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# A Fundamental Theorem of Multivariable Calculus

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## Introduction

Calculus is essentially the study of two operations and their relation to one another: integration and differentiation. The Fundamental Theorem of Calculus is the key to this study in that it gives us the inverse relationship of integratoin and differentiation.

**Theorem**. **Fundamental Theorem of Calculus**.

(Part I) Let G be a smooth function defined on a line. Let [a,b] be a closed interval on the line. Then,

$$G(x) = \frac{d}{dx}\int_a^x G(t)\,dt.$$

(Part II) Let F be a smooth function defined on a line. Let [a,b] be a closed interval on the line. Then,

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In this paper we seek to generalize part II of the Fundamental Theorem of Calculus to two dimensions. It is unknown to us whether or not the statement of our theorem is uniquely presented in this paper. But it is not common in the popular textbooks, (see [1] and [3]). We think that our theorem is more simply stated which makes it better than those stated in the "standard" text books. The statement of our theorem  relates derivatives and antiderivatives, in a way that closely resembles that of the FTC of one dimension.

**Theorem. Fundamental Theorem of Multivariable Calculus for a function**
Let *F* be a smooth function defined on a plane. Let *A* be a bounded simply connected region in the plane with smooth boundary *S*. Then

$$\int\int_A \nabla F\,dA = \int_S F\cdot\vec{n}\,dS$$

where $\vec{n}$ is a unit vector perpendicular to the boundary and directed outward the region *A*.

What we mean by more simply stated is that our theorem is stated in terms of vectors, the gradient (which we view as the derivative of a function), and integrals. By stating the theorem using these concepts it very closely mimics the statement of the FTC in one variable. It relates the integral of the derivative of a function to the function values at the boundary over which it is being integrated. The proof remains at the level of vectors never going to coordinate computations and therefore the proof is more conceptual, because of its emphasis on vector properties. We believe the conceptual nature of our presentation lends itself better to students just beginning to comprehend proofs and formal mathematics.

Generalizations of the FTC are seldom stated or discussed in many undergraduate Calculus texts. What is occasionally discussed is how Green's Theorem can be viewed as generalization of Part II of the FTC (For a brief overview of how this can be done see [2] in the section titled Green's Theorem). As is stated in [2], it is not at all obvious that Green's Theorem is a generalization of the FTC part II.

In the generalization of the FTC to n dimensions by Mutze (see [4]), the notation used almost exactly resembles that used in the one dimensional FTC. The theorem stated in [4] is limited to n-dimensional rectangular regions. In contrast, our theorem may be applied to any simply connected region in a plane.

Although the FTC was discovered a few hundred years ago, it is not a dormant topic. There are still papers being published about it. See [5], [6], and [7] for such examples. These papers discuss how more modern mathematical structures relate to the fundamental theorem of calculus.

# Dot Product

## Vectors in a plane

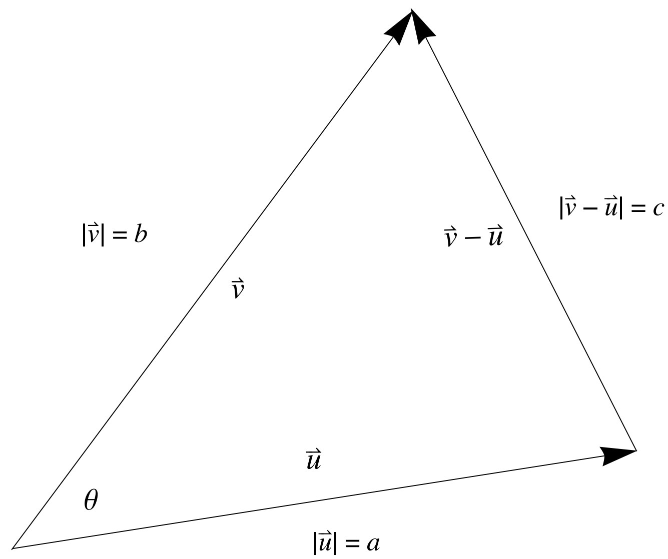
The Pythagoras Theorem states that if two sides of a triangle in a Euclidean plane are perpendicular, then the length of the third side can be computed as $c^2 = a^2 + b^2$.

If the two sides of the triangle are not perpendicular, then an additional term is needed to make the Pythagoras formula correct. The correction term is provided by the Cosine Law:

$$c^2 = a^2 + b^2 - 2\,a\,b\cos(\theta)$$

where $\theta$ is the angle between the given sides of length a and b.

Let us rewrite the Cosine Law using vectors




$$\left|\vec{v} - \vec{u}\right|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,|\vec{u}|\,|\vec{v}|\cos(\theta)$$

Suppose now that a Euclidean coordinate system is fixed in the plane. If the vectors $\vec{u}$ and $\vec{v}$ have coordinates $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v_1, v_2\rangle$, then $\vec{v} - \vec{u} = \langle v_1 - u_1, v_2 - u_2\rangle$ and the square of the magnitude of a vector is equal to the sum of squares of its coordinates:

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 = (u_1)^2 + (u_2)^2 + (v_1)^2 + (v_2)^2 - 2\,|\vec{u}|\,|\vec{v}|\cos(\theta)$$

Expanding the terms on the left side we get

$$(v_1 - u_1)^2 + (v_2 - u_2)^2 = (u_1)^2 + (u_2)^2 + (v_1)^2 + (v_2)^2 - 2\,v_1\,u_1 - 2\,v_2\,u_2$$

Therefore we have

$$|\vec{u}|\,|\vec{v}|\cos(\theta) = v_1\,u_1 + v_2\,u_2$$

The main conclusion here is that given two vectors $\vec{u}$ and $\vec{v}$, for any choice of a Euclidean coordinate system in the plane the number $v_1\,u_1 + v_2\,u_2$ does not depend on the choice of the system and carries the following geometric meaning: it is equal to $|\vec{u}|\,|\vec{v}|\cos(\theta)$.

