5-2006

Optimal Control of Partial Di®erential Equations and Variational Inequalities

Volodymyr Hrynkiv
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by Volodymyr Hrynkiv entitled "Optimal Control of Partial Di®erential Equations and Variational Inequalities." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Suzanne Lenhart, Major Professor

We have read this dissertation and recommend its acceptance:

Sam Jordan, John Chiasson, Vladimir Protopopescu

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

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Optimal Control of Partial Differential Equations and Variational Inequalities

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Volodymyr Hrynkiv

May 2006

Acknowledgments

I would like to express my deepest gratitude to my advisor, Dr. Suzanne Lenhart for her indispensable guidance and support. She has always been a source of inspiration during my studies. I would like to thank the other members of my committee, Dr. John Chiasson, Dr. G. Samuel Jordan, and Dr. Vladimir Protopopescu for their time and critical review of my work. I would also like to thank Dr. David Adams for the invaluable discussions with regard to this research. The first part of this dissertation was in collaboration with Dr. Vladimir Protopopescu and the second part was in collaboration with Dr. David Adams.

The financial support during my graduate study came from a teaching assistantship which I held in the Department of Mathematics at the University of Tennessee. I am most grateful to this department.

I also thank my parents for being there for me for all these years.

Abstract

This dissertation deals with optimal control of mathematical models described by partial differential equations and variational inequalities. It consists of two parts. In the first part, optimal control of a two dimensional steady state thermistor problem is considered. The thermistor problem is described by a system of two nonlinear elliptic partial differential equations coupled with some boundary conditions. The boundary conditions show how the thermistor is connected to its surroundings. Based on physical considerations, an objective functional to be minimized is introduced and the convective boundary coefficient is taken to be a control. Existence and uniqueness of the optimal control are proven. To characterize this optimal control, the optimality system consisting of the state and adjoint equations is derived.

In the second part we consider a variational inequality of the obstacle type where the underlying partial differential operator is biharmonic. This kind of variational inequality arises in plasticity theory. It concerns the small transverse displacement of a plate when its boundary is fixed and the whole plate is subject to a pressure to lie on one side of an obstacle. We consider an optimal control problem where the state of the system is given by the solution of the variational inequality and the obstacle is taken to be a control. For a given target profile we want to find an obstacle such that the corresponding solution to the variational inequality is close the target profile while the norm of the obstacle does not get too large in the appropriate space. We prove existence of an optimal control and derive the optimality system by using approximation techniques. Namely, the variational inequality and the objective functional are approximated by a semilinear partial differential equation and a corresponding approximating functional, respectively.

Contents

Part I

Introduction

Optimal control of partial differential equations

The development of optimal control theory began in the early 1950's in response to great demands for modelling of real life applications in different branches of engineering, economics and applied mathematics. Optimal control of ordinary differential equations (ODEs) was developed by Pontryagin and his students Boltyanskii, Gamkrelidze and Mishchenko (see [45]) and is a fruitful interaction between developments in pure mathematics and applications, a synthesis of concrete special problems with general theories. It culminated in what is now known as the Pontryagin Maximum Principle which is considered to be the cornerstone of deterministic optimal control for finite dimensional systems. This principle is a set of necessary conditions that must hold along an optimal trajectory. While generalization to distributed systems is not immediate, a general framework for optimal control of partial differential equations (PDEs) was developed during 1960's by J.-L. Lions where such problems can be properly formulated and their solution systematically attempted [39]. Even though, in general, there is no universal "maximum principle" for such systems, the techniques for studying optimal control of PDEs are somewhat similar to the ones used for ODEs [37].

In control problems, variables are divided into two categories, state variables and control variables. As a simple example, consider a car moving along a straight line. At each moment of time, the state of the car can be characterized by two numbers: distance s covered by the car and its velocity v. These two quantities change in time at the will of the driver who controls the car by increasing or decreasing the engine's power F. Hence, in this particular example, the quantities s and v are state variables and F is the control variable. In optimal control problems one is concerned with adjusting controls, within prescribed restrictions, to achieve the desired objective.

Now we consider, in general terms, a typical setting of an optimal control problem with a single PDE state equation [39].

(i) A control β belongs to a set of admissible controls U.

(ii) The state $u = u(\beta)$ of the system solves the equation

$$\mathcal{L}u = f(u, \beta)$$

in an appropriate weak sense, where $\mathcal{L}$ is a known PDE operator.

(iii) The objective is to maximize or minimize a functional $J(\beta)$ defined in terms of $u(\beta)$ and β, i.e., find $\beta^* \in U$ such that

$$J(\beta^*) = \inf_{\beta \in U} J(\beta) \text{ or } J(\beta^*) = \sup_{\beta \in U} J(\beta).$$

The solution technique can be outlined as follows. First, if the state system is nonlinear, the existence and uniqueness of the solution to the state system are proven using an iteration scheme or a fixed point theorem. After obtaining *a priori* estimates in an appropriate solution space, the existence of an optimal control is shown through a maximizing or minimizing sequence argument. The next step is to derive the optimality system which consists of the state system coupled with the adjoint system together with a characterization of the optimal control. To obtain necessary conditions for the optimality system, the objective functional is differentiated with respect to the control. We calculate the Gâteaux derivative of J with respect to β in the direction ℓ at β^*, i.e.

$$\lim_{\varepsilon \to 0^+} \frac{J(\beta^* + \varepsilon \ell) - J(\beta^*)}{\varepsilon}.$$

As the objective functional J may contain the state variable u, it follows that u must also be differentiated with respect to the control β. The difference quotient

$$\lim_{\varepsilon \to 0} \frac{u(\beta^* + \varepsilon \ell) - u(\beta^*)}{\varepsilon}$$

should be shown to converge to some function ψ, called sensitivity, in the appropriate solution space. *A priori* estimates are also needed for the convergence of this difference quotient to ψ. As a result, the "sensitivity" function ψ solves a PDE which is linear in ψ.

The adjoint of the operator of the sensitivity equation is determined. Having analyzed the objective functional and used the relationships between the state and adjoint equations, an explicit form of the optimal control is determined by standard techniques involving choices of the variation function ℓ. This representation gives necessary conditions for the optimal control in terms of the state and adjoint variables. Existence of solution to the optimality system follows from the existence of solution to the state and adjoint system, coupled with the existence of optimal control. Finally, uniqueness of solutions to the optimality system will imply uniqueness of the optimal control. In the parabolic case, uniqueness for the optimality system holds only for sufficiently small time (see [21, 32, 35, 36]) whereas in the elliptic case we will have to assume "smallness" of boundary data and/or "largeness" of other coefficients [33, 34].

In part II a two dimensional steady-state thermistor problem is considered

$$
\begin{aligned}
&\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega,\\
&\Delta u + \sigma(u)|\nabla\varphi|^2 = 0 \text{ in } \Omega,\\
&\frac{\partial u}{\partial n} + \beta u = 0 \text{ on } \partial\Omega,\\
&\varphi = \varphi_0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $\varphi(x)$ is electric potential, $u(x)$ represents temperature, and $\sigma(u)$ is electrical conductivity. Here n denotes the outward unit normal and $\partial/\partial n = n \cdot \nabla$ is the normal derivative on $\partial\Omega$. The first equation represents the conservation of charge and the second equation describes the steady diffusion of heat in the presence of Joule heating due to the electric current. Boundary conditions show how the thermistor is connected thermally and electrically to its surroundings.

It is known that large temperature gradients may cause thermistor to crack. Numerical experiments indicate (see [22, 54]) that low values of the heat transfer coefficient β will result in small temperature variations. On the other hand, low values of the heat transfer coefficient lead to high operating temperatures of a thermistor. This motivated us to take the heat transfer coefficient as a control and to consider the optimal control problem of minimizing the heat transfer coefficient while keeping the operating temperature of the thermistor not too high. These physical considerations lead us to the following objective functional

$$J(\beta) = \int_\Omega u\,dx + \int_{\partial\Omega} \beta^2\,ds.$$

Denoting the set of admissible controls by

$$U_M = \{\beta \in L^\infty(\partial\Omega) : 0 < \lambda \leq \beta \leq M\}$$

we arrive at the following optimal control problem:

$$\text{Find } \beta^* \in U_M \text{ such that } J(\beta^*) = \min_{\beta \in U_M} J(\beta).$$

Optimal control of variational inequalities

Since its birth in the sixties, the theory of variational inequalities has received a great deal of attention from many mathematicians. It has experienced a phenomenal growth in recent years and is now considered a field in its own right. Variational inequalities arise in continuum and fluid mechanics, free boundary value problems, optimal control problems etc [8–10,23,29,39,46,49]. The simplest meaningful model problem which leads to variational inequalities is that of finding the shortest path between two given points on the plane with obstacles. This shortest path is decomposed into two parts: rectilinear motion where the path comes up to and joins the obstacle and parts of moving along the obstacle. The theory of variational inequalities deals with multi- and infinite-dimensional generalizations of such minimization problems. The minimizing function in this case takes values on the sections of the obstacle which are not known in advance and satisfies some partial differential equation off the obstacle. This leads to consideration of the partial differential equation with a free boundary. Such situations are typical for many problems in physics, mechanics, and engineering. The theory of variational inequalities gives general methods and techniques for solving these kinds of problems. It was created at the meeting point of functional analysis, partial differential equations, calculus of variations, and convex analysis. As a result a deep, meaningful and elegant theory was developed which finds numerous applications in various branches of science and engineering.

As an illustrative example, we consider an obstacle problem which leads to a variational inequality (see [7, 46]). In this obstacle problem, we need to determine the equilibrium position of an elastic membrane which

(i) passes through a closed curve Γ, the boundary of a planar domain;

(ii) lies above an obstacle of height ψ; and

(iii) is subject to the action of a vertical force of density τf, where τ is the elastic tension of the membrane and f is a given force.

The unknown quantity here is the vertical displacement u of the membrane. We assume in this example that $\psi \leq 0$ on Γ. Physical considerations lead to the following set of admissible displacements

$$K = \{v \in H_0^1(\Omega) : \, v \geq \psi \text{ a.e. in } \Omega\}.$$

The principle of minimal energy from mechanics states that the displacement u is a minimizer of the total energy,

$$u \in K : E(u) \leq E(v), \quad \forall\, v \in K, \tag{1}$$

where the energy functional is given by

$$E(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right) dx.$$

It can be shown that the minimization problem (1) has a unique solution. Assume that u is the configuration solving (1) and let $v \in K$. Since $u + \alpha(v - u) = \alpha v + (1 - \alpha)u \geq \alpha\psi + (1 - \alpha)\psi = \psi \in K$ for any $\alpha \in [0, 1]$, the function

$$h(\alpha) = E(u + \alpha(v - u))$$

has a minimum at $\alpha = 0$. Consequently,

$$0 \leq \lim_{\alpha \to 0^+} \frac{E(u + \alpha(v-u)) - E(u)}{\alpha} = \int_\Omega \nabla u \cdot \nabla(v-u)\,dx - \int_\Omega f(v-u)\,dx.$$

Hence, if u is the minimizer of (1) then it must also solve

$$u \in K : \int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq \int_\Omega f(v-u)\,dx, \quad \forall\, v \in K. \tag{2}$$

The inequality (2) is an example of an elliptic variational inequality. Conversely, if u satisfies the variational inequality, then

$$\begin{aligned} E(v) = E(u + (v-u)) = E(u) + \int_\Omega \nabla u \cdot \nabla(v-u)\,dx - \int_\Omega f(v-u)\,dx \\ + \frac{1}{2}\int_\Omega |\nabla(v-u)|^2\,dx \geq E(u), \quad \forall\, v \in K. \end{aligned}$$

It is possible to derive the corresponding boundary value problem for the above variational inequality. For this, assume $f \in C(\Omega)$, $\psi \in C(\Omega)$, and the solution of (2) satisfies $u \in H_0^1(\Omega)$. We take $v = u + \phi$ in (2), where $\phi \in C_0^\infty(\Omega)$ and $\phi \geq 0$ to get

$$\int_\Omega (\nabla u \cdot \nabla \phi - f\phi)\,dx \geq 0.$$

Thus, we see that u must satisfy the differential inequality in the weak sense

$$-\Delta u - f \geq 0 \text{ a.e. in } \Omega.$$

Now suppose that for some $x_0 \in \Omega$, $u(x_0) > \psi(x_0)$. Then there exists a neighborhood $U(x_0) \subset \Omega$ of x_0 and a number $\delta > 0$ such that $u(x) > \psi(x) + \delta$ for $x \in U(x_0)$. In (2) we choose $v = u \pm \delta\phi$ with any $\phi \in C_0^\infty(U(x_0))$ satisfying $\|\phi\|_\infty \leq 1$ to get

$$(-\Delta u - f)(x_0) = 0.$$

Summarizing, we can write a corresponding "strong" formulation of (2) in the following form

$$u - \psi \geq 0, \quad (-\Delta u - f) \geq 0, \quad (u - \psi)(-\Delta u - f) = 0 \text{ a.e. in } \Omega.$$

Consequently, the domain Ω is decomposed into two parts. On the first part, denoted by N, we have

$$u > \psi \text{ and } \quad (-\Delta u - f) = 0 \text{ in } N,$$

where the membrane has no contact with the obstacle. On the second part, denoted by I, we have

$$u = \psi \text{ and } \quad (-\Delta u - f) > 0 \text{ in } I,$$

and there is contact between the membrane and the obstacle. Observe that the region of contact, $\{x \in \Omega : u(x) = \psi(x) \text{ a.e.}\}$ is not known *a priori*. For this reason, it is called the free boundary of the problem.

We can consider an optimal control problem in the framework of the given variational inequality. For example, we may take f as the control. This means that we are designing the shape of the membrane by choosing a suitable external force load f. In this case, one may try to minimize the following objective functional

$$J(f) = \int_\Omega \left\{ |u - z|^2 + f^2 \right\} dx,$$

where z is the desired profile. This means that we want to make the shape of the membrane close to the desired shape z in some average "least squares" sense while keeping the L^2 norm of f not too large. It is also possible to take the obstacle ψ as a control and formulate an appropriate optimal control problem. Optimal control of sources and various coefficients in variational inequalities can be found in [9].

The general ideology of optimal control of variational inequalities is similar to that of optimal control of PDEs [9,10]. The main difference arises when one tries to derive necessary conditions for optimality since one can not differentiate directly the map $f \mapsto u$, solution of the variational inequality. As a result, a family of approximation problems is introduced and approximate necessary conditions are derived. The semilinear PDE which is employed to approximate an obstacle type variational inequality is standard and was used to approximate variational inequalities before control of variational inequalities started [16,29]. Finally, one needs to show the convergence (in the appropriate space) of the optimal controls for the approximate problems to the optimal control for the original problem. This allows one to derive necessary conditions for the optimality system to the original problem. This method of deriving necessary conditions via approximations was first introduced by Barbu [9,10].

In part III we consider the following problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and let $z \in L^2(\Omega)$ be a given target profile. Denote $V \stackrel{\text{def}}{=} H^3(\Omega) \cap H_0^2(\Omega)$. Given $\psi \in V$ define the closed convex set

$$K(\psi) = \{v \in H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$$

and consider the following obstacle problem for the biharmonic operator

$$\text{find } u \in K(\psi) \text{ such that } \int_\Omega \Delta u \, \Delta(v-u)\, dx \geq 0 \quad \forall\, v \in K(\psi). \tag{3}$$

Equivalent strong formulation of (3) is

$$\begin{aligned}
&\text{find } u \in H_0^2(\Omega) : \\
&u \geq \psi \text{ a.e. in } \Omega, \\
&\Delta^2 u \geq 0 \text{ a.e. in } \Omega, \\
&(\Delta^2 u)(u-\psi) = 0 \text{ a.e. in } \Omega.
\end{aligned}$$

We consider an optimal control problem for (3) in which we view ψ as the control and $u \stackrel{\text{def}}{=} T(\psi)$ as the corresponding state. We introduce an objective functional

$$J(\psi) = \frac{1}{2} \int_\Omega \left\{ |T(\psi) - z|^2 + |\nabla \Delta \psi|^2 \right\} dx$$

and formulate the following optimal control problem:

$$\text{Find } \psi^* \in V \text{ such that } J(\psi^*) = \inf_{\psi \in V} J(\psi).$$

In other words, for a given target profile $z \in L^2(\Omega)$, we want to find an obstacle $\psi^* \in V$ such that the corresponding solution $u^* = T(\psi^*)$ is close to z in L^2 norm while ψ^* is not too large in $H^3(\Omega)$.

Part II

Optimal Control of a Thermistor Problem

Chapter 1

Introduction

Thermistor is a generic name for a device made from materials whose electrical conductivity is highly dependent on temperature. Thermistors are often used as temperature control elements in a wide variety of military and industrial equipment ranging from space vehicles to air conditioning controllers. They are also used in the medical field for localized and general body temperature measurement, in meteorology for weather forecasting as well as in chemical industries as process temperature sensors. The advantages of thermistors as temperature measurement devices are low cost, high resolution, and flexibility in size and shape. The applications of thermistors can be summarized as follows [30]:

(i) temperature sensing and control: thermistors provide inexpensive and reliable temperature sensing for a wide temperature range;

(ii) thermal relay and switch: voltage regulation, surge protection;

(iii) indirect measurement of other parameters: when a thermistor is heated its rate of change of temperature depends on its surroundings. This property can be used to monitor other quantities such as liquid level and fluid flow; and

(iv) long-wavelength detector.

History of thermistors and more detailed accounts of their applications in electronics and other related industries can be found in [42].

We consider a two dimensional steady-state thermistor problem

$$
\begin{aligned}
&\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \text{ in } \Omega,\\
&\Delta u + \sigma(u)|\nabla\varphi|^2 = 0 \text{ in } \Omega,\\
&\frac{\partial u}{\partial n} + \beta u = 0 \text{ on } \partial\Omega,\\
&\varphi = \varphi_0 \text{ on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $\varphi(x)$ is electric potential, $u(x)$ represents temperature, and $\sigma(u)$ is electrical conductivity. Here n denotes the outward unit normal and $\partial/\partial n = n \cdot \nabla$ is the normal derivative on $\partial\Omega$. The first equation represents the conservation of charge and the second equation describes the steady diffusion of heat in the presence of Joule heating due to the electric current. Boundary conditions show how the thermistor is connected thermally and electrically to its surroundings. For a more detailed discussion about the physical justification of equations (1.1) the reader is referred to [22, 25, 47].

It is known that large temperature gradients may cause thermistor to crack. Numerical experiments indicate (see [22, 54]) that low values of the heat transfer coefficient β will result in small temperature variations. On the other hand, low values of the heat transfer coefficient lead to high operating temperatures of a thermistor. This motivated us to take the heat transfer coefficient as a control and to consider the optimal control problem of minimizing the heat transfer coefficient while keeping the operating temperature of the thermistor not too high. These physical considerations lead us to the following objective functional

$$J(\beta) = \int_\Omega u\, dx + \int_{\partial\Omega} \beta^2\, ds.$$

Now denoting the set of admissible controls by

$$U_M = \{\beta \in L^\infty(\partial\Omega) : 0 < \lambda \le \beta \le M\}$$

we arrive at the following optimal control problem.

$$\textbf{Problem C}. \text{ Find } \beta^* \in U_M \text{ such that } J(\beta^*) = \min_{\beta \in U_M} J(\beta). \tag{1.2}$$

Everywhere below we use the standard notation for Sobolev spaces. In what follows we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for each $p \in [1, \infty]$ and other norms will be clearly marked.

Theoretical analysis of both the steady-state and time-dependent thermistor equations with different types of boundary and initial conditions has received a significant amount of attention. The reader is referred to [6, 17–19, 22, 25, 47, 50, 51], where existence of weak solutions, existence/nonexistence of classical solutions, uniqueness and related regularity results were established in different settings with various assumptions on the coefficients. For example, existence of a weak solution to a stationary problem with Dirichlet boundary conditions was proven by Cimatti and Prodi in [18] whereas the time-dependent case in two dimensions was first considered by Cimatti in [17]. This restriction on the space dimension was eliminated by Shi, Shillor, and Xu in [47]. Asymptotic results for the time dependent thermistor problem can be found in [22]. So far, the only known optimal control paper on a thermistor problem is a time dependent case by Lee and Shilkin in [31] where the source is taken to be the control. Thus our work is the first work on optimal control of the thermistor problem for the steady state case.

In chapter 2 we derive *a priori* estimates under assumptions of small boundary data. In chapter 3 we prove existence of an optimal control. Also, in chapter 3 we explain why the space dimension has to be restricted to $N = 2$. The optimality system is derived and an optimal control is characterized in chapter 4. Uniqueness of the optimal control is proven in chapter 5. In chapter 6 we solve the optimality system numerically for a simple case of optimization of a parameter, i.e., when β is a constant.

Chapter 2

A Priori Estimates

We make the following assumptions:

(i) $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary;

(ii) $\sigma(s) \in C^1(\mathbb{R})$, $0 < C_1 \leq \sigma(s) \leq C_2$ for all $s \in \mathbb{R}$;

(iii) $\sigma(s)$ is Lipschitz: $|\sigma(s_1) - \sigma(s_2)| \leq K\,|s_1 - s_2|$ for all $s_1, s_2 \in \mathbb{R}$;

(iv) $\varphi_0\Big|_{\partial\Omega} \in W^{1,\infty}(\partial\Omega)$; and

(v) extending φ_0 to the whole domain Ω and using the same notation, i.e., $\varphi_0 \in W^{1,\infty}(\Omega)$, we assume that $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ is sufficiently small.

We will need a result due to N. Meyers [44].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and suppose that $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an operator given by*

$$A = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)$$

with $a_{ij} \in L^\infty(\Omega)$ and $\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha \sum_{i,j} \xi_i\xi_j$ a.e. in Ω for some $\alpha > 0$.

Consider the Dirichlet problem

$$Av = f, v \in H_0^1(\Omega), f \in H^{-1}(\Omega). \tag{2.1}$$

Then there exists $r > 2$ (which depends on α, L^∞ norm of the coefficients a_{ij}'s, Ω, and on the dimension) such that if $f \in W_0^{-1,r}(\Omega)$ then $v \in W_0^{1,r}(\Omega)$ where v solves (2.1) and satisfies

$$\|v\|_{W_0^{1,r}(\Omega)} \leq C\|f\|_{W_0^{-1,r}(\Omega)} \tag{2.2}$$

and C depends on the same quantities as r.

Proof. See [12, 44, 48]. □

To derive a weak formulation for (1.1) we follow Lemma 1 from [25] and the discussion preceding it. We say that $\{u, \varphi\}$ is a weak solution to (1.1), if

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta uv \, ds = \int_\Omega \sigma(u)|\nabla\varphi|^2 v \, dx \quad \forall\, v \in H^1(\Omega), \\ &\int_\Omega \sigma(u)\nabla\varphi \cdot \nabla w \, dx = 0 \quad \varphi_0 - \varphi \in H_0^1(\Omega), \forall\, w \in H_0^1(\Omega). \end{aligned} \tag{2.3}$$

The quadratic term on the right hand side of the first equation in (2.3) creates a difficulty because $|\nabla\varphi|^2$ is only known to belong to $L^1(\Omega)$. Therefore, for an arbitrary $v \in C^1(\bar{\Omega})$, take $\zeta = (\varphi - \varphi_0)v \in H_0^1(\Omega)$ as a test function in the second equation of (2.3). Then

$$\int_\Omega \sigma(u)\big\{|\nabla\varphi|^2 v - v\nabla\varphi \cdot \nabla\varphi_0 + (\varphi - \varphi_0)\nabla\varphi \cdot \nabla v\big\} \, dx = 0.$$

By the density argument, the weak formulation of (1.1) is: find $u \in H^1(\Omega)$ and $\varphi \in H^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta uv \, ds = \int_\Omega (\varphi_0 - \varphi)\sigma(u)\nabla\varphi \cdot \nabla v \, dx + \int_\Omega (\sigma(u)\nabla\varphi \cdot \nabla\varphi_0)\, v \, dx \\ &\int_\Omega \sigma(u)\nabla\varphi \cdot \nabla w \, dx = 0 \qquad \varphi_0 - \varphi \in H_0^1(\Omega), \forall\, w \in H_0^1(\Omega), \forall\, v \in H^1(\Omega). \end{aligned} \tag{2.4}$$

We explain why the terms on the right hand side of the first equation in (2.4) make sense. By Theorem 2.1, there exists $r > 2$ such that $\varphi \in W^{1,r}(\Omega)$. Therefore, since $\nabla\varphi \in L^r(\Omega)$ and $\nabla\varphi_0 \in L^2(\Omega)$, it follows that there exists s' such that $\nabla\varphi \cdot \nabla\varphi_0 \in L^{s'}(\Omega)$ and

$$\frac{1}{s'} = \frac{1}{2} + \frac{1}{r}. \tag{2.5}$$

Next, since $\Omega \subset \mathbb{R}^2$ it follows $v \in H^1(\Omega) \subset\subset L^s(\Omega)$ for $s \in [1, \infty)$, conjugate of s' in (2.5). This implies that the integral

$$\int_\Omega (\sigma(u)\nabla\varphi \cdot \nabla\varphi_0)\, v\, dx$$

in (2.4) makes sense since $\sigma(u)$ is bounded and $(\sigma(u)\nabla\varphi \cdot \nabla\varphi_0) \in L^{s'}(\Omega)$, where s' is the conjugate of s:

$$\frac{1}{s'} + \frac{1}{s} = 1. \tag{2.6}$$

If we denote $\tilde{M} := \operatorname{ess\,sup}_{\partial\Omega} |\varphi_0|$ then by the weak maximum principle (see formula (16) in [25])

$$\sup_\Omega |\varphi| \leq \tilde{M}. \tag{2.7}$$

Therefore the integral

$$\int_\Omega (\varphi_0 - \varphi)\sigma(u)\nabla\varphi \cdot \nabla v\, dx$$

in (2.4) also makes sense. Thus, any solution of (2.3) is a solution to (2.4) and vice versa.

Note that $r > 2$ is chosen from the Meyers estimate, then s' is determined by (2.5), and finally s is obtained from (2.6). Also, observe that $s' < s$.

Existence of solution to (2.4) was proven by Howison, Rodrigues, and Shillor in [25] by using Schauder's fixed point theorem. It is also shown in [25] that the solution is unique provided the boundary data are sufficiently "small". Namely, $\lambda > 0$ implies (see [7, 53])

$$\hat{\kappa}\|u\|_s^2 \leq \int_\Omega |\nabla v|^2\, dx + \lambda \int_{\partial\Omega} v^2\, ds. \tag{2.8}$$

Assume that (u, φ) is a solution to the state system satisfying

$$\|\nabla\varphi\|_r \leq \Phi < \infty, \tag{2.9}$$

where Φ depends on $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$. If the boundary data are sufficiently small, i.e., if

$$K\Phi^2\Big(1 + \frac{2C_2}{C_1}\Big) < \hat{\kappa}, \tag{2.10}$$

where $\hat{\kappa} = \hat{\kappa}(s)$ is the constant from (2.8) and Φ is the constant from (2.9), then the solution is unique. Note that we will show next that φ satisfies (2.9) with a bound depending on $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$.

In what follows, given $\beta \in U_M$, we denote the solution of (2.4) by $u(\beta)$ and $\varphi(\beta)$. Note that solutions to (1.1) satisfy $u \geq 0$ on $\bar{\Omega}$ as a result of the maximum principle. Now we proceed to the derivation of *a priori* estimates.

Theorem 2.2. *Let $\beta \in U_M$ be given. Then $u(\beta)$ and $\varphi(\beta)$ solving* (2.4) *satisfy*

$$
\begin{aligned}
&\|\varphi\|_{W^{1,r}(\Omega)} \leq \Phi \text{ for some } r > 2, \text{ and} \\
&\|u\|_{H^1(\Omega)} \leq \tilde{C},
\end{aligned}
\tag{2.11}
$$

where Φ and $\tilde{C}$ are some positive constants.

Proof. We show the estimate for φ first. We are given the solution of a nonhomogeneous Dirichlet problem

$$
\begin{aligned}
\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \qquad &\text{in } \Omega, \\
\varphi = \varphi_0 \quad &\text{on } \partial\Omega.
\end{aligned}
\tag{2.12}
$$

Note that because of the assumption (i) we treat $\sigma(u)$ in (2.12) just as a bounded coefficient. Consider the following Dirichlet problem with zero boundary data

$$
\begin{aligned}
-\nabla \cdot (\sigma(u)\nabla\tilde{\varphi}) = \nabla \cdot (\sigma(u)\nabla\varphi_0) \quad &\text{in } \Omega, \\
\tilde{\varphi} = 0 \quad &\text{on } \partial\Omega.
\end{aligned}
\tag{2.13}
$$

Since the right hand side of (2.13) is in $H^{-1}(\Omega)$, by the standard theory for elliptic equations in divergence form, there exists $\tilde{\varphi} = (\varphi - \varphi_0) \in H_0^1(\Omega)$ that solves (2.13).

Since for any given $1 < p < \infty$, the mapping $\text{div}\colon (L^p(\Omega))^2 \to W^{-1,p}(\Omega)$ defined by

$$
\phi \mapsto \nabla \cdot \phi \tag{2.14}
$$

is onto, it provides $W^{-1,p}(\Omega)$ space with a "quotient" norm associated with (2.14), namely

$$\|f\|_{W^{-1,p}(\Omega)} = \inf_{\nabla\cdot g=f} \|g\|_{(L^p(\Omega))^2}, \tag{2.15}$$

(see expressions (4.10) and (4.11) in [12]). Taking into account the above facts it follows that $\nabla \cdot (\sigma(u)\nabla\varphi_0) \in W_0^{-1,q}(\Omega)$ for any $q \in [1,\infty)$ since $\nabla\varphi_0 \in L^\infty(\Omega)$.

By Theorem 2.1, there exists $r > 2$ such that $\tilde{\varphi} \in W_0^{1,r}(\Omega)$ and

$$\begin{aligned} \|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)} &\leq C\|\nabla \cdot (\sigma(u)\varphi_0)\|_{W_0^{-1,r}(\Omega)} \\ &\leq C\|\sigma(u)\nabla\varphi_0\|_r \\ &\leq CC_2\|\nabla\varphi_0\|_r \end{aligned}$$

where we have used (2.15). Since

$$\|\varphi\|_{W^{1,r}(\Omega)} \leq \|\varphi - \varphi_0\|_{W^{1,r}(\Omega)} + \|\varphi_0\|_{W^{1,r}(\Omega)}$$

and

$$\|\varphi - \varphi_0\|_{W^{1,r}(\Omega)} \equiv \|\tilde{\varphi}\|_{W^{1,r}(\Omega)} \leq C'\|\tilde{\varphi}\|_{W_0^{1,r}(\Omega)} \leq C'CC_2\|\nabla\varphi_0\|_r,$$

we obtain

$$\|\varphi\|_{W^{1,r}(\Omega)} \leq C'CC_2\|\nabla\varphi_0\|_r + \|\varphi_0\|_{W^{1,r}(\Omega)}.$$

We also have

$$\begin{aligned} \|\nabla\varphi_0\|_r &\leq C_3\|\nabla\varphi_0\|_\infty \\ \|\varphi_0\|_{W^{1,r}(\Omega)} &\leq C_4\|\varphi_0\|_{W^{1,\infty}(\Omega)} \end{aligned}$$

whence

$$\|\varphi\|_{W^{1,r}(\Omega)} \leq \Phi \text{ for some } r > 2, \tag{2.16}$$

where

$$\Phi \stackrel{\text{def}}{=} C'CC_2C_3\|\nabla\varphi_0\|_\infty + C_4\|\varphi_0\|_{W^{1,\infty}(\Omega)}. \tag{2.17}$$

Thus, (2.16) gives the desired estimate for φ.

We show that $\|u\|_{H^1(\Omega)} \leq \tilde{C}$. Use $u \in H^1(\Omega)$ as a test function in the first equation of the weak formulation (2.4) to get

$$\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} \beta u^2\,ds = \int_\Omega (\varphi_0 - \varphi)\sigma(u)\nabla\varphi\cdot\nabla u\,dx + \int_\Omega \sigma(u)u\nabla\varphi\cdot\nabla\varphi_0\,dx \tag{2.18}$$

Taking into account that $\beta(x) \geq \lambda > 0$ we have

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\,dx + \lambda\int_{\partial\Omega} u^2\,ds \leq \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} \beta u^2\,ds\\
&= \int_\Omega (\varphi_0 - \varphi)\sigma(u)\nabla\varphi\cdot\nabla u\,dx + \int_\Omega \sigma(u)u\nabla\varphi\cdot\nabla\varphi_0\,dx\\
&\leq \int_\Omega \sigma(u)|\varphi_0 - \varphi|\cdot|\nabla\varphi|\cdot|\nabla u|\,dx + \int_\Omega \sigma(u)|u|\cdot|\nabla\varphi|\cdot|\nabla\varphi_0|\,dx\\
&\leq C_2\int_\Omega |\varphi_0 - \varphi|\cdot|\nabla\varphi|\cdot|\nabla u|\,dx + C_2\int_\Omega |u|\cdot|\nabla\varphi|\cdot|\nabla\varphi_0|\,dx\\
&\leq 2C_2\tilde{M}\int_\Omega |\nabla\varphi|\cdot|\nabla u|\,dx + C_2\|\nabla\varphi_0\|_\infty\int_\Omega |u|\cdot|\nabla\varphi|\,dx\\
&\leq 2C_2\tilde{M}\|\nabla\varphi\|_2\cdot\|\nabla u\|_2 + C_2\|\nabla\varphi_0\|_\infty\|u\|_2\cdot\|\nabla\varphi\|_2\\
&\leq 2C_2\tilde{M}C_5\Phi\|\nabla u\|_2 + C_2\|\nabla\varphi_0\|_\infty C_5\Phi\|u\|_2\\
&\leq \tilde{C}_1\|u\|_{H^1(\Omega)}
\end{aligned}$$

where we used (2.7), $\|\nabla\varphi\|_2 \leq C_5\|\nabla\varphi\|_r$ and $\tilde{C}_1 \equiv \max(2C_2\tilde{M}C_5\Phi, C_2\|\nabla\varphi_0\|_\infty C_5\Phi)$. We obtained

$$\int_\Omega |\nabla u|^2\,dx + \lambda\int_{\partial\Omega} u^2\,ds \leq \tilde{C}_1\|u\|_{H^1(\Omega)}. \tag{2.19}$$

It can be shown (for example, see [7, 53]) that the quantity

$$\|v\|_*^2 \stackrel{\text{def}}{=} \int_\Omega |\nabla v|^2\,dx + \lambda\int_{\partial\Omega} v^2\,ds$$

defines a norm on $H^1(\Omega)$ which is equivalent to $\|\cdot\|_{H^1(\Omega)}$ norm. Therefore, there exists $k > 0$ such that

$$k\|u\|^2_{H^1(\Omega)} \leq \|u\|^2_* \leq \tilde{C}_1\|u\|_{H^1(\Omega)}, \tag{2.20}$$

whence (2.19) and (2.20) give

$$\|u\|_{H^1(\Omega)} \leq \tilde{C},$$

where $\tilde{C} \equiv \tilde{C}_1/k$. $\square$

Chapter 3

Existence of an Optimal Control

Having obtained *a priori* estimates we proceed to the proof of existence of an optimal control.

