizing functions for her problem (for certain boundary conditions).

We begin with the initial value problem. The existence and uniqueness of a quasi-continuous solution was established by Hinton [6], but we present a proof in order to establish certain bounds and continuity properties of the solution. In Chapter 4, we take sequences of coefficients, leading to a sequence of initial value problems. We prove that the sequence of solutions $\{y_n\}$, $\{z_n\}$ converges to the solution of the limit problem, with the $\{y_n\}$ converging uniformly and the $\{z_n\}$ converging pointwise.

The eigenvalue problem is addressed in Chapter 5 by including general boundary conditions. We show that the eigenvalues are the roots of an entire function. Taking a sequence of eigenvalue problems, we prove that the sequence of ith eigenvalues converges to the ith eigenvalue of the limit problem, where the eigenvalues are listed in increasing order. This establishes conditions under which each eigenvalue depends continuously on the coefficients and on the boundary data. Separate conditions are found which guarantee the continuous dependence on the endpoints of the interval.

Next we turn to the issue of differentiability of the eigenvalues as functions of the problem parameters. For this issue, we address the less general 2-dimensional Sturm-Liouville problem

$$\begin{cases} dy = dPz \\ dz = (dQ - \lambda dW)y \end{cases}$$

on $[a, b]$ with separated boundary conditions. We determine conditions under which each eigenvalue is differentiable as a function of the endpoints. For the dependence on the coefficients and boundary data, we find that the same hypotheses that guarantee continuity also guarantee differentiability. In all cases, we derive expressions of the derivatives of the eigenvalues with respect to the data.

We conclude with an application in Chapter 7 involving extremal properties of eigenvalues. We consider the fourth order problem $(ry'')'' + (py')' + qy = \lambda wy$ with boundary conditions $y(a) = y'(a) = y(b) = y'(b) = 0$, viewing the smallest eigenvalue λ_0 as a function of the coefficients. We pose the problem of determining if there exist functions that attain the supremum and infimum of λ_0 over a certain class of coefficient functions. We use the continuous dependence of λ_0 on the coefficients to find a sequence of functions that approaches such an extremizing function.

There are several areas we plan to explore in future work. One area is to extend our results to apply to singular problems. Also, the question of differentiability of the eigenvalues with respect to the problem parameters is addressed in this work for a less general problem, and we plan to extend these results to the original problem. More detailed study can be given to extremal eigenvalue problems to actually characterize the extremizing functions rather than only proving their existence.

Chapter 2

Preliminaries

Here we give a preliminary discussion of Stieltjes integrals and some previous results by D.B. Hinton [6].

Let N be a non-degenerate ring. Let $\|\cdot\|$ be a norm for N and suppose that N is complete with respect to this norm. In this work, we take N to be the ring of $2n \times 2n$ matrices and we define the norm as follows: First we define the vector norm by $\|\bar{x}\| := \sum_i |x_i|$ for $\bar{x} = (x_1, \ldots, x_{2n})^T$. Then define the norm on N by $\|A\| := \sum_{i,j} |a_{ij}|$ for $A = \{a_{ij}\}$. It follows that for A in N and $\bar{x}$ a $2n$-vector, $\|A\bar{x}\| \leq \|A\|\|\bar{x}\|$. Let $\mathbb{R}^m$ be the space of m-vectors.

For some number interval $[a, b]$ with $a < b$, a function $F : [a, b] \to N$ is said to be *quasi-continuous* if the left and right limits exist at each interior point of $[a, b]$ and the appropriate one-sided limit exists at each endpoint. A function $F : [a, b] \to N$ is said to be of *bounded variation* if there exists a number K such that

$$\sum_{i=1}^{m} \|F(t_i) - F(t_{i-1})\| \leq K$$

for any partition $a = t_0 < t_1 < \ldots < t_m = b$. The greatest lower bound of such constants K is called the *total variation* of F and is denoted $\bigvee_a^b F$ or $\int_a^b \|dF(t)\|$. If a real function F is nondecreasing on an interval $[a, b]$, then F is of bounded variation with $\bigvee_a^b F = F(b) - F(a)$. A function of bounded variation is also quasi-continuous. For $t \geq a$ and a function F of bounded variation on $[a, t]$, define the total variation function

$$v_F(t) := \bigvee_a^t F.$$

If F is of bounded variation and is continuous, then v_F is also continuous [18, Theorem 6.26].

If F is a function from $[a, b]$ to N, and if G is a function from $[a, b]$ to N or to $\mathbb{R}^{2n}$, then the *Cauchy-left integral*

$$(L) \int_a^b dF(x)G(x)$$

denotes an element Y of N or $\mathbb{R}^{2n}$ with the following property: for each positive number ϵ, there is a partition $s = \{s_i\}_0^n$ from a to b such that for any partition $t = \{t_i\}_0^m$ that is a refinement of s, then

$$\left\| Y - \sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_{i-1}) \right\| < \epsilon$$

The *Cauchy-right integral*

$$(R) \int_a^b dF(x)G(x)$$

denotes an element Y of N or $\mathbb{R}^{2n}$ with the following property: for each positive number ϵ, there is a partition $s = \{s_i\}_0^n$ from a to b such that for any partition $t = \{t_i\}_0^m$ that is a refinement of s, then

$$\left\| Y - \sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_i) \right\| < \epsilon$$

The ordinary *Stieltjes integral*

$$\int_a^b dF(x)G(x)$$

denotes an element Y of N or $\mathbb{R}^{2n}$ with the following property: for each positive number ϵ, there is a partition $s = \{s_i\}_0^n$ from a to b such that for any partition $t = \{t_i\}_0^m$ that is a refinement of s, then

$$\left\| Y - \sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_i^*) \right\| < \epsilon,$$

for all t_i^* such that $t_i^* \in [t_{i-1}, t_i]$. We can define similarly the Cauchy-left integral $(L)\int_a^b G(x)dF(x)$, the Cauchy-right integral $(R)\int_a^b G(x)dF(x)$, and the Stieltjes integral $\int_a^b G(x)dF(x)$. These integrals are known to be unique if they exist. The Cauchy integrals are known to exist if F is of bounded variation and G is quasi-continuous. If in addition, either F or G is continuous, the Stieltjes integral is known to exist. If the Stieltjes integral exists, then the corresponding left-Cauchy and right-Cauchy integrals exist and all three integrals are equal.

Now we state some theorems that will be referenced in this paper [6]. They are stated here without proof.

Theorem 2.1. *If F is of bounded variation from $[a, b]$ to N, G is quasi-continuous from $[a, b]$ to N or to $\mathbb{R}^{2n}$, and f and g are real functions on $[a, b]$ such that for $x \leq y$, $\bigvee_x^y F \leq f(y) - f(x)$, and $g(x) = \|G(x)\|$, then*

$$\left\| (L)\int_a^b dF(s)G(s) \right\| \leq \left| (L)\int_a^b df(s)g(s) \right| \leq \left(\bigvee_a^b f \right) \|g\|_{[a,b]},$$

where $\|g\|_{[a,b]} := \sup\{g(x) : x \in [a, b]\}$.

Theorem 2.2. *If $K \geq 0$, h is a real nondecreasing function on $[a, b]$, and if m is a real function on $[a, b]$ bounded above by some $T > 0$ and such that for each x,*

$$m(x) \leq K + (L) \int_a^x m(s)dh(s),$$

then

$$m(x) \leq Ke^{[h(x)-h(a)]}$$

for each x.

Let $\mathcal{F}$ be the set of all functions $F : [a, b] \times [a, b] \to N$ such that

1. $F(x, x) = I$ for all x,

2. F is quasi-continuous with respect to its first place, and

3. there is a real nondecreasing function g on $[a, b]$ such that $g(a) = 0$ and

   $$\|F(t, x) - F(t, y)\| \leq \|g(x) - g(y)\|$$

   for all t, x, and y. Such a function g is called a *super function* for F.

In this paper, we consider a problem that involves a function $F(t)$, which is not an element of $\mathcal{F}$ since it is a function of only one variable. However, we can define $\tilde{F}(t, x) := I + F(x) - F(t)$, which can be shown to be an element of $\mathcal{F}$. This allows the following two theorems to apply to our case.

Theorem 2.3. *If $F \in \mathcal{F}$, $Q : [a, b] \to N$ is quasi-continuous, $X = L$ or $X = R$, and P is defined on $[a, b]$ by*

$$P(t) = (X) \int_a^t d_s F(t, s) Q(s),$$

then P is quasi-continuous. Moreover, if F is continuous with respect to its first variable, then P is continuous.

Theorem 2.4. *Given $F \in \mathcal{F}$, there is a unique $M \in \mathcal{F}$ that is a solution of*

$$M(t,x) = I + (L)\int_x^t d_s F(t,s)M(s,x)$$

for all t and x. Moreover, if F is continuous with respect to its first place, then so is M.

Theorem 2.5. *If $F \in \mathcal{F}$ and G is a quasi-continuous function from $[a,b]$ to N or to $\mathbb{R}^{2n}$, then there is a unique quasi-continuous function Y on $[a,b]$ such that*

$$Y(t) = G(t) + (L)\int_x^t d_s F(t,s)Y(s).$$

Now we state two convergence theorems for Stieltjes integrals [5], followed by a convergence theorem for a sequence if functions.

Theorem 2.6. (Helly's Integral Convergence Theorem) *Let f be a continuous function on $[a,b]$ and let $\{g_n\}$ be a sequence of functions, uniformly of bounded variation on $[a,b]$, converging to a function g at every point of $[a,b]$. Then*

$$\lim_{n\to\infty}\int_a^b f dg_n = \int_a^b f dg.$$

Theorem 2.7. (Osgood's Theorem) *Let g be a function of bounded variation on $[a,b]$ and let $\{f_n\}$ be a sequence of functions which is uniformly bounded and converges pointwise to a function f on $[a,b]$. If $\int_a^b f_n dg$ and $\int_a^b f dg$ exist, then*

$$\lim_{n\to\infty}\int_a^b f_n dg = \int_a^b f dg.$$

Theorem 2.8. (Helly's Pointwise Convergence Theorem) *If f_n is a sequence of functions, uniformly of bounded variation on $[a,b]$ such that $f_n(a)$ is bounded in n, then there exists a subsequence f_{n_m} and a function f of bounded variation such that $\lim_{m\to\infty} f_{n_m} = f$ at every point of $[a,b]$.*

Here we introduce some notation that will be used throughout this paper. Let $\mathcal{P}[a,b]$ denote the set of all partitions of the interval $[a,b]$. Also, define $v_f(t) := \bigvee_a^t f$ and $I_f(t) := \int_a^t f$.

The final theorem stated here can be found in [4, p.105].

Theorem 2.9. *If $f \in L^1([a,b])$, then $I_f(t)$ is absolutely continuous and of bounded variation, and $\bigvee_a^b I_f = \int_a^b |f|$.*

Chapter 3

The Initial Value Problem

We consider the system of $2n$ equations

$$\begin{cases} dy = Aydt + dPz \\ dz = (dQ - \lambda dW)y + Dzdt \end{cases} \tag{3.1}$$

which can be written as the Stieltjes integral equation

$$\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix} = \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} + \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dM(s,\lambda) & D(s)ds \end{bmatrix} \begin{bmatrix} y(s,\lambda) \\ z(s,\lambda) \end{bmatrix} \tag{3.2}$$

on $[a,b] \times K$, where K is a compact set in $\mathbb{C}$, and $M(t,\lambda) = Q(t) - \lambda W(t)$. Here, y and z are n-vectors and A, P, Q, W, and D are $n \times n$ real matrices. We require the following conditions on the coefficients, where each condition holds for each element in the matrix:

$$\begin{cases} A, D \in L_1([a,b]); \\ P = P^T \text{ is continuous and nondecreasing with } P(a) = 0; \\ Q = Q^T \text{ is of bounded variation on } [a,b]; \\ W = W^T \text{ is nondecreasing with } W(a) = 0, \end{cases} \tag{3.3}$$

Remark 3.1. *1. When we say a matrix function is of bounded variation, we mean that each component is of bounded variation.*

2. When we say a real symmetric matrix A is positive, we mean that all of the eigenvalues of A are positive. This is equivalent to the condition that $\langle A\bar{x}, \bar{x}\rangle > 0$ *for all* $\bar{x} \neq \bar{0}$*. The meaning of a matrix function* $A(t)$ *being nondecreasing follows accordingly, i.e.,* $A(t)$ *is nondecreasing if* $A(t_2) - A(t_1)$ *is nonnegative for* $t_2 \geq t_1$*.*

3. The symmetry condition on Q*,* P*, and* W *is needed later for self-adjoint problems, but it is not needed to prove existence of a solution.*

In Reid's paper [16], the derivatives of P, Q, and W are required to be L_1 functions. We are allowing for more generality in the coefficients, not requiring a Banach space.

We define the following terms:

$$L := \left\| \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} \right\|;$$

$$F(t,\lambda) := \begin{bmatrix} \int_a^t A(s)ds & P(t) \\ M(t,\lambda) & \int_a^t D(s)ds \end{bmatrix};$$

$$f(t,\lambda) := \int_a^t \|A(s)\|ds + \int_a^t \|D(s)\|ds + \bigvee_a^t P + \bigvee_a^t M(\cdot,\lambda).$$

Note that $\bigvee_a^t M(\cdot,\lambda) \le \bigvee_a^t Q + |\lambda| \bigvee_a^t W$.

Suppose we fix λ and let $Y(t) := \begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}$. Then (3.2) can be written in the form $Y(t) = Y(a) + \int_a^t dF(s)Y(s)$. We now show that for fixed λ, $\tilde{F}(t,x) := I + F(x,\lambda) - F(t,\lambda)$ is an element of $\mathcal{F}$, which was defined in Chapter 2. The first condition, $\tilde{F}(x,x) = I$ is clearly satisfied. For the second condition, it suffices to show that $F(t,\lambda)$ is quasi-continuous in the first place. We know P is continuous by assumption. The remaining elements of F are now shown to be of bounded variation, which implies they are quasi-continuous. We know $\int_a^t A$ and $\int_a^t D$ are of bounded variation by Theorem 2.9, and Q is of bounded variation by assumption. P and W are of bounded variation because they are nondecreasing (the fact that a nondecreasing matrix is of bounded variation is proved later, in the proof of Lemma 4.1). The third condition is satisfied for $g(t) := \bigvee_a^t \tilde{F}$. Since $\tilde{F}(t,x) \in \mathcal{F}$, Theorems 2.3, 2.4, and 2.5 apply to our problem.

The existence and uniqueness of a quasi-continuous solution to the initial value problem (3.2) was established in Theorem 2.5. We repeat the proof to establish certain uniform bounds, to determine the continuity of the solution, and to establish a Lipschitz condition with respect to a spectral parameter.

Define successive approximations for $k = 1, 2, 3, \ldots$ as follows: Given initial conditions $y(a,\lambda)$ and $z(a,\lambda)$, let

$$\begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda) \end{bmatrix} = \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix}, \tag{3.4}$$

$$\begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} = \begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda) \end{bmatrix} + \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dM(s,\lambda) & D(s)ds \end{bmatrix} \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix}. \tag{3.5}$$

First we examine some properties of the successive approximations.

Lemma 3.2. *If successive approximations are defined as in (3.4), (3.5) subject to the conditions in (3.3), then*

$$\left\| \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k-1)}(t,\lambda) \\ z^{(k-1)}(t,\lambda) \end{bmatrix} \right\| \leq L \frac{f(t,\lambda)^k}{k!}$$

for $k = 1, 2, 3, \ldots$

Proof. The proof is by induction. First note that by definition,

$$\left\| \begin{bmatrix} y^{(1)}(t,\lambda) \\ z^{(1)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda) \end{bmatrix} \right\| = \left\| \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dM(s,\lambda) & D(s)ds \end{bmatrix} \begin{bmatrix} y^{(0)}(s,\lambda) \\ z^{(0)}(s,\lambda) \end{bmatrix} \right\|$$
$$\leq Lf(t,\lambda).$$

Assume the lemma is true for some k. Now

$$\begin{bmatrix} y^{(k+1)}(t,\lambda) \\ z^{(k+1)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix}$$

$$= \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dM(s,\lambda) & D(s)ds \end{bmatrix} \left(\begin{bmatrix} y^{(k)}(s,\lambda) \\ z^{(k)}(s,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix} \right)$$

$$= \int_a^t dF(s,\lambda) \left(\begin{bmatrix} y^{(k)}(s,\lambda) \\ z^{(k)}(s,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix} \right).$$

Now we will find a bound for this last integral. If $[c,d]$ is a subinterval of $[a,b]$, then over $[c,d]$ we have for $T = \{t_i\}_0^m \in \mathcal{P}[c,d]$,

$$\Delta v_F = \bigvee_c^d F = \sup_{T \in \mathcal{P}[c,d]} \sum_{i=1}^m \|F(t_i,\lambda) - F(t_{i-1},\lambda)\|$$

$$\leq \sup_{T \in \mathcal{P}[c,d]} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} [\|A(s)\|ds + \|D(s)\|]\, ds +$$

$$\sum_{i=1}^m \int_{t_{i-1}}^{t_i} [\|P(t_i) - P(t_{i-1})\| + \|M(t_i,\lambda) - M(t_{i-1},\lambda)\|]$$

$$= \int_c^d \|A(s)\|ds + \int_c^d \|D(s)\|ds + \bigvee_c^d P + \bigvee_c^d M(\cdot,\lambda)$$

$$= f(d,\lambda) - f(c,\lambda) = \Delta f.$$

Hence $\Delta v_F \leq \Delta f$.

Now our bound is

$$\left\| \begin{bmatrix} y^{(k+1)}(t,\lambda) \\ z^{(k+1)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix} \right\|$$

$$= \left\| \int_a^t dF(s,\lambda) \left(\begin{bmatrix} y^{(k)}(s,\lambda) \\ z^{(k)}(s,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix} \right) \right\|$$

$$\leq (L)\int_a^t dv_F(s,\lambda)\left\|\left(\begin{bmatrix} y^{(k)}(s,\lambda) \\ z^{(k)}(s,\lambda)\end{bmatrix} - \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda)\end{bmatrix}\right)\right\|$$
$$\leq (L)\int_a^t df(s,\lambda)L\frac{f(s,\lambda)^k}{k!}$$
$$\leq L\frac{f(t,\lambda)^{k+1}}{(k+1)!},$$

where the second to last inequality follows from induction hypothesis and the last inequality can be found in [6, p.318]. □

Lemma 3.3. *If successive approximations are defined as in (3.4), (3.5) subject to the conditions in (3.3), then (i)* $\left\|\begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda)\end{bmatrix}\right\|$ *and (ii)* $\bigvee_a^t \begin{bmatrix} y^{(k)}(\cdot,\lambda) \\ z^{(k)}(\cdot,\lambda)\end{bmatrix}$ *are bounded independently of* k *and* $(t,\lambda)\in[a,b]\times K$.