**Definition. Dot Product**

For any two vectors $\vec{u}$ and $\vec{v}$ in a plane the number $|\vec{u}|\,|\vec{v}|\cos(\theta)$ is called the dot product of the vectors $\vec{u}$ and $\vec{v}$. It is denoted $\vec{u}\cdot\vec{v}$ and can be computed in any Euclidean coordinate system in the plane using the formula $v_1\,u_1 + v_2\,u_2$.

It is obvious from the definition that the **dot product is commutative** (i.e. for any vectors $\vec{u}$ and $\vec{v}$ in a plane we have $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$).

**Proposition. Linearity of the dot product**

For any vectors $\vec{u}$ and $\vec{v}$ in a plane and for any number *a* we have

$$\vec{u}\cdot\left(a\,\vec{v}\right) = a\left(\vec{u}\cdot\vec{v}\right) = \left(a\,\vec{u}\right)\cdot\vec{v}$$

*Proof.*

Suppose $\vec{u}$ and $\vec{v}$ are vectors in a plane and a is in $\mathbb{R}$. Fix an arbitrary coordinate system. It is clear that we may write any vectors $\vec{u}$ and $\vec{v}$ in a plane as, $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ for some $u_1, u_2, v_1, v_2 \in \mathbb{R}$. Therefore, we may write

$$\vec{u} \cdot (a\,\vec{v}) = \langle u_1, u_2 \rangle \cdot [a\,\langle v_1, v_2 \rangle] =$$

$$\langle u_1, u_2 \rangle \cdot \langle a v_1, a v_2 \rangle \underset{\text{definition of dot product}}{=} a u_1\, v_1 + a u_2\, v_2 = a\,(u_1\, v_1 + u_2\, v_2) \underset{\text{definition of dot product}}{=} a\,(\vec{u} \cdot \vec{v}).$$

On the other hand we have,

$$a\,(\vec{u} \cdot \vec{v}) = a\,\langle u_1\, v_1, u_2\, v_2 \rangle = \langle a u_1\, v_1, a u_2\, v_2 \rangle \underset{\text{definition of dot product}}{=} \langle a u_1, a u_2 \rangle \cdot \langle v_1, v_2 \rangle = (a\,\vec{u}) \cdot \vec{v}.$$

Since equality is an equivalence relation we have, $\vec{u} \cdot (a\,\vec{v}) = a\,(\vec{u} \cdot \vec{v}) = (a\,\vec{u}) \cdot \vec{v}$.

**Proposition. Distributivity of the dot product**

For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in a plane we have

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

and

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

*Proof.*

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a plane and a is in $\mathbb{R}$. Fix an arbitrary coordinate system. It is clear that we may write any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ as, $\vec{u} = \langle u_1, u_2 \rangle$, $\vec{v} = \langle v_1, v_2 \rangle$, and $\vec{w} = \langle w_1, w_2 \rangle$ for some $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$. Therefore,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \langle v_1 + w_1, v_2 + w_2 \rangle \underset{\text{definition of dot product}}{=} \langle u_1\,(v_1 + w_1), u_2\,(v_2 + w_2) \rangle =$$

$$\langle u_1\, v_1 + u_1\, w_1, u_2\, v_2 + u_2\, w_2 \rangle = \langle u_1\, v_1, u_2\, v_2 \rangle + \langle u_1\, w_1, u_2\, w_2 \rangle = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$

In addition,

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \langle u_1 + v_1, u_2 + v_2 \rangle \cdot \vec{w} = \langle (u_1 + v_1)\, w_1, (u_2 + v_2)\, w_2 \rangle =$$

$$\langle u_1\, w_1 + v_1\, w_1, u_2\, w_2 + v_2\, w_2 \rangle = \langle u_1\, w_1, u_2\, w_2 \rangle + \langle v_1\, w_1, v_2\, w_2 \rangle = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}.$$

**Proposition. Triviality of the dot product**

For any vectors $\vec{u}$ and $\vec{v}$ in a plane the dot product $\vec{u} \cdot \vec{v}$ is equal to zero if and only if the vectors are perpendicular.

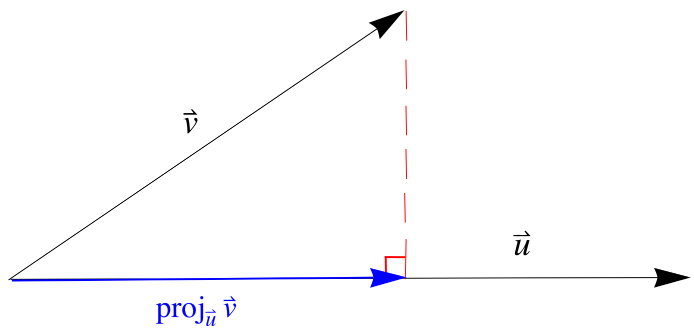
*Proof.*

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in a plane that are perpendicular. Let $\theta$ represent the angle between $\vec{u}$ and $\vec{v}$. Be definition of perpendicular, $\theta = \frac{\pi}{2}$. Therefore, we see that the dot product, $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\left(\frac{\pi}{2}\right) = 0$. Now suppose that $\vec{u} \cdot \vec{v} = 0$. By the definition of the dot product $0 = \vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos(\theta)$. Notice that $|\vec{u}| > 0$ and $|\vec{v}| > 0$. Therefore, $\cos(\theta) = 0$. But then $\theta = \frac{\pi}{2}$ and thus $\vec{u}$ and $\vec{v}$ are perpendicular.