Theorem 3.1. *There exists a solution to Problem C.*

Proof. Using the estimate from Theorem 2.2,

$$J(\beta) = \int_\Omega u\,dx + \int_{\partial\Omega} \beta^2\,ds \geq \int_\Omega u\,dx \geq -\int_\Omega |u|\,dx \geq -\mu,$$

where $\mu > 0$ is some constant, it follows that we can choose a minimizing sequence $\{\beta_n\}_{n=1}^{\infty} \subset U_M$ such that

$$\lim_{n\to\infty} J(\beta_n) = \inf_{\beta\in U_M} J(\beta).$$

Let $u_n = u(\beta_n)$ and $\varphi_n = \varphi(\beta_n)$ be the corresponding solutions to

$$\begin{aligned}
&\int_\Omega \nabla u_n \nabla v\,dx + \int_{\partial\Omega} \beta_n u_n v\,ds = \int_\Omega (\varphi_0 - \varphi_n)\,\sigma(u_n)\nabla\varphi_n\nabla v\,dx + \int_\Omega \sigma(u_n)\nabla\varphi_n\nabla\varphi_0\,v\,dx \\
&\int_\Omega \sigma(u_n)\nabla\varphi_n\cdot\nabla w\,dx = 0 \qquad \varphi_0 - \varphi_n \in H_0^1(\Omega), \forall\, w \in H_0^1(\Omega), \forall\, v \in H^1(\Omega).
\end{aligned} \tag{3.1}$$

By Theorem 2.2 we have

$$\|u_n\|_{H^1(\Omega)} \leq C,$$
$$\|\varphi_n\|_{W^{1,r}(\Omega)} \leq C \text{ for all } n,$$

where $C > 0$ denotes a generic constant which does not depend on n. Therefore, on a subsequence

$$u_n \overset{w}{\rightharpoonup} u^* \text{ in } H^1(\Omega),$$
$$\varphi_n \overset{w}{\rightharpoonup} \varphi^* \text{ in } W^{1,r}(\Omega).$$

Also, $\beta_n \in U_M$ for all n implies that $\beta_n \in L^\infty(\partial\Omega)$ and $\|\beta_n\|_{L^\infty(\partial\Omega)} \leq M$ for all n. Hence, on a subsequence $\beta_n \overset{w*}{\rightharpoonup} \tilde{\beta}$ in $L^\infty(\partial\Omega)$, i.e.,

$$\int_{\partial\Omega} \beta_n w \, ds \to \int_{\partial\Omega} \tilde{\beta} w \, ds \quad \forall\, w \in L^1(\partial\Omega) \text{ as } n \to \infty. \tag{3.2}$$

On the other hand, $\beta_n \in L^2(\partial\Omega)$ and $\|\beta_n\|_{L^2(\partial\Omega)} \leq M$ for each n implies $\beta_n \overset{w}{\rightharpoonup} \beta^*$ in $L^2(\partial\Omega)$ which means

$$\int_{\partial\Omega} \beta_n v \, ds \to \int_{\partial\Omega} \beta^* v \, ds \quad \forall\, v \in L^2(\partial\Omega) \text{ as } n \to \infty. \tag{3.3}$$

We justify that $\tilde{\beta} = \beta^*$ a.e. Indeed, let $v \in L^2(\partial\Omega)$ be arbitrary. Then by (3.2) and (3.3)

$$\Big| \int_{\partial\Omega} (\tilde{\beta} - \beta^*) v \, ds \Big| \leq \Big| \int_{\partial\Omega} (\tilde{\beta} - \beta_n) v \, ds \Big| + \Big| \int_{\partial\Omega} (\beta_n - \beta^*) v \, ds \Big| \to 0 \text{ as } n \to \infty.$$

Thus

$$\int_{\partial\Omega} (\tilde{\beta} - \beta^*) v \, ds = 0 \quad \forall\, v \in L^2(\partial\Omega)$$

whence $\tilde{\beta} = \beta^*$ a.e. Note that $\beta^* \in U_M$ because U_M being a closed convex set of a Hilbert space $\mathcal{H}$ implies that U_M is closed with respect to weak convergence.

Since $r > 2$, we have $W^{1,r}(\Omega) \subset\subset C(\bar{\Omega})$ and $W^{1,r}(\Omega) \subset H^1(\Omega) \subset\subset L^s(\Omega)$, where s is conjugate of s' (see (2.5) and (2.6)). Hence, on a subsequence

$$\varphi_n \xrightarrow{s} \varphi^* \text{ in } C(\bar{\Omega}),$$
$$u_n \xrightarrow{s} u^* \text{ in } L^s(\Omega).$$

All in all, we obtain

$$\begin{aligned} u_n &\xrightarrow{s} u^* \text{ in } L^s(\Omega), \\ \nabla u_n &\overset{w}{\rightharpoonup} \nabla u^* \text{ in } L^2(\Omega), \\ \varphi_n &\xrightarrow{s} \varphi^* \text{ in } C(\bar{\Omega}), \\ \nabla \varphi_n &\overset{w}{\rightharpoonup} \nabla \varphi^* \text{ in } L^r(\Omega), \\ \beta_n &\overset{w}{\rightharpoonup} \beta^* \text{ in } L^2(\partial\Omega), \text{ and} \\ \beta_n &\overset{w*}{\rightharpoonup} \beta^* \text{ in } L^\infty(\partial\Omega). \end{aligned} \tag{3.4}$$

Next, we want to show that $u^* = u(\beta^*)$ and $\varphi^* = \varphi(\beta^*)$ solve (2.4) with control β^*, i.e., pass to the limit as $n \to \infty$ in (3.1). Notice that (2.6) and (2.5) imply

$$\frac{1}{s} + \frac{1}{r} + \frac{1}{2} = 1. \tag{3.5}$$

From (3.4) it is immediate that

$$\int_\Omega \nabla u_n \cdot \nabla v \, dx \to \int_\Omega \nabla u^* \cdot \nabla v \, dx \quad \text{as } n \to \infty. \tag{3.6}$$

We show that

$$\int_{\partial\Omega} \beta_n u_n v \, ds \to \int_{\partial\Omega} \beta^* u^* v \, ds \quad \text{as } n \to \infty. \tag{3.7}$$

Indeed, by the trace inequality $u^* \in H^1(\Omega)$ implies $u^* \in L^2(\partial\Omega)$. Similarly, $v \in L^2(\partial\Omega)$. Hence $u^* v \in L^1(\partial\Omega)$ and we get

$$\begin{aligned}
&\Big| \int_{\partial\Omega} \beta_n u_n v\, ds - \int_{\partial\Omega} \beta^* u^* v\, ds \Big| \le \int_{\partial\Omega} |\beta_n u_n v - \beta_n u^* v|\, ds + \Big| \int_{\partial\Omega} \beta_n u^* v - \beta^* u^* v\, ds \Big| \\
&\le M \int_{\partial\Omega} |u_n - u^*| \cdot |v|\, ds + \Big| \int_{\partial\Omega} (\beta_n - \beta^*) u^* v\, ds \Big| \le M \|u_n - u^*\|_{L^2(\partial\Omega)} \cdot \|v\|_{L^2(\partial\Omega)} \\
&+ \Big| \int_{\partial\Omega} (\beta_n - \beta^*) u^* v\, ds \Big| \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

where we have taken into account that $H^1(\Omega) \subset\subset L^2(\partial\Omega)$ and $\beta_n \xrightarrow{w*} \beta^*$ in $L^\infty(\partial\Omega)$. This proves (3.7).

Next, we show

$$\int_\Omega \sigma(u_n)\, v \nabla\varphi_n \cdot \nabla\varphi_0\, dx \to \int_\Omega \sigma(u^*)\, v \nabla\varphi^* \cdot \nabla\varphi_0\, dx \quad \text{as } n \to \infty. \tag{3.8}$$

We have

$$\begin{aligned}
&\Big| \int_\Omega \sigma(u_n)\, v\nabla\varphi_n \cdot \nabla\varphi_0\, dx - \int_\Omega \sigma(u^*)\, v\nabla\varphi^* \cdot \nabla\varphi_0\, dx \Big| \le \Big| \int_\Omega [\sigma(u_n) - \sigma(u^*)]\, v\nabla\varphi_n \cdot \nabla\varphi_0\, dx \Big| \\
&+ \Big| \int_\Omega \sigma(u^*)\, v\, (\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla\varphi_0\, dx \Big| \le K \|\nabla\varphi_0\|_\infty \int_\Omega |u_n - u^*| \cdot |v| \cdot |\nabla\varphi_n|\, dx \\
&+ \Big| \int_\Omega \sigma(u^*)\, v\, (\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla\varphi_0\, dx \Big| \le K \|\nabla\varphi_0\|_\infty \cdot \|u_n - u^*\|_s \cdot \|v\|_2 \cdot \|\nabla\varphi_n\|_r \\
&+ \Big| \int_\Omega \sigma(u^*)\, v\, (\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla\varphi_0\, dx \Big| \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

since $u_n \xrightarrow{s} u^*$ in $L^s(\Omega)$, $\nabla\varphi \xrightarrow{w} \nabla\varphi^*$ in $L^2(\Omega)$, and $\|\nabla\varphi_n\|_r \le C$. This completes the proof of (3.8).

Now we show

$$\int_\Omega (\varphi_0 - \varphi_n)\, \sigma(u_n) \nabla\varphi_n \cdot \nabla v\, dx \to \int_\Omega (\varphi_0 - \varphi^*)\, \sigma(u^*) \nabla\varphi^* \cdot \nabla v\, dx \quad \text{as } n \to \infty. \tag{3.9}$$

Indeed, we can write

$$\begin{aligned}
&\Big|\int_\Omega (\varphi_0-\varphi_n)\,\sigma(u_n)\nabla\varphi_n\cdot\nabla v\,dx-\int_\Omega(\varphi_0-\varphi^*)\,\sigma(u^*)\nabla\varphi^*\cdot\nabla v\,dx\Big|\\
&\le\Big|\int_\Omega[\sigma(u_n)\,(\varphi_0-\varphi_n)\nabla\varphi_n\cdot\nabla v-\sigma(u^*)\,(\varphi_0-\varphi_n)\nabla\varphi^*\cdot\nabla v]\,dx\Big|\\
&+\Big|\int_\Omega[\sigma(u^*)\,(\varphi_0-\varphi_n)\nabla\varphi^*\cdot\nabla v-\sigma(u^*)\,(\varphi_0-\varphi^*)\nabla\varphi^*\cdot\nabla v]\,dx\Big|\stackrel{\text{def}}{=}A+B.
\end{aligned}$$

We deal with the term B first:

$$\begin{aligned}
B&\stackrel{\text{def}}{=}\Big|\int_\Omega[\sigma(u^*)\,(\varphi_0-\varphi_n)\nabla\varphi^*\cdot\nabla v-\sigma(u^*)\,(\varphi_0-\varphi^*)\nabla\varphi^*\cdot\nabla v]\,dx\Big|\\
&=\Big|\int_\Omega\sigma(u^*)\nabla\varphi^*\cdot\nabla v\,[(\varphi_0-\varphi_n)-(\varphi_0-\varphi^*)]\,dx\Big|\\
&=\Big|\int_\Omega\sigma(u^*)\nabla\varphi^*\cdot\nabla v\,(\varphi^*-\varphi_n)\,dx\Big|\le C_2\int_\Omega|\nabla\varphi^*|\cdot|\nabla v|\cdot|\varphi^*-\varphi_n|\,dx\\
&\le C_2\|\varphi^*-\varphi_n\|_{C(\bar\Omega)}\cdot\|\nabla v\|_2\cdot\|\nabla\varphi^*\|_2\to 0\quad\text{as } n\to\infty.
\end{aligned}$$

Now we look at the term A:

$$\begin{aligned}
A&\stackrel{\text{def}}{=}\Big|\int_\Omega[\sigma(u_n)\,(\varphi_0-\varphi_n)\nabla\varphi_n\cdot\nabla v-\sigma(u^*)\,(\varphi_0-\varphi_n)\nabla\varphi^*\cdot\nabla v]\,dx\Big|\\
&=\Big|\int_\Omega(\varphi_0-\varphi_n)(\sigma(u_n)\,\nabla\varphi_n-\sigma(u^*)\,\nabla\varphi^*)\cdot\nabla v\,dx\Big|\\
&\le\Big|\int_\Omega(\varphi_0-\varphi_n)(\sigma(u_n)\,\nabla\varphi_n-\sigma(u^*)\,\nabla\varphi_n)\cdot\nabla v\,dx\Big|\\
&+\Big|\int_\Omega(\varphi_0-\varphi_n)(\sigma(u^*)\,\nabla\varphi_n-\sigma(u^*)\,\nabla\varphi^*)\cdot\nabla v\,dx\Big|\\
&=\Big|\int_\Omega(\varphi_0-\varphi_n)(\sigma(u_n)-\sigma(u^*))\nabla\varphi_n\cdot\nabla v\,dx\Big|\\
&+\Big|\int_\Omega(\varphi_0-\varphi_n)\,\sigma(u^*)(\nabla\varphi_n-\nabla\varphi^*)\cdot\nabla v\,dx\Big|\stackrel{\text{def}}{=}A_1+A_2,
\end{aligned}$$

where

$$\begin{aligned}
A_1&\stackrel{\text{def}}{=}\Big|\int_\Omega(\varphi_0-\varphi_n)(\sigma(u_n)-\sigma(u^*))\nabla\varphi_n\cdot\nabla v\,dx\Big|\\
&\le\int_\Omega|\varphi_0-\varphi_n|\cdot|\sigma(u_n)-\sigma(u^*)|\cdot|\nabla\varphi_n|\cdot|\nabla v|\,dx\\
&\le 2\tilde MK\int_\Omega|u_n-u^*|\cdot|\nabla\varphi_n|\cdot|\nabla v|\,dx\\
&\le 2\tilde MK\|u_n-u^*\|_s\cdot\|\nabla\varphi_n\|_r\cdot\|\nabla v\|_2\to 0\quad\text{as } n\to\infty
\end{aligned}$$

because $\|u_n - u^*\|_s \to 0$ as $n \to \infty$ and $\|\nabla\varphi_n\|_r \le C$. For A_2 we get

$$\begin{aligned}
A_2 &\overset{\text{def}}{=} \Big| \int_\Omega (\varphi_0 - \varphi_n)\, \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla v \, dx \Big| \\
&= \Big| \int_\Omega (\varphi_0 - \varphi^* + \varphi^* - \varphi_n)\, \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla v \, dx \Big| \\
&\le \Big| \int_\Omega (\varphi_0 - \varphi^*)\, \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla v \, dx \Big| \\
&+ \Big| \int_\Omega (\varphi^* - \varphi_n)\, \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla v \, dx \Big| \overset{\text{def}}{=} A_{21} + A_{22}.
\end{aligned}$$

Notice that $\varphi_0 \in W^{1,\infty}(\Omega)$ implies $\varphi_0 \in C(\bar\Omega)$. Also, $\varphi^* \in L^r(\Omega)$, $\nabla\varphi^* \in L^r(\Omega)$ implies $\varphi^* \in W^{1,r}(\Omega)$. Since $r > 2$ it follows that $\varphi^* \in C(\bar\Omega)$. Thus we conclude $(\varphi_0 - \varphi^*)\, \sigma(u^*)\nabla v \in L^2(\Omega)$ and therefore

$$A_{21} \overset{\text{def}}{=} \Big| \int_\Omega (\varphi_0 - \varphi^*)\, \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla v \, dx \Big| \to 0 \quad \text{as } n \to \infty$$

by the weak convergence $\nabla\varphi_n \overset{w}{\rightharpoonup} \nabla\varphi^*$ in $L^2(\Omega)$. Similarly,

$$\begin{aligned}
A_{22} &\overset{\text{def}}{=} \Big| \int_\Omega (\varphi^* - \varphi_n)\, \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla v \, dx \Big| \\
&\le C_2 \|\varphi^* - \varphi_n\|_{C(\bar\Omega)} \cdot \|\nabla\varphi_n - \nabla\varphi^*\|_2 \cdot \|\nabla v\|_2 \\
&\le C_2 \|\varphi^* - \varphi_n\|_{C(\bar\Omega)} \cdot (\|\nabla\varphi_n\|_2 + \|\nabla\varphi^*\|_2) \cdot \|\nabla v\|_2 \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This proves (3.9).

Finally, we obtain

$$\begin{aligned}
&\Big| \int_\Omega \sigma(u_n)\nabla\varphi_n \cdot \nabla w \, dx - \int_\Omega \sigma(u^*)\nabla\varphi^* \cdot \nabla w \, dx \Big| \\
&\le \Big| \int_\Omega (\sigma(u_n)\nabla\varphi_n \cdot \nabla w - \sigma(u^*)\nabla\varphi_n \cdot \nabla w) \, dx \Big| \\
&+ \Big| \int_\Omega (\sigma(u^*)\nabla\varphi_n \cdot \nabla w - \sigma(u^*)\nabla\varphi^* \cdot \nabla w) \, dx \Big| \\
&= \Big| \int_\Omega (\sigma(u_n) - \sigma(u^*))\nabla\varphi_n \cdot \nabla w \, dx \Big| + \Big| \int_\Omega \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla w \, dx \Big| \\
&\le K \int_\Omega |u_n - u^*| \cdot |\nabla\varphi_n| \cdot |\nabla w| \, dx + \Big| \int_\Omega \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla w \, dx \Big| \\
&\le K\|u_n - u^*\|_s \cdot \|\nabla\varphi_n\|_r \cdot \|\nabla w\|_2 + \Big| \int_\Omega \sigma(u^*)(\nabla\varphi_n - \nabla\varphi^*) \cdot \nabla w \, dx \Big| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This proves that

$$\int_\Omega \sigma(u_n)\nabla\varphi_n \cdot \nabla w\, dx \to \int_\Omega \sigma(u^*)\nabla\varphi^* \cdot \nabla w\, dx \quad \text{as } n \to \infty.$$

We have verified the convergence of each term in (3.1) and letting $n \to \infty$ in (3.1), we obtain

$$\int_\Omega \nabla u^* \cdot \nabla v\, dx + \int_{\partial\Omega} \beta^* u^* v\, dx = \int_\Omega (\varphi_0 - \varphi^*)\sigma(u^*)\nabla\varphi^* \cdot \nabla v\, dx + \int_\Omega \sigma(u^*)\nabla\varphi^* \cdot \nabla\varphi_0\, v\, dx$$
$$\int_\Omega \sigma(u^*)\nabla\varphi^* \cdot \nabla w\, dx = 0 \qquad \varphi_0 - \varphi_n \in H_0^1(\Omega), \forall\, w \in H_0^1(\Omega), \forall\, v \in H^1(\Omega).$$

Therefore (u^*, φ^*) is a weak solution associated with β^*, i.e., $u^* = u(\beta^*)$ and $\varphi^* = \varphi(\beta^*)$.

We show that β^* is optimal. As $u_n \overset{w}{\rightharpoonup} u^*$ in $L^2(\Omega)$ it follows that $\lim_{n\to\infty} \int_\Omega u_n\, dx$ exists and

$$\lim_{n\to\infty} \int_\Omega u_n\, dx = \int_\Omega u^*\, dx.$$

Also, since $\lim_{n\to\infty} J(\beta_n)$ and $\lim_{n\to\infty} \int_\Omega u_n\, dx$ exist, we conclude that $\lim_{n\to\infty} \int_{\partial\Omega} \beta_n^2\, ds$ exists and

$$\int_{\partial\Omega} (\beta^*)^2\, ds \leq \lim_{n\to\infty} \int_{\partial\Omega} \beta_n^2\, ds.$$

Therefore, we can write

$$\begin{aligned} \inf_{\beta\in U_M} J(\beta) &= \lim_{n\to\infty} J(\beta_n) \\ &= \lim_{n\to\infty} \int_\Omega u_n\, dx + \lim_{n\to\infty} \int_{\partial\Omega} \beta_n^2\, ds \\ &\geq \int_\Omega u^*\, dx + \int_{\partial\Omega} (\beta^*)^2\, ds = J(\beta^*). \end{aligned} \tag{3.10}$$

This implies that β^* is an optimal control. $\square$

Chapter 4

Derivation of the Optimality System

In order to characterize an optimal control, we need to derive an optimality system which consists of the original state system coupled with an adjoint system. To obtain the necessary conditions for the optimality system, we differentiate the objective functional with respect to the control. Since the objective functional depends on u, and u is coupled to φ through the PDE, we will need to differentiate u and φ with respect to control β.

Theorem 4.1. (Sensitivities) *If the boundary data φ_0 are sufficiently small, i.e., if $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ is small enough, then the mapping $\beta \mapsto (u, \varphi)$ is differentiable in the following sense:*

$$\begin{aligned} &\frac{u(\beta + \varepsilon\ell) - u(\beta)}{\varepsilon} \stackrel{w}{\rightharpoonup} \psi_1 \ \text{in } H^1(\Omega), \\ &\frac{\varphi(\beta + \varepsilon\ell) - \varphi(\beta)}{\varepsilon} \stackrel{w}{\rightharpoonup} \psi_2 \ \text{in } H_0^1(\Omega) \ \text{as } \varepsilon \to 0 \end{aligned} \tag{4.1}$$

for any $\beta \in U_M$ and $\ell \in L^\infty(\partial\Omega)$ such that $(\beta + \varepsilon\ell) \in U_M$ for small ε. Moreover, the sensitivities, $\psi_1 \in H^1(\Omega)$ and $\psi_2 \in H_0^1(\Omega)$, satisfy

$$\begin{aligned}
&\Delta\psi_1 + \sigma'(u)|\nabla\varphi|^2\psi_1 + 2\sigma(u)\nabla\varphi\cdot\nabla\psi_2 = 0 \ \ in\ \Omega,\\
&\nabla\cdot[\sigma'(u)\psi_1\nabla\varphi + \sigma(u)\nabla\psi_2] = 0 \ \ in\ \Omega,\\
&\frac{\partial\psi_1}{\partial n} + \beta\psi_1 + \ell u = 0 \ \ on\ \partial\Omega,\\
&\psi_2 = 0 \ \ on\ \partial\Omega.
\end{aligned} \tag{4.2}$$

Proof. Recall that we had the following notation: $u = u(\beta)$ and $\varphi = \varphi(\beta)$. Now we also denote $u^\varepsilon = u(\beta^\varepsilon)$, $\varphi^\varepsilon = \varphi(\beta^\varepsilon)$, where $\beta^\varepsilon \overset{\text{def}}{=} \beta + \varepsilon\ell$. The weak formulation for $(u^\varepsilon, \varphi^\varepsilon)$ is

$$\begin{aligned}
&\int_\Omega \nabla u^\varepsilon\nabla v\,dx + \int_{\partial\Omega}\beta^\varepsilon u^\varepsilon v\,ds = \int_\Omega(\varphi_0 - \varphi^\varepsilon)\,\sigma(u^\varepsilon)\nabla\varphi^\varepsilon\nabla v\,dx + \int_\Omega \sigma(u^\varepsilon)\nabla\varphi^\varepsilon\nabla\varphi_0\,v\,dx\\
&\int_\Omega \sigma(u^\varepsilon)\nabla\varphi^\varepsilon\nabla w\,dx = 0 \qquad \varphi_0 - \varphi^\varepsilon \in H_0^1(\Omega), \forall\, w\in H_0^1(\Omega), \forall\, v\in H^1(\Omega).
\end{aligned} \tag{4.3}$$

Similarly, for (u, φ) we have

$$\begin{aligned}
&\int_\Omega \nabla u\cdot\nabla\tilde v\,dx + \int_{\partial\Omega}\beta u\tilde v\,ds = \int_\Omega(\varphi_0 - \varphi)\,\sigma(u)\nabla\varphi\cdot\nabla\tilde v\,dx + \int_\Omega \sigma(u)\nabla\varphi\cdot\nabla\varphi_0\,\tilde v\,dx\\
&\int_\Omega \sigma(u)\nabla\varphi\cdot\nabla\tilde w\,dx = 0 \qquad \varphi_0 - \varphi \in H_0^1(\Omega), \forall\, \tilde w\in H_0^1(\Omega), \forall\, \tilde v\in H^1(\Omega).
\end{aligned} \tag{4.4}$$

Take test functions $v = (u^\varepsilon - u)/\varepsilon$, $\tilde v = (u^\varepsilon - u)/\varepsilon$, $w = (\varphi^\varepsilon - \varphi)/\varepsilon$, and $\tilde w = (\varphi^\varepsilon - \varphi)/\varepsilon$, subtract corresponding equations in (4.4) from (4.3), and divide by ε:

$$\begin{aligned}
&\int_\Omega \nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\cdot\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\,dx + \int_{\partial\Omega}\beta\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\,ds\\
&= -\int_{\partial\Omega}\ell u^\varepsilon\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\,ds\\
&+ \frac{1}{\varepsilon}\int_\Omega\Big[(\varphi_0 - \varphi^\varepsilon)\,\sigma(u^\varepsilon)\nabla\varphi^\varepsilon - (\varphi_0 - \varphi)\,\sigma(u)\nabla\varphi\Big]\cdot\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\,dx\\
&+ \frac{1}{\varepsilon}\int_\Omega\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon - \sigma(u)\nabla\varphi\Big]\cdot\nabla\varphi_0\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\,dx,\\
&\frac{1}{\varepsilon}\int_\Omega\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon\cdot\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big) - \sigma(u)\nabla\varphi\cdot\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big]\,dx = 0.
\end{aligned} \tag{4.5}$$

We derive $H^1(\Omega)$ estimate for $(\varphi^\varepsilon - \varphi)/\varepsilon$ first. Since $(\varphi^\varepsilon - \varphi)/\varepsilon \in H_0^1(\Omega)$ it follows from Poincaré's inequality that it is sufficient to have a bound on $\|\nabla(\varphi^\varepsilon - \varphi)/\varepsilon\|_2$.

The second equation in (4.5) implies

$$\int_\Omega \sigma(u^\varepsilon)\nabla\Big(\frac{\varphi^\varepsilon}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx = \int_\Omega \sigma(u)\nabla\Big(\frac{\varphi}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx. \tag{4.6}$$

Taking into account (4.6) we can write

$$\begin{aligned}
&\int_\Omega \sigma(u)\Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|^2\,dx = \int_\Omega \sigma(u)\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx\\
&= \int_\Omega \sigma(u)\nabla\Big(\frac{\varphi^\varepsilon}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx - \int_\Omega \sigma(u)\nabla\Big(\frac{\varphi}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx\\
&= \int_\Omega \sigma(u)\nabla\Big(\frac{\varphi^\varepsilon}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx - \int_\Omega \sigma(u^\varepsilon)\nabla\Big(\frac{\varphi^\varepsilon}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx\\
&= \int_\Omega \Big(\sigma(u)-\sigma(u^\varepsilon)\Big)\nabla\Big(\frac{\varphi^\varepsilon}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx.
\end{aligned} \tag{4.7}$$

We obtain

$$\begin{aligned}
&C_1\int_\Omega \Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|^2\,dx \le \int_\Omega \sigma(u)\Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|^2\,dx\\
&= \int_\Omega \Big(\sigma(u)-\sigma(u^\varepsilon)\Big)\nabla\Big(\frac{\varphi^\varepsilon}{\varepsilon}\Big)\cdot\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\,dx\\
&\le \int_\Omega \Big|\frac{\sigma(u)-\sigma(u^\varepsilon)}{\varepsilon}\Big|\cdot|\nabla\varphi^\varepsilon|\cdot\Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|\,dx\\
&\le \int_\Omega \Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\cdot|\nabla\varphi^\varepsilon|\cdot\Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|\,dx\\
&\le K\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\Big\|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big\|_2
\end{aligned} \tag{4.8}$$

where we used (4.7). Thus

$$C_1\Big\|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big\|_2^2 \le K\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\Big\|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big\|_2$$

whence

$$\Big\|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big\|_2 \le \frac{K}{C_1}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\,. \tag{4.9}$$

Since $H^1(\Omega) \subset L^s(\Omega)$ we have

$$\Big\| \frac{u^\varepsilon - u}{\varepsilon} \Big\|_s \le M_1 \Big\| \frac{u^\varepsilon - u}{\varepsilon} \Big\|_{H^1(\Omega)}. \tag{4.10}$$

By the Meyers estimate, Theorem 2.1,

$$\|\nabla \varphi^\varepsilon\|_r \le \Phi. \tag{4.11}$$

Substituting (4.11) and (4.10) into (4.9) we get

$$\Big\| \nabla \Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon} \Big) \Big\|_2 \le \frac{K M_1 \Phi}{C_1} \Big\| \frac{u^\varepsilon - u}{\varepsilon} \Big\|_{H^1(\Omega)}. \tag{4.12}$$

Then Poincaré's inequality implies

$$\Big\| \frac{\varphi^\varepsilon - \varphi}{\varepsilon} \Big\|_{H^1(\Omega)} \le C_6 \Big\| \nabla \Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon} \Big) \Big\|_2 \le \frac{C_6 K M_1 \Phi}{C_1} \Big\| \frac{u^\varepsilon - u}{\varepsilon} \Big\|_{H^1(\Omega)}. \tag{4.13}$$

Now we proceed to H^1 norm estimate of $(u^\varepsilon - u)/\varepsilon$. We obtain from (4.5):

$$\begin{aligned}
&\int_\Omega \Big| \nabla \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) \Big|^2 dx + \lambda \int_{\partial\Omega} \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big)^2 ds \le \int_\Omega \Big| \nabla \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) \Big|^2 dx + \int_{\partial\Omega} \beta \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big)^2 ds \\
&= \Big| - \int_{\partial\Omega} \ell u^\varepsilon \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) ds + \frac{1}{\varepsilon} \int_\Omega \Big[(\varphi_0 - \varphi^\varepsilon)\, \sigma(u^\varepsilon) \nabla \varphi^\varepsilon - (\varphi_0 - \varphi)\, \sigma(u) \nabla \varphi \Big] \nabla \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) dx \\
&+ \frac{1}{\varepsilon} \int_\Omega \Big[\sigma(u^\varepsilon) \nabla \varphi^\varepsilon - \sigma(u) \nabla \varphi \Big] \cdot \nabla \varphi_0 \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) dx \Big| \\
&= \Big| - \int_{\partial\Omega} \ell u^\varepsilon \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) ds + \mathcal{C} + \mathcal{D} \Big|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C} &\stackrel{\text{def}}{=} \frac{1}{\varepsilon} \int_\Omega \Big[(\varphi_0 - \varphi^\varepsilon)\, \sigma(u^\varepsilon) \nabla \varphi^\varepsilon - (\varphi_0 - \varphi)\, \sigma(u) \nabla \varphi \Big] \cdot \nabla \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) dx \\
&= \int_\Omega \Big[(\varphi_0 - \varphi) \Big(\frac{\sigma(u^\varepsilon) \nabla \varphi^\varepsilon - \sigma(u) \nabla \varphi}{\varepsilon} \Big) + \Big(\frac{\varphi - \varphi^\varepsilon}{\varepsilon} \Big) \sigma(u^\varepsilon) \nabla \varphi^\varepsilon \Big] \cdot \nabla \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) dx,
\end{aligned}$$

and

$$\mathcal{D} \stackrel{\text{def}}{=} \frac{1}{\varepsilon} \int_\Omega \Big[\sigma(u^\varepsilon) \nabla \varphi^\varepsilon - \sigma(u) \nabla \varphi \Big] \cdot \nabla \varphi_0 \Big(\frac{u^\varepsilon - u}{\varepsilon} \Big) dx.$$

We have

$$
\begin{aligned}
|\mathcal{C}| &\le \int_\Omega |\varphi_0 - \varphi| \cdot \Big|\frac{\sigma(u^\varepsilon)\nabla\varphi^\varepsilon - \sigma(u)\nabla\varphi}{\varepsilon}\Big| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ \int_\Omega \Big|\frac{\varphi - \varphi^\varepsilon}{\varepsilon}\Big|\, \sigma(u^\varepsilon)|\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&\le 2\tilde{M} \int_\Omega \Big|\frac{\sigma(u^\varepsilon)\nabla\varphi^\varepsilon - \sigma(u)\nabla\varphi}{\varepsilon}\Big| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ C_2 \int_\Omega \Big|\frac{\varphi - \varphi^\varepsilon}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&= 2\tilde{M} \int_\Omega \Big|\frac{(\sigma(u^\varepsilon) - \sigma(u) + \sigma(u))\nabla\varphi^\varepsilon - \sigma(u)\nabla\varphi}{\varepsilon}\Big| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ C_2 \int_\Omega \Big|\frac{\varphi - \varphi^\varepsilon}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&\le 2\tilde{M} \int_\Omega \Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ 2\tilde{M} \int_\Omega \sigma(u) \cdot \Big|\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ C_2 \int_\Omega \Big|\frac{\varphi - \varphi^\varepsilon}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&\le 2\tilde{M}K \int_\Omega \Big|\frac{u^\varepsilon - u}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ 2\tilde{M}C_2 \int_\Omega \Big|\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx \\
&+ C_2 \int_\Omega \Big|\frac{\varphi - \varphi^\varepsilon}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big)\Big|\, dx.
\end{aligned}
$$

Similarly, for $\mathcal{D}$ we obtain

$$
\begin{aligned}
|\mathcal{D}| &\le \int_\Omega |\nabla\varphi_0| \cdot \Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\frac{u^\varepsilon - u}{\varepsilon}\Big|\, dx \\
&+ \int_\Omega \sigma(u)|\nabla\varphi_0| \cdot \Big|\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big| \cdot \Big|\frac{u^\varepsilon - u}{\varepsilon}\Big|\, dx \\
&\le K\|\nabla\varphi_0\|_\infty \int_\Omega \Big|\frac{u^\varepsilon - u}{\varepsilon}\Big| \cdot |\nabla\varphi^\varepsilon| \cdot \Big|\frac{u^\varepsilon - u}{\varepsilon}\Big|\, dx \\
&+ C_2\|\nabla\varphi_0\|_\infty \int_\Omega \Big|\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big| \cdot \Big|\frac{u^\varepsilon - u}{\varepsilon}\Big|\, dx.
\end{aligned}
$$

Thus we have

$$\int_{\Omega}\Big|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big|^2dx+\lambda\int_{\partial\Omega}\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)^2ds\le\int_{\partial\Omega}|\ell|\cdot|u^\varepsilon|\cdot\Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\,ds+|\mathcal{C}|+|\mathcal{D}|$$
$$\le M\int_{\partial\Omega}|u^\varepsilon|\cdot\Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\,ds+|\mathcal{C}|+|\mathcal{D}|\le M\|u^\varepsilon\|_{L^2(\partial\Omega)}\cdot\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{L^2(\partial\Omega)}+|\mathcal{C}|+|\mathcal{D}|$$
$$\le MM_2^2\|u^\varepsilon\|_{H^1(\Omega)}\cdot\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}+|\mathcal{C}|+|\mathcal{D}|,$$

where we have used the trace inequality $\|u^\varepsilon\|_{L^2(\partial\Omega)}\le M_2\|u^\varepsilon\|_{H^1(\Omega)}$.