Proof. (i) Using Lemma 3.2,

$$\left\|\begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda)\end{bmatrix}\right\| \leq \left\|\begin{bmatrix} y^{(0)}(t,\lambda) \\ z^{(0)}(t,\lambda)\end{bmatrix}\right\| + \sum_{i=1}^{k}\left\|\begin{bmatrix} y^{(i)}(t,\lambda) \\ z^{(i)}(t,\lambda)\end{bmatrix} - \begin{bmatrix} y^{(i-1)}(t,\lambda) \\ z^{(i-1)}(t,\lambda)\end{bmatrix}\right\|$$
$$\leq L + L\sum_{i=1}^{k}\frac{f(t,\lambda)^i}{i!}$$
$$\leq Le^{f(t,\lambda)},$$

Recall also that

$$f(t,\lambda)\leq \int_a^t \|A\| + \int_a^t \|D\| + \bigvee_a^t P + \bigvee_a^t Q + |\lambda|\bigvee_a^t W, \tag{3.6}$$

which is bounded independently of $(t,\lambda)\in[a,b]\times K$.

(ii) Let $T=\{t_i\}_0^m\in\mathcal{P}[a,t]$. Using Theorem 2.1 and letting C be a bound on

$$\left\|\begin{bmatrix} y^{(k)}(\cdot,\lambda) \\ z^{(k)}(\cdot,\lambda) \end{bmatrix}\right\| \text{ from part (i),}$$

$$\begin{aligned}
\left\|\bigvee_a^t \begin{bmatrix} y^{(k)}(\cdot,\lambda) \\ z^{(k)}(\cdot,\lambda) \end{bmatrix}\right\| &= \sup_{T\in P[a,t]} \sum_{i=1}^{m} \left\| \begin{bmatrix} y^{(k)}(t_i) \\ z^{(k)}(t_i) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t_{i-1}) \\ z^{(k)}(t_{i-1}) \end{bmatrix} \right\| \\
&= \sup_{T\in P[a,t]} \sum_{i=1}^{m} \left\| \int_{t_{i-1}}^{t_i} \begin{bmatrix} A(s)ds & dP(s) \\ dM(s,\lambda) & D(s)ds \end{bmatrix} \begin{bmatrix} y^{(k-1)}(s,\lambda) \\ z^{(k-1)}(s,\lambda) \end{bmatrix} \right\| \\
&\le \sup_{T\in P[a,t]} C \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} dv_F(s) \\
&= C\int_a^t dv_F(s) \le C\int_a^t df(s,\lambda) \\
&= Cf(t,\lambda),
\end{aligned}$$

since Δv_F is bounded by Δf. Independence follows from (3.6). □

Now we examine the continuity of solutions, starting by proving that each successive approximation is quasi-continuous in the first variable. This will be used in the following theorem to prove the solution is quasi-continuous.

Lemma 3.4. *As a function of the first variable, $y^{(k)}(t,\lambda)$ is continuous and $z^{(k)}(t,\lambda)$ is quasi-continuous on $[a,b]$, for $k = 0,1,2,\ldots$. If Q and W are continuous, then $z^{(k)}(t,\lambda)$ is continuous on $[a,b]$, for $k = 0,1,2,\ldots$.*

Proof. The quasi-continuity of $y^{(k)}(\cdot,\lambda)$ and $z^{(k)}(\cdot,\lambda)$ follows from the fact that they are of bounded variation (Lemma 3.3). Now we prove that $y^{(k)}(\cdot,\lambda)$ is actually continuous.

Fix $\lambda \in K$. By definition of successive approximations, for $k = 1,2,3,\ldots$

$$y^{(k)}(t,\lambda) = y(a,\lambda) + \int_a^t A(s)y^{(k-1)}(s,\lambda)ds + \int_a^t dP(s)z^{(k-1)}(s,\lambda).$$

Let $g(t) = \int_a^t A(s)y^{(k-1)}(s,\lambda)ds$, and $h(t) = \int_a^t dP(s)z^{(k-1)}(s,\lambda)$. To show that g is continuous, note that

$$\begin{aligned}\|g(t_2) - g(t_1)\| &= \left|\left|\int_{t_1}^{t_2} A(s)y^{(k-1)}(s,\lambda)ds\right|\right| \\ &\leq C\int_{t_1}^{t_2} ||A(s)||\, ds,\end{aligned}$$

where C is a uniform bound on $\left|\left|\begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda)\end{bmatrix}\right|\right|$. Since $A \in L_1([a,b])$, $I_{\|A\|}(t) := \int_a^t \|A(s)\|ds$ is absolutely continuous by Theorem 2.9. Then

$$I_{\|A\|}(t_2) - I_{\|A\|}(t_1) = \int_{t_1}^{t_2} \|A(s)\|ds$$

can be made arbitrarily small, and hence $\|g(t_2) - g(t_1)\|$ can also. Therefore g is continuous.

Now for h, we have

$$\begin{aligned}\|h(t_2) - h(t_1)\| &= \left|\left|\int_{t_1}^{t_2} dP(s)z^{(k-1)}(s,\lambda)\right|\right| \\ &\leq \int_{t_1}^{t_2} dv_P(s)\|z^{(k-1)}(s,\lambda)\| \\ &\leq C(v_P(t_2) - v_P(t_1)).\end{aligned}$$

Since P is continuous, v_P is also. Then $v_P(t_2) - v_P(t_1)$ can be made arbitrarily small and hence $\|h(t_2) - h(t_1)\|$ can also. Now both g and h are continuous, so $y^{(k)}(t,\lambda)$ is continuous as a function of t.

If Q and W are continuous, then $z^{(k)}(t,\lambda)$ can be shown to be continuous using a similar argument. $\square$

Theorem 3.5. *The initial value problem (3.2) has a unique solution in the space of quasi-continuous functions. This solution is bounded in norm and in total variation independently of t and λ on $[a, b] \times K$.*

Proof. We prove the existence of a solution by showing it is the limit of the sequence of successive approximations. By Lemma 3.2, we have for $p > k$,

$$\begin{aligned}\left\|\begin{bmatrix} y^{(p)}(t,\lambda) \\ z^{(p)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix}\right\| &\leq \sum_{i=k+1}^{p} \left\|\begin{bmatrix} y^{(i)}(t,\lambda) \\ z^{(i)}(t,\lambda) \end{bmatrix} - \begin{bmatrix} y^{(i-1)}(t,\lambda) \\ z^{(i-1)}(t,\lambda) \end{bmatrix}\right\| \\ &\leq L \sum_{i=k+1}^{p} \frac{f(t,\lambda)^i}{i!} \leq L \frac{f(t,\lambda)^{k+1}}{(k+1)!} e^{f(t,\lambda)} \\ &\leq L \frac{f(b,\lambda)^{k+1}}{(k+1)!} e^{f(b,\lambda)} \to 0 \text{ as } k \to \infty.\end{aligned}$$

Thus $\{[y^{(k)}, z^{(k)}]\}$ is uniformly Cauchy in $[a, b]$. Taking the limit as $k \to \infty$ in equation (3.5) and using the fact that the convergence of the successive approximations is uniform, we get that

$$\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix} := \lim_{k\to\infty} \begin{bmatrix} y^{(k)}(t,\lambda) \\ z^{(k)}(t,\lambda) \end{bmatrix}$$

is a solution. We know that each successive approximation is quasi-continuous in t and uniformly bounded in norm and in total variation by Lemma 3.3, so this solution has these same properties.

To prove uniqueness, assume that $\begin{bmatrix} y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} y_2 \\ z_2 \end{bmatrix}$ are both quasi-continuous solutions. For fixed λ, let

$$Y(t) := \begin{bmatrix} y_2(t,\lambda) \\ z_2(t,\lambda) \end{bmatrix} - \begin{bmatrix} y_1(t,\lambda) \\ z_1(t,\lambda) \end{bmatrix},$$

$$F(t,\lambda) := \begin{bmatrix} \int_a^t A(s)ds & P(t) \\ M(t,\lambda) & \int_a^t D(s)ds \end{bmatrix}$$

as before, so that

$$Y(t) = \int_a^t dF(s,\lambda)Y(s).$$

Then

$$\|Y(t)\| \leq (L)\int_a^t \|dF(s,\lambda)\|\|Y(s)\| \leq (L)\int_a^t dv_F(s,\lambda)\|Y(s)\|.$$

Then by Theorem 2.2,

$$\|Y(t)\| \leq 0e^{(v_F(t,\lambda)-v_F(a,\lambda))} = 0.$$

Hence $\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} y_2 \\ z_2 \end{bmatrix}$. □

The first part of the following corollary follows from the fact that the $y^{(k)}(t,\lambda)$ are continuous with respect to t and that they converge uniformly to $y(t,\lambda)$ as $k \to \infty$. The second part follows from Lemma 3.4 and the uniform convergence of the successive approximations.

Corollary 3.6. *1. $y(t,\lambda)$ is continuous with respect to t.*

2. If Q and W are continuous on $[a,b]$, then $z(t,\lambda)$ is continuous on $[a,b]$.

We conclude with some additional properties of solutions. First a Lipschitz condition in λ, and then a property concerning total variation of the solution.

Lemma 3.7. *For any λ_1 and λ_2 in K, we have the Lipschitz condition*

$$\left\| \begin{bmatrix} y(t,\lambda_2) \\ z(t,\lambda_2) \end{bmatrix} - \begin{bmatrix} y(t,\lambda_1) \\ z(t,\lambda_1) \end{bmatrix} \right\| \leq c|\lambda_2 - \lambda_1|,$$

where c is a constant independent of $t \in [a,b]$.

Proof. Let

$$\phi(t) := \left\| \begin{bmatrix} y(t,\lambda_2) \\ z(t,\lambda_2) \end{bmatrix} - \begin{bmatrix} y(t,\lambda_1) \\ z(t,\lambda_1) \end{bmatrix} \right\|$$
$$= \|y(t,\lambda_2) - y(t,\lambda_1)\| + \|z(t,\lambda_2) - z(t,\lambda_1)\|.$$

Using the equation $y(t,\lambda_i) = y(a,\lambda_i) + \int_a^t A(s)y(s,\lambda_i)ds + \int_a^t dP(s)z(s,\lambda_i)$ for $i = 1,2$ and the fact that $y(a,\lambda_1) = y(a,\lambda_2)$, we have

$$y(t,\lambda_2)-y(t,\lambda_1) = \int_a^t A(s)[y(s,\lambda_2)-y(s,\lambda_1)]ds+\int_a^t dP(s)[z(s,\lambda_2)-z(s,\lambda_1)].$$

Similarly, we have

$$z(t,\lambda_2) - z(t,\lambda_1) = \int_a^t dQ(s)[y(s,\lambda_2) - y(s,\lambda_1)]$$
$$- \int_a^t dW(s)[\lambda_2 y(s,\lambda_2) - \lambda_1 y(s,\lambda_1)] + \int_a^t D(s)[z(s,\lambda_2) - z(s,\lambda_1)]ds.$$

By adding and subtracting the term $\int_a^t dW(s)\lambda_1 y(s,\lambda_2)$, we get

$$z(t,\lambda_2)-z(t,\lambda_1) = \int_a^t dQ(s)[y(s,\lambda_2)-y(s,\lambda_1)]-\int_a^t dW(s)(\lambda_2-\lambda_1)y(s,\lambda_2)$$
$$- \int_a^t dW(s)\lambda_1[y(s,\lambda_2) - y(s,\lambda_1)] + \int_a^t D(s)[z(s,\lambda_2) - z(s,\lambda_1)]ds.$$

Let $C > 0$ be a bound on $\left\| \begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix} \right\|$, from Theorem 3.5. Then

$$\phi(t) \leq \int_a^t \|A(s)\|\|y(s,\lambda_2) - y(s,\lambda_1)\|ds + \int_a^t dv_P(s)\|z(s,\lambda_2) - z(s,\lambda_1)\|$$
$$+ \int_a^t dv_Q(s)\|y(s,\lambda_2) - y(s,\lambda_1)\| + |\lambda_2 - \lambda_1| \int_a^t dv_W(s)\|y(s,\lambda_2)\|$$
$$+ |\lambda_1| \int_a^t dv_W(s)\|y(s,\lambda_2) - y(s,\lambda_1)\| + \int_a^t \|D(s)\|\|z(s,\lambda_2) - z(s,\lambda_1)\|ds$$

$$\leq \int_a^t dv_A(s)\phi(s) + \int_a^t dv_P(s)\phi(s) + (L)\int_a^t dv_Q(s)\phi(s) + C|\lambda_2 - \lambda_1| \bigvee_a^b W$$
$$+ |\lambda_1|(L)\int_a^t dv_W(s)\phi(s) + \int_a^t dv_D(s)\phi(s)$$

By Theorem 2.2, it follows that

$$\phi(t) \leq \left(C|\lambda_2 - \lambda_1| \bigvee_a^b W \right) e^{m(t)},$$

where

$$m(t) = v_A(t) + v_P(t) + v_Q(t) + |\lambda_1| v_W(t) + v_D(t).$$

Then

$$\phi(t) \leq c|\lambda_2 - \lambda_1|,$$

where

$$c = C \bigvee_a^b W e^{m(b)}.$$

□

Theorem 3.8. *There is a nondecreasing function q on $[a, b]$ such that $\bigvee_{x_1}^{x_2} y(x, \lambda) \leq q(x_2) - q(x_1)$ and $\bigvee_{x_1}^{x_2} z(x, \lambda) \leq q(x_2) - q(x_1)$ for $x_1 < x_2$ and all $\lambda \in K$.*

Proof. Fix $\lambda \in K$. For any $T = \{t_i\}_0^m \in \mathcal{P}[x_1, x_2]$,

$$\begin{aligned}
\bigvee_{x_1}^{x_2} y(\cdot, \lambda) &= \sup_{T \in \mathcal{P}[x_1, x_2]} \sum_{i=1}^m \|y(t_i, \lambda) - y(t_{i-1}, \lambda)\| \\
&= \sup_{T \in \mathcal{P}[x_1, x_2]} \sum_{i=1}^m \left\| \int_{t_{i-1}}^{t_i} A(s)y(s, \lambda)ds + \int_{t_{i-1}}^{t_i} dP(s)z(s, \lambda) \right\| \\
&\leq \sup_{T \in \mathcal{P}[x_1, x_2]} \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \|A(s)y(s, \lambda)ds\| + \int_{t_{i-1}}^{t_i} \|dP(s)z(s, \lambda)\| \\
&= \int_{x_1}^{x_2} \|A(s)y(s, \lambda)ds\| + \int_{x_1}^{x_2} \|dP(s)z(s, \lambda)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{x_1}^{x_2}\|A(s)\|ds + C\bigvee_{x_1}^{x_2} P \\
&= q_1(x_2) - q_1(x_1),
\end{aligned}
$$

where C is a bound on $\left\|\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda)\end{bmatrix}\right\|$ and

$$
q_1(x) = C\int_a^x \|A(s)\|ds + C\bigvee_a^x P.
$$

Note that q_1 is continuous since P is continuous [18, Theorem 6.26] and since $A \in L_1([a,b])$ (see Theorem 2.9). Now

$$
\begin{aligned}
\bigvee_{x_1}^{x_2} z(x,\lambda) &= \sup_{T\in\mathcal{P}[x_1,x_2]} \sum_{i=1}^{m} \|z(t_i,\lambda) - z(t_{i-1},\lambda)\| \\
&= \sup_{T\in\mathcal{P}[x_1,x_2]} \sum_{i=1}^{m} \left\| \int_{t_{i-1}}^{t_i} (dQ(s) - \lambda dW(s))y(s,\lambda) + \int_{t_{i-1}}^{t_i} D(s)z(s,\lambda)ds \right\| \\
&\leq \sup_{T\in\mathcal{P}[x_1,x_2]} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \|(dQ(s) - \lambda dW(s))y(s,\lambda)\| + \int_{t_{i-1}}^{t_i} \|D(s)z(s,\lambda)ds\| \\
&\leq C\int_{x_1}^{x_2} \|dQ(s)\| + C(\max_{\lambda\in K}|\lambda|)\int_{x_1}^{x_2}\|dW(s)\| + C\int_{x_1}^{x_2}\|D(s)\|ds \\
&= q_2(x_2) - q_2(x_1),
\end{aligned}
$$

where

$$
q_2(x) = C\bigvee_a^x Q + C(\max_{\lambda\in K}\|\lambda\|)\bigvee_a^x W + C\int_a^x \|D(s)\|ds
$$

Note that q_2 is nondecreasing but may have discontinuities.

Then $q(x) := q_1(x) + q_2(x)$ satisfies the conclusion of the theorem. □

Chapter 4

A Sequence of Initial Value Problems

In chapter 1, we examined the continuity of a solution to the initial value problem as a function of t and of λ. Now we study continuity as a function of the problem coefficients. To accomplish this, we define a sequence of initial value problems satisfying the same conditions as the problem in the previous chapter (therefore the results from chapter one apply to each problem in the sequence). We will show that the sequence of solutions converges to the solution of the limit problem.

The sequence of problems is defined as follows, for $n = 1, 2, 3, \ldots$:

$$\begin{cases} dy_n = A_n y_n dt + dP_n z_n \\ dz_n = (dQ_n - \lambda dW_n) y_n + D_n z_n dt \end{cases} \tag{4.1}$$

on $[a, b] \times K$, which written as a Stieltjes integral equation has the form

$$\begin{bmatrix} y_n(t,\lambda) \\ z_n(t,\lambda) \end{bmatrix} = \begin{bmatrix} y_n(a,\lambda) \\ z_n(a,\lambda) \end{bmatrix} + \int_a^t \begin{bmatrix} A_n(s)ds & dP_n(s) \\ dM_n(s,\lambda) & D_n(s)ds \end{bmatrix} \begin{bmatrix} y_n(s,\lambda) \\ z_n(s,\lambda) \end{bmatrix}$$

where A_n, P_n, $M_n = Q_n - \lambda W_n$, and D_n satisfy the same conditions as A, P, M, and D, respectively, as stated in (3.3). We also require that Q_n and W_n be uniformly of bounded variation, stated below along with the types of convergence that are required:

$$\begin{cases} A_n \to A \text{ weakly in } L_1([a,b]); \\ D_n \to D \text{ weakly in } L_1([a,b]); \\ P_n \to P \text{ uniformly in } [a,b]; \\ Q_n \to Q \text{ pointwise in } [a,b]; \\ W_n \to W \text{ pointwise in } [a,b]; \\ y_n(a,\lambda) \to y(a,\lambda); \\ z_n(a,\lambda) \to z(a,\lambda); \\ \bigvee_a^b Q_n \leq \tilde{Q} \text{ for all } n; \\ \bigvee_a^b W_n \leq \tilde{W} \text{ for all } n, \end{cases} \tag{4.2}$$

for some constants $\tilde{Q}$ and $\tilde{W}$. The modes of convergence required here are more inclusive than in previous work [16], which is appropriate due to the increased generality in the classes of coefficients.

For the remainder of this section we fix λ and write $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}$, understanding that there is still a dependence on λ.

We define the following terms for use in the proof:

$$L = \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\|;$$

$$L_n = \left\| \begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix} \right\|;$$

$$f(t) = \int_a^t \|A(s)\| ds + \int_a^t \|D(s)\| ds + \bigvee_a^t P + \bigvee_a^t M;$$

$$f_n(t) = \int_a^t \|A_n(s)\| ds + \int_a^t \|D_n(s)\| ds + \bigvee_a^t P_n + \bigvee_a^t M_n.$$

Recall that in Lemma 3.3 we proved the existence of a uniform bound on all successive approximations for one initial value problem. Now we extend that uniform bound to all successive approximations for all of the initial value problems in the sequence.

