5-2012

Alexander and Conway polynomials of Torus knots

Katherine Ellen Louise Agle
University of Tennessee - Knoxville, kagle@utk.edu

To the Graduate Council:

I am submitting herewith a thesis written by Katherine Ellen Louise Agle entitled "Alexander and Conway polynomials of Torus knots." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

James R. Conant, Major Professor

We have read this thesis and recommend its acceptance:

Don B. Hinton, Morwen B. Thistlethwaite

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

Alexander and Conway polynomials of Torus knots

A Thesis Presented for
The Master of Science
Degree
The University of Tennessee, Knoxville

Katherine Ellen Louise Agle

May 2012

I dedicate this to my parents, who taught me to always pursue things that I enjoy.

Acknowledgements

My greatest gratitude goes towards my parents for supporting me, to my sister Lisa who has always motivated me to try harder, to my sister Becky for understanding research trumps dishes, and to my youngest sister Sara who spent many late nights with me when some of my most original work was created.

I would also like to thank Don Hinton who has been a great mentor to me throughout my college career and encouraged me to participate in the undergraduate research programs which led to my love of abstract mathematics, and more specifically knot theory. Thanks goes to Morwen Thistlethwaite for serving on my thesis committee.

Thanks also to James Conant for helping me develop this thesis, providing stimulating conversation that led to the development of the counterexample used in this thesis, and for helping me with all of the work that goes into finishing a graduate degree.

Abstract

We disprove the conjecture that if K is amphicheiral and K is concordant to K', then $C_{K'}(z)C_{K'}(iz)C_{K'}(z^2)$ is a perfect square inside the ring of power series with integer coefficients. The Alexander polynomial of (p,q)-torus knots are found to be of the form $A_{T(p,q)}(t) = \frac{f(t^q)}{f(t)}$ where $f(t) = 1 + t + t^2 + \cdots + t^{p-1}$. Also, for (p^n,q)-torus knots, the Alexander polynomial factors into the form $A_{T(p^n,q)} = f(t)f(t^p)f(t^{p^2})\cdots f(t^{p^{n-2}})f(t^{p^{n-1}})$. A new conversion from the Alexander polynomial to the Conway polynomial is discussed using the Lucas polynomial. This result is used to show that the Conway polynomial of $(2^n,q)$-torus knots are of the form $C_{T(2^n,q)}(z) = K_1K_2\cdots K_n$ where $K_1 = F_q(z)$, $F_q(z)$ being the Fibonacci polynomial, and $K_i(z) = K_{i-1}(\sqrt{z^4+4z^2})$.

Contents

List of Figures

Chapter 1

Introduction

First, it is necessary to define a few key terms to establish an adequate background. A *knot* is a simple, closed curve in Euclidean 3-space, and a *link* is two or more knots tangled together. A *knot diagram* or *projection* is an image that represents a particular knot. There are three types of *Reidemeister moves* that change the projection of a knot, but not the knot itself. They are shown Figure 1.1. Two knots are considered equivalent if and only if they differ by Reidemeister moves. A knot

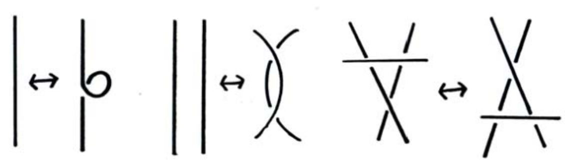


Figure 1.1: Reidemeister Moves

that is simply a circle is called the *unknot* or *trivial* knot. The *composition* of two knots, K_1 and K_2, is denoted by $K_1 \# K_2$ and is done by joining two knots at any arc as shown in Figure 1.2. A knot K composed with the unknot is again K. *Prime knots* are knots that are not the composition of two or more nontrivial knots. An *amphicheiral* or *achiral* knot may be either equivalent to its mirror image or its mirror image with reversed orientation. A knot not equivalent to its mirror image is called

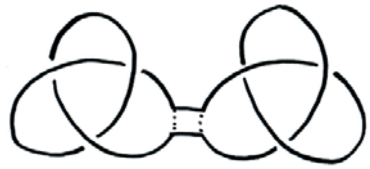


Figure 1.2: The composition of two trefoil knots

chiral. A knot is *invertible* if it can be deformed into itself, but with the orientation reversed.

