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Multiplicative Sets of Atoms

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To the Graduate Council:

I am submitting herewith a dissertation written by Ashley Nicole Rand entitled "Multiplicative Sets of Atoms." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Multiplicative Sets of Atoms

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I dedicate this dissertation and the completion of my graduate education to my father the late Timothy Rand, my mother Kimberly Jones, and my sisters Jackie Lee and Stephanie Yates.

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Abstract

It is possible for an element to have both an atom factorization and a factorization that will always contain a reducible element. This leads us to consider the multiplicatively closed set generated by the atoms and units of an integral domain. We start by showing that for a nice subset S of the atoms of R, there exists an integral domain containing R with set of atoms S. A multiplicatively closed set is saturated if the factors of each element in the set are also elements in the set. Considering polynomial and power series subrings, we find necessary and sufficient conditions for the set generated by the atoms and units to be saturated. We then generalize this to integral domains of the form $D + M$.

Contents

Chapter 1

Preliminaries

Factorization properties of integral domains have been studied extensively. The building blocks of factorization in integral domains are atoms and prime elements.

Definition 1.0.1. *Let R be an integral domain.*

1. *A nonzero nonunit $r \in R$ is an atom (irreducible element) in R if whenever $r = ab$ for $a, b \in R$, then a is a unit in R or b is a unit in R.*

2. *A nonzero nonunit $p \in R$ is prime in R if whenever $p \mid ab$ for $a, b \in R$, then $p \mid a$ or $p \mid b$.*

The nicest factorization structure occurs in a unique factorization domain (UFD), where every nonzero, nonunit element factors as a product of atoms and such factorizations are unique up to order and units. But there are many factorization properties that are much more interesting. For example, consider the following factorization properties.

Definition 1.0.2. *Let R be an integral domain.*

1. *R is atomic if each nonzero nonunit in R may be written as a finite product of atoms in R.*

2. *R satisfies the ascending chain condition on principal ideals (ACCP) if every strictly ascending chain of principal ideals of R is finite.*

Factorization properties between UFD and atomic have been studied extensively (see [2], [3], [10]). In this dissertation, we look at the set $\mathcal{A}(R)$ of all finite products of atoms and units in R. When R is atomic, $\mathcal{A}(R)$ is the set of all nonzero elements of R. It follows that for all integral domains between atomic domains and UFDs, we have $\mathcal{A}(R) = R\setminus\{0\}$, which is a relatively uninteresting set. This leads us to consider the set $\mathcal{A}(R)$ for integral domains which are not atomic.

Throughout this dissertation, R will denote an integral domain and $U(R)$ the group of units of R. We denote the integers, nonnegative integers, integers modulo n, rationals, reals, and complex numbers by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}_n, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$, respectively. We define a few properties of subsets of R.

Definition 1.0.3. *Let R be an integral domain.*

1. *A nonempty subset S of R is multiplicatively closed if $ab \in S$ for all $a, b \in S$.*

2. *A multiplicatively closed set S is generated by a nonempty set A if S is the set of all finite products of elements of A, i.e., $S = \{a_1 \cdots a_n \mid a_i \in A\}$.*

3. *A multiplicatively closed set S is saturated in R if whenever $ab \in S$ for $a, b \in R$, then $a, b \in S$.*

4. *A subset $S \subseteq R$ is unit closed if $us \in S$ for all $s \in S$ and $u \in U(R)$.*

Recall that $\mathcal{A}(R)$ is the set of all finite products of atoms and units in R, i.e., $\mathcal{A}(R) := \{ua_1 \cdots a_n \mid u \in U(R), a_i \text{ is an atom in } R\}$. Throughout, we will denote the set of atoms of R by $\mathcal{A}_R$, the set of prime elements of R by $\mathcal{P}_R$, and the set of all finite products of prime elements and units in R by $\mathcal{P}(R)$, i.e., $\mathcal{P}(R) := \{up_1 \cdots p_n \mid u \in U(R), p_1, \dots, p_n \in \mathcal{P}_R\}$. Notice that both $\mathcal{A}(R)$ and $\mathcal{P}(R)$ are multiplicatively closed and include the empty product of atoms and primes, respectfully. Thus $U(R) \subseteq \mathcal{A}(R)$ and $U(R) \subseteq \mathcal{P}(R)$. One of the goals of this dissertation is to answer the question of when is $\mathcal{A}(R)$ saturated in R? We work toward this goal by first recognizing the next well known theorem, which we prove for completeness.

Theorem 1.0.4. *Let R be an integral domain. Then all prime elements of R are atoms.*

Proof. Let $p \in R$ be prime and $x, y \in R$ such that $p = xy$. Then $p \mid xy$; so we may assume $p \mid x$. Therefore, there exists $a \in R$ such that $pa = x$. It follows that $p = xy = pay$, and thus $1 = ay$ by cancellation. Hence $y \in U(R)$; so p is an atom in R. $\square$

From this theorem, it follows that $\mathcal{P}(R) \subseteq \mathcal{A}(R)$. In general, atoms are not necessarily prime. For example, in the ring $R = \mathbb{Z}[X^2, X^3]$, the element X^2 is an atom. We can see that X^2 is not prime since $X^2 \mid X^6 = X^3 \cdot X^3$, but $X^2 \nmid X^3$ in $\mathbb{Z}[X^2, X^3]$. This is an example where $\mathcal{P}(R) \subsetneq \mathcal{A}(R)$.

We start by considering the properties of $\mathcal{P}(R)$. It is well know that a factorization into prime elements is unique, up to order and units. It follows that $\mathcal{P}(R)$ is saturated in R.

Proposition 1.0.5. *([13, Theorem 4]) Let R be an integral domain and $\mathcal{P}(R)$ be the multiplicatively closed set generated by the prime elements and units of R. Then $\mathcal{P}(R)$ is saturated in R.*

Proof. Let $ab \in \mathcal{P}(R)$. If $ab \in U(R)$, then $a, b \in U(R) \subseteq \mathcal{P}(R)$. So we may assume $ab \notin U(R)$. Then there exist primes $p_1, \ldots, p_n$ such that $ab = p_1 \cdots p_n$. We proceed by induction on n. If $n = 1$, then $ab = p_1$. Then we may assume $a \in U(R)$ and $b = a^{-1}p_1$. It follows that $a, b \in \mathcal{P}(R)$. If $n \geq 2$, then $p_1 \mid p_1 \cdots p_n = ab$. Thus we may assume $p_1 \mid a$, i.e., there exists $a_1 \in R$ such that $a = p_1 a_1$. It follows that $p_1 a_1 b = ab = p_1 \cdot p_2 \cdots p_n$, and so $a_1 b = p_2 \cdots p_n$. By the induction hypothesis, $a_1, b \in \mathcal{P}(R)$. Hence $a = p_1 a_1 \in \mathcal{P}(R)$. $\square$

Notice that the prime factorizations of a and b only include primes from the prime factorization of ab. The next corollary follows from this observation.

Corollary 1.0.6. *Let R be an integral domain and S be a nonempty subset of the prime elements of R that is unit closed. Then the set generated by S and $U(R)$ is saturated in R.*

We now have the tools to consider the set $\mathcal{A}(R)$ for an integral domain R which is not atomic, but where all the atoms of R are prime. For example, a greatest common divisor (GCD)-domain. Recall, for $a, b \in R$, we say d is a *greatest common divisor of* a, b, denoted $\gcd(a, b)$, if $d \mid a$ and $d \mid b$, and $e \mid d$ for all $e \in R$ such that $e \mid a$ and $e \mid b$. Then an integral domain R is a *GCD-domain* if any two elements of R have a greatest common divisor.

We will show that all GCD-domains are Schreier domains. Following Cohn [6], an integral domain R is a *Schreier domain* if R is integrally closed and if $x, y, z \in R$ with $x \mid yz$, then $x = x_1x_2$ where $x_1 \mid y$ and $x_2 \mid z$.

Let R be a GCD-domain and $x, y, z \in R$ such that $x \mid yz$. We may assume $x, y, z \neq 0$; so $a = \gcd(x, y) \neq 0$. It follows that $\frac{x}{a} \mid (\frac{y}{a})z$. Then $\frac{x}{a} \mid z$ since $1 = \gcd(\frac{x}{a}, \frac{y}{a})$. Thus $x = a(\frac{x}{a})$, where $a \mid y$ and $\frac{x}{a} \mid z$. Since a GCD-domain is integrally closed [13, Theorem 50], a GCD-domain is a Schreier domain.

From the definition, we see that all atoms in a Schreier domain are prime. For if $a \in R$ is an atom and $a \mid bc$, then there exist $a_1, a_2 \in R$ such that $a = a_1a_2$ with $a_1 \mid b$ and $a_2 \mid c$. But a is an atom, so we may assume a_1 is a unit. It follows that $a \mid c$. Thus a is prime. Now we have that $\mathcal{A}(R) = \mathcal{P}(R)$ for any Schreier domain or GCD-domain R. Hence $\mathcal{A}(R)$ is saturated in R.

An integral domain V is a *valuation domain* if for any $a, b \in V$ either $a \mid b$ or $b \mid a$. Note that a valuation domain V is a GCD-domain; so $\mathcal{A}(V) = \mathcal{P}(V)$. Let V be the valuation domain with value group $\mathbb{Q}$. Then V has no atoms. In this case, it follows that $\mathcal{A}(V) = \mathcal{P}(V) = U(V)$.

Unlike prime factorizations, a factorization into atoms need not be unique. We have already considered $\mathbb{Z}[X^2, X^3]$, where $X^2X^2X^2 = X^6 = X^3X^3$ are two distinct factorizations of X^6 into atoms. For the set generated by the atoms and units to be saturated, we need that whenever a product of two elements can be factored into a product of atoms, then each of the elements can also be factored into a product of atoms. Before giving an example where $\mathcal{A}(R)$ is not saturated in R, we turn to the definition of properties of monoids, so we can give a well-known result on graded domains that we will use several times.

Definition 1.0.7. *Let M be commutative monoid with binary operation $+$ and identity element 0, and let $a, b, c \in M$.*

1. *The monoid M is reduced if whenever $a + b = 0$, then $a = 0 = b$.*

2. *The monoid M is cancellative if whenever $a + b = a + c$, then $b = c$.*

3. *The monoid M is torsion-free if whenever $na = nb$ for a positive integer n, then $a = b$.*

Note that M is a commutative, cancellative, torsion-free monoid if and only if $\langle M \rangle := \{a - b \mid a, b \in M\}$, the group generated by M, is a torsion-free abelian group, i.e, $\langle M \rangle$ is an abelian group such that $na = 0$ implies $a = 0$ for all $a \in \langle M \rangle$ and $n \in \mathbb{Z} \setminus \{0\}$. In this case, it is also common to say M is torsionless. It is well-known that a torsionless monoid may be well-ordered, where the ordering is compatible with the operation ([15, Theorem 22]). The semi-group ring $R[X; M] = \{\sum r_m X^m \mid r_m \in R, m \in M\}$ is an integral domain if and only if R is an integral domain and M is a commutative, cancellative, torsion-free monoid ([11, Theorem 8.1]). Moreover, when $R[X; M]$ is an integral domain, $U(R[X; M]) = \{uX^m \mid u \in U(R), m \text{ invertible in } M\}$ ([11, Theorem 11.1].

We begin with a well-known result on graded domains, and we will include the proof for completeness. Recall that a graded domain is an integral domain $R = \oplus_{\alpha \in \Gamma} R_\alpha$, where Γ is a torsionless grading monoid (i.e., Γ is commutative, cancellative, and torsion-free), each R_α is an additive subgroup of R, and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$.

Lemma 1.0.8. *Let $R = \oplus R_\alpha$ be a graded domain. Then $ab \in R_\alpha$ for $0 \neq a, b \in R$ if and only if a, b are homogeneous and $\deg(a) = \alpha - \deg(b)$.*

Proof. ($\Rightarrow$) Let $ab \in R_\alpha$ for $0 \neq a, b \in R$. Then there exist homogeneous $a_1, \ldots, a_m \in R$ and homogeneous $b_1, \ldots, b_n \in R$ with $\deg(a_1) < \deg(a_2) < \cdots < \deg(a_m)$ and $\deg(b_1) < \deg(b_2) < \cdots < \deg(b_n)$ such that $a = a_1 + \cdots + a_m$ and $b = b_1 + \cdots + b_n$. Multiplying, we get $ab = (a_1 + \cdots + a_m)(b_1 + \cdots + b_n) = a_1b_1 + a_1b_2 + \cdots + a_mb_n$, with $\deg(a_1b_1) = \deg(a_1) + \deg(b_1) < \deg(a_m) + \deg(b_n) = \deg(a_mb_n)$. It follows that ab is homogeneous if and only if $m = 1 = n$. Thus a, b are homogeneous and $\deg(a) + \deg(b) = \deg(ab) = \alpha$, i.e., $\deg(a) = \alpha - \deg(b)$.

($\Leftarrow$) Assume a, b are homogeneous and $\deg(a) = \alpha - \deg(b)$. Then ab is homogeneous with $\deg(ab) = \deg(a) + \deg(b) = \alpha - \deg(b) + \deg(b) = \alpha$. Hence $ab \in R_\alpha$. $\square$

Now we can give an example where $\mathcal{A}(R)$ is not saturated in R.

Example 1.0.9. *Let $S = \{\frac{2n}{3^m} + k \mid n, m, k \in \mathbb{Z}^+\}$ be an additive submonoid of $\mathbb{Q}$, and let $R = \mathbb{Z}_3[X; S] = \{\sum a_i X^{s_i} \mid a_i \in \mathbb{Z}_3, s_i \in S\}$ be the monoid domain. First, we show that X is an atom. Let $a, b \in R$ be such that $X = ab$. Then a, b are monomials and $\deg(a) + \deg(b) = 1$. But $\deg(a), \deg(b) \in S$; thus we may assume $\deg(a) = 1$ and $\deg(b) = 0$. Hence $b \in U(R)$. Then $X^2 = X \cdot X$ is a product of atoms. On the other hand, $X^2 = X^{2/3}X^{4/3}$, and we show that any finite factorization of $X^{2/3}$ is not a product of atoms. Write $X^{2/3} = a_1 \cdots a_n$ for atoms $a_1, \ldots, a_n$ in R. It follows that a_i is a monomial for $1 \leq i \leq n$ since $X^{2/3}$ is a monomial. Then $\sum_{i=1}^n \deg(a_i) = 2/3$. For $1 \leq i \leq n$, $\deg(a_i) \in S$. Thus $\deg(a_i) \geq 0$. Hence $\deg(a_i) \leq 2/3$, and for $1 \leq i \leq n$, there exist n_i, m_i such that $\deg(a_i) = \frac{2n_i}{3^{m_i}}$ with $2n_i < 3^{m_i}$. Then $a_i = X^{\frac{2n_i}{3^{m_i}}} = (X^{\frac{2n_i}{3^{m_i+1}}})^3$ is not an atom. Thus $X^{2/3}$ cannot be factored as a product of atoms.*

Much of this dissertation is devoted to determining when $\mathcal{A}(R)$ is saturated in R for specific classes of integral domains. So we start by considering the saturated property. Chapter 2 is devoted to looking at the role of the units in the saturated definition. After defining two other types of saturated sets, we explore the varying levels of saturation. We conclude the chapter by showing the relationships between the three definitions.

In chapter 3, we consider the different forms that $\mathcal{A}(R)$ can take. We start by showing that any free monoid can be realized as the multiplicative set generated by the primes and units of an integral domain. We know that all prime elements are atoms, so for any free monoid M with infinitely many generators, there exists an integral domain R such that $\mathcal{A}(R)$ is isomorphic to M. But atoms need not be prime, so we conclude chapter 3 by showing the wider variety of sets that can be realized as $\mathcal{A}(R)$.

In chapter 4, we look at the multiplicatively closed set $\mathcal{A}(R)$ for polynomial and power series subrings of the form $R = A + B_1X + B_2X^2 + \cdots$, where $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ are integral domains. By considering different properties on $A, B_1, B_2, \ldots$, we determine necessary and sufficient conditions for $\mathcal{A}(R)$ to be saturated in R.

In chapter 5, we generalize the polynomial and power series subrings of the from $A + XK[X]$ and $A + XK[[X]]$ with the $D + M$ construction. For this construction, we start with an integral domain $T = K + M$, where K is a subfield of T, M is a nonzero maximal ideal of T, and D a subring of K. Then $R = D + M$ is an integral domain with elements of the form $d \in D$, $m \in M$, and $d + m \in R$. We will look at the relationship between $\mathcal{A}(R), \mathcal{A}(D)$, and $\mathcal{A}(T)$.

We conclude this dissertation by considering the set of atoms of an integral domain R and the larger integral domains that share the same set of atoms. It would be nice to be able to find the largest integral domain with a particular set of atoms. We work toward that goal by defining a set that looks at all quotients of atoms.

1.1 Inert Extensions

Let $R \subseteq T$ be an extension of integral domains. In general, we cannot say much about the relationship between $\mathcal{A}(R)$ and $\mathcal{A}(T)$. But in the remainder of this chapter, we look at inert extensions of rings. We first define both inert and strongly inert extensions.

Definition 1.1.1. *An extension of integral domains $R \subseteq T$ is an inert extension if whenever $xy \in R$ for nonzero $x, y \in T$, then $xu, yu^{-1} \in R$ for some $u \in U(T)$. An extension of integral domains $R \subseteq T$ is a strongly inert extension if whenever $xy \in R$ for nonzero $x, y \in T$, then $x, y \in R$.*

It is clear from the definitions that a strongly inert extension is an inert extension. The converse is not true. We show that the extension $\mathbb{Z} \subseteq \mathbb{Q}$ is inert, but not strongly inert. Let $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ be such that $\frac{a}{b}, \frac{c}{d}$ are reduced, i.e., $\gcd(a, b) = \gcd(c, d) = 1$, and $\frac{a}{b}\frac{c}{d} \in \mathbb{Z}$. It follows that $d \mid a$ and $b \mid c$; then $\frac{a}{b}\frac{b}{d} = \frac{a}{d} \in \mathbb{Z}$ and $\frac{c}{d}\frac{d}{b} = \frac{c}{b} \in \mathbb{Z}$. Note that $\frac{b}{d} \in U(\mathbb{Q})$ with $(\frac{b}{d})^{-1} = \frac{d}{b}$. Thus $\mathbb{Z} \subseteq \mathbb{Q}$ is an inert extension. It is not a strongly inert extension since $\frac{1}{2}(2) = 1 \in \mathbb{Z}$, but $\frac{1}{2} \notin \mathbb{Z}$. We now give a couple of technical propositions relating the units and atoms of a strongly inert extension $R \subseteq T$.

Proposition 1.1.2. *Let $R \subseteq T$ be a strongly inert extension of integral domains. Then $U(T) = U(R)$.*

Proof. It is clear that $U(R) \subseteq U(T)$ since $R \subseteq T$. Let $u \in U(T)$. Then there exists $s \in T$ such that $us = 1$. But $us = 1 \in R$ and so $u, s \in R$. Thus $u \in U(R)$. $\square$

Notice that this property does not hold for inert extensions. Recall $\mathbb{Z} \subseteq \mathbb{Q}$ is an inert extension that is not strongly inert. Here $U(\mathbb{Z}) = \mathbb{Z} \setminus \{0\} \subsetneq \mathbb{Q} \setminus \{0\} = U(\mathbb{Q})$. The atoms of a ring are very dependent on its units. We use the previous proposition to show that there is a nice relationship between the atoms of R and the atoms of T for a strongly inert extension $R \subseteq T$.

Proposition 1.1.3. *Let $R \subseteq T$ be a strongly inert extension of integral domains. Then $a \in R$ is an atom in R if and only if a is an atom in T.*

Proof. (⇒) Let $a \in R$ be an atom in R and $s, t \in T$ be such that $a = st$. Then $st = a \in R$; hence $s, t \in R$. But a is an atom in R. So we may assume $s \in U(R)$. Thus $s \in U(T)$, and hence a is an atom in T.

(⇐) Let $a \in R$ be an atom in T and $r, s \in R$ be such that $a = rs$. Then $r, s \in R \subseteq T$, and we may assume $r \in U(T)$. It follows from Proposition 1.1.2 that $r \in U(R)$. Thus a is an atom in R. □

These propositions help to establish a relationship between when $\mathcal{A}(T)$ is saturated in T and when $\mathcal{A}(R)$ is saturated in R.

Theorem 1.1.4. *Let $R \subseteq T$ be a strongly inert extension of integral domains. If $\mathcal{A}(T)$ is saturated in T, then $\mathcal{A}(R)$ is saturated in R.*

Proof. We first show that $\mathcal{A}(R) = \mathcal{A}(T) \cap R$. We have $\mathcal{A}(R) \subseteq \mathcal{A}(T) \cap R$ by Proposition 1.1.3. For the reverse inclusion, let $x \in \mathcal{A}(T) \cap R$. Then there exist atoms $a_1, \ldots, a_n$ in T such that $x = a_1 \cdots a_n$. It follows that $a_1, a_2 \cdots a_n \in R$ since $a_1(a_2 \cdots a_n) = x \in R$ and $R \subseteq T$ is strongly inert. Continuing this argument, we get $a_2, \ldots, a_n \in R$. Then we have $a_1, \ldots, a_n$ are atoms in R by Proposition 1.1.3. Hence $x \in \mathcal{A}(R)$, and so $\mathcal{A}(R) = \mathcal{A}(T) \cap R$. Now we show $\mathcal{A}(R)$ is saturated in R whenever $\mathcal{A}(T)$ is saturated in T. Let $xy \in \mathcal{A}(R)$ for $x, y \in R$. Then $xy \in \mathcal{A}(R) \subseteq \mathcal{A}(T)$. It follows that $x, y \in \mathcal{A}(T)$ since $\mathcal{A}(T)$ is saturated in T. Thus $x, y \in \mathcal{A}(T) \cap R = \mathcal{A}(R)$. □

We give an example to show that the converse is false.

Example 1.1.5. *Let $R = \mathbb{Z}_3$ and $T = \mathbb{Z}_3[X; S]$, where $S = \{\frac{2n}{3^m} + k \mid n, m, k \in \mathbb{Z}^+\}$ as in Example 1.0.9. We have shown that $\mathcal{A}(T)$ is not saturated in T, but R is a field and hence $\mathcal{A}(R) = R \setminus \{0\}$. Thus $\mathcal{A}(R)$ is saturated in R. Also S is reduced, so $R \subseteq T$ is a strongly inert extension by Lemma 1.0.8. The same result holds with $R = \mathbb{Z}$ and $T = \mathbb{Z}[X; S]$.*

We have the following result from [4, Lemma 1.1].

Lemma 1.1.6. *Let $R \subseteq T$ be an inert extension of integral domains. Then an atom of R is either an atom or a unit in T. Moreover, if $U(T) \cap R = U(R)$, then $a \in R$ is an atom in R if and only if it is an atom in T.*

Proof. Let $a \in R \subseteq T$ be an atom in R. Assume $a \notin U(T)$ and $b, c \in T$ such that $a = bc$. Then $bc = a \in R$, so there exists $u \in U(T)$ such that $ub, u^{-1}c \in R$. It follows that $ub \in U(R)$ or $u^{-1}c \in U(R)$ since $a = bc = (ub)(u^{-1}c)$. Hence $b \in U(T)$ or $c \in U(T)$, respectively, since $U(R) \subseteq U(T)$. Thus a is an atom in T.

Suppose that $U(T) \cap R = U(R)$. If a is an atom in R, then we have shown that a is an atom in T. For the converse, let $a \in R$ be an atom in T. Let $b, c \in R$ be such that $a = bc$. Then we may assume $b \in U(T)$ since $b, c \in R \subseteq T$. Hence $b \in U(T) \cap R = U(R)$. □

Either of these cases can occur. Consider the inert extension $\mathbb{Z} \subseteq \mathbb{Q}$. We know 2 is an atom of $\mathbb{Z}$, but a unit of $\mathbb{Q}$. Note that $U(\mathbb{Z}) = \{\pm 1\} \subsetneq \mathbb{Z} \setminus \{0\} = \mathbb{Q} \setminus \{0\} \cap \mathbb{Z} = U(T) \cap R$. For the other case, consider the inert extension $\mathbb{Z} \subseteq \mathbb{Z}[X]$. Here 2 is an atom in both $\mathbb{Z}$ and $\mathbb{Z}[X]$ and $U(T) \cap R = \{\pm 1\} \cap \mathbb{Z} = \{\pm 1\} = U(\mathbb{Z})$. As Lemma 1.1.6 suggests, for an inert extension $R \subseteq T$ of integral domains, we have $\mathcal{A}(T)$ is saturated in T precisely when $\mathcal{A}(R)$ is saturated in R if $U(T) \cap R = U(R)$.

Theorem 1.1.7. *Let $R \subseteq T$ be an inert extension of integral domains. If $U(T) \cap R = U(R)$ and $\mathcal{A}(T)$ is saturated in T, then $\mathcal{A}(R)$ is saturated in R.*

Proof. Let $ab \in \mathcal{A}(R)$ for $a, b \in R$. If $ab \in U(R)$, then $a, b \in U(R) \subseteq \mathcal{A}(R)$. Otherwise, $ab = a_1 \cdots a_n$ for some atoms $a_1, \ldots, a_n \in R$. So $a_1, \ldots, a_n$ are atoms in T by Lemma 1.1.6. Thus $ab \in \mathcal{A}(T)$. Then $a, b \in \mathcal{A}(T)$ by hypothesis. So there exist atoms $t_1, \ldots, t_k \in T$ such that $a = t_1 \cdots t_k$. Then there exists $u_1 \in U(T)$ such that $u_1 t_1, u_1^{-1} t_2 \cdots t_k \in R$ since $R \subseteq T$ is an inert extension and $t_1(t_2 \cdots t_k) = a \in R$. But $u_1 t_1$ is an atom in T, since t_1 is an atom in T. Hence ut_1 is an atom in R by Lemma 1.1.6. Proceeding inductively, we have $a = (u_1 t_1)(u_2 t_2) \cdots (u_k t_k)$ for atoms $u_1 t_1, u_2 t_2, \ldots, u_k t_k$ in R. Thus $a \in \mathcal{A}(R)$, and similarly, $b \in \mathcal{A}(R)$. □

Example 1.1.5 shows that it is possible for $U(T) \cap R = U(R)$ and $\mathcal{A}(R)$ to be saturated in R without $\mathcal{A}(T)$ saturated in T. Notice in this example, $U(R) = R \setminus \{0\} = \mathbb{Z}_3 \setminus \{0\}$ and $U(T) = \mathbb{Z}_3 \setminus \{0\}$. Thus $U(T) = U(T) \cap R = U(R)$. But we have shown that $\mathcal{A}(R)$ is saturated in R, and $\mathcal{A}(T)$ is not saturated in T.

Chapter 2

The Saturated Set

A saturated set behaves nicely because all divisors of an element in the set are also elements of the set. Recall that a saturated set S of an integral domain R is defined to be a multiplicatively closed set such that $xy \in S$ for $x, y \in R$ implies $x, y \in S$. In this chapter, we explore the role of units in the definition of a saturated set. We define two other types of saturated sets by modifying the saturated definition. We then show that while the definitions are comparable, no two of the definitions are equivalent. In the second section, we consider the smallest saturated set containing a multiplicatively closed set S and the equivalent notions for our new definitions. In the final section, we look at the additional conditions needed to make the definitions equivalent.

2.1 Generalized Definitions

Definition 2.1.1. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then*

1. *S is saturated if $x, y \in S$ whenever $xy \in S$ for $x, y \in R$.*

2. *S is II-saturated if $x, y \in S$ whenever $xy \in S$ for $x \in R \setminus U(R)$ and $y \in R$.*

3. *S is III-saturated if $x, y \in S$ whenever $xy \in S$ for $x, y \in R \setminus U(R)$.*

We start with a few examples of multiplicatively closed sets. For an atomic domain R, we have $S = R\backslash(U(R)\cup\{0\})$ is multiplicatively closed. Let (D, M) be a quasilocal domain, i.e., M is the unique maximal ideal of D. Then $T = M\backslash\{0\}$ is multiplicatively closed. Notice that both of these examples are III-saturated, but not saturated or II-saturated.