Lemma 4.1. *Let* $\begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix}$, $k = 1, 2, 3, \ldots$ *be the* kth *successive approximation for the* nth *problem in the sequence (4.1) for* $n = 1, 2, 3, \ldots,$ *defined in the same manner as for problem (3.1). Then under assumptions (4.2),* $\left\| \begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix} \right\|$ *and* $\bigvee_a^b \begin{bmatrix} y_n^{(k)} \\ z_n^{(k)} \end{bmatrix}$ *are bounded independently of* n, k, *and* t.

Proof. By the proof of Lemma 3.3, we have

$$\left\| \begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix} \right\| \leq L_n e^{f_n(b)};$$

$$\bigvee_a^t \begin{bmatrix} y_n^{(k)} \\ z_n^{(k)} \end{bmatrix} \leq L_n e^{f_n(b)} f_n(b).$$

Therefore the proof will be complete when we find a uniform bound on $\{L_n\}_1^\infty$ and $\{f_n(b)\}_1^\infty$. The sequence $\{L_n\}_1^\infty$ is bounded because of the assumption that $L_n \to L$.

For $f_n(b) = \int_a^b \|A_n(s)\| ds + \int_a^b \|D_n(s)\| ds + \bigvee_a^b P_n + \bigvee_a^b M_n$, first observe that the weak convergence of $\{A_n\}_1^\infty$ and $\{D_n\}_1^\infty$ guarantees a uniform bound on $\int_a^b \|A_n(s)\| ds$ and $\int_a^b \|D_n(s)\| ds$ for all n [16, p.430]. If $P(t) = \{P_{ij}\}$, then for $T = \{t_i\}_1^m \in \mathcal{P}[a,b]$,

$$\begin{aligned}\bigvee_a^b P &= \sup_{T\in\mathcal{P}[a,b]} \sum_{k=1}^m \|P(t_k) - P(t_{k-1})\| \\ &= \sup_{T\in\mathcal{P}[a,b]} \sum_{i,j=1}^n \sum_{k=1}^m |P_{ij}(t_k) - P_{ij}(t_{k-1})|.\end{aligned}$$

Since P is symmetric and nondecreasing, together with the fact that $P(t) \geq$

$P(a) = 0$, any component of P is bounded by the square root of the product of the corresponding diagonal entries (which are nonnegative), i.e., $|P_{ij}| \leq \sqrt{P_{ii}P_{jj}}$. Then using the fact that $\sqrt{ab} \leq \frac{1}{2}(a+b)$ for any positive numbers a and b, we have $|P_{ij}| \leq \frac{1}{2}(P_{ii} + P_{jj})$. Then continuing from above, we have

$$\begin{aligned}\bigvee_a^b P &\leq \frac{1}{2} \sum_{i,j=1}^n \sup_{T\in\mathcal{P}[a,b]} \sum_{k=1}^m |P_{ii}(t_k) - P_{ii}(t_{k-1}) + P_{jj}(t_k) - P_{jj}(t_{k-1})| \\ &= \frac{1}{2} \sum_{i,j=1}^n (\bigvee_a^b P_{ii} + \bigvee_a^b P_{jj}).\end{aligned}$$

Now define $\hat{P}(t) := \sum_{i=1}^n P_{ii}(t)$. Since the diagonal entries of P are nondecreasing and have initial value zero, we have $\sum_{i=1}^n \bigvee_a^b P_{ii} = \bigvee_a^b \hat{P}$. Therefore,

$$\begin{aligned}\bigvee_a^b P &\leq \frac{1}{2} \sum_{i,j=1}^n \left(\bigvee_a^b P_{ii} + \bigvee_a^b P_{jj} \right) \\ &= \frac{1}{2} \left(n \bigvee_a^b \hat{P} + n \bigvee_a^b \hat{P} \right)\end{aligned}$$

$$= n \bigvee_a^b \hat{P} = n\hat{P}(b),$$

with the last equality holding because $\hat{P}$ is continuous and nondecreasing with $\hat{P}(a) = 0$. Similarly,

$$\bigvee_a^b P_j \leq n\hat{P}_j(b).$$

Since $\hat{P}_j(b) \to \hat{P}(b)$, there exists a bound on $\{\hat{P}_j(b)\}_1^\infty$. Hence there exists a bound on $\bigvee_a^b P_j$ for all j.

Recall we have a uniform bound on $\bigvee_a^b M_n$ for all n by assumption. Therefore, $\{f_n(b)\}_{n=1}^\infty$ is bounded and the proof is completed. □

In preparation for showing the convergence of the sequence of solutions, we now show the same is true for each set of successive approximations.

Lemma 4.2. *Under the assumptions in (4.2), for each* $k = 0, 1, 2, ...$

1. $y_n^{(k)}(t) \to y^{(k)}(t)$ *uniformly on* $[a, b]$ *as* $n \to \infty$, *and*

2. $z_n^{(k)}(t) \to z^{(k)}(t)$ *pointwise on* $[a, b]$ *as* $n \to \infty$.

Proof. The proof is by induction on k. First, note that

$$\|y^{(0)}(t) - y_n^{(0)}(t)\| = \|y(a) - y_n(a)\| \to 0$$

and

$$\|z^{(0)}(t) - z_n^{(0)}(t)\| = \|z(a) - z_n(a)\| \to 0$$

by assumption. Assume the lemma to be true for some k, and now consider y and z separately to show the lemma is true for $k + 1$.

1. By definition of successive approximations,

$$y^{(k+1)}(t) - y_n^{(k+1)}(t) = y(a) + \int_a^t [A(s)y^{(k)}(s)ds + dP(s)z^{(k)}(s)]$$
$$- y_n(a) - \int_a^t [A_n(s)y_n^{(k)}(s)ds + dP_n(s)z_n^{(k)}(s)].$$

Then

$$y^{(k+1)}(t) - y_n^{(k+1)}(t) = [y(a) - y_n(a)] + \int_a^t [dP(s) - dP_n(s)]z^{(k)}(s)$$
$$- \int_a^t dP_n(s)[z_n^{(k)}(s) - z^{(k)}(s)] + \int_a^t [A(s) - A_n(s)]y^{(k)}(s)ds$$
$$- \int_a^t A_n(s)[y_n^{(k)}(s) - y^{(k)}(s)]ds. \quad (4.3)$$

We will show that each of these terms converges to zero uniformly in $[a, b]$.

First, $\|y(a) - y_n(a)\| \to 0$ by assumption.

For the second term in (4.3), we use integration by parts and the fact that $P(a) = P_n(a) = 0$ to write

$$\int_a^t [dP(s) - dP_n(s)]z^{(k)}(s) =$$
$$- \int_a^t [P(s) - P_n(s)]dz^{(k)}(s) + [P(t) - P_n(t)]z^{(k)}(t). \quad (4.4)$$

Since P and P_n are continuous and therefore bounded on $[a, b]$,

$$\left\| \int_a^t [dP(s) - dP_n(s)]z^{(k)}(s) \right\| \leq$$
$$\left(\max_{t \in [a,b]} \|P(t) - P_n(t)\| \right) \bigvee_a^b z^{(k)} + \|P(t) - P_n(t)\| \|z^{(k)}(t)\|.$$

This converges to zero as $n \to \infty$ since $\bigvee_a^b z^{(k)}$ and $\|z^{(k)}(t)\|$ are bounded (by Lemma 3.3) and since $P_n \to P$ uniformly by assumption.

Rewrite the third term in (4.3) as

$$\int_a^t dP_n(s)[z_n^{(k)}(s) - z^{(k)}(s)] = \int_a^t [dP_n(s) - dP(s)][z_n^{(k)}(s) - z^{(k)}(s)] + \int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]. \quad (4.5)$$

Now treat the first of the two integrals on the right-hand side of (4.5) the same way as in (4.4). Performing integration by parts and taking the norm, we get

$$\left\|\int_a^t [dP_n(s) - dP(s)][z_n^{(k)}(s) - z^{(k)}(s)]\right\| \leq \left(\max_{t\in[a,b]} \|P_n(t) - P(t)\|\right) \bigvee_a^b [z_n^{(k)} - z^{(k)}] + \|P_n(t) - P(t)\| \|z_n^{(k)}(t) - z^{(k)}(t)\|,$$

which converges uniformly to zero for the same reasons as in (4.4).

The pointwise convergence of the second integral on the right-hand side of (4.5) follows from Theorem 2.7. This theorem applies since P is of bounded variation, since $[z_n^{(k)}(s) - z^{(k)}(s)]$ is uniformly bounded, and since $z_n^{(k)}(t) \to z^{(k)}(t)$ pointwise by induction hypothesis. Now we use the Ascoli-Arzela Theorem to show the convergence is actually uniform and not only pointwise. The sequence $\{\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]\}_{n=1}^{\infty}$ is uniformly bounded on $[a, b]$ since

$$\left\|\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]\right\| \leq \left(\sup_{t\in[a,b]} \|z_n^{(k)}(t) - z^{(k)}(t)\|\right) \bigvee_a^b P,$$

which is finite since $z^{(k)}$ and $z_n^{(k)}$ are quasi-continuous and by Lemma 4.1 are uniformly bounded. To show the sequence is equicontinuous, fix n and k and let $g(t) = \int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]$. Now

$$\|g(t_2) - g(t_1)\| = \left\| \int_{t_1}^{t_2} dP(s)[z_n^{(k)}(s) - z^{(k)}(s)] \right\|$$
$$\leq \left(\sup_{t\in[a,b]} \|z_n^{(k)}(t) - z^{(k)}(t)\| \right) \bigvee_{t_1}^{t_2} P$$
$$= \left(\sup_{t\in[a,b]} \|z_n^{(k)}(t) - z^{(k)}(t)\| \right) (v_P(t_2) - v_P(t_1))$$

Now P being continuous implies that v_P is continuous, and therefore g is also continuous by this last inequality. Since $\|z_n^{(k)}(t) - z^{(k)}(t)\|$ is bounded independently of $n, k,$ and t by Lemma 4.1, the sequence is equicontinuous on $[a, b]$. Then the Ascoli-Arzela Theorem says that the sequence has a uniformly convergent subsequence. Since the sequence converges pointwise and has a uniformly convergent subsequence, the sequence itself converges uniformly.

We rewrite the fourth term on the right-hand side of (4.3) using the the function $I_A(t) = \int_a^t A(s)ds$. Then exactly as done in (4.4), integrate by parts and take the norm to get

$$\left\| \int_a^t [A(s) - A_n(s)]y^{(k)}(s)ds \right\| = \left\| \int_a^t [dI_A(s) - dI_{A_n}(s)]y^{(k)}(s)ds \right\|$$
$$\leq \left(\sup_{t\in[a,b]} \|I_A(s) - I_{A_n}(s)\| \right) \bigvee_a^b y^{(k)} + \|I_A(t) - I_{A_n}(t)\| \|y^{(k)}(t)\| \quad (4.6)$$

which converges uniformly to zero since $\bigvee_a^b y^{(k)}$ and $\|y^{(k)}(t)\|$ are bounded, and since $A_n \to A$ weakly in $L_1([a, b])$ implies that $I_{A_n} \to I_A$ uniformly [16, p.430].

For the last term in (4.3), we have

$$\left\| \int_a^t A_n(s)[y_n^{(k)}(s) - y^{(k)}(s)]ds \right\| \leq \left(\sup_{t\in[a,b]} \|y_n^{(k)}(t) - y^{(k)}(t)\| \right) \int_a^t \|A_n(s)\|ds,$$

Now $\int_a^t \|A_n(s)\|ds$ is uniformly bounded since $A_n \to A$ weakly in $L_1([a,b])$ [16, p.430], and $y_n^{(k)}(t) \to y^{(k)}(t)$ uniformly by induction hypothesis. Therefore the term on the right-hand side of (4.6) converges uniformly to zero.

We have shown that for $y^{(k+1)}(t) - y_n^{(k+1)}(t)$ written as the sum of five terms, each term converges to zero uniformly on $[a,b]$, completing the inductive step.

2. Similarly to part 1, we write

$$z^{(k+1)}(t) - z_n^{(k+1)}(t) = z(a) + \int_a^t [dM(s)y^{(k)}(s) + D(s)z^{(k)}(s)ds] \\ - z_n(a) - \int_a^t [dM_n(s)y_n^{(k)}(s) + D_n(s)z_n^{(k)}(s)ds].$$

Then

$$z^{(k+1)}(t) - z_n^{(k+1)}(t) = [z(a) - z_n(a)] + \int_a^t [dM(s) - dM_n(s)]y^{(k)}(s) \\ - \int_a^t dM_n(s)[y_n^{(k)}(s) - y^{(k)}(s)] + \int_a^t [D(s) - D_n(s)]z^{(k)}(s)ds \\ - \int_a^t D_n(s)[z_n^{(k)}(s) - z^{(k)}(s)]ds. \quad (4.7)$$

We will show that each of these terms converges to zero pointwise in $[a,b]$. First, $\|z(a) - z_n(a)\| \to 0$ by assumption.

The pointwise convergence to zero of the second term on the right-hand side of (4.7), $\int_a^t [dM(s) - dM_n(s)]y^{(k)}(s)$, follows from Theorem 2.6, noting that $y^{(k)}(t)$ is continuous, the $\{M_n\}_1^\infty$ are of uniform bounded variation, and $M_n \to M$ pointwise in $[a,b]$.

For the third term on the right-hand side of (4.7), we have

$$\left\|\int_a^t dM_n(s)[y_n^{(k)}(s) - y^{(k)}(s)]\right\| \leq \left(\sup_{t\in[a,b]} \|y_n^{(k)}(t) - y^{(k)}(t)\|\right)\left(\sup_n \bigvee_a^b M_n\right),$$

which converges uniformly to zero since $\bigvee_a^b M_n$ is bounded independently of n and since $y_n^{(k)}(t) \to y^{(k)}(t)$ uniformly by inductive hypothesis.

We rewrite the fourth term on the right-hand side of (4.7) using the function $I_D(t) = \int_a^t D(s)ds$ and the same method of integrating by parts, getting

$$\left|\left|\int_a^t [D(s) - D_n(s)]z^{(k)}(s)ds\right|\right| = \left|\left|\int_a^t [dI_D(s) - dI_{D_n}(s)]z^{(k)}(s)\right|\right|$$
$$\leq \left(\sup_{t\in[a,b]} \|I_D(s) - I_{D_n}(s)\|\right) \bigvee_a^b z^{(k)} + \|I_D(t) - I_{D_n}(t)\|\|z^{(k)}(t)\|.$$

The right-hand side converges uniformly to zero since $\bigvee_a^b z^{(k)}$ and $\|z^{(k)}(t)\|$ are bounded and since $D_n \to D$ weakly in $L_1([a,b])$ implies that $I_{D_n} \to I_D$ uniformly.

We write the last term in (4.7) as

$$\int_a^t D_n(s)[z_n^{(k)}(s) - z^{(k)}(s)]ds = \int_a^t [dI_{D_n}(s) - dI_D(s)][z_n^{(k)}(s) - z^{(k)}(s)]$$
$$+ \int_a^t dI_D(s)[z_n^{(k)}(s) - z^{(k)}(s)].$$

The first of these two integrals can be shown to converge uniformly to zero using integration by parts as seen before. Theorem 2.7 can be used to show pointwise convergence of the second integral. This theorem applies because $D \in L_1([a,b])$ implies that I_D is of bounded variation, and because $z_n^{(k)}(t) \to z^{(k)}(t)$ pointwise by induction hypothesis. The convergence can be shown to be uniform using the Ascoli-Arzela Theorem as in part 1, i.e., as in the last term of (4.5).

Having shown that $z^{(k+1)}(t) - z_n^{(k+1)}(t)$ can be written as the sum of five terms, one converging pointwise and four converging uniformly in $[a,b]$, the inductive step is complete.

□

Corollary 4.3. *The convergence* $\|z^{(k)}(t) - z_n^{(k)}(t)\| \to 0$ *as* $n \to \infty$ *is uniform on* $[a, b]$ *if* $M_n \to M$ *as* $n \to \infty$ *uniformly on* $[a, b]$*, for* $k = 1, 2, \ldots$ *.*

Proof. In the proof of Lemma 4.2 (2), $z^{(k+1)}(t) - z_n^{(k+1)}(t)$ was written as the sum of five terms, all of which converge uniformly to zero except for the term $\int_a^t [dM(s) - dM_n(s)]y^{(k)}(s)$, which converges pointwise. Under the assumption that $M_n \to M$ uniformly, we now show that this term converges uniformly. Using integration by parts in the same manner as used in the proof of the previous Lemma,

$$\left\|\int_a^t [dM(s) - dM_n(s)]y^{(k)}(s)\right\| \leq \left(\sup_{t\in[a,b]} \|M(t) - M_n(t)\|\right) \bigvee_a^b y^{(k)} + \|M(t) - M_n(t)\| \|y^{(k)(t)}\|,$$

which converges uniformly to zero since $\bigvee_a^b y^{(k)}$ and $\|y^{(k)}\|$ are bounded and since $M_n \to M$ uniformly. □

Theorem 4.4. *Let* $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ *be the solution of (3.1) and* $\begin{bmatrix} y_n(t) \\ z_n(t) \end{bmatrix}$ *be the solution of (4.1), and assume (4.2) holds. Then as* $n \to \infty$*,*

1. $y_n(t) \to y(t)$ *uniformly in* $[a, b]$*, and*

2. $z_n(t) \to z(t)$ *pointwise in* $[a, b]$*.*

Proof. 1. We write

$$\|y(t) - y_n(t)\| \leq \|y(t) - y^{(k)}(t)\| + \|y^{(k)}(t) - y_n^{(k)}(t)\| + \|y_n^{(k)}(t) - y_n(t)\|.$$

As in the proof of Thoerem 3.5, using Lemma 3.2, we have

$$\left\| \begin{bmatrix} y_n^{(k)}(t) \\ z_n^{(k)}(t) \end{bmatrix} - \begin{bmatrix} y_n(t) \\ z_n(t) \end{bmatrix} \right\| \leq L_n \sum_{i=k+1}^{\infty} \frac{f_n(b)^i}{i!},$$

which tends to zero as $k \to \infty$. Since the L_n and $f_n(b)$ are bounded independent of n, this convergence is independent of n, and it is clearly independent of t and k as well. So given any $\epsilon > 0$, there exists a $k_0 > 0$ such that if $k \geq k_0$, then

$$\|y_n^{(k)}(t) - y_n(t)\| < \frac{\epsilon}{3}$$

for all $n = 1, 2, 3, ...$ and $t \in [a, b]$.

Also, by the convergence of successive approximations to the solution as shown in the proof of Theorem 3.5, we can assume for the same k_0 that $k \geq k_0$ implies

$$\|y(t) - y^{(k)}(t)\| < \frac{\epsilon}{3}$$

for all $n = 1, 2, 3, \ldots$ and $t \in [a, b]$. By Lemma 4.2, we can assume that for the same k_0 that $k \geq k_0$ implies

$$\|y_n^{(k)}(t) - y^{(k)}(t)\| < \frac{\epsilon}{3}.$$

Therefore, for $k \geq k_0$,

$$\|y(t) - y_n(t)\| < \epsilon$$

for all $n = 1, 2, 3, \ldots$ and $t \in [a, b]$.