1.0.1 Knot Polynomials

The *Alexander polynomial*, denoted $\Delta_K(t)$, is a Laurent polynomial in t with integer coefficients satisfying the following properties, which determine it uniquely. One can show there is a polynomial satisfying these properties using algebraic topology Adams (2004).

1. $\Delta(\bigcirc) = 1$
2. $\Delta(L_+) - \Delta(L_-) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0) = 0$, where L_+, L_-, and L_0 are defined in Figure 1.3.

Example 1.1 shows the calculation of the Alexander polynomial of two unknots using

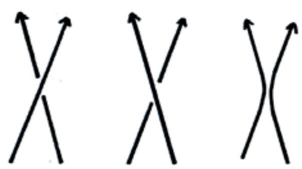
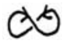


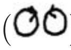
Figure 1.3: L_+, L_- and L_0 from left to right

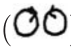
the skein relationship defined above.

Example 1.1.

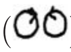
$\Delta($  $) - \Delta($  $) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta($  $) = 0$

But, we can see that  *and*  *are equivalent to the unknot, thus we have*

$$\Delta(\bigcirc) - \Delta(\bigcirc) + +(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta($$  $$) = 0$$

$$1 - 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta($$  $$) = 0$$

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta($$  $$) = 0$$

$$\Delta($$  $$) = 0$$

The Alexander polynomial is an example of a *knot invariant*. That is two knots are not equivalent if their Alexander polynomials differ. Two projections of a knot have the same Alexander polynomial, but two projections with the same Alexander polynomial need not be the same knot. For example, there are nontrivial knots with Alexander polynomial equal to 1. Also the Alexander polynomial cannot distinguish between a knot and its mirror image, thus $A_K(t) = A_{K^*}(t)$ where K^* is the mirror image of K. Also, the Alexander polynomial is symmetric, i.e. $\Delta_K(t^{-1}) = \Delta_K(t)$. The Alexander polynomial of a composition of two knots K_1 and K_2 is $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t)\Delta_{K_2}(t)$.

Another knot invariant is the *Conway polynomial*, denoted by $\nabla_L(Z)$ (where K is substituted for L when it is a knot), which satisfies the following properties Cromwell (2004):

1. Invariance: $\nabla_L(z)$ is invariant under ambient isotopy of L.
2. Normalization: if K is the trivial knot then $\nabla_K(z) = 1$.
3. Skein relations: $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$

Ambient isotopy simply means knots that only differ by Reidemeister moves Adams (2004). The relationship between the Alexander and Conway polynomials is $\Delta_L(x^2) = \nabla_L(x - x^{-1})$ Livingston (1993).

1.0.2 The Burau Representation

Crossings in a braid are denoted σ_i to represent the i^{th} string crossing over the $(i+1)^{th}$ string. Undercrossings are similarly represented by σ_i^{-1}. The σ_i generate a group known as the braid group on n strings, denoted B_n. An example is shown in Figure 1.4 for the braid group B_3 on 3 strings. A chain of several σ_i, for example $\sigma_1\sigma_3^{-1}\sigma_2$, is called a *braid word*, and represents a braid.

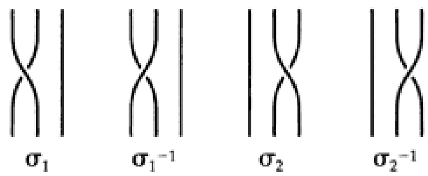


Figure 1.4: The generators for the braid group B_3 Adams (2004)

The braid group B_n is defined by the presentation $\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i^2 = 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, [\sigma_i, \sigma_j] = 1 \text{ for } |i-j| > 1\rangle$, which captures the topology of the situation. The *Burau representation*, defined by Equation 1.1, is an alternate computation of the Alexander polynomial, which is known to be unfaithful for $n \geq 5$ Bigelow (1999). Each generator σ_i or σ_i^{-1} is assigned a matrix which is generated by centering $\begin{bmatrix} -t \\ -t \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ -\frac{1}{t} \\ -\frac{1}{t} \end{bmatrix}$ in the (i,i) position of the $(n-1) \times (n-1)$ identity matrix. The matrices for the generators in the braid group B_4 are shown in Figure 1.5. Let Ψ be the matrix product of braid words of a knot, that is, Ψ is the product of the matrices of each braid generator. Then

$$\frac{det(I - \Psi)}{1 + t + ... + t^{n-1}} = \Delta_L, \tag{1.1}$$

where Δ_L is the Alexander polynomial and det is the determinant Weisstein (2012a).