**Definition. Projection**.

For any two vectors $\vec{u}$ and $\vec{v}$, the projection of $\vec{v}$ onto $\vec{u}$ is defined as

$\text{proj}_{\vec{u}}\, \vec{v} = \frac{\vec{v} \cdot \vec{u}}{|u|^2} \cdot \vec{u}$. Notice that $\text{proj}_{\vec{u}}\, \vec{v}$ and $\vec{u}$ have the same direction.




**Proposition**.
Let $\vec{a}$ and $\vec{b}$ be two linearly independent (i.e. not proportional) vectors in a plane. For any vectors $\vec{u}$, $\vec{v}$ in the plane if $\text{proj}_{\vec{a}}\,\vec{v} = \text{proj}_{\vec{a}}\,\vec{u}$ and $\text{proj}_{\vec{b}}\,\vec{v} = \text{proj}_{\vec{b}}\,\vec{u}$ , then $\vec{v} = \vec{u}$.

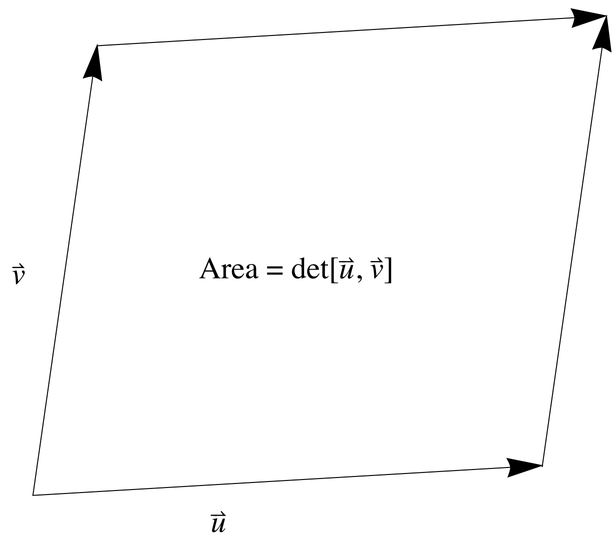
*Proof.*

The equality $\text{proj}_{\vec{a}}\,\vec{v} - \text{proj}_{\vec{a}}\,\vec{u} = 0$ means $\frac{\vec{v}\cdot\vec{a}}{|a|^2}\cdot\vec{a} - \frac{\vec{u}\cdot\vec{a}}{|a|^2}\cdot\vec{a} \underset{\text{Factoring}}{=} (\vec{v} - \vec{u})\cdot\vec{a}\,\frac{\vec{a}}{|a|^2} = 0$. Since $\vec{a} \neq 0$, it implies $(\vec{v} - \vec{u})\cdot\vec{a} = 0$ which means that the vectors $\vec{v} - \vec{u}$ and $\vec{a}$ are perpendicular. Analogously the equality $\text{proj}_{\vec{b}}\,\vec{v} - \text{proj}_{\vec{b}}\,\vec{u} = 0$ implies that the vectors $\vec{v} - \vec{u}$ and $\vec{b}$ are perpendicular. Since the vector $\vec{v} - \vec{u}$ is perpendicular to two linearly independent vectors in a (2-dimensional) plane, the vector $\vec{v} - \vec{u}$ is equal to $\vec{0}$.

# Determinant as a Product

## Area of a parallelogram

Let $\vec{u}$ and $\vec{v}$ be two vectors in a plane. If we draw these vectors so that they have common beginning, then these vectors determine a parallelogram.




We denote by det$[\vec{u}, \vec{v}]$ a number that we will think of as the area of the parallelogram. Our notation comes from the algebraic terminology where this number is called ‘determinant’. The rigorous definition of this number is going to be based on three very basic properties we would expect such a number to satisfy.

**Triviality of determinant.** If the vectors $\vec{u}$ and $\vec{v}$ are linearly dependent (in the case of two vectors it means the vectors are proportional), then the parallelogram degenerates and its area is equal to 0.

**Linearity of determinant.** If *a* is any number, then

$$\det[a\cdot\vec{u}, \vec{v}] = a\cdot\det[\vec{u}, \vec{v}] = \det[\vec{u}, a\cdot\vec{v}]$$

**Distributivity of determinant.** If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any three vectors in a plane, then

$$\det[\vec{u}, \vec{v}+\vec{w}] = \det[\vec{u}, \vec{v}] + \det[\vec{u}, \vec{w}]$$

and

$$\det[\vec{u}+\vec{v}, \vec{w}] = \det[\vec{u}, \vec{w}] + \det[\vec{v}, \vec{w}]$$

Notice that these three properties of determinant do not determine the value det$[\vec{u}, \vec{v}]$ uniquely. They only determine the value up to a scaling factor. Basically, we have to agree on what the area of a unit square is. The common agreement is that the area of a unit square is equal to 1. If we fix a coordinate system in the plane, this means:

**Choice of scale for determinant.** $\det[\langle 1, 0\rangle, \langle 0, 1\rangle] = 1$.

Let us show now how other properties of the determinant follow from those listed above.

**Lemma. Antisymmetry of determinant**

For any two vectors $\vec{u}$ and $\vec{v}$ in a plane we have

$$\det[\vec{u}, \vec{v}] = -\det[\vec{v}, \vec{u}]$$

*Proof.*

$$0 \underset{\text{Triviality}}{=} \det[\vec{u}+\vec{v}, \vec{u}+\vec{v}] \underset{\text{Distributivity}}{=} \det[\vec{u}, \vec{u}+\vec{v}] + \det[\vec{v}, \vec{u}+\vec{v}] \underset{\text{Distributivity}}{=}$$
$$\det[\vec{u}, \vec{u}] + \det[\vec{u}, \vec{v}] + \det[\vec{v}, u] + \det[v, v] \underset{\text{Triviality}}{=} \det[u, v] + \det[v, u]$$

It is clear that,

$0 = \det[u, v] + \det[v, u] \Longleftrightarrow \det[u, v] = -\det[v, u]$.