2. Write

$$\|z(t) - z_n(t)\| \leq \|z(t) - z^{(k)}(t)\| + \|z^{(k)}(t) - z_n^{(k)}(t)\| + \|z_n^{(k)}(t) - z_n(t)\|,$$

and in a similar manner obtain

$$\|z(t) - z_n(t)\| < \epsilon,$$

but this time the convergence of the middle term is pointwise rather than uniform (see Lemma 4.2).

□

The next corollary follows directly from Corollary 4.3 and Theorem 4.2 (2).

Corollary 4.5. *The convergence* $z_n(t) \to z(t)$ *is uniform if* $M_n \to M$ *is uniform.*

Chapter 5

The Eigenvalue Problem

In this section, we add boundary conditions to the problem (3.1) and examine the eigenvalue problem:

$$\begin{cases} dy(t,\lambda) = A(t)y(t,\lambda)dt + dP(t)z(t,\lambda) \\ dz(t,\lambda) = (dQ(t) - \lambda dW(t))y(t,\lambda) + D(t)z(t,\lambda)dt \end{cases} \tag{5.1}$$

on $[a,b] \times K$, assuming hypotheses (3.3), with boundary conditions

$$\Gamma \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} + \Omega \begin{bmatrix} y(b,\lambda) \\ z(b,\lambda) \end{bmatrix} = 0. \tag{5.2}$$

Here we assume the Γ and Ω are $2n \times 2n$ constant matrices such that $[\Gamma|\Omega]$ has full rank $2n$. For the self-adjoint problem, we require that

$$\Gamma E \Gamma^* = \Omega E \Omega^*, \tag{5.3}$$

where

$$E = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

and I_n is the $n \times n$ identity matrix. This condition will be discussed further later, but self-adjointness is not assumed for this chapter. Even when self-adjointness is assumed, the number of eigenvalues λ of (5.1)-(5.2) may actually be finite since difference equations are included in this formulation.

Let $\Phi(t,\lambda)$ be the $2n \times 2n$ matrix whose columns are the solutions of (5.1) such that $\Phi(a,\lambda) = I_{2n}$. The existence of such a Φ is guaranteed by Theorem 3.5. Then the solution of (5.1), when initial conditions are given, is $\Phi(t,\lambda)\begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix}$.

Now we will see that the eigenvalues of this problem are the zeros of an entire function. This fact will be used later in proving the convergence of a sequence of eigenvalues to the eigenvalue of the limit problem. Define the function $d : \mathbb{C} \to \mathbb{R}$ by

$$d(b,\lambda) = \det\left[\Gamma + \Omega\Phi(b,\lambda)\right].$$

Lemma 5.1. *For fixed* b, *the eigenvalues of (5.1), (5.2) are the roots* λ *of* $d(b,\lambda)$.

Proof. The eigenvalue problem has a nontrivial solution $\phi(t,\lambda) = \begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}$ if and only if $\phi(t) = \Phi(t,\lambda)\phi(a)$, where $\phi(a) \neq 0$, and $\Gamma\phi(a) + \Omega\phi(b) = 0$. But this holds if and only if $\left[\Gamma + \Omega\Phi(b,\lambda)\right]\phi(a) = 0$. Since $\phi(a) \neq 0$, this equation has a solution λ if and only if $\det\left[\Gamma + \Omega\Phi(b,\lambda)\right] = 0$. □

Lemma 5.2. *For fixed* b, $d(b,\lambda)$ *is an entire function in* λ.

Proof. Let K be a compact subset of $\mathbb{C}$ and fix λ in K. Define successive approximations to $\Phi(t,\lambda)$ as follows:

$$\Phi^{(0)}(t,\lambda) = I,$$

$$\Phi^{(k+1)}(t,\lambda) = I + \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dQ(s) - \lambda dW(s) & D(s)ds \end{bmatrix} \Phi^{(k)}(s,\lambda).$$

Now $\{\Phi^{(k)}(t,\lambda)\}_{k=0}^{\infty}$ is uniformly Cauchy in $[a,b] \times K$ and converges uniformly to

$$\Phi(t,\lambda) = I + \int_a^t \begin{bmatrix} A(s)ds & dP(s) \\ dQ(s) - \lambda dW(s) & D(s)ds \end{bmatrix} \Phi(s,\lambda),$$

as shown in the proof of Theorem 3.5. In particular, $\Phi^{(k)}(b,\lambda) \to \Phi(b,\lambda)$ as $k \to \infty$, and hence

$$\det\left[\Gamma + \Omega\Phi^{(k)}(b,\lambda)\right] \to \det\left[\Gamma + \Omega\Phi(b,\lambda)\right] = d(b,\lambda)$$

as $k \to \infty$ uniformly for $\lambda \in K$.

A proof by induction shows that each component of $\Phi^{(k)}(b,\lambda)$ is a polynomial in λ. Hence, $\det\left[\Gamma + \Omega\Phi^{(k)}(b,\lambda)\right]$ is a polynomial in λ. Since λ is in a compact set K and since $d(b,\lambda)$ is the uniform limit of polynomials in λ, $d(b,\lambda)$ is analytic on the interior of K and is therefore entire since K is arbitrary [2, Thm 2.1]. $\square$

For the initial value problem, we took sequences of coefficients to examine the dependence of solutions on the coefficients. Now we take sequences of coefficients and also sequences of boundary data to examine the dependence of solutions and the dependence of eigenvalues on these parameters. We assume the sequences $\{A_n\}$, $\{D_n\}$, $\{P_n\}$, $\{Q_n\}$, and $\{W_n\}$ satisfy the same conditions as A, D, P, Q, and W, respectively, as stated in (3.3). We also require the same hypotheses as stated in (4.2).

The sequence of eigenvalue problems is given by

$$\begin{cases} dy_n(t,\lambda) = A_n(t)y_n(t,\lambda)dt + dP_n(t)z_n(t,\lambda) \\ dz_n(t,\lambda) = (dQ_n(t) - \lambda dW_n(t))y_n(t,\lambda) + D_n(t)z_n(t,\lambda)dt \end{cases} \tag{5.4}$$

on $[a, b] \times K$, with boundary conditions

$$\Gamma_n \begin{bmatrix} y_n(a,\lambda) \\ z_n(a,\lambda) \end{bmatrix} + \Omega_n \begin{bmatrix} y_n(b,\lambda) \\ z_n(b,\lambda) \end{bmatrix} = 0, \tag{5.5}$$

for $n = 0, 1, 2, \ldots$. Here, $\Gamma_n \to \Gamma$, $\Omega_n \to \Omega$, and Γ_n, Ω_n satisfy the same conditions as Γ, Ω, respectively. Also, each coefficient in this problem satisfies the same conditions as the corresponding coefficient in (3.3), and each sequence of coefficients satisfies the conditions given in (4.2).

Let $\Phi_n(t, \lambda)$ be the $2n \times 2n$ matrix whose columns satisfy (5.4) such that $\Phi_n(a, \lambda) = I$, and let

$$d_n(b, \lambda) = \det\left[\Gamma_n + \Omega_n \Phi_n(b, \lambda)\right].$$

By Lemma 5.1, the roots λ of $d_n(b, \lambda) = 0$ are the eigenvalues of the nth problem in the sequence.

Lemma 5.3. *Assume (4.2) holds and that* $\Gamma_n \to \Gamma$, $\Omega_n \to \Omega$ *as* $n \to \infty$*. Then for fixed* b, $d_n(b, \lambda) \to d(b, \lambda)$ *uniformly on compact subsets of* $\mathbb{C}$ *as* $n \to \infty$.

Proof. Let K be a compact subset of $\mathbb{C}$ and fix $\lambda \in K$. We know that $\Phi_n(t, \lambda) \to \Phi(t, \lambda)$ pointwise as shown in the proof of Theorem 4.4, and we know that $\Gamma_n \to \Gamma$ and $\Omega_n \to \Omega$ by assumption. Therefore, $d_n(b, \lambda) \to d(b, \lambda)$ pointwise. To show this convergence is uniform, we appeal to Montel's Theorem [2, Thm 2.9].

First we verify that $\{d_n(b, \lambda)\}_{n=0}^{\infty}$ is a uniformly bounded sequence. Let

$$c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \text{ or } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Using the bound in the proof of Lemma 3.3 and the fact that $\Phi_n(t,\lambda)$ has columns satisfying (5.4),

$$\|\Phi_n(t,\lambda)\cdot c\| \leq \|\Phi_n(a,\lambda)\|\cdot ce^{f_n(b)}.$$

Recall that

$$f_n(t) = \int_a^t \|A_n(s)\|ds + \int_a^t \|D_n(s)\|ds + \bigvee_a^t P_n + \bigvee_a^t M_n$$

was shown to be bounded independently of n and $\lambda \in K$ in the proof of Lemma 4.1. Then since $\Phi_n(a,\lambda) = I$, $\{d_n(b,\lambda)\}_{n=0}^{\infty}$ is uniformly bounded.

We also know that each $d_n(b,\lambda)$ is analytic by Lemma 5.2. Then by Montel's Theorem, there exists a subsequence that converges uniformly on K. Since $\{d_n(b,\lambda)\}_{n=0}^{\infty}$ converges pointwise and every subsequence of $\{d_n(b,\lambda)\}_{n=0}^{\infty}$ has a uniformly convergent subsequence on K, the sequence itself must be uniformly convergent on K. □

The continuity of $d(b,\lambda)$ and $d_n(b,\lambda)$ is discussed in the following lemma.

Lemma 5.4. *For fixed λ, $d(b,\lambda)$ ($d_n(b,\lambda)$) is quasi-continuous in b. If Q and W (Q_n and W_n), are continuous, then $d(b,\lambda)$ ($d_n(b,\lambda)$) is continuous in b and λ.*

Proof. We give the proof for d. The proof for d_n is similar. By definition of $d(b,\lambda)$, it has the same type of continuity as $\Phi(b,\lambda)$. For fixed λ, $\Phi(b,\lambda)$ is quasi-continuous in b by Theorem 3.5, and hence $d(b,\lambda)$ is also.

By Corollary 3.6, $y(b,\lambda)$ is actually continuous in b. If we assume that Q and W are continuous, we can show that $z(b,\lambda)$ is continuous in b. This follows by first observing that the successive approximations $z^{(k)}(b,\lambda)$ are continuous in b, using a parallel argument as the proof of the continuity of $y^{(k)}(t,\lambda)$ in Lemma

3.4. The proof for $y^{(k)}(t,\lambda)$ depended on the continuity of P. Likewise, the proof for $z^{(k)}(t,\lambda)$ will depend on the assumption that Q and W are continuous. The continuity of $z^{(k)}(b,\lambda)$ along with the uniform convergence to $z(b,\lambda)$ as $k \to \infty$ gives the continuity of $z(b,\lambda)$. Therefore, $\Phi(b,\lambda)$ is continuous in b, assuming that Q and W are continuous. For fixed b, $\Phi(b,\lambda)$ is continuous in λ because of the Lipschitz condition from Lemma 3.7. Therefore, $d(b,\lambda)$ is continuous in both b and λ if Q and W are continuous. □

In the next theorem, we prove the continuous dependence of each eigenvalue and the solution on the coefficients and on the boundary data, under certain conditions. One condition is that all eigenvalues, which are the roots of d and d_n, are real. This condition will always be true in the self-adjoint case. The other condition is that there exists a uniform lower bound on the eigenvalues.

Theorem 5.5. *Given the sequence of eigenvalue problems (5.4), (5.5) under the hypotheses of Lemma 5.3, let $\{\lambda_i\}_i$ and $\{\lambda_i^{(n)}\}_i$ be the roots of $d(b,\lambda)$ and $d_n(b,\lambda)$, respectively, for fixed b. Assume that all of these roots are real and listed in increasing order, and that there exists a constant m such that $\lambda_1 \geq m$ and $\lambda_1^{(n)} \geq m$ for all n. Then $\lambda_i^{(n)} \to \lambda_i$ as $n \to \infty$, for each λ_i. Moreover, taking the normalized eigenfunctions such that*

$$\left\| \begin{bmatrix} y_n(a,\lambda_i^{(n)}) \\ z_n(a,\lambda_i^{(n)}) \end{bmatrix} \right\| = 1,$$

and as $n \to \infty$, for each i,

$$\begin{bmatrix} y_n(a,\lambda_i^{(n)}) \\ z_n(a,\lambda_i^{(n)}) \end{bmatrix} \to \begin{bmatrix} y(a,\lambda_i) \\ z(a,\lambda_i) \end{bmatrix}$$

then for $i = 0, 1, 2, \ldots$ *.*

1. $y_n(t, \lambda_i^{(n)}) \to y(t, \lambda_i)$ *uniformly in* $[a, b]$*, and*

2. $z_n(t, \lambda_i^{(n)}) \to z(t, \lambda_i)$ *pointwise in* $[a, b]$

   as $n \to \infty$*.*

Proof. Note that by Lemma 5.1, λ_i is the ith eigenvalue of (5.1)-(5.2), and $\lambda_i^{(n)}$ is the ith eigenvalue of (5.4)-(5.5).

First consider $i = 0$. There exists a symmetric convex contour C_0 that encloses the numbers m and λ_0 on the real line, but no other eigenvalues $\{\lambda_i : i \neq 0\}$ are on or enclosed by C_0, except possibly multiplicities of λ_0. Such a C_0 exists because the eigenvalues are real and are the roots of an entire function (by Lemma 5.2) and therefore have no finite point of accumulation.

Now we will appeal to Rouche's Theorem to compare the number of roots of $d(b, \lambda)$ and $d_n(b, \lambda)$ inside the countour C_0. Since no eigenvalue lies on C_0, the points $\lambda \in C_0$ form a compact set such that if

$$n_0 := \min\{|d(b, \lambda)| : \lambda \in C_0\},$$

then $n_0 > 0$. Let ϵ be an arbitrary positive number. Without loss of generality, we may assume $0 < \epsilon < n_0$. Since $d_n(b, \lambda) \to d(b, \lambda)$ uniformly on compact sets by Lemma 5.3, there exists a positive number N_0 such that $|d_n(b, \lambda) - d(b, \lambda)| < \epsilon$ for all $n \geq N_0$, and all $\lambda \in C_0$. Since $\epsilon < n_0$, we have

$$|d_n(b, \lambda) - d(b, \lambda)| < |d(b, \lambda)|$$

for all $\lambda \in C_0$. We also know that d and d_n are entire and hence analytic on and inside C_0. Therefore, for $n \geq N_0$, d and d_n have the same number of roots inside

C_0 by Rouche's Theorem. By the construction of C_0, the number of roots of d_n inside C_0 is equal to the multiplicity m_0 of the first eigenvalue, λ_0. Due to the lower bound m on all the eigenvalues, the roots of d_n that lie inside C_0 must be the first m_0 eigenvalues of the nth problem, $\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots, \lambda_{m_0-1}^{(n)}$.

Now let r_0 be a positive number. We may assume that

$$0 < r_0 < \min\{|\lambda - \lambda_0| : \lambda \in C_0\}.$$

Form the contour $\tilde{C}_0$ with center λ_0 and radius r_0, so $\tilde{C}_0 \subset C_0$. By the same argument used for C_0, d_n has m_0 roots inside the contour $\tilde{C}_0$ for large n, where m_0 is the multiplicity of λ_0. These roots must be $\lambda_0^{(n)}, \lambda_1^{(n)}, \ldots, \lambda_{m_0-1}^{(n)}$. Since all of these values lie in $\tilde{C}_0$ and since $\lambda_0 = \lambda_1 = \ldots = \lambda_{m_0-1}$, we conclude

$$|\lambda_i - \lambda_i^{(n)}| < r_0,$$

for $i = 0, 1, \ldots, m_0 - 1$ and large n. Hence $\lambda_i^{(n)} \to \lambda_i$ as $n \to \infty$, for $i = 0, 1, \ldots, m_0 - 1$.

This proves convergence for the first m_0 eigenvalues. To prove the same for further eigenvalues, construct another contour C_1. Let p_0 be the point of intersection of C_0 and $\mathbb{R}$ such that $p_0 > \lambda_0$. Let C_1 be a symmetric convex contour that passes through p_0, encloses λ_{m_0}, and does not contain or enclose any other eigenvalue $\{\lambda_i : i \neq m_0\}$, except possibly multiplicities of λ_{m_0}. The points $\lambda \in C_1$ form a compact set such that if

$$n_1 := \min\{|d(b, \lambda)| : \lambda \in C_1\},$$

then $n_1 > 0$. Let ϵ be an arbitrary positive number, and we may assume $0 < \epsilon < n_1$. There exists a positive number N_1 such that $|d_n(b, \lambda) - d(b, \lambda)| < \epsilon$ for all

$n \geq N_1$ and for all $\lambda \in C_1$. Using Rouche's Theorem as before, d_n must have exactly m_1 roots enclosed by C_1, where m_1 is the multiplicity of λ_{m_0}. Form $\tilde{C}_1$ in the same manner as $\tilde{C}_0$, to reach the conclusion that $\lambda_i^{(n)} \to \lambda_i$ as $n \to \infty$, for $i = m_0, m_0+1, \ldots, m_0+m_1-1$. Continue in this manner to prove the convergence for all i.

The proof of 1 and 2 follows from a slight modification of the proof of Theorem 4.4. □

Having found conditions under which each eigenvalue depends continuously on the coefficients and on the boundary data, we now consider the dependence on the endpoints a and b. Here we will examine the dependence on b, denoted $\lambda = \lambda(b)$, noting there are similar results for the dependence on a.