$$\sigma_1 = \begin{bmatrix} -t & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & -t \end{bmatrix}$$

$$\sigma_1^{-1} = \begin{bmatrix} -\frac{1}{t} & 0 & 0 \\ -\frac{1}{t} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \sigma_2^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -\frac{1}{t} & 0 \\ 0 & -\frac{1}{t} & 1 \end{bmatrix}, \text{ and } \sigma_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -\frac{1}{t} \end{bmatrix}$$

Figure 1.5: The matrices of the generators of B_4

Example 1.2 shows a calculation of the Alexander polynomial of the figure-eight knot using the Burau representation.

Example 1.2. *Consider the figure-eight knot which is equivalent to the closure of the braid word* $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$*. Then*

$$\begin{aligned}
\Psi &= \begin{bmatrix} -t & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -\frac{1}{t} \end{bmatrix} \begin{bmatrix} -t & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -\frac{1}{t} \end{bmatrix} \\
&= \begin{bmatrix} t^2 - t & -t^2 + t - 1 \\ t - 1 + \frac{1}{t} & -t + (1 - \frac{1}{t})^2 \end{bmatrix} \\
I - \Psi &= \begin{bmatrix} 1 - t^2 + t & t^2 - t + 1 \\ -t + 1 - \frac{1}{t} & 1 + t - (1 - \frac{1}{t})^2 \end{bmatrix} \\
det(I - \Psi) &= (1 - t^2 + t)(1 + t - (1 - \frac{1}{t})^2) - (t^2 - t + 1)(-t + 1 - \frac{1}{t}) \\
&= -t^2 - \frac{1}{t^2} + 2t + \frac{2}{t} + 1 \\
\Delta_K &= \frac{det(I - \Psi)}{1 + t + ... + t^{n-1}} \\
&= \frac{-t^2 - \frac{1}{t^2} + 2t + \frac{2}{t} + 1}{1 + t + t^2} \\
&= \frac{-t^2 + 3t - 1}{t^2} \\
&= -t - \frac{1}{t} + 3
\end{aligned}$$

which is the correct result.

Chapter 2

Conjectures

Initially, two conjectures concerning the Conway polynomial were investigated. Specifically these conjectures dealt with the splitting property of the Conway polynomial, i.e. $C(z) = F(z)F(-z)$.

2.0.3 Conant's Conjecture

Conjecture 2.0.1 (Generalized Kawauchi Conjecture)**.** *The Conway polynomial $C(z)$ of any achiral knot has the splitting property, i.e. $C(z) = F(z)F(-z)$ for a polynomial $F(z)$ with integer coefficients Ermotti et al. (2011).*

The following conjecture concerning braids was made by J. Conant originating from the Generalized Kawauchi Conjecture.

Conjecture 2.0.2. *All braids of the form ww^*ww^*, where w^* is the braid w with all crossings reversed, such that ww^*ww^* closes to a knot, have Conway polynomial of the form $f(z)f(-z)$ Conant (2006).*

The Generalized Kawauchi conjecture was disproven by Ermotti et al in Ermotti et al. (2011) with the knot shown in Figure 2.1. This knot has the Conway polynomial

$$C_K(z) = (4z^8 + 16z^6 + 12z^4 - 16z^2 + 1)(1 + z)(1 - z)(2z^4 - 1)^2$$

which does not satisfy the splitting property. However, this is the closure of a tangle

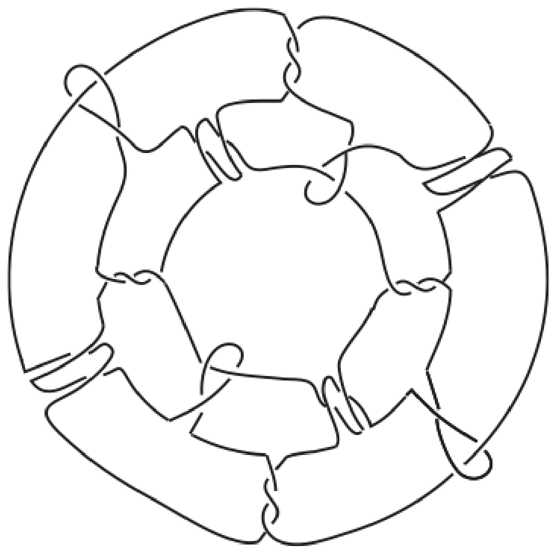


Figure 2.1: A counterexample to the Kawauchi conjecture Ermotti et al. (2011)

and not a braid, which means that Conjecture 2.0.2 remains to be shown. This counterexample does raise suspicion that the conjecture is untrue.