**Lemma. Coordinate formula for determinant**

Suppose a coordinate system is fixed in a plane. Let $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$ be arbitrary vectors in the plane. Then $\det[\vec{v}, \vec{w}] = v_1 \cdot w_2 - v_2 \cdot w_1$.

*Proof.*

$$\det[\vec{v}, \vec{w}] = \det[v_1 \langle 1, 0\rangle + v_2 \langle 0, 1\rangle, w_1 \langle 1, 0\rangle + w_2 \langle 0, 1\rangle] \underset{\text{Distributivity}}{=} \det[v_1 \langle 1, 0\rangle, w_1 \langle 1, 0\rangle] +$$
$$\det[v_1 \langle 1, 0\rangle, w_2 \langle 0, 1\rangle] + \det[v_2 \langle 0, 1\rangle, w_1 \langle 1, 0\rangle] + \det[v_2 \langle 0, 1\rangle, w_2 \langle 0, 1\rangle] \underset{\text{Triviality}}{=}$$
$$0 + \det[v_1 \langle 1, 0\rangle, w_2 \langle 0, 1\rangle] + \det[v_2 \langle 0, 1\rangle, w_1 \langle 1, 0\rangle] + 0 \underset{\text{Linearity}}{=}$$
$$v_1 \cdot w_2 \cdot \det[\langle 1, 0\rangle, \langle 0, 1\rangle] + v_2 \cdot w_1 \cdot \det[\langle 0, 1\rangle, \langle 1, 0\rangle] \underset{\text{Antisymmetry}}{=}$$
$$(v_1 \cdot w_2 - v_2 \cdot w_1) \cdot \det[\langle 1, 0\rangle, \langle 0, 1\rangle] \underset{\text{Choice of scale}}{=} v_1 \cdot w_2 - v_2 \cdot w_1$$

The coordinate formula for the determinant suggests that the determinant $\det[\vec{v}, \vec{w}]$ can be considered as a dot product of the vector $\vec{u}$ with a vector $\vec{\gamma} = \langle w_2, -w_1\rangle$.

**Key algebraic property of the vector $\vec{\gamma}$:**

for any vector $\vec{u}$ we have $\det[\vec{u}, \vec{w}] = \vec{u} \cdot \vec{\gamma}$

This key algebraic property of the vector $\vec{\gamma}$ inspired the following notation for this vector: $\det[*, \vec{w}]$. Let us describe now the key geometric properties of the vector $\det[*, \vec{w}]$.

**Proposition. Determinant as a Dot Product.**

The vector $\vec{\gamma}$ satisfies the following important geometric properties:

1) $\det[*, \vec{w}]$ is perpendicular to $\vec{w}$
2) $|\det[*, \vec{w}]| = |\vec{w}|$
3) If $\vec{w} \neq 0$, then $\det[\det[*, \vec{w}], \vec{w}] > 0$.

*Proof.*

1) Suppose $\vec{u} = \vec{w}$. Then, $\vec{w} \cdot \det[*, \vec{w}] \underset{\text{Definition}}{=} \det[\vec{w}, \vec{w}] \underset{\text{Triviality}}{=} 0$. Therefore, $\det[*, \vec{w}]$ is perpendicular to $\vec{w}$.

2) In part 1 we showed that $\det[*, \vec{w}]$ is perpendicular to $\vec{w}$, therefore, the parallelogram that the vectors $\det[*, \vec{w}]$ and $\vec{w}$ create is a rectangle. It is clear that, we may find the area of this rectangle by multiplying the magnitudes of the vectors which make up its sides. Recall that $\det[\det[*, \vec{w}], \vec{w}]$ represents the area of the parrallelogram created by $\det[*, \vec{w}]$ and $\vec{w}$. We will use this in the step denoted by $\diamond$.

$$|\det[*, \vec{w}]|^2 = \det[*, \vec{w}]\cdot\det[*, \vec{w}] = \det[\det[*, \vec{w}], \vec{w}] \underset{\diamond}{=} |\det[*, \vec{w}]| * |\vec{w}|$$

By cancellation, $|\det[*, \vec{w}]| = |\vec{w}|$.

3) Suppose that $\vec{w} \neq 0$. Recall that $|\vec{w}| = 0$ if and only if $\vec{w}$ is the zero vector. Since $\vec{w} \neq 0$, $|\vec{w}|>0$.
$\det[\det[*, \vec{w}], \vec{w}] \underset{\text{key algebraic property}}{=} \det[*, \vec{w}]\cdot\det[*, \vec{w}] = |\det[*, \vec{w}]|^2 > 0.$

# Integration

## Line integral

We are going to define a concept that is useful for adding quantities distributed along a path in a plane. Let $f$ be a smooth function from an interval $[a, b]$ to a plane. The image of the function $f$ can be thought of as a path in the plane. Suppose that a continuous function $F$ is defined on the path $f([a, b])$. The concept of line integral is designed to make the following computation rigorous. Assume we travel along a path in the plane by making finitely many short steps and keep adding the following products for every step: the value of the function $F$ at the beginning of the step times the length of the step. Then the total sum remains approximately the same ones the length of each step becomes sufficiently small.

Let us explain this idea of summation using notations. We think of the interval $[a, b]$ as a time interval during which we travel along the path. Choose a large natural number N and subdivide the time interval into N short subintervals of duration $\mathrm{dt} = \frac{b-a}{N}$. During the i-th time interval $[t_i, t_{i+1}]$ we travel from the point $f(t_i)$ to the point $f(t_{i+1})$ of the plane. Then the length of the i-th step is approximately equal to the product of our speed $\left|\frac{\mathrm{df}}{\mathrm{dt}}\right|$ at the time $t_i$ and the time dt spent on making the i-th step. Thus the sum we are computing will be

$$\sum_{i=1}^{N} F(f(t_i))\cdot\left|\frac{\mathrm{df}}{\mathrm{dt}}(t_i)\right|\cdot\mathrm{dt}$$

which is a Riemann sum of the integral $\int_a^b F(f(t))\cdot\left|\frac{\mathrm{df}}{\mathrm{dt}}(t)\right| dt$

Thus the concept of line integral can be defined using the concept of the usual single-variable integral as follows.