Theorem 5.6. *Under the assumptions of Theorem 5.5, if $\lambda(b)$ is one of the eigenvalues of (5.1)-(5.2), then $\lambda(b)$ is continuous if $d(b, \lambda)$ is continuous in b. Otherwise, $\lambda(b)$ is quasi-continuous.*

Proof. Consider the sequence of boundary value problems for $n = 1, 2, 3, \ldots,$

$$\begin{cases} dy_n(t,\lambda) = A(t)y_n(t,\lambda)dt + dP(t)z_n(t,\lambda) \\ dz_n(t,\lambda) = (dQ(t) - \lambda dW(t))y_n(t,\lambda) + D(t)z_n(t,\lambda)dt \end{cases} \tag{5.6}$$

on $[a, b_n] \times K$, where $b_n \to b$. Again, assume the hypotheses (3.3) on the coefficients. We also have boundary conditions

$$\Gamma \begin{bmatrix} y_n(a,\lambda) \\ z_n(a,\lambda) \end{bmatrix} + \Omega \begin{bmatrix} y_n(b_n,\lambda) \\ z_n(b_n,\lambda) \end{bmatrix} = 0. \tag{5.7}$$

Similarly to an earlier discussion, define $\Phi_n(t,\lambda)$ to be the $2n \times 2n$ matrix whose columns satisfy (5.6) such that $\Phi_n(a,\lambda) = I_{2n}$, and let

$$d(b_n,\lambda) := \det\left[\Gamma + \Omega\Phi_n(b_n,\lambda)\right].$$

Let $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\lambda_i^{(n)}\}_{i=1}^{\infty}$ be the roots of $d(b,\lambda)$ and $d(b_n,\lambda)$, respectively, for $n = 1,2,3,\ldots$. As seen earlier, these roots are the eigenvalues of the corresponding problems. An argument similiar to that in the proof of Theorem 5.5 shows that $\lambda_i^{(n)} \to \lambda_i$ as $n \to \infty$, for each $i = 0,1,2,\ldots$. The proof in Theorem 5.5 appeals to Lemma 5.3, so we must show that the Lemma's conclusion holds in this case. If $d(b,\lambda)$ is continuous in b, then

$$d(b_n,\lambda) \to d(b,\lambda),$$

i.e., the conclusion of Lemma 5.3 is valid. Otherwise, $d(b,\lambda)$ is quasi-continuous in b, because by definition it has the same type of continuity as Φ, which was shown earlier to be quasi-continuous. □

Theorem 5.6 together with Lemma 5.4 yield the following corollary:

Corollary 5.7. *Under the assumptions of Theorem 5.5, if $\lambda(b)$ is an eigenvalue of (5.1)-(5.2), then $\lambda(b)$ is continuous if Q and W are continuous.*

By reworking Theorem 5.6 and Lemma 5.4 in terms of one-sided continuity, we can generalize the previous corollary to read

Corollary 5.8. *Under the assumptions of Theorem 5.5, if $\lambda(b)$ is an eigenvalue of (5.1)-(5.2), then $\lambda(b)$ is right (left) continuous if Q and W are right (left) continuous.*

Chapter 6

Stieltjes Sturm-Liouville Equations

In this chapter, we let $n = 1$ and we eliminate the coefficients A and D, resulting in the Sturm-Liouville equations

$$\begin{cases} y(t,\lambda) = y(a,\lambda) + \int_a^t dP(s)z(s,\lambda) \\ z(t,\lambda) = z(a,\lambda) + \int_a^t [dQ(s) - \lambda dW(s)]y(s,\lambda) \end{cases} \tag{6.1}$$

on $[a, b]$, assuming the same conditions on P, Q, and W as before, i.e.,

$$\begin{cases} P \text{ is continuous and nondecreasing with } P(a) = 0; \\ Q \text{ is of bounded variation on } [a,b]; \\ W \text{ is nondecreasing with } W(a) = 0. \end{cases} \tag{6.2}$$

In addition, assume that these coefficients are defined on some interval $[a_0, b_0]$ such that $[a, b] \subset [a_0, b_0]$. We attach separated boundary conditions

$$(\cos\alpha)y(a,\lambda) - (\sin\alpha)z(a,\lambda) = 0, \tag{6.3}$$

$$(\cos\beta)y(b,\lambda) - (\sin\beta)z(b,\lambda) = 0 \tag{6.4}$$

where $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$.

Finally, we assume that if $\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}$ is a nontrivial solution of (6.1) and $[c,d] \subset [a_0, b_0]$, then

$$\int_c^d |y(s,\lambda)|^2 dW(s) > 0. \tag{6.5}$$

In the previous chapter, we proved the continuous dependence of each eigenvalue on the coefficients, boundary terms, and endpoints, under certain assumptions. For (6.1)-(6.4), we will determine conditions under which each eigenvalue is differentiable as a function of all problem data. First we will show that under the mild additional assumption (6.5), all eigenvalues of (6.1)-(6.4) are real and that the problem is self-adjoint.

Theorem 6.1. *Under assumption (6.5), all eigenvalues of (6.1)-(6.4) are real and simple.*

Proof. Let λ be an eigenvalue of (6.1)-(6.4) with eigenfunctions y and z. Using integration by parts and a Leibniz rule for Stieltjes integrals, we have

$$\begin{aligned} z(t,\lambda)\bar{y}(t,\lambda)|_a^b &= \int_a^b d(z\bar{y})(s,\lambda) \\ &= \int_a^b \bar{y}(s,\lambda) dz(s,\lambda) + z(s,\lambda) d\bar{y}(s,\lambda) \\ &= \int_a^b \bar{y}(s,\lambda) \left[dQ(s) - \lambda dW(s)\right] y(s,\lambda) + z(s,\lambda) dP(s) \bar{z}(s,\lambda) \\ &= \int_a^b |y(s,\lambda)|^2 \left[dQ(s) - \lambda dW(s)\right] + |z(s,\lambda)|^2 \, dP(s). \end{aligned}$$

Then solving for λ yields

$$\lambda = \frac{-z(t,\lambda)\bar{y}(t,\lambda)|_a^b + \int_a^b |y(s,\lambda)|^2 \, dQ(s) + \int_a^b |z(s,\lambda)|^2 \, dP(s)}{\int_a^b |y(s,\lambda)|^2 \, dW(s)}. \tag{6.6}$$

The denominator and the last two terms in the numerator in (6.6) are clearly real-valued. We can show that $z(s,\lambda)\bar{y}(s,\lambda)|_a^b$ is also real-valued by examining the boundary conditions (6.3) and (6.4):

$$(z\bar{y})(b,\lambda) = \begin{cases} 0, & \text{if } \beta = \pi \\ \cot\beta\,|y(b,\lambda)|^2, & \text{if } 0 < \beta < \pi, \end{cases}$$

$$(z\bar{y})(a,\lambda) = \begin{cases} 0, & \text{if } \alpha = \pi \\ \cot\alpha\,|y(a,\lambda)|^2, & \text{if } 0 < \alpha < \pi. \end{cases}$$

Since every term on the right-hand side of (6.6) is real, λ is also real.

Now we show that all of the eigenvalues are simple if P is not identically zero by considering two cases.

First, if $\alpha = \frac{\pi}{2}$, then the initial conditions (6.3) can be written

$$\begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

for any nonzero real number c. Let $\begin{bmatrix} \phi(t,\lambda) \\ \psi(t,\lambda) \end{bmatrix}$ be the eigenfunction corresponding to the eigenvalue λ in the case where $c = 1$. Then the eigenfunction corresponding to λ in any other case is $c \begin{bmatrix} \phi(t,\lambda) \\ \psi(t,\lambda) \end{bmatrix}$, due to the uniqueness of the solution to the initial value problem. Hence, in the case where $\alpha = \frac{\pi}{2}$, the eigenvalue λ has a one-dimensional family of eigenfunctions.

Second, if $\alpha \neq \frac{\pi}{2}$, then then the initial conditions (6.3) can be written

$$\begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} = \begin{bmatrix} c \\ c\cot\alpha \end{bmatrix}$$

for any nonzero real number c. A similar argument to the first case shows that the eigenvalue λ has a one-dimensional family of eigenfunctions. Thus, in both cases, the eigenvalue λ is simple. □

Using the notation from the more general problem in the previous chapter, the boundary conditions (6.3)-(6.4) have the matrix form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(a,\lambda) \\ z(a,\lambda) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix} \begin{bmatrix} y(b,\lambda) \\ z(b,\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It can be verified that (5.3) holds with

$$\Gamma = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix}. \tag{6.7}$$

Therefore, (6.3)-(6.4) is a self-adjoint problem.

Consider the eigenvalues of (6.1)-(6.4) as a function of the endpoint b and assume that all the eigenvalues have a uniform lower bound. Then Corollary 5.7 applies since the eigenvalues are known to be real by Theorem 6.1. This corollary indicates that the eigenvalues depend continuously on the right endpoint if Q and W are continuous. We now show that for Dirichlet boundary conditions at the right endpoint, the continuity of Q and W is not needed. Writing the fundamental matrix as

$$\Phi(t,\lambda) = \begin{bmatrix} y_1(t,\lambda) & y_2(t,\lambda) \\ z_1(t,\lambda) & z_2(t,\lambda) \end{bmatrix},$$

a calculation using (6.7) shows that

$$\begin{aligned} d(b,\lambda) &= \det[\Gamma + \Omega\Phi(b,\lambda)] \\ &= \cos\alpha[\cos\beta y_2(b,\lambda) - \sin\beta z_2(b,\lambda)] - \sin\alpha[\cos\beta y_1(b,\lambda) - \sin\beta z_1(b,\lambda)]. \end{aligned}$$

Then for Dirichlet boundary conditions at the right endpoint ($\beta = \pi$),

$$d(b, \lambda) = -\cos\alpha y_2(b, \lambda) + \sin\alpha y_1(b, \lambda),$$

which is continuous in b since $y(t, \lambda)$ is continuous in the first parameter. Then $\lambda(b)$ is continuous by Theorem 5.6.

The following example illustrates these continuity properties as well as some differentiability properties.

Example 6.2. *By letting $z = y'$, $a = 0$, $P(t) = t$, $W(t) = t$, and*
$Q(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 1, & t > 1, \end{cases}$ *problem (6.1) becomes*

$$-y'' + \delta(t-1)y = \lambda y \qquad \text{on } [0, b], \tag{6.8}$$

where δ is the delta function.

Letting λ_0 denote the smallest eigenvalue, we examine the continuity and differentiability of λ_0 as a function of the right endpoint b for two different boundary conditions.

1. *First consider Dirichlet boundary conditions $y(0, \lambda) = y(b, \lambda) = 0$. We first show that $\lambda_0 > 0$, by multiplying both sides of equation (6.8) by y and integrating from 0 to b, yielding*

   $$\int_0^b -y''(s)y(s, \lambda_0)ds + \int_0^b \delta(s-1)y^2(s, \lambda_0)ds = \lambda_0 \int_0^b y^2(s, \lambda_0)ds.$$

   Using integration by parts and taking into account the Dirichlet boundary conditions,

   $$\lambda_0 = \frac{\int_0^b (y'(s, \lambda_0))^2 ds + \int_0^b \delta(s-1)y^2(s, \lambda_0)ds}{\int_0^b y^2(s, \lambda_0)ds} \ge 0.$$

Now assuming that $\lambda_0 = 0$, *the previous equation indicates that* $y'(t, \lambda_0) \equiv 0$ *on* $[0, b]$. *Then* $y(t) \equiv 0$ *on* $[0, b]$ *since* $y(0, \lambda_0) = 0$, *which is a contradiction since* y *is a nontrivial solution. Therefore* $\lambda_0 > 0$.

We compute the formula for $\lambda_0(b)$ *separately for* $b < 1$ *and* $b > 1$. *For* $b < 1$, $\delta(t-1) = 0$ *on* $[0, b]$. *Then equation (6.8) becomes* $-y'' = \lambda y$. *Since* $\lambda > 0$, *the solution is*

$$y(t, \lambda_0) = c_1 \sin(\theta t) + c_2 \cos(\theta t),$$

where c_1 *and* c_2 *are constants and* $\theta = \sqrt{\lambda_0}$. *The boundary condition* $y(0, \lambda) = 0$ *yields the solution to be*

$$y(t, \lambda_0) = c_1 \sin(\theta t),$$

and the boundary condition $y(b, \lambda) = 0$ *along with the fact that the eigenvalues are positive requires that*

$$\theta = \frac{k\pi}{b}, \qquad k = 1, 2, 3, \ldots .$$

Therefore, the smallest eigenvalue λ_0 *has the formula*

$$\lambda_0 = \left(\frac{\pi}{b}\right)^2$$

for $b < 1$.

For $b > 1$, *the formula for the solution is*

$$y(t, \lambda) = \begin{cases} \sin(\theta t), & 0 \leq t < 1 \\ c \sin(\theta(t-b)), & 1 < t \leq b \end{cases}, \tag{6.9}$$

where c *is a constant. Since* $y(\cdot, \lambda)$ *is continuous, we observe that*

$$y(1, \lambda) = \sin\theta = c \sin(\theta(1-b)). \tag{6.10}$$

By integrating equation (6.8) from 0 *to* t*, the problem can be written as*

$$-y'(t,\lambda)+y'(0,\lambda)+\int_0^t dQ(s)y(s,\lambda)=\lambda\int_0^t y(s,\lambda)ds. \tag{6.11}$$

Taking the limit of equation (6.11) as $t \to 1^-$*, we get*

$$-y'(1^-,\lambda)+y'(0,\lambda)+\int_0^1 dQ(s)y(s,\lambda)=\lambda\int_0^1 y(s,\lambda)ds. \tag{6.12}$$

Taking the limit of equation (6.11) as $t \to 1^+$*, we get*

$$-y'(1^+,\lambda)+y'(0,\lambda)+\int_0^1 dQ(s)y(s,\lambda)+y(1,\lambda)=\lambda\int_0^1 y(s,\lambda)ds. \tag{6.13}$$

By subtracting (6.13) from (6.12),

$$y'(1^+,\lambda)=y'(1^-,\lambda)+y(1,\lambda). \tag{6.14}$$

From (6.9),

$$y'(t,\lambda)=\begin{cases}\theta\cos(\theta t), & 0\le t<1\\ c\theta\cos(\theta(t-b)), & 1<t\le b\end{cases},$$

and substituting this into (6.14), we get

$$\theta c\cos(\theta(1-b))=\theta\cos\theta+\sin\theta. \tag{6.15}$$

Equations (6.10) and (6.15) together give

$$\frac{\theta c\cos(\theta(1-b))}{c\sin(\theta(1-b))}=\frac{\theta\cos\theta+\sin\theta}{\sin\theta},$$

and hence

$$\cot(\theta(1-b))=\cot\theta+\frac{1}{\theta}. \tag{6.16}$$

This implies that

$$\theta(1-b) = arccot\left[\cot\theta + \frac{1}{\theta}\right] - \pi$$

(we subtract π *to obtain the correct branch of the cotangent), which leads to the parametric equations*

$$\begin{cases} b = 1 - \frac{1}{\theta}\left[arccot\left(\cot\theta + \frac{1}{\theta}\right) - \pi\right] \\ \lambda_0 = \theta^2 \end{cases}$$

for $b > 1$.

To determine if $\lambda_0(b)$ *is continuous at* $b = 1$, *note that* $\theta \to \pi$ *as* $b \to 1^+$, *which can be shown by examining the parametric equations above. Then*

$$\lambda_0(1^+) = \lim_{b \to 1^+} \lambda_0(b) = \lim_{\theta \to \pi} \theta^2 = \pi^2.$$

Referring to the solution for $b < 1$, *we have*

$$\lambda_0(1^-) = \lim_{b \to 1^-} \lambda_0(b) = \lim_{b \to 1^-} \left(\frac{\pi}{b}\right)^2 = \pi^2.$$

Therefore, $\lambda_0(b)$ *is continuous at* $b = 1$.

To determine if $\lambda_0(b)$ *is differentiable at* $b = 1$, *we calculate the left and right-sided derivatives of* $\theta(b)$, *because* $\lambda_0(b)$ *is differentiable if* $\theta(b)$ *is differentiable. For the left-sided derivative, recall that for* $b < 1$

$$\theta(b) = \frac{\pi}{b}.$$

Then

$$\theta'(1^-) = \lim_{b \to 1^-} \frac{-\pi}{b^2} = -\pi.$$

For the right-sided derivative, differentiate (6.16) with respect to b :

$$(-\csc^2(\theta(1-b)))\left[(1-b)\frac{d\theta}{db}-\theta\right] = (-\csc^2\theta)\frac{d\theta}{db} - \frac{1}{\theta^2}\frac{d\theta}{db},$$

which simplifies to

$$(1-b)\frac{d\theta}{db}-\theta = \frac{\sin^2(\theta(1-b))}{\sin^2\theta}\frac{d\theta}{db} + \frac{\sin^2(\theta(1-b))}{\theta^2}\frac{d\theta}{db}.$$

Take the limit of both sides of this equation as $b \to 1^+$, *which means that* $\theta \to \pi$, *using L'Hôpital's Rule to find the limit of the first term on the right-hand side. The result is*

$$-\pi = \left(\frac{\pi}{\theta'(1^+)}\right)^2 \theta'(1^+),$$

which implies that $\theta'(1^+) = -\pi$. *Since* $\theta'(1^-) = \theta'(1^+)$, $\theta(b)$ *is differentiable at* $b = 1$, *and hence* $\lambda_0(b)$ *is also. Figure 6.1 contains the graph of* $\lambda_0(b)$ *for the cases* $b < 1$ *and* $b > 1$, *illustrating the continuity and differentiability at* $b = 1$.

2. *Now consider mixed Dirichlet and Neumann boundary conditions* $y(0,\lambda) = y'(b,\lambda) = 0$. *We can show that* $\lambda_0 > 0$ *by a similar argument as in the Dirichlet case.*

   For $b < 1$, *the argument used in the Dirichlet case shows the solution is*

$$y(t,\lambda_0) = c\sin(\theta t),$$

   for some constant c. *The boundary condition* $y'(b,\lambda) = 0$ *requires that* $c\theta\cos(\theta b) = 0$, *and hence*

$$\theta b = \frac{\pi}{2} + k\pi, \qquad k = 0, 1, 2, \ldots .$$

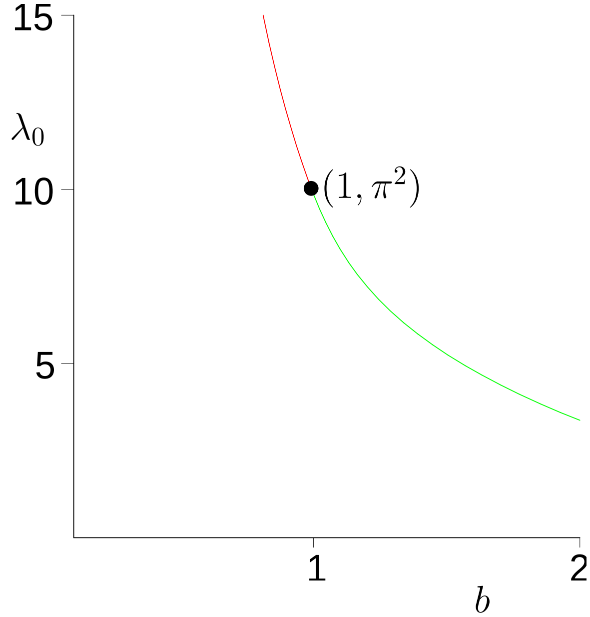

Figure 6.1: Dirichlet Boundary Conditions

Therefore, the smallest eigenvalue has the formula

$$\lambda_0(b) = \left(\frac{\pi}{2b}\right)^2$$

for $b < 1$.

For $b > 1$, *the solution is*

$$y(t, \lambda) = \begin{cases} \sin(\theta t), & 0 \le t < 1 \\ c\cos(\theta(t-b)), & 1 < t \le b \end{cases}, \tag{6.17}$$

where c *is a constant. Since* $y(\cdot, \lambda)$ *is continuous, we observe that*

$$y(1, \lambda) = \sin\theta = c\cos(\theta(1-b)). \tag{6.18}$$

Note that (6.14) holds for any boundary conditions, and in this case we use

(6.14) and (6.17) to get

$$-c\theta\sin(\theta(1-b)) = \theta\cos\theta + \sin\theta. \tag{6.19}$$

Equations (6.18) and (6.19) together give

$$\frac{-c\theta\sin(\theta(1-b))}{c\cos(\theta(1-b))} = \frac{\theta\cos\theta + \sin\theta}{\sin\theta},$$

and hence

$$\tan(\theta(1-b)) = -\cot\theta - \frac{1}{\theta}. \tag{6.20}$$

This leads to the parametric equations

$$\begin{cases} b = 1 - \frac{1}{\theta} arctan\left(-\cot\theta - \frac{1}{\theta}\right) \\ \lambda_0 = \theta^2 \end{cases}$$

for $b > 1$.