It is easy to see from the definitions that saturated ⇒ II-saturated ⇒ III-saturated. We give a few more examples showing that neither of the converses hold.

Example 2.1.2. *Let $R = \mathbb{Z}$ and $S = \{\pm 2^n \mid n \geq 1\}$. Then S is multiplicatively closed and $U(R) = \{\pm 1\}$. We know all integer factors of 2^n are of the form $\pm 2^k$ for $0 \leq k \leq n$. Thus S is III-saturated since $\pm 2^n \in S$ for all $n \geq 1$. But $1(2) = 2 \in S$ with $2 \in R \setminus U(R)$, $1 \in R$, and $1 \notin S$; so S is not II-saturated.*

Example 2.1.3. *Let $R = \mathbb{Z}$ and $S = \{1\}$. Note that $U(R) = \{\pm 1\}$. Trivially S is multiplicatively closed and II-saturated since all divisors of 1 are units. But S is not saturated since $-1(-1) = 1 \in S$ with $-1 \in R$ and $-1 \notin S$.*

The saturated, II-saturated, and III-saturated definitions only differ by which elements are allowed to be units; so we consider when $U(R) \subseteq S$. It is known that when S is saturated, $U(R) \subseteq S$. For if $u \in U(R)$ and $x \in S$, then $u(u^{-1}x) = x \in S$. Thus $u \in S$ since S is saturated. This is not the case for II-saturated and III-saturated sets as we can see in the previous examples. We have $U(\mathbb{Z}) = \{\pm 1\} \nsubseteq \{\pm 2^n \mid n \geq 1\}$ in Example 2.1.2 and $U(\mathbb{Z}) = \{\pm 1\} \nsubseteq \{1\}$ in Example 2.1.3. The relationship between the generalized definitions and the unit closed property is also of interest. Recall that a multiplicatively closed set S is unit closed if $ux \in S$ for all $x \in S$ and $u \in U(R)$. In the third section of this chapter, we will show that the unit closed property is the missing link between III-saturated and saturated sets, but for now we just show that if S is saturated, then S is unit closed.

Proposition 2.1.4. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. If S is saturated, then S is unit closed.*

Proof. Let $x \in S \subseteq R$ and $u \in U(R)$. Then $u^{-1}(ux) = x \in S$. It follows that $ux \in S$ since S is saturated. Thus S is unit closed. $\square$

Multiplicatively closed sets that are II-saturated and III-saturated are not necessarily unit closed. Consider Example 2.1.3 with $R = \mathbb{Z}$ and $S = \{1\}$. We have shown that S is II-saturated, and hence III-saturated. But $-1(1) = -1 \notin S$ for $-1 \in U(R)$ and $1 \in S$. Thus S is not unit closed.

2.2 Smallest Saturated Sets

Let R be an integral domain and S a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. It is known that the set $S^I := \bigcap\{T \mid T \subseteq R \text{ is saturated and } S \subseteq T\}$ is the smallest saturated set containing S. In this section, we find that a similar set can be defined for both the II-saturated and III-saturated definitions. We start by showing that an arbitrary intersection of II-saturated (resp., III-saturated) sets is II-saturated (resp., III-saturated). Then there exists a smallest set that contains S and is II-saturated (resp., III-saturated) since S^I contains S and is both II-saturated and III-saturated. In particular, the intersection of all II-saturated (resp., III-saturated) sets containing S is the smallest II-saturated (resp., III-saturated) set containing S. It follows that $S \subseteq S^{III} \subseteq S^{II} \subseteq S^I$, where S^{III} is the smallest III-saturated set containing S and S^{II} is the smallest II-saturated set containing S.

Proposition 2.2.1. *Let R be an integral domain and $\{S_i\}$ be a family of nonempty multiplicatively closed subsets of $R \setminus \{0\}$ with nonempty intersection.*

1. *If every S_i is II-saturated, then $\bigcap S_i$ is II-saturated.*

2. *If every S_i is III-saturated, then $\bigcap S_i$ is III-saturated.*

Proof. First, we show that if every S_i is II-saturated, then $\bigcap S_i$ is II-saturated. Let $x \in R \setminus U(R), y \in R$ be such that $xy \in \bigcap S_i$. Then $xy \in S_i$ for every i. Thus $x, y \in S_i$ for every i since S_i is II-saturated. Hence $x, y \in \bigcap S_i$, and thus $\bigcap S_i$ is II-saturated.

Now we show that if every S_i is III-saturated, then $\bigcap S_i$ is III-saturated. Let $xy \in \bigcap S_i$ for $x, y \in R \setminus U(R)$. Then $xy \in S_i$ for every i. Thus $x, y \in S_i$ for every i since S_i is III-saturated. Hence $x, y \in \bigcap S_i$, and it follows that $\bigcap S_i$ is III-saturated. $\square$

So $S^{II} := \bigcap\{T \mid T \subseteq R$ is II-saturated and $S \subseteq T\}$ and $S^{III} := \bigcap\{T \mid T \subseteq R$ is III-saturated and $S \subseteq T\}$ are the smallest II-saturated and III-saturated sets containing S. As with the smallest saturated set containing S, we can alternately define S^{II} and S^{III} in terms of elements. It is known that $S^I = \{x \in R \mid xy \in S \text{ for some } y \in R\}$. The next two propositions show that there are analogous "element definitions" for both the smallest II-saturated and III-saturated sets.

Proposition 2.2.2. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then $S' := \{x \in R \mid xy \in S \text{ for some } y \in R \setminus U(R)\} \cup S$ is the smallest II-saturated multiplicatively closed set containing S, i.e., $S' = S^{II}$.*

Proof. Let $T = \{x \in R \mid xy \in S \text{ for some } y \in R \setminus U(R)\}$. Then $S' = T \cup S$. We start by showing that S' is multiplicatively closed. Let $a, b \in S'$.

Case 1: If $a, b \in S$, then $ab \in S \subseteq S'$ since S is multiplicatively closed.

Case 2: If $a \in T, b \in S$, then there exists $c \in R \setminus U(R)$ such that $ac \in S$. Thus $(ab)c = (ac)b \in S$ since S is multiplicatively closed. But $c \in R \setminus U(R)$ and $ab \in R$; so $ab \in T \subseteq S'$ by the definition of T.

Case 3: If $a, b \in T$, then there exist $c, d \in R \setminus U(R)$ such that $ac, bd \in S$. Thus $(ab)(cd) = (ac)(bd) \in S$ with $ab \in R$ and $cd \in R \setminus U(R)$ since $c, d \in R \setminus U(R)$. Hence $ab \in T \subseteq S'$.

Now we show that $S \setminus T \subseteq U(R)$. Let $x \in S \setminus T$. Then $xy \notin S$ for every $y \in R \setminus U(R)$ since $x \notin T$. But $x^2 \in S$ since S is multiplicatively closed. Thus $x \in U(R)$.

With this fact, we show that S' is II-saturated. Let $x \in R, y \in R \setminus U(R)$ be such that $xy \in S'$. If $xy \in S \setminus T$, then we have shown that $xy \in U(R)$. But this is a contradiction since $y \notin U(R)$. Hence $xy \in T$. So there exists $z \in R \setminus U(R)$ such that $(xy)z \in S$. Then $xz \notin U(R)$ and $y(xz) = (xy)z \in S$; so $y \in S'$. Similarly, $x(yz) = (xy)z \in S$ and $yz \notin U(R)$. Thus $x \in S'$. Hence S' is II-saturated.

Finally, we show S' is the smallest II-saturated set containing S. Note that S' contains S since $S \subseteq T \cup S = S'$. Let W be a II-saturated multiplicatively closed set containing S. We show that $S' \subseteq W$. Let $x \in S'$. If $x \in S$, then $x \in W$ since $S \subseteq W$. If $x \in T$, then there exists $y \in R \setminus U(R)$ such that $xy \in S \subseteq W$. Thus $x, y \in W$ since W is II-saturated. So $S' \subseteq W$, and it follows that $S' = S^{II}$. □

Recall Example 2.1.2, where $R = \mathbb{Z}$ and $S = \{\pm 2^n \mid n \geq 1\}$. Then $S^{II} = \{x \in R \mid xy \in S \text{ for some } y \in R \setminus U(R)\} \cup S = \{x \in \mathbb{Z} \mid xy = \pm 2^n \text{ for some } y \in \mathbb{Z} \setminus \{\pm 1\}, n \geq 1\} \cup \{\pm 2^n \mid n \geq 1\}$. It follows that $S^{II} = \{\pm 2^n \mid n \geq 0\}$. We now turn to the III-saturated definition.

Proposition 2.2.3. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then $S' := \{x \in R \setminus U(R) \mid xy \in S \text{ for some } y \in R \setminus U(R)\} \cup S$ is the smallest III-saturated multiplicatively closed set containing S, i.e., $S' = S^{III}$.*

Proof. Let $T = \{x \in R \setminus U(R) \mid xy \in S \text{ for some } y \in R \setminus U(R)\}$. Then $S' = T \cup S$. First, we show that S' is multiplicatively closed. Let $a, b \in S'$.

Case 1: If $a, b \in S$, then $ab \in S \subseteq S'$ since S is multiplicatively closed.

Case 2: If $a \in T, b \in S$, then there exists $c \in R \setminus U(R)$ such that $ac \in S$. Thus $(ab)c = (ac)b \in S$ since S is multiplicatively closed. We know $a \in R \setminus U(R)$ since $a \in T$; so $ab \in R \setminus U(R)$. Then $ab \in T \subseteq S'$ since $c \in R \setminus U(R)$.

Case 3: If $a, b \in T$, then $a, b \in R \setminus U(R)$ and there exist $c, d \in R \setminus U(R)$ such that $ac, bd \in S$. So $(ab)(cd) = (ac)(bd) \in S$ with $ab, cd \in R \setminus U(R)$ since $a, b, c, d \in R \setminus U(R)$. Thus $ab \in T \subseteq S'$.

Next, we show that $S \setminus T \subseteq U(R)$, and it follows that $S \setminus T = S \cap U(R)$ since $T \cap U(R) = \emptyset$. Let $x \in S \setminus T$. Then there does not exist $y \in R \setminus U(R)$ such that $xy \in S$ since $x \notin T$. But $x^2 \in S$ since S is multiplicatively closed. Thus $x \in U(R)$.

Now we show that S' is III-saturated. Let $x, y \in R \setminus U(R)$ be such that $xy \in S'$. Then $xy \notin U(R)$, and so $xy \in T$ since $S' \setminus T = S \setminus T \subseteq U(R)$. So there exists $z \in R \setminus U(R)$ such that $(xy)z \in S$. Hence $y \in S'$ since $xz \notin U(R)$ and $y(xz) = xyz \in S$. Similarly, $x(yz) \in S$ with $yz \notin U(R)$ implies $x \in S'$. Thus S' is III-saturated

Finally, we show S' is the smallest III-saturated set containing S. We know $S \subseteq T \cup S = S'$; so let W be a III-saturated multiplicatively closed set containing S and $x \in S'$. Then $x \in S \setminus T \subseteq W$, or $x \in T$ and there exists $y \in R \setminus U(R)$ such that $xy \in S \subseteq W$. It follows that $x, y \in W$ since $x \in T \subseteq R \setminus U(R)$ and W is III-saturated. Thus $S' \subseteq W$; so $S' = S^{III}$. $\square$

We now give an example where $S \subsetneq S^{III}$. Let $R = \mathbb{Z}$ and $S = \{4^n \mid n \geq 0\}$. Then $S^{III} = \{x \in R \setminus U(R) \mid xy \in S \text{ for some } y \in R \setminus U(R)\} \cup S = \{x \in \mathbb{Z} \setminus \{\pm 1\} \mid xy = 4^n$ for some $y \in \mathbb{Z} \setminus \{\pm 1\}, n \geq 0\} \cup \{4^n \mid n \geq 0\}$. It follows that $S^{III} = \{\pm 2^n \mid n \geq 1\} \cup \{1\}$.

From these “element definitions”, it is clear that $S \subseteq S^{III} \subseteq S^{II} \subseteq S^{I}$ for any multiplicatively closed set S. Revisiting the earlier examples, we see that these containments cannot be reversed.

Example 2.2.4. *Let $R = \mathbb{Z}$ and $S = \{\pm 2^n \mid n \geq 1\}$. We have shown that S is III-saturated, i.e., $S^{III} = S$. We have also shown that S is not II-saturated; thus $S \subsetneq S^{II}$. In fact, $S^{II} = S \cup \{\pm 1\}$. Hence $S^{III} \subsetneq S^{II}$.*

Example 2.2.5. *Let $R = \mathbb{Z}$ and $S = \{1\}$. We have shown that S is II-saturated, and $S^{I} = \{\pm 1\}$. Thus $S^{II} = S \subsetneq S^{I}$.*

Notice that in Example 2.2.4, we have $S = S^{III} \subsetneq S^{II} = S^{I}$ and in Example 2.2.5, we have $S = S^{III} = S^{II} \subsetneq S^{I}$. In the next section, we will show that it is not possible to have $S \subsetneq S^{III} \subsetneq S^{II} \subsetneq S^{I}$ by showing that $S^{III} \cup U(R) = S^{II} \cup U(R) = S^{I}$, $S^{III} \cap U(R) = S \cap U(R)$, and if $S \setminus U(R) \neq \emptyset$, then $U(R) \subseteq S^{II}$.

2.3 Saturated Equivalences

We conclude this chapter by considering the additional hypotheses needed to have the reverse implications. Knowing the equivalences will assist us in the remainder of this dissertation, where we will examine when the multiplicatively closed set generated by the atoms and units of a ring is saturated.

Proposition 2.3.1. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then S is saturated if and only if S is III-saturated and $U(R) \subseteq S$.*

Proof. The set S being saturated implies S is III-saturated and $U(R) \subseteq S$. So it suffices to show that if S is III-saturated and $U(R) \subseteq S$, then S is saturated. Let $x, y \in R$ be such that $xy \in S$. If $x, y \in U(R)$, then $x, y \in S$ since $U(R) \subseteq S$. If $x, y \in R \setminus U(R)$, then $x, y \in S$ since S is III-saturated. So we may assume $x \in R \setminus U(R)$ and $y \in U(R)$. Then $xy^2 \in R \setminus U(R)$ and $x(xy^2) = (xy)^2 \in S$ since S is multiplicatively closed. It follows that $x \in S$ since S is III-saturated. Also, $y \in U(R) \subseteq S$. Hence S is saturated. $\square$

Recall that saturated $\Rightarrow$ II-saturated $\Rightarrow$ III-saturated. So, it follows from Proposition 2.3.1 that S is saturated if and only if S is II-saturated and $U(R) \subseteq S$. We also know that $U(R) \subseteq S^I$ since S^I is saturated. But $U(R)$ need not be contained in either S^{II} or S^{III}. In fact, from the "element definitions" we have shown that $U(R) \cap S = U(R) \cap S^{III}$. Note that it is possible for $U(R) \cap S \subsetneq U(R) \cap S^{II}$. Recall Example 2.1.2, where $R = \mathbb{Z}$ and $S = \{\pm 2^n \mid n \geq 1\}$. We have shown that $S^{II} = \{\pm 2^n \mid n \geq 0\}$. Thus $U(R) \cap S = \emptyset \subsetneq \{\pm 1\} = U(R) \cap S^{II}$. The next proposition shows that $U(R) \subseteq S^{II}$ whenever $S \nsubseteq U(R)$.

Proposition 2.3.2. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. If $S \setminus U(R) \neq \emptyset$, then $U(R) \subseteq S^{II}$.*

Proof. Let $u \in U(R)$ and $x \in S \setminus U(R)$. Then $u^{-1}x \notin U(R)$ and $u(u^{-1}x) = x \in S$. Thus $u \in S^{II}$, and so $U(R) \subseteq S^{II}$. $\square$

We have shown that $U(R) \subseteq S^I$ for all nonempty multiplicatively closed sets S. In Example 2.1.3, we have seen this is not the case for S^{II} when $S = \{1\} \subsetneq U(\mathbb{Z})$. In fact, all multiplicatively closed subsets of $U(R)$ are II-saturated since $\{x \in R \mid xy \in S$ for some $y \in R \setminus U(R)\} = \emptyset$ when $S \subseteq U(R)$. Thus for $S \subsetneq U(R)$, we have $S^{II} \subsetneq S^I$. We show that if $S \nsubseteq U(R)$, then $S^{II} = S^I$.

Proposition 2.3.3. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. If $S \setminus U(R) \neq \emptyset$, then $S^{II} = S^I$.*

Proof. It suffices to show $S^I \subseteq S^{II}$ since S^{II} is always contained in S^I. First, $U(R) \subseteq S^{II}$ by Proposition 2.3.2. We show S^{II} is saturated. Let $x, y \in R$ be such that $xy \in S^{II}$. If $x, y \in U(R)$, then $x, y \in S^{II}$ since $U(R) \subseteq S^{II}$. So we may assume, without loss of generality, that $x \in R \backslash U(R)$. Then $x, y \in S^{II}$ since S^{II} is II-saturated. Thus S^{II} is saturated. □

Unlike S^{II}, we know that S^{III} contains only the units that are in S. Thus $U(R) \subseteq S^{III}$ if and only if $U(R) \subseteq S$. Looking back to Proposition 2.3.1, we see that more is true when $U(R) \subseteq S$ and S is III-saturated. In fact, in the next theorem we prove that $S^I = S^{III}$ if and only if $U(R) \subseteq S$.

Proposition 2.3.4. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then $S^I = S^{III}$ if and only if $U(R) \subseteq S$.*

Proof. ($\Rightarrow$) Let $u \in U(R)$. Then $u \in S^I = S^{III}$. It follows that $u \in S$ since $S^{III} = \{x \in R \setminus U(R) \mid xy \in S$ for some $y \in R \setminus U(R)\} \cup S$ by Proposition 2.2.3. Thus $U(R) \subseteq S$.

($\Leftarrow$) We have shown that $S^{III} \subseteq S^I$ for all multiplicatively closed $\emptyset \neq S \subseteq R \backslash \{0\}$. Let $x \in S^I$. Then there exists $y \in R$ such that $xy \in S$. If $x \in U(R)$, then $x \in S \subseteq S^{III}$ by hypothesis. Suppose $x \notin U(R)$. Then $xy^2 \notin U(R)$ and $x(xy^2) = (xy)^2 \in S$ since S is multiplicatively closed. Thus $x \in S^{III}$ and $S^I = S^{III}$. □

The above theorem shows that $S^{III} = S^{II} = S^I$ when $U(R) \subseteq S$. We now consider the relationship between these three sets when $\emptyset \neq S \subseteq U(R)$.

Proposition 2.3.5. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R\setminus\{0\}$. Then $S^{III} = S^{II} = S$. Moreover, $S^{III} = S^{II} = S = S^I$ if and only if $S = U(R)$.*

Proof. Recall that $S^{II} = \{x \in R \mid xy \in S$ for some $y \in R \setminus U(R)\} \cup S$. Let $x \in \{x \in R \mid xy \in S$ for some $y \in R \setminus U(R)\}$. Then there exists $y \in R \setminus U(R)$ such that $xy \in S \subseteq U(R)$. This is a contradiction since $y \in R \setminus U(R) \Rightarrow xy \in R \setminus U(R)$. Hence $S^{II} = S$. Then $S^{III} = S$ since $S \subseteq S^{III} \subseteq S^{II} = S$.

For the "moreover" statement, if S is saturated, i.e., $S = S^I$, then $U(R) \subseteq S$. Thus $S = U(R)$ since $S \subseteq U(R)$ by hypothesis. The converse is simply that the set of units is saturated; this is known. □

Note that it is possible for $S^{III} = S^{II} \subsetneq U(R)$. Consider Example 2.1.3, where $R = \mathbb{Z}$ and $S = \{1\} \subsetneq \{\pm 1\} = U(R)$. We have shown that S is II-saturated. Thus $S^{III} = S^{II} = \{1\}$. It is clear that $S^I = \{\pm 1\} = U(R)$ in this example. Notice $S^{III} \cup U(R) = S^{II} \cup U(R) = S^I$. This relationship holds for all multiplicatively closed sets.

Theorem 2.3.6. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then $S^{III} \cup U(R) = S^I$.*

Proof. It suffices to show that $S^I \subseteq S^{III} \cup U(R)$ since the other containment is clear. Let $x \in S^I \setminus U(R)$. Then there exists $y \in R$ such that $xy \in S$. But S is multiplicatively closed, thus $x(xy^2) = (xy)^2 \in S$ with $xy^2 \in R \setminus U(R)$. Hence $x \in S^{III}$ and $S^{III} \cup U(R) = S^I$. □

We have shown that $S^{III} \subseteq S^{II}$. So the next corollaries follow directly from the theorem.

Corollary 2.3.7. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then $S^{II} \cup U(R) = S^I$.*

Corollary 2.3.8. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then $S^I = S^{II} = S^{III}$ if and only if $U(R) \subseteq S$.*

We now show that it is not possible for $S \subsetneq S^{III} \subsetneq S^{II} \subsetneq S^{I}$.

Theorem 2.3.9. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. Then either $S = S^{III} = S^{II}$ or $S^{II} = S^{I}$.*

Proof. If $S \subseteq U(R)$, then $S = S^{III} = S^{II}$ by Proposition 2.3.5. If $S \not\subseteq U(R)$, then $S \setminus U(R) \neq \emptyset$, and it follows that $S^{II} = S^{I}$ by Proposition 2.3.3 and Corollary 2.3.7. $\square$

We conclude this chapter with two results that will allow us to check only the nonunits to determine if a set is saturated. Recall that a multiplicatively closed set S is unit closed if $ux \in S$ for all $x \in S$ and $u \in U(R)$.

Corollary 2.3.10. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. If S is III-saturated, $S \cap U(R) \neq \emptyset$, and S is unit closed, then S is saturated.*

Proof. Let $u \in S \cap U(R)$. Then $1 = u^{-1}u \in S$ since S is unit closed. Thus $v = v(1) \in S$ for all $v \in U(R)$, i.e., $U(R) \subseteq S$. Hence $S^{III} = S^{III} \cup U(R) = S^{I}$ by Theorem 2.3.6. $\square$

Corollary 2.3.11. *Let R be an integral domain and S be a nonempty, multiplicatively closed subset of $R \setminus \{0\}$. If $S \cap U(R) \neq \emptyset$ and S is unit closed, then $S^{III} = S^{I}$.*

Proof. Note that $\emptyset \neq S \cap U(R) \subseteq S^{III} \cap U(R)$. We show that S^{III} is unit closed; the result then follows from Corollary 2.3.10. Let $u \in U(R)$ and $x \in S^{III}$. If $x \in S$, then $ux \in S \subseteq S^{III}$ since S is unit closed. If $x \in S^{III} \setminus S \subseteq R \setminus U(R)$, then there exists $y \in R \setminus U(R)$ such that $xy \in S$. Thus $uxy \in S$ by hypothesis. It follows that $ux \in S^{III}$ since $(ux)y \in S$ and $ux, y \in R \setminus U(R)$. Hence S^{III} is unit closed. $\square$

Chapter 3

Construction

Coykendall and Mammenga [8] take a "nice" monoid M and construct an integral domain T with atomic factorization structure isomorphic to M. By atomic factorization structure isomorphic to M, they mean $(\mathcal{A}(T)/\sim) \cong M$, where $\sim$ is the associates relation and $\mathcal{A}(T)$ is the multiplicatively closed set generated by the atoms and units of T. The integral domain T is constructed by first embedding M in an integral domain R such that M is isomorphic to a subset S of $\mathcal{A}_R$, the set of atoms of R. Then, using Roitman's construction (see [16], [17]), they adjoin X and r/X for each atom r in R that is not in S. This makes the atom r reducible since $r = X(r/X)$. In this chapter, we generalize their result. Starting with an integral domain R and a "nice" subset S of $\mathcal{A}_R$, we construct an integral domain T containing R with $\mathcal{A}_T$ isomorphic to S. Note that Coykendall and Zafrullah use this construction in [9] to construct an integral domain with a single atom, up to associates, which is not prime.

We will show that Coykendall and Mammenga's result is a special case of our theorem by constructing an integral domain T with $\mathcal{A}(T) \cong M$ for any "nice" monoid M. We start by considering the set $\mathcal{P}_R$ of prime elements of an integral domain R. We have shown that the multiplicatively closed set $\mathcal{P}(R)$ generated by the primes and units of an integral domain R is saturated in Proposition 1.0.5. Here, we show that

$\mathcal{P}(R)$ can take on the structure of any free monoid with infinitely many generators. Recall that M is a free monoid with generating set T if $M \cong \oplus_{x\in T}\mathbb{Z}^+$. We also reference a few results from [4] concerning splitting multiplicative sets. We define a (saturated) multiplicatively closed subset S of R to be a *splitting multiplicative set* if for each $x \in R$, $x = as$ for some $a \in R$ and $s \in S$ such that $aR \cap tR = atR$ for all $t \in S$. We will use [4, Corollary 1.4], which says that if R is an integral domain with splitting multiplicative set S and $0 \neq x \in R$ such that $x = as$ with $a \in R$, $s \in S$ and $aR \cap tR = atR$ for all $t \in S$, then x is prime in R_S if and only if a is prime in R. In the second section of this chapter, we consider the multiplicatively closed set $\mathcal{A}(R)$ generated by the atoms and units of an integral domain R. We show that for any set S that can be realized as a "nice" subset of $\mathcal{A}_R$ there exists an integral domain T containing R such that $S = \mathcal{A}_T$. We finish this chapter by constructing, for a fixed integral domain R, a function that organizes into a partially ordered set all of the integral domains constructed using our main theorem. We will show that the ordering is inversely related to the ordering of the subsets of $\mathcal{A}_R$.

3.1 Set of Prime Elements

We start by showing that any UFD with finitely many primes, up to associates, is a PID. This result is a fairly well-known special case of the fact that a Krull domain with only a finite number of height-one primes is a PID.

Proposition 3.1.1. *A UFD with finitely many primes, up to associates, is a PID.*

Proof. Let R be a UFD with distinct primes $p_1, p_2, \ldots, p_n$ up to associates. Let M be a maximal ideal of R, and let $x \in M$. Then x is not a unit and so can be written as a product of primes. Hence $x \in (p_i)$ for some $i = 1, 2, \ldots, n$. It follows that $M \subseteq (p_1) \cup \cdots \cup (p_n)$. Then the prime avoidance lemma says that for some $i = 1, 2, \ldots, n$, $M \subseteq (p_i)$. This implies $M = (p_i)$ since M is maximal. Thus $\{(p_1), \ldots, (p_n)\}$ is the complete set of maximal ideals of R. It follows that $R_{(p_i)}$ is a discrete valuation ring for $i = 1, \ldots, n$ and $R = \bigcap_i R_{(p_i)}$. Then R is a Bézout domain by [13, Theorem 107], and R is atomic since UFD implies atomic. Hence R is a PID [6, Proposition 1.2]. $\square$

Now we show that for any finite number n, there exists a PID R with exactly n nonassociate primes.

Proposition 3.1.2. *Let $n \in \mathbb{Z}^+$ and γ be an infinite cardinal. Then there exists a PID R with $|\mathcal{P}_R/\sim| = n$ and $|R| = |U(R)| = \gamma$.*

Proof. Let K be a field such that $|K| = \gamma$. Then $K[X]$ is a PID with an infinite number of distinct, nonassociate prime elements by Euclid's proof. Let $p_1, \ldots, p_n$ be distinct, nonassociate prime elements of $K[X]$. The $(p_1), \ldots, (p_n)$ are distinct maximal ideals of $K[X]$. Let $S = K[X] \setminus ((p_1) \cup \cdots \cup (p_n))$. Then $R = K[X]_S$ is a PID with distinct maximal ideals $(p_1)_S, \ldots, (p_n)_S$. So the prime elements of R are $p_1, \ldots, p_n$, up to associates.