**Definition. Line integral defined computationally**
Let $f$ be a smooth function from an interval $[a, b]$ to a plane. Suppose that a continuous function $F$ is defined on the path $S = f([a, b])$. Then the following number is called a line integral of the function $F$ along the path $f$:

$$\int_a^b F(f(t))\cdot\left|\frac{\mathrm{df}}{\mathrm{dt}}(t)\right| dt$$

We will use the following equivalent definition of the line integral. Our main goal of using this definition is to have a purely geometric description of a line integral (without any parameterization function $f$ involved).

**Definition. Line integral defined geometrically**
Let $S$ be a smooth path in a plane. Suppose that a continuous function $F$ is defined on the

plane. Choose a large natural number N and subdivide the path $S$ into N short subpaths $S_i$. Then the length of the i-th path (denote it by $|S_i|$) is approximately equal to the distance between its endpoints. Denote by $M_i$ the midpoint of the straight line segment joining the endpoints of the i-th path $S_i$. Then the limit

$$\lim_{N\to\infty} \sum_{i=1}^{N} F(M_i)\cdot|S_i|$$

is called the line integral of the function $F$ along the path $S$ and it is denoted by

$$\int_S F\, dS$$

The value $F(M_i)$ in the definition above can actually be replaced with an approximation without affecting the value of the limit. Various smoothness conditions come into play when one tries to replace the value $F(M_i)$ with its approximation. The key question to address here is: "What kind of approximation will guarantee that the limit is the same?". In our main theorem we approximate the value $F(M_i)$ with the value $F_i(M_i)$, where $F_i$ is the linear function having the same values as $F$ at both endpoints of the i-th path $S_i$. If the function $F$ is continuous, then $F_i(M_i)$ is a good approximation for $F(M_i)$ (in the sense that $\lim_{N\to\infty} \sum_{i=1}^{N} F(M_i)\cdot|S_i| = \lim_{N\to\infty} \sum_{i=1}^{N} F_i(M_i)\cdot|S_i|$).

## Double integral

Suppose that a continuous function $F$ is defined on the plane. Let $A$ be a bounded simply connected (i.e. without holes) region in a plane. Then its boundary is a simple closed curve (we will denote it by $S$). Subdivide the path $S$ into short subpaths $S_i$. Join the endpoints of each subpath $S_i$ by a straight line segment $L_i$ and consider the piecewise linear path $L$ made by all the segments $L_i$. The path $L$ bounds a region P of the plane. Triangulate the region P (that means subdivide it into triangular subregions so that if two triangles intersect, then the intersection is either their common vertex or their common side) into large number N of small triangles. For every triangle $T_i$ of the subdivision denote by $|T_i|$ the area of this triangle. Also, for every triangle $T_i$ choose a point $M_i$ in the triangle. Then the limit

$$\lim_{N\to\infty} \sum_{i=1}^{N} F(M_i)\cdot|T_i|$$

is called the double integral of the function $F$ over the region $A$ and it is denoted by

$$\iint_A F\, dA$$

The value $F(M_i)$ in the definition above can actually be replaced with an approximation without affecting the value of the limit. Various smoothness conditions come into play when one tries to replace the value $F(M_i)$ with its approximation. The key question to address here is: "What kind of approximation will guarantee that the limit is the same?". In our main theorem we integrate the gradient of some function $\nabla F$ and we approximate the value $\nabla F(M_i)$ with the value $\nabla F_i$, where $F_i$ is the linear function having the same values as $F$ at all the vertices of the i-th triangle $T_i$. If the function $F$ is smooth enough, then $\nabla F_i$ is a good approximation for $\nabla F(M_i)$ (in the sense that $\lim_{N\to\infty} \sum_{i=1}^{N} F(M_i)\cdot|T_i| = \lim_{N\to\infty} \sum_{i=1}^{N} \nabla F_i\cdot|T_i|$).

# Function on a planar region

If P is a point in the plane, we may represent a translation of P as P + $\vec{w}$, where $\vec{w}$ is a vector in the plane. Suppose P has coordinates $(p_1, p_2)$, and suppose $\vec{w} = \langle w_1, w_2 \rangle$. Then P + $\vec{w}$ represents the translation of P by $w_1$ units in the horizontal direction and $w_2$ units in the verticle direction. In other words, P + $\vec{w}$ = $(p_1, p_2) + \langle w_1, w_2 \rangle = (p_1 + w_1, p_2 + w_2)$. Geometrically, the sum of a point and a vector is the translation of a point in the plane. If $F$ is a function $\mathbb{R}^2$ to $\mathbb{R}$ then it takes points as its inputs. Since P + $\vec{w}$ also represents a point, $F$ may take P + $\vec{w}$ as its input.