Taking into account the expressions for $|\mathcal{C}|$ and $|\mathcal{D}|$ we get

$$\int_{\Omega}\Big|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big|^2dx+\lambda\int_{\partial\Omega}\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)^2ds\le MM_2^2\|u^\varepsilon\|_{H^1(\Omega)}\cdot\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}$$
$$+2\tilde{M}K\int_{\Omega}\Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\cdot|\nabla\varphi^\varepsilon|\cdot\Big|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big|\,dx+2\tilde{M}C_2\int_{\Omega}\Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|\cdot\Big|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big|\,dx$$
$$+C_2\int_{\Omega}\Big|\frac{\varphi-\varphi^\varepsilon}{\varepsilon}\Big|\cdot|\nabla\varphi^\varepsilon|\cdot\Big|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big|\,dx+K\|\nabla\varphi_0\|_\infty\int_{\Omega}\Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\cdot|\nabla\varphi^\varepsilon|\cdot\Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\,dx$$
$$+C_2\|\nabla\varphi_0\|_\infty\int_{\Omega}\Big|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big|\cdot\Big|\frac{u^\varepsilon-u}{\varepsilon}\Big|\,dx.$$

Using Hölder's inequality, *a priori* bounds (2.11), (4.10), and (4.13) we have

$$\begin{aligned}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_*^2&\equiv\int_{\Omega}\Big|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big|^2dx+\lambda\int_{\partial\Omega}\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)^2ds\\
&\le MM_2^2\tilde{C}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}\\
&+2\tilde{M}K\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\Big\|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big\|_2\\
&+2\tilde{M}C_2\Big\|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big\|_2\cdot\Big\|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big\|_2\\
&+C_2\Big\|\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\Big\|\nabla\Big(\frac{u^\varepsilon-u}{\varepsilon}\Big)\Big\|_2\\
&+K\|\nabla\varphi_0\|_\infty\cdot\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_2\\
&+C_2\|\nabla\varphi_0\|_\infty\cdot\Big\|\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\Big\|_2\cdot\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_2\\
&\le MM_2^2\tilde{C}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}+2\tilde{M}KM_1\Phi\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}^2\\
&+\frac{2\tilde{M}C_2KM_1\Phi}{C_1}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}^2+\frac{C_2C_6KM_1\Phi^2}{C_1}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}^2\\
&+K\|\nabla\varphi_0\|_\infty M_1\Phi\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}^2+C_2\|\nabla\varphi_0\|_\infty\frac{KM_1\Phi}{C_1}\Big\|\frac{u^\varepsilon-u}{\varepsilon}\Big\|_{H^1(\Omega)}^2.\end{aligned}$$

Due to the equivalence of norms expressed by (2.20) we get

$$k\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_{H^1(\Omega)} \le \Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_* \le MM_2^2\tilde{C}\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|_{H^1(\Omega)}$$
$$+ 2\tilde{M}KM_1\Phi\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_{H^1(\Omega)} + \frac{2\tilde{M}C_2KM_1\Phi}{C_1}\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_{H^1(\Omega)} + \frac{C_2C_6KM_1\Phi^2}{C_1}\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_{H^1(\Omega)}$$
$$+ K\|\nabla\varphi_0\|_\infty M_1\Phi\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_{H^1(\Omega)} + C_2\|\nabla\varphi_0\|_\infty\frac{KM_1\Phi}{C_1}\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|^2_{H^1(\Omega)}.$$

By definition, Φ includes $\|\nabla\varphi_0\|_\infty$ and $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ (see (2.17)). Therefore, if $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ is chosen sufficiently small so that

$$\begin{aligned} k_1 \equiv k - 2\tilde{M}KM_1\Phi - \frac{2\tilde{M}C_2KM_1\Phi}{C_1} - \frac{C_2C_6KM_1\Phi^2}{C_1} \\ - K\|\nabla\varphi_0\|_\infty M_1\Phi - C_2\|\nabla\varphi_0\|_\infty\frac{KM_1\Phi}{C_1} > 0 \end{aligned} \tag{4.14}$$

then

$$\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|_{H^1(\Omega)} \le \frac{MM_2^2\tilde{C}}{k_1}, \tag{4.15}$$

where the constant in (4.15) does not depend on ε. Consequently, (4.13) yields

$$\Big\|\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big\|_{H^1(\Omega)} \le \frac{C_6KM_1\Phi}{C_1}\Big\|\frac{u^\varepsilon - u}{\varepsilon}\Big\|_{H^1(\Omega)} \le \frac{C_6KM_1\Phi}{C_1}\cdot\frac{MM_2^2\tilde{C}}{k_1}. \tag{4.16}$$

These *a priori* estimates justify the existence of ψ_1 and ψ_2, and the convergences in (4.1).

Note that $\|\varphi^\varepsilon\|_{W^{1,r}(\Omega)} \le \Phi$, and $W^{1,r}(\Omega) \subset\subset C(\bar{\Omega})$ imply $\varphi^\varepsilon \to \hat{\varphi}$ in $C(\bar{\Omega})$. On the other hand,

$$\Big\|\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big\|_{H^1(\Omega)} \le c$$

implies that $\|\varphi^\varepsilon - \varphi\|_{L^2(\Omega)} \le \|\varphi^\varepsilon - \varphi\|_{H^1(\Omega)} \to 0$ as $\varepsilon \to 0$. Therefore,

$$\begin{aligned} \|\hat{\varphi} - \varphi\|_{L^2(\Omega)} &\le \|\hat{\varphi} - \varphi^\varepsilon\|_{L^2(\Omega)} + \|\varphi^\varepsilon - \varphi\|_{L^2(\Omega)} \\ &\le c\|\hat{\varphi} - \varphi^\varepsilon\|_{C(\bar{\Omega})} + \|\varphi^\varepsilon - \varphi\|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

This justifies that $\hat{\varphi} = \varphi$ a.e.

Next, we show that the sensitivities satisfy system (4.2). Observe that now we have

$$
\begin{aligned}
\frac{u^\varepsilon - u}{\varepsilon} &\overset{w}{\rightharpoonup} \psi_1 \text{ in } H^1(\Omega),\\
\frac{u^\varepsilon - u}{\varepsilon} &\overset{s}{\to} \psi_1 \text{ in } L^s(\Omega),\\
\frac{\varphi^\varepsilon - \varphi}{\varepsilon} &\overset{w}{\rightharpoonup} \psi_2 \text{ in } H^1(\Omega),\\
\frac{\varphi^\varepsilon - \varphi}{\varepsilon} &\overset{s}{\to} \psi_2 \text{ in } L^s(\Omega),\\
u^\varepsilon &\overset{s}{\to} u \text{ in } L^s(\Omega),\\
\nabla u^\epsilon &\overset{w}{\rightharpoonup} \nabla u \text{ in } L^2(\Omega),\\
\varphi^\varepsilon &\overset{s}{\to} \varphi \text{ in } C(\bar{\Omega}),\\
\nabla \varphi^\varepsilon &\overset{w}{\rightharpoonup} \nabla \varphi \text{ in } L^r(\Omega),\\
\frac{u^\varepsilon - u}{\varepsilon} &\overset{w}{\rightharpoonup} \psi_1 \text{ in } L^2(\partial\Omega),\\
\beta^\varepsilon &\overset{w}{\rightharpoonup} \beta \text{ in } L^2(\partial\Omega), \text{ and}\\
\beta^\varepsilon &\overset{w*}{\rightharpoonup} \beta \text{ in } L^\infty(\partial\Omega)
\end{aligned}
\tag{4.17}
$$

as $\varepsilon \to 0$. Note that $(\varphi - \varphi^\varepsilon)/\varepsilon \overset{s}{\to} -\psi_2$ in $L^s(\Omega)$.

Subtract (4.4) from (4.3), and divide by ε:

$$
\begin{aligned}
&\int_\Omega \nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big) \cdot \nabla v \, dx + \int_{\partial\Omega} \beta\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big) v \, ds + \int_{\partial\Omega} \ell u^\varepsilon \, ds\\
&= \frac{1}{\varepsilon} \int_\Omega \Big[(\varphi_0 - \varphi^\varepsilon)\, \sigma(u^\varepsilon) \nabla \varphi^\varepsilon \cdot \nabla v - (\varphi_0 - \varphi)\, \sigma(u) \nabla \varphi \cdot \nabla v\Big] \, dx\\
&+ \frac{1}{\varepsilon} \int_\Omega \Big[\sigma(u^\varepsilon) \nabla \varphi^\varepsilon - \sigma(u) \nabla \varphi\Big] \cdot \nabla \varphi_0 v \, dx \quad \forall\, v \in H^1(\Omega),
\end{aligned}
\tag{4.18}
$$

$$
\frac{1}{\varepsilon} \int_\Omega \Big[\sigma(u^\varepsilon) \nabla \varphi^\varepsilon \cdot \nabla w - \sigma(u) \nabla \varphi \cdot \nabla w\Big] \, dx = 0 \quad \forall\, w \in H_0^1(\Omega).
\tag{4.19}
$$

It is immediate that for the left hand side of (4.18) we get

$$
\begin{aligned}
\int_\Omega \nabla\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big) \cdot \nabla v \, dx &\to \int_\Omega \nabla \psi_1 \cdot \nabla v \, dx,\\
\int_{\partial\Omega} \beta\Big(\frac{u^\varepsilon - u}{\varepsilon}\Big) v \, ds &\to \int_{\partial\Omega} \beta \psi_1 v \, ds, \text{ and}\\
\int_{\partial\Omega} \ell u^\varepsilon v \, ds &\to \int_{\partial\Omega} \ell u v \, ds \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

We write the first term on the right hand side of (4.18) in the form

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_\Omega \Big[(\varphi_0-\varphi^\varepsilon)\,\sigma(u^\varepsilon)\nabla\varphi^\varepsilon\cdot\nabla v-(\varphi_0-\varphi)\,\sigma(u)\nabla\varphi\cdot\nabla v\Big]\,dx\\
&=\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon-\sigma(u)\nabla\varphi\Big]\cdot\nabla v\,dx+\frac{1}{\varepsilon}\int_\Omega(\varphi-\varphi^\varepsilon)\sigma(u)\nabla\varphi\cdot\nabla v\,dx \qquad (4.20)\\
&=\mathcal{G}+\mathcal{F},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}&\overset{\text{def}}{=}\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon-\sigma(u)\nabla\varphi\Big]\cdot\nabla v\,dx\\
\mathcal{F}&\overset{\text{def}}{=}\frac{1}{\varepsilon}\int_\Omega(\varphi-\varphi^\varepsilon)\sigma(u)\nabla\varphi\cdot\nabla v\,dx.
\end{aligned}$$

First, we show

$$\mathcal{F}\to-\int_\Omega\psi_2\,\sigma(u)\nabla\varphi\cdot\nabla v\,dx \text{ as } \varepsilon\to 0. \qquad (4.21)$$

Indeed, we can write

$$\begin{aligned}
&\Big|\int_\Omega\Big(\frac{\varphi-\varphi^\varepsilon}{\varepsilon}\Big)\sigma(u)\nabla\varphi\cdot\nabla v\,dx-\int_\Omega(-\psi_2)\sigma(u)\nabla\varphi\cdot\nabla v\,dx\Big|\\
&\le C_2\int_\Omega\Big|\frac{\varphi^\varepsilon-\varphi}{\varepsilon}-\psi_2\Big|\cdot|\varphi|\cdot|\nabla v|\,dx\le C_2\Big\|\frac{\varphi^\varepsilon-\varphi}{\varepsilon}-\psi_2\Big\|_s\cdot\|\nabla\varphi\|_r\cdot\|\nabla v\|_2\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}$$

This proves (4.21).

Next, we have

$$\begin{aligned}
\mathcal{G}&=\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon-\sigma(u)\nabla\varphi\Big]\cdot\nabla v\,dx\\
&=\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\Big[\sigma(u^\varepsilon)-\sigma(u)\Big]\nabla\varphi^\varepsilon\cdot\nabla v\,dx\\
&+\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\,\sigma(u)\Big[\nabla\varphi^\varepsilon-\nabla\varphi\Big]\cdot\nabla v\,dx=\mathcal{G}_1+\mathcal{G}_2
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_1&\overset{\text{def}}{=}\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\Big[\sigma(u^\varepsilon)-\sigma(u)\Big]\nabla\varphi^\varepsilon\cdot\nabla v\,dx,\\
\mathcal{G}_2&\overset{\text{def}}{=}\frac{1}{\varepsilon}\int_\Omega(\varphi_0-\varphi^\varepsilon)\,\sigma(u)\Big[\nabla\varphi^\varepsilon-\nabla\varphi\Big]\cdot\nabla v\,dx.
\end{aligned}$$

First, we show

$$\mathcal{G}_2 \to \int_\Omega (\varphi_0 - \varphi)\,\sigma(u)\nabla\psi_2 \cdot \nabla v\,dx \quad \text{as } \varepsilon \to 0. \tag{4.22}$$

We write

$$\begin{aligned}\mathcal{G}_2 &= \frac{1}{\varepsilon}\int_\Omega (\varphi_0 - \varphi)\,\sigma(u)\nabla(\varphi^\varepsilon - \varphi)\cdot\nabla v\,dx \\ &\quad + \frac{1}{\varepsilon}\int_\Omega (\varphi - \varphi^\varepsilon)\,\sigma(u)\nabla(\varphi^\varepsilon - \varphi)\cdot\nabla v\,dx = \mathcal{G}_{21} + \mathcal{G}_{22}\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}\mathcal{G}_{21} &\stackrel{\text{def}}{=} \frac{1}{\varepsilon}\int_\Omega (\varphi_0 - \varphi)\,\sigma(u)\nabla(\varphi^\varepsilon - \varphi)\cdot\nabla v\,dx \\ \mathcal{G}_{22} &\stackrel{\text{def}}{=} \frac{1}{\varepsilon}\int_\Omega (\varphi - \varphi^\varepsilon)\,\sigma(u)\nabla(\varphi^\varepsilon - \varphi)\cdot\nabla v\,dx.\end{aligned}$$

We show

$$\mathcal{G}_{21} \to \int_\Omega (\varphi_0 - \varphi)\,\sigma(u)\nabla\psi_2\cdot\nabla v\,dx, \tag{4.24}$$

$$\mathcal{G}_{22} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.25}$$

Indeed, for (4.24) we can write

$$\begin{aligned}&\Big|\mathcal{G}_{21} - \int_\Omega (\varphi_0 - \varphi)\,\sigma(u)\nabla\psi_2\cdot\nabla v\,dx\Big| \\ &= \Big|\int_\Omega \Big[(\varphi_0 - \varphi)\,\sigma(u)\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\cdot\nabla v - (\varphi_0 - \varphi)\,\sigma(u)\nabla\psi_2\cdot\nabla v\Big]\,dx \\ &= \Big|\int_\Omega \Big[\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big) - \nabla\psi_2\Big](\varphi_0 - \varphi)\,\sigma(u)\nabla v\,dx\Big| \to 0 \quad \text{as } \varepsilon \to 0\end{aligned}$$

since $\nabla(\varphi^\varepsilon - \varphi)/\varepsilon \xrightarrow{w} \nabla\psi_2$ in $L^2(\Omega)$ and $(\varphi_0 - \varphi)\,\sigma(u)\nabla v \in L^2(\Omega)$ because $r > 2$ implies $\varphi_0 - \varphi \in W^{1,r}(\Omega) \subset\subset C(\bar{\Omega})$.

Now we prove (4.25). We have

$$
\begin{aligned}
|\mathcal{G}_{22}| &= \Big| \int_\Omega (\varphi - \varphi^\varepsilon)\, \sigma(u) \nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big) \cdot \nabla v \, dx \Big| \\
&\le C_2 \int_\Omega |\varphi - \varphi^\varepsilon| \cdot \Big|\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big| \cdot |\nabla v| \, dx \\
&\le C_2 \|\varphi - \varphi^\varepsilon\|_{C(\bar\Omega)} \cdot \Big\|\nabla\Big(\frac{\varphi^\varepsilon - \varphi}{\varepsilon}\Big)\Big\|_2 \cdot \|\nabla v\|_2 \to 0 \quad \text{as } \varepsilon \to 0
\end{aligned}
$$

since $\|\varphi - \varphi^\varepsilon\|_{C(\bar\Omega)} \to 0$ as $\varepsilon \to 0$, and $\|\nabla(\varphi^\varepsilon - \varphi)/\varepsilon)\|_2$ is bounded. Hence (4.25) follows. This ends the proof of (4.22).

Now we deal with the term $\mathcal{G}_1$. First, we need to prove the following claim:

$$
\frac{\sigma(u^\varepsilon) - \sigma(u)}{\varepsilon} \to \sigma'(u)\, \psi_1 \text{ in } L^s(\Omega) \text{ as } \varepsilon \to 0\ . \tag{4.26}
$$

We know that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} = \sigma'(u), \\
&\frac{u^\varepsilon - u}{\varepsilon} \xrightarrow{s} \psi_1 \text{ in } L^s(\Omega), \text{ and} \\
&\Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u}\Big| \le K.
\end{aligned} \tag{4.27}
$$

Note that the first limit in (4.27) is pointwise a.e. We can write

$$
\begin{aligned}
&\Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{\varepsilon} - \sigma'(u)\, \psi_1\Big| = \Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} \cdot \frac{u^\varepsilon - u}{\varepsilon} - \sigma'(u)\, \psi_1\Big| \\
&= \Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} \cdot \Big(\frac{u^\varepsilon - u}{\varepsilon} - \psi_1 + \psi_1\Big) - \sigma'(u)\, \psi_1\Big| \\
&= \Big|\frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} \cdot \Big(\frac{u^\varepsilon - u}{\varepsilon} - \psi_1\Big) + \Big(\frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} - \sigma'(u)\Big)\psi_1\Big| = |f + g|
\end{aligned} \tag{4.28}
$$

where we denoted

$$
\begin{aligned}
f &\stackrel{\text{def}}{=} \frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} \cdot \Big(\frac{u^\varepsilon - u}{\varepsilon} - \psi_1\Big), \\
g &\stackrel{\text{def}}{=} \Big(\frac{\sigma(u^\varepsilon) - \sigma(u)}{u^\varepsilon - u} - \sigma'(u)\Big)\psi_1.
\end{aligned}
$$

We have $\|f+g\|_s \le \|f\|_s + \|g\|_s$. We show that $\|f\|_s \to 0$ and $\|g\|_s \to 0$ as $\varepsilon \to 0$. Indeed,

$$\begin{aligned}\|f\|_s^s &= \int_\Omega \Big| \frac{\sigma(u^\varepsilon)-\sigma(u)}{u^\varepsilon - u} \cdot \Big(\frac{u^\varepsilon - u}{\varepsilon} - \psi_1 \Big) \Big|^s \, dx = \int_\Omega \Big| \frac{\sigma(u^\varepsilon)-\sigma(u)}{u^\varepsilon - u} \Big|^s \cdot \Big| \frac{u^\varepsilon - u}{\varepsilon} - \psi_1 \Big|^s \, dx \\ &\le K^s \int_\Omega \Big| \frac{u^\varepsilon - u}{\varepsilon} - \psi_1 \Big|^s \, dx = K^s \Big\| \frac{u^\varepsilon - u}{\varepsilon} - \psi_1 \Big\|_s^s \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

For the $\|g\|_s$ term we use Lebesgue Dominated Convergence Theorem. Observe that

$$|\sigma'(u)| \le K.$$

It is clear

$$\int_\Omega \Big| \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{u^\varepsilon - u} - \sigma'(u) \Big) \psi_1 \Big| \, dx \le C$$

where $C > 0$ denotes a generic constant, and we also have

$$\Big| \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{u^\varepsilon - u} - \sigma'(u) \Big) \psi_1 \Big| \le 2K|\psi_1| \quad \forall\, x, \forall\, \varepsilon > 0,$$

where $\psi_1 \in L^1(\Omega)$. Therefore we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \Big| \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{u^\varepsilon - u} - \sigma'(u) \Big) \psi_1 \Big|^s \, dx = \int_\Omega \lim_{\varepsilon\to 0} \Big| \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{u^\varepsilon - u} - \sigma'(u) \Big) \psi_1 \Big|^s \, dx = 0$$

whence $\|g\|_s \to 0$. This proves the claim.

Now we are ready to show that

$$\mathcal{G}_1 \equiv \int_\Omega (\varphi_0 - \varphi^\varepsilon) \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon} \Big) \nabla\varphi^\varepsilon \cdot \nabla v \, dx \to \int_\Omega (\varphi_0 - \varphi)\, \sigma'(u) \psi_1 \nabla\varphi \cdot \nabla v \, dx. \quad (4.29)$$

Indeed,

$$
\begin{aligned}
&\Big|\int_\Omega (\varphi_0-\varphi^\varepsilon)\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla v\,dx-\int_\Omega(\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v\,dx\Big|\\
&=\Big|\int_\Omega\Big\{(\varphi_0-\varphi)\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla v\,dx+(\varphi-\varphi^\varepsilon)\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla v\\
&-(\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v\Big\}\,dx\Big|\le\Big|\int_\Omega(\varphi_0-\varphi)\Big\{\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\\
&-\sigma'(u)\psi_1\nabla\varphi\Big\}\cdot\nabla v\,dx\Big|+\Big|\int_\Omega(\varphi-\varphi^\varepsilon)\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla v\,dx\Big|\\
&\le\Big|\int_\Omega(\varphi_0-\varphi)\Big\{\Big[\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1+\sigma'(u)\psi_1\Big]\nabla\varphi^\varepsilon-\sigma'(u)\psi_1\nabla\varphi\Big\}\cdot\nabla v\,dx\Big|\\
&+\Big|\int_\Omega(\varphi-\varphi^\varepsilon)\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1+\sigma'(u)\psi_1\Big)\nabla\varphi^\varepsilon\cdot\nabla v\,dx\Big|\\
&=\Big|\int_\Omega\Big\{(\varphi_0-\varphi)\Big[\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1\Big]\nabla\varphi^\varepsilon\cdot\nabla v\\
&+(\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla(\varphi^\varepsilon-\varphi)\cdot\nabla v\Big\}\,dx\Big|\\
&+\Big|\int_\Omega(\varphi-\varphi^\varepsilon)\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1\Big)\nabla\varphi^\varepsilon\cdot\nabla v+(\varphi-\varphi^\varepsilon)\,\sigma'(u)\psi_1\nabla\varphi^\varepsilon\cdot\nabla v\,dx\Big|\\
&\le\|\varphi_0-\varphi\|_\infty\int_\Omega\Big|\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1\Big|\cdot|\nabla\varphi^\varepsilon|\cdot|\nabla v|\,dx\\
&+\Big|\int_\Omega\{(\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla(\varphi^\varepsilon-\varphi)\cdot\nabla v\}\,dx\Big|\\
&+\|\varphi-\varphi^\varepsilon\|_{C(\bar\Omega)}\int_\Omega\Big|\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1\Big|\cdot|\nabla\varphi^\varepsilon|\cdot|\nabla v|\,dx\\
&+\|\varphi-\varphi^\varepsilon\|_{C(\bar\Omega)}\int_\Omega|\sigma'(u)\psi_1|\cdot|\nabla\varphi^\varepsilon|\cdot|\nabla v|\,dx\\
&\le\|\varphi_0-\varphi\|_\infty\cdot\Big\|\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\|\nabla v\|_2\\
&+\Big|\int_\Omega(\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla(\varphi^\varepsilon-\varphi)\cdot\nabla v\,dx\Big|\\
&+\|\varphi-\varphi^\varepsilon\|_{C(\bar\Omega)}\cdot\Big\|\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}-\sigma'(u)\psi_1\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\|\nabla v\|_2\\
&+\|\varphi-\varphi^\varepsilon\|_{C(\bar\Omega)}\cdot\|\sigma'(u)\psi_1\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\|\nabla v\|_2\to 0 \text{ as } \varepsilon\to 0,
\end{aligned}
$$

since

$$\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon} \xrightarrow{s} \sigma'(u)\psi_1 \text{ in } L^s(\Omega)\ ,$$
$$\varphi^\varepsilon \xrightarrow{s} \varphi \text{ in } C(\bar\Omega),$$
$$\nabla\varphi^\varepsilon \xrightarrow{w} \nabla\varphi \text{ in } L^r(\Omega) \text{ as } \varepsilon\to 0,$$
$$\|\nabla\varphi^\varepsilon\|_r \le \Phi.$$

We also used the fact that $(\varphi_0-\varphi)\in C(\bar\Omega)$, $\sigma'(u)\psi_1\in L^s(\Omega)$, $\nabla v\in L^2(\Omega)$ will imply $(\varphi_0-\varphi)\sigma'(u)\psi_1\nabla v\in L^{r'}(\Omega)$ (because $\frac{1}{r}+\frac{1}{s}+\frac{1}{2}=1$ implies $\frac{1}{s}+\frac{1}{2}=\frac{1}{r'}$). This completes the proof of (4.29). Note that without the assumption $\Omega\subset\mathbb{R}^2$ we would not be able to justify convergence of the integral $\int_\Omega(\varphi-\varphi^\varepsilon)\,\sigma'(u)\psi_1\nabla\varphi^\varepsilon\cdot\nabla v\,dx$ to zero as we need the imbedding $W^{1,r}(\Omega)\subset\subset C(\bar\Omega)$.

Now we show that the second term on the right hand side of (4.18):

$$\frac{1}{\varepsilon}\int_\Omega\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon-\sigma(u)\nabla\varphi\Big]\cdot\nabla\varphi_0 v\,dx \to \int_\Omega\sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\,dx+\int_\Omega\sigma(u)\nabla\psi_2\cdot\nabla\varphi_0 v\,dx. \tag{4.30}$$

To this end, we write

$$\begin{aligned}&\frac{1}{\varepsilon}\int_\Omega\Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon-\sigma(u)\nabla\varphi\Big]\cdot\nabla\varphi_0 v\,dx\\&=\int_\Omega\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla\varphi_0 v\,dx+\int_\Omega\sigma(u)\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\cdot\nabla\varphi_0 v\,dx\end{aligned}$$

and then we show

$$\int_\Omega\Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla\varphi_0 v\,dx\to\int_\Omega\sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\,dx, \tag{4.31}$$
$$\int_\Omega\sigma(u)\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\cdot\nabla\varphi_0 v\,dx\to\int_\Omega\sigma(u)\nabla\psi_2\cdot\nabla\varphi_0 v\,dx. \tag{4.32}$$

We show the limit in (4.31):

$$
\begin{aligned}
&\Big| \int_\Omega \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla\varphi_0 v\,dx - \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\,dx\Big| \\
&\le \Big| \int_\Omega \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon} - \sigma'(u)\psi_1\Big)\nabla\varphi^\varepsilon\cdot\nabla\varphi_0 v\,dx\Big| + \Big|\int_\Omega \Big[\sigma'(u)\psi_1\nabla\varphi^\varepsilon\cdot\nabla\varphi_0 v \\
&- \sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\Big]\,dx\Big| \le \|\nabla\varphi_0\|_\infty\cdot\Big\|\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon} - \sigma'(u)\psi_1\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\|v\|_2 \\
&+ \Big|\int_\Omega \sigma'(u)\psi_1\nabla(\varphi^\varepsilon-\varphi)\cdot\nabla\varphi_0 v\,dx\Big| \to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

This proves (4.31). Note that (4.32) is obvious from the weak convergence of $(\varphi^\varepsilon-\varphi)/\varepsilon$. This completes the proof of (4.30).