We now show that $|U(R)| = \gamma$. We have shown that $(p_1)_S, \ldots, (p_n)_S$ are all of the maximal ideals of R. Hence the Jacobson radical is $J(R) = (p_1)_S \cap \cdots \cap (p_n)_S = (p_1p_2\cdots p_n)_S$. Consider the map $\varphi : R \to U(R)$ defined by $r \mapsto 1 - rp_1p_2\cdots p_n$. Then φ is one-to-one since $1 - sp_1p_2\cdots p_n = 1 - rp_1p_2\cdots p_n$ if and only if $s = r$. It follows that $|R| \leq |U(R)|$; thus $\gamma = |R| = |U(R)|$ since $U(R) \subseteq R$. $\square$

We also have that the set $\mathcal{P}(R)$ with infinitely many generators can have the structure of any free monoid.

Proposition 3.1.3. *Let M be a free monoid with infinite generating set T. Then there exists a UFD R with $\mathcal{P}(R) \cong M$.*

Proof. It suffices to show that $|\mathcal{P}_R| = |T|$ for a UFD such that $U(R) = \{1\}$ since $\mathcal{P}_R \cup U(R)$ generates $\mathcal{P}(R)$ and T generates M. We look at the various different cases for the cardinality of T. Note that R must be infinite since a finite UFD is a field and thus has no prime elements.

Case 1: Suppose T is countable. We show $R = \mathbb{Z}_2[X]$, where X is an indeterminate, is the desired UFD. First, X is prime in R, and R has countably many primes by Euclid's proof. These primes are not associates of one another since $U(R) = \{1\}$. Conversely, R is countable since $|\mathbb{Z}_2[X]| = \max\{|\mathbb{Z}_2| = 2, |\{X\}| = 1, \aleph_0\} = \aleph_0$. Thus there are at most countably many prime elements. Then $\mathcal{P}_R$ is countably infinite.

Case 2: Suppose T is uncountable. Let $R = \mathbb{Z}_2[\{X_\alpha\}_{\alpha \in T}]$, where $\{X_\alpha\}_{\alpha \in T}$ is a set of independent indeterminates. Then each X_α is prime; thus $\{X_\alpha\}_{\alpha \in T} \subseteq \mathcal{P}_R$. Note that $U(R) = \{1\}$. Thus $|T| \leq |\mathcal{P}_R|$ and $|R| = |T|$; so $|\mathcal{P}_R| = |T|$. □

3.2 Set of Atoms

We now consider the structure of the multiplicatively closed set $\mathcal{A}(R)$ generated by the atoms and units of R. Coykendall and Mammenga prove in [8] that for any reduced, cancellative, torsion-free monoid M, there exists an integral domain with atomic factorization structure isomorphic to M. Recall the following definitions.

Definition 3.2.1. *Let M be a commutative monoid with binary operation $+$ and identity element 0, and let $a, b, c \in M$.*

1. *The monoid M is reduced if whenever $a + b = 0$, then $a = 0 = b$.*
2. *The monoid M is cancellative if whenever $a + b = a + c$, then $b = c$.*
3. *The monoid M is torsion-free if whenever $na = nb$ for a positive integer n, then $a = b$.*

To generalize the result in [8], we start with an integral domain R and a subset S of $\mathcal{A}_R$ that is unit closed, meaning that all associates of elements of S are also elements of S. Then we construct an integral domain T containing R such that $\mathcal{A}_T = S$. At the end of this section, we show that the result in [8] is a special case of our theorem.

We start by generalizing two results from [1]. The first result [1, Proposition 1] states that $U(R[X, r/X]) = U(R)$ if and only if $r \notin U(R)$. The second result [1, Lemma 7] states that $s \in R$ is an atom in $R[X, r/X]$ if and only if s is an atom in R and not an associate of r. We want to consider the ring $R[\{Y_i, r_i/Y_i \mid r_i \in S\}]$, where $S \subseteq R \setminus \{0\}$ has arbitrarily many elements and Y_i is an indeterminate for each $r_i \in S$. The generalizations will help us understand the form of both the atoms and units in $R[\{Y_i, r_i/Y_i \mid r_i \in S\}]$. Note that $R[\{Y_i, r_i/Y_i \mid r_i \in S\}]$ can be graded by $\oplus_{r_i \in S}\mathbb{Z}$, with $\deg(r \prod Y_j^{m_j} \prod (r_i/Y_i)^{n_i}) = (m_j - n_j)$ for $r \neq 0$. Almost all entries in this sequence are zero since $r \prod Y_j^{m_j} \prod (r_i/Y_i)^{n_i}$ has only finitely many factors. We first generalize [1, Proposition 1] to show we have control on the units of $R[\{Y_i, r_i/Y_i \mid r_i \in S\}]$.

Lemma 3.2.2. *Let R be an integral domain and S be a nonempty subset of $R \setminus \{0\}$. Then $U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]) = U(R)$ if and only if $S \cap U(R) = \emptyset$.*

Proof. We prove the finite case by induction. The result holds when $|S| = 1$ by [1, Proposition 1]. For $S = \{\alpha_1, \ldots, \alpha_n\}$, we have $R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}] = R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S \setminus \{\alpha_n\}\}][Y_n, \alpha_n/Y_n]$. Then $U(R) = U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S \setminus \{\alpha_n\}\}])$ if and only if $(S \setminus \{\alpha_n\}) \cap U(R) = \emptyset$ by the induction hypothesis. Also, $U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]) = U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S \setminus \{\alpha_n\}\}])$ if and only if $\alpha_n \notin U(R)$. Hence $U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]) = U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S \setminus \{\alpha_n\}\}]) = U(R)$ if and only if $S \cap U(R) = \emptyset$.

For the infinite case, let $r \in U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}])$. Then $r \in U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in T\}])$ for some finite $T \subseteq S$. By the finite case, we have $U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in T\}]) = U(R)$ since $T \cap U(R) \subseteq S \cap U(R) = \emptyset$. Thus $r \in U(R)$. For the reverse implication, assume to the contrary that there exists $\alpha \in S \cap U(R)$. Let $\beta \in R$ be such that $\alpha\beta = 1$. Then $\beta Y_\alpha, \frac{\alpha}{Y_\alpha} \in R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}] \setminus R$ with $(\beta Y_\alpha)(\frac{\alpha}{Y_\alpha}) = \beta\alpha = 1$. Hence $\beta Y_\alpha, \frac{\alpha}{Y_\alpha} \in U(R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]) \setminus U(R)$. $\square$

In the same way, we generalize [1, Lemma 7]. Then we know the form of the atoms when we adjoin arbitrarily many elements of the form $Y_i, \alpha_i/Y_i$ to R.

Lemma 3.2.3. *Let R be an integral domain and S be a nonempty subset of $R \setminus \{0\}$. Then $r \in R$ is an atom in $T = R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]$ if and only if r is an atom in R and r is not an associate in R of any $\alpha_i \in S$.*

Proof. ($\Rightarrow$) Let $T = R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]$. We start by showing that $U(T) \cap R = U(R)$. Let $a \in U(T) \cap R$. Then there exists $b \in T$ such that $ab = 1 \in R$. Thus a, b are monomials and $\deg(b) = -\deg(a) = 0$ by Lemma 1.0.8. It follows that $b \in R$, and hence $a \in U(R)$. Let $r \in R$ be an atom in T. Write $r = xy$ for $x, y \in R$. We may assume $x \in U(T)$ since r is an atom in T. It follows that $x \in U(T) \cap R = U(R)$. Also, if r is an associate of some $\alpha_i \in S$, then there exists $u \in U(R)$ such that $r = u\alpha_i = (uY_i)(\alpha_i/Y_i)$ is a nontrival factorization in T. This is a contradiction; so r is not an associate of any $\alpha_i \in S$.

($\Leftarrow$) Let $r \in R$ be an atom in R. Write $r = ab$ for $a, b \in T$. First, notice that both a and b must be monomials and $\deg(a) = -\deg(b)$ by Lemma 1.0.8 since $ab = r \in R$. Then there exist $0 \neq r_1, r_2 \in R$, $\alpha_i \in S$ and indeterminates Y_i such that $a = r_1 Y_{j_1}^{k_1} \cdots Y_{j_n}^{k_n} (\frac{\alpha_{m_1}}{Y_{m_1}})^{a_1} \cdots (\frac{\alpha_{m_l}}{Y_{m_l}})^{a_l}$ and $b = r_2 (\frac{\alpha_{j_1}}{Y_{j_1}})^{k_1} \cdots (\frac{\alpha_{j_n}}{Y_{j_n}})^{k_n} Y_{m_1}^{a_1} \cdots Y_{m_l}^{a_l}$. Note the choice of exponents follows from $\deg(a) = -\deg(b)$. If $j_i = m_s$, then $Y_{j_i}^{k_i} (\frac{\alpha_{m_s}}{Y_{m_s}})^{a_s} = \alpha_{m_k}^{k_i} (\frac{\alpha_{m_k}}{Y_{m_k}})^{a_s - k_i}$ when $a_s \geq k_i$, or $Y_{j_i}^{k_i} (\frac{\alpha_{m_s}}{Y_{m_s}})^{a_s} = \alpha_{m_k}^{a_s} Y_{j_i}^{k_i - a_s}$ when $a_s < k_i$. So we may assume $j_i \neq m_k$ for $1 \leq i \leq n$ and $1 \leq k \leq l$. Thus $r = ab = (r_1 Y_{j_1}^{k_1} \cdots Y_{j_n}^{k_n} (\frac{\alpha_{m_1}}{Y_{m_1}})^{a_1} \cdots (\frac{\alpha_{m_l}}{Y_{m_l}})^{a_l})(r_2 (\frac{\alpha_{j_1}}{Y_{j_1}})^{k_1} \cdots (\frac{\alpha_{j_n}}{Y_{j_n}})^{k_n} Y_{m_1}^{a_1} \cdots Y_{m_l}^{a_l}) = r_1 r_2 \alpha_{j_1}^{k_1} \cdots \alpha_{j_n}^{k_n} \alpha_{m_1}^{a_1} \cdots \alpha_{m_l}^{a_l}$. But $r_1, r_2, \alpha_{j_i}, \alpha_{m_b} \in R$ for $i = 1, \ldots, n$ and $b = 1, \ldots, l$. Regrouping, $r = (r_1 \alpha_{j_1}^{k_1} \cdots \alpha_{j_n}^{k_n})(r_2 \alpha_{m_1}^{a_1} \cdots \alpha_{m_l}^{a_l}) \in R$ with $r_1 \alpha_{j_1}^{k_1} \cdots \alpha_{j_n}^{k_n}$, $r_2 \alpha_{m_1}^{a_1} \cdots \alpha_{m_l}^{a_l - 1} \in R$. Then we may assume $r_1 \alpha_{j_1}^{k_1} \cdots \alpha_{j_n}^{k_n} \in U(R)$ since r is an atom in R. Let $\alpha \in R$ be such that $(\alpha)(r_1 \alpha_{j_1}^{k_1} \cdots \alpha_{j_n}^{k_n}) = 1$. It follows that $(\alpha)(r_1 Y_{j_1}^{k_1} \cdots Y_{j_n}^{k_n} (\frac{\alpha_{j_1}}{Y_{j_1}})^{k_1} \cdots (\frac{\alpha_{j_n}}{Y_{j_n}})^{k_n}) = (\alpha)(r_1 \alpha_{j_1}^{k_1} \cdots \alpha_{j_n}^{k_n}) = 1$. Thus $a = r_1 Y_{j_1}^{k_1} \cdots Y_{j_n}^{k_n} (\frac{\alpha_{j_1}}{Y_{j_1}})^{k_1} \cdots (\frac{\alpha_{j_n}}{Y_{j_n}})^{k_n} \in U(T)$, and hence r is an atom in T. $\square$

Notice that in the previous theorem, if S is unit closed, then $r \in R$ is an atom in $T = R[\{Y_i, \alpha_i/Y_i \mid \alpha_i \in S\}]$ if and only if r is an atom in R and $r \notin S$. We know r

is not associates with any $\alpha \in S$ since all associates of $\alpha \in S$ are also in S since S is unit closed.

Now we prove our main theorem of this chapter. As mentioned earlier, the idea is to adjoin elements $Y_r, r/Y_r$ for each atom r that we want to be reducible. Then $r = Y(r/Y)$ can be factored as a product of two atoms. But we really want no finite atom factorizations of r, so we adjoin more elements of the same form $Y_{Y_r}, Y_r/Y_{Y_r}$ and $Y_{r/Y_r}, (r/Y_r)/(Y_{r/Y_r})$. Now $r = Y(r/Y) = (Y_{Y_r})(Y_r/Y_{Y_r})(Y_{r/Y_r})((r/Y_r)/(Y_{r/Y_r}))$ is the product of four atoms. Continuing in this manner, for countably many steps, causes r to have no finite atom factorization. We will rely on both Lemma 3.2.2 and Lemma 3.2.3 to keep track of all the units and atoms.

Theorem 3.2.4. *Let R be an integral domain and S a subset of $\mathcal{A}_R$ that is unit closed. Then there exists an integral domain T containing R such that $\mathcal{A}_T = S$.*

Proof. Let $T_0 = R$. Inductively define $T_{n+1} = T_n[\{Y_{\alpha_i^{(n)}}, \alpha_i^{(n)}/Y_{\alpha_i^{(n)}} \mid \alpha_i^{(n)} \in \mathcal{A}_{T_n} \setminus S\}]$, where $Y_{\alpha_i^{(n)}}$ are indeterminates and $\mathcal{A}_{T_n}$ is the set of atoms of T_n. Then for $n \geq 0$, $U(T_n) = U(T_{n+1})$ by Lemma 3.2.2 since $\mathcal{A}_{T_n} \cap U(T_n) = \emptyset$. It follows that $U(T_n) = U(R)$. Also, the atoms in S are not associates of any element of $\mathcal{A}_{T_n} \setminus S$ since S is unit closed. Thus all elements of S are atoms in T_n by Lemma 3.2.3. Let $T = \bigcup T_n$. Then T is an integral domain since each T_n is an integral domain and $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$. We have also shown that $U(T_n) = U(R)$ for $n \geq 0$; thus $U(T) = U(R)$.

Finally, we show $\mathcal{A}_T = S$. Let $r \in \mathcal{A}_T$. Then there exists $n \geq 0$ such that $r \in T_n$ since $r \in T$. If r is not an atom in T_n, then there exist $a, b \in T_n \setminus U(T_n)$ such that $r = ab$. This is a contradiction since $U(T_n) = U(T)$ implies $r = ab$ is a nontrivial factorization in T. Thus $r \in \mathcal{A}(T_n)$. It follows that $r \in S$, otherwise $r = Y_r(\frac{r}{Y_r})$ is a nontrivial factorization in $T_{n+1} \subseteq T$. For the reverse inclusion, suppose $r \in S$ and $a, b \in T$ such that $r = ab$. Then there exists $n \geq 0$ such that $a, b \in T_n$. So we may assume $a \in U(T_n) = U(T)$. Hence $r \in \mathcal{A}_T$. $\square$

We conclude this section with two examples to demonstrate the construction from Theorem 3.2.4 and [8, Theorem 3.3] as a corollary. The first example shows that any integral domain is contained in an integral domain with no atoms. This type of integral domain has been studied extensively and is called an antimatter domain. In [7, Theorem 2.13], Coykendall, Dobbs, and Mullins also show that every integral domain can be embedded as a subring of some antimatter domain which is not a field. Notice that their proof is completely different than the proof given here.

Example 3.2.5. *1. Let R be an integral domain with set of atoms $\mathcal{A}_R$. We construct an integral domain containing R with no atoms. Let $T_0 = R$ and $T_1 = R[\{Y_\alpha, \alpha/Y \mid \alpha \in \mathcal{A}_R\}]$, where Y_α is an indeterminate for each $\alpha \in \mathcal{A}_R$. Define $T_n = T_{n-1}[\{Y_{\alpha_i^{(n-1)}}, \alpha_i^{(n-1)}/Y_{\alpha_i^{(n-1)}} \mid \alpha_i^{(n-1)} \in \mathcal{A}_{T_{n-1}}\}]$. Let $T = \bigcup_n T_n$. Then there are no atoms in T since at each step T_n there is a nontrivial factorization $r = Y_r(\frac{r}{Y_r})$ induced for each atom r of the previous step T_{n-1}.*

2. Let R be an integral domain with set of atoms $\mathcal{A}_R$. Let M be a monoid such that $M \cong \mathcal{A}(R)/\sim$, where $\sim$ is the associate relation. Note that the equivalence class of an atom in $\mathcal{A}(R)$ is represented by an atom of M. Assume $M \cong \mathcal{A}(R)/\sim$ is torsion-free. Then we can construct an integral domain T containing R such that $\mathcal{A}(T) \cong M$.

It is clear that M is commutative, so we show that M is reduced and cancellative. Let $a, b, c \in \mathcal{A}(R)$ be such that $\bar{a}\bar{b} = 1$. Then there exists $u \in U(R)$ such that $uab = 1$. It follows that $ua, b \in U(R)$ and $\bar{u}\bar{a} = \bar{a} = 1 = \bar{b}$. Thus M is reduced. Suppose $\bar{a}\bar{b} = \bar{a}\bar{c}$. Then there exists $u \in U(R)$ such that $ab = uac$. Hence $b = uc$ by cancellation; so $\bar{b} = \bar{u}\bar{c} = \bar{c}$. Thus M is cancellative. It follows that $T_0 = \mathbb{Z}_2[X; M]$ is an integral domain with $U(T_0) = \{1\}$. We have shown that each of the atoms $m \in M$ correspond to the atom $X^m \in T_0$. Let $S = \{X^m \mid m \text{ atom in } M\}$, and note that this set is unit closed. Use Theorem 3.2.4 to construct T such that $\mathcal{A}_T = S$. It follows that $\mathcal{A}(T) \cong M$.

This type of construction was also used by Coykendall and Zafrullah [9] to construct an integral domain with a single, non-prime atom, up to associates. Now we consider [8, Theorem 3.3], which follows directly from Theorem 3.2.4. We strengthen their result slightly and for a "nice" monoid M, we construct an integral domain T such that $M \cong \mathcal{A}(T)$ by the map $m \mapsto X^m$. Using the coefficient ring $\mathbb{Z}_2$, we ensure that $U(T) = \{1\}$. Thus $(\mathcal{A}(T)/\sim) = \mathcal{A}(T) \cong M$.

Corollary 3.2.6. *Let M be a commutative, reduced, cancellative, torsion-free monoid. Then there exists an integral domain with $\mathcal{A}(T) \cong M$.*

Proof. Let M be a commutative, reduced, cancellative, torsion-free monoid. Then $R = \mathbb{Z}_2[X; M]$ is an integral domain, and $U(R) = U(\mathbb{Z}_2[X; M]) = \{rX^m \mid r \in U(\mathbb{Z}_2), m \text{ invertible in } M\} = \{1\}$ [11, Theorem 11.1]. We first show that $m \in \mathcal{A}_M$ implies $X^m \in \mathcal{A}_R$. Let $m \in M$ be an atom in M. Write $X^m = ab$ for $a, b \in R$. Then a, b are monomials, so there exist $m_1, m_2 \in M$ such that $a = X^{m_1}$ and $b = X^{m_2}$. It follows that $X^m = ab = X^{m_1}X^{m_2} = X^{m_1+m_2}$ and equivalently $m = m_1 + m_2$. We may assume $m_1 = 0$ since m is an atom in M and M is reduced. Then $a = X^{m_1} = 1$ is a unit in R, and so X^m is an atom in R. Let $S = \{X^m \mid m \in \mathcal{A}_M\} \subseteq \mathcal{A}_R$, and note that S is unit closed since $U(R) = \{1\}$. Construct the integral domain T from the proof of Theorem 3.2.4 with $\mathcal{A}_T = S$. Then $M \cong \mathcal{A}(T)$ by the monoid isomorphism $m \mapsto X^m$. $\square$

This corollary gives a partial answer to the following question: If M is a commutative, cancellative monoid, when does there exist an integral domain R such that $\mathcal{A}(R) \cong M$. In general, $\mathcal{A}(R)$ is a commutative, cancellative monoid, that need not be reduced or torsion-free. It is easy to see that $\mathcal{A}(R)$ is reduced if and only if $U(R) = \{1\}$, and $\mathcal{A}(R)$ is not torsion-free if the characteristic of R is not 2 since $(-1)^2 = 1^2$, but $-1 \neq 1$. For example in the integral domain $R = \mathbb{R} + X\mathbb{C}[[X]]$, we have $\mathcal{A}(R)$ is a commutative, cancellative monoid that is neither reduced nor torsion-free.

3.3 Partial ordering of constructed integral domains

For an integral domain R and a subset S of $\mathcal{A}_R$ that is unit closed, we use the construction in Theorem 3.2.4 to create an integral domain T^S. If we are consistent in the way we choose indeterminates, then we can construct integral domains that are partially ordered. So, if we have $A \subseteq B$ subsets of $\mathcal{A}_R$ that are unit closed, then $T^B \subseteq T^A$. The containment reverses because we adjoin more indeterminates for the integral domain with set of atoms A since $A \subseteq B$.

Let $B_0 = R$ and $B_n = B_{n-1}[\{Y_r, r/Y_r \mid r \in B_{n-1}\}]$. Then $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ are integral domains, and thus $\bigcup B_n$ is an integral domain. Note that all of the integral domains we construct are contained in $\bigcup B_n$.

To make this relationship explicit, we define a function f such that for $A \subseteq B$, we have $f(B) \subseteq f(A)$. Let R be an integral domain and

$$f : \{S \subseteq \mathcal{A}_R \mid S \text{ is unit closed}\} \to \{T \mid R \subseteq T \subseteq \bigcup B_n \text{ integral domain}\}$$

be defined by $S \mapsto T^S$, where T^S is the integral domain constructed similarly to the construction in Theorem 3.2.4 as follows. Let $T_0^S = R$ and $T_n^S = T_{n-1}^S[\{Y_r, r/Y_r \mid r \in T_{n-1}^S \setminus (S \cup U(R))\}]$. Then $T^S = \bigcup T_n^S$ is an integral domain since each T_n^S is an integral domain and $T_0^S \subseteq T_1^S \subseteq T_2^S \subseteq \cdots$. Also from Theorem 3.2.4, we have $\mathcal{A}_{T^S} = S$. It suffices to show that f meets the desired condition, that is if $A \subseteq B$, for $A, B \in \{S \subseteq \mathcal{A} \mid S \text{ is unit closed}\}$, then $f(B) = T^B \subseteq T^A = f(A)$.

Lemma 3.3.1. *Let R be an integral domain with set of atoms $\mathcal{A}_R$ and $A, B \in \{S \subseteq \mathcal{A}_R \mid S$ is unit closed$\}$ with $A \subseteq B$. Then $T_n^B \subseteq T_n^A$ for all $n \geq 1$.*

Proof. We proceed by induction on n. For $A \subseteq B$, we have $R \setminus (B \cup U(R)) \subseteq R \setminus (A \cup U(R))$. Thus the result holds for $n = 1$ and $T_1^B \subseteq T_1^A$ by construction. Assume the result holds for $n - 1$. Then $T_{n-1}^B \subseteq T_{n-1}^A$. It follows that $T_{n-1}^B \setminus (B \cup$

$U(R)) \subseteq T^A_{n-1} \setminus (A \cup U(R))$. Hence $T^B_n = T^B_{n-1}[\{Y_r, r/Y_r \mid r \in T^B_{n-1} \setminus (B \cup U(R))\}] \subseteq T^A_{n-1}[\{Y_r, r/Y_r \mid r \in T^A_{n-1} \setminus (A \cup U(R))\}] = T^A_n$. □

It follows from this lemma that $f(B) \subseteq f(A)$ since $T^B = \bigcup T^B_n \subseteq \bigcup T^A_n = T^A$. Now that we can construct an integral domain with a particular set of atoms, we turn our focus to the other aspect of this dissertation. We are concerned about when the set generated by atoms and units is saturated. For an integral domain R, the multiplicatively closed set $\mathcal{A}(R)$ is saturated if whenever ab can be factored as a finite product of atoms for $a, b \in R$, then both a and b can be factored as a finite product of atoms. We conclude this section with a result that again adjoins elements $Y_r, r/Y_r$ only now our intent is to have a factorization of r into atoms. We construct an integral domain T containing R such that $\mathcal{A}(T)$ is saturated and contains $\mathcal{A}(R)$.

Theorem 3.3.2. *Let R be an integral domain. Then there exists an integral domain T such that $R \subseteq T$, $U(R) = U(T)$, $\mathcal{A}(R) \subseteq \mathcal{A}(T)$, and $\mathcal{A}(T)$ is saturated.*

Proof. Let $T_0 = R$ and S_0 be the saturation of $\mathcal{A}(T_0) = \mathcal{A}(R)$ in R. Let $A_0 = S_0 \setminus \mathcal{A}(T_0)$ be the set of nonunit elements in the saturation of $\mathcal{A}(T_0)$ that cannot be factored as a finite product of atoms. Let $T_1 = R[\{Y^{(0)}_i, a^{(0)}_i/Y^{(0)}_i \mid a^{(0)}_i \in A_0\}]$, where $Y^{(0)}_i$ are indeterminates for each i. Then $A_0 \subseteq \mathcal{A}(T_1)$ since $a^{(0)}_i = Y^{(0)}_i(a^{(0)}_i/Y^{(0)}_i)$ for all $a^{(0)}_i \in A_0$, and $Y^{(0)}_i, a^{(0)}_i/Y^{(0)}_i \in \mathcal{A}(T_1)$. It follows from Lemma 3.2.2 that $U(T_1) = U(T_0)$ since $A_0 \cap U(R) = \emptyset$. Also, $\mathcal{A}(T_0) \subseteq \mathcal{A}(T_1)$ by Lemma 3.2.3. Hence $S_0 \subseteq \mathcal{A}(T_1)$. Inductively define S_n to be the saturation of $\mathcal{A}(T_n)$, $A_n = S_n \setminus \mathcal{A}(T_n)$, and $T_{n+1} = T_n[\{Y^{(n)}_i, a^{(n)}_i/Y^{(n)}_i \mid a^{(n)}_i \in A_n\}]$. Then T_n is an integral domain for $n \geq 0$ and $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$; thus $\bigcup_n T_n$ is an integral domain. Finally, $\mathcal{A}(T_n) \subseteq S_n \subseteq \mathcal{A}(T_{n+1})$ and $U(T_n) = U(R)$ for all $n \geq 0$. Hence $\mathcal{A}(T) = \bigcup_n S_n = \bigcup_n \mathcal{A}(T_n)$. We show $\mathcal{A}(T) = \bigcup_n \mathcal{A}(T_n)$ is saturated. Let $ab \in \mathcal{A}(T)$ for $a, b \in T$. Then there exists an n such that $ab \in \mathcal{A}(T_n) \subseteq S_n$. Hence $a, b \in S_n \subseteq \mathcal{A}(T_{n+1}) \subseteq \mathcal{A}(T)$ since S_n is saturated. □

Chapter 4

Polynomials and Power Series

In this chapter, we investigate the multiplicatively closed set $\mathcal{A}(R)$ generated by the atoms and units of polynomial and power series subrings of the form $R = A + B_1X + B_2X^2 + \cdots$. We will first look at the power series ring case and generalize a result from Mammenga's dissertation [14]. Then we turn to the polynomial ring case and find that the analogous results hold. In the final section, we look at several examples showing that the property of $\mathcal{A}(R)$ being saturated is not stable for the polynomial or power series extension, nor is this property stable in the localization case.

4.1 Power Series Rings

Recall that the units in a power series ring are all the power series in which the constant term is a unit in its respective ring. For example, $U(R[[X]]) = \{f(X) \in R[[X]] \mid f(0) \in U(R)\}$ and for $R = A + B_1X + B_2X^2 + \cdots$ a power series ring, we have $U(R) = \{f(X) \in R \mid f(0) \in U(A)\}$. We start with a technical proposition that shows the atoms of an integral domain A are directly related to the atoms of $A + XK[[X]]$, where K is the quotient field of A.