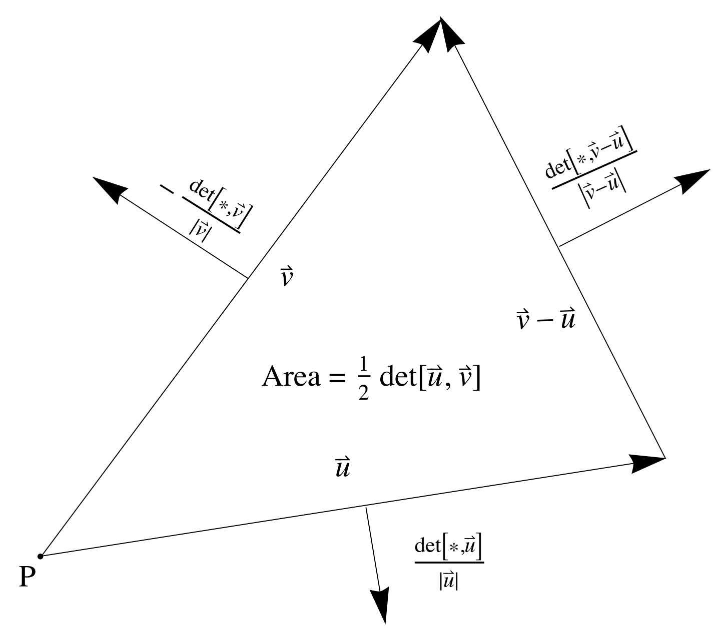
**Lemma**.

Let T be a triangle on a plane. Let f be any linear function on the triangle T. If E is one of the three sides of the triangle, we denote by $M_E$ the midpoint of the side, by $|E|$ the length of the side, and by $\vec{n}_E$ the unit vector perpendicular to the side E and directed outward of the triangle T. Then

$$\nabla f * |T| = \sum_{\text{E side of T}} f(M_E) * \vec{n}_E * |E|$$

where $|T|$ denotes the area of the triangle T

*Proof.*




Let P be a vertex of the triangle T. Let $\vec{u}$ and $\vec{v}$ be the vectors from P to the other two vertices of T, as in the picture above. Notice that the third side of the triangle can be written in terms of $\vec{u}$ and $\vec{v}$, that is, $\vec{v} - \vec{u}$. The area of the triangle T, |T|, can be written as $\frac{1}{2}\det[\vec{u}, \vec{v}]$.

By proposition, determinant as a dot product, for each side of the triangle $\vec{w}$, we may write a normal vector as $\frac{\det[*,\vec{w}]}{|\vec{w}|}$. Choose the sign for $\frac{\det[*,\vec{w}]}{|\vec{w}|}$ so that it is directed outward from the triangle, as on the picture above. If we replace each normal unit vector in the right hand side of the lemma with its representation in determinant notation we obtain,

$$\sum_{E \text{ side of } T} f(M_E) * \vec{n}_E * |E| = f(M_u) * \frac{\det[*, \vec{u}]}{|\vec{u}|} * |\vec{u}| - f(M_v) * \frac{\det[*, \vec{v}]}{|\vec{v}|} * |\vec{v}| + f(M_{v-u}) * \frac{\det[*, \vec{v}-\vec{u}]}{|\vec{v}-\vec{u}|} * |\vec{v}-\vec{u}| =$$

$$\det[*, \vec{u}]\, f(M_u) - \det[*, v]\, f(M_v) + \det[e, \vec{v}-\vec{u}]\, f(M_{v-u}) =$$

$$\det[*, \vec{u}]\, f\left(P + \frac{\vec{u}}{2}\right) - \det[*, \vec{v}]\, f\left(P + \frac{\vec{v}}{2}\right) + \det[*, \vec{v}-\vec{u}]\, f\left(P + \frac{\vec{u}+\vec{v}}{2}\right) =$$

Recall that f is linear. Since $f$ is a linear function, for all $\vec{w}$ in $\mathbb{R}^2$, $f(P + \vec{w}) = f(P) + \nabla f \cdot \vec{w}$.
Therefore

$$\underset{\text{linearity of } f}{=}$$

$$\det[*, \vec{u}]\left(f(P) + \frac{1}{2}(\nabla f \cdot \vec{u})\right) - \det[*, \vec{v}]\left(f(P) + \frac{1}{2}(\nabla f \cdot \vec{v})\right) + \det[*, \vec{v}-\vec{u}]\left(f(P) + \frac{1}{2}\left(\nabla f \cdot \frac{\vec{u}+\vec{v}}{2}\right)\right) =$$

$$\underset{\text{distributivity}}{=} \det[*, \vec{u}]\, f(P) + \frac{1}{2}\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\, f(P) -$$

$$\frac{1}{2}\det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) + \det[*, \vec{v}-\vec{u}]\, f(P) + \frac{1}{2}\det[*, \vec{v}-\vec{u}]\left(\nabla f \cdot \frac{\vec{u}+\vec{v}}{2}\right)$$

$$\underset{\text{factorization}}{=} \{\det[*, \vec{u}] - \det[*, \vec{v}] + \det[*, \vec{v}-\vec{u}]\}\, f(P) +$$

$$\frac{1}{2}\{\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) + \det[*, \vec{v}-\vec{u}]\,(\nabla f \cdot (\vec{u}+\vec{v}))\}$$

Notice that,

$$\det[*, \vec{u}] - \det[*, \vec{v}] + \det[*, \vec{v}-\vec{u}] \underset{\text{distributivity of determinant}}{=}$$

$$\det[*, \vec{u}] - \det[*, \vec{v}] + \det[*, \vec{v}] - \det[*, \vec{u}] = 0$$

Then,

$$\{\det[*, \vec{u}] - \det[*, \vec{v}] + \det[*, \vec{v}-\vec{u}]\}\, f(P) +$$

$$\frac{1}{2}\{\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) + \det[*, \vec{v}-\vec{u}]\,(\nabla f \cdot (\vec{u}+\vec{v}))\} =$$

$$\frac{1}{2}\{\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) + \det[*, \vec{v}-\vec{u}]\,(\nabla f \cdot (\vec{u}+\vec{v}))\} =$$