Finally, we show that the terms of (4.19) satisfy

$$
\frac{1}{\varepsilon}\int_\Omega \Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon\cdot\nabla w - \sigma(u)\nabla\varphi\cdot\nabla w\Big]\,dx \to \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla w\,dx + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla w\,dx. \tag{4.33}
$$

Indeed, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_\Omega \Big[\sigma(u^\varepsilon)\nabla\varphi^\varepsilon\cdot\nabla w - \sigma(u)\nabla\varphi\cdot\nabla w\Big]\,dx \\
&= \int_\Omega \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla w\,dx + \int_\Omega \sigma(u)\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\cdot\nabla w\,dx,
\end{aligned}
$$

and therefore it suffices to prove

$$
\int_\Omega \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla w\,dx \to \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla w\,dx, \tag{4.34}
$$

$$
\int_\Omega \sigma(u)\nabla\Big(\frac{\varphi^\varepsilon-\varphi}{\varepsilon}\Big)\cdot\nabla w\,dx \to \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla w\,dx. \tag{4.35}
$$

Note that (4.35) is immediate. To show (4.34) we write

$$
\begin{aligned}
&\Big|\int_\Omega \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon}\Big)\nabla\varphi^\varepsilon\cdot\nabla w\,dx - \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla w\,dx\Big|\\
&\le \Big|\int_\Omega \Big(\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon} - \sigma'(u)\psi_1\Big)\nabla\varphi^\varepsilon\cdot\nabla w\,dx\Big|\\
&+\Big|\int_\Omega \Big(\sigma'(u)\psi_1\nabla\varphi^\varepsilon\cdot\nabla w - \sigma'(u)\psi_1\nabla\varphi\cdot\nabla w\Big)\,dx\Big|\\
&\le \Big\|\frac{\sigma(u^\varepsilon)-\sigma(u)}{\varepsilon} - \sigma'(u)\psi_1\Big\|_s\cdot\|\nabla\varphi^\varepsilon\|_r\cdot\|\nabla w\|_2\\
&+\Big|\int_\Omega \sigma'(u)\psi_1\nabla(\varphi^\varepsilon-\varphi)\cdot\nabla w\,dx\Big| \to 0 \text{ as } \varepsilon\to 0,
\end{aligned}
$$

by (4.26) for the first term and $\nabla\varphi^\varepsilon \overset{w}{\rightharpoonup} \nabla\varphi$ in $L^r(\Omega)$ and $\sigma'(u)\psi_1\nabla w \in L^{r'}(\Omega)$ (because $(1/s)+(1/2)=(1/r')$) for the second term. This completes the proof of (4.33). Consequently, letting $\varepsilon\to 0$ in (4.18) and (4.19) we obtain

$$
\begin{aligned}
&\int_\Omega \nabla\psi_1\cdot\nabla v\,dx + \int_{\partial\Omega}(\beta\psi_1+\ell u)v\,ds\\
&=\int_\Omega(\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v\,dx + \int_\Omega(\varphi_0-\varphi)\,\sigma(u)\nabla\psi_2\cdot\nabla v\,dx\\
&-\int_\Omega \psi_2\,\sigma(u)\nabla\varphi\cdot\nabla v\,dx + \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\,dx + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi_0 v\,dx,\\
&\int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla w + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla w\,dx = 0 \quad \forall\, v\in H^1(\Omega), \forall\, w\in H_0^1(\Omega).
\end{aligned}
\tag{4.36}
$$

To determine the "strong" formulation corresponding to (4.36) we differentiate (1.1) with respect to β (remembering that in a Gâteaux sense $(\partial u/\partial\beta)=\psi_1$, $(\partial\varphi/\partial\beta)=\psi_2$, and $(\partial\sigma(u)/\partial\beta)=\sigma'(u)\psi_1$)

$$
\begin{aligned}
&\Delta\psi_1 + \sigma'(u)\psi_1|\nabla\varphi|^2 + 2\sigma(u)\nabla\varphi\cdot\nabla\psi_2 = 0 \text{ in } \Omega,\\
&\nabla\cdot(\sigma'(u)\psi_1\nabla\varphi + \sigma(u)\nabla\psi_2) = 0 \text{ in } \Omega,\\
&\psi_2 = 0 \text{ on } \partial\Omega,\\
&\frac{\partial\psi_1}{\partial n} + \beta\psi_1 + \ell u = 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{4.37}
$$

and write a weak formulation for (4.37). Multiply by test functions, integrate over Ω, and integrate by parts to get

$$\begin{aligned}
&\int_\Omega \nabla\psi_1 \cdot \nabla v\,dx + \int_{\partial\Omega} (\beta\psi_1 + \ell u)v\,ds \\
&= \int_\Omega \sigma'(u)\psi_1 |\nabla\varphi|^2 v\,dx + 2\int_\Omega \sigma(u)\nabla\varphi\cdot\nabla\psi_2 v\,dx \quad \forall\, v\in H^1(\Omega), \qquad (4.38)\\
&\int_\Omega (\sigma'(u)\psi_1\nabla\varphi + \sigma(u)\nabla\psi_2)\cdot\nabla w\,dx = 0 \quad \forall\, w\in H^1_0(\Omega). \qquad (4.39)
\end{aligned}$$

To deal with the quadratic term on the right hand side of (4.38) we again follow Lemma 1 from [25]. Take $w = (\varphi_0 - \varphi)v \in H^1_0(\Omega)$ in (4.39), where $v\in C^1(\bar\Omega)$ is arbitrary. Then $\nabla w = v\nabla\varphi - v\nabla\varphi_0 + (\varphi-\varphi_0)\nabla v$ and we obtain

$$\begin{aligned}
0 &= \int_\Omega \Big[\sigma'(u)\psi_1\nabla\varphi + \sigma(u)\nabla\psi_2\Big]\cdot\nabla w\,dx \\
&= \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\Big[v\nabla\varphi - v\nabla\varphi_0 + (\varphi-\varphi_0)\nabla v\Big]\,dx \\
&+ \int_\Omega \sigma(u)\nabla\psi_2\cdot\Big[v\nabla\varphi - v\nabla\varphi_0 + (\varphi-\varphi_0)\nabla v\Big]\,dx \\
&= \int_\Omega \Big[\sigma'(u)\psi_1|\nabla\varphi|^2 v - \sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\Big]\,dx \\
&+ \int_\Omega \Big[(\varphi-\varphi_0)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v + \sigma(u)\nabla\psi_2\cdot\nabla\varphi v\Big]\,dx \\
&+ \int_\Omega \Big[-\sigma(u)\nabla\psi_2\cdot\nabla\varphi_0 v + (\varphi-\varphi_0)\,\sigma(u)\nabla\psi_2\cdot\nabla v\Big]\,dx,
\end{aligned}$$

whence

$$\begin{aligned}
&\int_\Omega \sigma'(u)\psi_1|\nabla\varphi|^2 v\,dx + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi\, v\,dx = \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\,dx \\
&+ \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi_0 v\,dx + \int_\Omega (\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v\,dx + \int_\Omega (\varphi_0-\varphi)\,\sigma(u)\nabla\psi_2\cdot\nabla v\,dx.
\end{aligned}$$

Hence, adding "$\sigma(u)$" term to both sides gives

$$\begin{aligned}
&\int_\Omega \sigma'(u)\psi_1|\nabla\varphi|^2 v\,dx + 2\int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi\, v\,dx = \int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla\varphi_0 v\,dx \\
&+ \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi_0 v\,dx + \int_\Omega (\varphi_0-\varphi)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v\,dx \qquad (4.40)\\
&+ \int_\Omega (\varphi_0-\varphi)\,\sigma(u)\nabla\psi_2\cdot\nabla v\,dx + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi\, v\,dx.
\end{aligned}$$

By the density argument and taking into account (4.40), we can rewrite (4.38) and (4.39) as follows

$$\begin{aligned}
&\int_\Omega \nabla\psi_1 \cdot \nabla v\,dx + \int_{\partial\Omega} (\beta\psi_1 + \ell u)v\,ds = \int_\Omega \sigma'(u)\psi_1 \nabla\varphi \cdot \nabla\varphi_0 v\,dx \\
&+ \int_\Omega \sigma(u)\nabla\psi_2 \cdot \nabla\varphi_0 v\,dx + \int_\Omega (\varphi_0 - \varphi)\,\sigma'(u)\psi_1\nabla\varphi\cdot\nabla v\,dx \\
&+ \int_\Omega (\varphi_0 - \varphi)\,\sigma(u)\nabla\psi_2 \cdot \nabla v\,dx + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla\varphi\, v\,dx \quad \forall\, v \in H^1(\Omega), \\
&\int_\Omega (\sigma'(u)\psi_1\nabla\varphi + \sigma(u)\nabla\psi_2)\cdot\nabla w\,dx = 0 \quad \forall\, w \in H_0^1(\Omega).
\end{aligned} \tag{4.41}$$

If we compare (4.41) with (4.36) we see that these two weak formulations coincide since

$$\int_\Omega \sigma(u)v\nabla\psi_2 \cdot \nabla\varphi\,dx = -\int_\Omega \sigma(u)\psi_2\nabla v\cdot\nabla\varphi\,dx$$

if we integrate by parts to get the gradient off ψ_2 and use the fact that $\nabla\cdot(\sigma(u)\nabla\varphi) = 0$ in Ω. $\square$

In order to derive the optimality system and to characterize optimal control, we need to introduce adjoint functions and the adjoint operator associated with ψ_1 and ψ_2.

We have formally

$$\begin{aligned}
\int_\Omega p\,\Delta\psi_1\,dx &= -\int_\Omega \nabla p\cdot\nabla\psi_1\,dx + \int_{\partial\Omega} \frac{\partial\psi_1}{\partial n} p\,ds \\
&= \int_\Omega \psi_1 \Delta p\,dx + \int_{\partial\Omega} \frac{\partial\psi_1}{\partial n} p\,ds - \int_{\partial\Omega} \frac{\partial p}{\partial n}\psi_1\,ds \\
&= \int_\Omega \psi_1 \Delta p\,dx - \int_{\partial\Omega} \psi_1\Big(\frac{\partial p}{\partial n} + \beta^* p\Big)\,ds - \int_{\partial\Omega} \ell u p\,ds.
\end{aligned}$$

Therefore, if we set $\frac{\partial p}{\partial n} + \beta^* p = 0$ on $\partial\Omega$ we will get $\int_\Omega p\,\Delta\psi_1\,dx = \int_\Omega \psi_1\Delta p\,dx - \int_{\partial\Omega} \ell u p\,ds$. Next, we have

$$\int_\Omega p\,\sigma'(u)|\nabla\varphi|^2\psi_1\,dx = \int_\Omega \psi_1\,\sigma'(u)|\nabla\varphi|^2 p\,dx.$$

Now, we look at the term

$$\begin{aligned}\int_\Omega p\,\sigma(u)\nabla\varphi\cdot\nabla\psi_2\,dx &= \int_\Omega p\,\sigma(u)\varphi_{x_1}(\psi_2)_{x_1}\,dx + \int_\Omega p\,\sigma(u)\varphi_{x_2}(\psi_2)_{x_2}\,dx \\ &= -\int_\Omega (p\,\sigma(u)\varphi_{x_1})_{x_1}\psi_2\,dx - \int_\Omega (p\,\sigma(u)\varphi_{x_2})_{x_2}\psi_2\,dx \\ &+ \int_{\partial\Omega} p\,\sigma(u)\varphi_{x_1}\psi_2\eta_1\,ds + \int_{\partial\Omega} p\,\sigma(u)\varphi_{x_2}\psi_2\eta_2\,ds \\ &= -\int_\Omega \psi_2\nabla\cdot(p\,\sigma(u)\nabla\varphi)\,dx + \int_{\partial\Omega}\psi_2\,\sigma(u)p\frac{\partial\varphi}{\partial n}\,ds \\ &= -\int_\Omega \psi_2\nabla\cdot(p\,\sigma(u)\nabla\varphi)\,dx\end{aligned}$$

since $\psi_2 = 0$ on $\partial\Omega$. Hence

$$2\int_\Omega p\,\sigma(u)\nabla\varphi\cdot\nabla\psi_2\,dx = -2\int_\Omega \psi_2\nabla\cdot(p\,\sigma(u)\nabla\varphi)\,dx.$$

Next, we have

$$\begin{aligned}\int_\Omega q\nabla\cdot[\sigma(u)\nabla\psi_2]\,dx &= \int_\Omega q\big[\sigma(u)(\psi_2)_{x_1}\big]_{x_1}\,dx + \int_\Omega q\big[\sigma(u)(\psi_2)_{x_2}\big]_{x_2}\,dx \\ &= -\int_\Omega q_{x_1}\sigma(u)(\psi_2)_{x_1}\,dx - \int_\Omega q_{x_2}\sigma(u)(\psi_2)_{x_2}\,dx \\ &+ \int_{\partial\Omega} q\sigma(u)(\psi_2)_{x_1}\eta_1\,ds + \int_{\partial\Omega} q\sigma(u)(\psi_2)_{x_2}\eta_2\,ds \\ &= \int_\Omega (q_{x_1}\sigma(u))_{x_1}\psi_2\,dx + \int_\Omega (q_{x_2}\sigma(u))_{x_2}\psi_2\,dx \\ &+ \int_{\partial\Omega} q\sigma(u)\frac{\partial\psi_2}{\partial n}\,ds - \int_{\partial\Omega}\sigma(u)\psi_2\frac{\partial q}{\partial n}\,ds \\ &= \int_\Omega \psi_2\nabla\cdot[\sigma(u)\nabla q]\,dx + \int_{\partial\Omega} q\sigma(u)\frac{\partial\psi_2}{\partial n}\,ds,\end{aligned}$$

since $\psi_2 = 0$ on $\partial\Omega$. Thus, if we set $q = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega q\nabla\cdot[\sigma(u)\nabla\psi_2]\,dx = \int_\Omega \psi_2\nabla\cdot[\sigma(u)\nabla q]\,dx. \tag{4.42}$$

Finally, using $q = 0$ on $\partial\Omega$, we can write

$$\int_\Omega q\nabla\cdot[\sigma'(u)\psi_1\nabla\varphi]\,dx = \int_\Omega q[\sigma'(u)\psi_1\varphi_{x_1}]_{x_1}\,dx + \int_\Omega q[\sigma'(u)\psi_1\varphi_{x_2}]_{x_2}\,dx$$
$$= -\int_\Omega q_{x_1}\sigma'(u)\psi_1\varphi_{x_1}\,dx - \int_\Omega q_{x_2}\sigma'(u)\psi_1\varphi_{x_2}\,dx + \int_{\partial\Omega} q\,\sigma'(u)\psi_1\varphi_{x_1}\eta_1\,ds$$
$$+ \int_{\partial\Omega} q\,\sigma'(u)\psi_1\varphi_{x_2}\eta_2\,ds = -\int_\Omega \psi_1\sigma'(u)\nabla\varphi\cdot\nabla q\,dx.$$

All in all, using the notation $\mathcal{L}$ for the operator in the sensitivity system acting on ψ_1, ψ_2

$$\int_\Omega \begin{pmatrix} p & q \end{pmatrix} \mathcal{L} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dx = \int_\Omega p\,\Delta\psi_1\,dx + \int_\Omega p\,\sigma'(u)|\nabla\varphi|^2\psi_1\,dx$$
$$+ 2\int_\Omega p\,\sigma(u)\nabla\varphi\cdot\nabla\psi_2\,dx + \int_\Omega q\nabla\cdot[\sigma(u)\nabla\psi_2]\,dx + \int_\Omega q\nabla\cdot[\sigma'(u)\psi_1\nabla\varphi]\,dx$$
$$= \int_\Omega \psi_1\Delta p\,dx + \int_\Omega \psi_1\sigma'(u)|\nabla\varphi|^2 p\,dx - 2\int_\Omega \psi_2\nabla\cdot[p\,\sigma(u)\nabla\varphi]\,dx$$
$$- \int_\Omega \psi_1\sigma'(u)\nabla\varphi\cdot\nabla q\,dx + \int_\Omega \psi_2\nabla\cdot[\sigma(u)\nabla q]\,dx = \int_\Omega \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} \mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} dx$$

where

$$\mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \Delta p + \sigma'(u)|\nabla\varphi|^2 p - \sigma'(u)\nabla\varphi\cdot\nabla q \\ \nabla\cdot[-2p\,\sigma(u)\nabla\varphi + \sigma(u)\nabla q] \end{pmatrix}$$
$$\frac{\partial p}{\partial n} + \beta^* p = 0 \text{ on } \partial\Omega,$$
$$q = 0 \text{ on } \partial\Omega.$$

Thus, the adjoint system is given by

$$\begin{aligned} &\Delta p + \sigma'(u)|\nabla\varphi|^2 p - \sigma'(u)\nabla\varphi\cdot\nabla q = 1 \text{ in } \Omega, \\ &\nabla\cdot[-2p\,\sigma(u)\nabla\varphi + \sigma(u)\nabla q] = 0 \text{ in } \Omega, \\ &\frac{\partial p}{\partial n} + \beta^* p = 0 \text{ on } \partial\Omega, \\ &q = 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.43}$$

where the nonhomogeneous term "1" comes from differentiating the integrand of $J(\beta)$ with respect to the state u.

We have the following

Theorem 4.2. *Let $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ be sufficiently small. Then, given an optimal control $\beta^* \in U_M$ and corresponding states u, φ, there exists a solution $(p, q) \in H^1(\Omega) \times H_0^1(\Omega)$ to the adjoint system*

$$
\begin{aligned}
&\Delta p + \sigma'(u)|\nabla\varphi|^2 p - \sigma'(u)\nabla\varphi \cdot \nabla q = 1 \ \textit{in}\ \Omega,\\
&\nabla \cdot [-2p\,\sigma(u)\nabla\varphi + \sigma(u)\nabla q] = 0 \ \textit{in}\ \Omega,\\
&\frac{\partial p}{\partial n} + \beta^* p = 0 \ \textit{on}\ \partial\Omega,\\
&q = 0 \ \textit{on}\ \partial\Omega.
\end{aligned}
\tag{4.44}
$$

Furthermore, β^* can be characterized by

$$\beta^*(x) = \min\left(\max\left(-\frac{up}{2}, \lambda\right), M\right).$$

Proof. We rewrite the operator in (4.2) with zero Robin boundary conditions as follows:

$$\mathcal{L}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

where

$$
\begin{aligned}
&\mathcal{L}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}\Delta\Phi_1 + \sigma'(u)|\nabla\varphi|^2\Phi_1 + 2\sigma(u)\nabla\varphi\cdot\nabla\Phi_2\\ \nabla\cdot[\sigma'(u)\Phi_1\nabla\varphi + \sigma(u)\nabla\Phi_2]\end{pmatrix}\\
&\frac{\partial\Phi_1}{\partial n} + \beta^*\Phi_1 = 0 \text{ on } \partial\Omega,\\
&\Phi_2 = 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{4.45}
$$

Note that the boundary conditions in (4.45) are part of the definition of the operator $\mathcal{L}$. We define the adjoint system as

$$\mathcal{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.46}$$

where $\mathcal{L}^*$ is a formal Hilbert space adjoint of $\mathcal{L}$ and the components on the right hand side of (4.46) are the derivatives of the objective functional with respect to each state, respectively.

The weak formulation of (4.43) is

$$\begin{aligned} &-\int_\Omega \nabla p \cdot \nabla v \, dx - \int_{\partial\Omega} \beta^* p \, v \, ds + \int_\Omega \sigma'(u)|\nabla\varphi|^2 p \, v \, dx \\ &-\int_\Omega \sigma'(u)(\nabla\varphi \cdot \nabla q) v \, dx = \int_\Omega v \, dx \quad \forall\, v \in H^1(\Omega), \\ &2\int_\Omega p \, \sigma(u)\nabla\varphi \cdot \nabla w \, dx - \int_\Omega \sigma(u)\nabla q \cdot \nabla w \, dx = 0 \quad \forall\, w \in H_0^1(\Omega). \end{aligned} \tag{4.47}$$

We justify that the integral $\int_\Omega \sigma'(u)|\nabla\varphi|^2 p \, v \, dx$ in (4.47) makes sense. Recall (2.6) and (2.5):

$$\begin{aligned} \frac{1}{s'} + \frac{1}{s} &= 1, \\ \frac{1}{s'} &= \frac{1}{2} + \frac{1}{r}\,. \end{aligned} \tag{4.48}$$

Then obviously,

$$\frac{1}{s} + \frac{1}{r} + \frac{1}{2} = 1,$$

whence

$$\frac{1}{s} + \frac{1}{r} = \frac{1}{2}\,. \tag{4.49}$$

Adding $1/r$ on both sides of (4.49), we obtain

$$\frac{1}{s} + \frac{1}{r/2} = \frac{1}{2} + \frac{1}{r}\,. \tag{4.50}$$

Now, comparing the second equation in (4.48) with (4.50) we get

$$\frac{1}{s'} = \frac{1}{s} + \frac{1}{r/2}\,. \tag{4.51}$$

Since $p \in H^1(\Omega)$ implies $p \in L^s(\Omega)$ and taking into account that $|\nabla\varphi|^2 \in L^{r/2}(\Omega)$ it follows from (4.51) that $\sigma'(u)|\nabla\varphi|^2 p \in L^{s'}(\Omega)$. Therefore, the integral $\int_\Omega \sigma'(u)|\nabla\varphi|^2 p\, v\, dx$ makes sense as $v \in H^1(\Omega) \subset L^s(\Omega)$.

Next, we show that the solution to the adjoint system (4.44) exists. Define a map $F : H^1(\Omega) \times H_0^1(\Omega) \mapsto H^1(\Omega) \times H_0^1(\Omega)$ with $F(V, W) = (p, q)$ in the following way

$$
\begin{aligned}
&\Delta p + \sigma'(u)|\nabla\varphi|^2 V - \sigma'(u)\nabla\varphi \cdot \nabla W = 1 \text{ in } \Omega, \\
&\nabla \cdot [-2V\, \sigma(u)\nabla\varphi + \sigma(u)\nabla q] = 0 \text{ in } \Omega, \\
&\frac{\partial p}{\partial n} + \beta^* p = 0 \text{ on } \partial\Omega, \\
&q = 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{4.52}
$$

Standard L^2-theory for elliptic problems and the Lax-Milgram Lemma imply that this is a well-defined map. The weak formulation of (4.52) is

$$
\begin{aligned}
&-\int_\Omega \nabla p \cdot \nabla\Theta\, dx - \int_{\partial\Omega} \beta^* p\, \Theta\, ds + \int_\Omega \sigma'(u)|\nabla\varphi|^2 V\, \Theta\, dx \\
&-\int_\Omega \sigma'(u)(\nabla\varphi \cdot \nabla W)\, \Theta\, dx = \int_\Omega \Theta\, dx \quad \forall\, \Theta \in H^1(\Omega), \\
&2\int_\Omega V\, \sigma(u)\nabla\varphi \cdot \nabla\Psi\, dx = \int_\Omega \sigma(u)\nabla q \cdot \nabla\Psi\, dx \quad \forall\, \Psi \in H_0^1(\Omega).
\end{aligned}
$$

We use the Banach fixed point theorem. Let $F(V_1, W_1) = (p_1, q_1)$ and $F(V_2, W_2) = (p_2, q_2)$. We show the contraction property

$$
\|p_1 - p_2\|_{H^1(\Omega)} + \|q_1 - q_2\|_{H_0^1(\Omega)} \leq \delta\Big(\|V_1 - V_2\|_{H^1(\Omega)} + \|W_1 - W_2\|_{H_0^1(\Omega)}\Big) \tag{4.53}
$$

for some $0 < \delta < 1$. Indeed, for $F(V_1, W_1) = (p_1, q_1)$, we have

$$
\begin{aligned}
&-\int_\Omega \nabla p_1 \cdot \nabla\Theta\, dx - \int_{\partial\Omega} \beta^* p_1\, \Theta\, ds + \int_\Omega \sigma'(u)|\nabla\varphi|^2 V_1\, \Theta\, dx \\
&-\int_\Omega \sigma'(u)(\nabla\varphi \cdot \nabla W_1)\, \Theta\, dx = \int_\Omega \Theta\, dx \quad \forall\, \Theta \in H^1(\Omega), \\
&2\int_\Omega V_1\, \sigma(u)\nabla\varphi \cdot \nabla\Psi\, dx = \int_\Omega \sigma(u)\nabla q_1 \cdot \nabla\Psi\, dx \quad \forall\, \Psi \in H_0^1(\Omega).
\end{aligned}
\tag{4.54}
$$

Similarly, for $F(V_2, W_2) = (p_2, q_2)$, we can write

$$
\begin{aligned}
&-\int_\Omega \nabla p_2 \cdot \nabla \Theta \, dx - \int_{\partial\Omega} \beta^* p_2 \Theta \, ds + \int_\Omega \sigma'(u)|\nabla\varphi|^2 V_2 \Theta \, dx \\
&-\int_\Omega \sigma'(u)(\nabla\varphi \cdot \nabla W_2)\Theta \, dx = \int_\Omega \Theta \, dx \quad \forall\, \Theta \in H^1(\Omega), \\
&2\int_\Omega V_2 \, \sigma(u)\nabla\varphi \cdot \nabla\Psi \, dx = \int_\Omega \sigma(u)\nabla q_2 \cdot \nabla\Psi \, dx \quad \forall\, \Psi \in H_0^1(\Omega).
\end{aligned}
\tag{4.55}
$$

Let us denote $\bar{p} \overset{\text{def}}{=} p_1 - p_2$, $\bar{q} \overset{\text{def}}{=} q_1 - q_2$, $\bar{W} \overset{\text{def}}{=} W_1 - W_2$, and $\bar{V} \overset{\text{def}}{=} V_1 - V_2$. Take $\Theta = \bar{p}$ in (4.54) and (4.55)

$$
\begin{aligned}
\int_\Omega \nabla p_1 \cdot \nabla\bar{p}\, dx + \int_{\partial\Omega} \beta^* p_1 \,\bar{p}\, ds + \int_\Omega \bar{p}\, dx &= \int_\Omega \sigma'(u)|\nabla\varphi|^2 V_1 \bar{p}\, dx - \int_\Omega \sigma'(u)\nabla\varphi \cdot \nabla W_1 \,\bar{p}\, dx, \\
\int_\Omega \nabla p_2 \cdot \nabla\bar{p}\, dx + \int_{\partial\Omega} \beta^* p_2 \,\bar{p}\, ds + \int_\Omega \bar{p}\, dx &= \int_\Omega \sigma'(u)|\nabla\varphi|^2 V_2 \bar{p}\, dx - \int_\Omega \sigma'(u)\nabla\varphi \cdot \nabla W_2 \,\bar{p}\, dx,
\end{aligned}
$$

and subtract the second equation from the first one to get

$$
\int_\Omega |\nabla\bar{p}|^2 dx + \int_{\partial\Omega} \beta^* \bar{p}^2 \, ds = \int_\Omega \sigma'(u)|\nabla\varphi|^2 \,\bar{V}\bar{p}\, dx - \int_\Omega \sigma'(u)\nabla\varphi \cdot \nabla\bar{W}\,\bar{p}\, dx. \tag{4.56}
$$

Thus, we obtain

$$
\int_\Omega |\nabla\bar{p}|^2 \, dx + \lambda \int_{\partial\Omega} \bar{p}^2 \, ds \le \left| \int_\Omega \sigma'(u)|\nabla\varphi|^2 \,\bar{V}\bar{p}\, dx \right| + \left| \int_\Omega \sigma'(u)\nabla\varphi \cdot \nabla\bar{W}\,\bar{p}\, dx \right|. \tag{4.57}
$$

We estimate the first term on the right hand side of (4.57). Note that adding $1/s$ to both sides of (4.51) gives

$$
1 = \frac{1}{s} + \frac{1}{s} + \frac{1}{r/2}\,. \tag{4.58}
$$

Now taking into account that $\bar{p} \in H^1(\Omega) \subset L^s(\Omega)$, $|\nabla\varphi|^2 \in L^{r/2}(\Omega)$, and (4.58) we have

$$
\begin{aligned}
\left| \int_\Omega \sigma'(u)|\nabla\varphi|^2 \,\bar{W}\bar{p}\, dx \right| &\le K \int_\Omega |\nabla\varphi|^2 \,\bar{V}\bar{p}\, dx \\
&\le K \||\nabla\varphi|^2\|_{r/2} \|\bar{V}\|_s \|\bar{p}\|_s \\
&\le K M_1^2 \Phi^2 \|\bar{p}\|_{H^1(\Omega)} \|\bar{V}\|_{H^1(\Omega)}
\end{aligned}
\tag{4.59}
$$

where we have used $\|\bar{p}\|_s \leq M_1\|\bar{p}\|_{H^1(\Omega)}$ and $\||\nabla\varphi|^2\|_{r/2} = \|\nabla\varphi\|_r^2 \leq \Phi^2$. Recall that Φ depends on the boundary data φ_0 from (2.17). We also estimate:

$$\left|\int_\Omega \sigma'(u)(\nabla\varphi\cdot\nabla\bar{W})\,\bar{p}\,dx\right| \leq K\int_\Omega |\nabla\varphi|\cdot|\nabla\bar{W}|\cdot|\bar{p}|\,dx$$
$$\leq K\|\nabla\varphi\|_r\|\nabla\bar{W}\|_2\|\bar{p}\|_s \tag{4.60}$$
$$\leq KM_1\Phi\|\nabla\bar{W}\|_2\|\bar{p}\|_{H^1(\Omega)}$$
$$\leq KM_1\Phi\|\bar{W}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}.$$

Taking into account (2.20), (4.59), and (4.60) we rewrite (4.57) as

$$k\|\bar{p}\|^2_{H^1(\Omega)} \leq KM_1^2\Phi^2\|\bar{V}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)} + KM_1\Phi\|\bar{W}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}.$$

This implies

$$\|\bar{p}\|_{H^1(\Omega)} \leq \frac{KM_1^2\Phi^2}{k}\|\bar{V}\|_{H^1(\Omega)} + \frac{KM_1\Phi}{k}\|\bar{W}\|_{H^1(\Omega)}. \tag{4.61}$$

To derive an estimate for $\|\bar{q}\|_{H_0^1(\Omega)}$, take $\Psi = \bar{q}$ in (4.54) and (4.55) to get

$$\begin{aligned} 2\int_\Omega V_1\,\sigma(u)\nabla\varphi\cdot\nabla\bar{q}\,dx &= \int_\Omega \sigma(u)\nabla q_1\cdot\nabla\bar{q}\,dx, \\ 2\int_\Omega V_2\,\sigma(u)\nabla\varphi\cdot\nabla\bar{q}\,dx &= \int_\Omega \sigma(u)\nabla q_2\cdot\nabla\bar{q}\,dx. \end{aligned} \tag{4.62}$$

Subtract the second equation in (4.62) from the first one

$$2\int_\Omega \bar{V}\,\sigma(u)\nabla\varphi\cdot\nabla\bar{q}\,dx = \int_\Omega \sigma(u)|\nabla\bar{q}|^2\,dx,$$

whence

$$C_1\int_\Omega |\nabla\bar{q}|^2\,dx \leq \int_\Omega \sigma(u)|\nabla\bar{q}|^2\,dx = 2\int_\Omega \bar{V}\,\sigma(u)\nabla\varphi\cdot\nabla\bar{q}\,dx \leq 2C_2\|\bar{V}\|_s\|\nabla\varphi\|_r\|\nabla\bar{q}\|_2. \tag{4.63}$$

Consequently,

$$\|\nabla \bar{q}\|_2 \le \frac{2C_2\Phi}{C_1}\|\bar{V}\|_s \le \frac{2C_2\Phi M_1}{C_1}\|\bar{V}\|_{H^1(\Omega)}. \tag{4.64}$$

Recall Poincaré's inequality

$$\|\bar{q}\|_{H_0^1(\Omega)} \le C_6\|\nabla \bar{q}\|_2. \tag{4.65}$$

Thus, taking into consideration (4.65), we obtain from (4.64)

$$\|\bar{q}\|_{H_0^1(\Omega)} \le C_6\|\nabla \bar{q}\|_2 \le \frac{2C_6C_2\Phi M_1}{C_1}\|\bar{V}\|_{H^1(\Omega)}. \tag{4.66}$$

Now if we add (4.61) and (4.66) we get

$$\begin{aligned}\|\bar{p}\|_{H^1(\Omega)} + \|\bar{q}\|_{H_0^1(\Omega)} &\le \left(\frac{KM_1^2\Phi^2}{k} + \frac{2C_6C_2\Phi M_1}{C_1}\right)\|\bar{V}\|_{H^1(\Omega)} + \frac{KM_1\Phi}{k}\|\bar{W}\|_{H^1(\Omega)} \\ &\le \max\left\{\frac{KM_1^2\Phi^2}{k} + \frac{2C_6C_2\Phi M_1}{C_1}, \frac{KM_1\Phi}{k}\right\}\Big(\|\bar{W}\|_{H^1(\Omega)} + \|\bar{V}\|_{H^1(\Omega)}\Big).\end{aligned} \tag{4.67}$$

Recalling that $\Phi = c_1\|\nabla\varphi_0\|_\infty + c_2\|\varphi_0\|_{W^{1,\infty}(\Omega)}$, where $c_1, c_2 > 0$ denote generic constants (see (2.17)) and choosing φ_0 so that $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ is small, we obtain

$$\delta = \max\left\{\frac{KM_1^2\Phi^2 + KM_1\Phi}{k}, \frac{2C_6C_2\Phi M_1}{C_1}\right\} < 1.$$

This proves (4.53), the contraction property, which gives the desired fixed point of the map and therefore the existence of solutions to the adjoint system.