Proposition 4.1.1. *Let A be an integral domain with quotient field K and $A \neq K$. Let $R = A + XK[[X]]$ and $r \in R$. Then r is an atom in R if and only if there exists an atom $a \in A$ and $u \in U(R)$ such that $r = ua$.*

Proof. ($\Rightarrow$) Let r be an atom in R. Then there exists $a_0 \in A$ and $a_1, a_2, \ldots \in K$ such that $r = a_0 + a_1X + a_2X^2 + \cdots$. If $a_0 = 0$, then $r = a_1X + a_2X^2 + \cdots = a(\frac{a_1}{a}X + \frac{a_2}{a}X^2 + \cdots)$ is not at atom. So $a_0 \neq 0$, and $\frac{a_i}{a_0} \in K$ for all $i \geq 1$. Thus $r = a_0(1+\frac{a_1}{a_0}X+\frac{a_2}{a_0}X^2+\cdots)$ with a_0 an atom in R since $1+\frac{a_1}{a_0}X+\frac{a_2}{a_0}X^2+\cdots \in U(R)$. We show that a_0 is an atom in A. Let $a, b \in A$ be such that $a_0 = ab$. Then $a, b \in R$; so we may assume $b \in U(R)$. Hence $b \in U(A)$ since $b \in A$. Thus $r = ua_0$ for $u = 1 + \frac{a_1}{a_0}X + \frac{a_2}{a_0}X^2 + \cdots \in U(R)$ and a_0 an atom in A.

($\Leftarrow$) Let a be an atom in A and $u \in U(R)$. Let $r = ua$, and write $r = (b_0 + b_1X + \cdots)(c_0 + c_1X + \cdots)$ for $b_0 + b_1X + \cdots, c_0 + c_1X + \cdots \in R$. We may assume that $u = 1 + u_1X + \cdots$ for $u_1, u_2, \ldots \in K$. Then $b_0c_0 + (b_0c_1 + b_1c_0)X + \cdots = r = ua = a + au_1X + \cdots$. So $a = b_0c_0$, and we may assume $b_0 \in U(A)$ by hypothesis. It follows that $b_0 + b_1X + \cdots \in U(R)$ and $r = ua$ is an atom in R. $\square$

Now we state the result from Mammenga's dissertation [14, Theorem 2.3.3]. We include the proof because it reveals the keys needed in the generalizations. We shall see that the relationship proved in Proposition 4.1.1 is the key for the $A + XK[[X]]$ case.

Theorem 4.1.2. *([14, Theorem 2.3.3]) Let A be an integral domain with quotient field K and $R = A + AX + \cdots + AX^{n-1} + X^nK[[X]]$. If A is atomic, then $\mathcal{A}(R)$ is saturated in R if and only if $A = K$ or $n = 1$.*

Proof. For $A = K$, we have $R = K[[X]]$. Then R is a PID and $\mathcal{A}(R) = R \setminus \{0\}$ is saturated in R.

For $n = 1$, we have $R = A + XK[[X]]$. Let $\mathcal{R} = \{ua_1 \cdots a_n \mid u \in U(R), a_i$ an atom in $A\}$. We show $\mathcal{A}(R) = \mathcal{R}$. Note that $U(R) \subseteq \mathcal{R}$. Let r be an atom in R. Then there exists an atom $a \in A$ and $u \in U(R)$ such that $r = ua$ by Proposition

4.1.1. It follows that $r = ua \in \mathcal{R}$ and $\mathcal{A}(R) \subseteq \mathcal{R}$ since the generators of $\mathcal{A}(R)$ are in $\mathcal{R}$ and $\mathcal{R}$ is multiplicatively closed. For the reverse inclusion, we start by noting that $U(R) \subseteq \mathcal{A}(R)$. Let a be an atom in A. Then a is an atom in R by Proposition 4.1.1. Thus $\mathcal{R} \subseteq \mathcal{A}(R)$. We now show that $\mathcal{A}(R) = \mathcal{R}$ is saturated in R. By Proposition 2.3.1, we show that this set is $III-$saturated in R since $U(R) \subseteq \mathcal{R} = \mathcal{A}(R)$. Suppose $f(X), g(X) \in R \setminus U(R)$ such that $f(X)g(X) \in \mathcal{R}$. It follows that $f(0)g(0) \neq 0$, and hence $f(0), g(0) \neq 0$. Write $f(X) = r_0 + r_1X + r_2X^2 + \cdots, g(X) = s_0 + s_1X + s_2X^2 + \cdots \in R$ with $0 \neq r_0, s_0 \in A$ and $r_i, s_i \in K$ for $i \geq 1$. Then $f(X) = r_0(1 + \frac{r_1}{r_0}X + \frac{r_2}{r_0}X^2 + \cdots) = ur_0$ for $u = 1 + \frac{r_1}{r_0}X + \frac{r_2}{r_0}X^2 + \cdots \in U(R)$, and similarly, $g(X) = vs_0$ for some $v \in U(R)$. We have assumed A is atomic, so there exist atoms $a_1, \ldots, a_k, b_1, \ldots, b_l$ in A such that $f(X) = ur_0 = ua_1 \cdots a_k \in \mathcal{R}$ and $g(X) = vs_0 = vb_1 \cdots b_l \in \mathcal{R}$. Thus $\mathcal{A}(R) = \mathcal{R}$ is saturated in R.

For $n \geq 2$, assume that $\mathcal{A}(R)$ is saturated in R by way of contradiction. First, we show X is an atom in R. Write $X = rs$ for $r, s \in R$. Then $\text{ord}(r) = 1-$ $\text{ord}(s)$. So we may assume $\text{ord}(r) = 0$ and $\text{ord}(s) = 1$, and there exist $r_0, r_1, \ldots, r_{n-1}, s_1, s_2, \ldots, s_{n-1} \in A$ and $r_i, s_j \in K$ for $i, j \geq n$, such that $r = r_0 + r_1X + \cdots$ and $s = s_1X + s_2X^2 + \cdots$. Therefore, $X = rs = (r_0 + r_1X + \cdots)(s_1X + s_2X^2 + \cdots) = r_0s_1X + \cdots$. It follows that $r_0s_1 = 1$, and so $r_0 \in U(A)$. Thus $r \in U(R)$ and X is an atom in R. Then $X^{2n} = X \cdot X \cdots X \in \mathcal{A}(R)$. Let $0 \neq a \in A \setminus U(A)$. Then $X^{2n} = (aX^n)(a^{-1}X^n)$ with $aX^n, a^{-1}X^n \in R$. We have assumed $\mathcal{A}(R)$ is saturated in R and shown $(aX^n)(a^{-1}X^n) = X^{2n} \in \mathcal{A}(R)$; thus $aX^n, a^{-1}X^n \in \mathcal{A}(R)$. It follows that there exist atoms $f_1(X), \ldots, f_k(X) \in R$ such that $a^{-1}X^n = f_1(X) \cdots f_k(X)$. For $1 \leq i \leq k$, write $f_i(X) = a_{n_i}X^{n_i} + a_{n_i+1}X^{n_i+1} + \cdots$. Then $a^{-1}X^n = f_1(X) \cdots f_k(X) = a_{n_1} \cdots a_{n_k}X^{n_1+\cdots+n_k} + \cdots$ with $a^{-1} = a_1 \cdots a_k$ and $n = n_1 + n_2 + \cdots + n_k$. Note that $n_i \leq n$ for $1 \leq i \leq k$. If there exists $1 \leq i \leq k$ such that $n_i = n$, then $f_i(X) = a_{n_i}X^{n_i} + a_{n_i+1}X^{n_i+1} + \cdots = a(\frac{a_{n_i}}{a}X^{n_i} + \frac{a_{n_i+1}}{a}X^{n_i+1} + \cdots)$ is not an atom in R. Thus $n_i < n$ and $a_i \in A$ for all $1 \leq i \leq k$. It follows that $a^{-1} = a_1 \cdots a_k \in A$, but this is a contradiction since $a \in A \setminus U(A)$. Hence $a^{-1}X^n \notin \mathcal{A}(R)$ and $\mathcal{A}(R)$ is not saturated in R. $\square$

The $n = 1$ case of Theorem 4.1.2 is shown by characterizing the atoms and units of $A + XK[[X]]$. We have shown that $\mathcal{A}(R)$ can be viewed as simply all finite products of atoms in A and units in R. The next theorem also uses this simplifying technique to prove that $\mathcal{A}(R)$ is saturated in R. Notice in both of these proofs that X is not an atom nor can X be factored as a product of atoms.

Theorem 4.1.3. *Let A be a PID with quotient field K and B a proper overring of A, i.e., $A \subsetneq B \subseteq K$. Let $R = A + BX + X^2K[[X]]$. Then $\mathcal{A}(R)$ is saturated in R.*

Proof. Let $\mathcal{R} = \{u\pi_1 \cdots \pi_n(a_1^{n_1} + b_1X) \cdots (a_k^{n_k} + b_kX) \mid u \in U(R), \pi_i, a_j$ atoms in A, $n_j \geq 1, a_j \nmid b_j$ in $B\}$. First, we show that $\mathcal{A}(R) = \mathcal{R}$. It is clear that $\mathcal{R}$ is multiplicatively closed and $U(R) \subseteq \mathcal{R}$ by definition. Choose $\frac{a}{b} \in B \setminus A$ such that $a, b \in A$ with $\gcd(a, b) = 1$. Then $(a, b) = R$; so there exist $\alpha, \beta \in A$ such that $\alpha a + \beta b = 1$. Dividing by b, we have $\frac{1}{b} = \alpha(\frac{a}{b}) + \beta \in B$. Note that $\frac{1}{b} \notin A$ since $\frac{a}{b} \notin A$. Let $r \in R$ be an atom. Write $r = r_0 + r_1X + X^2f(X) \in R$ with $r_0 \in A, r_1 \in B$, and $f(X) \in K[[X]]$. If $r_0 = 0$, then $r = r_1X + X^2f(X) = b(\frac{r_1}{b}X + \frac{f(X)}{b}X^2)$ is reducible. So we may assume $r_0 \neq 0$. Thus $r = r_0 + r_1X + X^2f(X) = (r_0 + r_1X)(1 + \frac{f(X)}{r_0+r_1X}X^2)$ with $\frac{f(X)}{r_0+r_1X} \in K[[X]]$ since $r_0 + r_1X \in U(K[[X]])$. Hence $r = u(r_0 + r_1X)$ for $u = 1 + \frac{f(X)}{r_0+r_1X}X^2 \in U(R)$. Then there exist distinct (nonassociate) primes $a_1, \ldots, a_n \in A$ and $v \in U(A)$ such that $r_0 = va_1^{k_1} \cdots a_n^{k_n}$ for $k_i \geq 1$ since A is a PID. We may assume $v = 1$, and consider $a_1^{k_1} \cdots a_n^{k_n} + v^{-1}r_1X = v^{-1}r_0 + v^{-1}r_1X = v^{-1}(r_0 + r_1X)$. If there exists $1 \leq i \leq n$ such that $a_i | r_1$ in B, then $r_0 + r_1X = a_i(\frac{r_0}{a_i} + \frac{r_1}{a_i}X)$. It follows that $\frac{r_0}{a_i} + \frac{r_1}{a_i}X \in U(R)$ since $r_0 + r_1X$ is an atom in R. Thus $r = u(r_0 + r_1X) = wa_i \in \mathcal{R}$ for some $w \in U(R)$. Now, if $a_i \nmid r_1$ in B for $1 \leq i \leq n$, we choose $a_1, \ldots, a_n$ to be distinct prime factors of r_0. Then $\gcd(a_1^{k_1}, a_2^{k_2} \cdots a_n^{k_n}) = 1$ and there exist $\alpha, \beta \in A$ such that $a_1^{k_1}\alpha + (a_2^{k_2} \cdots a_n^{k_n})\beta = 1$ since A is a PID. Multiplying by r_1, we have $r_1 = r_1(1) = r_1(a_1^{k_1}\alpha + (a_2^{k_2} \cdots a_n^{k_n})\beta) = a_1^{k_1}(r_1\alpha) + a_2^{k_2} \cdots a_n^{k_n}(r_1\beta)$ with $r_1\alpha, r_1\beta \in B$. Now $r_0 + r_1X = a_1^{k_1} \cdots a_n^{k_n} + (a_1^{k_1}(r_1\alpha) + a_2^{k_2} \cdots a_n^{k_n}(r_1\beta))X = (a_1^{k_1} + (r_1\beta)X)(a_2^{k_2} \cdots a_n^{k_n} + (r_1\alpha)X + X^2g(X))$ for some $g(X) \in K[[X]]$. It follows that $a_2^{k_2} \cdots a_n^{k_n} \in U(R)$ since $r_0 + r_1X$ is an atom. Thus $r = u(r_0 + r_1X) = uv(a_1^{k_1} + r_1\beta X) \in \mathcal{R}$ for some $u, v \in U(R)$. So $\mathcal{A}(R) \subseteq \mathcal{R}$ since all generators of $\mathcal{A}(R)$ are in $\mathcal{R}$ and $\mathcal{R}$ is multiplicatively closed.

For the reverse inclusion, we know $U(R) \subseteq \mathcal{A}(R)$, and we show that if π, a are atoms of A, $n \geq 1$, and $b \in B \setminus \{0\}$ such that $a \nmid b$, then $\pi, a^n + bX$ are atoms in R. It then follows that $\mathcal{R} \subseteq \mathcal{A}(R)$ since $\mathcal{A}(R)$ is multiplicatively closed. Let π be an atom in A and $f(X), g(X) \in R$ such that $\pi = f(X)g(X)$. Then $f(0), g(0) \in A$ and $f(0)g(0) = \pi$; so we may assume $f(0) \in U(A)$. Thus $f(X) \in U(R)$, and so π is an atom in R. Let $a^n + bX$ be as above and $a^n + bX = f(X)g(X)$ for some $f(X), g(X) \in R$. Write $f(X) = ua^l + c_1X + c_2X^2 + \cdots$ and $g(X) = va^k + d_1X + d_2X^2 + \cdots$ for $u, v \in A$, $l, k \geq 0$, $c_1, d_1 \in B$, and $c_i, d_i \in K$ for $i \geq 2$. Then $a^n + bX = f(X)g(X) = uva^{l+k} + (a^l d_1 + a^k c_1)X + \cdots$. Hence $uv = 1$, $l + k = n$, and $b = a^l d_1 + a^k c_1$. It follows that $l = 0$ or $k = 0$ since $a \nmid b$. Thus $f(X) = u + c_1X + \cdots \in U(R)$ or $g(X) = v + d_1X + \cdots \in U(R)$ since $u, v \in U(A)$. Then $a^n + bX$ is an atom in R. Hence $\mathcal{R} \subseteq \mathcal{A}(R)$.

We now show $\mathcal{A}(R) = \mathcal{R}$ is saturated in R. By Proposition 2.3.1, it suffices to show that $\mathcal{A}(R)$ is $III-$saturated in R since $U(R) \subseteq \mathcal{A}(R)$. Suppose $f(X)g(X) \in \mathcal{R}$ for $f(X), g(X) \in R \setminus U(R)$. It is clear that $f(0), g(0) \neq 0$ by definition of $\mathcal{R}$. Then $f(X) = a + bX + X^2h(X)$ for $a \in A \setminus \{0\}, b \in B$, and $h(X) \in K[[X]]$. We know A is atomic, so there exist $a_1, \ldots, a_k$ distinct atoms in A and $u \in U(A)$ such that $a = ua_1^{n_1} \cdots a_k^{n_k}$ for some $n_i \geq 1$. Let $c \in A$ be such that $c|a$ in A, $c|b$ in B, and if $ca_i|a$, then $ca_i \nmid b$ in B for all $1 \leq i \leq k$. Thus $f(X) = a + bX + X^2h(X) = c(a_1^{m_1} \cdots a_k^{m_k} + \frac{b}{c}X + \frac{h(X)}{c}X^2)$ for $0 \leq m_i \leq n_i$. Note that $c(a_1^{m_1} \cdots a_k^{m_k} + \frac{b}{c}X) \in U(K[[X]])$. Thus $f(X) = uc(a_1^{m_1} \cdots a_k^{m_k} + \frac{b}{c}X)$ for $u = 1 + \frac{h(X)}{c(a_1^{m_1} \cdots a_k^{m_k} + \frac{b}{c}X)}X^2 \in U(R)$. Now, as above, we factor $a_1^{m_1} \cdots a_k^{m_k} + \frac{b}{c}X = u'(a_1^{m_1} + b_1X) \cdots (a_k^{m_k} + b_kX)$ for some $u' \in U(R)$ and $b_i \in B$ since A is a PID. Also, $c = c_1 \cdots c_l$ for atoms $c_1, \ldots, c_l \in A$ since A is atomic. Thus $f(X) = uc(a_1^{m_1} + b_1X) \cdots (a_k^{m_k} + b_kX) = uc_1 \cdots c_l(a_1^{m_1} + b_1X) \cdots (a_k^{m_k} + b_kX) \in \mathcal{R}$. Similarly, $g(X) \in \mathcal{R}$. □

This proof is constructive and shows that all atoms of $R = A+BX+X^2K[[X]]$ are associates of atoms of the form π or $a^n + bx$ with π and a atoms in A. When we prove the more general results, we will characterize the elements of $\mathcal{A}(R)$ without knowing much about the form of the atoms. Before doing that, we look back to the ring $R = A + XK[[X]]$. In an effort to loosen the hypotheses on A, we recall Proposition 4.1.1, which states that all atoms of R are of the form ua for an atom $a \in A$. So $\mathcal{A}(A)$ and $\mathcal{A}(R)$ are very intimately connected. In fact, we have shown in the proof of Theorem 4.1.2 that $\mathcal{A}(R) = \{ua_1 \cdots a_n \mid a_i \text{ atoms in } A, u \in U(R)\} = U(R)\mathcal{A}(A)$. When A is atomic, $\mathcal{A}(R)$ is saturated in R and $\mathcal{A}(A) = A \setminus \{0\}$ is saturated in A. The next theorem shows that even when A is not atomic, $\mathcal{A}(R)$ is saturated in R and $\mathcal{A}(A)$ is saturated in A at precisely the same time.

Theorem 4.1.4. *Let A be an integral domain with quotient field K and $R = A + XK[[X]]$. Then $\mathcal{A}(A)$ is saturated in A if and only if $\mathcal{A}(R)$ is saturated in R.*

Proof. If $A = K$, then $\mathcal{A}(A) = K \setminus \{0\}$ and $\mathcal{A}(R) = R \setminus \{0\}$ since $R = K[[X]]$ is a PID. Thus $\mathcal{A}(A)$ is saturated in A and $\mathcal{A}(R)$ is saturated in R. So we assume $A \subsetneq K$. For both implications, we will use Proposition 2.3.1 and simply show that the set is $III-$saturated since $U(A) \subseteq \mathcal{A}(A)$ and $U(R) \subseteq \mathcal{A}(R)$.

($\Rightarrow$) Let $rs \in \mathcal{A}(R)$ for $r, s \in R \setminus U(R)$. Then there exist atoms $a_1, \ldots, a_n$ in A and $u \in U(R)$ such that $rs = ua_1 \cdots a_n$ by Proposition 4.1.1. We may assume $u = 1 + u_1X + u_2X^2 + \cdots$. Write $r = r_0 + r_1X + \cdots$ and $s = s_0 + s_1X + \cdots$ for $r_0, s_0 \in A$ and $r_i, s_i \in K$ for $i \geq 1$. Then $r_0s_0 + (r_1s_0 + r_0s_1)X + \cdots = rs = ua_1 \cdots a_n = a_1 \cdots a_n + u_1a_1 \cdots a_nX + \cdots$. Hence $r_0s_0 = a_1 \cdots a_n \in \mathcal{A}(A)$ with $r_0, s_0 \in A$. It follows that $r_0, s_0 \in \mathcal{A}(A)$ since $\mathcal{A}(A)$ is saturated in A. Then $r = wr_0$ and $s = vs_0$ for some $w, v \in U(R)$ by Proposition 4.1.1, and hence $r = wr_0, s = vs_0 \in \mathcal{A}(R)$.

($\Leftarrow$) Let $a, b \in A \setminus U(A)$ be such that $ab \in \mathcal{A}(A)$. Then $ab \in \mathcal{A}(R)$ since $\mathcal{A}(A) \subseteq \mathcal{A}(R)$. Also, $a, b \in A \subseteq R$, so $a, b \in \mathcal{A}(R)$ by hypothesis. Then there exist atoms $a_1, \ldots, a_n, b_1, \ldots, b_k \in A$ and units $u, v \in U(R)$ such that $a = ua_1 \cdots a_n$, $b = vb_1 \cdots b_k \in \mathcal{A}(A)$ by Proposition 4.1.1. $\square$

Now we consider the case where $\mathcal{A}(R)$ is not saturated in R. In Theorem 4.1.2, we showed that $\mathcal{A}(R)$ is not saturated in R for $R = A + AX + \cdots + AX^{n-1} + X^nK[[X]]$ when $n \geq 2$. The important properties in the proof are first that X is an atom, and hence $X^n = X \cdot X \cdots X$ is a product of atoms. Second, for $0 \neq r \in A \setminus U(A)$, we have $rX^n, r^{-1}X^n \in R$ but $r^{-1}X^m \notin R$ for all $m \leq n-1$. Then $r^{-1}X^n = r(r^{-2}X^n)$ is neither an atom nor can it be factored into a product of atoms since $\frac{1}{r} \notin A$. For an integral domain $R = A + AX + \cdots + AX^{n-1} + X^nK[X]$, there may be several such $r \in A \setminus U(A) \subseteq K$, but all we need is that at least one such r exists. For R to maintain an appropriate $r \in A \setminus U(A)$, instead of K we can take any ring that contains a unit, which is a nonunit of A. The $n \geq 2$ hypothesis remains to ensure that X is an atom in R.

Theorem 4.1.5. *Let $A \subseteq B_n \subseteq B_{n+1} \subseteq \cdots$ be integral domains and $R = A + AX + \cdots + AX^{n-1} + B_nX^n + B_{n+1}X^{n+1} + \cdots$ be a power series ring. If $n \geq 2$ such that $U(A) \subsetneq U(B_n) \cap A$, then $\mathcal{A}(R)$ is not saturated in R.*

Proof. First, we show that X is an atom in R. Suppose $f(X), g(X) \in R$ such that $X = f(X)g(X)$. Then $1 = \text{ord}(X) = \text{ord}(f(X)g(X)) = \text{ord}(f(X)) + \text{ord}(g(X))$. So we may assume $\text{ord}(f(X)) = 1$ and $\text{ord}(g(X)) = 0$. Write $f(X) = a_1X + a_2X^2 + \cdots$ and $g(X) = b_0 + b_1X + b_2X^2 + \cdots$ with $b_0, a_i, b_i \in A$, for $1 \leq i < n$ and $a_i, b_i \in B_i$ for $i \geq n$. Then $X = f(X)g(X) = a_1b_0X + (a_1b_1 + b_0a_2)X^2 + \cdots$. It follows that $a_1b_0 = 1$. Therefore, $b_0 \in U(A)$, and hence $g(X) = b_0 + b_1X + \cdots \in U(R)$. Thus X is an atom in R.

Assume $\mathcal{A}(R)$ is saturated in R. Let $b \in (U(B_n) \cap A) \setminus U(A)$. So $b(b^{-1}X^n) = X^n = X \cdot X \cdots X \in \mathcal{A}(R)$ with $b, b^{-1}X^n \in R$. Thus $b^{-1}X^n \in \mathcal{A}(R)$ since $\mathcal{A}(R)$ is saturated in R. Then there exist atoms $f_1(X), \ldots, f_k(X) \in R$ such that $b^{-1}X^n = f_1(X)f_2(X)\cdots f_k(X)$. It follows that $\sum_{i=1}^{k} \text{ord}(f_i(X)) = \text{ord}(f_1(X)\cdots f_k(X)) = \text{ord}(b^{-1}X^n) = n$. Assume there exists $1 \leq j \leq k$ such that $f_j(X)$ has order n. Write $f_j(X) = r_nX^n + r_{n+1}X^{n+1} + \cdots$ for $r_l \in B_l$. Then $f_j(X) = r_nX^n + r_{n+1}X^{n+1} + \cdots = b(\frac{r_n}{b}X^n + \frac{r_{n+1}}{b}X^{n+1} + \cdots)$ is not an atom. Thus $\text{ord}(f_i(X)) < n$ for $1 \leq i \leq k$, and

$f_i(X) = a_{l_i}X^{l_i} + a_{l_i+1}X^{l_i+1} + \cdots \in R$ with $a_{l_i} \in A$ since $l_i = \mathrm{ord}(f_i(X)) < n$. Hence $b^{-1}X^n = f_1(X)f_2(X)\cdots f_k(X) = a_{l_1}a_{l_2}\cdots a_{l_k}X^n + (\sum_{i=1}^k a_{l_i+1}\prod_{j\neq i} a_{l_j})X^{n+1} + \cdots$. It follows that $b^{-1} = a_{l_1}a_{l_2}\cdots a_{l_k} \in A$. But this is a contradiction of our choice of b; thus $b^{-1}X^n \notin \mathcal{A}(R)$. Therefore, $\mathcal{A}(R)$ is not saturated in R. □

Notice again that $X^n = X \cdot X \cdots X$ can be factored as a finite product of atoms, if X is an atom. Also, X^n can be factored as $X^n = b(b^{-1}X^n)$, where $0 \neq b \in A \setminus U(A)$ and $b^{-1}X^n$ cannot be factored as a product of atoms. When we generalize, we need to maintain both factorizations. This means that B_1, the coefficient ring of the X term, can properly contain A as long as no elements of $A \setminus U(A)$ are units in B_1. Consider the next example.

Example 4.1.6. *Let $R = \mathbb{Z}+\mathbb{Z}[\sqrt{2}]X+X^2\mathbb{Q}[\sqrt{2}][[X]]$. We can see that X is an atom in R by the same argument as in the proof of Theorem 4.1.5. Thus $X^2 = X \cdot X \in \mathcal{A}(R)$. But $X^2 = n(\frac{X^2}{n})$ for all $n \in \mathbb{Z} \setminus \{0\}$, and $\frac{X^2}{n}$ cannot be factored as a finite product of atoms for $n \neq \pm 1$. Hence $\mathcal{A}(R)$ is not saturated in R.*

Notice that in this example, $U(A) = \{\pm 1\}$ and $U(B_1) \cap A = \{\pm 1\}$. Thus X is still an atom. We also have $U(A) = \{\pm 1\} \subsetneq \mathbb{Z} \setminus \{0\} = (\mathbb{Q} \setminus \{0\}) \cap \mathbb{Z} = U(B_2) \cap A$; so $X^2 = n(\frac{X^2}{n})$ is a non-trivial factorization in R for all $n \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Theorem 4.1.7. *Let $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains and $R = A + B_1X + B_2X^2 + \cdots$ be a power series ring. If there exists an $n \geq 2$ such that $U(A) \subsetneq U(B_n) \cap A$ and $U(A) = U(B_i) \cap A$ for $1 \leq i < n$, then $\mathcal{A}(R)$ is not saturated in R.*

Proof. By the proof of Theorem 4.1.5, we see that X is an atom in R and for $b \in (U(B_n) \cap A) \setminus U(A)$, we have $b(b^{-1}X^n) = X^n = X \cdot X \cdots X \in \mathcal{A}(R)$. So we may assume there exist atoms $f_1(X), \ldots f_k(X)$ in R such that $b^{-1}X^n = f_1(X)\cdots f_k(X)$ with $\mathrm{ord}(f_i(X)) < n$. Then $f_i(X) = a_{l_i}X^{l_i} + a_{l_i+1}X^{l_i+1} + \cdots \in R$ with $a_{l_i} \in B_{l_i} \subseteq B_{n-1}$ for $1 \leq i \leq k$. Then $b^{-1}X^n = f_1(X)\cdots f_k(X) = a_{l_1}\cdots a_{l_k}X^{l_1+\cdots+l_k} + \cdots$; hence $b^{-1} = a_{l_1}\cdots a_{l_k} \in B_{n-1}$. It follows that $b \in U(B_{n-1}) \cap A = U(A)$, but this is a contradiction. Therefore, $\mathcal{A}(R)$ is not saturated in R. □

Now we switch our focus to when $\mathcal{A}(R)$ is saturated in R. Recall that for $R = A + XK[[X]]$ and $0 \neq a \in A \setminus U(A)$, we had $X = a(\frac{X}{a})$ is reducible. This property allowed us to classify all the atoms of $A + XK[[X]]$ as unit multiples of atoms of A. So for power series rings of the form $R = A + B_1X + B_2X^2 + \cdots$, we require $U(A) \subsetneq U(B_1) \cap A$. Then X is not an atom in R. The next theorem extends this idea and shows that any product of atoms is not divisible by X.