To determine if $\lambda_0(b)$ *is continuous at* $b = 1$, *we use a computer to approximate*

$$\lambda_0(1^+) = \lim_{b\to 1^+} \lambda_0 \approx 4.12,$$

which can be shown by examining the parametric equations above. Referring to the solution for $b < 1$, *we have*

$$\lambda_0(1^-) = \lim_{b\to 1^-} \lambda_0(b) = \lim_{b\to 1^-} \left(\frac{\pi}{2b}\right)^2 = \frac{\pi^2}{4} \approx 2.47.$$

Therefore, $\lambda_0(b)$ *is discontinuous at* $b = 1$. *The graph of* $\lambda(b)$ *is given in Figure 6.2 for* $b < 1$ *and* $b > 1$, *illustrating the discontinuity at* $b = 1$.

In this example, $Q(t)$ has a point of discontinuity at $t = 1$. This discontinuity does not prevent $\lambda(b)$ from being continuous in the case of Dirichlet boundary

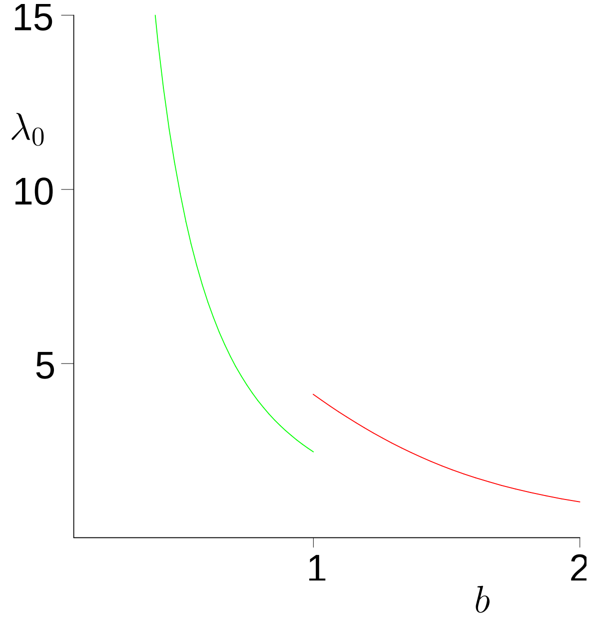

Figure 6.2: Mixed Boundary Conditions

conditions, as explained immediately before this example. This example illustrates that the discontinuity of Q can result in $\lambda(b)$ being discontinuous for other boundary conditions. In this chapter, the conditions for continuity and differentiability of each eigenvalue will be compatible with the results of this example.

The identity in the following lemma will be used throughout the remainder of this chapter.

Lemma 6.3. *If* $\begin{cases} dy_i = dP_i z_i \\ dz_i = (dQ_i - \lambda_i dW_i) y_i \end{cases}$ *for* $i = 1, 2$, *then*

$$-z_1 y_2 + z_2 y_1 |_a^b = \int_a^b (\lambda_1 dW_1 - \lambda_2 dW_2 - dQ_1 + dQ_2) y_1 y_2 + \int_a^b (dP_1 - dP_2) z_1 z_2.$$

Proof. Using integration by parts followed by a Leibniz rule for Stieltjes integrals, we have

$$
\begin{aligned}
-z_1y_2+z_2y_1|_a^b &= \int_a^b d(-z_1y_2+z_2y_1)\\
&= \int_a^b -dz_1y_2 - z_1dy_2 + dz_2y_1 + z_2dy_1\\
&= \int_a^b -(dQ_1-\lambda_1dW_1)y_1y_2 - z_1dP_2z_2 + (dQ_2-\lambda_2dW_2)y_2y_1 + z_2dP_1z_1\\
&= \int_a^b (\lambda_1dW_1-\lambda_2dW_2-dQ_1+dQ_2)y_1y_2 + \int_a^b (dP_1-dP_2)z_1z_2.
\end{aligned}
$$

□

First we examine the eigenvalue dependence on the endpoints a and b. We write $\lambda=\lambda(a)$ and $\lambda=\lambda(b)$ to represent this dependence. In Theorem 5.6 and Corollaries 5.7 and 5.8, we found conditions under which the eigenvalues depend continuously on the right endpoint. Now we explore further the eigenvalue as a function of the endpoints and find conditions under which this function is of bounded variation, absolutely continuous, and differentiable. The result on absolute continuity is of particular interest when examining differentiability, because any absolutely continuous function is differentiable almost everywhere. All of these results have a corresponding result for a in place of b. First we state some Lemmas and general results.

Let $\lambda(a)$ be an eigenvalue of (6.1)-(6.5) with eigenfunction $\begin{bmatrix} y(t,\lambda(a)) \\ z(t,\lambda(a)) \end{bmatrix}$, and let $\lambda(b)$ be an eigenvalue of (6.1)-(6.5) with eigenfunction $\begin{bmatrix} y(t,\lambda(b)) \\ z(t,\lambda(b)) \end{bmatrix}$. From the identity in Lemma 6.3, with $i=1$ corresponding to the right endpoint $b+h$ and $i=2$ corresponding to the right endpoint b, we have

$$[\lambda(b+h)-\lambda(b)]\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b)) =$$
$$-z(t,\lambda(b+h))y(t,\lambda(b))+z(t,\lambda(b))y(t,\lambda(b+h))|_a^b. \quad (6.21)$$

Note that the identity is simplified here because the only difference between the two problems ($i = 1, 2$) is $\lambda_1 = \lambda(b+h)$ and $\lambda_2 = \lambda(b)$, while the coefficients are the same between the two problems. We can use the following lemma to simplify the right-hand side of this equation.

Lemma 6.4. *For any f and h in the same class of functions, with α and β as in (6.3)-(6.4),*

1. *if $\alpha(f) = \alpha(f+h)$ and $y(t,\lambda)$, $z(t,\lambda)$ satisfy (6.3) for $\lambda = \lambda(f)$ and for $\lambda = \lambda(f+h)$, then $z(a,\lambda(f+h))y(a,\lambda(f)) - z(a,\lambda(f))y(a,\lambda(f+h)) = 0$;*
2. *if $\beta(f) = \beta(f+h)$ and $y(t,\lambda)$, $z(t,\lambda)$ satisfy (6.4) for $\lambda = \lambda(f)$ and for $\lambda = \lambda(f+h)$, then $z(b,\lambda(f+h))y(b,\lambda(f)) - z(b,\lambda(f))y(b,\lambda(f+h)) = 0$.*

Proof.

1. Let $\alpha := \alpha(f) = \alpha(f+h)$. We consider two cases. First, if $\alpha = \frac{\pi}{2}$, then the boundary conditions (6.3) indicate that $z(a,\lambda(f)) = z(a,\lambda(f+h)) = 0$, proving the lemma in this case. Second, if $\alpha \neq \frac{\pi}{2}$, then $y(a,\lambda) = (\tan\alpha)z(a,\lambda)$ for any λ. So

$$z(a,\lambda(f+h))y(a,\lambda(f)) - z(a,\lambda(f))y(a,\lambda(f+h)) =$$
$$z(a,\lambda(f+h))(\tan\alpha)z(a,\lambda(f)) - z(a,\lambda(f))(\tan\alpha)z(a,\lambda(f+h)) = 0.$$

2. The proof is similar.

□

Throughout the remainder of this chapter, we choose y and z by the initial conditions

$$y(a,\lambda) = \sin\alpha,$$

$$z(a,\lambda) = \cos\alpha$$

so that (6.3) holds.

Lemma 6.5. *For any* $b \in [a, b_0]$, *let* $\lambda(b)$ *be an eigenvalue of (6.1)-(6.5) with eigenfunction* $\begin{bmatrix} y(t,\lambda(b)) \\ z(t,\lambda(b)) \end{bmatrix}$. *Assume that for* h *such that* $b+h \in [a, b_0]$, $\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b)) \neq 0$. *Then*

$$\lambda(b+h) - \lambda(b) = \frac{y(b,\lambda(b))\int_b^{b+h}[dQ(s) - \lambda(b+h)dW(s)]y(s,\lambda(b+h))}{\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b))} - \frac{z(b,\lambda(b))\int_b^{b+h} dP(s)z(s,\lambda(b+h))}{\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b))}.$$

Proof. Lemma 6.4 allows us to simplify (6.21) to

$$\lambda(b+h) - \lambda(b) = \frac{-z(b,\lambda(b+h))y(b,\lambda(b)) + z(b,\lambda(b))y(b,\lambda(b+h))}{\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b))}. \quad (6.22)$$

We can rewrite the terms

$$\begin{aligned} y(b,\lambda(b+h)) &= y(b+h,\lambda(b+h)) - \int_b^{b+h} dy(s,\lambda(b+h)) \\ &= y(b+h,\lambda(b+h)) - \int_b^{b+h} dP(s)z(s,\lambda(b+h)), \end{aligned}$$

$$z(b,\lambda(b+h)) = z(b+h,\lambda(b+h)) - \int_b^{b+h} dz(s,\lambda(b+h))$$

$$= z(b+h, \lambda(b+h)) - \int_b^{b+h} [dQ(s) - \lambda(b+h)dW(s)]y(s, \lambda(b+h)).$$

These substitutions into (6.22) yield

$$\lambda(b+h) - \lambda(b) = \frac{[-z(b+h, \lambda(b+h)) + \int_b^{b+h}[dQ(s) - \lambda(b+h)dW(s)]y(s, \lambda(b+h))]y(b, \lambda(b))}{\int_a^b dW(s)y(s, \lambda(b+h))y(s, \lambda(b))} + \frac{z(b, \lambda(b))[y(b+h, \lambda(b+h)) - \int_b^{b+h} dP(s)z(s, \lambda(b+h))]}{\int_a^b dW(s)y(s, \lambda(b+h))y(s, \lambda(b))}. \quad (6.23)$$

We consider two cases. First assume that $\beta = \frac{\pi}{2}$. In this case, $z(b, \lambda(b)) = z(b+h, \lambda(b+h)) = 0$, and this substitution into (6.23) completes the proof for this case. For the case when $\beta \neq \frac{\pi}{2}$, we can write the boundary conditions at b as

$$z(b, \lambda(b)) = (\tan\beta)y(b, \lambda(b)),$$

$$z(b+h, \lambda(b+h)) = (\tan\beta)y(b+h, \lambda(b+h)).$$

This implies that

$$-z(b+h, \lambda(b+h))y(b, \lambda(b)) + z(b, \lambda(b))y(b+h, \lambda(b+h)) =$$
$$-(\tan\beta)y(b+h, \lambda(b+h))y(b, \lambda(b)) + (\tan\beta)y(b, \lambda(b))y(b+h, \lambda(b+h)) = 0.$$

This substitution into (6.23) completes the proof for the second case. □

Lemma 6.6. *Assume $\lambda(b)$ is continuous in b, and let $a < c < d < b_0$. Then there exist positive constants c_1 and h_0 such that if $c \leq t, b \leq d$ and $|h| \leq h_0$, then*

$$\int_a^t dW(s)y(s, \lambda(b+h))y(s, \lambda(b)) \geq c_1.$$

Proof. Our strategy is to prove that this term is arbitrarily close to $\int_a^t dW(s)y(s,\lambda(b))^2$, which for $t = c$ we now prove has a positive lower bound independent of $b \in [c, d]$.

If we suppose not, then we can assume there exists a sequence $b_n \subset [c, d]$ such that

$$\int_a^c dW(s)y(s,\lambda(b_n))^2 \to 0 \tag{6.24}$$

as $n \to \infty$. By the Bolzano-Weirstrass Theorem, there exists a subsequence $\{b_{n_j}\}$ such that $b_{n_j} \to b$ for some $b \in [c, d]$. Since $\lambda(b)$ is continuous by assumption and $y(t, \lambda)$ satisfies a Lipschitz condition in λ,

$$\int_a^c dW(s)y(s,\lambda(b_{n_j}))^2 \to \int_a^c dW(s)y(s,\lambda(b))^2.$$

By (6.24) we conclude that

$$\int_a^c dW(s)y(s,\lambda(b))^2 = 0.$$

This contradicts (6.5), so $\int_a^c dW(s)y(s,\lambda(b))^2$ must have a positive uniform lower bound independent of b.

Now we will prove that $\int_a^c dW(s)y(s,\lambda(b+h))y(s,\lambda(b))$ is arbitrarily close to $\int_a^c dW(s)y(s,\lambda(b))^2$. We have

$$\left|\int_a^t y(s,\lambda(b))^2 dW(s) - \int_a^t y(s,\lambda(b+h))y(s,\lambda(b))dW(s)\right| = \left|\int_a^t y(s,\lambda(b))\left[y(s,\lambda(b)) - y(s,\lambda(b+h))\right] dW(s)\right|.$$

The right-hand side approaches zero as $h \to 0$ uniformly in b because y is bounded and because $y(s,\lambda(b)) - y(s,\lambda(b+h)) \to 0$ uniformly in b due to the continuity of $\lambda(b)$ and the Lipschitz condition in λ of $y(t,\lambda)$. Since $\int_a^t y(s,\lambda(b))^2 dW(s)$ and $\int_a^t y(s,\lambda(b+h))y(s,\lambda(b))dW(s)$ can be made arbitrarily close by taking small enough values of h and since $\int_a^c y(s,\lambda(b))^2 dW(s)$ has a positive uniform lower bound, the proof is complete, as $\int_a^t y(s,\lambda(b))^2 dW(s) \geq \int_a^c y(s,\lambda(b))^2 dW(s)$ for $t \geq c$. □

Theorem 6.7. *Assume that* $\lambda(b)$ *is continuous on* $[a_0, b_0]$. *Then for fixed* $a \in (a_0, b_0)$, $\lambda(b)$ *is of bounded variation on any interval* $[c,d]$ *such that* $a < c < d < b_0$. *For fixed* $b \in (a_0, b_0)$, $\lambda(a)$ *is of bounded variation on any interval* $[c,d]$ *such that* $a_0 < c < d < b$.

Proof. We give the proof for $\lambda(b)$, noting that the proof for $\lambda(a)$ is similar. Fix $a \in (a_0, b_0)$ and let $a < c < d < b_0$. Let $T = \{t_i\}_1^m$ be any partition of $[c,d]$ with mesh small enough so that for each i,

$$\int_a^{t_{i-1}} dW(s) y(s,\lambda(t_i)) y(s,\lambda(t_{i-1})) \geq c_1$$

by Lemma 6.6. From Lemma 6.5, we have

$$\begin{aligned}
\bigvee_c^d \lambda &= \sup_{T\in\mathcal{P}[c,d]} \sum_{i=1}^m |\lambda(t_i) - \lambda(t_{i-1})| \\
&= \sup_{T\in\mathcal{P}[c,d]} \sum_{i=1}^m \Bigg| \frac{y(t_i,\lambda(t_i)) \int_{t_{i-1}}^{t_i} [dQ(s) - \lambda(t_i) dW(s)] y(s,\lambda(t_i))}{\int_a^{t_{i-1}} dW(s) y(s,\lambda(t_i)) y(s,\lambda(t_{i-1}))} \\
&+ \frac{z(t_{i-1},\lambda(t_{i-1})) \int_{t_{i-1}}^{t_i} dP(s) z(s,\lambda(t_i))}{\int_a^{t_{i-1}} dW(s) y(s,\lambda(t_i)) y(s,\lambda(t_{i-1}))} \Bigg| \\
&\leq K \sup_{T\in\mathcal{P}[c,d]} \sum_{i=1}^m \left[\int_{t_{i-1}}^{t_i} |dQ(s)| + \int_{t_{i-1}}^{t_i} dW(s) + \int_{t_{i-1}}^{t_i} dP(s) \right]
\end{aligned}$$

Here, K is a positive constant, which exists by the following argument. We have already shown that $y(t,\lambda)$ and $z(t,\lambda)$ are uniformly bounded in t. We have also shown that $\lambda(b)$ is continuous and is therefore bounded on $[a, b_0]$. This fact, together with the fact that $y(t,\lambda)$ and $z(t,\lambda)$ are continuous in λ, implies that y and z are bounded in λ. Finally, we use the fact that the denominator is uniformly bounded below by a positive constant as shown above. Now we can write

$$\bigvee_c^d \lambda \leq K \left[\bigvee_c^d Q + \bigvee_c^d W + \bigvee_c^d P \right],$$

which is finite since Q, W, and P are of bounded variation. Recall that Q is of bounded variation by hypothesis, while W and P are of bounded variation since they are nondecreasing. Therefore, $\lambda(b)$ is of bounded variation on $[c, d]$. $\square$

The following theorem gives conditions under which $\lambda(a)$ and $\lambda(b)$ are absolutely continuous.