Recall that for variation $\ell \in L^\infty(\Omega)$, with $\beta^* + \varepsilon\ell \in U_M$ for ε small, the weak formulation of the sensitivity system (4.2) is given by

$$
\begin{aligned}
&-\int_\Omega \nabla\psi_1\cdot\nabla v\,dx - \int_{\partial\Omega}\beta\psi_1 v\,ds + \int_\Omega \sigma'(u)\psi_1|\nabla\varphi|^2 v\,dx \\
&+2\int_\Omega v\sigma(u)\nabla\psi_2\cdot\nabla\varphi\,dx = \int_{\partial\Omega}\ell uv\,ds \quad \forall\, v\in H^1(\Omega), \qquad (4.68)\\
&\int_\Omega \sigma'(u)\psi_1\nabla\varphi\cdot\nabla w\,dx + \int_\Omega \sigma(u)\nabla\psi_2\cdot\nabla w\,dx = 0 \quad \forall\, w\in H_0^1(\Omega). \qquad (4.69)
\end{aligned}
$$

Now we are ready to characterize the optimal control. Since the minimum of J is achieved at β^* and for small $\varepsilon > 0$, $\beta^* + \varepsilon\ell \in U_M$, and denoting $u^\varepsilon = u(\beta^* + \varepsilon\ell), \varphi^\varepsilon = \varphi(\beta^* + \varepsilon\ell)$, we obtain

$$
\begin{aligned}
0 &\le \lim_{\varepsilon\to 0^+} \frac{J(\beta^*+\varepsilon\ell) - J(\beta^*)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\int_\Omega u^\varepsilon\,dx + \int_{\partial\Omega}(\beta^*+\varepsilon\ell)^2\,ds - \int_\Omega u^*\,dx - \int_{\partial\Omega}(\beta^*)^2\,ds\right] \\
&= \lim_{\varepsilon\to 0^+}\left[\int_\Omega \frac{u^\varepsilon - u^*}{\varepsilon}\,dx + \int_{\partial\Omega}\frac{(\beta^*+\varepsilon\ell)^2 - (\beta^*)^2}{\varepsilon}\,ds\right] \\
&= \lim_{\varepsilon\to 0^+}\int_\Omega \frac{u^\varepsilon - u^*}{\varepsilon}\,dx + \lim_{\varepsilon\to 0^+}\int_{\partial\Omega}\frac{(\beta^*)^2 + 2\beta^*\varepsilon\ell + \varepsilon^2\ell^2 - (\beta^*)^2}{\varepsilon}\,ds \\
&= \int_\Omega \psi_1\,dx + \int_{\partial\Omega} 2\beta^*\ell\,ds \\
&= \int_\Omega \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\,dx + \int_{\partial\Omega} 2\beta^*\ell\,ds \\
&= -\int_\Omega \nabla p\cdot\nabla\psi_1\,dx - \int_{\partial\Omega}\beta^* p\,\psi_1\,ds + \int_\Omega \psi_1\,\sigma'(u)|\nabla\varphi|^2 p\,dx - \int_\Omega \psi_1\,\sigma'(u)\nabla\varphi\cdot\nabla q\,dx \\
&+ 2\int_\Omega p\,\sigma(u)\nabla\varphi\cdot\nabla\psi_2\,dx - \int_\Omega \sigma(u)\nabla q\cdot\nabla\psi_2\,dx + \int_{\partial\Omega} 2\beta^*\ell\,ds \\
&= \Big\{ -\int_\Omega \nabla p\nabla\psi_1\,dx - \int_{\partial\Omega}\beta^* p\,\psi_1\,ds + \int_\Omega \psi_1\,\sigma'(u)|\nabla\varphi|^2 p\,dx + 2\int_\Omega p\,\sigma(u)\nabla\varphi\nabla\psi_2\,dx\Big\} \\
&+ \Big[-\int_\Omega \psi_1\,\sigma'(u)\nabla\varphi\cdot\nabla q\,dx - \int_\Omega \sigma(u)\nabla q\cdot\nabla\psi_2\,dx\Big] + \int_{\partial\Omega} 2\beta^*\ell\,ds \\
&= \int_{\partial\Omega}\ell up\,ds + \int_{\partial\Omega} 2\beta^*\ell\,ds,
\end{aligned}
$$

where we integrated by parts and used the sensitivity system (4.68), (4.69) with test functions p, q. Thus, we obtain

$$\int_{\partial\Omega} \ell(2\beta^* + up)\, ds \geq 0. \tag{4.70}$$

i) Take the variation ℓ to have support on the set $\{x \in \partial\Omega : \lambda < \beta^*(x) < M\}$. The variation $\ell(x)$ can be of any sign, therefore we obtain

$$2\beta^* + up = 0$$

whence

$$\beta^* = -\frac{up}{2}\,.$$

ii) On the set $\{x \in \partial\Omega : \beta^*(x) = M\}$, the variation must satisfy $\ell(x) \leq 0$ and therefore we get

$$2\beta^* + up \leq 0$$

implying

$$M = \beta^*(x) \leq -\frac{up}{2}\,.$$

iii) On the set $\{x \in \partial\Omega : \beta^*(x) = \lambda\}$, the variation must satisfy $\ell(x) \geq 0$. This implies

$$2\beta^* + up \leq 0$$

and hence

$$\lambda = \beta^*(x) \geq -\frac{up}{2}\,.$$

Combining cases (i), (ii), and (iii) gives

$$\beta^*(x) = \begin{cases} -\frac{up}{2}, \text{ if } \lambda < -\frac{up}{2} < M, \\ M, \text{ if } -\frac{up}{2} \geq M, \\ \lambda, \text{ if } -\frac{up}{2} \leq \lambda. \end{cases} \tag{4.71}$$

We can write (4.71) compactly as

$$\beta^*(x) = \min\left(\max\left(-\frac{up}{2}, \lambda\right), M\right). \tag{4.72}$$

This yields our desired characterization of optimal control β^*. $\square$

Substituting (4.72) into the state system (1.1) and the adjoint equations (4.43) we obtain the following optimality system:

$$\begin{aligned}
&\Delta u^* + \sigma(u^*)|\nabla\varphi^*|^2 = 0 \text{ in } \Omega,\\
&\nabla\cdot(\sigma(u^*)\nabla\varphi^*) = 0 \text{ in } \Omega,\\
&\Delta p + \sigma'(u^*)|\nabla\varphi^*|^2 p - \sigma'(u^*)\nabla\varphi^*\cdot\nabla q = 1 \text{ in } \Omega,\\
&\nabla\cdot[-2p\,\sigma(u^*)\nabla\varphi^* + \sigma(u^*)\nabla q] = 0 \text{ in } \Omega,\\
&\frac{\partial p}{\partial n} + \min(\max(-u^*p/2, \lambda), M)p = 0 \text{ on } \partial\Omega,\\
&q = 0 \text{ on } \partial\Omega,\\
&\frac{\partial u^*}{\partial n} + \min(\max(-u^*p/2, \lambda), M)u^* = 0 \text{ on } \partial\Omega,\\
&\varphi^* = \varphi_0 \text{ on } \partial\Omega.
\end{aligned} \tag{4.73}$$

Remarks. We make the following observations concerning the system (4.73).

(1) If we take into account the second equation in (4.73) then the last equation in (4.73) can be rewritten as

$$-2\sigma(u^*)\nabla\varphi^*\cdot\nabla p + \nabla\cdot[\sigma(u^*)\nabla q] = 0.$$

(2) We show that if $\varphi_0 \equiv$ constant then $\beta^* = \lambda$. Indeed, the maximum principle for weak solutions implies that $\varphi^* \equiv$ constant and therefore $\nabla\varphi^* = 0$. This simplifies (4.73) significantly. Namely, we obtain

$$
\begin{aligned}
&\Delta u^* = 0 \text{ in } \Omega, \\
&\Delta p = 1 \text{ in } \Omega, \\
&\nabla \cdot [\sigma(u^*) \nabla q] = 0 \text{ in } \Omega, \\
&\frac{\partial u^*}{\partial n} + \beta^* u^* = 0 \text{ on } \partial\Omega, \\
&\frac{\partial p}{\partial n} + \beta^* p = 0 \text{ on } \partial\Omega, \\
&q = 0 \text{ on } \partial\Omega.
\end{aligned} \tag{4.74}
$$

Applying the maximum principle to q gives $q \equiv 0$ in $\bar{\Omega}$, and the system becomes

$$
\begin{aligned}
&\Delta u^* = 0 \text{ in } \Omega, \\
&\Delta p = 1 \text{ in } \Omega, \\
&\frac{\partial u^*}{\partial n} + \beta^* u^* = 0 \text{ on } \partial\Omega, \\
&\frac{\partial p}{\partial n} + \beta^* p = 0 \text{ on } \partial\Omega.
\end{aligned} \tag{4.75}
$$

Now if we integrate the first equation in (4.75) over Ω and integrate by parts we obtain $\int_{\partial\Omega} \beta^* u^* \, ds = 0$. Since $\beta^*(x) > 0$ and $u^*(x) \geq 0$ for all $x \in \partial\Omega$ it follows that $u^* = 0$ a.e. on $\partial\Omega$. Hence, we can write

$$
\begin{aligned}
&\Delta u^* = 0 \text{ in } \Omega, \\
&u^* = 0 \text{ a.e. on } \partial\Omega.
\end{aligned} \tag{4.76}
$$

This implies that $u^* = 0$ a.e. in Ω. Thus, if we take into account (4.72) we get $\beta^* = \lambda$ and

$$
\begin{aligned}
&\Delta p = 1 \text{ in } \Omega, \\
&\frac{\partial p}{\partial n} + \lambda p = 0 \text{ on } \partial\Omega.
\end{aligned} \tag{4.77}
$$

Observe that for (4.77) to be solvable we have to have a compatibility condition

$$\int_{\partial\Omega} \frac{\partial p}{\partial n}\, ds = \int_{\Omega} dx$$

which implies

$$-\lambda \int_{\partial\Omega} p\, ds = meas(\Omega).$$

Chapter 5

Uniqueness of the Optimal Control

In this chapter we prove that the solution to the optimality system (4.73) is unique under some additional assumptions. We assume that $\sigma' \in C^{0,1}(\mathbb{R})$ with Lipschitz constant $R > 0$,

$$|\sigma'(s_1) - \sigma'(s_2)| \leq R|s_1 - s_2| \quad \text{for all } s_1, s_2 \in \mathbb{R}. \tag{5.1}$$

We have the following

Theorem 5.1. *In addition to all the standard assumptions from chapter 2, let* (5.1) *hold. If λ is large enough while $\|\varphi_0\|_{W^{1,\infty}(\Omega)}$ is sufficiently small, then solution to the optimality system* (4.73) *with u and p components being in $L^\infty(\Omega)$, is unique, and therefore the optimal control β^* is unique.*

Proof. Weak formulation of the first two equations in (4.73) is

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta u v \, ds = \int_\Omega \sigma(u)|\nabla\varphi|^2 v \, dx \quad \forall\, v \in H^1(\Omega), \\
&\int_\Omega \sigma(u)\nabla\varphi \cdot \nabla w \, dx = 0, \; \varphi - \varphi_0 \in H_0^1(\Omega), \, \forall\, w \in H_0^1(\Omega).
\end{aligned}$$

The weak formulation for the adjoint system is given by (4.47):

$$\begin{aligned}
-\int_\Omega \nabla p \cdot \nabla \tilde{v} \, dx - \int_{\partial\Omega} \beta p \tilde{v} \, ds + \int_\Omega \sigma'(u)|\nabla\varphi|^2 p \, \tilde{v} \, dx - \int_\Omega \sigma'(u)\nabla\varphi \cdot \nabla q \, \tilde{v} \, dx = \int_\Omega \tilde{v} \, dx, \\
2\int_\Omega p \, \sigma(u)\nabla\varphi \cdot \nabla\tilde{w} \, dx - \int_\Omega \sigma(u)\nabla q \cdot \nabla \tilde{w} \, dx = 0 \quad \forall\, \tilde{v} \in H^1(\Omega), \forall\, \tilde{w} \in H_0^1(\Omega).
\end{aligned}$$

Observe that since $p \in L^\infty(\Omega)$ it follows that $\nabla q \in L^r(\Omega)$ by the Meyers estimate. Let $(u_1, \varphi_1, p_1, q_1)$ and $(u_2, \varphi_2, p_2, q_2)$ be two solutions to the optimality system. We have

$$\begin{aligned}
&\int_\Omega \nabla u_1 \cdot \nabla v\, dx + \int_{\partial\Omega} \beta_1 u_1 v\, ds = \int_\Omega \sigma(u_1)|\nabla\varphi_1|^2 v\, dx, \quad \forall\, v \in H^1(\Omega),\\
&\int_\Omega \sigma(u_1)\nabla\varphi_1 \cdot \nabla w\, dx = 0,\ \varphi_0 - \varphi_1 \in H_0^1(\Omega),\ \forall\, w \in H_0^1(\Omega),\\
&\int_\Omega \nabla p_1 \cdot \nabla\tilde{v}\, dx + \int_{\partial\Omega} \beta_1 p_1 \tilde{v}\, ds + \int_\Omega \tilde{v}\, dx = \int_\Omega \sigma'(u_1)|\nabla\varphi_1|^2 p_1\, \tilde{v}\, dx \\
&- \int_\Omega \sigma'(u_1)\nabla\varphi_1 \cdot \nabla q_1\, \tilde{v}\, dx, \quad \forall\, \tilde{v} \in H^1(\Omega),\\
&2\int_\Omega p_1\, \sigma(u_1)\nabla\varphi_1 \cdot \nabla\tilde{w}\, dx = \int_\Omega \sigma(u_1)\nabla q_1 \cdot \nabla\tilde{w}\, dx \quad \forall\, \tilde{w} \in H_0^1(\Omega),
\end{aligned} \tag{5.2}$$

where $\beta_1 = \min(\max(-u_1 p_1/2, \lambda), M)$. We write a similar system for $(u_2, \varphi_2, p_2, q_2)$:

$$\begin{aligned}
&\int_\Omega \nabla u_2 \cdot \nabla v\, dx + \int_{\partial\Omega} \beta_2 u_2 v\, ds = \int_\Omega \sigma(u_2)|\nabla\varphi_2|^2 v\, dx, \quad \forall\, v \in H^1(\Omega),\\
&\int_\Omega \sigma(u_2)\nabla\varphi_2 \cdot \nabla w\, dx = 0,\ \varphi_0 - \varphi_2 \in H_0^1(\Omega),\ \forall\, w \in H_0^1(\Omega),\\
&\int_\Omega \nabla p_2 \cdot \nabla\tilde{v}\, dx + \int_{\partial\Omega} \beta_2 p_2 \tilde{v}\, ds + \int_\Omega \tilde{v}\, dx = \int_\Omega \sigma'(u_2)|\nabla\varphi_2|^2 p_2\, \tilde{v}\, dx \\
&- \int_\Omega \sigma'(u_2)\nabla\varphi_2 \cdot \nabla q_2\, \tilde{v}\, dx, \quad \forall\, \tilde{v} \in H^1(\Omega),\\
&2\int_\Omega p_1\, \sigma(u_2)\nabla\varphi_2 \cdot \nabla\tilde{w}\, dx = \int_\Omega \sigma(u_2)\nabla q_2 \cdot \nabla\tilde{w}\, dx \quad \forall\, \tilde{w} \in H_0^1(\Omega).
\end{aligned} \tag{5.3}$$

We set $\bar{u} = u_1 - u_2$, $\bar{\varphi} = \varphi_1 - \varphi_2$, $\bar{p} = p_1 - p_2$, and $\bar{q} = q_1 - q_2$. Now, taking $w = \varphi_1 - \varphi_2$ in (5.2) and $w = \varphi_1 - \varphi_1$ in (5.3) yields

$$\int_\Omega \sigma(u_1)\nabla\varphi_1 \cdot \nabla(\varphi_1 - \varphi_2)\, dx = \int_\Omega \sigma(u_2)\nabla\varphi_2 \cdot \nabla(\varphi_1 - \varphi_2)\, dx. \tag{5.4}$$

Adding $-\int_\Omega \sigma(u_1)\nabla\varphi_2 \cdot \nabla(\varphi_1 - \varphi_2)\, dx$ to both sides of (5.4), it is easy to check that the following relation holds

$$\int_\Omega \sigma(u_1)|\nabla\bar{\varphi}|^2\, dx = \int_\Omega (\sigma(u_2) - \sigma(u_1))\nabla\varphi_2 \cdot \nabla\bar{\varphi}\, dx.$$

Hence,

$$
\begin{aligned}
C_1\|\nabla\bar{\varphi}\|_2^2 &= C_1\int_\Omega |\nabla\bar{\varphi}|^2\,dx \\
&\le \int_\Omega \sigma(u_1)|\nabla\bar{\varphi}|^2\,dx \\
&= \int_\Omega (\sigma(u_2)-\sigma(u_1))\nabla\varphi_2\cdot\nabla\bar{\varphi}\,dx \\
&\le K\int_\Omega |\bar{u}|\;|\nabla\varphi_2|\;|\nabla\bar{\varphi}|\,dx \\
&\le K\Phi\|\bar{u}\|_s\|\nabla\bar{\varphi}\|_2.
\end{aligned}
$$

This implies the following estimate

$$
\|\nabla\bar{\varphi}\|_2 \le \frac{K\Phi}{C_1}\|\bar{u}\|_s \tag{5.5}
$$

which will be used throughout the proof.

Take $v=\bar{u}$ and $\tilde{v}=\bar{u}$ in (5.2) and (5.3) to get

$$
\begin{aligned}
\int_\Omega \nabla u_1\cdot\nabla\bar{u}\,dx + \int_{\partial\Omega}\beta_1 u_1\bar{u}\,ds &= \int_\Omega \sigma(u_1)|\nabla\varphi_1|^2\bar{u}\,dx \\
\int_\Omega \nabla u_2\cdot\nabla\bar{u}\,dx + \int_{\partial\Omega}\beta_2 u_2\bar{u}\,ds &= \int_\Omega \sigma(u_2)|\nabla\varphi_2|^2\bar{u}\,dx
\end{aligned} \tag{5.6}
$$

Subtract second equation in (5.6) from the first

$$
\int_\Omega |\nabla\bar{u}|^2\,dx + \int_{\partial\Omega}(\beta_1u_1-\beta_2u_2)\bar{u}\,ds = \int_\Omega\Big(\sigma(u_1)|\nabla\varphi_1|^2-\sigma(u_2)|\nabla\varphi_2|^2\Big)\bar{u}\,dx.
$$

We can write

$$
\int_\Omega |\nabla\bar{u}|^2\,dx + \int_{\partial\Omega}\beta_1\bar{u}^2\,ds + \int_\Omega(\beta_1-\beta_2)u_2\bar{u}\,ds = \int_\Omega\Big(\sigma(u_1)|\nabla\varphi_1|^2-\sigma(u_2)|\nabla\varphi_2|^2\Big)\bar{u}\,dx.
$$

Thus, we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla \bar{u}|^2\,dx + \lambda \int_{\partial\Omega} \bar{u}^2\,ds \le \int_\Omega |\nabla \bar{u}|^2\,dx + \int_{\partial\Omega} \beta_1 \bar{u}^2\,ds \\
&= \left| \int_{\partial\Omega} (\beta_2-\beta_1)u_2\bar{u}\,ds + \int_\Omega \Big(\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big)\bar{u}\,dx \right| \\
&\le \int_{\partial\Omega} |\beta_1-\beta_2|\cdot|u_2|\cdot|\bar{u}|\,ds + \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big|\cdot|\bar{u}|\,dx \\
&\le \frac{1}{2}\int_{\partial\Omega} |u_1p_1-u_2p_2|\cdot|u_2|\cdot|\bar{u}|\,ds + \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big|\cdot|\bar{u}|\,dx \\
&= \frac{1}{2}\int_{\partial\Omega} |u_1p_1-u_1p_2+u_1p_2-u_2p_2|\cdot|u_2|\cdot|\bar{u}|\,ds + \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big||\bar{u}|\,dx \\
&= \frac{1}{2}\int_{\partial\Omega} |u_1\bar{p}+\bar{u}p_2|\cdot|u_2|\cdot|\bar{u}|\,ds + \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big|\cdot|\bar{u}|\,dx \\
&\le \frac{1}{2}\int_{\partial\Omega} |u_1|\cdot|u_2||\bar{u}|\cdot|\bar{p}|\,ds + \frac{1}{2}\int_{\partial\Omega} |p_2|\bar{u}^2|u_2|\,ds + \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big||\bar{u}|\,dx \\
&\le \frac{1}{2}\|u_1\|_\infty\|u_2\|_\infty \int_{\partial\Omega} |\bar{u}|\cdot|\bar{p}|\,ds + \frac{1}{2}\|p_2\|_\infty\|u_2\|_\infty \int_{\partial\Omega} \bar{u}^2\,ds \\
&+ \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big|\cdot|\bar{u}|\,dx \\
&\le \frac{1}{2}\|u_1\|_\infty\|u_2\|_\infty\|\bar{u}\|_{L^2(\partial\Omega)}\|\bar{p}\|_{L^2(\partial\Omega)} + \frac{1}{2}\|p_2\|_\infty\|u_2\|_\infty\|\bar{u}\|^2_{L^2(\partial\Omega)} \\
&+ \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big|\cdot|\bar{u}|\,dx,
\end{aligned}
$$

where $|\beta_1-\beta_2| \le \frac{1}{2}|u_1p_1-u_2p_2|$. Denote $\alpha_1 \stackrel{\text{def}}{=} \|u_1\|_\infty\|u_2\|_\infty$ and $\alpha_2 \stackrel{\text{def}}{=} \|p_2\|_\infty\|u_2\|_\infty$. Taking into account Cauchy's inequality, for any $a, b \in \mathbb{R}$

$$
a\cdot b \le \frac{a^2}{2} + \frac{b^2}{2} \tag{5.7}
$$

we can write

$$
\begin{aligned}
&\int_\Omega |\nabla\bar{u}|^2\,dx + \Big(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4}\Big)\int_{\partial\Omega} \bar{u}^2\,ds \\
&\le \frac{\alpha_1}{4}\|\bar{p}\|^2_{L^2(\partial\Omega)} + \int_\Omega \Big|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\Big|\cdot|\bar{u}|\,dx.
\end{aligned}
$$

We deal with the last term

$$\int_\Omega \left|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\right| \cdot |\bar{u}|\, dx \le \|\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2\|_{s'}\|\bar{u}\|_s$$
$$\le \|\sigma(u_1)\nabla\bar{\varphi}\cdot\nabla(\varphi_1+\varphi_2)\|_{s'}\|\bar{u}\|_s + \|\big(\sigma(u_1)-\sigma(u_2)\big)|\nabla\varphi_2|^2\|_{s'}\|\bar{u}\|_s$$
$$\le C_2\|\nabla\bar{\varphi}\|_2\cdot\|\nabla(\varphi_1+\varphi_2)\|_r\cdot\|\bar{u}\|_s + K\|\bar{u}\,|\nabla\varphi_2|^2\|_{s'}\cdot\|\bar{u}\|_s$$
$$\le \frac{C_2K\Phi}{C_1}\|\nabla(\varphi_1+\varphi_2)\|_r\cdot\|\bar{u}\|_s^2 + K\||\nabla\varphi_2|^2\|_{r/2}\|\bar{u}\|_s^2$$
$$\le \frac{2C_2K\Phi^2}{C_1}\|\bar{u}\|_s^2 + K\Phi^2\|\bar{u}\|_s^2$$
$$\le \frac{2C_2K\Phi^2M_1^2}{C_1}\|\bar{u}\|_{H^1(\Omega)}^2 + K\Phi^2M_1^2\|\bar{u}\|_{H^1(\Omega)}^2,$$

where we used (4.48), (4.51), (5.5), and where we have taken into account that $\|\nabla\varphi\|_r \le \Phi$ and $\|\bar{u}\|_s \le M_1\|\bar{u}\|_{H^1(\Omega)}$.

Consequently,

$$\int_\Omega |\nabla\bar{u}|^2\, dx + \Big(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4}\Big)\int_{\partial\Omega}\bar{u}^2\, ds$$
$$\le \frac{\alpha_1}{4}\|\bar{p}\|_{L^2(\partial\Omega)}^2 + \frac{2C_2K\Phi^2M_1^2}{C_1}\|\bar{u}\|_{H^1(\Omega)}^2 + K\Phi^2M_1^2\|\bar{u}\|_{H^1(\Omega)}^2.$$

Therefore,

$$\int_\Omega |\nabla\bar{u}|^2\, dx + \Big(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4}\Big)\int_{\partial\Omega}\bar{u}^2\, ds \le \frac{\alpha_1}{4}\|\bar{p}\|_{L^2(\partial\Omega)}^2 + M_3\|\bar{u}\|_{H^1(\Omega)}^2 \tag{5.8}$$

where $M_3 \stackrel{\text{def}}{=} \frac{2C_2K\Phi^2M_1^2}{C_1} + K\Phi^2M_1^2$.

To derive an estimate for $\bar{p}$ we take test functions in the equations for p to be $\bar{p}$, namely:

$$\int_\Omega \nabla p_1\nabla\bar{p}\, dx + \int_{\partial\Omega}\beta_1p_1\bar{p}\, ds + \int_\Omega \bar{p}\, dx = \int_\Omega \sigma'(u_1)|\nabla\varphi_1|^2p_1\,\bar{p}\, dx - \int_\Omega \sigma'(u_1)\nabla\varphi_1\nabla q_1\,\bar{p}\, dx$$
$$\int_\Omega \nabla p_2\nabla\bar{p}\, dx + \int_{\partial\Omega}\beta_2p_2\bar{p}\, ds + \int_\Omega \bar{p}\, dx = \int_\Omega \sigma'(u_2)|\nabla\varphi_2|^2p_2\,\bar{p}\, dx - \int_\Omega \sigma'(u_2)\nabla\varphi_2\nabla q_2\,\bar{p}\, dx.$$

Subtracting the second equation from the first yields

$$\int_\Omega |\nabla \bar{p}|^2\,dx + \int_{\partial\Omega} (\beta_1 p_1 - \beta_2 p_2)\bar{p}\,ds = \int_\Omega \big(\sigma'(u_1)|\nabla\varphi_1|^2 p_1 - \sigma'(u_2)|\nabla\varphi_2|^2 p_2\big)\bar{p}\,dx$$
$$+ \int_\Omega \big(\sigma'(u_2)\nabla\varphi_2\cdot\nabla q_2 - \sigma'(u_1)\nabla\varphi_1\cdot\nabla q_1\big)\bar{p}\,dx.$$

We have

$$\int_\Omega |\nabla \bar{p}|^2\,dx + \int_{\partial\Omega} (\beta_1 p_1 - \beta_1 p_2 + \beta_1 p_2 - \beta_2 p_2)\bar{p}\,ds$$
$$= \int_\Omega \big(\sigma'(u_1)|\nabla\varphi_1|^2 p_1 - \sigma'(u_2)|\nabla\varphi_2|^2 p_2\big)\bar{p}\,dx + \int_\Omega \big(\sigma'(u_2)\nabla\varphi_2\nabla q_2 - \sigma'(u_1)\nabla\varphi_1\nabla q_1\big)\bar{p}\,dx,$$

whence

$$\int_\Omega |\nabla \bar{p}|^2\,dx + \int_{\partial\Omega} \beta_1\bar{p}^2\,ds + \int_{\partial\Omega}(\beta_1-\beta_2)p_2\bar{p}\,ds$$
$$= \int_\Omega \big(\sigma'(u_1)|\nabla\varphi_1|^2 p_1 - \sigma'(u_2)|\nabla\varphi_2|^2 p_2\big)\bar{p}\,dx + \int_\Omega \big(\sigma'(u_2)\nabla\varphi_2\nabla q_2 - \sigma'(u_1)\nabla\varphi_1\nabla q_1\big)\bar{p}\,dx.$$

Therefore, we obtain

$$\begin{aligned}\int_\Omega |\nabla \bar{p}|^2\,dx + \lambda\int_{\partial\Omega}\bar{p}^2\,ds &\le \int_\Omega |\nabla \bar{p}|^2\,dx + \int_{\partial\Omega}\beta_1\bar{p}^2\,ds \\ &\le \int_{\partial\Omega}|\beta_1-\beta_2|\cdot|p_2|\cdot|\bar{p}|\,ds + \mathcal{A} + \mathcal{B},\end{aligned} \tag{5.9}$$

where

$$\mathcal{A} \stackrel{\text{def}}{=} \int_\Omega \big|\sigma'(u_1)|\nabla\varphi_1|^2 p_1 - \sigma'(u_2)|\nabla\varphi_2|^2 p_2\big|\cdot|\bar{p}|\,dx,$$
$$\mathcal{B} \stackrel{\text{def}}{=} \int_\Omega |\sigma'(u_1)\nabla\varphi_1\cdot\nabla q_1 - \sigma'(u_2)\nabla\varphi_2\cdot\nabla q_2|\cdot|\bar{p}|\,dx.$$

Denote $\alpha_3 \stackrel{\text{def}}{=} \|u_1\|_\infty\|p_2\|_\infty$ and $\alpha_4 \stackrel{\text{def}}{=} \|p_2\|_\infty$. We estimate the first term on the right hand side of (5.9):

$$\begin{aligned}
\int_{\partial\Omega} |\beta_1 - \beta_2| \cdot |p_2| \cdot |\bar{p}| \, ds &\le \frac{1}{2} \int_{\partial\Omega} |u_1 p_1 - u_2 p_2| \cdot |p_2| \cdot |\bar{p}| \, ds \\
&= \frac{1}{2} \int_{\partial\Omega} |u_1 \bar{p} + \bar{u} p_2| \cdot |p_2| \cdot |\bar{p}| \, ds \\
&\le \frac{1}{2} \int_{\partial\Omega} |u_1| \cdot |p_2| \, \bar{p}^2 \, ds + \frac{1}{2} \int_{\partial\Omega} |\bar{u}| \cdot |\bar{p}| \, p_2^2 \, ds \qquad (5.10) \\
&\le \frac{\alpha_3}{2} \|\bar{p}\|^2_{L^2(\partial\Omega)} + \frac{\alpha_4^2}{2} \int_{\partial\Omega} |\bar{u}| \cdot |\bar{p}| \, ds \\
&\le \frac{\alpha_3}{2} \|\bar{p}\|^2_{L^2(\partial\Omega)} + \frac{\alpha_4^2}{4} \|\bar{u}\|^2_{L^2(\partial\Omega)} + \frac{\alpha_4^2}{4} \|\bar{p}\|^2_{L^2(\partial\Omega)}.
\end{aligned}$$

Let us estimate $\mathcal{A}$:

$$\begin{aligned}
\mathcal{A} &\stackrel{\text{def}}{=} \int_{\Omega} \left| \sigma'(u_1) |\nabla\varphi_1|^2 p_1 - \sigma'(u_2) |\nabla\varphi_2|^2 p_2 \right| \cdot |\bar{p}| \, dx \\
&\le \|\sigma'(u_1)|\nabla\varphi_1|^2 p_1 - \sigma'(u_2)|\nabla\varphi_2|^2 p_2\|_{s'} \|\bar{p}\|_s \\
&= \|\sigma'(u_1)|\nabla\varphi_1|^2 p_1 - \sigma'(u_1)|\nabla\varphi_1|^2 p_2 + \sigma'(u_1)|\nabla\varphi_1|^2 p_2 - \sigma'(u_2)|\nabla\varphi_2|^2 p_2\|_{s'} \|\bar{p}\|_s \\
&\le \|\sigma'(u_1)|\nabla\varphi_1|^2 \bar{p}\|_{s'} \|\bar{p}\|_s + \|\big(\sigma'(u_1)|\nabla\varphi_1|^2 - \sigma'(u_2)|\nabla\varphi_2|^2\big) p_2\|_{s'} \|\bar{p}\|_s \\
&\le K \big\| |\nabla\varphi_1|^2 \bar{p} \big\|_{s'} \|\bar{p}\|_s + \alpha_4 \|\sigma'(u_1) \nabla\bar{\varphi} \cdot \nabla(\varphi_1 + \varphi_2)\|_{s'} \|\bar{p}\|_s \\
&+ \alpha_4 \|\big(\sigma'(u_1) - \sigma'(u_2)\big) |\nabla\varphi_2|^2\|_{s'} \|\bar{p}\|_s \\
&\le K \|\bar{p}\|_s \big\| |\nabla\varphi_1|^2 \big\|_{\frac{r}{2}} \|\bar{p}\|_s + \frac{2C_2 K \Phi^2 \alpha_4}{C_1} \|\bar{u}\|_s \|\bar{p}\|_s + R\,\Phi^2 \alpha_4 \|\bar{u}\|_s \|\bar{p}\|_s \\
&\le K\Phi^2 \|\bar{p}\|_s^2 + \left(\frac{2C_2 K \Phi^2 \alpha_4}{C_1} + R\,\Phi^2 \alpha_4 \right) \|\bar{u}\|_s \|\bar{p}\|_s \\
&\le K\Phi^2 M_1^2 \|\bar{p}\|^2_{H^1(\Omega)} + \left(\frac{2C_2 K \Phi^2 \alpha_4 M_1^2}{C_1} + R\,\Phi^2 \alpha_4 M_1^2 \right) \|\bar{u}\|_{H^1(\Omega)} \|\bar{p}\|_{H^1(\Omega)},
\end{aligned}$$

where we have taken into account that (4.51) allows us to write

$$\big\| |\nabla\varphi_1|^2 \bar{p} \big\|_{s'} \le \|\bar{p}\|_s \big\| |\nabla\varphi_1|^2 \big\|_{\frac{r}{2}}$$

and $s > s'$. Thus, we obtain

$$\mathcal{A} \le K\Phi^2 M_1^2 \|\bar{p}\|^2_{H^1(\Omega)} + \left(\frac{2C_2 K \Phi^2 \alpha_4 M_1^2}{C_1} + R\,\Phi^2 \alpha_4 M_1^2 \right) \|\bar{u}\|_{H^1(\Omega)} \|\bar{p}\|_{H^1(\Omega)}. \qquad (5.11)$$

Finally, we estimate $\mathcal{B}$:

$$
\begin{aligned}
\mathcal{B} &\overset{\text{def}}{=} \int_\Omega |\sigma'(u_1)\nabla\varphi_1 \cdot \nabla q_1 - \sigma'(u_2)\nabla\varphi_2 \cdot \nabla q_2| \cdot |\bar{p}|\,dx \\
&\le \|\sigma'(u_1)\nabla\varphi_1 \cdot \nabla q_1 - \sigma'(u_2)\nabla\varphi_2 \cdot \nabla q_2\|_{s'}\|\bar{p}\|_s \\
&= \|\sigma'(u_1)\nabla\varphi_1 \cdot \nabla q_1 - \sigma'(u_1)\nabla\varphi_1 \cdot \nabla q_2 + \sigma'(u_1)\nabla\varphi_1 \cdot \nabla q_2 - \sigma'(u_2)\nabla\varphi_2 \cdot \nabla q_2\|_{s'}\|\bar{p}\|_s \\
&\le \|\sigma'(u_1)\nabla\varphi_1 \cdot \nabla\bar{q}\|_{s'}\|\bar{p}\|_s + \|(\sigma'(u_1)\nabla\varphi_1 - \sigma'(u_2)\nabla\varphi_2) \cdot \nabla q_2\|_{s'}\|\bar{p}\|_s \\
&\le K\|\nabla\varphi_1\nabla\bar{q}\|_{s'}\|\bar{p}\|_s + \|(\sigma'(u_1)\nabla\varphi_1 - \sigma'(u_1)\nabla\varphi_2 + \sigma'(u_1)\nabla\varphi_2 - \sigma'(u_2)\nabla\varphi_2)\nabla q_2\|_{s'}\|\bar{p}\|_s \\
&\le K\|\nabla\varphi_1 \cdot \nabla\bar{q}\|_{s'}\|\bar{p}\|_s + \|\sigma'(u_1)\nabla\bar{\varphi} \cdot \nabla q_2\|_{s'}\|\bar{p}\|_s + \|(\sigma'(u_1) - \sigma'(u_2))\nabla\varphi_2 \cdot \nabla q_2\|_{s'}\|\bar{p}\|_s.
\end{aligned}
$$