Theorem 4.1.8. *Let A be an integral domain that satisfies ACCP and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_1) \cap A$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a power series ring. Then $f(X) \in R$ is a nonempty product of atoms in R if and only if $f(0)$ is a nonzero nonunit in A.*

Proof. ($\Rightarrow$) Let $f(X)$ be a nonempty product of atoms in R. Then $f(X)$ is not a unit, i.e., $f(0) \notin U(A)$, since atoms do not divide units. By way of contradiction, suppose $f(0) = 0$. Then there exist atoms $f_1(X), \ldots, f_n(X) \in R$ such that $f(X) = f_1(X) \cdots f_n(X)$. It follows that $0 = f(0) = f_1(0) \cdots f_n(0)$; so there exists $1 \leq i \leq n$ such that $f_i(0) = 0$. Write $f_i(X) = b_1X + b_2X^2 + \cdots$ for some $b_j \in B_j$. Choose $0 \neq a \in (U(B_1) \cap A) \setminus U(A)$. Then $f_i(X) = a(\frac{b_1}{a}X + \frac{b_2}{a}X^2 + \cdots)$, which is a contradiction of the choice of $f_i(X)$. Thus $f(0) \neq 0$.

($\Leftarrow$) Let $f(X) \in R$ be such that $f(0)$ is a nonzero nonunit in A. We will show that there exists an atom $g(x)$ in R such that $g(X) \mid f(X)$. Suppose to the contrary that there does not exist such an atom. Then there exist nonatoms, nonunits $\{g_i(X)\}_{i=1}^{\infty}$ such that $f(X) = g_0(X)h_0(X)$ and $g_n(X) = g_{n+1}(X)h_{n+1}(X)$ for nonunits $h_i(X) \in R$. Then $f(X) = g_0(X)h_0(X) = g_1(X)h_1(X)h_0(X) = \cdots$. It follows that $(g_0(X)) \subsetneq (g_1(X)) \subsetneq (g_2(X)) \subsetneq \cdots$, and hence $(g_0(0)) \subsetneq (g_1(0)) \subsetneq (g_2(0)) \subsetneq \cdots$. This is a contradiction since A satisfies ACCP. So, if $g_i(X)$ is an atom in R, then we are done. If there exists a $g_i(X) \in U(R)$, then $g_{i-1}(X)$ is an atom.

Now, let $g_0(X), g_1(X), \ldots$ be atoms in R such that $f(X) = g_0(X)h_0(X) = g_0(X)g_1(X)h_1(X) = g_0(X)g_1(X)g_2(X)h_2(X) = \cdots$ for some $h_0(X), h_1(X), \ldots \in R$. Then $f(0) = g_0(0)h_0(0) = g_0(0)g_1(0)h_1(0) = g_0(0)g_1(0)g_2(0)h_2(0) = \cdots$, and so

$(h_0(0)) \subseteq (h_1(0)) \subseteq \cdots$ is an ascending chain in A. Therefore, there exists an $n \geq 0$ such that $(h_i(0)) = (h_n(0))$ for all $i \geq n$ since A satisfies ACCP. Hence $g_i(0) \in U(A)$ for all $i \geq n+1$. Then $g_i(X) \in U(R)$ for $i \geq n+1$, and $f(X) = g_0(X)g_1(X)\cdots g_n(X)$ is a finite product of atoms. $\square$

Note the necessity of the $U(A) \subsetneq U(B_1) \cap A$ hypothesis by Example 4.1.6. Theorem 4.1.8 allows us to categorize the elements of $\mathcal{A}(R)$ as those power series in R with nonzero constant term. We use this fact to show that $\mathcal{A}(R)$ is saturated in R since $ab = 0 \Leftrightarrow a = 0$ or $b = 0$.

Corollary 4.1.9. *Let A be an integral domain that satisfies ACCP and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_1) \cap A$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a power series ring. Then $\mathcal{A}(R)$ is saturated in R.*

Proof. We know that $f(X) \in R$ is a unit if and only if $f(0) \in U(A) \subseteq A \setminus \{0\}$. Then $f(X) \in \mathcal{A}(R)$ if and only if $f(0) \neq 0$ by Theorem 4.1.8. Thus $\mathcal{A}(R)$ is saturated in R. $\square$

In Theorems 4.1.8 and Corollary 4.1.9 we have assumed A satisfies ACCP, but recall Theorem 4.1.2 only requires A to be atomic. The next example shows that if A is atomic but does not satisfy ACCP, then $\mathcal{A}(R)$ need not be saturated in R for $R = A + B_1X + B_2X^2 + \cdots$, where $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ and $U(A) \subsetneq U(B_1) \cap A$. So the stronger hypothesis on A is necessary when we have fewer restrictions on the integral domains $B_1, B_2, \ldots$.

We first follow Roitman's [17] construction of an atomic domain A such that both $A[X]$ and $A[[X]]$ are not atomic. Then we modify $A[[X]]$ to fit all of the hypotheses of Theorem 4.1.8 except A will be atomic, but not satisfy ACCP.

Let Y_1, Y_2, Z be independent indeterminates over a field k. Let $R = k[Z, \{\frac{Y_1}{Z^n}, \frac{Y_2}{Z^n} \mid n \geq 0\}]$. Now we create atom factorizations for all reducible elements. So in the spirit of the construction in Theorem 3.3.2, define $R_0 = R$ and $R_{n+1} = R_n[\{T_s, s/T_s \mid s \in \mathcal{R}_n\}]$, where $\mathcal{R}_n$ is the set of reducible elements of R_n and T_s is an indeterminate for each $s \in \mathcal{R}_n$. Let $A = \bigcup_{n=0}^{\infty} R_n$. We first show that A is atomic. If $s \in A$ is a reducible element in A, then it is reducible in some T_n. By construction $s = T_s(s/T_s)$ is the product of two atoms in $T_{n+1} \subseteq A$. Roitman shows that both $A[X]$ and $A[[X]]$ are not atomic since $Y_1 + Y_2X$ is not a product of atoms in either $A[X]$ or $A[[X]]$.

Let K be the quotient field of A and $R = A + A[\frac{1}{Z}]X + X^2K[[X]]$. We will first show that $Y_1 + Y_2X$ is a nonunit with nonzero constant term but cannot be factored as a product of atoms in R. Then we show that $\mathcal{A}(R)$ is not saturated in R. Hence the ACCP hypothesis on A is necessary for both Theorem 4.1.8 and Corollary 4.1.9.

Example 4.1.10. *Let A be the atomic domain constructed in Roitman [17], where $A[X]$ and $A[[X]]$ are not atomic. Notice that A does not satisfy ACCP since $(Y_1) \subsetneq (Y_1/Z) \subsetneq (Y_1/Z^2) \subsetneq \cdots$ is a strictly ascending chain in A. Let $R = A + A[\frac{1}{Z}]X + X^2K[[X]]$, where K is the quotient field of A.*

Roitman shows that any finite factorization of $Y_1 + Y_2X$ in $A[[X]]$ has a factor that is divisible by Z^n for all $n \geq 1$. Notice that $A[[X]] \subseteq R$ and $Y_2/Z^n \in A$ for all $n \geq 0$; so there are no new factorizations in R. It follows that any factorization of $Y_1 + Y_2X$ in $R = A + A[\frac{1}{Z}]X + X^2K[[X]]$ also has a factor that is divisible by Z^n for $n \geq 1$. But Z is not a unit in R; thus $Y_1 + Y_2X$ cannot be factored as a finite product of atoms in R. Also, $Y_1 + Y_2X$ is a nonunit with nonzero constant term since $Y_1 \notin U(A)$. Thus Theorem 4.1.8 needs the hypothesis that A satisfies ACCP.

We also have that $Y_1^2 - Y_2^2X^2 = Y_1^2(1 - \frac{Y_2^2}{Y_1^2}X^2)$ is a factorization of $Y_1^2 - Y_2^2X^2$ into a finite product of atoms since $Y_1^2 = T_{Y_1}^2(\frac{Y_1}{T_{Y_1}})^2$ with atoms $T_{Y_1}, \frac{Y_1}{T_{Y_1}}$ in R and $1 - \frac{Y_2^2}{Y_1^2}X^2 \in U(R)$. Thus $(Y_1 + Y_2X)(Y_1 - Y_2X) = Y_1^2 - Y_2^2X^2 \in \mathcal{A}(R)$, but we have shown that $Y_1 + Y_2X \notin \mathcal{A}(R)$. Hence we need that A satisfies ACCP in Corollary 4.1.9.

We give a corollary combining Theorem 4.1.5 and Corollary 4.1.9 to give necessary and sufficient conditions for $\mathcal{A}(R)$ to be saturated in R.

Corollary 4.1.11. *Let A be an integral domain that satisfies ACCP and let $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_n) \cap A$ and $U(A) = U(B_i) \cap A$ for*

$i < n$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a power series ring. Then $\mathcal{A}(R)$ saturated in R if and only if $n = 1$.

4.2 Polynomial Rings

Now we consider the polynomial ring case and the analogues of the power series rings results. We start by recalling that for an integral domain R and a polynomial ring $R[X]$ the set of units are all the constant polynomials in $U(R)$, i.e., $U(R[X]) = U(R)$. Then for $R = A + B_1X + B_2X^2 + \cdots$, we have $U(R) = \{f(X) = u \in R \mid u \in U(A)\}$. Because there are fewer units than in the power series case, the atoms, up to units, are more diverse. Luckily many of the analogous results still hold. The first theorem shows that Mammenga's result (Theorem 4.1.2) holds in the polynomial case.

Theorem 4.2.1. *Let A be an integral domain with quotient field K and $R = A + AX + \cdots + AX^{n-1} + X^nK[X]$. If A is atomic, then $\mathcal{A}(R)$ is saturated in R if and only if $A = K$ or $n = 1$.*

Proof. For $A = K$, we have $R = K[X]$. Then R is a PID and hence is atomic. Thus $\mathcal{A}(R) = R \setminus \{0\}$ is saturated in R.

For $n = 1$, we have $R = A + XK[X]$. Let $\mathcal{R} = \{ua_1 \cdots a_k(1 + Xf_1(X)) \cdots (1 + Xf_n(X)) \mid u \in U(R) = U(A), a_i \text{ atom in } A, 1 + Xf_j(X) \text{ atom in } K[X] \text{ with } f_j(X) \in K[X]\}$. It is clear that $\mathcal{R}$ is multiplicatively closed and $U(R) \subseteq \mathcal{R}$. We show $\mathcal{A}(R) = \mathcal{R}$. Let r be an atom in R. Write $r = \alpha + Xg(X)$ for some $\alpha \in A$ and $g(X) \in K[X]$. If $\alpha = 0$, then $r = Xg(X) = a(\frac{g(X)}{a}X)$ for $0 \neq a \in A \setminus U(A)$. Thus $\alpha \neq 0$. Then $r = \alpha(1 + \frac{g(X)}{\alpha}X)$, and either $1 + \frac{g(X)}{\alpha}X \in U(R)$ or $\alpha \in U(R)$ since r is an atom in R. If $1 + \frac{g(X)}{\alpha}X$ is a unit in R, then α is an atom in R. We know $A \subseteq R$ and $U(A) = U(R)$, and so it follows that α is an atom in A. If α is a unit in R, then $1 + \frac{g(X)}{\alpha}X$ is an atom in R. We show $1 + \frac{g(X)}{\alpha}X$ is an atom in $K[X]$. Let $a_0 + a_1X + \cdots + a_nX^n, b_0 + b_1X + \cdots + b_kX^k \in K[X]$ be such that $1 + \frac{g(X)}{\alpha}X = (a_0 + a_1X + \cdots + a_nX^n)(b_0 + b_1X + \cdots + b_kX^k) = a_0b_0 + \cdots + a_nb_kX^{n+k}$. Then

$a_0b_0 = 1$, and in particular $a_0 \neq 0$. Therefore $1+\frac{g(X)}{\alpha}X = a_0(1+\frac{a_1}{a_0}X+\cdots+\frac{a_n}{a_0}X^n)(b_0+b_1X+\cdots+b_kX^k) = (1+a_1/a_0X+\cdots+a_n/a_0X^n)(a_0b_0+a_0b_1X+\cdots+a_0b_kX^k)$ with $1+\frac{a_1}{a_0}X+\cdots+\frac{a_n}{a_0}X^n$ and $a_0b_0+a_0b_1X+\cdots+a_0b_kX^k = 1+a_0b_1X+\cdots+a_0b_kX^k$ elements of R. Hence $1+\frac{a_1}{a_0}X+\cdots+\frac{a_n}{a_0}X^n \in U(R)$ or $a_0b_1X+\cdots+a_0b_kX^k \in U(R)$. If $1+\frac{a_1}{a_0}X+\cdots+\frac{a_n}{a_0}X^n$ is a unit in R, then $\frac{a_1}{a_0} = \frac{a_2}{a_0} = \cdots = \frac{a_n}{a_0} = 0$ since $U(R) = U(A)$. It follows that $a_1 = a_2 = \cdots = a_n = 0$. Thus $a_0+a_1X+\cdots+a_nX^n = a_0 \in U(K[X])$. Similarly, if $a_0(b_0+b_1X+\cdots+b_kX^k)$ is a unit in R, then $b_0+b_1X+\cdots+b_kX^k = b_0 \in U(K[X])$. Hence $1+\frac{g(x)}{\alpha}X$ is an atom in $K[X]$ and $r = \alpha(1+\frac{g(X)}{\alpha}X) \in \mathcal{R}$. Then $\mathcal{A}(R) \subseteq \mathcal{R}$.

For the other inclusion, let $a \in A$. If a is an atom in A, then it is an atom in R by Lemma 1.0.8, which says that all divisors of a are elements of A. Let $1+Xf(X)$ be an atom in $K[X]$ and $g(X), h(X) \in R$ be such that $1+Xf(X) = g(X)h(X)$. Then we may assume $g(X)$ is a unit in $K[X]$ since $R \subseteq K[X]$. It follows that $g(X) = k \in K \setminus \{0\}$. Thus $k = g(0) \in A$ since $g(X) \in R$. Also, $k(h(0)) = g(0)h(0) = 1$, and hence $g(X) = k$ is a unit in R. Therefore, $1+Xf(X)$ is an atom in R. It follows that $\mathcal{R} \subseteq \mathcal{A}(R)$ since $U(R) \subseteq \mathcal{A}(R)$ and $\mathcal{A}(R)$ is multiplicatively closed.

Now we show $\mathcal{A}(R) = \mathcal{R}$ is $III-$saturated in R, and it follows that $\mathcal{A}(R)$ is saturated in R by Proposition 2.3.1. Let $g(X)h(X) \in \mathcal{A}(R)$ for $g(X), h(X) \in R \setminus U(R)$. Then there exist $\alpha, \beta \in A$ and $g_1(X), h_1(X) \in K[X]$ such that $g(X) = \alpha + Xg_1(X)$, and $h(X) = \beta + Xh_1(X)$. Note $\alpha, \beta \neq 0$ since $g(X)h(X) \in \mathcal{A}(R) = \mathcal{R}$. Then $g(X) = \alpha + Xg_1(X) = \alpha(1+\frac{g_1(X)}{\alpha}X)$ with $\alpha \in A$ and $1+\frac{g_1(X)}{\alpha}X \in K[X]$. We know $\alpha = b_1\cdots b_k$ for atoms $b_1, \ldots, b_k$ in A since A is atomic. Also, $1+X\frac{g_1(X)}{\alpha} = (a_1+Xf_1(X))\cdots(a_n+Xf_n(X))$ for atoms $a_1+Xf_1(X), \ldots, a_n+Xf_n(X) \in K[X]$ since $K[X]$ is a UFD. We may assume $a_i = 1$ for $1 \leq i \leq n$ since $a_i+Xf_i(X) = a_i(1+\frac{f_i(X)}{a_i}X)$ and $a_1\cdots a_n = 1$. Thus $g(X) = \alpha(1+\frac{g_1(X)}{\alpha}X) = b_1\cdots b_k(1+\frac{f_1(X)}{a_1}X)\cdots(1+\frac{f_n(X)}{a_n}X) \in \mathcal{A}(R)$. Similarly, $h(X) \in \mathcal{A}(R)$. Thus $\mathcal{A}(R)$ is saturated in R.

For $n \geq 2$, we show $\mathcal{A}(R)$ is not saturated in R by way of contradiction. Assume $\mathcal{A}(R)$ is saturated in R. First, we show that X is an atom in R. Let $f(X), g(X) \in R$ be such that $X = f(X)g(X)$. Then $f(X)$ and $g(X)$ are monomials with $1 = \deg(X) =$

$\deg(f(X)g(X)) = \deg(f(X)) + \deg(g(X))$ by Lemma 1.0.8. Without loss of generality, assume $f(X) = aX$ and $g(X) = b$ for some $a, b \in R$. It follows that $X = f(X)g(X) = (aX)(b) = abX$, and hence $ab = 1$. Therefore, $g(X) = b \in U(A) = U(R)$, and X is an atom in R.

Let $0 \neq a \in A \setminus U(A)$. Then $aX^n, a^{-1}X^n \in R$ and $(aX^n)(a^{-1}X^n) = X^{2n} = X \cdot X \cdots X \in \mathcal{A}(R)$. Thus $a^{-1}X^n \in \mathcal{A}(R)$. Let $f_1(X), \ldots, f_l(X) \in R$ be atoms such that $a^{-1}X^n = f_1(X) \cdots f_l(X)$. For $1 \leq i \leq l$, $f_i(X)$ is a monomial; so there exists $0 \neq a_i \in K$, $j_i \geq 0$, such that $f_i(X) = a_i X^{j_i}$ with $j_1 + \cdots + j_l = \sum \deg(f_i(X)) = \deg(f_1(X) \cdots f_l(X)) = \deg(kX^n) = n$. If there exists $1 \leq i \leq l$ such that $j_i = n$, then $a_i X^n = a(\frac{a_i}{a} X^n)$ is not an atom. Thus $j_i < n$, and so $a_i \in A$, for $1 \leq i \leq l$. Then $a^{-1}X^n = f_1(X) \cdots f_l(X) = \prod_i a_i X^{j_i} = a_1 \cdots a_l X^n$. It follows that $a^{-1} = a_1 \cdots a_l \in A$, but this contradicts our choice of a. Hence $a^{-1}X^n \notin \mathcal{A}(R)$ and $\mathcal{A}(R)$ is not saturated in R. □

Like the power series case, $\mathcal{A}(R)$ is not saturated in R when X is an atom in R. So, when we generalize we need that X is an atom and that there exists an element $0 \neq r \in A \setminus U(A)$ such that $rX^n \in R$ for some $n \geq 2$. Then we can again use the fact that rX^n is not an atom and $X \nmid rX^n$ by our choice of r.

Theorem 4.2.2. *Let $A \subseteq B_n \subseteq B_{n+1} \subseteq \cdots$ be integral domains and $R = A + AX + \cdots + AX^{n-1} + B_nX^n + B_{n+1}X^{n+1} + \cdots$ be a polynomial ring. If $n \geq 2$ such that $U(A) \subsetneq U(B_n) \cap A$, then $\mathcal{A}(R)$ is not saturated in R.*

Proof. First, we show that X is an atom in R. Let $f(X), g(X) \in R$ be such that $X = f(X)g(X)$. Then we may assume $f(X) = aX$ for some $0 \neq a \in A$. It follows that $g(X) = a^{-1}$. Thus $g(X) = a^{-1} \in U(R)$ and X is an atom in R.

Assume $\mathcal{A}(R)$ is saturated in R and choose $a \in (U(B_n) \cap A) \setminus U(A)$. Then $a(a^{-1}X^n) = X^n = X \cdot X \cdots X \in \mathcal{A}(R)$. Hence $a^{-1}X^n \in \mathcal{A}(R)$. Write $a^{-1}X^n = f_1(X)f_2(X) \cdots f_k(X)$ for atoms $f_1(X), \ldots, f_k(X) \in R$. Then $n = \deg(a^{-1}X^n) = \deg(f_1(X) \cdots f_k(X)) = \sum \deg(f_i(X))$; so $\deg(f_i(X)) \leq n$ for $1 \leq i \leq k$. Also, $f_i(X)$ is a monomial, and so $f_i(X) = a_i X^{k_i}$ with $0 \leq k_i \leq n$. If $k_i = n$, then

$f_i(X) = a_iX^n = a(\frac{a_i}{a}X)$ is reducible. Thus $k_i < n$ and $a_i \in A$ for $1 \leq i \leq k$. It follows that $a^{-1}X^n = f_1(X)f_2(X)\cdots f_k(X) = a_1a_2\cdots a_kX^n$; thus $a^{-1} = a_1a_2\cdots a_k \in A$. This contradicts the choice of a. Hence $\mathcal{A}(R)$ is not saturated in R. $\square$

Again, we can generalize this result by allowing the coefficient domains B_i for $i < n$ to properly contain A, but we require that $U(A) = U(B_i) \cap A$, thus ensuring that X is an atom. This generalization is illustrated by the integral domain $R = \mathbb{Z} + \mathbb{Z}[\sqrt{2}]X + X^2\mathbb{Q}[\sqrt{2}][X]$. In this case, X is an atom since all divisors of X are of the form a or aX for $a \in \mathbb{Z}$. But $X^2 = 2(\frac{X^2}{2})$, where $\frac{X^2}{2}$ cannot be factored as a finite product of atoms in R.

Theorem 4.2.3. *Let $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains and $R = A + B_1X + B_2X^2 + \cdots$ be a polynomial ring. If there exists an $n \geq 2$ such that $U(A) \subsetneq U(B_n) \cap A$ and $U(A) = U(B_i) \cap A$ for all $i < n$, then $\mathcal{A}(R)$ is not saturated in R.*

Proof. The proof of Theorem 4.2.2 shows that X is an atom in R. Let $0 \neq a \in (U(B_n) \cap A) \setminus U(A)$ and $f_1(X), \ldots, f_k(X)$ be atoms in R such that $a^{-1}X^n = f_1(X)\cdots f_k(X)$. For $1 \leq i \leq k$, we know $k_i = \deg(f_i(X)) < n$ from the proof of the previous theorem. Then $f_i(X) = a_iX^{k_i}$ with $a_i \in B_{k_i} \subseteq B_{n-1}$. Then $a^{-1} = a_1\cdots a_k \in B_{n-1}$. It follows that $a \in U(B_{n-1}) \cap A = U(A)$. But this is a contradiction. $\square$

Now we turn to the case where $\mathcal{A}(R)$ is saturated in R. It suffices to have $0 \neq a \in A \setminus U(A)$ such that $\frac{X}{a} \in R$. Then $X = a(\frac{X}{a})$ is not an atom in R. It follows that all of the atoms in R have a nonzero constant term.

Theorem 4.2.4. *Let A be an integral domain that satisfies ACCP with quotient field K and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_1) \cap A$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a polynomial ring. Then $f(X) \in R$ is a nonempty product of atoms if and only if $f(0) \neq 0$ and $f(X) \notin U(R) = U(A)$.*

Proof. ($\Rightarrow$) Let $f(X) \in R$ be a nonempty product of atoms. Then $f(X) \notin U(R)$ since units are not divisible by atoms. By way of contradiction, suppose $f(0) = 0$. Write

$f(X) = f_1(X) \cdots f_k(X)$ for atoms $f_1(X), \dots, f_k(X)$ in R. Then there exists $1 \leq i \leq k$ such that $f_i(0) = 0$ since $f(0) = f_1(0) \cdots f_k(0)$. Let $f_i(X) = b_1 X + b_2 X^2 + \cdots + b_n X^n$ for $b_j \in B_j$. Choose $0 \neq a \in (U(B_1) \cap A) \setminus U(A)$. Then $f(X) = b_1 X + b_2 X^2 + \cdots + b_n X^n = a(\frac{b_1}{a} X + \frac{b_2}{a} X^2 + \cdots + \frac{b_n}{a} X^n)$ is not an atom in R. But this is a contraction; thus $f(0) \neq 0$.

($\Leftarrow$) Let $f(X) \in R \setminus U(R)$ be such that $f(0) \neq 0$. Then there exist $f_1(X), \dots, f_k(X) \in R$ such that $f(X) = f_1(X) \cdots f_k(X)$ with $0 < \deg(f_i(X)) \leq \deg(f(X))$ for $1 \leq i \leq k$. Choose $f_1(X), \dots, f_k(X)$ such that if there exists $g(X) \in R$, such that $g(X) \mid f_i(X)$ and $\deg(g(X)) < \deg(f_i(X))$, then $g(X) = a \in A$. Notice that $k \leq \sum_{i=1}^{k} \deg(f_i(X)) = \deg(f(X)) < \infty$ and $f_i(0) \neq 0$ since $0 \neq f(0) = f_1(0) \cdots f_k(0)$.

CLAIM: There exists $a_i \in A$ such that $f_i(X) = a_i g_i(X)$ for some atom $g_i(X) \in R$. If not, there exists a sequence $\{b_j\}$ of nonunits in A such that $f_i(X) = b_1 h_1(X) = b_1 b_2 h_2(X) = \cdots$ for $h_j(X) \in R$. Then $h_j(X) = b_{j+1} h_{j+1}(X)$ for $j \geq 1$, and it follows that $h_j(0) = b_{j+1} h_{j+1}(0)$. Hence $(f_i(0)) \subsetneq (h_1(0)) \subsetneq (h_2(0)) \subsetneq \cdots$ is a strictly ascending chain of principal ideals in A. This is a contradiction since A satisfies ACCP. So for $i \geq 1$, we may choose $a_i \in A$ such that $f_i(X) = a_i g_i(X)$ for atom $g_i(X) \in R$.

Then $f(X) = f_1(X) \cdots f_k(X) = a_1 \cdots a_k g_1(X) \cdots g_k(X)$. Finally, there exist atoms $b_j \in A$ such that $a_1 \cdots a_k = b_1 \cdots b_n$ since A satisfies ACCP. Hence $f(X) = a_1 \cdots a_k g_1(X) \cdots g_k(X) = b_1 \cdots b_n g_1(X) \cdots g_k(X)$ is a nonempty product of atoms. $\square$

Note that the proof of the statement "if $f(X)$ is a nonempty product of atoms, then $f(0) \neq 0$ and $f(X) \notin U(R) = U(A)$," does not require A to be atomic, much less satisfy ACCP. However, we do need the hypothesis that $U(A) \subsetneq U(B_1) \cap A$. We have seen that for $R = \mathbb{Z} + \mathbb{Z}[\sqrt{2}]X + X^2\mathbb{Q}[\sqrt{2}][X]$, we have $\mathcal{A}(R)$ is not saturated in R. Also, $U(A) = \{\pm 1\} = U(B_1) \cap A$. So the $U(A) \subsetneq U(B_1) \cap A$ hypothesis is necessary for both the previous theorem and the next corollary. Now for a polynomial ring $R = A + B_1X + B_2X^2 + \cdots$, we have $\mathcal{A}(R)$ is the set of all elements with nonzero constant term. It is an immediate consequence that $\mathcal{A}(R)$ saturated in R.

Corollary 4.2.5. *Let A be an integral domain that satisfies ACCP and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_1) \cap A$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a polynomial ring. Then $\mathcal{A}(R)$ is saturated in R.*

Proof. We know that $f(X) \in R$ is a unit if and only if $f(X) = u \in U(A)$. Then $f(X) \in \mathcal{A}(R)$ if and only if $f(0) \neq 0$ by Theorem 4.2.4. Thus $\mathcal{A}(R)$ is saturated in R. $\square$

We again use the integral domain constructed by Roitman in [17] to show that when A is atomic but does not satisfy ACCP, Theorem 4.2.4 and Corollary 4.2.5 need not hold.