2.0.4 Collins's Conjecture

A knot is *slice* if it bounds a locally flat disk in the 4-ball, where locally flat means that every point of the disc has a neighborhood around it which looks like the standard embedding of a disc into the 4-ball. A *ribbon knot* is a knot which bounds a smooth disc in the 4-ball such that the singularities of the ribbon disc are either minima or saddle points. Two knots K_1 and K_2 are *concordant* if $K_1\# - K_2$ is slice, where $-K$ is the mirror image of K with reversed orientation Collins (2011). The following conjecture is made by Collins.

Conjecture 2.0.3. *If K is amphicheiral and K is concordant to K', then*

$$C_{K'}(z)C_{K'}(iz)C_{K'}(z^2)$$

is a perfect square inside the ring of power series with integer coefficients Collins (2011).

Every ribbon knot is a slice knot, and it is conjectured that every smoothly slice knot is a ribbon knot Weisstein (2012c). The knot 8_8 is a ribbon knot, thus it is a slice knot. Then 4_1 is concordant to $4_1\#8_8$. However, the Conway polynomial of $4_1\#8_8$ is

$$C_{4_1\#8_8}(z) = (1 - z^2)(1 + 2z^2 + 2z^4),$$

so then

$$\begin{aligned} &C_{K'}(z)C_{K'}(iz)C_{K'}(z^2) \\ &\qquad = (1 - z^2)(1 + 2z^2 + 2z^4)(1 + z^2)(1 - 2z^2 + 2z^4)(1 - z^4)(1 + 2z^4 + 2z^8) \\ &\qquad = 8z^{24} - 8z^{20} - 2z^{16} - 2z^{12} + 3z^8 + 1 \end{aligned}$$

Then taking the Taylor polynomial of the square root, we get

$$1 + \frac{3z^8}{2} - z^{12} - \frac{17z^{16}}{8} - \frac{5z^{20}}{2} + \frac{107z^{24}}{16} + \dots$$

which does not have integer coefficients, and thus is not a square inside the ring of power series with integer coefficients. Therefore this is a counterexample to the above conjecture.

Chapter 3

Torus Knots

A (p, q)-*torus knot*, where $p, q \geq 2$ and p and q are relatively prime, is created by traveling p times vertically and q times horizontally around the torus. Torus knots are prime, invertible, and chiral. Despite being chiral, the symmetry exists where $T(p, q)$ knots are equivalent to $T(q, p)$ knots Livingston (1993). Torus knots have unique braid representations in which there are p strands where the first strand overlaps the remaining strands to the last position repeated q times and closed to a knot. An example of the braid for a $(3, 2)-$torus knot can be seen in Figure 3.1.

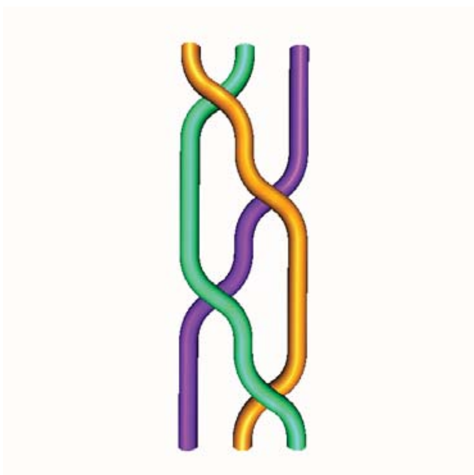


Figure 3.1: A braid for the (3,2)-torus knot

3.0.5 Alexander polynomial

The Alexander polynomial of the (p, q)-torus knot is defined by equation (3.1) Livingston (1993).

$$\Delta_{T(p,q)}(t) = \frac{(t^{|pq|} - 1)(t - 1)}{(t^{|p|} - 1)(t^{|q|} - 1)} \tag{3.1}$$

For the case when $p = 2$, this simplifies to Equation (3.2).