By the distributivity of the dot product, $\nabla f \cdot (\vec{u}+\vec{v}) = (\nabla f \cdot \vec{u}) + (\nabla f \cdot \vec{v})$ . Using this, we obtain,

$$= \frac{1}{2}\{\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) + \det[*, \vec{v}-\vec{u}][(\nabla f \cdot \vec{u}) + (\nabla f \cdot \vec{v})]\} \underset{\text{distributivity over scalar addition}}{=}$$

$$\frac{1}{2}\{\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) +$$

$$\det[*, \vec{v}-\vec{u}]\,(\nabla f \cdot \vec{u}) + \det[*, \vec{v}-\vec{u}]\,(\nabla f \cdot \vec{v})\} \underset{\text{distributivity of the determinant}}{=}$$

$$\frac{1}{2}\{\det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) + \det[*, \vec{v}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{u}]\,(\nabla f \cdot \vec{u}) +$$

$$\det[*, \vec{v}]\,(\nabla f \cdot \vec{v}) - \det[*, \vec{u}]\,(\nabla f \cdot \vec{v})\} \underset{\text{cancellation}}{=} \frac{1}{2}\{\det[*, \vec{v}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{u}]\,(\nabla f \cdot \vec{v})\}$$

We have now reduced the right hand side of the equality in the statement of the lemma to $\frac{1}{2}\{\det[*, \vec{v}]\,(\nabla f \cdot \vec{u}) - \det[*, \vec{u}]\,(\nabla f \cdot \vec{v})\}$.

Notice that, $\nabla f * |T| = \nabla f * \frac{1}{2} \det[\vec{u}, \vec{v}]$. To verify the equality we will show that the vectors $\nabla f * \frac{1}{2} \det[\vec{u}, \vec{v}]$ and $\frac{1}{2} \{\det[*, \vec{v}] (\nabla f \cdot \vec{u}) - \det[*, \vec{u}] (\nabla f \cdot \vec{v})\}$ are the same by showing that their projections onto $\vec{u}$ and $\vec{v}$ are the same. It is clear that we may multiply each by 2 and this will still work. First we project $\nabla f \cdot \det[\vec{u}, \vec{v}]$ and $\det[*, \vec{v}] (\nabla f \cdot \vec{u}) - \det[*, \vec{u}] (\nabla f \cdot \vec{v})$ onto $\vec{u}$.

1) $\text{proj}_{\vec{u}}(\nabla f * \det[\vec{u}, \vec{v}]) = \frac{1}{|\vec{u}|} (\nabla f * \det[\vec{u}, \vec{v}]) \cdot \vec{u} = \frac{1}{|\vec{u}|} \det[\vec{u}, \vec{v}] (\nabla f \cdot \vec{u})$

2) $\text{proj}_{\vec{u}}[\det[*, \vec{v}] (\nabla f \cdot \vec{u}) - \det[*, \vec{u}] (\nabla f \cdot \vec{v})] =$

$$\frac{1}{|\vec{u}|}[\det[*, \vec{v}] (\nabla f \cdot \vec{u}) - \det[*, \vec{u}] (\nabla f \cdot \vec{v})] \cdot \vec{u} \underset{\text{distributivity of dot product}}{=}$$

$$\frac{1}{|\vec{u}|} \det[*, \vec{v}] (\nabla f \cdot \vec{u}) \cdot \vec{u} - \frac{1}{|\vec{u}|} \det[*, \vec{u}] (\nabla f \cdot \vec{v}) \cdot \vec{u} \underset{\text{algebraic property}}{=}$$

$$\frac{1}{|\vec{u}|} \det[\vec{u}, \vec{v}] (\nabla f \cdot \vec{u}) - \frac{1}{|\vec{u}|} \det[\vec{u}, \vec{u}] (\nabla f \cdot \vec{v}) \underset{\text{triviality of determinant}}{=} \frac{1}{|\vec{u}|} \det[\vec{u}, \vec{v}] (\nabla f \cdot \vec{u})$$

Now we project each vector onto $\vec{v}$.

1) $\text{proj}_{\vec{v}}(\nabla f * \det[\vec{u}, \vec{v}]) = \frac{1}{|\vec{v}|} (\nabla f * \det[\vec{u}, \vec{v}]) \cdot \vec{v} = \frac{1}{|\vec{v}|} \det[\vec{u}, \vec{v}] (\nabla f \cdot \vec{v})$

2) $\text{proj}_{\vec{v}}[\det[*, \vec{v}] (\nabla f \cdot \vec{u}) - \det[*, \vec{u}] (\nabla f \cdot \vec{v})] =$

$$\frac{1}{|\vec{v}|}[\det[*, \vec{v}] (\nabla f \cdot \vec{u}) - \det[*, \vec{u}] (\nabla f \cdot \vec{v})] \cdot \vec{v} \underset{\text{distributivity of dot product}}{=}$$

$$\frac{1}{|\vec{v}|} \det[*, \vec{v}] (\nabla f \cdot \vec{u}) \cdot \vec{v} - \frac{1}{|\vec{v}|} \det[*, \vec{u}] (\nabla f \cdot \vec{v}) \cdot \vec{v} \underset{\text{algebraic property}}{=}$$

$$\frac{1}{|\vec{v}|} \det[\vec{u}, \vec{u}] (\nabla f \cdot \vec{u}) - \frac{1}{|\vec{v}|} \det[\vec{v}, \vec{u}] (\nabla f \cdot \vec{v}) \underset{\text{triviality of determinant}}{=}$$

$$-\frac{1}{|\vec{v}|} \det[\vec{v}, \vec{u}] (\nabla f \cdot \vec{v}) \underset{\text{antisymmetry of determinnant}}{=} \frac{1}{|\vec{v}|} \det[\vec{u}, \vec{v}] (\nabla f \cdot \vec{v})$$

Since each is the same when projected onto both $\vec{u}$ and $\vec{v}$ we may conclude that,

$\nabla f * |T| = \sum_{E \text{ side of } T} f(M_E) * \vec{n}_E * |E|$

**Theorem. Fundamental Theorem of Multivariable Calculus for a function**
Let *F* be a smooth function defined on a plane. Let *A* be a bounded simply connected region in the plane with smooth boundary *S*. Then

$$\int\int_A \nabla F \, dA = \int_S F \cdot \vec{n} \, dS$$

where $\vec{n}$ is a unit vector perpendicular to the boundary and directed outward the region *A*.