Example 4.2.6. *Following Example 4.1.10, let A be the atomic domain constructed by Roitman [17]. Recall that A does not satisfy ACCP since $(Y_1) \subsetneq (Y_1/Z) \subsetneq (Y_1/Z^2) \subsetneq \cdots$. Let K be the quotient field of A, and let $R = A + A[\frac{1}{Z}]X + X^2K[X]$. It is clear that $Y_1 + X$ is not a unit in R; we will also show that it cannot be factored as a finite product of atoms. We know $Y_1 = Z(\frac{Y_1}{Z})$ is reducible; so let T_{Y_1} be the indeterminate such that $Y_1 = T_{Y_1}(Y_1/T_{Y_1})$. Also, $Y_1/Z^n = Z(Y_1/Z^{n+1})$ is reducible; so there exists T_{Y_1/Z^n} such that $Y_1/Z^n = T_{Y_1/Z^n}(\frac{Y_1}{T_{Y_1/Z^n}Z^n})$ for all $n \geq 0$. It follows that $Y_1 + X = Z^n(Y_1/Z^n + X/Z^n)$ for all n. Let $f_1(X), \ldots, f_k(X)$ be atoms such that $Y_1 + X = f_1(X) \cdots f_k(X)$. Then $1 = \deg(Y_1 + X) = \deg(f_1(X) \cdots f_k(X)) = \sum \deg(f_i(X))$. So we may assume $\deg(f_1(X)) = 1$ and $\deg(f_2(X)) = \cdots = \deg(f_k(X)) = 0$. Let $b_0, a_2, \cdots, a_k \in A$ and $b_1 \in A[\frac{1}{Z}]$ be such that $f_1(X) = b_0 + b_1X$ and $f_i(X) = a_i$ for $2 \leq i \leq k$. Define $a = a_2 \cdots a_k$. Then $Y_1 + X = f_1(X) \cdots f_k(X) = ab_0 + ab_1X$; hence $ab_0 = Y_1$ and $ab_1 = 1$. Thus $a \in U(A[\frac{1}{Z}]) = U(A) \cup \{uZ^n, \frac{u}{Z^n} \mid n \geq 0, u \in U(A)\}$. Note that $a \neq \frac{u}{Z^n}$ since $\frac{u}{Z^n} \notin A$. If $a \in U(A)$, then $a^{-1} \in A$, and if $a = uZ^n$, then $a^{-1}Y_1 \in A$ by construction. It follows that $f_1(X) = b_0 + b_1X = a^{-1}Y_1 + a^{-1}X = Z(a^{-1}\frac{Y_1}{Z} + \frac{a^{-1}}{Z}X)$. But this is a* *contradiction since we assumed $f_1(X)$ was an atom. Thus any finite factorization of $Y_1 + X$ contains at least one reducible factor.*

Also, $Y_1^2(1 - \frac{X^2}{Y_1^2}) = T_{Y_1}^2(\frac{Y_1}{T_{Y_1}})^2(1 - \frac{X^2}{Y_1^2})$ with T_{Y_1} and $\frac{Y_1}{T_{Y_1}}$ atoms in R. We show $1 - \frac{X^2}{Y_1^2}$ is an atom in R. Let $f(X), g(X) \in R$ be such that $1 - \frac{X^2}{Y_1^2} = f(X)g(X)$. Then $\deg(f(X)) + \deg(g(X)) = \deg(f(X)g(X)) = 2$. The only degree 1 factors are $1 + \frac{X}{Y_1}$ and $1 - \frac{X}{Y_1}$, but $\frac{1}{Y_1} \notin A[\frac{1}{Z}]$; so $\deg(f(X)), \deg(g(X)) \neq 1$. We may assume $\deg(f) = 0$; then there exists $a \in A$ such that $f(X) = a$. Thus $a(g(0)) = f(0)g(0) = 1$, and so $f(X) = a \in U(A) = U(R)$. Hence $1 - \frac{X^2}{Y_1^2}$ is an atom in R. Now $(Y_1 + X)(Y_1 - X) = Y_1^2 - X^2 = T_{Y_1}^2(\frac{Y_1}{T_{Y_1}})^2(1 - \frac{X^2}{Y_1^2}) \in \mathcal{A}(R)$, but we have shown that $Y_1 + X \notin \mathcal{A}(R)$. So ACCP is a necessary hypothesis on A in Corollary 4.2.5.

This example shows that A being an atomic domain is not a strong enough hypothesis, but unlike the power series case, we have control over the degrees of the factors in any polynomial factorization. This allows us to generalize Theorem 4.2.4 and require A to be only atomic if we strengthen the hypothesis on the other integral domains.

Theorem 4.2.7. *Let A be an atomic domain and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_1) \cap A$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a polynomial ring. If for every properly ascending chain $(a_0) \subsetneq (a_1) \subsetneq \cdots$ in A and $b \in \cup B_i$, there exists $n \geq 1$ such that $\frac{a_0}{a_n} \nmid b$, then $f(X) \in R$ is a nonempty product of atoms if and only if $f(0) \neq 0$ and $f(X) \notin U(R) = U(A)$.*

Proof. $(\Rightarrow)$ As mentioned earlier, we only need that $U(A) \subsetneq U(B_1) \cap A$. So this proof follows from the proof of Theorem 4.2.4.

$(\Leftarrow)$ Let $f(X) \in R \setminus U(R)$ be such that $f(0) \neq 0$. If $f(X) = a \in A$, then $f(X)$ is a nonempty product of atoms in A. So, we assume $\deg(f(X)) \geq 1$. Then there exist $f_1(X), \ldots, f_k(X) \in R$ such that $f(X) = f_1(X) \cdots f_k(X)$ with $0 < \deg(f_i(X)) \leq \deg(f(X))$ for $1 \leq i \leq k$, and furthermore, if there exists a nonunit $g(X) \in R$ such that $g(X) \mid f_i(X)$ and $\deg(g(X)) < \deg(f_i(X))$, then $g(X) = a \in A$. Note that

$f_i(0) \neq 0$ since $0 \neq f(0) = f_1(0) \cdots f_k(0)$, and $k \leq \sum_{i=1}^{k} \deg(f_i(X)) = \deg(f(X)) < \infty$.

CLAIM: There exists $a_i \in A$ such that $f_i(X) = a_i g_i(X)$ for some atom $g_i(X) \in R$. Write $f_i(X) = r_0 + r_1 X + \cdots + r_n X^n$ with $r_0 \in A$ and $r_j \in B_j$, for $1 \leq j \leq n$ and $r_n \neq 0$. If $r_0 = f_i(0) \mid f(X)$, then $f_i(X) = r_0(1 + \frac{r_1}{r_0}X + \cdots + \frac{r_n}{r_0}X^n)$. It follows that $1 + \frac{r_1}{r_0}X + \cdots + \frac{r_n}{r_0}X^n$ is an atom in R by choice of $f_i(X)$ since we assumed $f_i(X)$ only has factors of degree zero and n. If $r_0 \nmid f(X)$, then suppose there exists a sequence $\{b_j\}$ of atoms in A such that $f_j(X) = b_1 h_1(X) = b_1 b_2 h_2(X) = \cdots$ for $h_j(X) \in R$. Then $h_n(X) = b_{n+1} h_{n+1}(X)$ for all $n \geq 1$. Hence $(f_i(0)) \subsetneq (h_1(0)) \subsetneq (h_2(0)) \subsetneq \cdots$ is an ascending chain of principal ideals in A that does not terminate. Let $j \geq 1$ be the smallest integer such that $r_j \neq 0$. Then there exists $m \geq 1$ such that $\frac{f_i(0)}{h_m(0)} \nmid r_j$ by hypothesis. Let $h_m(X) = s_0 + s_1 X + \cdots + s_k X^k$. Note that $b_1 \cdots b_m = \frac{f_i(0)}{h_1(0)} \frac{h_1(0)}{h_2(0)} \cdots \frac{h_{m-1}(0)}{h_m(0)} = \frac{f_i(0)}{h_m(0)}$. Then $r_0 + r_1 X + \cdots + r_n X^n = f_i(X) = \frac{f_i(0)}{h_m(0)} h_m(X) = \frac{f_i(0)}{h_m(0)}(s_0 + s_1 X + \cdots + s_k X^k)$. This is a contradiction since $r_j = \frac{f_i(0)}{h_m(0)} s_j$, but $\frac{f_i(0)}{h_m(0)} \nmid r_j$.

So for each i, let $a_i \in A$ be such that $f_i(X) = a_i g_i(X)$ for atom $g_i(X) \in R$. Then $f(X) = f_1(X) \cdots f_k(X) = a_1 \cdots a_k g_1(X) \cdots g_k(X)$ and there exist atoms $b_1, \ldots, b_n \in A$ such that $a_1 \cdots a_k = b_1 \cdots b_n$ since A is atomic. Hence $f(X) = a_1 \cdots a_k g_1(X) \cdots g_k(X) = b_1 \cdots b_n g_1(X) \cdots g_k(X)$ is a nonempty product of atoms. □

It follows immediately that $\mathcal{A}(R)$ is saturated in R since again $\mathcal{A}(R) = \{f(X) \in R \mid f(0) \neq 0\}$.

Corollary 4.2.8. *Let A be an atomic domain and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_1) \cap A$. Let $R = A + B_1 X + B_2 X^2 + \cdots$ be a polynomial ring. If for every properly ascending chain $(a_0) \subsetneq (a_1) \subsetneq \cdots$ in A and $b \in \cup B_i$, there exists $n \geq 1$ such that $\frac{a_0}{a_n} \nmid b$, then $\mathcal{A}(R)$ is saturated in R.*

We conclude this section with a corollary combining Theorem 4.2.2 and Corollary 4.2.5 that gives necessary and sufficient conditions for $\mathcal{A}(R)$ to be saturated in R.

Corollary 4.2.9. *Let A be an integral domain that satisfies ACCP and $A \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be integral domains such that $U(A) \subsetneq U(B_n) \cap A$ and $U(A) = U(B_i) \cap A$ for $i < n$. Let $R = A + B_1X + B_2X^2 + \cdots$ be a polynomial ring. Then $\mathcal{A}(R)$ saturated in R if and only if $n = 1$.*

4.3 Stability of the $\mathcal{A}(R)$ Saturated Property

It is well-know that if R is an integral domain, then R satisfies ACCP if and only if $R[X]$ satisfies ACCP, if and only if $R[[X]]$ satisfies ACCP. Thus, whenever $R, R[X]$, or $R[[X]]$ satisfies ACCP then $\mathcal{A}(R), \mathcal{A}(R[X])$, and $\mathcal{A}(R[[X]])$ are all saturated. The next example shows that neither $\mathcal{A}(R[X])$ nor $\mathcal{A}(R[[X]])$ need to be saturated if R is atomic, but does not satisfy ACCP. We will first state a lemma from [17] that we will use in the example.

Lemma 4.3.1. *([17, Lemma 1.3]) Let R be any integral domain, and let $A = \bigcup_{n=0}^{\infty} R_n$, where $R_0 = R$ and $R_n = R_{n-1}[\{Y_\alpha, \frac{\alpha}{Y_\alpha} \mid \alpha$ reducible in $R_{n-1}\}]$. For $a \in A$ and $r \in R$ such that $a \mid r$ in A, there exists $r_a \in R$ such that $r_a \mid r$ in R. In particular, $a \mid r_a$ in A.*

Example 4.3.2. *Let A be the integral domain constructed by Roitman [17] with neither $A[X]$ nor $A[[X]]$ atomic. We will show that $\mathcal{A}(A[X])$ is not saturated in $A[X]$ and $\mathcal{A}(A[[X]])$ is not saturated in $A[[X]]$. But A is atomic; so $\mathcal{A}(A) = A \setminus \{0\}$ is saturated in A.*

First, we show $\mathcal{A}(A[X])$ is not saturated in $A[X]$. Let T_{Y_1} be the indeterminate from the construction that corresponds to Y_1. Then we show $Y_1 + T_{Y_1}Y_2X = T_{Y_1}(Y_1/T_{Y_1} + Y_2X)$ is a product of atoms. By construction, $T_{Y_1}, Y_1/T_{Y_1}$ are atoms in A, and thus in $A[X]$ by Lemma 1.0.8. We show that $Y_1/T_{Y_1} + Y_2X$ is an atom in $A[X]$. Let $f(X), g(X) \in A[X]$ be such that $Y_1/T_{Y_1} + Y_2X = f(X)g(X)$. We may assume $\deg(f(X)) = 0$ and $\deg(g(X)) = 1$ since $1 = \deg(f(X)g(X)) = \deg(f(X)) + \deg(g(X))$. Let $f(X) = a$ and $g(X) = b_0 + b_1X$ with $0 \neq a, b_0, b_1 \in A$.

Then $ab_0 = Y_1/T_{Y_1}$*; thus* $a \in U(A)$ *or* $b_0 \in U(A)$*. If* $a \in U(A)$*, then we are done. If* $b_0 \in U(A)$*, then* $a = b_0^{-1}Y_1/T_{Y_1}$*. It follows that* $Y_2 = ab_1 = (b_0^{-1}Y_1/T_{Y_1})b_1$*, but* $b_1 = b_0Y_2(\frac{T_{Y_1}}{Y_1}) \notin A$*. Thus* $f(X) = a \in U(A) = U(A[X])$*. We also have* $Y_1 + T_{Y_1}Y_2X = Z(Y_1/Z + T_{Y_1}(Y_2/Z)X)$*. All of the divisors of* Y_1/Z *are of the forms* $Z^n, T_{Y_1/Z^n}$*, and* $(Y_1/Z^n)/T_{Y_1/Z^n}$ *for* $n \geq 0$*. Thus the only common divisors of* Y_1/Z *and* $T_{Y_1}(Y_2/Z)$ *are of the form* Z^n*. Hence* $Y_1/Z + T_{Y_1}(Y_2/Z)X$ *cannot be factored as a finite product of atoms. Thus* $Z(Y_1/Z + T_{Y_1}(Y_2/Z)X) = Y_1 + T_{Y_1}Y_2X \in \mathcal{A}(A[X])$*. But* $Y_1/Z + T_{Y_1}(Y_2/Z)X \notin \mathcal{A}(A[X])$*. So* $\mathcal{A}(A[X])$ *is not saturated in* $A[X]$*.*

We also have that $\mathcal{A}(A[[X]])$ *is not saturated in* $A[[X]]$*. First,* $Y_1 + T_{Y_1}Y_2X = T_{Y_1}(Y_1/T_{Y_1} + Y_2X)$ *is a product of atoms. Again,* T_{Y_1} *is an atom in both* A *and* $A[[X]]$*, and* Y_1/T_{Y_1} *is an atom in* A*. Then for* $f(X), g(X) \in A[[X]]$ *such that* $Y_1/T_{Y_1} + Y_2X = f(X)g(X)$*, we have* $f(0)g(0) = Y_1/T_{Y_1}$*. Thus* $f(0)$ *or* $g(0)$ *is a unit in* A *and hence* $f(X)$ *or* $g(X)$ *is a unit in* $A[[X]]$*. On the other hand, we have* $Y_1 + T_{Y_1}Y_2X = Z(Y_1/Z + T_{Y_1}(Y_2/Z)X)$*. We will show that any finite factorization of* $Y_1/Z + T_{Y_1}(Y_2/Z)X$ *in* $A[[X]]$ *has at least one factor that is divisible by* Z^n *for all* $n \geq 1$*. Assume* $Y_1/Z + T_{Y_1}(Y_2/Z)X = f_1(X)\cdots f_k(X)$ *for* $f_i(X) \in A[[X]]$ *and* $k \geq 2$*. Then* $Y_1/Z = f_1(0)\cdots f_k(0)$*. Let* $\mathbf{T}$ *be the set of all indeterminates used in the definition of* A *and* L *be the quotient field of* $k[Z, \mathbf{T}]$*. Then* $A \subseteq L[Y_1, Y_2]$*, and the set* $\{Y_1/Z, T_{Y_1}(Y_2/Z)\}$ *is algebraically independent over* L*. Thus* $f_1(0) = l(Y_1/Z)$ *for some* $l \in L$ *and* $f_i(0) \in L$ *for* $2 \leq i \leq n$*. Define* $g(X) = f_2(X)\cdots f_k(X)$*. Let* $f_1(X) = \sum_{n=0}^{\infty} a_nX^n$*, where* $a_0 = lY_1/Z$*. Then* $g(X) = \sum_{n=0}^{\infty} b_nX^n$*, where* $b_0 = f_2(0)\cdots f_k(0) \in A$ *since* $f_i(X) \in A[[X]]$ *for* $1 \leq i \leq k$*. It follows that* $Y_1/Z = f_1(0)g(0) = l(Y_1/Z)b_0$ *and* $T_{Y_1}(Y_2/Z) = l(Y_1/Z)b_1 + a_1b_0$*. Let* $\varphi : L[Y_1, Y_2] \to L[Y_2]$ *be the evaluation homomorphism defined by* $\varphi(Y_1) = 0$*. Then* $T_{Y_1}(Y_2/Z) = \varphi(T_{Y_1}(Y_2/Z)) = \varphi(l(Y_1/Z)b_1 + a_1b_0) = b_0\varphi(a_1)$*. Thus* b_0 *divides both* $Y_1/Z = l(Y_1/Z)b_0 = f_1(0)b_0$ *and* $T_{Y_1}(Y_2/Z) = b_0\varphi(a_1)$ *in* A *since* $f_i(X) \in A[[X]]$ *for* $1 \leq i \leq n$ *and* $\varphi(A) \subseteq A$*.*

Let $R' = R[T_{Y_1}, Y_1/T_{Y_1}] = k[Z, \{\frac{Y_1}{Z^n}, \frac{Y_2}{Z^n} \mid n \geq 0\}, T_{Y_1}, Y_1/T_{Y_1}]$*, and let* A' *be the integral domain constructed using Roitman [17] to make all reducible elements the* *product of two atoms. Note that $A' = A$, so Lemma 4.3.1 holds for A using R'. Thus there exists $r = r_{b_0} \in R'$ such that r divides both $Y_1/Z = l(Y_1/Z)b_0$ and $T_{Y_1}(Y_2/Z) = b_0\varphi(a_1)$ in A with r divisible by b_0 in A. But the only common divisors of $Y_1/Z = l(Y_1/Z)b_0$ and $T_{Y_1}(Y_2/Z) = b_0\varphi(a_1)$ in R' are the nonnegative powers of Z, up to associates. Hence b_0 divides some power Z^m in A and so we may assume $b_0 = Z^m$ for some $m \geq 0$. Now we show that $Z^i \mid f_1(X) = \sum_{n=0}^{\infty} a_n X^n$ by showing $Z^i \mid a_n$ in A for all $i \geq 0$ by induction on n. The $n = 0$ case holds since $Z_i \mid l(Y_1/Z)b_0 = a_0 Z^m$ for all $i \geq 1$. Let $n \geq 1$. We have shown the coefficient of X^n in $Y_1/Z + T_{Y_1}((Y_2/Z)X) = f_1 g$ is divisible in A by Z^i for all i. But the coefficient of X^n is $b_0 a_n + b_1 a_{n-1} + \cdots + b_{n-1}a_1 + b_n a_0$. Hence by the induction hypothesis $Z^i \mid b_0 a_n = Z^m a_n$ for all i. It follows that a_n is divisible by all powers of Z. Thus $f_1(X)$ is not an atom in $A[[X]]$. But then $\mathcal{A}(A[[X]])$ is not saturated in $A[[X]]$ since $Z(Y_1/Z + T_{Y_1}(Y_2/Z)X) = Y_1 + T_{Y_1}Y_2X \in \mathcal{A}(A[[X]])$ and $Y_1/Z + T_{Y_1}(Y_2/Z)X \notin \mathcal{A}(A[[X]])$.*

It is not always the case that $\mathcal{A}(R)$ is saturated in R, but $\mathcal{A}(R[X])$ is not saturated in $R[X]$. In many cases, is is true that $\mathcal{A}(R)$ is saturated in R and $\mathcal{A}(R[X])$ is saturated in $R[X]$, for example when R is Noetherian or satisfies ACCP. We give an example where this relationship holds, even though R is not atomic.

Example 4.3.3. *Let R be a valuation domain with no atoms. Then $\mathcal{A}(R) = U(R)$ is saturated in R. Also $R[X]$ is a GCD-domain by [13, p. 42] since a valuation domain is a GCD-domain. Note $U(R) \subsetneq \mathcal{A}(R[X])$, in particular $X \in \mathcal{A}(R[X]) \setminus U(R)$. Since $R[X]$ is a GCD-domain, if $f \in R[X]$ is an atom, then f is prime. Thus $\mathcal{A}(R[X])$ is also the set generated by the prime elements of $R[X]$. Hence $\mathcal{A}(R[X])$ is saturated in $R[X]$ by Proposition 1.0.5.*

We conclude this chapter with various examples of integral domains and their multiplicatively closed sets generated by the atoms and units. First we look at localizations. The next example shows that $\mathcal{A}(R)$ can be saturated in R while $\mathcal{A}(R_S)$ is not saturated in R_S.

Example 4.3.4. *Let* $R = \mathbb{Z} + X\mathbb{Z}[1/3] + X^2\mathbb{Q}[[X]]$*. We have shown that* $\mathcal{A}(R)$ *is saturated in* R*. Let* S *be the multiplicative set generated by* $\{\pm 1, \pm 3\}$*. Then* $R_S = \mathbb{Z}[1/3] + X\mathbb{Z}[1/3] + X^2\mathbb{Q}[[X]]$*. We know* $\mathcal{A}(\mathbb{Z}[1/3] + X\mathbb{Z}[1/3] + X^2\mathbb{Q}[[X]])$ *is not saturated in* R_S *by Theorem 4.1.5.*

The next examples look at the relationship between $\mathcal{A}(A) \cap \mathcal{A}(B)$ and $\mathcal{A}(A \cap B)$, for integral domains A and B We start with an example that shows there may exist atoms in $A \cap B$ that are not atoms in A or B and conversely there exist atoms in A or B which are not atoms in $A \cap B$.

Example 4.3.5. *Let* $A = \mathbb{Z}[\frac{1}{3}]$ *and* $B = \mathbb{Z}[\frac{1}{5}]$*. Then* $A \cap B = \mathbb{Z}$*. We know that* 3 *and* 5 *are atoms in* $\mathbb{Z}$*, but* 3 *is a unit in* A *and* 5 *is a unit in* B*. It follows that* $15 = 3(5)$ *is an atom in both* A *and* B*, but it is not an atom in* $A \cap B$*.*

The next is an example where both $\mathcal{A}(A)$ and $\mathcal{A}(B)$ are saturated, but $\mathcal{A}(A \cap B)$ is not saturated.

Example 4.3.6. *Let* $A = \mathbb{Z} + X\mathbb{Z}[\frac{1}{3}] + X^2\mathbb{Q}[X], B = \mathbb{Z} + X\mathbb{Z}[\frac{1}{5}] + X^2\mathbb{Q}[X]$*. Then* $A \cap B = \mathbb{Z} + X\mathbb{Z} + X^2\mathbb{Q}[X]$*. We have shown that both* $\mathcal{A}(A)$ *and* $\mathcal{A}(B)$ *are saturated by Corollary 4.2.5 and that* $\mathcal{A}(A \cap B)$ *is not saturated by Theorem 4.2.2.*

Finally, an example where $\mathcal{A}(A)$ and $\mathcal{A}(B)$ are not saturated, but $\mathcal{A}(A \cap B)$ is saturated.

Example 4.3.7. *Let* $A = \mathbb{Z}_2[X; \{\frac{2m}{3^n} + k \mid m, n, k \in \mathbb{Z}^+\}]$ *and* $B = \mathbb{Z}_2[X; \{\frac{2m}{5^n} + k \mid m, n, k \in \mathbb{Z}^+\}]$*. Then* $A \cap B = \mathbb{Z}_2[X]$*. We have shown that* $\mathcal{A}(A)$ *is not saturated, and similarly,* $\mathcal{A}(B)$ *is not saturated. But* $A \cap B$ *is a PID, and hence atomic, i.e.,* $\mathcal{A}(A \cap B) = (A \cap B) \setminus \{0\}$*. Thus* $\mathcal{A}(A \cap B)$ *is saturated.*

Chapter 5

$D + M$ construction

In this chapter, we will explore the structure of the multiplicatively closed set generated by the atoms and units of integral domains of the form $R = D + M$. In this construction, R is a subring of an integral domain $T = K + M$, where K is a subfield of T, M is a nonzero maximal ideal of T, and D is a subring of K. Note that $K \cap M = \emptyset$; so for every element $x \in T$, there exists a unique $k \in K$ and $m \in M$ such that $x = k + m$. When we write $x = d + m$ for $x \in R$, it is always assumed that $d \in D$ and $m \in M$. This construction has been classically studied when T is a valuation domain or an integral domain of the form $K[X]$ or $K[[X]]$, where K is a field and $M = XT$ is the nonzero maximal ideal generated by X. We have studied both the polynomial ring and power series ring cases in the previous chapter. Recall that for an integral domain A with quotient field K, we have $\mathcal{A}(A + XK[[X]])$ is saturated if and only if $\mathcal{A}(A)$ is saturated by Theorem 4.1.4. Also, for an atomic domain A, in which case $\mathcal{A}(A) = A \setminus \{0\}$ is saturated, then $\mathcal{A}(A + XK[[X]])$ and $\mathcal{A}(A + XK[X])$ are saturated by Theorem 4.1.2 and Theorem 4.2.1, respectfully. In the first section of this chapter, we will look at the generalization of these results by first considering the relationship between atoms in T and the corresponding elements in R. We then work to eliminate the atomic condition on A and discover a nice relationship between $\mathcal{A}(T)$, $\mathcal{A}(R)$, and $\mathcal{A}(D)$. The second section is devoted to the classical valuation domain case, where it suffices to focus on the atoms in R and D. We conclude this chapter with a section considering the pullbacks case.

5.1 General Construction

Let T be an integral domain of the form $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. In this section, we work toward showing the relationship between $\mathcal{A}(T)$, $\mathcal{A}(R)$, and $\mathcal{A}(D)$. These relationships are defined by the three types of elements of R, which are d, $d+m$, and m. When D is a field, $\mathcal{A}(D) = D \setminus \{0\}$ is saturated, and we will show that $\mathcal{A}(R)$ is saturated in R if and only if $\mathcal{A}(T)$ is saturated in T. On the other hand, when D is not a field, none of the elements of M are atoms in R since $m = d(m/d)$ for all $0 \neq d \in D$. Furthermore, when D is not a field, the elements of M cannot be factored into a finite product of atoms. In this case, we will show that $\mathcal{A}(R)$ is saturated in R precisely when $\mathcal{A}(D)$ is saturated in D and $\mathcal{A}(T) \setminus M$ is saturated in T. To prove these results, we start with a few technical propositions showing that there is a strong connection between when $d \in D$ is an atom in D and an atom in R. We also consider when an element $d + m \in R$ is an atom in R and an atom in T. We start by proving a few technical propositions, some of which follow from the fact that $D + M \subseteq K + M$ is an inert extension. Many of these technical propositions can be found throughout the literature in the $D + M$ context, but we include them here for completeness.

Proposition 5.1.1. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If $d + m \in R$ is a unit in R, then d is a unit in D. Moreover, if $m = 0$, then d is a unit in R if and only if d is a unit in D.*

Proof. Let $d+m \in U(R)$. Then there exists $a+n \in R$ such that $1 = (d+m)(a+n) = da + (ma + nd + mn)$ with $da \in D$ and $ma + nd + mn \in M$. It follows that $da = 1$ for $a, d \in D$. Hence $d \in U(D)$.

The "moreover" statement follows since $D \subseteq R$. $\square$

There are many examples where the converse fails when $m \neq 0$. For example, if $R = \mathbb{Z} + X\mathbb{Q}[X] \subseteq \mathbb{Q}[X]$, then 1 is a unit in $\mathbb{Z}$. But $1 + X$ is certainly not a unit in R. The next proposition considers when $d + m$ is an atom in R.