$$\Delta_{T(2,q)}(t) = \frac{(t^q + 1)}{(t + 1)} \tag{3.2}$$

Theorem 3.1. *The Alexander polynomial of the torus knot can be written as* $\Delta_{T(p,q)}(t) = \frac{f(t^q)}{f(t)}$ *where* $f(t) = 1 + \cdots + t^{p-1}$ *for* $p, q \geq 2$ *and* p, q *relatively prime.*

The first few examples of this calculation are shown below:

$$\begin{aligned}
\Delta_{T(2,q)}(t) &= \frac{(t^{2q} - 1)(t - 1)}{(t^2 - 1)(t^q - 1)} = \frac{(t^q - 1)(t^q + 1)(t - 1)}{(t - 1)(t + 1)(t^q - 1)} \\
&= \frac{t^q + 1}{t + 1} \\
\Delta_{T(3,q)}(t) &= \frac{(t^{3q} - 1)(t - 1)}{(t^3 - 1)(t^q - 1)} = \frac{(t^q - 1)(t^{2q} + t^q - 1)(t - 1)}{(t - 1)(t^2 + t + 1)(t^q - 1)} \\
&= \frac{t^{2q} +^q +1}{t^2 + t + 1} \\
\Delta_{T(4,q)}(t) &= \frac{(t^{4q} - 1)(t - 1)}{(t^4 - 1)(t^q - 1)} = \frac{(t^q - 1)(t^{3q} + t^{2q} + t^q - 1)(t - 1)}{(t - 1)(t^3 + t^2 + t + 1)(t^q - 1)} \\
&= \frac{t^{3q} + t^{2q} +^q +1}{t^3 + t^2 + t + 1} \\
\Delta_{T(5,q)}(t) &= \frac{(t^{5q} - 1)(t - 1)}{(t^5 - 1)(t^q - 1)} = \frac{(t^q - 1)(t^{4q} + t^{3q} + t^{2q} +^q -1)(t - 1)}{(t - 1)(t^4 + t^3 + t^2 + t + 1)(t^q - 1)} \\
&= \frac{t^{4q} + t^{3q} + t^{2q} +^q +1}{t^4 + t^3 + t^2 + t + 1}
\end{aligned}$$

Proof. First it should be noted that a polynomial of the form $t^{pq} - 1$ can be factored into the form $(t^q - 1)(t^{(p-1)q} + t^{(p-2)q} + \cdots + 1)$. Similarly, $t^p - 1$ can be factored into the form $(t - 1)(t^{p-1} + t^{p-2} + \cdots + 1)$. Then from (3.1) we can apply these factorizations, which yield Equation (3.3).

$$\begin{aligned}
\Delta_{T(p,q)}(t) &= \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \\
&= \frac{(t^q-1)(t^{(p-1)q}+t^{(p-2)q}+\cdots+1)(t-1)}{(t-1)(t^{p-1}+t^{p-2}+\cdots+1)(t^q-1)} \\
&= \frac{(t^{(p-1)q}+t^{(p-2)q}+\cdots+1)}{(t^{p-1}+t^{p-2}+\cdots+1)} \\
&= \frac{f(t^q)}{f(t)} \text{ where } f(t) = t^{p-1}+t^{p-2}+\cdots+1 \qquad (3.3)
\end{aligned}$$

□

It should be noted that the braid expression for torus knots can be generalized to $(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$, but since $T(p,q) = T(q,p)$, it can also be written as $(\sigma_1\sigma_2\cdots\sigma_{q-1})^p$. Thus, with this second form we see that $T(p,q)$ knots have braids of the form w^p where w is the braid $\sigma_1\sigma_2\cdots\sigma_{q-1}$. This is interesting, because the Alexander polynomial turns out to be $\Delta_{T(p^n,q)} = f(t^{p^{n-1}})f(t^{p^{n-2}})\cdots f(t)$.