Note that $\nabla\varphi_1 \in L^r(\Omega)$ and $\nabla\bar{q} \in L^2(\Omega)$ implies

$$\nabla\varphi_1 \cdot \nabla\bar{q} \in L^{s'}(\Omega). \tag{5.12}$$

since $\frac{1}{s'} = \frac{1}{2} + \frac{1}{r}$. Also, $\nabla\varphi_2 \in L^r(\Omega)$, $\nabla q_2 \in L^r(\Omega)$ implies

$$\nabla\varphi_2 \cdot \nabla q_2 \in L^{r/2}(\Omega) \tag{5.13}$$

because $\frac{1}{r/2} = \frac{2}{r} = \frac{1}{r} + \frac{1}{r}$. Hence, we conclude $\bar{u}\nabla\varphi_2 \cdot \nabla q_2 \in L^{s'}(\Omega)$ (since $\frac{1}{s'} = \frac{1}{s} + \frac{1}{r/2}$). By the Meyers estimate

$$\|\nabla q\|_r \le \tilde{\Phi} < \infty. \tag{5.14}$$

Therefore, taking into account (5.5), (5.12), (5.13), and (5.14) we obtain

$$
\begin{aligned}
\mathcal{B} &\le K\|\nabla\varphi_1\|_r\|\nabla\bar{q}\|_2\|\bar{p}\|_s + K\|\nabla\bar{\varphi}\|_2\|\nabla q_2\|_r\|\bar{p}\|_s + R\|\bar{u}\|_s\|\nabla\varphi_2 \cdot \nabla q_2\|_{\frac{r}{2}}\|\bar{p}\|_s \\
&\le K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla\bar{q}\|_2 + K\tilde{\Phi}M_1\|\nabla\bar{\varphi}\|_2\|\bar{p}\|_{H^1(\Omega)} + R\|\bar{u}\|_s\|\nabla\varphi_2\|_r\|\nabla q_2\|_r\|\bar{p}\|_s \\
&\le K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla\bar{q}\|_2 + K\tilde{\Phi}M_1\frac{K\Phi}{C_1}\|\bar{u}\|_s\|\bar{p}\|_{H^1(\Omega)} + RM_1^2\Phi\tilde{\Phi}\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}
\end{aligned}
$$

whence

$$\mathcal{B} \le K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla\bar{q}\|_2 + \left(\frac{K^2\Phi\tilde{\Phi}M_1^2}{C_1} + RM_1^2\Phi\tilde{\Phi}\right)\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}. \tag{5.15}$$

Collecting all the terms from (5.10), (5.11), and (5.15) gives

$$\int_\Omega |\nabla\bar{p}|^2\,dx + \lambda\int_{\partial\Omega}\bar{p}^2\,ds \le \int_{\partial\Omega}|\beta_1-\beta_2|\cdot|p_2|\cdot|\bar{p}|\,ds + \mathcal{A} + \mathcal{B}$$
$$\le \frac{\alpha_3}{2}\|\bar{p}\|^2_{L^2(\partial\Omega)} + \frac{\alpha_4^2}{4}\|\bar{u}\|^2_{L^2(\partial\Omega)} + \frac{\alpha_4^2}{4}\|\bar{p}\|^2_{L^2(\partial\Omega)} + K\Phi^2M_1^2\|\bar{p}\|^2_{H^1(\Omega)}$$
$$+\left(\frac{2C_2K\Phi^2\alpha_4M_1^2}{C_1} + R\,\Phi^2\alpha_4M_1^2\right)\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)} + K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla\bar{q}\|_2$$
$$+\left(\frac{K^2\Phi\tilde{\Phi}M_1^2}{C_1} + RM_1^2\Phi\tilde{\Phi}\right)\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}$$

which implies

$$\int_\Omega |\nabla\bar{p}|^2\,dx + \left(\lambda - \frac{\alpha_3}{2} - \frac{\alpha_4^2}{4}\right)\int_{\partial\Omega}\bar{p}^2\,ds$$
$$\le \frac{\alpha_4^2}{4}\|\bar{u}\|^2_{L^2(\partial\Omega)} + K\Phi^2M_1^2\|\bar{p}\|^2_{H^1(\Omega)} + K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla\bar{q}\|_2 \tag{5.16}$$
$$+\left(\frac{2C_2K\Phi^2\alpha_4M_1^2}{C_1} + R\,\Phi^2\alpha_4M_1^2 + \frac{K^2\Phi\tilde{\Phi}M_1^2}{C_1} + RM_1^2\Phi\tilde{\Phi}\right)\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}.$$

Setting $M_4 \overset{\text{def}}{=} K\Phi^2M_1^2$ and $M_5 \overset{\text{def}}{=} \frac{2C_2K\Phi^2\alpha_4M_1^2}{C_1} + R\,\Phi^2\alpha_4M_1^2 + \frac{K^2\Phi\tilde{\Phi}M_1^2}{C_1} + RM_1^2\Phi\tilde{\Phi}$ we rewrite (5.16) as follows

$$\int_\Omega |\nabla\bar{p}|^2\,dx + \left(\lambda - \frac{\alpha_3}{2} - \frac{\alpha_4^2}{4}\right)\int_{\partial\Omega}\bar{p}^2\,ds \le \frac{\alpha_4^2}{4}\|\bar{u}\|^2_{L^2(\partial\Omega)} + M_4\|\bar{p}\|^2_{H^1(\Omega)}$$
$$+ K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla\bar{q}\|_2 + M_5\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}. \tag{5.17}$$

This gives the desired estimate for $\bar{p}$.

Note that the estimate for $\bar{\varphi}$ is given by (5.5):

$$\|\nabla\bar{\varphi}\|_2 \le \frac{KM_1\Phi}{C_1}\|\bar{u}\|_{H^1(\Omega)}, \tag{5.18}$$

where we have taken into account that $\|\bar{u}\|_s \le M_1\|\bar{u}\|_{H^1(\Omega)}$. We proceed to the derivation of an estimate for $\|\nabla\bar{q}\|_2$. If we use $\bar{q} = \tilde{w}$ as a test function in the equations for q we will have

$$2\int_\Omega p_1\,\sigma(u_1)\nabla\varphi_1\cdot\nabla\bar q\,dx = \int_\Omega \sigma(u_1)\nabla q_1\cdot\nabla\bar q\,dx$$
$$2\int_\Omega p_2\,\sigma(u_2)\nabla\varphi_2\cdot\nabla\bar q\,dx = \int_\Omega \sigma(u_2)\nabla q_2\cdot\nabla\bar q\,dx.$$

After subtracting we get

$$2\int_\Omega (p_1\,\sigma(u_1)\nabla\varphi_1 - p_2\,\sigma(u_2)\nabla\varphi_2)\cdot\nabla\bar q\,dx = \int_\Omega (\sigma(u_1)\nabla q_1 - \sigma(u_2)\nabla q_2)\cdot\nabla\bar q\,dx.$$

Claim:

$$\begin{aligned}
&2\int_\Omega (p_1\,\sigma(u_1)\nabla\varphi_1 - p_2\,\sigma(u_2)\nabla\varphi_2)\cdot\nabla\bar q\,dx\\
&= 2\int_\Omega \{p_1\sigma(u_1)\nabla\bar\varphi + p_1(\sigma(u_1)-\sigma(u_2))\nabla\varphi_2 + \bar p\,\sigma(u_2)\nabla\varphi_2\}\cdot\nabla\bar q\,dx.
\end{aligned}$$

Indeed,

$$\begin{aligned}
&2\int_\Omega (p_1\,\sigma(u_1)\nabla\varphi_1 - p_2\,\sigma(u_2)\nabla\varphi_2)\cdot\nabla\bar q\,dx\\
&= 2\int_\Omega (p_1\,\sigma(u_1)\nabla\varphi_1 - p_1\,\sigma(u_1)\nabla\varphi_2 + p_1\,\sigma(u_1)\nabla\varphi_2 - p_2\,\sigma(u_2)\nabla\varphi_2)\cdot\nabla\bar q\,dx\\
&= 2\int_\Omega \{p_1\,\sigma(u_1)\nabla\bar\varphi + [p_1\,\sigma(u_1) - p_2\,\sigma(u_2)]\nabla\varphi_2\}\cdot\nabla\bar q\,dx\\
&= 2\int_\Omega \{p_1\,\sigma(u_1)\nabla\bar\varphi + [p_1\,\sigma(u_1) - p_1\,\sigma(u_2) + p_1\,\sigma(u_2) - p_2\,\sigma(u_2)]\nabla\varphi_2\}\cdot\nabla\bar q\,dx\\
&= 2\int_\Omega \{p_1\,\sigma(u_1)\nabla\bar\varphi + p_1[\sigma(u_1) - \sigma(u_2)]\nabla\varphi_2 + \bar p\,\sigma(u_2)\nabla\varphi_2\}\cdot\nabla\bar q\,dx.
\end{aligned}$$

This proves the claim. On the right hand side of the equation for $\bar q$, we have

$$\begin{aligned}
&\int_\Omega (\sigma(u_1)\nabla q_1 - \sigma(u_2)\nabla q_2)\cdot\nabla\bar q\,dx\\
&= \int_\Omega (\sigma(u_1)\nabla q_1 - \sigma(u_1)\nabla q_2 + \sigma(u_1)\nabla q_2 - \sigma(u_2)\nabla q_2)\cdot\nabla\bar q\,dx\\
&= \int_\Omega \sigma(u_1)\nabla\bar q\cdot\nabla\bar q\,dx + \int_\Omega (\sigma(u_1)-\sigma(u_2))\nabla q_2\cdot\nabla\bar q\,dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&2\int_\Omega \{p_1\,\sigma(u_1)\nabla\bar\varphi + p_1[\sigma(u_1)-\sigma(u_2)]\nabla\varphi_2 + \bar p\,\sigma(u_2)\nabla\varphi_2\}\cdot\nabla\bar q\,dx\\
&= \int_\Omega \sigma(u_1)|\nabla\bar q|^2\,dx + \int_\Omega(\sigma(u_1)-\sigma(u_2))\nabla q_2\cdot\nabla\bar q\,dx.
\end{aligned}$$

This implies

$$\begin{aligned}
C_1\|\nabla\bar q\|_2^2 &= C_1\int_\Omega|\nabla\bar q|^2\,dx \le \int_\Omega \sigma(u_1)|\nabla\bar q|^2\,dx\\
&= \Big|2\int_\Omega\{p_1\,\sigma(u_1)\nabla\bar\varphi + p_1[\sigma(u_1)-\sigma(u_2)]\nabla\varphi_2 + \bar p\,\sigma(u_2)\nabla\varphi_2\}\cdot\nabla\bar q\,dx\\
&+ \int_\Omega(\sigma(u_2)-\sigma(u_1))\nabla q_2\cdot\nabla\bar q\,dx\Big|\\
&\le 2\int_\Omega\sigma(u_1)|p_1|\cdot|\nabla\bar\varphi|\cdot|\nabla\bar q|\,dx + 2\int_\Omega|p_1|\cdot|\sigma(u_1)-\sigma(u_2)|\cdot|\nabla\varphi_2|\cdot|\nabla\bar q|\,dx\\
&+ 2\int_\Omega\sigma(u_2)|\bar p|\cdot|\nabla\varphi_2|\cdot|\nabla\bar q|\,dx + \int_\Omega|\sigma(u_1)-\sigma(u_2)|\cdot|\nabla q_2|\cdot|\nabla\bar q|\,dx\\
&\le 2C_2\int_\Omega|p_1|\cdot|\nabla\bar\varphi|\cdot|\nabla\bar q|\,dx + 2K\int_\Omega|p_1|\cdot|\bar u|\cdot|\nabla\varphi_2|\cdot|\nabla\bar q|\,dx\\
&+ 2C_2\int_\Omega|\bar p|\cdot|\nabla\varphi_2|\cdot|\nabla\bar q|\,dx + K\int_\Omega|\bar u|\cdot|\nabla q_2|\cdot|\nabla\bar q|\,dx\\
&\le 2C_2\|p_1\|_\infty\|\nabla\bar\varphi\|_2\|\nabla\bar q\|_2 + 2K\|p_1\|_\infty\|\bar u\|_s\|\nabla\varphi_2\|_r\|\nabla\bar q\|_2\\
&+ 2C_2\|\bar p\|_s\|\nabla\varphi_2\|_r\|\nabla\bar q\|_2 + K\|\bar u\|_s\|\nabla q_2\|_r\|\nabla\bar q\|_2\\
&\le 2C_2\|p_1\|_\infty\frac{KM_1\Phi}{C_1}\|\bar u\|_{H^1(\Omega)}\|\nabla\bar q\|_2 + 2KM_1\Phi\|p_1\|_\infty\|\bar u\|_{H^1(\Omega)}\|\nabla\bar q\|_2\\
&+ 2C_2M_1\Phi\|\bar p\|_{H^1(\Omega)}\|\nabla\bar q\|_2 + KM_1\tilde\Phi\|\bar u\|_{H^1(\Omega)}\|\nabla\bar q\|_2.
\end{aligned}$$

Thus

$$\|\nabla\bar q\|_2 \le \left(\frac{2\alpha_5C_2KM_1\Phi}{C_1^2} + \frac{2\alpha_5KM_1\Phi}{C_1} + \frac{KM_1\tilde\Phi}{C_1}\right)\|\bar u\|_{H^1(\Omega)} + \frac{2C_2M_1\Phi}{C_1}\|\bar p\|_{H^1(\Omega)} \tag{5.19}$$

where we have set $\alpha_5 \stackrel{\text{def}}{=} \|p_1\|_\infty$. Now put $M_6 \stackrel{\text{def}}{=} \frac{2\alpha_5C_2KM_1\Phi}{C_1^2} + \frac{2\alpha_5KM_1\Phi}{C_1} + \frac{KM_1\tilde\Phi}{C_1}$ and rewrite (5.19) in the following form

$$\|\nabla\bar q\|_2 \le M_6\|\bar u\|_{H^1(\Omega)} + \frac{2C_2M_1\Phi}{C_1}\|\bar p\|_{H^1(\Omega)}. \tag{5.20}$$

This gives us the estimate for $\nabla \bar{q}$.

All in all, we have

$$\begin{aligned}
&\int_\Omega |\nabla \bar{u}|^2\,dx + \left(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4}\right)\int_{\partial\Omega} \bar{u}^2\,ds \le \frac{\alpha_1}{4}\|\bar{p}\|^2_{L^2(\partial\Omega)} + M_3\|\bar{u}\|^2_{H^1(\Omega)}\\
&\int_\Omega |\nabla \bar{p}|^2\,dx + \left(\lambda - \frac{\alpha_3}{2} - \frac{\alpha_4^2}{4}\right)\int_{\partial\Omega} \bar{p}^2\,ds \le \frac{\alpha_4^2}{4}\|\bar{u}\|^2_{L^2(\partial\Omega)} + M_4\|\bar{p}\|^2_{H^1(\Omega)}\\
&+ K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla \bar{q}\|_2 + M_5\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}\\
&\|\nabla \bar{q}\|_2 \le \frac{2C_2M_1\Phi}{C_1}\|\bar{p}\|_{H^1(\Omega)} + M_6\|\bar{u}\|_{H^1(\Omega)}\\
&\|\nabla \bar{\varphi}\|_2 \le \frac{KM_1\Phi}{C_1}\|\bar{u}\|_{H^1(\Omega)}
\end{aligned} \tag{5.21}$$

Adding the first two estimates in (5.21) gives

$$\begin{aligned}
&\int_\Omega |\nabla \bar{u}|^2\,dx + \left(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4} - \frac{\alpha_4^2}{4}\right)\int_{\partial\Omega} \bar{u}^2\,ds + \int_\Omega |\nabla \bar{p}|^2\,dx\\
&+ \left(\lambda - \frac{\alpha_3}{2} - \frac{\alpha_4^2}{4} - \frac{\alpha_1}{4}\right)\int_{\partial\Omega} \bar{p}^2\,ds\\
&\le M_3\|\bar{u}\|^2_{H^1(\Omega)} + M_4\|\bar{p}\|^2_{H^1(\Omega)} + K\Phi M_1\|\bar{p}\|_{H^1(\Omega)}\|\nabla \bar{q}\|_2 + M_5\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}.
\end{aligned} \tag{5.22}$$

Using the estimate for $\nabla \bar{q}$ from (5.21) allows us to rewrite (5.22) in the following way

$$\begin{aligned}
&\int_\Omega |\nabla \bar{u}|^2\,dx + \left(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4} - \frac{\alpha_4^2}{4}\right)\int_{\partial\Omega} \bar{u}^2\,ds + \int_\Omega |\nabla \bar{p}|^2\,dx\\
&+ \left(\lambda - \frac{\alpha_3}{2} - \frac{\alpha_4^2}{4} - \frac{\alpha_1}{4}\right)\int_{\partial\Omega} \bar{p}^2\,ds\\
&\le M_3\|\bar{u}\|^2_{H^1(\Omega)} + M_4\|\bar{p}\|^2_{H^1(\Omega)} + \frac{2C_2KM_1^2\Phi^2}{C_1}\|\bar{p}\|^2_{H^1(\Omega)}\\
&+ K\Phi M_1M_6\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)} + M_5\|\bar{u}\|_{H^1(\Omega)}\|\bar{p}\|_{H^1(\Omega)}.
\end{aligned} \tag{5.23}$$

Using Cauchy's inequality, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla \bar{u}|^2\,dx + \Big(\lambda - \frac{\alpha_2}{2} - \frac{\alpha_1}{4} - \frac{\alpha_4^2}{4}\Big)\int_{\partial\Omega} \bar{u}^2\,ds + \int_\Omega |\nabla \bar{p}|^2\,dx \\
&+ \Big(\lambda - \frac{\alpha_3}{2} - \frac{\alpha_4^2}{4} - \frac{\alpha_1}{4}\Big)\int_{\partial\Omega} \bar{p}^2\,ds \\
&\le M_3\|\bar{u}\|^2_{H^1(\Omega)} + M_4\|\bar{p}\|^2_{H^1(\Omega)} + \frac{2C_2KM_1^2\Phi^2}{C_1}\|\bar{p}\|^2_{H^1(\Omega)} + \frac{K\Phi M_1M_6}{2}\|\bar{u}\|^2_{H^1(\Omega)} \\
&+ \frac{K\Phi M_1M_6}{2}\|\bar{p}\|^2_{H^1(\Omega)} + \frac{M_5}{2}\|\bar{u}\|^2_{H^1(\Omega)} + \frac{M_5}{2}\|\bar{p}\|^2_{H^1(\Omega)}.
\end{aligned} \tag{5.24}$$

From (5.24) we conclude

$$\begin{aligned}
&k\|\bar{u}\|^2_{H^1(\Omega)} + k\|\bar{p}\|^2_{H^1(\Omega)} \le M_3\|\bar{u}\|^2_{H^1(\Omega)} + M_4\|\bar{p}\|^2_{H^1(\Omega)} \\
&+ \frac{2C_2KM_1^2\Phi^2}{C_1}\|\bar{p}\|^2_{H^1(\Omega)} + \frac{K\Phi M_1M_6}{2}\|\bar{u}\|^2_{H^1(\Omega)} + \frac{K\Phi M_1M_6}{2}\|\bar{p}\|^2_{H^1(\Omega)} \\
&+ \frac{M_5}{2}\|\bar{u}\|^2_{H^1(\Omega)} + \frac{M_5}{2}\|\bar{p}\|^2_{H^1(\Omega)},
\end{aligned}$$

and consequently,

$$\begin{aligned}
&\Big(k - M_3 - \frac{K\Phi M_1M_6}{2} - \frac{M_5}{2}\Big)\|\bar{u}\|^2_{H^1(\Omega)} \\
&+ \Big(k - M_4 - \frac{2C_2KM_1^2\Phi^2}{C_1} - \frac{K\Phi M_1M_6}{2} - \frac{M_5}{2}\Big)\|\bar{p}\|^2_{H^1(\Omega)} \le 0
\end{aligned}$$

We remind the reader the definitions of some constants.

$$\begin{aligned}
&M_6 \stackrel{\text{def}}{=} \frac{2\alpha_5C_2KM_1\Phi}{C_1^2} + \frac{2\alpha_5KM_1\Phi}{C_1} + \frac{KM_1\tilde{\Phi}}{C_1}, \\
&M_5 \stackrel{\text{def}}{=} \frac{2C_2K\Phi^2\alpha_4M_1^2}{C_1} + R\,\Phi^2\alpha_4M_1^2 + \frac{K^2\Phi\tilde{\Phi}M_1^2}{C_1} + RM_1^2\Phi\tilde{\Phi}, \\
&M_4 \stackrel{\text{def}}{=} K\Phi^2M_1^2, \\
&M_3 \stackrel{\text{def}}{=} \frac{2C_2K\Phi^2M_1^2}{C_1} + K\Phi^2M_1^2, \\
&\alpha_5 \stackrel{\text{def}}{=} \|p_1\|_\infty, \alpha_4 \stackrel{\text{def}}{=} \|p_2\|_\infty, \alpha_3 \stackrel{\text{def}}{=} \|u_1\|_\infty\|p_2\|_\infty, \\
&\alpha_2 \stackrel{\text{def}}{=} \|p_2\|_\infty\|u_2\|_\infty, \alpha_1 \stackrel{\text{def}}{=} \|u_1\|_\infty\|u_2\|_\infty,
\end{aligned} \tag{5.25}$$

Therefore if the boundary data φ_0 are chosen so that

$$k - M_3 - \frac{K\Phi M_1 M_6}{2} - \frac{M_5}{2} > 0, \text{ and}$$
$$k - M_4 - \frac{2C_2 K M_1^2 \Phi^2}{C_1} - \frac{K\Phi M_1 M_6}{2} - \frac{M_5}{2} > 0,$$

then $\bar{u} = 0, \bar{p} = 0, \nabla\bar{q} = 0$, and $\nabla\bar{\varphi} = 0$. Then it follows that $\bar{q} = 0$ and $\bar{\varphi} = 0$ (since $\nabla\bar{q} = 0$, $\nabla\bar{\varphi} = 0$ and $\bar{q} = 0$, $\bar{\varphi} = 0$ on the boundary) which gives the uniqueness of solutions of the optimality system. This uniqueness implies the uniqueness of the optimal control, since we have the existence of an optimal control and corresponding states and adjoints, which satisfy the optimality system. □

Chapter 6

Case of a Constant Heat Transfer Coefficient

In this chapter we look at optimization of a constant heat transfer coefficient in the thermistor problem. Namely, let the state system be described by (1.1) and the objective functional be given by the following expression

$$J(\beta) = \int_\Omega u\,dx + \beta^2, \tag{6.1}$$

where now the set of admissible controls is given by $U_M = \{\beta \in \mathbb{R} : 0 < \lambda \leq \beta \leq M\}$. Note that now the system (1.1) has a constant heat transfer coefficient β. We have the following

$$\textbf{Problem P}. \text{ Find } \beta^* \in U_M \text{ such that } J(\beta^*) = \min_{\beta \in U_M} J(\beta). \tag{6.2}$$

In other words, we need to adjust the parameter $\beta \in U_M$ in such a way that the functional (6.1) is minimized. For other examples of this technique applied to optimizing a parameter, see [28]. Observe that all the theory from the previous sections (existence of optimal control, derivation of optimality system, etc.) carries over to this case. As for the characterization of optimal control, if we repeat all the steps from chapter 4 and take into account the new form of the objective functional (6.1), we obtain

$$
\begin{aligned}
0 &\le \lim_{\varepsilon\to 0^+} \frac{J(\beta^*+\varepsilon\ell)-J(\beta^*)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[\int_\Omega u(\beta^*+\varepsilon\ell)\,dx + (\beta^*+\varepsilon\ell)^2 - \int_\Omega u(\beta^*)\,dx - (\beta^*)^2\right] \\
&= \lim_{\varepsilon\to 0^+} \left[\int_\Omega \frac{u(\beta^*+\varepsilon\ell)-u(\beta^*)}{\varepsilon}\,dx + \frac{(\beta^*+\varepsilon\ell)^2-(\beta^*)^2}{\varepsilon}\right] \\
&= \lim_{\varepsilon\to 0^+} \int_\Omega \frac{u^\varepsilon - u^*}{\varepsilon}\,dx + \lim_{\varepsilon\to 0^+} \frac{(\beta^*)^2 + 2\beta^*\varepsilon\ell + \varepsilon^2\ell^2 - (\beta^*)^2}{\varepsilon} \\
&= \int_\Omega \psi_1\,dx + 2\beta^*\ell \\
&= \int_\Omega \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\,dx + 2\beta^*\ell \\
&= -\int_\Omega \nabla p\cdot\nabla\psi_1\,dx - \int_{\partial\Omega} \beta^* p\,\psi_1\,ds + \int_\Omega \psi_1\,\sigma'(u)|\nabla\varphi|^2 p\,dx - \int_\Omega \psi_1\,\sigma'(u)\nabla\varphi\cdot\nabla q\,dx \\
&+ 2\int_\Omega p\,\sigma(u)\nabla\varphi\cdot\nabla\psi_2\,dx - \int_\Omega \sigma(u)\nabla q\cdot\nabla\psi_2\,dx + 2\beta^*\ell \\
&= \Big\{-\int_\Omega \nabla p\cdot\nabla\psi_1\,dx - \int_{\partial\Omega} \beta^* p\,\psi_1\,ds + \int_\Omega \psi_1\,\sigma'(u)|\nabla\varphi|^2 p\,dx + 2\int_\Omega p\,\sigma(u)\nabla\varphi\cdot\nabla\psi_2\,dx\Big\} \\
&+ \Big[-\int_\Omega \psi_1\,\sigma'(u)\nabla\varphi\cdot\nabla q\,dx - \int_\Omega \sigma(u)\nabla q\cdot\nabla\psi_2\,dx\Big] + 2\beta^*\ell \\
&= \ell\int_{\partial\Omega} up\,ds + 2\beta^*\ell,
\end{aligned}
$$

where we integrated by parts and used the sensitivity system (4.68), (4.69) with test functions p, q. Thus, we obtained

$$
\ell\left(\int_{\partial\Omega} up\,ds + 2\beta^*\right) \ge 0. \tag{6.3}
$$

Now repeating similar arguments as those leading to formula (4.72), the optimal parameter characterization is

$$
\beta^* = \min\left(\max\left(\lambda, -\frac{1}{2}\int_{\partial\Omega} up\,ds\right), M\right). \tag{6.4}
$$

Hence, optimality system is given by (1.1), (4.43) and (6.4).

We present a numerical example for a test problem. We consider a square domain with a side of unit length which is approximated by a uniform mesh of size h and the optimality system is discretized by finite differences. In what follows, $j \in \mathbb{N}$ marks a grid point on the $x-$axis and is changing from from 0 to $\tilde{M}+1$ with increment 1, where $h(\tilde{M}+1)=1$ (that is, the interval from 0 to 1 on the $x-$axis is divided into $\tilde{M}+1$ subintervals). Similarly, for $k \in \mathbb{N}$ along the $y-$axis. The grid points with $j=0$, $j=\tilde{M}+1$, $k=0$, and $k=\tilde{M}+1$ correspond to the boundaries of the square.

For each internal grid point (j,k) the state equation for u is discretized as follows

$$\begin{aligned}
&\frac{1}{h^2}(u_{j+1,k}+u_{j-1,k}+u_{j,k-1}-4u_{j,k})\\
&+\sigma(u_{j,k})\left(\frac{\varphi_{j+1,k}-\varphi_{j-1,k}}{2h}\right)^2+\sigma(u_{j,k})\left(\frac{\varphi_{j,k+1}-\varphi_{j,k-1}}{2h}\right)^2=0.
\end{aligned}$$

Thus, for $j=1,\ldots,\tilde{M}$ and $k=1,\ldots,\tilde{M}$, we have

$$\begin{aligned}
&u_{j+1,k}+u_{j-1,k}+u_{j,k+1}+u_{j,k-1}-4u_{j,k}\\
&+\frac{1}{4}\sigma(u_{j,k})(\varphi_{j+1,k}-\varphi_{j-1,k})^2+\frac{1}{4}\sigma(u_{j,k})(\varphi_{j,k+1}-\varphi_{j,k-1})^2=0
\end{aligned}$$

For $j=1,\ldots,\tilde{M}$ and $k=1,\ldots,\tilde{M}$, the rest of the discrete equations approximating the optimality system looks like this

$$\begin{aligned}
&(\sigma_{j+1,k}+\sigma_{j,k})\varphi_{j+1,k}-(\sigma_{j+1,k}+2\sigma_{j,k}+\sigma_{j-1,k})\varphi_{j,k}+(\sigma_{j-1,k}+\sigma_{j,k})\varphi_{j-1,k}\\
&+(\sigma_{j,k+1}+\sigma_{j,k})\varphi_{j,k+1}-(\sigma_{j,k+1}+2\sigma_{j,k}+\sigma_{j,k-1})\varphi_{j,k}+(\sigma_{j,k-1}+\sigma_{j,k})\varphi_{j,k-1}=0,\\
&p_{j+1,k}+p_{j-1,k}+p_{j,k+1}+p_{j,k-1}-4p_{j,k}+\frac{1}{4}\sigma'_{j,k}\cdot(\varphi_{j+1,k}-\varphi_{j-1,k})^2p_{j,k}\\
&+\frac{1}{4}\sigma'_{j,k}\cdot(\varphi_{j,k+1}-\varphi_{j,k-1})^2p_{j,k}-\frac{1}{4}\sigma'_{j,k}\cdot(\varphi_{j+1,k}-\varphi_{j-1,k})(q_{j+1,k}-q_{j-1,k})\\
&-\frac{1}{4}\sigma'_{j,k}\cdot(\varphi_{j,k+1}-\varphi_{j,k-1})(q_{j,k+1}-q_{j,k-1})=h^2,\\
&-\sigma_{j,k}(\varphi_{j+1,k}-\varphi_{j-1,k})(p_{j+1,k}-p_{j-1,k})-\sigma_{j,k}(\varphi_{j,k+1}-\varphi_{j,k-1})(p_{j,k+1}-q_{j,k-1})\\
&+(\sigma_{j+1,k}+\sigma_{j,k})q_{j+1,k}-(\sigma_{j+1,k}+2\sigma_{j,k}+\sigma_{j-1,k})q_{j,k}+(\sigma_{j-1,k}+\sigma_{j,k})q_{j-1,k}\\
&+(\sigma_{j,k+1}+\sigma_{j,k})q_{j,k+1}-(\sigma_{j,k+1}+2\sigma_{j,k}+\sigma_{j,k-1})q_{j,k}+(\sigma_{j,k-1}+\sigma_{j,k})q_{j,k-1}=0.
\end{aligned} \tag{6.5}$$

We illustrate discretization of the boundary conditions along the boundary $y = 0$. We have

$$
\begin{aligned}
&-\frac{\partial u}{\partial y} + \beta u = 0 \text{ on } y = 0,\\
&-\frac{\partial p}{\partial y} + \beta p = 0 \text{ on } y = 0,\\
&q = 0 \text{ on } y = 0,\\
&\varphi = \varphi_0(x, 0) \equiv f_1(x) \text{ on } y = 0.
\end{aligned}
$$

We use the first order approximation for the derivatives:

$$\left.\frac{\partial u}{\partial y}\right|_{j,0} = \frac{u_{j,1} - u_{j,0}}{h}.$$

Therefore, we get

$$-\frac{u_{j,1} - u_{j,0}}{h} + \beta u_{j,0} = 0$$

which allows us to solve for $u_{j,0}$:

$$u_{j,0} = \frac{u_{j,1}}{1 + \beta h}.$$

Thus, on the boundary $y = 0$, we obtain

$$
\begin{aligned}
&u_{j,0} = \frac{u_{j,1}}{1 + \beta h},\\
&p_{j,0} = \frac{p_{j,1}}{1 + \beta h},\\
&q_{j,0} = 0,\\
&\varphi_{j,0} = \varphi(x_j, 0) \equiv f_1(x_j),
\end{aligned}
\tag{6.6}
$$

where $x_j = jh$ and $j = 1, \ldots, \tilde{M}$. Also, for example, $\sigma_{j,0} \overset{\text{def}}{=} \sigma(u_{j,0}) = \sigma\left(\frac{u_{j,1}}{1+\beta h}\right)$.

Similarly to (6.6), we can write for the boundary $x = 0$:

$$
\begin{aligned}
u_{0,k} &= \frac{u_{1,k}}{1+\beta h}, \\
p_{0,k} &= \frac{p_{1,k}}{1+\beta h}, \\
q_{0,k} &= 0, \\
\varphi_{0,k} &= \varphi(0, y_k) \equiv g_1(y_k),
\end{aligned} \tag{6.7}
$$

where $y_k = kh$ and $k = 1, \ldots, \tilde{M}$. On $y = 1$:

$$
\begin{aligned}
u_{j,\tilde{M}+1} &= \frac{u_{j,\tilde{M}}}{1+\beta h}, \\
p_{j,\tilde{M}+1} &= \frac{p_{j,\tilde{M}}}{1+\beta h}, \\
q_{j,\tilde{M}+1} &= 0, \\
\varphi_{j,\tilde{M}+1} &= \varphi(x_j, 1) \equiv f_2(x_j),
\end{aligned} \tag{6.8}
$$

where $x_j = jh$ and $j = 1, \ldots, \tilde{M}$. On $x = 1$:

$$
\begin{aligned}
u_{\tilde{M}+1,k} &= \frac{u_{\tilde{M},k}}{1+\beta h}, \\
p_{\tilde{M}+1,k} &= \frac{p_{\tilde{M},k}}{1+\beta h}, \\
q_{\tilde{M}+1,k} &= 0, \\
\varphi_{\tilde{M}+1,k} &= \varphi(1, y_k) \equiv g_2(y_k),
\end{aligned} \tag{6.9}
$$

where $y_k = kh$ and $k = 1, \ldots, \tilde{M}$.