Proposition 5.1.2. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If $d + m$ is an atom in R and $d \neq 0$, then d is either an atom or a unit in D.*

Proof. Let $d+m$ be an atom in R. Then $d+m = d(1+m/d)$ with $d, 1+m/d \in R$, so either d or $1 + m/d$ is a unit in R. If d is a unit in R, then $d \in U(D)$ by Proposition 5.1.1. If $1 + m/d$ is a unit in R, then d is an atom in R. Let $a, b \in D$ be such that $d = ab$. Then we may assume $a \in U(R)$ since $a, b \in D \subseteq R$ and d is an atom in R. It follows from Proposition 5.1.1 that $a \in U(D)$. Thus d is an atom in D. $\square$

Both of the cases in Proposition 5.1.2 can occur. Again, consider the integral domain $R = \mathbb{Z} + X\mathbb{Q}[X]$. We have that $2 + X$ is an atom in R with 2 an atom in $\mathbb{Z}$. Also, $1+X$ is an atom in R with $1 \in U(D)$. Similar to the $m = 0$ case of Proposition 5.1.1, where d is a unit in D if and only if d is a unit in R, we have the following proposition.

Proposition 5.1.3. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. Then $d \in D$ is an atom in R if and only if d is an atom in D.*

Proof. We have shown in the proof of Proposition 5.1.2 that if $d \in D$ is an atom in R, then d is an atom in D. Conversely, assume d is an atom in D. Let $a+m, b+n \in R$ be such that $d = (a+m)(b+n) = ab+(an+bm+mn)$ with $ab \in D$ and $an+bm+mn \in M$.

Then $d = ab$ for $a, b \in D$; so $a, b \neq 0$ and we may assume $a \in U(D)$. Thus $a \in U(R)$ by Proposition 5.1.1. Also, $ab = d = (a + m)(b + n) = a(1 + m/a)b(1 + n/b) = ab(1+m/a)(1+n/b)$; so $1 = (1+m/a)(1+n/b)$ by cancellation. Then $1+m/a$ and $1 + n/b$ are units in R. It follows that $a + m = a(1 + m/a) \in U(R)$. Hence d is an atom in R. $\square$

We know that each atom in D is an atom in R and $U(D) \subseteq U(R)$. Then for $R = D+M$, it follows that $\mathcal{A}(D) \subseteq \mathcal{A}(R)$. The reverse inclusion is more complicated since $D \subsetneq R$. The next proposition gives the strongest result we could hope for.

Proposition 5.1.4. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If $d \neq 0$ and $d + m \in \mathcal{A}(R)$, then $d \in \mathcal{A}(D)$.*

Proof. Let $d + m \in \mathcal{A}(R)$. If $d + m \in U(R)$, then $d \in U(D) \subseteq \mathcal{A}(D)$ by Proposition 5.1.1. So we may assume $d+m \in \mathcal{A}(R)\backslash U(R)$. Write $d+m = (d_1+m_1)\cdots(d_k+m_k)$ for atoms $d_1+m_1, \ldots, d_k+m_k$ in R. Then each d_i is an atom or unit in D by Proposition 5.1.2. Also, there exists an $n \in M$ such that $d + m = (d_1 + m_1)\cdots(d_k + m_k) = d_1d_2\cdots d_k + n$, and it follows that $d = d_1 \cdots d_k$. Thus $d = d_1 \cdots d_k \in \mathcal{A}(D)$. $\square$

From these propositions, we can see that the elements of $\mathcal{A}(D)$ are heavily dependent on the elements of $\mathcal{A}(R)$. In fact, the next theorem shows that $\mathcal{A}(D)$ is saturated in D whenever $\mathcal{A}(R)$ is saturated in R.

Theorem 5.1.5. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If $\mathcal{A}(R)$ is saturated in R, then $\mathcal{A}(D)$ is saturated in D.*

Proof. Let $ab \in \mathcal{A}(D)$ with $a, b \in D$. Then $ab \in \mathcal{A}(R)$ by Proposition 5.1.3, with $a, b \in D \subseteq R$. So $a, b \in \mathcal{A}(R)$ by hypothesis. The result now follows from Proposition 5.1.4 since $a, b \in D$. $\square$

We show in the next example that the converse does not hold in general.

Example 5.1.6. *Let $S = \{\frac{2n}{3^m} + k \mid n, m, k \in \mathbb{Z}^+\}$ and $T = \mathbb{R}[X; S]$, where X is an indeterminate. Then $T = \mathbb{R} + M$, where M is the maximal ideal generated by $\{X^s\}_{s \in S}$. For $D = \mathbb{Q}$, we have $R = \mathbb{Q} + M$. We know that $D = \mathbb{Q}$ is a field, thus $\mathcal{A}(D) = D \setminus \{0\}$ is saturated. We now show that X is an atom in R. Let $a, b \in R$ be such that $X = ab$. Then a, b are monomials and $1 = \deg(ab) = \deg(a) + \deg(b)$. So we may assume $\deg(a) = 0$, and it follows that $a \in \mathbb{Q} \setminus \{0\}$ is a unit in R. Hence $X^2 = X \cdot X \in \mathcal{A}(R)$. On the other hand $X^2 = X^{2/3}X^{4/3}$, and we will show that $X^{2/3}$ does not factor as a finite product of atoms in R. First note that $X^{2m/3^n} = X^{2m/3^{n+1}}X^{4m/3^{n+1}}$ is not an atom for all $n \geq 0$, $m \geq 1$. Let $a_1, \ldots, a_k$ be atoms in R such that $X^{2/3} = a_1 \cdots a_k$. Then $2/3 = \deg(a_1 \cdots a_k) = \deg(a_1) + \cdots + \deg(a_k)$. Thus there exists $1 \leq i \leq k$ such that $0 < \deg(a_i) \leq 2/3$ since $\deg(f) \geq 0$ for all $f \in R$. It follows that $\deg(a_i) = 2m/3^n \leq 2/3$ for some $m \geq 1, n \geq 0$. Then a_i is not an atom by above. Thus $\mathcal{A}(R)$ is not saturated.*

Now we shift our focus to the relationship between $\mathcal{A}(R)$ and $\mathcal{A}(T)$. To this end, we consider elements of the form $k + m \in T$. Note that for $0 \neq k \in K$, $k + m = k(1 + m/k) \in T$, where $1 + m/k$ is an element in R. The next proposition shows that there is a nice relationship between atoms of this form in R and in T.

Proposition 5.1.7. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. For $m \in M$ and $0 \neq k \in K$, $k + m$ is an atom in T if and only if $1 + m/k$ is an atom in R.*

Proof. If $m = 0$, then $0 \neq k \in K$ is a unit in T and 1 is a unit in R. Thus we may assume $m \neq 0$.

($\Rightarrow$) Let $0 \neq k \in K$ and $m \in M$ be such that $k + m$ is an atom in T. Then $1 + m/k$ is an atom in T since $k + m = k(1 + m/k)$ with $k \in U(T)$. Write $1 + m/k = (d_1 + m_1)(d_2 + m_2) = d_1d_2 + (d_2m_1 + d_1m_2 + m_1m_2)$ for $d_1 + m_1, d_2 + m_2 \in R$. It follows that $d_1d_2 = 1$. Also, we may assume $d_1 + m_1 \in U(T)$ since $d_1 + m_1, d_2 + m_2 \in R \subseteq T$. So there exists $c + l \in T$ such that $1 = (d_1 + m_1)(c + l) = d_1c + (cm_1 + d_1l + m_1l)$ with $cm_1 + d_1l + m_1l \in M$. Consequently, $d_1d_2 = 1 = d_1c$; so $c = d_2 \in D$ by cancellation.

Hence $c + l = d_2 + l \in R$ and $d_1 + m_1 \in U(R)$. Thus $1 + m/k$ is an atom in R by definition.

($\Leftarrow$) Let $1 + m$ be an atom in R and $k_1 + m_1, k_2 + m_2 \in T$ be such that $1 + m = (k_1 + m_1)(k_2 + m_2)$. Then $1 + m = (k_1 + m_1)(k_2 + m_2) = k_1k_2 + (k_1m_2 + k_2m_1 + m_1m_2)$ with $k_1k_2 \in K$ and $k_1m_2 + k_2m_1 + m_1m_2 \in M$, and so $k_1k_2 = 1$. Note that $k_1, k_2 \neq 0$. On the other hand, $1 + m = (k_1 + m_1)(k_2 + m_2) = k_1k_2(1 + m_1/k_1)(1 + m_2/k_2) = (1 + m_1/k_1)(1 + m_2/k_2)$ with $1 + m_1/k_1, 1 + m_2/k_2 \in R$. Thus, we may assume $1 + m_1/k_1 \in U(R)$. It follows that $1 + m_1/k_1 \in U(T)$ since $R \subseteq T$. Hence $k_1 + m_1 = k_1(1 + m_1/k_1) \in U(T)$. Finally, if $1 + m$ is an atom in T, then $k + km$ is an atom in T for all $0 \neq k \in K$ since $K \setminus \{0\} \subseteq U(T)$. $\square$

Note that $1 + m$ is an atom in T if and only if it is an atom in R is a direct consequence of Proposition 5.1.7. Finally, we look at the elements of M, and consider when $m \in M$ is an atom in R and when it is an atom in T.

Proposition 5.1.8. *Let $T = K + M$, where K is a subfield and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If D is a field, then $m \in M$ is an atom in R if and only if m is an atom in T.*

Proof. ($\Rightarrow$) Let $m \in M$ be an atom in R and $k_1 + m_1, k_2 + m_2 \in T$ be such that $m = (k_1 + m_1)(k_2 + m_2)$. Then $m = (k_1 + m_1)(k_2 + m_2) = k_1k_2 + (k_1m_2 + k_2m_1 + m_1m_2)$; so we may assume $k_1 = 0$ and $k_2 \neq 0$. Also, $M \cap U(R) = \emptyset$; so $k_2 \neq 0$ since $m = m_1m_2$ is a nontrivial factorization in R. Therefore $m = m_1(k_2 + m_2) = k_2m_1(1 + m_2/k_2)$ with $k_2m_1, 1 + m_2/k_2 \in R$. Thus $1 + m_2/k_2 \in U(R) \subseteq U(T)$ by hypothesis. It follows that $k_2 + m_2 = k_2(1 + m_2/k_2) \in U(T)$, and so m is an atom in T by definition.

($\Leftarrow$) Let $m \in M$ be an atom in T and $d_1 + m_1, d_2 + m_2 \in R$ be such that $m = (d_1 + m_1)(d_2 + m_2)$. Then, without loss of generality, $d_1 + m_1 \in U(T)$ since $d_1 + m_1, d_2 + m_2 \in R \subseteq T$. So there exists $t + n \in T$ such that $1 = (d_1 + m_1)(t + n) = d_1t + (d_1n + tm_1 + m_1n)$. Thus $d_1t = 1$. It follows that $t \in D$ since D is a field and $t = d_1^{-1} \in D$. This implies $t + n \in R$, and so $d_1 + m_1 \in U(R)$. Thus m is an atom in R. $\square$

This result relies on the fact that D is a field. For when D is not a field, choose $0 \neq d \in D \setminus U(D)$. Then $m = d(m/d)$ is not an atom for all $m \in M$. Before considering the alternative case where D is not a field, we give the next theorem. We use Proposition 5.1.7 for elements of the form $d + m \in R$ and Proposition 5.1.8 for elements of the form $m \in M$. Note that for $d \neq 0$, we have $d + m = d(1 + m/d)$ is an atom in R if and only if it is an atom in T by Proposition 5.1.7 since D is a field.

Theorem 5.1.9. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If D is a field, then $\mathcal{A}(T)$ is saturated in T if and only if $\mathcal{A}(R)$ is saturated in R.*

Proof. ($\Rightarrow$) Let $ab \in \mathcal{A}(R)$ for $a, b \in R$. We may assume $ab \notin U(R)$ since $U(R) \subseteq \mathcal{A}(R)$. Then $ab = (d_1+n_1) \cdots (d_l+n_l)$ for atoms $d_1+n_1, \ldots, d_l+n_l$ of R. For $1 \leq i \leq l$, if $d_i \neq 0$, then $d_i + n_i$ is an atom in T by Proposition 5.1.7, and if $d_i = 0$, then m_i is an atom in T by Proposition 5.1.8. Hence $ab = (d_1 + n_1) \cdots (d_l + n_l) \in \mathcal{A}(T)$. Therefore $a, b \in \mathcal{A}(T)$ by hypothesis. Write $a = l_1 \cdots l_j(k_{j+1} + l_{j+1}) \cdots (k_n + l_n)$ for $l_j \in M$, $k_i + l_i \in T$ atoms of T with $k_i \neq 0$ for $j + 1 \leq i \leq n$. Then $a = l_1 \cdots l_j(k_{j+1} + l_{j+1}) \cdots (k_n + l_n) = k_{j+1} \cdots k_n l_1 \cdots l_j(1 + l_{j+1}/k_{j+1}) \cdots (1 + l_n/k_n) \in \mathcal{A}(R)$ when $j \geq 1$ since $k_{j+1} \cdots k_n l_1 \in M$ is an atom in T, and thus an atom in R by Proposition 5.1.8 and $1 + m_i/l_i$ is an atom in R by Proposition 5.1.7. When $j = 0$, we have $a = (k_1 + l_1) \cdots (k_n + l_n) = k_1 \cdots k_n + m$ for some $m \in M$. Then $k_1 \cdots k_n \in D \setminus \{0\} = U(D)$ since $a \in R \setminus M$. Thus $a = (k_1 + l_1) \cdots (k_n + l_n) = k_1 \cdots k_n(1 + l_1/k_1) \cdots (1 + l_n/k_n) \in \mathcal{A}(R)$ by Proposition 5.1.7. Similarly, $b \in \mathcal{A}(R)$.

($\Leftarrow$) Let $ab \in \mathcal{A}(T)$ for $a, b \in T$. We may assume $ab \notin U(T)$ since $U(T) \subseteq \mathcal{A}(T)$. Then there exist atoms $m_1, \ldots, m_j \in M$ and atoms $k_{j+1}+m_{j+1}, \ldots, k_n+m_n$ in T with $k_i \neq 0$ for $j+1 \leq i \leq n$ such that $ab = m_1 \cdots m_j(k_{j+1}+m_{j+1}) \cdots (k_n+m_n)$. We have shown above that $ab = m_1 \cdots m_j(k_{j+1}+m_{j+1}) \cdots (k_n+m_n) = k_{j+1} \cdots k_n m_1 \cdots m_j(1+ m_{j+1}/k_{j+1}) \cdots (1 + m_n/k_n) \in \mathcal{A}(R)$. Thus $a, b \in \mathcal{A}(R)$ by hypothesis. So there exist atoms $a_1, \ldots, a_l \in M$ and $d_{l+1} + a_{l+1}, \ldots, d_k + a_k$ atoms in R such that $a =$

$a_1 \cdots a_l(d_{l+1} + a_{l+1}) \cdots (d_k + a_k)$. Therefore $a \in \mathcal{A}(T)$ by Proposition 5.1.8 and Proposition 5.1.7. Similarly, $b \in \mathcal{A}(T)$. $\square$

The next example shows that it is necessary for D to be a field in the previous theorem. We have $\mathcal{A}(T)$ saturated in T, but $\mathcal{A}(R)$ is not saturated in R with D not a field.

Example 5.1.10. *Let $S = \{\frac{2n}{3^m} + k \mid n, m, k \in \mathbb{Z}^+\}$ and $D = \mathbb{Z}_3[X; S]$, where X is an indeterminate. Let K be the quotient field of D and $T = K[[Y]] = K + YK[[Y]]$. Then T is a PID, and hence atomic. Thus $\mathcal{A}(T) = T \setminus \{0\}$ is saturated. Let $R = D + YK[[Y]]$. In Example 1.0.9, we have shown that $\mathcal{A}(D)$ is not saturated in D. It follows from Theorem 5.1.5 that $\mathcal{A}(R)$ is not saturated in R.*

Now we consider the relationship between $\mathcal{A}(T)$, $\mathcal{A}(R)$, and $\mathcal{A}(D)$ when D is not a field.

Theorem 5.1.11. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. If T is atomic, then $\mathcal{A}(R)$ is saturated in R if and only if $\mathcal{A}(D)$ is saturated in D.*

Proof. When D is a field, we use [3, Proposition 1.2] which says that R is atomic if and only if T is atomic and D is a field. Then $\mathcal{A}(R) = R \setminus \{0\}$ and $\mathcal{A}(D) = D \setminus \{0\}$ are both saturated. For the remainder of the proof, we assume D is not a field.

We always have that $\mathcal{A}(R)$ saturated in R implies $\mathcal{A}(D)$ is saturated in D by Theorem 5.1.5. So it suffices to show that $\mathcal{A}(R)$ is saturated in R when $\mathcal{A}(D)$ is saturated in D and T is atomic.

Let $ab \in \mathcal{A}(R)$ for $a = d_1 + m_1, b = d_2 + m_2 \in R$. First we show that $d_1, d_2 \neq 0$. Let $m \in M \subseteq R$. Then $M \cap \mathcal{A}(R) = \emptyset$ since $m = d(m/d)$ for all $m \in M$ and $0 \neq d \in D$, in particular this is a nontrivial factorization for $0 \neq d \in D \setminus U(D)$. Thus $d_1 d_2 \neq 0$, and it follows that $d_1, d_2 \neq 0$. Then $ab = (d_1 + m_1)(d_2 + m_2) = d_1 d_2 + m$ for $m = d_1 m_2 + d_2 m_1 + m_1 m_2 \in M$. Hence $d_1 d_2 \in \mathcal{A}(D)$ by Proposition 5.1.4. Thus $d_1, d_2 \in \mathcal{A}(D)$ by hypothesis. The result follows from Proposition 5.1.7 since

$a = d_1 + m_1 = d_1(1 + m_1/d_1)$ with $d_1 \in \mathcal{A}(D) \subseteq \mathcal{A}(R)$, and $1 + m_1/d_1 \in T \setminus \{0\} = \mathcal{A}(T)$. $\square$

This theorem is a generalization of Theorem 4.1.4 which said for an integral domain A with quotient field K and $R = A + XK[[X]]$, we have $\mathcal{A}(A)$ is saturated in A if and only if $\mathcal{A}(R)$ is saturated in R. Recall that Theorem 4.1.4 was our attempt to eliminate the atomic condition from the integral domain A. In Theorem 5.1.11, we do not have the atomic condition on D, which is the analog for A. Instead, we have imposed the atomic condition on T. To completely eliminate the atomic hypothesis, we are forced to include a hypothesis concerning atoms of the form $1 + m$ since all atoms in R are associates of atoms of the form $d \in D$ or $1 + m \in T$. We will also require that D is not a field so that $m \in M$ factors as $m = d(m/d)$, and thus is not an atom in R.

Theorem 5.1.12. *Let $T = K + M$ be an integral domain, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subring of K that is not a field and $R = D + M$. Then $\mathcal{A}(R)$ is saturated in R if and only if $\mathcal{A}(D)$ is saturated in D and $\mathcal{A}(T) \setminus M$ is saturated in T.*

Proof. Let $0 \neq d \in D \setminus U(D)$. Then $m = d(m/d)$ for all $m \in M$. Hence M contains no atoms of R. Also, note that $\mathcal{A}(T) \setminus M$ is multiplicatively closed since M is a prime ideal of T.

($\Rightarrow$) First, $\mathcal{A}(R)$ saturated in R implies $\mathcal{A}(D)$ is saturated in D by Theorem 5.1.5. So it suffices to show $\mathcal{A}(T) \setminus M$ is saturated in T. Let $ab \in \mathcal{A}(T) \setminus M$, for $a = k_a + m_a, b = k_b + m_b \in T$. If $k_a = 0$, then $ab = m_a(k_b + m_b) \in M$. Thus $k_a, k_b \neq 0$ since $ab \notin M$; so $k_a^{-1}a = 1 + m_a/k_a, k_b^{-1}b = 1 + m_b/k_b \in R$. Write $ab = (k_1 + m_1) \cdots (k_n + m_n)$ for atoms $k_1 + m_1, \ldots, k_n + m_n$ in T and $k_1, \ldots, k_n \in K \setminus \{0\}$. Then $k_a k_b + (k_a m_b + k_b m_a + m_a m_b) = ab = k_1 \cdots k_n + m$ for some $m \in M$. It follows that $k_a k_b = k_1 \cdots k_n$ and $(k_a^{-1}a)(k_b^{-1}b) = k_a^{-1}k_b^{-1}ab = (k_a k_b)^{-1}(k_1 + m_1) \cdots (k_n + m_n) = (k_a k_b)^{-1} k_1 \cdots k_n (1 + m_1/k_1) \cdots (1 + m_n/k_n) = (1 + m_1/k_1) \cdots (1 + m_n/k_n)$ with $1 + m_1/k_1, \ldots, 1 + m_n/k_n$ atoms in R for $1 \leq i \leq n$ by Proposition 5.1.7.

Hence $(k_a^{-1}a)(k_b^{-1}b) \in \mathcal{A}(R)$, and so $k_a^{-1}a, k_b^{-1}b \in \mathcal{A}(R)$ by hypothesis. Then there exist atoms $d_1 + n_1, \ldots, d_l + n_l \in R$ such that $k_a^{-1}a = (d_1 + n_1) \cdots (d_l + n_l)$. Hence $1 + m_a/k_a = k_a^{-1}a = (d_1 + n_1) \cdots (d_l + n_l) = d_1 \cdots d_l + n$ for some $n \in N$ and it follows that $1 = d_1 \cdots d_l$. For $1 \leq i \leq l$, we have $1 + n_i/d_i$ is either an atom or a unit in R since $d_i + n_i = d_i(1 + n_i/d_i)$ is an atom in R. If $1 + n_i/d_i$ is an atom in R, then it is also an atom in T by Proposition 5.1.7. On the other hand, if $1 + n_i/d_i$ is a unit in R, then it is a unit in T since $R \subseteq T$. Thus $k_a^{-1}a \in \mathcal{A}(T) \setminus M$, and it follows that $a = k_a(k_a^{-1}a) \in \mathcal{A}(T) \setminus M$ since $k_a \in U(T) \subseteq \mathcal{A}(T) \setminus M$. Similarly, $b \in \mathcal{A}(T) \setminus M$.

($\Leftarrow$) Let $ab \in \mathcal{A}(R)$ with $a = d_a + m_a, b = d_b + m_b \in R$. Then $ab \notin M$ and $d_a d_b \neq 0$ since D is not a field. Hence $d_a, d_b \neq 0$. It follows from Proposition 5.1.4 that $d_a d_b \in \mathcal{A}(D)$ since $d_a d_b + (d_a m_b + d_b m_a + m_a m_b) = ab \in \mathcal{A}(R)$. Hence $d_a, d_b \in \mathcal{A}(D)$. There also exist atoms $d_1 + m_1, \ldots, d_n + m_n$ in R such that $d_a d_b + (m_a d_b + m_b d_a + m_a m_b) = ab = (d_1 + m_1) \cdots (d_n + m_n) = d_1 \cdots d_n + m$ for some $m \in M$. Then $d_a d_b = d_1 \cdots d_n$; so $d_1, \ldots, d_n \neq 0$ and $(d_a^{-1}a)(d_b^{-1}b) = d_a^{-1}d_b^{-1}(d_1 + m_1) \cdots (d_n + m_n) = (d_a d_b)^{-1} d_1 \cdots d_n (1 + m_1/d_1) \cdots (1 + m_n/d_n) = (1 + m_1/d_1) \cdots (1 + m_n/d_n)$. Furthermore, for $1 \leq k \leq n$, we know $1 + m_k/d_k$ is either an atom or a unit in R since $d_i + m_i = d_i(1 + m_i/d_i)$ is an atom in R. It follows that $1 + m_i/d_i$ is either an atom in T by Proposition 5.1.7 or a unit in T since $R \subseteq T$. Thus $(d_a^{-1}a)(d_b^{-1}b) \in \mathcal{A}(T)$, and so $d_a^{-1}a, d_b^{-1}b \in \mathcal{A}(T)$ by hypothesis. Write $d_a^{-1}a = (k_1 + n_1) \cdots (k_l + n_l)$ for atoms $k_1 + n_1, \ldots, k_l + n_l \in T$ with $k_1, \ldots, k_l \in K \setminus \{0\}$ since $a \notin M$. Then $1 + m_a/d_a = d_a^{-1}a = (k_1 + n_1) \cdots (k_l + n_l) = k_1 \cdots k_l + n$ for some $n \in M$; so we may assume $k_1 \cdots k_l = 1$. Also, $1 + m_1/k_1, \ldots, 1 + m_l/k_l$ are atoms in T and hence atoms in R by Proposition 5.1.7. Thus $d_a^{-1}a \in \mathcal{A}(R)$. We have shown that $d_a \in \mathcal{A}(R)$, and so $a = d_a(d_a^{-1}a) \in \mathcal{A}(R)$. Similarly, $b \in \mathcal{A}(R)$. $\square$

Note that this theorem applies to the polynomial ring $R = A + XK[X]$, where K is the quotient field of A and $A \neq K$. Recall that $\mathcal{A}(A + XK[X]) = \{ua_1 \cdots a_k(1 + Xf_1(X)) \cdots (1 + Xf_n(X)) \mid u \in U(R) = U(A), a_i \text{ atom in } A, 1 + Xf_j(X) \text{ atom in } K[X] \text{ with } f_j(X) \in K[X]\}$ from Theorem 4.2.1. As in the previous theorem, all of the products of atoms can be simplified into products of atoms of the forms $a \in A$ and $1 + m \in K[X]$.

To simplify the $D + M$ construction, it can be assumed that T is quasilocal with unique maximal ideal M. In this setting, the previous theorem can be strengthened in the following way.

Corollary 5.1.13. *Let $T = K + M$ be a quasilocal integral domain, where K is a subfield of T and M is the unique maximal ideal of T. Let D be a subring of K that is not a field and $R = D + M$. Then $\mathcal{A}(R)$ is saturated in R if and only if $\mathcal{A}(D)$ is saturated in D.*

Proof. First note that $M \cap U(R) = \emptyset$ and $1 + m \in U(T)$ for all $m \in M$ since T is quasilocal. Then $U(T) \subseteq \mathcal{A}(T) \setminus M \subseteq U(T)$. It follows that $\mathcal{A}(T) \setminus M = U(T)$ is saturated. The result follows from Theorem 5.1.12 since D is not a field. $\square$

The next example shows that $\mathcal{A}(T) \setminus M$ need not be saturated.