Theorem 3.2. *The Alexander polynomial of the torus knot has the factorization* $\Delta_{T(p^n,q)} = f(t)f(t^p)f(t^{p^2})\cdots f(t^{p^{n-2}})f(t^{p^{n-1}})$ *for* $p,q \geq 2$ *and* p,q *relatively prime.*

Proof. Proof by induction. The beginning function is $f(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$. We want to show that $\Delta_{T(p^2,q)}(t) = f(t)f(t^p)$ as the base case for when $n=2$ (since $n=1$ is the initial function).

$$\begin{aligned}
f(t)\cdot f(t^p) &= \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \cdot \frac{(t^{p^2q}-1)(t^p-1)}{(t^{p^2}-1)(t^{pq}-1)} \\
&= \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \\
&= \Delta_{T(p^2,q)}(t)
\end{aligned}$$

Next we want to show that $\Delta_{T(p^n,q)} = f(t^{p^{n-1}})f(t^{p^{n-2}})\cdots f(t)$ implies $\Delta_{T(p^{n+1},q)} = f(t^{p^n})f(t^{p^{n-1}})\cdots f(t)$.

$$\begin{aligned} f(t)\cdots f(t^{p^{n-1}})f(t^p) &= \Delta_{T(p^n,q)}(t)\cdot f(t^p) \\ &= \frac{(t^{p^n q}-1)(t-1)}{(t^{p^n}-1)(t^q-1)} \cdot \frac{(t^{p^{n+1}q}-1)(t^{p^n}-1)}{(t^{p^{n+1}}-1)(t^{p^n q}-1)} \\ &= \frac{(t^{p^{n+1}q}-1)(t-1)}{(t^{p^{n+1}}-1)(t^q-1)} \\ &= \Delta_{T(p^{n+1},q)}(t) \end{aligned}$$

Therefore $\Delta_{T(p^{n+1},q)}(t) = f(t)\cdots f(t^{p^{n-1}})f(t^{p^n})$. □

3.0.6 Conway polynomial

D. Rowland noticed several patterns in Rowland (2008) for $(2,q)$-torus links. It should be noted that when q is odd a torus knot is formed, and when q is even a torus link is formed. The first pattern found was that the degree of the Conway polynomial is $(n-1)$, and it is *monic* meaning that the highest term has 1 as its coefficient. Interestingly, the polynomials form Pascal's triangle along the diagonals when lined up as seen in Figure 3.2. It can also be seen that these equations form the sequence of Fibonacci polynomials where $F_1(x) = 1$, $F_2(x) = x$ and $F_n = xF_{n-1} + F_{n-2}$, and when evaluated at $z = 1$ they form the Fibonacci sequence. This led to the formulation of Equation (3.4) for the Conway polynomial of $(2, 2n+1)$-torus knots.

$$\nabla_{T(2,2n+1)}(z) = \binom{n}{0} + \binom{n+1}{2}z^2 + \cdots + \binom{2n}{2n}z^{2n} = \sum_{j=0}^{n}\binom{n+j}{2j}z^{2j} \tag{3.4}$$

As was mentioned in the introduction there is an identity $\Delta_L(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \nabla_L(z)$ to convert the Alexander polynomial into the Conway polynomial. This relationship is simple for converting from the Conway polynomial to the Alexander polynomial, since all that is required is substituting $t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ for z, but as is shown in Example

$$
\begin{aligned}
\nabla_{T(2,1)} &= 1 \\
\nabla_{T(2,2)} &= \ \ 1z \\
\nabla_{T(2,3)} &= 1 + 1z^2 \\
\nabla_{T(2,4)} &= \ \ 2z + 1z^3 \\
\nabla_{T(2,5)} &= 1 + 3z^2 + 1z^4 \\
\nabla_{T(2,6)} &= \ \ 3z + 4z^3 + 1z^5 \\
\nabla_{T(2,7)} &= 1 + 6z^2 + 5z^4 + 1z^6 \\
\nabla_{T(2,8)} &= \ \ 4z + 10z^3 + 6z^5 + 1z^7 \\
\nabla_{T(2,9)} &= 1 + 10z^2 + 15z^4 + 7z^6 + 1z^8
\end{aligned}
$$

Figure 3.2: The Conway polynomial of $(2, q)$-torus links

3.3, this method is complicated when converting from the Alexander polynomial to the Conway polynomial.

Example 3.3. *Here is an example converting the Alexander polynomial of the* $(2, 3)$-*torus knot to the Conway polynomial.*

$$
\begin{aligned}
\Delta_{T(2,3)}(t) &= t + t^{-1} - 1 \\
&= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 + 2 - 1 \\
&= z^2 + 1 \\
&= \nabla_{T(2,3)}(z)
\end{aligned}
$$

A more complicated example can be shown with the $(2,5)$*-torus knot.*

$$\begin{aligned}\Delta_{T(2,5)}(t) &= t^2 + t^{-2} - t - t^{-1} + 1\\ &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^4 + 4t + 4t^{-1} - 6 - t - t^{-1} + 1\\ &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^4 + 3(t + t^{-1}) - 5\\ &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^4 + 3(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 + 6 - 5\\ &= z^4 + 3z^2 + 1\\ &= \nabla_{T(2,5)}(z)\end{aligned}$$

It is easy to see that as polynomials grow larger, the substitution becomes more complicated with middle terms that must be taken care of. This led to the motivation to find a conversion using the form $t^n + t^{-n}$ since the Alexander polynomial is symmetric across t.