Theorem 6.8. *For fixed $a \in (a_0, b_0)$ and $a < c < d < b_0$, $\lambda(b)$ is absolutely continuous on $[c, d]$ if $P(t)$, $Q(t)$, and $W(t)$ are absolutely continuous on $[c, d]$. For fixed $b \in (a_0, b_0)$ and $a_0 < c < d < b$, $\lambda(a)$ is absolutely continuous on $[c, d]$ if $P(t)$, $Q(t)$, and $W(t)$ are absolutely continuous on $[c, d]$.*

Proof. We give the proof for $\lambda(b)$, noting that the proof for $\lambda(a)$ is similar. Fix $a \in (a_0, b_0)$ and let $a < c < d < b_0$. Let $T = \{(c_i, d_i)\}_1^m$ be any collection of disjoint intervals in $[c, d]$ with the length of each interval small enough so that $\int_a^{c_i} dW(s)y(s, \lambda(d_i))y(s, \lambda(c_i))$ is uniformly bounded below by a positive constant, as in the proof of Theorem 6.7. Since Q and W are assumed to be absolutely continuous, we know that $\lambda(b)$ is continuous by Corollary 5.7. Referring to Lemma 6.6, we see that the conditions of Lemma 6.5 are satisfied. Thus,

$$\sum_{i=1}^m |\lambda(d_i) - \lambda(c_i)| \leq \sum_{i=1}^m \left| \frac{y(d_i, \lambda(d_i)) \int_{c_i}^{d_i} [dQ(s) - \lambda(d_i) dW(s)] y(s, \lambda(d_i))}{\int_a^{c_i} dW(s) y(s, \lambda(d_i)) y(s, \lambda(c_i))} \right.$$
$$\left. + \frac{z(c_i, \lambda(c_i)) \int_{c_i}^{d_i} dP(s) z(s, \lambda(d_i))}{\int_a^{c_i} dW(s) y(s, \lambda(d_i)) y(s, \lambda(c_i))} \right|$$

$$\leq K \sum_{i=1}^{m} \int_{c_i}^{d_i} [|dQ| + dW + dP]$$

for some constant K, which exists since $\lambda(b)$ is bounded and $y(t,\lambda)$ and $z(t,\lambda)$ are uniformly bounded in both t and λ, and since the denominator is uniformly bounded below by a positive constant. Then

$$\sum_{i=1}^{m} |\lambda(d_i) - \lambda(c_i)| \leq$$
$$K \sum_{i=1}^{m} \left[(v_Q(d_i) - v_Q(c_i)) + (W(d_i) - W(c_i)) + (P(d_i) - P(c_i)) \right]. \quad (6.25)$$

By recalling that $v_Q(t) = \int_a^t |Q'(t)|$, we see that Q is absolutely continuous implies that v_Q is also absolutely continuous. Thus, applying the definition of absolute continuity, we see that $\lambda(b)$ is absolutely continuous since v_Q, W, and P are absolutely continuous (recall assumption (6.25)). □

We introduce the notation for a right-sided limit and left-sided limit:

$$f(c^+) := \lim_{t \to c^+} f(t)$$

$$f(c^-) := \lim_{t \to c^-} f(t).$$

Theorem 6.9. *For fixed* $a \in (a_0, b_0)$, $\lambda(b)$ *is right (left) differentiable at a point* $b \in (a, b_0)$ *if* $P(t)$, $Q(t)$, *and* $W(t)$ *are right (left) differentiable at* b; *for fixed* $b \in (a_0, b_0)$, $\lambda(a)$ *is right (left) differentiable at a point* $a \in (a_0, b)$ *if* $P(t)$, $Q(t)$, *and* $W(t)$ *are right (left) differentiable at* a. *In this case,*

$$\lambda'(b^+) = \frac{y(b,\lambda(b))^2[Q'(b^+) - \lambda(b)W'(b^+)] - z(b,\lambda(b))z(b^+,\lambda(b))P'(b^+)}{\int_a^b y(s,\lambda(b))^2 dW(s)},$$
$$\lambda'(b^-) = \frac{y(b,\lambda(b))^2[Q'(b^-) - \lambda(b)W'(b^-)] - z(b,\lambda(b))z(b^-,\lambda(b))P'({}^-b)}{\int_a^b y(s,\lambda(b))^2 dW(s)},$$

$$\lambda'(a^+) = \frac{-y(a,\lambda(a))^2[Q'(a^+) - \lambda(a)W'(a^+)] + z(a,\lambda(a))z(a,\lambda(a^+))P'(a^+)}{\int_a^b y(s,\lambda(a))^2 dW(s)},$$
$$\lambda'(a^-) = \frac{-y(a,\lambda(a))^2[Q'(a^-) - \lambda(a)W'(a^-)] + z(a,\lambda(a))z(a,\lambda(a^-))P'(a^-)}{\int_a^b y(s,\lambda(a))^2 dW(s)}.$$

Proof. We give the proof for $\lambda(b^+)$, noting that the proof for the other cases are similar.

First observe that the hypotheses of Lemma 6.5 are satisfied due to the continuity of Q and W (see argument in the proof for Theorem 6.8). Dividing both sides of equation in Lemma 6.5 by h, we get

$$\begin{aligned}\frac{\lambda(b+h)-\lambda(b)}{h} &= \\ &\frac{y(b,\lambda(b))\int_b^{b+h}[dQ(s)-\lambda(b+h)dW(s)]y(s,\lambda(b+h))}{h\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b))} \\ &\qquad - \frac{z(b,\lambda(b))\int_b^{b+h} dP(s)z(s,\lambda(b+h))}{h\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b))}. \end{aligned} \quad (6.26)$$

We calculate the limit of the right-hand side of this equation as $h \to 0^+$, which will prove the existence of the derivative $\lambda'(b^+)$. Now $\lambda(b)$ is right continuous by Corollary 5.8, and $y(t,\lambda)$ satisfies a Lipschitz condition in λ. Therefore, as $h \to 0^+$,

$$\int_a^b dW(s)y(s,\lambda(b+h))y(s,\lambda(b)) \to \int_a^b dW(s)y(s,\lambda(b))^2 > 0. \quad (6.27)$$

Next we show that $\frac{1}{h}\int_b^{b+h} dP(s)z(s,\lambda(b+h)) \to P'(b)z(b^+,\lambda(b))$ as $h \to 0^+$. First we write

$$\begin{aligned}\frac{1}{h}\int_b^{b+h} dP(s)z(s,\lambda(b+h)) &= \frac{1}{h}\int_b^{b+h} dP(s)[z(s,\lambda(b+h)) - z(b^+,\lambda(b+h))] \\ &\quad + \frac{1}{h}\int_b^{b+h} dP(s)z(b^+,\lambda(b+h)), \end{aligned} \quad (6.28)$$

and we now show that the first of the two terms on the right-hand side converges to zero. We find a bound on $z(t,\lambda(b+h)) - z(b^+,\lambda(b+h))$ for $b < t \le b+h$, recalling that

$$z(t,\lambda) = z(a,\lambda) + \int_a^t [dQ(s) - \lambda dW(s)]y(s,\lambda).$$

For $b \le t \le b+h$,

$$\begin{aligned}|z(t,\lambda(b+h)) - z(b^+,\lambda(b+h))| &= \left| \int_{b^+}^t [dQ(s) - \lambda(b+h)dW(s)]y(s,\lambda(b+h)) \right| \\ &\le \max_{b\le t\le b+h} |y(t,\lambda(b+h))| \bigvee_{b^+}^{b+h} [Q + |\lambda(b+h)|W].\end{aligned}$$

Then

$$\begin{aligned}&\left| \frac{1}{h} \int_b^{b+h} dP(s)[z(s,\lambda(b+h)) - z(b^+,\lambda(b+h))] \right| \le \\ &\max_{b\le t\le b+h} |y(t,\lambda(b+h))| \bigvee_{b^+}^{b+h} [Q + |\lambda(b+h)|W] \frac{1}{h} \bigvee_b^{b+h} P \\ &= \max_{b\le t\le b+h} |y(t,\lambda(b+h))| \bigvee_{b^+}^{b+h} [Q + |\lambda(b+h)|W] \frac{P(b+h) - P(b)}{h}\end{aligned}$$

since P is continuous and nondecreasing, and we now show that this converges to zero. Since Q and W are of bounded variation,

$$\bigvee_{b^+}^{b+h} [Q + |\lambda(b+h)|W] \to 0$$

as $h \to 0^+$. Since y is bounded and

$$\frac{P(b+h) - P(b)}{h} \to P'(b)$$

($P'(b)$ exists by assumption), we conclude that

$$\frac{1}{h} \int_b^{b+h} dP(s)[z(s,\lambda(b+h)) - z(b^+,\lambda(b+h))] \to 0$$

as $h \to 0^+$.

For the second term in (6.28),

$$\frac{1}{h}\int_b^{b+h} dP(s)z(b^+,\lambda(b+h)) = z(b^+,\lambda(b+h))\frac{1}{h}\bigvee_b^{b+h} P$$
$$= z(b^+,\lambda(b+h))\frac{P(b+h)-P(b)}{h}$$

since P is continuous and nondecreasing. Note that

$$z(b^+,\lambda(b+h)) \to z(b^+,\lambda(b))$$

since λ is continuous and $z(t,\lambda)$ satisfies a Lipschitz condition in λ. Therefore as $h \to 0^+$,

$$\frac{1}{h}\int_b^{b+h} dP(s)z(b^+,\lambda(b+h)) \to P'(b^+)z(b^+,\lambda(b)). \quad (6.29)$$

All that remains is to show that the second term in (6.26) converges, i.e., that

$$\frac{1}{h}\int_b^{b+h}[dQ(s)-\lambda(b+h)dW(s)]y(s,\lambda(b+h)) \to [Q'(b^+)-\lambda(b)W'(b^+)]y(b,\lambda(b))$$

as $h \to 0^+$. First we write

$$\frac{1}{h}\int_b^{b+h}[dQ(s)-\lambda(b+h)dW(s)]y(s,\lambda(b+h)) =$$
$$\frac{1}{h}\int_b^{b+h}[dQ(s)-\lambda(b+h)dW(s)][y(s,\lambda(b+h))-y(b,\lambda(b+h))]+$$
$$\frac{1}{h}\int_b^{b+h}[dQ(s)-\lambda(b+h)dW(s)]y(b,\lambda(b+h)), \quad (6.30)$$

and we now show that the first of the two terms on the right-hand side converges to zero. We find a bound on $y(t,\lambda(b+h))-y(b,\lambda(b+h))$ for $b<t\le b+h$, recalling that

$$y(t,\lambda) = y(a,\lambda) + \int_a^t dP(s)z(s,\lambda).$$

For $b \leq t \leq b+h$,

$$\begin{aligned} |y(t,\lambda(b+h)) - y(b,\lambda(b+h))| &= \left| \int_b^t dP(s) z(s,\lambda(b+h)) \right| \\ &\leq \max_{b \leq t \leq b+h} |z(t,\lambda(b+h))| \bigvee_b^{b+h} P \\ &= [P(b+h) - P(b)] \max_{b \leq t \leq b+h} |z(t,\lambda(b+h))|. \end{aligned}$$

Then

$$\left| \frac{1}{h} \int_b^{b+h} [dQ(s) - \lambda(b+h) dW(s)][y(s,\lambda(b+h)) - y(b,\lambda(b+h))] \right| \leq$$
$$\frac{[P(b+h) - P(b)]}{h} \max_{b \leq t \leq b+h} |z(t,\lambda(b+h))| \bigvee_b^{b+h} [Q + |\lambda(b+h)| W] \quad (6.31)$$

and we now show that this converges to zero. Since P is differentiable,

$$\frac{P(b+h) - P(b)}{h} \to P'(b).$$

It has been shown that $\max_{b \leq t \leq b+h} |z(t,\lambda(b+h))|$ is bounded. We have also shown that

$$\bigvee_b^{b+h} [Q + |\lambda(b+h)| W] \to 0$$

as $h \to 0^+$. Now we conclude that

$$\frac{1}{h} \int_b^{b+h} [dQ(s) - \lambda(b+h) dW(s)][y(s,\lambda(b+h)) - y(b,\lambda(b+h))] \to 0$$

as $h \to 0^+$.

For the second term on the right-hand side of (6.30), use the assumption that Q and W are right differentiable and hence right continuous to conclude that for small h,

$$\frac{1}{h}\int_b^{b+h}[dQ(s)-\lambda(b+h)dW(s)]y(b,\lambda(b+h)) =$$

$$y(b,\lambda(b+h))\frac{1}{h}[(Q(b+h)-Q(b))-\lambda(b+h)(W(b+h)-W(b))]$$

$$\to y(b,\lambda(b))[Q'(b^+)-\lambda(b)W'(b^+)]$$

as $h \to 0^+$ since $\lambda(b)$ is continuous and $y(t,\lambda)$ satisfies a Lipschitz condition in λ. □

Remark 6.10. *1. Theorem 6.9 has an analogous result if the hypotheses are changed so that P, Q, and W are differentiable at $t=a$ or $t=b$. By Corallary 3.6, $z(t,\lambda)$ is continuous at a or b so that $z(b,\lambda)=z(b^+,\lambda)=z(b^-,\lambda)$ and $z(a,\lambda)=z(a^+,\lambda)=z(a^-,\lambda)$.*

2. We can find specific results of this Theorem for Dirichlet and Neumann boundary conditions. Referring to (6.26), we see that the differentiability of $\lambda(b)$ depends only on the differentiability of P in the Dirichlet case, since $y(b,\lambda(b))=0$. In the Neumann case, the differentiability of $\lambda(b)$ depends only on the differentiability of W, Q, and v_Q since $z(b,\lambda(b))=0$. If we add the assumption that v_Q is differentiable, we could rewrite the right-hand side of (6.31) as

$$[P(b+h)-P(b)]\max_{b\le t\le b+h}|z(t,\lambda(b+h))|\frac{1}{h}\bigvee_b^{b+h}[Q+|\lambda(b+h)|W].$$

Using the assumption that P is continuous, we know that

$$P(b+h)-P(b)\to 0$$

as $h\to 0$. Now $v_w=W$ since W is nondecreasing with $W(a)=0$. So v_W is differentiable since W is assumed to be differentiable. Then using the added *assumption that v_Q is differentiable, we conclude that*

$$\bigvee_b^{b+h}[Q + |\lambda(b+h)|W] \to v_Q'(b) + \lambda(b)W'(b)$$

as $h \to 0$. Then we can conclude that $\lambda(b)$ is differentiable in the Neumann case only assuming that W, Q, and v_Q are differentiable (differentiability of P is not needed).

Having found conditions guaranteeing $\lambda(a)$ and $\lambda(b)$ to be continuous and differentiable, we now turn to $\lambda = \lambda(\alpha)$ and $\lambda = \lambda(\beta)$. In Theorem 5.5, λ was shown to depend continuously on α and β under the stated hypotheses. Now we prove differentiability under the same hypotheses, and again we derive differential expressions for the eigenvalues.

Theorem 6.11. *Under the hypotheses of Theorem 5.5, $\lambda(\alpha)$ and $\lambda(\beta)$ are differentiable. Moreover,*

$$\lambda'(\alpha) = -\frac{y(a,\lambda(\alpha))^2 + z(a,\lambda(\alpha))^2}{\int_a^b y(s,\lambda(\alpha))^2 dW(s)},$$

and

$$\lambda'(\beta) = \frac{y(b,\lambda(\beta))^2 + z(b,\lambda(\beta))^2}{\int_a^b y(s,\lambda(\beta))^2 dW(s)}.$$

Proof. We give the proof for $\lambda(\beta)$ and note that the proof is similar for $\lambda(\alpha)$.

Assume $\beta \neq \frac{\pi}{2}$ (the proof when $\beta = \frac{\pi}{2}$ is similar). The boundary conditions (6.3)- (6.4) can now be written as

$$y(b,\lambda(\beta)) = (\tan\beta)z(b,\lambda(\beta)), \tag{6.32}$$

$$y(b,\lambda(\beta+h)) = (\tan(\beta+h))z(b,\lambda(\beta+h)).$$

By Lemma 6.3 together with Lemma 6.4 (1),

$$- z(b,\lambda(\beta+h))y(b,\lambda(\beta)) + z(b,\lambda(\beta))y(b,\lambda(\beta+h)) =$$
$$[\lambda(\beta+h)-\lambda(\beta)]\int_a^b y(s,\lambda(\beta+h))y(s,\lambda(\beta))dW(s).$$

Substitution from (6.32) yields

$$- z(b,\lambda(\beta+h))(\tan\beta)z(b,\lambda(\beta)) + z(b,\lambda(\beta))(\tan(\beta+h))z(b,\lambda(\beta+h)) =$$
$$[\lambda(\beta+h)-\lambda(\beta)]\int_a^b y(s,\lambda(\beta+h))y(s,\lambda(\beta))dW(s).$$

The integral in this equation is nonzero since $\lambda(\beta)$ is continuous (Theorem 5.5 and Lemma 6.6). Therefore,

$$\frac{\lambda(\beta+h)-\lambda(\beta)}{h} = \frac{\tan(\beta+h)-\tan\beta}{h}\frac{z(b,\lambda(\beta+h))z(b,\lambda(\beta))}{\int_a^b y(s,\lambda(\beta+h))y(s,\lambda(\beta))dW(s)}. \tag{6.33}$$

We now prove that the limit of the right-hand side exists as h approaches zero, which implies that $\lambda'(\beta)$ exists. The denominator

$$\int_a^b y(s,\lambda(\beta+h))y(s,\lambda(\beta))dW(s) \to \int_a^b y^2(s,\lambda(\beta))dW(s) > 0$$

since $\lambda(\beta)$ is continuous and $y(t,\lambda)$ satisfies a Lipschitz condition in λ. Similarly, the numerator

$$z(b,\lambda(\beta+h))z(b,\lambda(\beta)) \to z^2(b,\lambda(\beta)).$$

Also, since $\frac{d}{dx}\tan x = \sec^2 x$,

$$\frac{\tan(\beta+h)-\tan\beta}{h} \to \sec^2\beta.$$

Then by (6.33), $\lambda'(\beta)$ exists, and

$$\lambda'(\beta) = \frac{(\sec^2\beta)z(b,\lambda(\beta))^2}{\int_a^b y(s,\lambda(\beta))^2 dW(s)}$$

$$= \frac{(1+\tan^2\beta) z(b,\lambda(\beta))^2}{\int_a^b y(s,\lambda(\beta))^2 dW(s)}$$
$$= \frac{y(b,\lambda(\beta))^2 + z(b,\lambda(\beta))^2}{\int_a^b y(s,\lambda(\beta))^2 dW(s)},$$

where the final equality follows from the boundary conditions (6.32). □

Finally, we examine the dependence of eigenvalues on the coefficients P, Q, and W. In Theorem 5.5, the continuous dependence was proven for each eigenvalue under the stated assumptions. Now we will prove that each eigenvalue is differentiable with respect to the coefficients, under the same hypotheses. Since the coefficients are functions, we use the Gateaux derivative, defined by

$$d\lambda_f(h) := \lim_{\epsilon\to 0} \frac{\lambda(f+\epsilon h) - \lambda(f)}{\epsilon}$$

for any f and h that are in the same class of functions.

Theorem 6.12. *Under the hypotheses of Theorem 5.5, the Gateaux derivatives for P, Q, and W exist, and*

1. $d\lambda_P(h) = -\dfrac{\int_a^b z(s,\lambda(P))^2 dh(s)}{\int_a^b y(s,\lambda(P))^2 dW(s)}$;
2. $d\lambda_Q(h) = \dfrac{\int_a^b y(s,\lambda(Q))^2 dh(s)}{\int_a^b y(s,\lambda(Q))^2 dW(s)}$;
3. $d\lambda_W(h) = -\lambda(W)\dfrac{\int_a^b y(s,\lambda(W))^2 dh(s)}{\int_a^b y(s,\lambda(W))^2 dW(s)}$.

Proof. 1. By the identity in Lemma 6.3,

$$-z(t,\lambda(P+\epsilon h))y(t,\lambda(P)) + z(t,\lambda(P))y(t,\lambda(P+\epsilon h))|_a^b =$$
$$[(\lambda(P+\epsilon h)) - \lambda(P)] \int_a^b dW(s) y(s,\lambda(P+\epsilon h)) y(s,\lambda(P))$$

$$+\int_a^b [d(P(s)+\epsilon h(s)) - dP(s)] z(s,\lambda(P+\epsilon h)) z(s,\lambda(P)).$$

Because α and β are constant, the left-hand side of this equation is zero by Lemma 6.4. So

$$[(\lambda(P+\epsilon h)) - \lambda(P)] \int_a^b dW(s) y(s,\lambda(P+\epsilon h)) y(s,\lambda(P)) +$$
$$\epsilon \int_a^b dh(s) z(s,\lambda(P+\epsilon h)) z(s,\lambda(P)) = 0.$$

The term $\int_a^b dW(s)y(s,\lambda(P+\epsilon h))y(s,\lambda(P))$ is nonzero for sufficiently small ϵ since $\lambda(P)$ is continuous (by Theorem 5.5) and $y(b,\lambda)$ satisfies a Lipschitz condition in λ. It follows that

$$\frac{\lambda(P+\epsilon h) - \lambda(P)}{\epsilon} = -\frac{\int_a^b dh(s) z(s,\lambda(P+\epsilon h)) z(s,\lambda(P))}{\int_a^b dW(s) y(s,\lambda(P+\epsilon h)) y(s,\lambda(P))}.$$

Letting $\epsilon \to 0$,

$$d\lambda_P(h) = -\frac{\int_a^b z(s,\lambda(P))^2 dh(s)}{\int_a^b y(s,\lambda(P))^2 dW(s)},$$

since $\lambda(P)$ is continuous and $z(t,\lambda)$ satisfies a Lipschitz condition in λ.