*Proof.*

The plan of our proof is to describe Riemann sums for each integral and then show that our Riemann approximation for the double integral equals the Riemann approximation for the line integral. The equality of integrals then follows after passing to limits in the equality of Riemann approximations.

**Riemann sum approximations**

Subdivide the path *S* into a large number M of short subpaths $S_i$. Join the endpoints of each subpath $S_i$ by a straight line segment $L_i$ and consider the piecewise linear path *L* made by all the segments $L_i$. The path *L* bounds a region P of the plane. Triangulate the region P (that means subdivide it into triangular subregions so that if two triangles intersect, then the intersection is either their common vertex or their common side) into large number N of small triangles $T_j$.

Consider a piecewise linear function $\tilde{F}$ on the region P defined on each triangle $T_j$ as the unique linear function taking the same values as the function $F$ at all the vertices of the triangle. Notice that the function $\tilde{F}$ is well-defined. Indeed, if a point Q of the polygon P belongs to two different triangles, then (by the definition of triangulation) it is either their common vertex (in which case $\tilde{F}(Q) = F(Q)$ by definition of $\tilde{F}$) or it belongs to a common side of the triangles (in which case the value $\tilde{F}(Q)$ is equal to the value of the unique linear function on the common side taking the same values at both endpoints as the function $F$).

$$\int\int_A \nabla F \, dA = \lim_{N\to\infty} \sum_{j=1}^{N} \nabla \tilde{F}\left(M_j\right) \cdot \left|T_j\right|$$

Since the function $\tilde{F}$ is linear on each triangle $T_j$, its gradient $\nabla \tilde{F}$ is constant and the value $\nabla \tilde{F}\left(M_j\right)$ does not depend on the choice of the point $M_j$ in the triangle $T_j$.

$$\int_S F \cdot \vec{n} \, dS = \lim_{N\to\infty} \sum_{i=1}^{M} \tilde{F}\,(M_i) \cdot \vec{n_i} \cdot |S_i|$$

Here $\vec{n_i}$ is a unit vector perpendicular to the straight line segment $L_i$ and directed outward the region $P$. In the rest of the proof we show the (exact, not approximate) equality of the Riemann sum approximations

$$\sum_{j=1}^{N} \nabla \tilde{F}\left(M_j\right) \cdot \left|T_j\right| = \sum_{i=1}^{M} \tilde{F}\,(M_i) \cdot \vec{n_i} \cdot |S_i|$$

**Equality of Riemann sum approximations**

By Lemma, we have

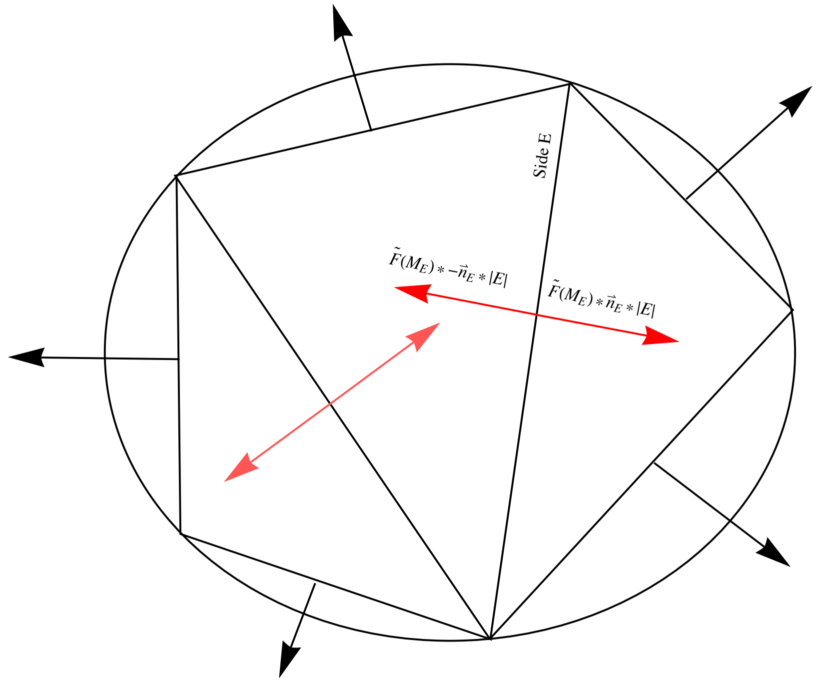
$$\nabla \tilde{F}\left(M_j\right) \cdot \left|T_j\right| = \sum_{E \text{ side of } T_j} \tilde{F}\,(M_E) * \vec{n}_E * |E|$$

For any side E of a triangle $T_j$ we have the following two possibilities:

1) the side E is also a side of another triangle $T_k$ (thus E does not belong to the boundary);
2) the side E is not a side of any other triangle (thus E belongs to the boundary).

In case a side E of a triangle $T_j$ is also a side of another triangle $T_k$, then the sum $\sum_{j=1}^{N} \nabla \tilde{F}\left(M_j\right) \cdot \left|T_j\right|$ will contain the term like $\tilde{F}(M_E) * \vec{n}_E * |E|$ exactly twice and the only difference between the two terms will be the direction of the unit normal vector: the two terms will have opposite directions. Therefore the two terms will cancel.

A Picture of the cancellation




In case a side E of a triangle $T_j$ is not a side of any other triangle, the term $\tilde{F}(M_E) * \vec{n}_E * |E|$ remains in the sum. Obviously, all the remaining terms will make together the sum $\sum_{i=1}^{M} \tilde{F}(M_i) \cdot \vec{n_i} \cdot |S_i|$.

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