The *Lucas polynomial* is a variation of the Fibonacci polynomial where $L_0 = 2$, $L_1 = x$, and $L_n = xL_{n-1} + L_{n-2}$. There is also a closed form for the Lucas polynomial defined by Equation (3.5) Weisstein (2012b).

$$L_n(x) = 2^{-n}[(x - \sqrt{x^2+4})^n + (x + \sqrt{x^2+4})^n] \tag{3.5}$$

Theorem 3.4. *There exists a relationship between the Alexander and Conway polynomials such that the substitution* $t^n + t^{-n} = L_{2n}(z)$*, where* L_n *is the Lucas polynomial, into the Alexander polynomial yields the Conway polynomial.*

Proof. Using the skein relationship $t^{\frac{1}{2}} - t^{-\frac{1}{2}} = z$, we can solve for t using the quadratic equation and get $t^{\pm 1} = \frac{z^2+2\pm z\sqrt{z^2+4}}{2}$. Then using the equation for the

Lucas polynomial,

$$
\begin{aligned}
L_{2n}(z) &= 2^{-2n}[(z-\sqrt{z^2+4})^{2n}+(z+\sqrt{z^2+4})^{2n}] \\
&= \left(\frac{z-\sqrt{z^2+4}}{2}\right)^{2n}+\left(\frac{z+\sqrt{z^2+4}}{2}\right)^{2n} \\
&= \left(\frac{(z-\sqrt{z^2+4})^2}{4}\right)^{n}+\left(\frac{(z+\sqrt{z^2+4})^2}{4}\right)^{n} \\
&= \left(\frac{2z^2+4-2z\sqrt{z^2+4}}{4}\right)^{n}+\left(\frac{2z^2+4+2z\sqrt{z^2+4}}{4}\right)^{n} \\
&= \left(\frac{z^2+2-z\sqrt{z^2+4}}{2}\right)^{n}+\left(\frac{z^2+2+z\sqrt{z^2+4}}{2}\right)^{n} \\
&= t^n+t^{-n}
\end{aligned}
$$

□

Example 3.5. *Below is an example of a conversion from the Alexander polynomial to the Conway polynomial using the substitution* $t^n - t^{-n} = L_{2n}(z)$*.*

$$
\begin{aligned}
\Delta_{T(2,3)}(t) &= t+t^{-1}-1 \\
&= L_2(z)-1 \\
&= z^2+2-1 \\
&= z^2+1 \\
&= \nabla_{T(2,3)}(z)
\end{aligned}
$$

A more complicated example made easier can be shown with the $(2,5)$-torus knot.

$$\begin{aligned}\Delta_{T(2,5)}(t) &= t^2 + t^{-2} - t - t^{-1} + 1 \\ &= L_4(z) - L_2(z) + 1 \\ &= z^4 + 4z^2 + 2 - (z^2 + 2) + 1 \\ &= z^4 + 3z^2 + 1 \\ &= \nabla_{T(2,5)}(z)\end{aligned}$$

Theorem 3.6. *The Conway polynomial of $(2^n, q)$-torus knots factors into the form $\nabla_{T(2^n,q)}(z) = K_1 K_2 \cdots K_n$ where $K_1 = F_q(z)$, $F_q(z)$ being the Fibonacci polynomial, and $K_i(z) = K_{i-1}(\sqrt{z^4 + 4z^2})$.*

Proof. From Theorem 3.2 we know that the Alexander polynomial of $(2^n, q)$-torus knots factor into the form $\Delta_{T(2^n,q)}(t) = f(t)f(t^2)f(t^4)\cdots f(t^{2^{n-1}})$. Thus, it suffices to show that $L_{4n}(z) = L_{2n}(\sqrt{z^4 + 4z^2})$.