2. The proof is similar to the proof in part 3 below.

3. By the identity in Lemma 6.3,

$$-z(t,\lambda(W+\epsilon h))y(t,\lambda(W)) + z(t,\lambda(W))y(t,\lambda(W+\epsilon h))|_a^b =$$
$$\int_a^b [\lambda(W+\epsilon h) d(W(s)+\epsilon h(s)) - \lambda(W) dW(s)] y(s,\lambda(W+\epsilon h)) y(s,\lambda(W)).$$

Again, the left-hand side of this equation is zero. So

$$[\lambda(W+\epsilon h)) - \lambda(W)] \int_a^b dW(s) y(s,\lambda(W+\epsilon h)) y(s,\lambda(W)) =$$

$$- \lambda(W + \epsilon h)\epsilon \int_a^b dh(s) y(s, \lambda(W + \epsilon h)) y(s, \lambda(W)),$$

which implies that

$$\frac{\lambda(W + \epsilon h)) - \lambda(W)}{\epsilon} = -\lambda(W+\epsilon h) \frac{\int_a^b dh(s) y(s, \lambda(W + \epsilon h)) y(s, \lambda(W))}{\int_a^b dW(s) y(s, \lambda(W + \epsilon h)) y(s, \lambda(W))}.$$

Reasoning as in case 1, we have

$$d\lambda_W(h) = -\lambda(W) \frac{\int_a^b y(s, \lambda(W))^2 dh(s)}{\int_a^b y(s, \lambda(W))^2 dW(s)}.$$

□

Chapter 7

Eigenvalue Extremal Values

We now consider an application of Theorem 5.5 to problems of finding extremal values of eigenvalues. In such a problem, an eigenvalue is viewed as a function of one or more problem parameters, and we examine the supremum and infimum of the eigenvalue where the parameters vary over a predetermined set. The goal is to determine the existence of an element, that may or may not be included in this set, that attains the extremal values of the eigenvalue. Here we consider a specific example, the fourth order problem

$$\begin{cases} (ry'')'' + (py')' + qy = \lambda wy \qquad \text{on } [a,b] \\ y(a) = y'(a) = y(b) = y'(b) = 0 \end{cases}, \tag{7.1}$$

where the real, measurable coefficients p, q, r, and w satisfy

$$\begin{cases} p, q, w, \in L^1(a,b), \\ w, r \geq m \text{ on } [a,b], \\ r \leq \hat{m} \text{ on } [a,b] \end{cases} \tag{7.2}$$

for some constants $m, \hat{m} > 0$. We will consider the dependence of the smallest eigenvalue λ_0 on p, q, r, and w. We will show that under certain conditions there exist functions that achieve the extremal values of $\lambda_0(p, q, r, w)$.

First we rewrite problem (7.1) in the form used in the previous chapters. Define

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad Z = \begin{bmatrix} (ry'')' + py' \\ ry'' \end{bmatrix}.$$

Then we can rewrite (7.1) as

$$Y' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} Z, \tag{7.3}$$

$$Z' = \begin{bmatrix} \lambda w - q & 0 \\ 0 & -p \end{bmatrix} Y + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} Z \tag{7.4}$$

with boundary conditions

$$Y(a) = Y(b) = 0.$$

From this form, the problem can readily be written in the form of problem (5.1), with

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$D(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$P(t) = \begin{bmatrix} 0 & 0 \\ 0 & \int_a^t \frac{1}{r} \end{bmatrix},$$

$$Q(t) = \begin{bmatrix} -\int_a^t q & 0 \\ 0 & -\int_a^t p \end{bmatrix},$$

$$W(t) = \begin{bmatrix} -\int_a^t w & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easily verified that these coefficients satisfy the hypotheses given in (3.3).

The smallest eigenvalue for problem (7.1) is the minimum of a Rayleigh quotient [17, Theorem 11.1], defined as follows: Let

$$\mathcal{Q} = \{y : y, y' \in AC[a,b], y(a) = y'(a) = y(b) = y'(b) = 0, \int_a^b r(y'')^2 < \infty\},$$

where $AC[a,b]$ is defined to be the set of all functions that are absolutely continuous on $[a,b]$. Then

$$\lambda_0 = \min_{y \in \mathcal{Q}} \frac{\int_a^b [r(y'')^2 - p(y')^2 + qy^2]}{\int_a^b wy^2}. \tag{7.5}$$

Our goal is to prove the existence of extremizing functions for λ_0 over the set of coefficients with the condition that the integral of the coefficients have a uniform bound. We consider $\lambda_0(p,q)$ and $\lambda_0(r,w)$ as separate cases.

For the first case, fix w and r, and let

$$\mathcal{S}_1 = \{(p,q) : p, q \text{ satisfy } (7.2), \int_a^b |p| \le M_1, \int_a^b |q| \le M_2\}$$

for some positive constants M_1 and M_2. We will prove the existence of extremizing functions for $\lambda_0(p,q)$ over $\mathcal{S}_1$. We begin by finding bounds on λ_0.

To find an upper bound for λ_0 over $\mathcal{S}_1$, consider the function $y(t) = (t-a)^2(t-b)^2$. Notice that y satisfies the boundary conditions in (7.1). Since y is a polynomial, there exists a constant $k > 0$ such that

$$|y|, |y'|, |y''| \le k$$

on $[a,b]$. Then

$$\int_a^b r(y'')^2 \le k^2 \int_a^b r \le k^2 \hat{m}(b-a),$$

proving that $y \in \mathcal{Q}$. Now from (7.5),

$$\lambda_0 \le \frac{\int_a^b [r(y'')^2 - p(y')^2 + qy^2]}{\int_a^b wy^2}$$
$$\le \frac{k^2 \int_a^b r + k^2 \int_a^b [|p| + |q|]}{\int_a^b wy^2},$$
$$\le \frac{k^2 \hat{m}(b-a) + k^2(M_1 + M_2)}{\int_a^b wy^2},$$

which is an upper bound since $w > 0$ and y is not identically zero implies that $\int_a^b wy^2 > 0$.

To find a lower bound for λ_0 over $\mathcal{S}_1$, we first find a bound on y and y' using a result from Brown and Hinton [1, Theorem 2.3]. This theorem applies to a problem on an infinite interval, so we extend problem (7.3) to the interval $[a, \infty)$ by defining

$$y(t) = \begin{cases} y(t), & \text{if } a \le t \le b \\ 0, & \text{if } t > b \end{cases},$$

$$p(t) = \begin{cases} p(t), & \text{if } a \le t \le b \\ p(b), & \text{if } t > b \end{cases},$$

$$w(t) = \begin{cases} w(t), & \text{if } a \le t \le b \\ w(b), & \text{if } t > b \end{cases}.$$

We rewrite the theorem here without proof in a version that applies to this particular problem:

Theorem 7.1. *Let*

$\mathcal{D} = \{y : y$ *is real and measurable,* y' *is locally absolutely continuous on* $[a, \infty)$,

$$\int_a^\infty wy^2 < \infty, \int_a^\infty r(y'')^2 < \infty, \int_a^\infty r(y'')^2 \neq 0.\}$$

Let $j = 0$ *or* $j = 1$, *and let* $\beta = 3/4 - j$. *If*

$$\sup_{t \geq a, \epsilon > 0} \left(\frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{1}{w} \right)^{1/2} < \infty \tag{7.6}$$

and

$$\sup_{t \geq a, \epsilon > 0} \left(\frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{1}{r} \right)^{1/2} < \infty, \tag{7.7}$$

then there exists a constant K *such that for all* $y \in \mathcal{D}$,

$$\left| y^{(j)}(t) \right| \leq K \left(\int_a^\infty wy^2 \right)^{\beta/2} \left(\int_a^\infty r(y'')^2 \right)^{(1-\beta)/2}$$

for all $t \geq a$.

First we prove that (7.6) and (7.7) are satisfied for our problem. To prove that (7.6) holds, we have

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} \frac{1}{w} \leq \frac{1}{\epsilon} \left(\frac{\epsilon}{m} \right) = \frac{1}{m}$$

since $w \geq m$. The same argument proves that (7.7) is satisfied since $r \geq m$.

Since (7.6) and (7.7) hold, we conclude from Theorem 7.1, using $j = 0$ and $j = 1$, that there exists a constant K such that

$$\begin{cases} |y(t)| \leq K \left(\int_a^b wy^2 \right)^{3/8} \left(\int_a^b r(y'')^2 \right)^{1/8}, \\ |y'(t)| \leq K \left(\int_a^b wy^2 \right)^{1/8} \left(\int_a^b r(y'')^2 \right)^{3/8}. \end{cases} \tag{7.8}$$

Note that we need only integrate over $[a, b]$ since $y(t) = 0$ for $t \geq b$.

Before continuing to find a lower bound for λ_0, we normalize so that $\int_a^b wy^2 = 1$ for all solutions y. Now

$$\int_a^b [r(y'')^2 - p(y')^2 + qy^2] \geq$$

$$\int_a^b r(y'')^2 - \left(\int_a^b p_+\right) \max_{a \le t \le b} (y')^2 - \left(\int_a^b |q_-|\right) \max_{a \le t \le b} y^2, \quad (7.9)$$

where

$$p_+(t) = \max\{0, p(t)\},$$

$$q_-(t) = \min\{0, q(t)\}.$$

Now by (7.8) and (7.9), using the normalization $\int_a^b wy^2 = 1$, we have

$$\int_a^b [r(y'')^2 - p(y')^2 + qy^2] \ge$$

$$\int_a^b r(y'')^2 - \left(\int_a^b p_+\right) K^2 \left(\int_a^b r(y'')^2\right)^{3/4} - \left(\int_a^b |q_-|\right) K^2 \left(\int_a^b r(y'')^2\right)^{1/4} \ge$$

$$\int_a^b r(y'')^2 - m_1 \left(\int_a^b r(y'')^2\right)^{3/4} - m_2 \left(\int_a^b r(y'')^2\right)^{1/4},$$

where $m_1 = K^2 M_1$ and $m_2 = K^2 M_2$. Then

$$\int_a^b [r(y'')^2 - p(y')^2 + qy^2] \ge F(X),$$

where

$$X = \int_a^b r(y'')^2$$

and

$$F(X) = X - m_1 X^{3/4} - m_2 X^{1/4}.$$

All that remains to prove that λ_0 has a lower bound is to prove that $F(X)$ has a lower bound for $0 < X < \infty$. Observe that

$$F'(X) = 1 - \frac{3}{4} m_1 X^{-1/4} - \frac{1}{4} m_2 X^{-3/4}.$$

Then F is increasing when

$$\frac{3}{4} m_1 X^{-1/4} + \frac{1}{4} m_2 X^{-3/4} < 1.$$

This must be true for $X \geq c$ for some $c > 0$ since

$$\frac{3}{4}m_1 X^{-1/4} + \frac{1}{4}m_2 X^{-3/4} \to 0$$

as $X \to \infty$ (recall that $m_1, m_2 > 0$). Also note that $F(X)$ is continuous on $[0, c]$. So $F(X)$ is bounded on $[0, c]$ and is increasing on $[c, \infty)$. Therefore, $F(X)$ is bounded below on $[0, \infty)$.

Now that we know λ_0 is bounded over $\mathcal{S}_1$, we are ready to prove the existence of functions that attain the extremal values of λ_0 over $\mathcal{S}_1$. Let $(p_n, q_n) \subset \mathcal{S}_1$ be such that

$$\lambda_0(p_n, q_n) \to \sup_{\mathcal{S}_1} \lambda_0(p, q).$$

Let

$$Q_n = \begin{bmatrix} -\int_a^t q_n & 0 \\ 0 & -\int_a^t p_n \end{bmatrix}.$$

Then

$$\bigvee_a^b Q_n = \int_a^b [|p_n| + |q_n|] \leq M_1 + M_2.$$

Since the $\{Q_n\}$ are of uniform bounded variation, there exists a subsequence, which we take to be $\{Q_n\}$ itself, such that $Q_n \to \bar{Q}$ for some function $\bar{Q}$ of bounded variation, by Theorem 2.8. Now to use the same notation as in the previous chapters, we write $\lambda_0(Q_n) = \lambda_0(p_n, q_n)$. We will apply Theorem 5.5 after verifying that all of the hypothesis are satisfied. We already showed that $Q_n \to \bar{Q}$, satisfying condition (4.2). We also showed that λ_0 has a lower bound over $\mathcal{S}_1$. Also, the hypothesis of Lemma 5.3 are satisfied since the boundary conditions are the same for each problem in the sequence of problems. Then by Theorem 5.5 we conclude that

$$\lambda_0(Q_n) \to \lambda_0(\bar{Q}).$$

This proves the supremum of λ_0 over $\mathcal{S}_1$ is attained by some function $\bar{Q}$. However, we do not know that $\bar{Q}$ has the form

$$\bar{Q} = \begin{bmatrix} -\int_a^t \bar{q} & 0 \\ 0 & -\int_a^t \bar{p} \end{bmatrix}$$

for some $\bar{p}$ and $\bar{q}$ in $\mathcal{S}_1$. We know that $\bar{Q}$ is of bounded variation by Theorem 2.8, but we do not know that it is absolutely continuous. We do not expect the extremizing function to be an element of $\mathcal{S}_1$ without generalizing the set $\mathcal{S}_1$ to include a larger class of functions. A similar argument shows the existence of functions that attain the infimum.

For the second case, fix p and q, and let

$$\mathcal{S}_2 = \{(r, w) : \int_a^b \frac{1}{r} \leq M_3, \int_a^b w \leq M_4\}$$

for some positive constants M_3 and M_4. We will prove the existence of functions that attain the extremal values of λ_0 over $\mathcal{S}_2$. As in the first case, we begin by finding bounds on λ_0.

To find an upper bound for λ_0, consider the function $y(t) = (t-a)^2(t-b)^2$, which we already showed to be an element of $\mathcal{Q}$. From the first case, there exists a constant k such that

$$|y|, |y'|, |y''| \leq k$$

on $[a, b]$. Then from (7.5),

$$\begin{aligned} \lambda_0 &\leq \frac{\int_a^b [r(y'')^2 - p(y')^2 + qy^2]}{\int_a^b wy^2} \\ &\leq \frac{k^2 \int_a^b [r + |p| + |q|]}{m \int_a^b y^2}. \end{aligned}$$

Now we show this gives a finite bound. Since $r \leq \hat{m}$, $\int_a^b r \leq \hat{m}(b-a)$. Since $p, q \in L^1(a,b)$ are fixed, $\int_a^b [|p| + |q|]$ is finite. Finally, since y is not identically zero,$\int_a^b y^2 > 0$.

To find a lower bound for λ_0, we again use Theorem 7.1. As before, we extend the problem to the interval $[a, \infty)$, and we use the same argument to obtain

$$\int_a^b [r(y'')^2 - p(y')^2 + qy^2] \geq$$
$$\int_a^b r(y'')^2 - \left(\int_a^b p_+\right) K^2 \left(\int_a^b r(y'')^2\right)^{3/4} - \left(\int_a^b |q_-|\right) K^2 \left(\int_a^b r(y'')^2\right)^{1/4}.$$

Then we have

$$\int_a^b [r(y'')^2 - p(y')^2 + qy^2] \geq$$
$$\int_a^b r(y'')^2 - m_1 \left(\int_a^b r(y'')^2\right)^{3/4} - m_2 \left(\int_a^b r(y'')^2\right)^{1/4},$$

where, $m_1 = K^2 \int_a^b p_+$ and $m_2 = K^2 \int_a^b |q_-|$ (recall that p and q are fixed in this case). The remainder of the argument is the same as in the first case.

Now that we know λ_0 is bounded over $\mathcal{S}_2$, we are ready to prove the existence of functions that attain the extremal values of λ_0 over $\mathcal{S}_2$. Let $\{(r_n, w_n)\} \subset \mathcal{S}_2$ be such that

$$\lambda_0(r_n, w_n) \to \sup_{\mathcal{S}_2} \lambda_0(r, w).$$

Let

$$P_n = \begin{bmatrix} 0 & 0 \\ 0 & \int_a^t 1/r_n \end{bmatrix},$$

$$W_n = \begin{bmatrix} -\int_a^t w_n & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\bigvee_a^b P_n = \int_a^b \frac{1}{r_n} \leq M_3,$$

$$\bigvee_a^b W_n = \int_a^b w_n \leq M_4.$$

Since the $\{P_n\}$ and $\{W_n\}$ are of uniform bounded variation, there exists subsequences, which we take to be the sequences themselves, such that $P_n \to \bar{P}$ and $W_n \to \bar{W}$ for some functions $\bar{P}$ and $\bar{W}$ of bounded variation, by Theorem 2.8. Now to use the same notation as in the previous chapters, we write $\lambda_0(P_n, W_n) = \lambda_0(r_n, w_n)$.

We will apply Theorem 5.5 after verifying that all of the hypothesis are satisfied. For condition (4.2), we already showed that $P_n \to \bar{P}$ and $W_n \to \bar{W}$, but we must show that the convergence $P_n \to \bar{P}$ is uniform. We will use the Ascoli-Arzela Theorem, so we must show that the sequence $\{P_n\}$ is uniformly bounded and equicontinuous. The sequence is uniformly bounded since

$$|P_n(t)| = \int_a^t \frac{1}{r_n} \leq \frac{1}{m}$$

by assumption. To prove the sequence is equicontinuous, let $\epsilon > 0$, and let $\delta = m\epsilon$, where m is the uniform lower bound on r. Then for any t_1 and t_2 such that $|t_2 - t_1| < \delta$, we have

$$|P_n(t_2) - P_n(t_1)| = \left| \int_{t_1}^{t_2} \frac{1}{r_n} \right| \leq \frac{1}{m} |t_2 - t_1| < \epsilon$$

for all n. Since we already know that $P_n \to \bar{P}$ pointwise, we conclude that the convergence is uniform by the Ascoli-Arzela Theorem.

For the remaining hypotheses of Theorem 5.5, we already showed that λ_0 has a lower bound. Also, the hypothesis of Lemma 5.3 are satisfied since the boundary conditions are the same for each problem in the sequence of problems. Then by Theorem 5.5 we conclude that

$$\lambda_0(P_n, W_n) \to \lambda_0(\bar{P}, \bar{W}).$$

As in the first case, this proves the existence of functions that attain the supremum of λ_0 over $\mathcal{S}_2$. Again, we do not expect the extremizing functions to be elements of $\mathcal{S}_2$ without enlarging the class $\mathcal{S}_2$. A similar argument shows the existence of functions that attain the infimum.

Remark 7.2. *The hypothesis that the coefficient function r must have a uniform positive lower bound was used to prove that the convergence of the $\{P_n\}$ is uniform. We could have replaced this hypothesis with the assumption that there exists a sequence $\{r_n\}$ converging to the supremum of λ_0 over $\mathcal{S}_2$ such that for all i,*

$$\frac{1}{r_{i+1}} > \frac{1}{r_i}.$$

Then Dini's Theorem guarantees that the convergence of $\{P_n\}$ is uniform [2, p.150].

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Vita

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