We solve the system consisting of (6.5), (6.6), (6.7), (6.8), and (6.9) by an iteration scheme with an initial guess for the values of β, $u_{j,k}$, $\varphi_{j,k}$, $p_{j,k}$, and $q_{j,k}$. With $\tilde{M} = 50$, $\lambda = 1$, $M = 10$, $\sigma(u) = e^{-10u}$, $f_1(x) = x$, $f_2(x) = 1$, $g_1(y) = y$, and $g_2(y) = 1$ the consecutive values of β converge to the lower bound $\lambda = 1$.

Part III

Optimal Control of a Biharmonic Obstacle Problem

Chapter 7

Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and let $z \in L^2(\Omega)$ be a given target profile. Denote $V \stackrel{\text{def}}{=} H^3(\Omega) \cap H_0^2(\Omega)$. Given $\psi \in V$ define the closed convex set

$$K(\psi) = \{v \in H_0^2(\Omega) : \, v \geq \psi \text{ a.e. in } \Omega\}$$

and consider the following obstacle problem for the biharmonic operator

$$\text{find } u \in K(\psi) \text{ such that} \int_\Omega \Delta u \, \Delta(v-u)\, dx \geq 0 \quad \forall \, v \in K(\psi). \tag{7.1}$$

Equivalent strong formulation of (7.1) is

$$\begin{aligned}
&\text{find } u \in H_0^2(\Omega) : \\
&u \geq \psi \text{ a.e. in } \Omega, \\
&\Delta^2 u \geq 0 \text{ a.e. in } \Omega, \\
&(\Delta^2 u)(u-\psi) = 0 \text{ a.e. in } \Omega.
\end{aligned}$$

The bilinear form

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \Delta u \, \Delta(v-u)\, dx$$

can be shown to satisfy the coercivity condition and consequently (see [29]), problem (1) admits a unique solution $u \in H_0^2(\Omega)$ which is also the minimizer of the functional

$$v \mapsto \int_\Omega |\Delta v|^2 \, dx$$

over the set $K(\psi)$. We will show in the next section that if u solves (7.1), then there exists a nonnegative Borel measure $\mu \in H^{-2}(\Omega)$ such that $\Delta^2 u = \mu$ in the sense of distributions.

We consider an optimal control problem for (7.1) in which we view ψ as the control and $u \stackrel{\text{def}}{=} T(\psi)$ as the corresponding state. We introduce an objective functional

$$J(\psi) = \frac{1}{2} \int_\Omega \left\{ |T(\psi) - z|^2 + |\nabla \Delta \psi|^2 \right\} dx \tag{7.2}$$

and formulate the following

$$\textbf{Problem C.} \text{ Find } \psi^* \in V \text{ s.t. } J(\psi^*) = \inf_{\psi \in V} J(\psi). \tag{7.3}$$

In other words, for a given target profile $z \in L^2(\Omega)$, we want to find an obstacle $\psi^* \in V$ such that the corresponding solution $u^* = T(\psi^*)$ is close to z in L^2 norm while ψ^* is not too large in $H^3(\Omega)$.

Any ψ^* satisfying (7.3) is called an optimal control, the corresponding state $u^* = T(\psi^*)$ is called an optimal state and (u^*, ψ^*) is referred to as an optimal pair.

Theoretical analysis of variational inequalities has received a significant amount of attention, see [8, 9, 16, 23, 29, 46, 49]. The motivation for this paper is threefold. First, the obstacle problem (7.1) arises in elasticity theory when, for example, a two dimensional plate is bent so that it must stay above the obstacle ψ. More specifically, it concerns the small transverse displacement of a plate when its boundary is fixed and the whole plate is simultaneously subject to a pressure to lie on one side of an obstacle. Secondly, use of the obstacle as the control is a novel feature for this operator. Thirdly, a very "rough" obstacle in the framework of the given optimal control problem leads to the consideration of fine pointwise properties of functions from the Sobolev spaces, in particular, the need for "capacity of a set" [2].

The fundamental work on optimal control of variational inequalities was done by Barbu [9, 10]. Various coefficients and source terms were taken as the control but not the obstacle. The first work to consider the obstacle as a control was by Adams, Lenhart and Yong in [3]. It was found there that the optimal obstacle is equal to its corresponding state, i.e., $\psi^* = T(\psi^*)$. The results from [3] were generalized in [4, 5, 13–15, 40, 41] to include source terms, bilateral problems as well as semilinear, quasilinear, and parabolic operators. Recent work on semilinear elliptic variational inequalities with control entering in lower order terms can be found in [14, 20]. A general framework for analysis of optimal control of elliptic variational inequalities using an augmented Lagrangian technique was developed by Ito and Kunisch in [26, 27]. For results on other types of control of variational inequalities, see [9–11].

To derive necessary conditions on an optimal control, we would like to differentiate the map $\psi \mapsto J(\psi)$, which would require differentiation of the map $\psi \mapsto T(\psi)$. Since the map $\psi \mapsto T(\psi)$ is not directly differentiable, an approximation problem with a semilinear PDE is introduced. The approximation map $\psi \mapsto T_\delta(\psi)$ will be then differentiable and approximate necessary conditions will be derived. This method of deriving necessary conditions through this approximation was first used by Barbu [9].

This part of the dissertation is organized as follows. In chapter 8 we review some results from the theory of Sobolev spaces and derive an estimate that shows explicitly the dependence of u on ψ in the H^2 norm. Existence of an optimal control is proven in chapter 9. Chapter 10 introduces and deals with a family of approximation problems which is the main tool in the derivation of optimality system. These problems have the regularized state equations with the same cost functional (7.2). We prove that the optimal control for the approximation problems converges strongly in $H^2(\Omega)$ to the optimal control for Problem C. This allows us to obtain necessary conditions for the optimal control which is given in chapter 11.

Chapter 8

Preliminaries

Recall the definition of $C_{2,2}$-capacity. For a given compact subset $e \subset \mathbb{R}^n$, define the closed convex set

$$K_e = \{v \in C_0^\infty(\Omega) : v \geq 1 \text{ on } e\}.$$

Then the $C_{2,2}$-capacity of e is defined by

$$C_{2,2}(e) = \inf_{v \in K_e} \|\Delta v\|^2_{L^2(\Omega)}.$$

For history and relevant information on capacity, the reader is referred to [2, 29, 43, 46]. We will need the following result concerning pointwise comparison of functions from Sobolev spaces.

Lemma 8.1. *Let $u, v \in H^2(\Omega)$. If $u \geq v$ a.e. (with respect to Lebesgue measure) then $\hat{u} \geq \hat{v}$ $C_{2,2}$-a.e., where $\hat{u}$ denotes a precise representative for u defined $C_{2,2}$-a.e. and*

$$\hat{u}(x) = \lim_{R \to 0} \frac{1}{|B(x,R)|} \int_{B(x,R)} u \, dy, \tag{8.1}$$

where the limit exists $C_{2,2}$-a.e.

Proof. See [2]. □

In what follows, we will use the precise representatives and for the sake of notational convenience we drop the "hats" which denote precise representatives.

Lemma 8.2. *Let $u = T(\psi)$ solve (7.1). Then there exists a nonnegative Borel measure $\mu \in H^{-2}(\Omega)$ such that*

$$\Delta^2 u = \mu \text{ in } \Omega, \tag{8.2}$$

where u denotes a precise representative defined $C_{2,2}$-a.e. in Ω, and $\Delta^2 u = \mu$ is understood in the sense of distributions, i.e.

$$\int_\Omega \Delta u \Delta \xi \, dx = \int_\Omega \xi \, d\mu \quad \forall \xi \in H_0^2(\Omega). \tag{8.3}$$

Proof. We follow [16, 29]. Let $\xi \in C_0^\infty(\Omega)$, $\xi \geq 0$ be arbitrary. Clearly, for an arbitrary $\varepsilon > 0$, we have $v = u + \varepsilon\xi \in K(\psi)$. Substituting this v into (7.1) we obtain

$$\int_\Omega \Delta u \Delta \xi \, dx \geq 0. \tag{8.4}$$

This implies that there exists a nonnegative Borel measure $\mu \in H^{-2}(\Omega)$ (see [38]) such that

$$\mu = \Delta^2 u,$$

where $\Delta^2 u$ is taken in the sense of distributions. By density of $C_0^\infty(\Omega)$ in $H_0^2(\Omega)$, (8.3) also holds for any $\xi \in H_0^2(\Omega)$. □

Lemma 8.3. *Let $u = T(\psi)$ solve (7.1) and let $\mu \in H^{-2}(\Omega)$ be the Borel measure from Lemma 8.2. Then:*

(i) there exists a positive constant such that $\mu(e) \leq constant \cdot C_{2,2}(e)$ for each set $e \subset \Omega$ whose distance from the boundary $\partial\Omega$ is positive, and

(ii) $u = \psi$ μ-a.e., where u and ψ are the representatives defined $C_{2,2}$-a.e., respectively.

Proof. (i) We follow Kinderlehrer and Stampacchia (see [29], Theorem 6.11), with appropriate modifications. Take an arbitrary $\zeta \in C_0^\infty(\Omega)$ with $\zeta \geq 0$ on Ω and $\zeta \geq 1$ on e. Since

$\Delta^2 u = \mu$ in Ω in the sense of distributions, we can write

$$\mu(e) = \int_\Omega \chi_e \, d\mu \le \int_\Omega \zeta \, d\mu = \int_\Omega \Delta\zeta \Delta u \, dx \le \|\Delta\zeta\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}.$$

Taking infimum over all ζ we obtain

$$\mu(e) \le const \cdot \inf_{\zeta \in K_e} \|\Delta\zeta\|_{L^2(\Omega)} = const \cdot C_{2,2}(e)$$

with $const = \|\Delta u\|_{L^2(\Omega)} < \infty$.

(ii) Take $v = \psi$ in (7.1) to get $\int_\Omega \Delta u \, \Delta(\psi - u) \, dx \ge 0$. This implies

$$\int_\Omega (\psi - u) \, d\mu \ge 0. \tag{8.5}$$

It follows from (8.5) that

$$\mu\{x \in \Omega : u > \psi \text{ a.e.}\} = 0, \tag{8.6}$$

and we will show

$$\mu\{x \in \Omega : u > \psi \; C_{2,2}\text{-a.e.}\} = 0. \tag{8.7}$$

Indeed, to show (8.7) write

$$\{x \in \Omega : u > \psi \; C_{2,2}\text{-a.e.}\} = \{x \in \Omega : u > \psi \text{ a.e.}\} \cup Q,$$

where

$$C_{2,2}(Q) = 0 \text{ and } Q \cap \{x \in \Omega : u > \psi \text{ a.e.}\} = \emptyset.$$

Therefore, taking into account (8.6) and part (i) of Lemma 8.3, we obtain

$$\mu\{x \in \Omega : u > \psi \; C_{2,2}\text{-a.e.}\} = \mu\{x \in \Omega : u > \psi \text{ a.e.}\} + \mu(Q) = 0.$$

This proves (8.7) which gives (ii). $\square$

Remarks.

1. Let $N = \{x \in \Omega : u(x) > \psi(x) \text{ a.e.}\}$ and $I = \{x \in \Omega : u(x) = \psi(x) \text{ a.e.}\}$ denote the non-coincidence set and the coincidence set, respectively. Then by Lemma 8.1 we obtain $I = \{x \in \Omega : u(x) = \psi(x)\ C_{2,2}\text{-a.e.}\}$ and $C_{2,2}\big(\Omega\backslash(N \cup I)\big) = 0$.

2. Observe that $C_{2,2}\{x \in \Omega : u(x) = \psi(x)\ C_{2,2}\text{-a.e.}\} > 0$ if $\psi > 0$ on a set of positive capacity.

   Indeed, if $C_{2,2}\{x \in \Omega : u(x) = \psi(x)\ C_{2,2}\text{-a.e.}\} = 0$ then since $\operatorname{supp}\mu \subset \{u = \psi\ C_{2,2}\text{-a.e.}\}$ it follows that $C_{2,2}\{\operatorname{supp}\mu\} = 0$. Therefore, by part (i) of Lemma 8.3 we get $\mu\{\operatorname{supp}\mu\} = 0$. This would imply that $\mu \equiv 0$ which gives $\Delta^2 u = 0$ on Ω. Hence, we conclude that $u \equiv 0$ which implies $\psi \le 0\ C_{2,2}$-a.e. Thus, if $\psi > 0$ on a set of positive capacity we get that u is nonzero and $C_{2,2}\{x \in \Omega : u(x) = \psi(x)\ C_{2,2}\text{-a.e.}\} > 0$.

The next lemma gives the key estimate which will be used in the proof of the existence of solution to Problem C. Even though this estimate (for a smooth obstacle) can be found in [1], a more detailed proof is presented here. Also, in what follows, we denote by $\|\cdot\|_2$, the L^2 norm on Ω.

Lemma 8.4. *Let u solve* (7.1) *with obstacle ψ and u_k solve* (7.1) *with obstacle ψ_k. Then there exists $\tilde{C} > 0$ such that*

$$\|u - u_k\|_{H^2(\Omega)} \le \tilde{C}\|\psi - \psi_k\|_{H^2(\Omega)}. \tag{8.8}$$

Proof. Since u and u_k are the solutions to (7.1) with obstacles ψ and ψ_k, respectively, it follows that there exist nonnegative measures $\mu, \mu_k \in H^{-2}(\Omega)$ such that

$$\begin{aligned}
&\Delta^2 u = \mu \text{ in } \Omega,\\
&u = \psi\ \mu\text{-a.e.},\\
&\Delta^2 u_k = \mu_k \text{ in } \Omega, \text{ and}\\
&u_k = \psi_k\ \mu_k\text{-a.e.}
\end{aligned} \tag{8.9}$$

By Lemma 8.1, we have $u \geq \psi$ $C_{2,2}$-a.e. Thus, $C_{2,2}\{x \in \Omega : u < \psi \text{ a.e.}\} = 0$. Hence, by part (i) of Lemma 8.3, we can write

$$\mu_k\{x \in \Omega : u < \psi \text{ a.e.}\} = 0. \tag{8.10}$$

Therefore,

$$-\int_\Omega u\, d\mu_k \leq -\int_\Omega \psi\, d\mu_k. \tag{8.11}$$

Similarly, we obtain

$$-\int_\Omega u_k\, d\mu \leq -\int_\Omega \psi_k\, d\mu. \tag{8.12}$$

Taking into account (8.9), (8.11), and (8.12) we have

$$\begin{aligned}
\|u - u_k\|^2_{H^2(\Omega)} &= \int_\Omega |\Delta(u - u_k)|^2 dx \\
&= \int_\Omega \Delta(u - u_k)\Delta(u - u_k)\, dx \\
&= \int_\Omega (u - u_k)\, d(\mu - \mu_k) \\
&= \int_\Omega u\, d\mu + \int_\Omega u_k\, d\mu_k - \int_\Omega u_k\, d\mu - \int_\Omega u\, d\mu_k \\
&= \int_\Omega \psi\, d\mu + \int_\Omega \psi_k\, d\mu_k - \int_\Omega u_k\, d\mu - \int_\Omega u\, d\mu_k \\
&\leq \int_\Omega \psi\, d\mu + \int_\Omega \psi_k\, d\mu_k - \int_\Omega \psi_k\, d\mu - \int_\Omega \psi\, d\mu_k \\
&= \int_\Omega (\psi - \psi_k)\, d(\mu - \mu_k) \\
&= \int_\Omega \Delta(\psi - \psi_k)\, \Delta(u - u_k)\, dx \\
&\leq \tilde{C}\|\Delta\psi - \Delta\psi_k\|_2 \|\Delta u - \Delta u_k\|_2 \\
&\leq \tilde{C}\|\psi - \psi_k\|_{H^2(\Omega)} \|u - u_k\|_{H^2(\Omega)}. \quad \square
\end{aligned}$$

Chapter 9

Existence of an Optimal Control

Now we are ready to prove the main theorem of part III.

Theorem 9.1. *There exists a solution to Problem C.*

Proof. Since the objective functional $J(\psi) \geq 0$ for all $\psi \in V$ it follows that there exists a minimizing sequence $\{\psi_n\} \subset V$ such that

$$\lim_{n \to \infty} J(\psi_n) = \inf_{\psi \in V} J(\psi).$$

Let $u_n = T(\psi_n)$ be the corresponding solutions to the problem:

$$\text{find}\, u_n \in K(\psi_n) \;\text{ s. t.} \int_\Omega \Delta u_n \, \Delta(v - u_n)\, dx \geq 0 \quad \forall\, v \in K(\psi_n). \tag{9.1}$$

We have (from (9.1) with $v = \psi_n$):

$$\|u_n\|_{H^2(\Omega)} \leq \|\psi_n\|_{H^2(\Omega)} \tag{9.2}$$

Since the sequence $\{\psi_n\}$ is bounded in $H^3(\Omega)$ from the format of $J(\psi)$, we can write

$$\|\psi_n\|_{H^2(\Omega)} \leq C_1 \|\psi_n\|_{H^3(\Omega)} \leq C_2 \quad \forall n. \tag{9.3}$$

Thus

$$\|u_n\|_{H^2(\Omega)} \leq \|\psi_n\|_{H^2(\Omega)} \leq C_3 \quad \forall n. \tag{9.4}$$

The estimate in (9.3) implies that there exists $\psi^* \in V$ such that on a subsequence $\psi_n \overset{w}{\rightharpoonup} \psi^*$ in $H^3(\Omega)$. Since $H^3(\Omega) \subset\subset H^2(\Omega)$ it follows $\psi_n \overset{s}{\to} \psi^*$ in $H^2(\Omega)$. By Lemma 8.4 for $u^* = T(\psi^*)$

$$\|u^* - u_n\|_{H^2(\Omega)} \leq \tilde{C}\|\psi^* - \psi_n\|_{H^2(\Omega)}. \tag{9.5}$$

We have

$$\begin{aligned} &\psi_n \overset{s}{\to} \psi^* \text{ in } H^2(\Omega), \\ &u_n \overset{s}{\to} u^* \text{ in } H^2(\Omega). \end{aligned} \tag{9.6}$$

We show that ψ^* is optimal. Indeed, since $\psi_n \overset{w}{\rightharpoonup} \psi^*$ in $H^3(\Omega)$ and $H^3(\Omega) \subset\subset L^2(\Omega)$ it follows that

$$\varliminf_{n\to\infty} \int_\Omega |\nabla\Delta\psi_n|^2\,dx \geq \int_\Omega |\nabla\Delta\psi^*|^2\,dx.$$

Also, $T(\psi_n) = u_n \overset{s}{\to} u^* = T(\psi^*)$ in $H^2(\Omega)$ implies that

$$\lim_{n\to\infty} \int_\Omega |T(\psi_n) - z|^2\,dx$$

exists and

$$\lim_{n\to\infty} \int_\Omega |T(\psi_n) - z|^2\,dx = \int_\Omega |T(\psi^*) - z|^2\,dx.$$

Since

$$\lim_{n\to\infty} \frac{1}{2}\left(\int_\Omega |T(\psi_n) - z|^2\,dx + \int_\Omega |\nabla\Delta\psi_n|^2\,dx \right)$$

exists and $\lim_{n\to\infty} \int_\Omega |T(\psi_n) - z|^2\,dx$ exists it follows that $\lim_{n\to\infty} \int_\Omega |\nabla\Delta\psi_n|^2\,dx$ exists. Therefore we obtain

$$\begin{aligned}\inf_{\psi\in V} J(\psi) = \lim_{n\to\infty} J(\psi_n) &= \lim_{n\to\infty} \frac{1}{2}\int_\Omega |T(\psi_n) - z|^2\,dx + \lim_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla\Delta\psi_n|^2\,dx \\ &\geq \lim_{n\to\infty} \frac{1}{2}\int_\Omega |T(\psi_n) - z|^2\,dx + \varliminf_{n\to\infty} \frac{1}{2}\int_\Omega |\nabla\Delta\psi_n|^2\,dx \\ &\geq \frac{1}{2}\int_\Omega |T(\psi^*) - z|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\Delta\psi^*|^2\,dx = J(\psi^*). \quad \square\end{aligned}$$

Chapter 10

Approximation Problems

To derive necessary conditions for the optimal control, we need to differentiate the map $\psi \longmapsto J(\psi)$ which depends on $T(\psi)$. The map $\psi \longmapsto u = T(\psi)$ cannot be differentiated directly thereby necessitating introduction of a family of approximation problems. Appropriate necessary conditions will be derived as in [9]. We follow the basic structure of the derivation of the necessary conditions in [3, 16] with appropriate modifications. We define a non-negative function γ

$$\gamma(r) = \begin{cases} 0, \ r \in [0, \infty), \\ -\frac{1}{3}r^3, \ r \in [-\frac{1}{2}, 0], \\ \frac{1}{2}r^2 + \frac{1}{4}r + \frac{1}{24}, \ r \in (-\infty, -\frac{1}{2}), \end{cases} \tag{10.1}$$

and its derivative β (which is non-positive)

$$\beta(r) \stackrel{\text{def}}{=} \gamma'(r) = \begin{cases} 0, \ r \in [0, \infty), \\ -r^2, \ r \in [-\frac{1}{2}, 0], \\ r + \frac{1}{4}, \ r \in (-\infty, -\frac{1}{2}). \end{cases} \tag{10.2}$$

For an arbitrary $\delta > 0$, the problem

$$\underset{v \in H_0^2(\Omega)}{\text{minimize}} \int_\Omega \left[\frac{1}{2}|\Delta v|^2 + \frac{1}{\delta}\gamma(v-\psi) \right] dx \tag{10.3}$$

has a unique solution u^δ. It can be shown that u^δ solves

$$\begin{cases} \Delta^2 u^\delta + \dfrac{1}{\delta}\beta(u^\delta - \psi) = 0 \ \text{ in } \Omega, \\ u^\delta = 0 \text{ on } \partial\Omega, \\ \dfrac{\partial u^\delta}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{10.4}$$

Taking into account that $\beta(u^\delta - \psi) \in H_0^1(\Omega)$, by the standard elliptic theory for this semilinear case (see [24]), it follows that there exists a solution $u^\delta \in H_0^2(\Omega) \cap H^4(\Omega)$ to (10.4). Set $T_\delta(\psi) = u^\delta$ and define the following approximate objective functional

$$J_\delta(\psi) := \frac{1}{2}\int_\Omega \left\{ |T_\delta(\psi) - z|^2 + |\nabla\Delta\psi|^2 \right\} dx.$$

Now we can state our approximation problem.

Problem C$_\delta$. Find $\psi^\delta \in V$ s.t. $J_\delta(\psi^\delta) = \inf_{\psi \in V} J_\delta(\psi)$.

With the help of this approximation, we can derive necessary conditions for corresponding optimal controls. As it was already mentioned in chapter 7, the existence and uniqueness of solution to the variational inequality (7.1) follows from the coercivity of the bilinear form $a(u,v) = \int_\Omega \Delta u \, \Delta(v-u)\, dx$ (see [29]). For the sake of completeness, we give a more constructive proof of the existence of solution to (7.1) by incorporating approximation techniques. Also, it will be used in the proof of Theorem 10.2.

Theorem 10.1. *Let $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$ be arbitrary. If $u^\delta = T_\delta(\psi)$ is a solution to* (10.4) *then there exists $u \in H_0^2(\Omega)$ such that*

$$T_\delta(\psi) \stackrel{w}{\rightharpoonup} u \ \textit{in}\ H_0^2(\Omega),$$

as $\delta \to 0^+$, where $u = T(\psi)$. Furthermore, the following estimate holds

$$\|\Delta T_\delta(\psi)\|_2 \le \|\Delta\psi\|_2. \tag{10.5}$$

Proof. First, we derive (10.5). We know that given $\psi \in V$ there exists a solution $T_\delta(\psi) = u^\delta \in H_0^2(\Omega) \cap H^4(\Omega)$ to (10.4). Let $v \in K(\psi)$ be arbitrary. Multiply (10.4) by $v - u^\delta$, integrate by parts and use properties of β to get

$$\int_\Omega \Delta u^\delta \Delta(v - u^\delta)\, dx = -\frac{1}{\delta}\int_\Omega \beta(u^\delta - \psi)(v - u^\delta)\, dx \ge 0.$$

Thus we can write

$$\int_\Omega \Delta u^\delta \Delta(v - u^\delta)\, dx \ge 0. \tag{10.6}$$

Now if we take $v = \psi$ in (10.6) we obtain

$$\|\Delta u^\delta\|_2^2 = \int_\Omega (\Delta u^\delta)^2\, dx \le \int_\Omega \Delta u^\delta \Delta\psi\, dx \le \|\Delta u^\delta\|_2 \|\Delta\psi\|_2$$

whence

$$\|\Delta T_\delta(\psi)\|_2 \le \|\Delta\psi\|_2. \tag{10.7}$$

This proves (10.5).

By *a priori* estimate (10.7) there exists $u \in H_0^2(\Omega)$ such that on a subsequence

$$u^\delta \stackrel{w}{\rightharpoonup} u \text{ in } H_0^2(\Omega) \tag{10.8}$$

as $\delta \to 0^+$. For an arbitrary $\varphi \in H_0^2(\Omega)$ we obtain from (10.4)

$$-\frac{1}{\delta}\int_\Omega \beta(u^\delta - \psi)\varphi\, dx = \int_\Omega \Delta u^\delta \Delta\varphi\, dx \le \|\Delta u^\delta\|_2 \|\Delta\varphi\|_2 \le C\|\Delta\psi\|_2 \|\Delta\varphi\|_2. \tag{10.9}$$

First, we show that $u \ge \psi$ a.e. in Ω. Take an arbitrary $\varphi \in H_0^2(\Omega)$, $\varphi \ge 0$ a.e. in Ω. Rewrite (10.9) as

$$0 \le -\int_\Omega \beta(u^\delta - \psi)\varphi\, dx \le C\delta\|\Delta\psi\|_2 \|\Delta\varphi\|_2. \tag{10.10}$$

Letting $\delta \to 0^+$ in (10.10) we obtain

$$\int_\Omega \beta(u-\psi)\varphi \, dx = 0 \quad \forall \varphi \in H_0^2(\Omega), \varphi \geq 0 \text{ a.e. in } \Omega.$$

This implies $u \geq \psi$ a.e. in Ω. Hence, $u \in K(\psi)$.

Next, we show that u satisfies

$$\int_\Omega \Delta u \, \Delta(v-u) \, dx \geq 0 \quad \forall \, v \, \in K(\psi), \tag{10.11}$$

i.e., we need to show that u minimizes

$$\int_\Omega |\Delta v|^2 \, dx, \quad v \in K(\psi).$$

Indeed, let $v \in K(\psi)$ be arbitrary. Taking into account (10.3) and the definition of γ we can write

$$\begin{aligned} \int_\Omega \frac{1}{2}|\Delta v|^2 \, dx &= \int_\Omega \left[\frac{1}{2}|\Delta v|^2 + \frac{1}{\delta}\gamma(v-\psi)\right] dx \\ &\geq \int_\Omega \left[\frac{1}{2}|\Delta u^\delta|^2 + \frac{1}{\delta}\gamma(u^\delta-\psi)\right] dx \\ &\geq \int_\Omega \frac{1}{2}|\Delta u^\delta|^2 \, dx. \end{aligned} \tag{10.12}$$

Using (10.8), (10.12), and lower semicontinuity of weak convergence we get

$$\int_\Omega |\Delta v|^2 \, dx \geq \varliminf_{\delta \to 0^+} \int_\Omega |\Delta u^\delta|^2 \, dx \geq \int_\Omega |\Delta u|^2 \, dx. \tag{10.13}$$

Hence, u satisfies (10.11). Thus, we have shown that $u = T(\psi)$, i.e., u is a solution to (7.1) associated with the obstacle ψ. $\square$

Now we prove the following

Proposition 10.1. *Problem C_δ admits an optimal pair $(\bar{u}^\delta, \psi^\delta)$, where $\bar{u}^\delta = T_\delta(\psi^\delta)$. Moreover, we have*

$$\|\Delta \bar{u}^\delta\|_2^2 \leq C \int_\Omega |\nabla \Delta \psi^\delta|^2 \, dx \leq C\|z\|_2^2. \tag{10.14}$$

Proof. Let $\{\psi_k\}_{k=1}^\infty \subset V$ be a minimizing sequence for J_δ. Since $\int_\Omega |\nabla \Delta \psi_k|^2 \, dx$ is bounded (because $\{\psi_k\}$ is a minimizing sequence) it follows that $\|\psi_k\|_{H^3(\Omega)} \leq C$. This implies that there exists $\psi^\delta \in V$ such that on a subsequence

$$\psi_k \overset{w}{\rightharpoonup} \psi^\delta \text{ in } H^3(\Omega).$$

Let $u_k^\delta = T_\delta(\psi_k)$ be the corresponding solution to (10.4) (with $\psi = \psi_k$). Then by (10.5)

$$\|\Delta T_\delta(\psi_k)\|_2 \leq \|\Delta \psi_k\|_2 \leq C_4 \tag{10.15}$$

and it follows that there exists $\bar{u}^\delta \in H_0^2(\Omega)$ such that on a subsequence $u_k^\delta \overset{w}{\rightharpoonup} \bar{u}^\delta$ in $H_0^2(\Omega)$ as $k \to \infty$. Thus we obtain on a subsequence

$$\begin{cases} \psi_k \overset{w}{\rightharpoonup} \psi^\delta \text{ in } H^3(\Omega), \\ u_k^\delta \overset{w}{\rightharpoonup} \bar{u}^\delta \text{ in } H_0^2(\Omega) \text{ as } k \to \infty. \end{cases} \tag{10.16}$$

Note that convergences in (10.16) imply that on a subsequence

$$\begin{cases} \psi_k \overset{s}{\rightarrow} \psi^\delta \text{ in } H^1(\Omega), \\ u_k^\delta \overset{s}{\rightarrow} \bar{u}^\delta \text{ in } H_0^1(\Omega) \text{ as } k \to \infty. \end{cases} \tag{10.17}$$

After multiplying (10.4) by a test function $v \in H_0^2(\Omega)$ and integrating by parts we get

$$\int_\Omega \Delta u_k^\delta \Delta v \, dx = -\frac{1}{\delta} \int_\Omega \beta(u_k^\delta - \psi_k) v \, dx. \tag{10.18}$$

Taking into account (10.17) and the fact that β is continuous we have

$$\int_\Omega \beta(u_k^\delta - \psi_k) v\, dx \to \int_\Omega \beta(\bar{u}^\delta - \psi^\delta) v\, dx$$

as $k \to \infty$. Now, letting $k \to \infty$ in (10.18) we obtain

$$\int_\Omega \Delta \bar{u}^\delta \Delta v\, dx = -\frac{1}{\delta} \int_\Omega \beta(\bar{u}^\delta - \psi^\delta) v\, dx. \tag{10.19}$$

This implies that $\bar{u}^\delta$ is the solution associated with ψ^δ, i.e., $\bar{u}^\delta = T_\delta(\psi^\delta)$. We show that ψ^δ is optimal for J_δ. Indeed, we have

$$\begin{aligned}\inf_{\psi \in V} J_\delta(\psi) = \lim_{k\to\infty} J_\delta(\psi_k) &\geq \frac{1}{2}\int_\Omega \big(T_\delta(\psi^\delta) - z\big)^2\, dx + \frac{1}{2} \varliminf_{k\to\infty} \int_\Omega |\nabla \Delta \psi_k|^2\, dx \\ &\geq \frac{1}{2}\int_\Omega \big(T_\delta(\psi^\delta) - z\big)^2\, dx + \frac{1}{2}\int_\Omega |\nabla \Delta \psi^\delta|^2\, dx = J_\delta(\psi^\delta).\end{aligned}$$

This proves that ψ^δ is optimal. Now we prove (10.14). Denoting $d\mu^\delta \stackrel{\text{def}}{=} -\frac{1}{\delta}\beta(\bar{u}^\delta - \psi^\delta)\, dx$, we estimate

$$\begin{aligned}\int_\Omega |\Delta \bar{u}^\delta|^2\, dx &= \int_\Omega \Delta \bar{u}^\delta \Delta \bar{u}^\delta\, dx = -\int_\Omega \bar{u}^\delta \frac{1}{\delta}\beta(\bar{u}^\delta - \psi^\delta)\, dx \\ &= \int_\Omega \bar{u}^\delta\, d\mu^\delta \leq \int_\Omega \psi^\delta\, d\mu^\delta \qquad (10.20) \\ &= \int_\Omega \Delta \psi^\delta \Delta \bar{u}^\delta\, dx = -\int_\Omega \nabla \bar{u}^\delta \cdot \nabla \Delta \psi^\delta\, dx \\ &\leq \left| \int_\Omega \nabla \bar{u}^\delta \cdot \nabla \Delta \psi^\delta\, dx \right| \leq \|\nabla \bar{u}^\delta\|_2 \cdot \|\nabla \Delta \psi^\delta\|_2 \qquad (10.21) \\ &\leq C_5 \|\Delta \bar{u}^\delta\|_2 \cdot \|\nabla \Delta \psi^\delta\|_2\end{aligned}$$

where we used properties of β in (10.20) and the Poincaré inequality in (10.21). Hence,

$$\begin{aligned}\|\Delta \bar{u}^\delta\|_2^2 \leq C \|\nabla \Delta \psi^\delta\|_2^2 &= C \int_\Omega |\nabla \Delta \psi^\delta|^2\, dx \\ &\leq C J_\delta(\psi^\delta) \leq C J_\delta(0) = C\|z\|_2^2. \quad \square\end{aligned}$$

Now we prove the main result of this chapter, namely the convergence of the optimal controls for the approximation problems to the minimizer for the original problem.