$$L_{n=2j}(\sqrt{z^4+4z^2}) =$$

$$=2^{-2j}\left(\left(\sqrt{z^4+4z^2}-\sqrt{\sqrt{z^4+4z^2}^2+4}\right)^{2j}\right.$$

$$\left.+\left(\sqrt{z^4+4z^2}+\sqrt{\sqrt{z^4+4z^2}^2+4}\right)^{2j}\right)$$

$$=2^{-2j}\left(\left(\sqrt{z^4+4z^2}-\sqrt{z^4+4z^2+4}\right)^{2j}+\left(\sqrt{z^4+4z^2}+\sqrt{z^4+4z^2+4}\right)^{2j}\right)$$

$$=2^{-2j}\left(\left(\sqrt{z^4+4z^2}-\sqrt{(z^2+2)^2}\right)^{2j}+\left(\sqrt{z^4+4z^2}+\sqrt{(z^2+2)^2}\right)^{2j}\right)$$

$$=2^{-2j}\left(\left(\sqrt{z^4+4z^2}-(z^2+2)\right)^{2j}+\left(\sqrt{z^4+4z^2}+(z^2+2)\right)^{2j}\right)$$

$$=2^{-4j}2^{2j}\left(\left(\sqrt{z^4+4z^2}-(z^2+2)\right)^{2j}+\left(\sqrt{z^4+4z^2}+(z^2+2)\right)^{2j}\right)$$

$$=2^{-4j}\left(2^{2j}\left(\sqrt{z^4+4z^2}-(z^2+2)\right)^{2j}+2^{2j}\left(\sqrt{z^4+4z^2}+(x^2+2)\right)^{2j}\right)$$

$$=2^{-4j}\left(\left(2\sqrt{z^4+4z^2}-2z^2-4\right)^{2j}+\left(2\sqrt{z^4+4z^2}+2z^2+4\right)^{2j}\right)$$

$$=\left(\left(-2\sqrt{z^4+4z^2}+2z^2+4\right)^{2j}+\left(2\sqrt{z^4+4z^2}+2z^2+4\right)^{2j}\right)$$

$$=2^{-4j}[(2z^2-2\sqrt{z^2+4}z+4)^{2j}+(2z^2+2\sqrt{z^2+4}z+4)^{2j}]$$

$$=2^{-4j}[(z-\sqrt{z^2+4})^{4j}+(x+\sqrt{z^2+4})^{4j}]$$

$$=L_{2n=4j}$$

$\square$

Bibliography

Bibliography

Adams, C. C. (2004). *The knot book*. American Mathematical Society, Providence, RI. An elementary introduction to the mathematical theory of knots, Revised reprint of the 1994 original. vii, 2, 3, 4

Bigelow, S. (1999). The burau representation is not faithful for n = 5. *Geom. Topol. 3*, pages 397–404. 4

Collins, J. (2011). *On the concordance orders of knots*. PhD thesis, University of Edinburgh. 7, 8

Conant, J. (2006). Chirality and the conway polynomial. *Topology Proc. 30 (2006), no. 1*, pages 153–162. 6

Cromwell, P. R. (2004). *Knots and links*. Cambridge University Press, Cambridge. 3

Ermotti, N., Hongler, C. V. Q., and Weber, C. (2011). On the kawauchi conjecture about the conway polynomial of achiral knots. pages 1–8. vii, 6, 7

Livingston, C. (1993). *Knot theory*, volume 24 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC. 3, 9, 10

Rowland, D. (2008). Twisting with fibonacci. 12

Weisstein, E. W. (2012a). Burau representation. 4

Weisstein, E. W. (2012b). Lucas polynomial.

Weisstein, E. W. (2012c). Ribbon knot. 8

Vita

Katherine Agle was born in Oak Ridge, Tennessee, the oldest daughter of Chuck and Angi Agle. She attended Linden Elementary, Robertsville Middle, and Oak Ridge High School. After graduation, she matriculated to the University of Tennessee, where she earned a Bachelor of Science in Mathematics in May in 2010. During her undergraduate study, she participated in summer research internships at the University of Georgia and Cal State San Bernardino. It was through the second program that she developed a passion for knot theory, which led to her presentation at the Joint Mathematical Meetings in San Francisco in the Spring of 2010. In 2012, she completed her Master of Science in Mathematics at the University of Tennessee, under the expert guidance of Professors Conant, Hinton, and Thistlethwaite.