Example 5.1.14. *Let $S = \{\frac{2n}{3^m} + k \mid n, m, k \in \mathbb{Z}^+\}$ and $T = \mathbb{Z}_3[X; S]$, where X is an indeterminate. Then $T = \mathbb{Z}_3 + M$, where M is the ideal generated by $\{X^s\}_{s \in S}$. So M is a nonzero maximal ideal and $T = K + M$ for $K = \mathbb{Z}_3$. We first show that $1 + X$ is an atom. Let $a = a_0 + a_1X^{s_1} + \cdots + a_nX^{s_n}, b = b_0 + b_1X^{t_1} + \cdots b_kX^{t_k} \in T$ be such that $0 \neq a_n, b_k \in K$, $0 < s_1 < \cdots < s_n$, $0 < t_1 < \cdots < t_k$, and $1 + X = ab = a_0b_0 + \cdots + a_nb_kX^{s_n+t_k}$. Then $a_nb_k \neq 0$, so $a_nb_kX^{s_n+t_k} = X$. Thus $s_n + t_k = 1$, and so we may assume $s_n = 1$ and $t_k = 0$. Hence $b = b_0 \in K$ is a unit in T and $1 + X$ is an atom in T. Similarly $1 - X$ is an atom in T. Thus $1 - X^2 = (1 + X)(1 - X) \in \mathcal{A}(T) \setminus M$. On the other hand, $1 - X^2 = (1 - X^{2/3})(1 + X^{2/3} + X^{4/3})$. We show $1 - X^{2/3} \notin \mathcal{A}(T) \setminus M$. Assume there exist atoms $a_1 = r_{0,a_1} + \cdots + r_{n_1,a_1}X^{s_{n_1,a_1}}, \ldots, a_k = r_{0,a_k} + \cdots + r_{n_k,a_k}X^{s_{n_k,a_k}} \in T$ with $0 < s_{1,a_i} < s_{2,a_i} < \cdots < s_{n_i,a_i}$ and $r_{n_i,a_i} \neq 0$ for each i such that $1 - X^{2/3} = a_1 \cdots a_k = r_{0,a_1} \cdots r_{0,a_k} + \cdots + r_{n_1,a_1} \cdots r_{n_k,a_k}X^{s_{n_1,a_1}+\cdots+s_{n_k,a_k}}$. Then $2/3 = s_{n_1,a_1} + \cdots + s_{n_k,a_k}$. Thus $s_{n_i,a_i} \leq 2/3$ for each i and there exists $1 \leq i \leq k$ such that $s_{j,a_i} < 2/3$ for*

$1 \leq j \leq n_i$. *Then* $\frac{s_{j,a_i}}{3} \in S$ *for each* j. *Also, for each* r_{j,a_i} *there exists* $b_j \in \mathbb{Z}_3$ *such that* $b_j^3 = r_{j,a_i}$. *Thus* $a_i = b_0^3 + \cdots + b_{n_i}^3 X^{s_{n_i,a_i}} = (b_0 + \cdots + b_{n_i} X^{\frac{s_{n_i,a_i}}{3}})^3$ *is not an atom since* $b_0 + \cdots + b_{n_i} X^{\frac{s_{n_i,a_i}}{3}}, (b_0 + \cdots + b_{n_i} X^{\frac{s_{n_i,a_i}}{3}})^2 \notin \mathbb{Z}_3 = U(T)$. *Hence* $\mathcal{A}(T) \setminus M$ *is not saturated in* T.

In the previous example, we actually have that $\mathcal{A}(T)$ is not saturated in T as well. We show that this will always be the case.

Theorem 5.1.15. *Let* $T = K + M$ *be an integral domain, where* K *is a subfield of* T *and* M *is a nonzero maximal ideal of* T. *Let* D *be a subring of* K *and* $R = D + M$. *If* $\mathcal{A}(T)$ *is saturated in* T, *then* $\mathcal{A}(T) \setminus M$ *is saturated in* T.

Proof. Let $ab \in \mathcal{A}(T) \setminus M$ for $a, b \in T$. Then $a, b \in \mathcal{A}(T)$ since $\mathcal{A}(T) \setminus M \subseteq \mathcal{A}(T)$ and $\mathcal{A}(T)$ is saturated in T. If $a \in M$, then $ab \in M$ since M is an ideal. It follows that $a, b \notin M$. Thus $a, b \in \mathcal{A}(T) \setminus M$. □

5.2 The Classical Valuation Domain Case

We now consider the classical valuation domain case. In this case, M is the unique maximal ideal of T since valuation domains are quasilocal. Our first result follows naturally from Theorem 5.1.12.

Theorem 5.2.1. *Let* $T = K + M$ *be an integral domain, where* K *is a subfield of* T *and* M *is a nonzero maximal ideal of* T. *Let* D *be a subring of* K *and* $R = D + M$. *If* T *is a valuation domain, then* $\mathcal{A}(R)$ *is saturated in* R *if and only if* $\mathcal{A}(D)$ *is saturated in* D.

Proof. We know that $\mathcal{A}(T)$ is saturated in T by Proposition 1.0.5 since $\mathcal{A}(T) = \mathcal{P}(T)$. Then $\mathcal{A}(T) \setminus M$ is saturated by Theorem 5.1.15. So the result follows from Theorem 5.1.12. □

We have already looked at a special case of Theorem 5.2.1 with power series rings. Recall Theorem 4.1.4, which states that for an integral domain A with quotient field

K, $A \neq K$, and $R = A + XK[[X]]$, we have $\mathcal{A}(A)$ is saturated in A if and only if $\mathcal{A}(R)$ is saturated in R. Note that in this case $T = K[[X]]$, $D = A$, and $M = XK[[X]]$.

5.3 Pullbacks

Pullbacks are a generalization of the $D+M$ construction, where we let T be an integral domain with nonzero maximal ideal M and residue field $K = T/M$. Define $\varphi : T \to K$ to be the natural projection. Then for a subring D of K, let $R = \varphi^{-1}(D) \subseteq T$ be the pullback of D in T. We first consider the case where T is quasilocal with unique maximal ideal M. Concluding this chapter, we give examples showing little can be said in the general case.

Proposition 5.3.1. *Let T be a quasilocal integral domain with unique maximal ideal M, residue field $K = T/M$, $\varphi : T \to K$ the natural projection, D a subring of K, and $R = \varphi^{-1}(D)$. If $a \in R$ is an atom in R, then a is either an atom or a unit in T.*

Proof. Suppose a is an atom in R such that $a \in R \setminus U(T)$. Then $a \in M$ since T is quasilocal. Let $x, y \in T$ be such that $a = xy$. Assume $x, y \notin U(T)$. Then $x, y \in M \subseteq R \setminus U(R)$. This is a contradiction since a is an atom in R. Thus either $x \in U(T)$ or $y \in U(T)$, and a is an atom in T. □

Both of the cases in Proposition 5.3.1 can occur. Let $T = \mathbb{C}[[X]] = \mathbb{C} + X\mathbb{C}[[X]]$ and $\varphi : T \to \mathbb{C}$ be the evaluation map $\varphi(X) = 0$. Let $D = \mathbb{R}$ and $R = \varphi^{-1}(D) = \mathbb{R} + X\mathbb{C}[[X]]$. Then $X \in R$ is an atom in both R and T. For the other case, let $T = \mathbb{C}[[X]]$ and $\varphi : T \to \mathbb{C}$ be as above. Let $D = \mathbb{Z}$. Then $R = \varphi^{-1}(D) = \mathbb{Z} + X\mathbb{C}[[X]]$. Now 2 is an atom in R, but a unit in T.

When D is a field, we get a stronger result that mirrors Proposition 5.1.8, which says that $m \in M$ is an atom in $R = D + M$ if and only if m is an atom in $T = K + M$.

Proposition 5.3.2. *Let T be a quasilocal integral domain with unique maximal ideal M, residue field $K = T/M$, $\varphi : T \to K$ the natural projection, D a subring of K,* *and $R = \varphi^{-1}(D)$. If D is a field, then $a \in R$ is an atom in T if and only if a is an atom in R.*

Proof. Note that D is a field, so we have $U(R) = U(T) \cap R$ by [5, Lemma 6.2].

($\Rightarrow$) Let a be an atom in T. Then $a \in M$ since T is quasilocal. Notice $M \subseteq R \setminus U(R)$; so $a \notin U(R)$. Let $a = xy$ for some $x, y \in R$. Assume that $x, y \notin U(R) = U(T) \cap R$. It follows that $x, y \in M$ since $M = T \setminus U(T)$. This is a contradiction since a is an atom in T. Hence $x \in U(R)$ or $y \in U(R)$.

($\Leftarrow$) Let a be an atom in R. Then $a \notin U(R) = U(T) \cap R$. So $a \notin U(T)$ and thus is a atom in T by Proposition 5.3.1. $\square$

For T quasilocal, the previous proposition implies that $\mathcal{A}(R) = \mathcal{A}(T) \cap R$ when D is a field since $U(R) = U(T) \cap R$ from [5]. This relationship allows us to show $\mathcal{A}(R)$ is saturated in R precisely when $\mathcal{A}(T)$ is saturated in T.

Theorem 5.3.3. *Let T be a quasilocal integral domain with unique maximal ideal M, residue field $K = T/M$, $\varphi : T \to K$ the natural projection, D a subring of K, and $R = \varphi^{-1}(D)$. If D is a field, then $\mathcal{A}(R)$ is saturated in R if and only if $\mathcal{A}(T)$ is saturated in T.*

Proof. ($\Rightarrow$) Let $xy \in \mathcal{A}(T)$ for $x, y \in T$. Then we may assume $x, y \notin U(T)$ since $U(T) \subseteq \mathcal{A}(T)$. Hence $x, y \in M \subseteq R$. We know $\mathcal{A}(R) = \mathcal{A}(T) \cap R$, and it follows that $xy \in \mathcal{A}(R)$. Therefore $x, y \in \mathcal{A}(R)$ by hypothesis. Thus $x, y \in \mathcal{A}(R) \subseteq \mathcal{A}(T)$, and so $\mathcal{A}(T)$ is saturated in T.

($\Leftarrow$) Let $xy \in \mathcal{A}(R)$ for $x, y \in R$. Then $xy \in \mathcal{A}(R) \subseteq \mathcal{A}(T)$ and $x, y \in R \subseteq T$. Thus $x, y \in \mathcal{A}(T)$ by hypothesis. It follows that $x, y \in \mathcal{A}(T) \cap R = \mathcal{A}(R)$. Hence $\mathcal{A}(R)$ is saturated in R. $\square$

We conclude this chapter with a few examples showing that the above results do not hold when T is not quasilocal. The first example shows that there is a pullback R where $\mathcal{A}(R)$ is saturated in R, but $\mathcal{A}(T)$ is not saturated in T.

Example 5.3.4. *Let* $T = \mathbb{Q}[\pi] + \mathbb{Q}[\pi]X + X^2\mathbb{R}[X]$ *and define* $\varphi : T \to \mathbb{C}$ *by* $\varphi(f(x)) = f(i)$*. Then* φ *is surjective since* $\varphi(-ax^2) = a$ *and* $\varphi(-ax^3) = ai$*. Let* $R = \varphi^{-1}(\mathbb{Q})$*. Then* $R \subseteq \mathbb{R}[X]$ *with* $U(R) = \mathbb{Q} \setminus \{0\} = R \cap U(\mathbb{R}[X])$*. It follows that* R *satisfies ACCP since* $\mathbb{R}[X]$ *is a UFD and thus satisfies ACCP by [12, Proposition 2.1]. Hence* R *is atomic and* $\mathcal{A}(R) = R \setminus \{0\}$ *is saturated in* R*. But* $\mathcal{A}(T)$ *is not saturated in* T *by Theorem 4.1.5.*

The next example is a pullback where $\mathcal{A}(T)$ is saturated in T, but $\mathcal{A}(R)$ is not saturated in R.

Example 5.3.5. *Let* $S = \{\frac{2n}{3^m} + k \mid n, m, k \in \mathbb{Z}^+\}$ *and* $D = \mathbb{Z}_3[X; S]$*, where* X *is an indeterminate. Let* K *be the quotient field of* D*,* $T = K[[Y]]$*, and* $\varphi : T \to K$ *be the surjective evaluation homomorphism defined by* $\varphi(Y) = 0$*. Then* $R = \varphi^{-1}(D) = \mathbb{Z}_3[X; S] + YK[[Y]]$*. We have shown in Example 5.1.10 that* $\mathcal{A}(T) = T \setminus \{0\}$ *is saturated in* T*, but* $\mathcal{A}(R)$ *is not saturated in* R*.*

In Example 5.1.6, we have that $\mathcal{A}(D)$ can be saturated in D while $\mathcal{A}(R)$ is not saturated in R. This example can be generalized to the pullbacks case where $\varphi : T \to \mathbb{R}$ is the evaluation map $\varphi(X) = 0$. So the final example in the pullbacks case is where $\mathcal{A}(R)$ is saturated in R, but $\mathcal{A}(D)$ is not saturated in D. Consider [5, Example 6.6 (a)] where R is atomic, D is not a field, and R is not dependent on D. In this example, we choose D such that $\mathcal{A}(D)$ is not saturated.

Example 5.3.6. *Let* K *be a field with subring* D*. Then for some set of indeterminates* $\{X_\alpha\}$ *and* $T = \mathbb{Z}[\{X_\alpha\}]$*, we have an epimorphism* $\varphi : T \to K$*. Let* $R = \varphi^{-1}(D)$*. Then* $U(R) = U(T) = \{\pm 1\}$*, and hence* $U(T) \cap R = U(R)$*. Thus* R *satisfies ACCP and hence is atomic for any subring* D *of* K*. It follows that* $\mathcal{A}(R)$ *is saturated.*

Chapter 6

Largest Integral Domain

We have looked at different properties of the set $\mathcal{A}(R)$ generated by the atoms and units of an integral domain R. In this chapter, we focus on the set $\mathcal{A}_R$ of atoms of R. It is possible for an extension $R \subseteq T$ of integral domains to share this set, i.e., $\mathcal{A}_R = \mathcal{A}_T$. For example, the integral domains $\mathbb{Q} + X\mathbb{C}[[X]] \subseteq \mathbb{R} + X\mathbb{C}[[X]] \subseteq \mathbb{C}[[X]]$ all have the same set of atoms. Notice that the group of units of each of these integral domains differ significantly. We explore the relationship between the atoms and the units of an integral domain, and define a set U_R in an attempt to determine the largest group of units for the set of atoms to remained unchanged.

6.1 Largest Set of Units

Let R be an integral domain with quotient field K and $\mathcal{A}_R \neq \emptyset$. Let $U_R = (\bigcap_{a \in \mathcal{A}_R} \left\{\frac{a}{b} \mid b \in \mathcal{A}_R\right\}) \cap (\bigcap_{b \in \mathcal{A}_R} \left\{\frac{a}{b} \mid a \in \mathcal{A}_R\right\})$. To simplify the notation for U_R, we will consider $U_1 = \bigcap_{a \in \mathcal{A}_R} \left\{\frac{a}{b} \mid b \in \mathcal{A}_R\right\}$ and $U_2 = \bigcap_{b \in \mathcal{A}_R} \left\{\frac{a}{b} \mid a \in \mathcal{A}_R\right\}$. Notice that $U_R = U_1 \cap U_2$. It is well-known that the set of units $U(R)$ for an integral domain R forms a group under multiplication. We show that U_R also has this property.

Proposition 6.1.1. *Let R be an integral domain with set of atoms $\mathcal{A}_R \neq \emptyset$. Let $U_R = (\bigcap_{a \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid b \in \mathcal{A}_R \right\}) \cap (\bigcap_{b \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid a \in \mathcal{A}_R \right\}) = U_1 \cap U_2$. Then U_R is a group under multiplication.*

Proof. First we show that U_R is closed under multiplication. Let $u, v \in U_R$ and $a \in \mathcal{A}_R$. Then there exists $b \in \mathcal{A}_R$ such that $u = \frac{a}{b}$ since $u \in U_1$. Also, there exists $c \in \mathcal{A}_R$ such that $v = \frac{b}{c}$ since $v \in U_1$. Multiplying, we get $uv = (\frac{a}{b})(\frac{b}{c}) = \frac{a}{c}$. Thus $uv \in U_1$. Similarly, $uv \in U_2$. It is clear that $1 \in U_R$ since $1 = a/a \in U_R$ for all $a \in \mathcal{A}_R$. Finally, we show U_R is closed under inverses. Let $u \in U_R$. We show $u^{-1} \in U_R$. Let $a \in \mathcal{A}_R$. Then there exist $b, c \in \mathcal{A}_R$ such that $u = \frac{a}{b}$ and $u = \frac{c}{a}$. Then $u^{-1} = \frac{a}{c} \in U_1$ and $u^{-1} = \frac{b}{a} \in U_2$. Thus $u^{-1} \in U_1 \cap U_2 = U_R$. Hence U_R is a group. □

Notice U_R is finding the units to have all possible associate relationships between the atoms. Our goal is to show that U_R is the largest group of units an integral domain could have if $\mathcal{A}_R$ is its set of atoms. We start by showing that the group of units of an integral domain T containing R with $\mathcal{A}_T = \mathcal{A}_R$ is contained in $U_R = (\bigcap_{a \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid b \in \mathcal{A}_R \right\}) \cap (\bigcap_{b \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid a \in \mathcal{A}_R \right\})$.

Proposition 6.1.2. *Let $R \subseteq T$ be integral domains with $\mathcal{A}_T = \mathcal{A}_R$. Then $U(T) \subseteq U_R = (\bigcap_{a \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid b \in \mathcal{A}_R \right\}) \cap (\bigcap_{b \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid a \in \mathcal{A}_R \right\}) = U_1 \cap U_2$.*

Proof. Let $u \in U(T)$ and $a \in \mathcal{A}_R$. Then $a \in \mathcal{A}_T$ and $ua, u^{-1}a \in \mathcal{A}_T = \mathcal{A}_R$. Thus $u = \frac{a}{u^{-1}a} \in U_1$ and $u = \frac{ua}{a} \in U_2$. Hence $u \in U_1 \cap U_2 = U_R$. □

We now consider U_R for different types of rings. First for a UFD R. All of the structure of a UFD ensures we have all possible associate relationships for atoms.

Theorem 6.1.3. *Let R be a UFD. Then $U_R = (\bigcap_{a \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid b \in \mathcal{A}_R \right\}) \cap (\bigcap_{b \in \mathcal{A}_R} \left\{ \frac{a}{b} \mid a \in \mathcal{A}_R \right\}) = U(R)$.*

Proof. We know $U(R) \subseteq U_R$ from Proposition 6.1.2. For the reverse inclusion, let $u \in U_R$ and $a \in \mathcal{A}_R$. Then there exist $x, y \in \mathcal{A}_R$ such that $u = \frac{a}{x}$ and $u = \frac{y}{a}$. It follows that $\frac{a}{x} = u = \frac{y}{a}$. Thus $a^2 = xy$. It follows that $a = wx$ and $a = vy$ for $v, w \in U(R)$ since R is a UFD. Therefore, $u = \frac{a}{x} = \frac{wx}{x} = w \in U(R)$. □

It would be nice if the converse of this theorem were true, and we could show that the largest integral domain with a given set of atoms is a UFD. However, as we show with the next example, it is possible for $U(R) = U_R$ when R is not a UFD.

Example 6.1.4. *Let $R = \mathbb{Z}_2[[X^2, X^3]]$. It is easy to check that the only atoms of R are $X^2, X^3, X^2 + X^3, X^3 + X^4$, up to associates. Also, R is not a UFD since $X^2 \cdot X^2 \cdot X^2 = X^3 \cdot X^3$. Now we calculate U_R by looking at the quotients of atoms, up to associates.*

Let $u \in U(R)$. Then $U_1 = \bigcap_{a \in \mathcal{A}_R} \{\frac{a}{b} \mid b \in \mathcal{A}_R\} = \{\frac{uX^2}{X^2} = u, \frac{uX^2}{X^3} = \frac{u}{X}, \frac{uX^2}{X^2+X^3} = \frac{u}{1+X}, \frac{uX^2}{X^3+X^4} = \frac{u}{X+X^2}\} \cap \{\frac{uX^3}{X^2} = uX, \frac{uX^3}{X^3} = u, \frac{uX^3}{X^2+X^3} = \frac{uX}{1+X}, \frac{uX^3}{X^3+X^4} = \frac{u}{1+X}\} \cap \{\frac{u(X^2+X^3)}{X^2} = u(1+X), \frac{u(X^2+X^3)}{X^3} = \frac{u(1+X)}{X}, \frac{u(X^2+X^3)}{X^2+X^3} = u, \frac{u(X^2+X^3)}{X^3+X^4} = \frac{u}{X}\} \cap \{\frac{u(X^3+X^4)}{X^2} = u(X + X^2), \frac{u(X^3+X^4)}{X^3} = u(1 + X), \frac{u(X^3+X^4)}{X^2+X^3} = uX, \frac{u(X^3+X^4)}{X^3+X^4} = u\}$ This intersection is $\{u\}$. So $U_1 = U(R)$.

$U_2 = \bigcap_{b \in \mathcal{A}_R} \{\frac{a}{b} \mid a \in \mathcal{A}_R\}$ Note that we are just looking at the reciprocals of the elements in the above sets. $U_2 = \{u, uX, u(1 + X), u(X + X^2)\} \cap \{\frac{u}{X}, u, \frac{u(1+X)}{X}, u(1 + X)\} \cap \{\frac{u}{1+X}, \frac{uX}{1+X}, u, uX\} \cap \{\frac{u}{X+X^2}, \frac{u}{1+X}, \frac{u}{X}, u\}$ The intersection is also $\{u\}$; so $U_2 = U(R)$

The calculation in the previous example also shows that $U_1 = U_2$. It would be helpful if this relationship always held. We show in the next theorem that it does when there are only finitely many atoms, up to associates.

Theorem 6.1.5. *Let R be an integral domain with only finitely many atoms, up to associates. Then $U_1 = U_2$.*

Proof. Let $A = \{a_1, \ldots, a_n\}$ be a maximal set of distinct nonassociate atoms. Then $\bigcap_{b \in A} \{a/b \mid a \in \mathcal{A}_R\} = U_2$ since $\{a/b \mid b \in \mathcal{A}_R\} = \{a/(ub) \mid b \in \mathcal{A}_R\}$ for $u \in U(R)$. Let $x \in U_1$. Then there exist $b_1, b_2, \ldots, b_n \in \mathcal{A}_R$ such that $x = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$. If $b_i = ub_j$ for $i \neq j$ and $u \in U(R)$, then $\frac{a_j}{b_j} = \frac{a_i}{b_i} = \frac{a_i}{ub_j}$. It follows that $ua_j = a_i$. This is a contradiction. Thus $\{b_1, \ldots, b_n\}$ is a maximal set of distinct (nonassociate) atoms. Hence for $b \in A$, there exists $a \in \mathcal{A}_R$ such that $x = \frac{a}{b}$, i.e., $x \in U_2$. Therefore, $U_1 \subseteq U_2$, and a similar argument shows $U_2 \subseteq U_1$. $\square$

Unfortunately, this theorem does not hold when there are infinitely many nonassociate atoms.

Example 6.1.6. *Let M be the submonoid of $\mathbb{Z} \times \mathbb{Z}$ generated by $\{(2-n, n-1)\}_{n \geq 1}$. It is clear that M is commutative, cancellative, and torsion-free since it inherits these properties from the additive structure of $\mathbb{Z}$. Also, M is reduced since the second component is strictly positive whenever the first component is negative. We show that $(2-n, n-1)$ is an atom in M for $n \geq 1$. Let $n \geq 1$ and $(a_1, a_2), (b_1, b_2) \in M$ be such that $(a_1, a_2) + (b_1, b_2) = (2-n, n-1)$. Then we have $a_1 + b_1 = 2 - n$ and $a_2 + b_2 = n - 1$. Solving each of these equations for n and equating them, we get $2 - a_1 - b_1 = a_2 + b_2 + 1$. Thus $1 = (a_1 + a_2) + (b_1 + b_2)$. By construction, M is generated by $(2-n, n-1)$ and $(2-n) + (n-1) = 1$; so any finite sum of these generators will have positive sum of the components. It follows that $(a_1, a_2) = (0, 0)$ or $(b_1, b_2) = (0, 0)$. Let $R = \mathbb{Z}_2[X; M]$ and $S = \{X^{(2-n,n-1)}\}_{n\geq 1} \subseteq \mathcal{A}_R$. Then S is unit closed; so use Theorem 3.2.4 to construct the integral domain T with set of atoms S. Then $u = \frac{X^{(1-n,n)}}{X^{(2-n,n-1)}} = X^{(-1,1)}$, for $n \geq 1$. Thus $u \in U_2$. But $u \notin U_1$ since there does not exist $X^{(2-n,n-1)}$ such that $u = \frac{X^{(1,0)}}{X^{(2-n,n-1)}}$. For if there was such an n, then $\frac{X^{(1,0)}}{X^{(2-n,n-1)}} = u = \frac{X^{(1-n,n)}}{X^{(2-n,n-1)}}$. But this is a contradiction since $X^{(1,0)} \neq X^{(1-n,n)}$, for $n \geq 1$.*

We now consider an atomic quasilocal integral domain R with maximal ideal M and quotient field K. We consider $M : M = \{k \in K \mid kM \subseteq M\}$, i.e., the largest ring contained in K such that M is an ideal.

Proposition 6.1.7. *Let R be an atomic quasilocal domain with maximal ideal M. Then $U_R \subseteq M : M$.*

Proof. It is clear that U_R is contained in the quotient field of R. Let $x \in U_R$. We will show that $xM \subseteq M$. First $\mathcal{A}_R \subseteq M$ since R is quasilocal. Let $m \in M$. Then there exist atoms $a_1, \ldots, a_n$ such that $m = a_1 \cdots a_n$ since R is atomic and m is not a unit. Also, there exists $a \in \mathcal{A}_R \subseteq M$ such that $x = \frac{a}{a_1}$ since $x \in U_R$. It follows that $xm = (\frac{a}{a_1})(a_1 \cdots a_n) = aa_2 \cdots a_n \in M$. Thus $U_R \subseteq M : M$. $\square$

It follows that $U_R \subseteq U(M : M)$ since U_R forms a multiplicative group. We consider integral domains of the from $D + M$.

Proposition 6.1.8. *Let $T = K + M$ be a quasilocal integral domain, where K is a subfield of T and M is the nonzero maximal ideal of T. Let D be a subfield of K and $R = D + M$. Then $U_T = U_R$.*

Proof. Notice that for $d \neq 0$, the elements of the forms d or $d + m$ are units in R, and $\mathcal{A}_T \subseteq M$ since T is quasilocal. Then $m \in M$ is a atom in T if and only if m is an atom of R by Proposition 5.1.8. It follows that $\mathcal{A}_R = \mathcal{A}_T$. Thus $U_T = U_R$. □

The non-quasilocal case is similar, but our result is not as strong.

Proposition 6.1.9. *Let $T = K + M$, where K is a subfield of T and M is a nonzero maximal ideal of T. Let D be a subfield of K and $R = D + M$. Then $\mathcal{A}_R \subseteq \mathcal{A}_T$.*

Proof. We look at each of the three types of elements. Note that all of the elements in $D \setminus \{0\}$ are units in R and all of the elements in $K \setminus \{0\}$ are units in T. For $k \neq 0$, we know $k + m \in T$ is an atom in T if and only if $1 + m/k$ is an atom in R by Proposition 5.1.7. It follows that all of the atoms $d + m \in R$ with $d \neq 0$ are atoms in T. Finally, $m \in M$ is an atom in R if and only if m is an atom in T by Proposition 5.1.8. It follows that $\mathcal{A}_R \subseteq \mathcal{A}_T$. □

The next example shows that the containment in the previous proposition can be proper in the non-quasilocal case.

Example 6.1.10. *Let $T = \mathbb{R}[X] = \mathbb{R} + X\mathbb{R}[X]$ and $R = \mathbb{Q} + X\mathbb{R}[X]$. We show that the polynomial $1 + X$ is an atom in both R and T. Notice that $\deg(1 + X) = 1$, and so any factorization in R or T must have a nonzero factor of degree zero. We know $\pi \in U(T)$; so $\pi(1 + X) = \pi + \pi X \in \mathcal{A}_T$. But $\pi \notin \mathbb{Q}$, and hence cannot be an atom of R. Then $\pi \notin \{\frac{a}{1+X} \mid a \text{ atom in } R\}$. It follows that $\pi \notin U_R$, but $\pi \in U(T) \subseteq U_T$ by Proposition 6.1.2.*

We conclude this chapter by again considering the extension $R \subseteq T$ and the relationship between $U(T)$ and U_R, when $\mathcal{A}_R = \mathcal{A}_T$.

Proposition 6.1.11. *Let $R \subseteq T$ be integral domains. If $\mathcal{A}_R = \mathcal{A}_T$, then $U_R \cap T = U(T)$.*

Proof. We know $U(T) \subseteq U_R \cap T$ by Proposition 6.1.2. Let $x \in U_R \cap T$ and $a \in \mathcal{A}_R$. Then there exists $b \in \mathcal{A}_R$ such that $x = \frac{a}{b}$ since $x \in U_R$. It follows that $bx = a$ with $b \in \mathcal{A}_R \subseteq R \subseteq T$ and $x \in T$ by hypothesis. Then $x \in U(T)$ since $a \in \mathcal{A}_R = \mathcal{A}_T$ and b is not a unit in T. $\square$

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