Theorem 10.2. *Let $(\bar{u}^\delta, \psi^\delta)$ be an optimal pair for Problem C_δ. Then there exist $\bar{u} \in H_0^2(\Omega)$ and $\bar{\psi} \in V$ such that on a subsequence*

$$\psi^\delta \xrightarrow{s} \bar{\psi} \textit{ in } H^2(\Omega),$$

$$\bar{u}^\delta \xrightarrow{s} \bar{u} \textit{ in } H_0^2(\Omega),$$

where $\bar{u} = T(\bar{\psi})$ and $(\bar{u}, \bar{\psi})$ is an optimal pair for Problem C.

Proof. By *a priori* estimates (10.14), there exist $\bar{u} \in H_0^2(\Omega)$ and $\bar{\psi} \in V$ such that on a subsequence

$$\begin{cases} \psi^\delta \overset{w}{\rightharpoonup} \bar{\psi} \text{ in } H^3(\Omega), \\ \bar{u}^\delta \overset{w}{\rightharpoonup} \bar{u} \text{ in } H_0^2(\Omega) \end{cases} \tag{10.22}$$

as $\delta \to 0^+$. Since $H^3(\Omega) \subset\subset H^2(\Omega)$ it follows that on a subsequence

$$\psi^\delta \xrightarrow{s} \bar{\psi} \text{ in } H^2(\Omega). \tag{10.23}$$

For an arbitrary $\varphi \in H_0^2(\Omega)$ we obtain from (10.4) (with $\psi = \psi^\delta$)

$$-\frac{1}{\delta}\int_\Omega \beta(\bar{u}^\delta - \psi^\delta)\varphi\, dx = \int_\Omega \Delta\bar{u}^\delta \Delta\varphi\, dx \le \|\Delta\bar{u}^\delta\|_2 \|\Delta\varphi\|_2 \le C\|z\|_2\|\Delta\varphi\|_2. \tag{10.24}$$

This implies that there exists $\bar{\mu} \in H^{-2}(\Omega)$ such that

$$\mu^\delta = \Delta^2\bar{u}^\delta \overset{*}{\rightharpoonup} \bar{\mu} \text{ in } H^{-2}(\Omega). \tag{10.25}$$

Convergence in (10.25) means that for an arbitrary function $f \in H_0^2(\Omega)$

$$\int_\Omega f\, d\mu^\delta \to \int_\Omega f\, d\bar{\mu}. \tag{10.26}$$

First, we show that $\bar{u} \geq \bar{\psi}$ a.e. in Ω. Take an arbitrary $\varphi \in H_0^2(\Omega)$, $\varphi \geq 0$ a.e. in Ω. We can rewrite (10.24) as follows

$$0 \leq -\int_\Omega \beta(\bar{u}^\delta - \psi^\delta)\varphi \, dx \leq C\delta \|z\|_2 \|\Delta\varphi\|_2. \tag{10.27}$$

Letting $\delta \to 0^+$ in (10.27) we obtain

$$\int_\Omega \beta(\bar{u} - \bar{\psi})\varphi \, dx = 0 \quad \forall \varphi \in H_0^2(\Omega), \varphi \geq 0 \text{ a.e. in } \Omega.$$

This implies $\bar{u} \geq \bar{\psi}$ a.e. in Ω. Hence, $\bar{u} \in K(\bar{\psi})$.

Next, we show that $\bar{u}$ satisfies

$$\int_\Omega \Delta\bar{u} \, \Delta(v - \bar{u}) \, dx \geq 0 \quad \forall \, v \, \in K(\bar{\psi}), \tag{10.28}$$

i.e., we need to show that $\bar{u}$ minimizes

$$\int_\Omega |\Delta v|^2 \, dx, \quad v \in K(\bar{\psi}).$$

Indeed, let $v \in K(\bar{\psi})$ be arbitrary. Taking into account (10.3) and the definition of γ we can write

$$\begin{aligned} \int_\Omega \frac{1}{2}|\Delta v|^2 \, dx &= \int_\Omega \left[\frac{1}{2}|\Delta v|^2 + \frac{1}{\delta}\gamma(v - \bar{\psi})\right] dx \\ &\geq \int_\Omega \left[\frac{1}{2}|\Delta \bar{u}^\delta|^2 + \frac{1}{\delta}\gamma(\bar{u}^\delta - \bar{\psi})\right] dx \\ &\geq \int_\Omega \frac{1}{2}|\Delta \bar{u}^\delta|^2 \, dx. \end{aligned} \tag{10.29}$$

Using (10.22), (10.29), and lower semicontinuity of weak convergence we get

$$\int_\Omega |\Delta v|^2 \, dx \geq \varliminf_{\delta \to 0^+} \int_\Omega |\Delta \bar{u}^\delta|^2 \, dx \geq \int_\Omega |\Delta \bar{u}|^2 \, dx. \tag{10.30}$$

Hence, $\bar{u}$ satisfies (10.28). Thus, we have shown that $\bar{u} = T(\bar{\psi})$. Let us show that $\bar{\psi}$ minimizes $J(\psi)$. To this end, denote $T_\delta(\psi^*) := u_*^\delta$ where ψ^* is an optimal control from Theorem 9.1 (see formula (9.6)). Note that by Theorem 10.1

$$u_*^\delta \stackrel{w}{\rightharpoonup} u^* = T(\psi^*) \text{ in } H_0^2(\Omega) \text{ as } \delta \to 0^+, \tag{10.31}$$

and the optimality of ψ^δ implies

$$J_\delta(\psi^\delta) \le J_\delta(\psi^*). \tag{10.32}$$

Taking into account (10.31), lower semicontinuity of weak convergence, Theorem 10.1, and (10.32) we can write

$$\begin{aligned}
J(\bar{\psi}) &\le \varliminf_{\delta\to 0^+} J_\delta(\psi^\delta) \le \varlimsup_{\delta\to 0^+} J_\delta(\psi^\delta) \le \varlimsup_{\delta\to 0^+} J_\delta(\psi^*) \\
&= \frac{1}{2} \varlimsup_{\delta\to 0^+} \int_\Omega \left\{ |T_\delta(\psi^*) - z|^2 + |\nabla\Delta\psi^*|^2 \right\} dx \\
&= \frac{1}{2} \int_\Omega \left\{ |T(\psi^*) - z|^2 + |\nabla\Delta\psi^*|^2 \right\} dx \\
&= J(\psi^*) = \inf_{\psi\in V} J(\psi).
\end{aligned}$$

Thus

$$J(\bar{\psi}) = \inf_{\psi\in V} J(\psi).$$

This proves that $\bar{\psi}$ minimizes $J(\psi)$.

Recollect that we obtained (see (10.25))

$$\mu^\delta \stackrel{*}{\rightharpoonup} \bar{\mu} \text{ in } H^{-2}(\Omega).$$

Taking $f \in H_0^2(\Omega)$ in (10.26) we get

$$\int_\Omega f\, d\bar{\mu} = \lim_{\delta\to 0^+} \int_\Omega f\, d\mu^\delta = \lim_{\delta\to 0^+} \int_\Omega \Delta f \cdot \Delta \bar{u}^\delta\, dx = \int_\Omega \Delta f \cdot \Delta \bar{u}\, dx.$$

This implies

$$\Delta^2 \bar{u} = \bar{\mu} \text{ in } \Omega. \tag{10.33}$$

Since μ^δ is a nonnegative Borel measure for each $\delta > 0$ and the set of all nonnegative Borel measures supported in Ω is convex and weak-$*$ closed in $H^{-2}(\Omega)$ it follows that $\bar{\mu} \geq 0$ (as a weak-$*$ limit of (μ^δ)'s).

Before we proceed to the proof of strong convergence of $\bar{u}^\delta$ to $\bar{u}$ in $H_0^2(\Omega)$ we make the following observation. Since $\bar{u} = T(\bar{\psi})$ solves the variational inequality (7.1) it follows that there exists a nonnegative measure $\mu^\dagger \in H^{-2}(\Omega)$ such that

$$\Delta^2 \bar{u} = \mu^\dagger \text{ in } \Omega,$$

$$\bar{u} = \bar{\psi} \;\; \mu^\dagger\text{-a.e.}$$

That is, we have

$$\int_\Omega \xi \, d\mu^\dagger = \int_\Omega \Delta \bar{u} \Delta \xi \, dx \quad \forall \, \xi \in H_0^2(\Omega).$$

On the other hand, we conclude from (10.33)

$$\int_\Omega \xi \, d\bar{\mu} = \int_\Omega \Delta \bar{u} \Delta \xi \, dx \quad \forall \, \xi \in H_0^2(\Omega).$$

Hence,

$$\int_\Omega \xi \, d\bar{\mu} = \int_\Omega \xi \, d\mu^\dagger \quad \forall \xi \in H_0^2(\Omega).$$

Therefore, we can write

$$\|\mu^\dagger - \bar{\mu}\|_{H^{-2}(\Omega)} = \sup |(\mu^\dagger - \bar{\mu})(\xi)| = 0,$$

where the supremum is taken over all $\xi \in H_0^2(\Omega)$ with $\|\xi\|_{H_0^2(\Omega)} = 1$. Thus $\mu^\dagger = \bar{\mu}$.

Now we are ready to prove that, in fact, $\bar{u}^\delta$ converges strongly to $\bar{u}$ in $H_0^2(\Omega)$ as $\delta \to 0^+$. Indeed,

$$
\begin{aligned}
\int_\Omega (\Delta \bar{u}^\delta - \Delta \bar{u})^2 dx &= \int_\Omega \Delta(\bar{u}^\delta - \bar{u})\Delta(\bar{u}^\delta - \bar{u})\, dx \\
&= \int_\Omega (\bar{u}^\delta - \bar{u})\, d(\mu^\delta - \bar{\mu}) \\
&= \int_\Omega \bar{u}^\delta\, d\mu^\delta + \int_\Omega \bar{u}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\mu^\delta - \int_\Omega \bar{u}^\delta\, d\bar{\mu} \\
&= \int_\Omega (\bar{u}^\delta - \psi^\delta)\, d\mu^\delta + \int_\Omega \psi^\delta\, d\mu^\delta + \int_\Omega \bar{u}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\mu^\delta - \int_\Omega \bar{u}^\delta\, d\bar{\mu} \\
&\leq \int_\Omega \psi^\delta\, d\mu^\delta + \int_\Omega \bar{\psi}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\mu^\delta - \int_\Omega \bar{u}^\delta\, d\bar{\mu}
\end{aligned}
\tag{10.34}
$$

where we have taken into account that $\int_\Omega (\bar{u}^\delta - \psi^\delta)\, d\mu^\delta \leq 0$ (by properties of β) and $\bar{u} = \bar{\psi}$ $\bar{\mu}$-a.e. Letting $\delta \to 0^+$ in (10.34) and taking into account convergences in (10.23) and (10.25), we get

$$
\begin{aligned}
\lim_{\delta \to 0} \|\bar{u}^\delta - \bar{u}\|^2_{H^2(\Omega)} &\leq \int_\Omega \bar{\psi}\, d\bar{\mu} + \int_\Omega \bar{\psi}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\bar{\mu} \\
&\leq \int_\Omega \bar{u}\, d\bar{\mu} + \int_\Omega \bar{u}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\bar{\mu} - \int_\Omega \bar{u}\, d\bar{\mu} = 0.
\end{aligned}
$$

This shows that $\bar{u}^\delta \xrightarrow{s} \bar{u}$ in $H_0^2(\Omega)$ and the proof of the theorem is complete. □

We will use this convergence in the next chapter (actually, we will need only strong L^2-convergence of $\bar{u}^\delta$ to $\bar{u}$).

Chapter 11

Characterization of the Optimal Control

In order to characterize an optimal control we need to derive necessary conditions which include the original state system coupled with an adjoint system. We must first obtain necessary conditions for the approximate problems. To this end, we differentiate the objective functional $J_\delta(\psi)$ with respect to the control. Since the objective functional depends on u^δ we will need to differentiate u^δ with respect to control ψ. We will make use of the following:

Lemma 11.1. (Sensitivity) *The map $\psi \longmapsto u^\delta \equiv u^\delta(\psi)$ is differentiable in the following sense: given $\psi, \ell \in V$, there exists a $\xi^\delta \in H_0^2(\Omega)$ such that*

$$\frac{u^\delta(\psi + \varepsilon\ell) - u^\delta(\psi)}{\varepsilon} \xrightarrow{w} \xi^\delta \text{ in } H_0^2(\Omega) \text{ as } \varepsilon \to 0. \tag{11.1}$$

Moreover, ξ^δ solves

$$\begin{cases} \Delta^2 \xi^\delta + \dfrac{1}{\delta}\beta'(u^\delta - \psi)(\xi^\delta - \ell) = 0 \text{ in } \Omega, \\ \xi^\delta = 0 \text{ on } \partial\Omega, \\ \dfrac{\partial \xi^\delta}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{11.2}$$

Proof. The proof is similar to the one in [3], Lemma 5.1. We have

$$\begin{cases} \Delta^2 u^\delta(\psi+\varepsilon\ell) + \dfrac{1}{\delta}\beta(u^\delta(\psi+\varepsilon\ell)-\psi-\varepsilon\ell) = 0 \text{ in } \Omega, \\ u^\delta(\psi+\varepsilon\ell) = 0 \text{ on } \partial\Omega, \\ \dfrac{\partial u^\delta(\psi+\varepsilon\ell)}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{11.3}$$

and

$$\begin{cases} \Delta^2 u^\delta(\psi) + \dfrac{1}{\delta}\beta(u^\delta(\psi)-\psi) = 0 \text{ in } \Omega, \\ u^\delta(\psi) = 0 \text{ on } \partial\Omega, \\ \dfrac{\partial u^\delta(\psi)}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{11.4}$$

Subtracting (11.4) from (11.3) gives

$$\begin{cases} \Delta^2 \Big(u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi)\Big) = -\dfrac{1}{\delta}\beta(u^\delta(\psi+\varepsilon\ell)-\psi-\varepsilon\ell) + \dfrac{1}{\delta}\beta(u^\delta(\psi)-\psi) \text{ in } \Omega, \\ u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi) = 0 \text{ on } \partial\Omega, \\ \dfrac{\partial(u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi))}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{11.5}$$

If we multiply (11.5) by the test function $\phi = u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi)$, integrate over Ω, and integrate by parts twice we get

$$\begin{aligned} &\int_\Omega \Big(\Delta[u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi)]\Big)^2 dx = \\ &= -\frac{1}{\delta}\int_\Omega \int_0^1 \beta'[\theta(u^\delta(\psi+\varepsilon\ell)-\psi-\varepsilon\ell) + (1-\theta)(u^\delta(\psi)-\psi)]\, d\theta \,\cdot \\ &\cdot (u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi) - \varepsilon\ell)(u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi))\, dx \\ &\le -\frac{1}{\delta}\int_\Omega \int_0^1 \beta'(\cdot)\, d\theta (u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi))^2\, dx \\ &+ \frac{\varepsilon}{\delta}\int_\Omega \ell \int_0^1 \beta'(\cdot)\, d\theta\, [u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi)]\, dx \\ &\le \frac{\varepsilon}{\delta}\|\ell\|_2 \|u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi)\|_2 \\ &\le \frac{C'\varepsilon}{\delta}\|\ell\|_2 \|\nabla(u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi))\|_2 \\ &\le \frac{C\varepsilon}{\delta}\|\ell\|_2 \|\Delta(u^\delta(\psi+\varepsilon\ell) - u^\delta(\psi))\|_2 \end{aligned}$$

where we used the Poincaré inequality twice. Thus we obtain

$$\Big\|\Delta\Big(\frac{u^\delta(\psi+\varepsilon\ell)-u^\delta(\psi)}{\varepsilon}\Big)\Big\|_2 \le \frac{C}{\delta}\|\ell\|_2$$

which gives the desired *a priori* estimate of the difference quotients and therefore justifies the weak convergence on a subsequence to a limit which we call ξ^δ. To see that ξ^δ satisfies (11.2), take an arbitrary $\varphi \in H_0^2(\Omega)$ and let $\varepsilon \to 0$ in the equation below

$$\int_\Omega \Delta\Big(\frac{u^\delta(\psi+\varepsilon\ell)-u^\delta(\psi)}{\varepsilon}\Big)\Delta\varphi\,dx = -\frac{1}{\delta}\int_\Omega\int_0^1 \beta'(\cdot)\,d\theta\Big(\frac{u^\delta(\psi+\varepsilon\ell)-u^\delta(\psi)}{\varepsilon}-\ell\Big)\varphi\,dx. \quad \square$$

We prove a theorem which gives necessary conditions for optimal pairs of Problem C_δ.

Theorem 11.1. *Let $(\bar{u}^\delta, \psi^\delta)$ be an optimal pair for problem C_δ. Then there exists an adjoint function $p^\delta \in H_0^2(\Omega)$ such that the triple $(\bar{u}^\delta, p^\delta, \psi^\delta)$ satisfies the following system of equations:*

$$\begin{aligned}
\Delta^2\bar{u}^\delta + \frac{1}{\delta}\beta(\bar{u}^\delta-\psi^\delta) &= 0 \text{ in } \Omega,\\
\Delta^2 p^\delta + \frac{1}{\delta}\beta'(\bar{u}^\delta-\psi^\delta)p^\delta &= \bar{u}^\delta - z \text{ in } \Omega,\\
-\Delta^3\psi^\delta + \frac{1}{\delta}\beta'(\bar{u}^\delta-\psi^\delta)p^\delta &= 0 \text{ in } \Omega,\\
\bar{u}^\delta = \frac{\partial \bar{u}^\delta}{\partial n} &= 0 \text{ on } \partial\Omega,\\
p^\delta = \frac{\partial p^\delta}{\partial n} &= 0 \text{ on } \partial\Omega,\\
\psi^\delta = \frac{\partial\psi^\delta}{\partial n} = \frac{\partial(\Delta\psi^\delta)}{\partial n} &= 0 \text{ on } \partial\Omega.
\end{aligned} \tag{11.6}$$

Proof. Let $(\bar{u}^\delta, \psi^\delta)$ be an optimal pair for problem C_δ. Take an arbitrary $\ell \in V$. First, we derive a system which is adjoint to (11.2). Rewrite (11.2) as follows (with ψ replaced by ψ^δ)

$$\mathcal{L}\xi^\delta = \frac{1}{\delta}\,\beta'(\bar{u}^\delta-\psi^\delta)\ell \tag{11.7}$$

where

$$
\begin{aligned}
\mathcal{L}\xi^\delta &\stackrel{\text{def}}{=} \Delta^2\xi^\delta + \frac{1}{\delta}\,\beta'(\bar{u}^\delta - \psi^\delta)\xi^\delta, \\
\xi^\delta &= 0 \text{ on } \partial\Omega, \\
\frac{\partial \xi^\delta}{\partial n} &= 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{11.8}
$$

Note that the boundary conditions in (11.8) are part of the definition of operator $\mathcal{L}$. We determine a Hilbert space adjoint of $\mathcal{L}$. If we set $p^\delta = 0$, $\frac{\partial p^\delta}{\partial n} = 0$ on $\partial\Omega$ we will have

$$
\begin{aligned}
\int_\Omega p^\delta\,(\Delta^2\xi^\delta)\,dx &= \int_\Omega \sum_{i=1}^n p^\delta\,(\Delta\xi^\delta)_{x_i x_i}\,dx \\
&= -\int_\Omega \sum_{i=1}^n p^\delta_{x_i}\,(\Delta\xi^\delta)_{x_i}\,dx + \int_{\partial\Omega} \sum_{i=1}^n p^\delta\,(\Delta\xi^\delta)_{x_i}\eta_i\,ds \\
&= -\int_\Omega \sum_{i=1}^n p^\delta_{x_i}\,(\Delta\xi^\delta)_{x_i}\,dx \\
&= \int_\Omega \sum_{i=1}^n p^\delta_{x_i x_i}\,\Delta\xi^\delta\,dx - \int_{\partial\Omega} \sum_{i=1}^n p^\delta_{x_i}\,\Delta\xi^\delta\eta_i\,ds \\
&= \int_\Omega \Delta p\,\Delta\xi^\delta\,dx - \int_{\partial\Omega} \frac{\partial p^\delta}{\partial n}\Delta\xi^\delta\,ds = \int_\Omega \Delta p\,\Delta\xi^\delta\,dx,
\end{aligned}
$$

where we integrated by parts twice. Similarly, taking into account that $\xi = 0$, $\frac{\partial \xi^\delta}{\partial n} = 0$ on $\partial\Omega$, one can show

$$
\int_\Omega \Delta p^\delta\,\Delta\xi^\delta\,dx = \int_\Omega (\Delta^2 p^\delta)\,\xi^\delta\,dx.
$$

Consequently, the adjoint function p^δ will satisfy

$$
\begin{aligned}
\Delta^2 p^\delta + \frac{1}{\delta}\beta'(\bar{u}^\delta - \psi^\delta)p^\delta &= \bar{u}^\delta - z \text{ in } \Omega, \\
p^\delta = \frac{\partial p^\delta}{\partial n} &= 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{11.9}
$$

where on the right hand side of (11.9) is the derivative of the integrand of the objective functional $J_\delta(\psi^\delta)$ with respect to the state $\bar{u}^\delta$. The standard elliptic theory (for example, see [24]) implies that there exists a solution $p^\delta \in H_0^2(\Omega)\cap H^4(\Omega)$ satisfying (11.9). We have

$$\begin{aligned}
J_\delta(\psi^\delta+\varepsilon\ell) &= \frac{1}{2}\int_\Omega \Big\{|\bar u^\delta(\psi^\delta+\varepsilon\ell)-z|^2+|\nabla\Delta(\psi^\delta+\varepsilon\ell)|^2\Big\}\,dx\\
&= \frac{1}{2}\int_\Omega \Big\{\Big(\bar u^\delta(\psi^\delta+\varepsilon\ell)\Big)^2-2\bar u^\delta(\psi^\delta+\varepsilon\ell)z+z^2+|\nabla\Delta\psi^\delta|^2\Big\}\,dx\\
&+\frac{1}{2}\int_\Omega \Big\{2\varepsilon\nabla\Delta\psi^\delta\cdot\nabla\Delta\ell+\varepsilon^2|\nabla\Delta\ell|^2\Big\}\,dx,\ \text{and}
\end{aligned}$$

$$\begin{aligned}
J_\delta(\psi^\delta) &= \frac{1}{2}\int_\Omega \Big\{|\bar u^\delta(\psi^\delta)-z|^2+|\nabla\Delta(\psi^\delta)|^2\Big\}\,dx\\
&= \frac{1}{2}\int_\Omega \Big\{\Big(\bar u^\delta(\psi^\delta)\Big)^2-2\bar u^\delta(\psi^\delta)z+z^2+|\nabla\Delta\psi^\delta|^2\Big\}\,dx,
\end{aligned}$$

whence

$$\begin{aligned}
0 &\le \varliminf_{\varepsilon\to0^+}\frac{J_\delta(\psi^\delta+\varepsilon\ell)-J_\delta(\psi^\delta)}{\varepsilon}\\
&= \varliminf_{\varepsilon\to0^+}\frac{1}{2}\int_\Omega\Bigg\{\left(\frac{\bar u^\delta(\psi^\delta+\varepsilon\ell)-\bar u^\delta(\psi^\delta)}{\varepsilon}\right)\Big(\bar u^\delta(\psi^\delta+\varepsilon\ell)+\bar u^\delta(\psi^\delta)\Big)\\
&-2\left(\frac{\bar u^\delta(\psi^\delta+\varepsilon\ell)-\bar u^\delta(\psi^\delta)}{\varepsilon}\right)z+2\nabla\Delta\psi^\delta\cdot\nabla\Delta\ell+\varepsilon|\nabla\Delta\ell|^2\Bigg\}\,dx\\
&= \int_\Omega\Big\{(\bar u^\delta-z)\xi^\delta+\nabla\Delta\psi^\delta\cdot\nabla\Delta\ell\Big\}\,dx\\
&= \int_\Omega\Big\{\Big(\Delta^2p^\delta+\frac{1}{\delta}\beta'(\bar u^\delta-\psi^\delta)p^\delta\Big)\xi^\delta+\nabla\Delta\psi^\delta\cdot\nabla\Delta\ell\Big\}\,dx\\
&= \int_\Omega\Big\{\Delta p^\delta\Delta\xi^\delta+\frac{1}{\delta}\beta'(\bar u^\delta-\psi^\delta)p^\delta\xi^\delta+\nabla\Delta\psi^\delta\cdot\nabla\Delta\ell\Big\}\,dx\\
&= \int_\Omega\left\{\frac{1}{\delta}\beta'(\bar u^\delta-\psi^\delta)\,\ell p^\delta+\nabla\Delta\psi^\delta\cdot\nabla\Delta\ell\right\}\,dx
\end{aligned}$$

where we have taken into account (11.7) and (11.9). Since $\ell\in V$ is arbitrary, taking $\partial(\Delta\psi^\delta)/\partial n=0$ on $\partial\Omega$ and using $\ell=\frac{\partial\ell}{\partial n}=0$ on $\partial\Omega$, we obtain

$$
\begin{aligned}
\int_\Omega \nabla\Delta\psi^\delta \cdot \nabla\Delta\ell\, dx &= \int_\Omega \sum_{i=1}^n (\Delta\psi^\delta)_{x_i} (\Delta\ell)_{x_i}\, dx \\
&= -\int_\Omega \sum_{i=1}^n (\Delta\psi^\delta)_{x_i x_i} \Delta\ell\, dx + \int_{\partial\Omega} \sum_{i=1}^n (\Delta\psi^\delta)_{x_i} (\Delta\ell)\eta_i\, ds \\
&= -\int_\Omega (\Delta^2\psi^\delta)\Delta\ell\, dx + \int_{\partial\Omega} \frac{\partial(\Delta\psi^\delta)}{\partial n}\Delta\ell\, ds \\
&= -\int_\Omega (\Delta^2\psi^\delta)\Delta\ell\, dx \\
&= \int_\Omega \sum_{i=1}^n (\Delta^2\psi^\delta)_{x_i} \ell_{x_i}\, dx - \int_{\partial\Omega} \sum_{i=1}^n (\Delta^2\psi^\delta)\ell_{x_i}\eta_i\, ds \\
&= \int_\Omega \sum_{i=1}^n (\Delta^2\psi^\delta)_{x_i} \ell_{x_i}\, dx - \int_{\partial\Omega} (\Delta^2\psi^\delta)\frac{\partial\ell}{\partial n}\, ds \\
&= \int_\Omega \sum_{i=1}^n (\Delta^2\psi^\delta)_{x_i} \ell_{x_i}\, dx \\
&= -\int_\Omega \sum_{i=1}^n (\Delta^2\psi^\delta)_{x_i x_i} \ell\, dx + \int_{\partial\Omega} \sum_{i=1}^n (\Delta^2\psi^\delta)_{x_i} \ell\eta_i\, ds \\
&= -\int_\Omega (\Delta^3\psi^\delta)\ell\, dx.
\end{aligned}
$$

Therefore, we conclude

$$
\begin{aligned}
-\Delta^3\psi^\delta + \frac{1}{\delta}\beta'(\bar{u}^\delta - \psi^\delta)p^\delta = 0 \text{ in } \Omega, \\
\psi^\delta = \frac{\partial\psi^\delta}{\partial n} = \frac{\partial(\Delta\psi^\delta)}{\partial n} = 0 \text{ on } \partial\Omega.
\end{aligned}
\tag{11.10}
$$

This completes the proof of the theorem. □

Next, we proceed to a theorem which gives necessary conditions for an optimal control for our original Problem C.

Theorem 11.2. *Let $(\bar{u}, \bar{\psi}) \in H_0^2(\Omega) \times V$ be an optimal pair for Problem C. Then there exist $\bar{p} \in H_0^2(\Omega)$ and $\bar{q} \in H^{-2}(\Omega)$ such that:*

$$\begin{aligned}
&\bar{u} = T(\bar{\psi}),\\
&\Delta^2 \bar{p} + \bar{q} = \bar{u} - z \ \ in\ \Omega,\\
&\Delta^3 \bar{\psi} + \bar{q} = 0 \ \ in\ \Omega,\\
&\bar{\psi} = \frac{\partial \bar{\psi}}{\partial n} = \frac{\partial (\Delta \bar{\psi})}{\partial n} = 0 \ \ on\ \partial\Omega,\\
&\bar{u} = \frac{\partial \bar{u}}{\partial n} = 0 \ \ on\ \partial\Omega,\\
&\bar{p} = \frac{\partial \bar{p}}{\partial n} = 0 \ \ on\ \partial\Omega,\\
&\left\langle \bar{q}, \bar{p} \right\rangle_{H^{-2},\, H_0^2} \geq 0.
\end{aligned} \tag{11.11}$$

Proof. In order to pass to a limit in (11.6) we need to show an *a priori* estimate for p^δ first. Multiply (11.9) by a test function $v = p^\delta$, integrate over Ω, and integrate by parts twice to get

$$\int_\Omega (\Delta p^\delta)^2\, dx = -\frac{1}{\delta} \int_\Omega \beta'(\bar{u}^\delta - \psi^\delta)(p^\delta)^2\, dx + \int_\Omega (\bar{u}^\delta - z) p^\delta\, dx.$$

Since $\beta'(\bar{u}^\delta - \psi^\delta) \geq 0$ we can write

$$\int_\Omega (\Delta p^\delta)^2\, dx = -\frac{1}{\delta} \int_\Omega \beta'(\bar{u}^\delta - \psi^\delta)(p^\delta)^2\, dx + \int_\Omega (\bar{u}^\delta - z) p^\delta\, dx \leq \int_\Omega (\bar{u}^\delta - z) p^\delta\, dx, \tag{11.12}$$

and, therefore

$$\|\Delta p^\delta\|_2^2 = \int_\Omega (\Delta p^\delta)^2\, dx \leq \|\bar{u}^\delta - z\|_2 \cdot \|p^\delta\|_2. \tag{11.13}$$

The fact that $p^\delta \in H_0^2(\Omega)$ and (10.14) give us the desired H^2 estimate on p^δ. It follows from (11.13) that there exists $\bar{p} \in H_0^2(\Omega)$ such that on a subsequence

$$p^\delta \xrightarrow{w} \bar{p} \text{ in } H_0^2(\Omega).$$

Denote $q^\delta := \frac{1}{\delta}\beta'(\bar{u}^\delta - \psi^\delta)p^\delta$ and let $\varphi \in H_0^2(\Omega)$ be arbitrary. From (11.9) we obtain

$$\begin{aligned} |q^\delta(\varphi)| &= \left|\frac{1}{\delta}\int_\Omega \beta'(\bar{u}^\delta - \psi^\delta)p^\delta\varphi\,dx\right| = \left|-\int_\Omega (\Delta^2 p^\delta)\varphi\,dx + \int_\Omega (\bar{u}^\delta - z)\varphi\,dx\right| \le \\ &= \left|-\int_\Omega \Delta p^\delta \Delta\varphi\,dx + \int_\Omega (\bar{u}^\delta - z)\varphi\,dx\right| \le \left|\int_\Omega \Delta p^\delta \Delta\varphi\,dx\right| + \left|\int_\Omega (\bar{u}^\delta - z)\varphi\,dx\right| \\ &\le \|\Delta p^\delta\|_2 \cdot \|\Delta\varphi\|_2 + \|\bar{u}^\delta - z\|_2 \cdot \|\varphi\|_2 \le C\|\bar{u}^\delta - z\|_2 \cdot \|\varphi\|_{H^2(\Omega)} \end{aligned}$$

whence

$$\|q^\delta\|_{H^{-2}(\Omega)} = \sup |q^\delta(\varphi)| \le C\|z\|_2$$

where supremum is taken over all $\varphi \in H_0^2(\Omega)$ such that $\|\varphi\|_{H^2(\Omega)} = 1$. This implies that there exists $\bar{q} \in H^{-2}(\Omega)$ such that

$$q^\delta \overset{w*}{\rightharpoonup} \bar{q} \text{ in } H^{-2}(\Omega).$$

We know by Theorem 10.2 that $(\bar{u}^\delta, \psi^\delta)$ converges to an optimal pair $(\bar{u}, \bar{\psi})$ as $\delta \to 0^+$. Letting $\delta \to 0^+$ in (11.6) gives the first six lines in (11.11). To show that $\left\langle \bar{q}, \bar{p} \right\rangle_{H^{-2}, H_0^2} \ge 0$ we note the following. By the lower semi-continuity of L^2-norm with respect to weak convergence we obtain from (11.12)

$$\int_\Omega (\Delta\bar{p})^2\,dx \le \int_\Omega (\bar{u} - z)\bar{p}\,dx. \tag{11.14}$$

Using $\bar{p}$ as a test function in the weak formulation of the second equation in (11.11) we have

$$\int_\Omega (\Delta\bar{p})^2\,dx + \left\langle \bar{q}, \bar{p} \right\rangle_{H^{-2}, H_0^2} = \int_\Omega (\bar{u} - z)\bar{p}\,dx. \tag{11.15}$$

Now combining (11.14) and (11.15) gives $\left\langle \bar{q}, \bar{p} \right\rangle_{H^{-2}, H_0^2} \ge 0$. $\square$

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